Problem 1 (Explaining Away, 10 pts)

In this problem, you will carefully work out a basic example with the "explaining away" effect. There are many derivations of this problem available in textbooks. We emphasize that while you may refer to textbooks and other online resources for understanding how to do the computation, you should do the computation below from scratch, by hand. Show your work.

We have three binary variables, rain R, grass-wet G, and sprinkler S. We assume the following factorization of the joint distribution:

$$\Pr(R, S, G) = \Pr(R)\Pr(S)\Pr(G \mid R, S).$$

The conditional probability tables look like the following:

$$\begin{aligned}
\Pr(R = 1) &= 0.25 \\
\Pr(S = 1) &= 0.5 \\
\Pr(G = 1|R = 0, S = 0) &= 0 \\
\Pr(G = 1|R = 1, S = 0) &= .75 \\
\Pr(G = 1|R = 0, S = 1) &= .75 \\
\Pr(G = 1|R = 1, S = 1) &= 1
\end{aligned}$$

1. Draw the graphical model corresponding to the factorization of the joint distribution. Are R and S independent? [Feel free to use facts you have learned about studying independence in graphical models.]

2. You check on the sprinkler without checking on the rain or the grass. What is the probability that the sprinkler is on?

3. You notice it is raining and check on the sprinkler without checking the grass. What is the probability that the sprinkler is on?

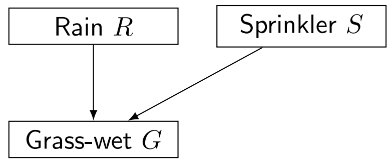
4. You notice that the grass is wet and go to check on the sprinkler (without checking if it is raining). What is the probability that the sprinkler is on?

5. You notice that it is raining and the grass is wet. You go check on the sprinkler. What is the probability that the sprinkler is on?

6. What is the "explaining away" effect that is shown above?

Solution

1. According to the factorization of the joint distribution, R and S are both parents of G, so the graphical model is represented as



According to this graphical model, R and S are independent if G is not observed.

2. The probability that the sprinkler is on is $\Pr(S=1) = 0.5$

3. We are seeking $\Pr(S=1|R=1)$. We can rewrite this as

$$\Pr(S=1|R=1) = \frac{\Pr(S=1,R=1)}{\Pr(R=1)} = \frac{\Pr(S=1,R=1,G=1)+\Pr(S=1,R=1,G=0)}{0.25}$$

Using the assumed joint distribution factorization, we find that

$$\Pr(S=1,R=1,G=1) = \Pr(R=1)\Pr(S=1)\Pr(G=1|R=1,S=1) = 0.25(0.5)(1) = 0.125$$

$$\Pr(S=1,R=1,G=0) = \Pr(R=1)\Pr(S=1)\Pr(G=0|R=1,S=1) = 0.25(0.5)(0) = 0$$

Therefore $\Pr(S=1|R=1) = \frac{0.125+0}{0.25} = 0.5$

4. We are seeking $\Pr(S=1|G=1)$. We can rewrite this as

$$\Pr(S=1|G=1) = \frac{\Pr(S=1,G=1)}{\Pr(G=1)} = \frac{\Pr(S=1,G=1,R=1)+\Pr(S=1,G=1,R=0)}{\Pr(S=1,G=1)+\Pr(S=0,G=1)}$$

$$= \frac{\Pr(S=1,G=1,R=1)+\Pr(S=1,G=1,R=0)}{\Pr(R=1,S=1,G=1)+\Pr(R=0,S=1,G=1)+\Pr(R=1,S=0,G=1)+\Pr(R=0,S=0,G=1)}$$

We can use the assumed joint factorization to simplify all of these joint probabilities:

$$\Pr(S=1,R=1,G=1) = \Pr(R=1)\Pr(S=1)\Pr(G=1|R=1,S=1) = 0.25(0.5)(1) = 0.125$$

$$\Pr(S=1,G=1,R=0) = \Pr(R=0)\Pr(S=1)\Pr(G=1|R=0,S=1) = 0.75(0.5)(0.75) = 0.28125$$

$$\Pr(R=1,S=0,G=1) = \Pr(R=1)\Pr(S=0)\Pr(G=1|R=1,S=0) = 0.25(0.5)(0.75) = 0.09375$$

$$\Pr(R=0,S=0,G=1) = \Pr(R=0)\Pr(S=0)\Pr(G=1|R=0,S=0) = 0.75(0.5)(0) = 0$$

Therefore $\Pr(S=1|G=1) = \frac{0.125+0.28125}{0.125+0.28125+0.09375+0} = 0.8125$

5. We are seeking $\Pr(S=1|R=1,G=1)$. We can rewrite this as

$$\Pr(S=1|R=1,G=1) = \frac{\Pr(S=1,R=1,G=1)}{\Pr(R=1,G=1)} = \frac{\Pr(S=1,R=1,G=1)}{\Pr(S=1,R=1,G=1)+\Pr(S=0,R=1,G=1)}$$

We can use the assumed joint factorization to simplify all of these joint probabilities:

$$\Pr(S=1,R=1,G=1) = \Pr(R=1)\Pr(S=1)\Pr(G=1|R=1,S=1) = 0.25(0.5)(1) = 0.125$$

$$\Pr(R=1,S=0,G=1) = \Pr(R=1)\Pr(S=0)\Pr(G=1|R=1,S=0) = 0.25(0.5)(0.75) = 0.09375$$

Therefore $\Pr(S=1|R=1,G=1) = \frac{0.125}{0.125+0.09375} \approx 0.571$

6. From parts 2, 4, and 5, we found that $\Pr(S = 1) = 0.5$, $\Pr(S = 1|G = 1) = 0.8125$, and $\Pr(S = 1|R = 1, G = 1) = 0.571$. This demonstrates the "explaining away" effect in the same way as was done in Lecture 18. Observing that the grass is wet (without observing any information about the rain) increases the probability of the sprinklers being on because the sprinklers are a likely explanation for why the grass is wet. However, observing that it is raining in addition to observing that the grass is wet lowers the probability that the sprinklers are on almost back to the prior (marginal) probability of $\Pr(S = 1) = 0.5$. The fact that it is raining "explains away" the effect of the sprinklers being on, because observing that it is raining explains why we observe that the grass is wet, reducing the chance that the sprinklers are on. Moreover, the results of problem 3 demonstrate the d-separation rule that when the grass is unobserved the sprinklers and the rain are conditionally independent, since $\Pr(S = 1) = \Pr(S = 1|R = 1)$.