Q1a)

$$Z = \frac{4+3i}{3-4i} = \frac{(4+3i)(3+4i)}{(3-4i)(3+4i)}$$

$$= \frac{12+16i+9i+12i^2}{9-16i^2}$$

$$= \frac{25i}{25}$$

$$= i$$

$\therefore Re(Z) = 0$, $Im(Z) = 1$

Q1b) $Z = e^{i\theta} - e^{-i\theta}$

$$= (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)$$

$$= i2\sin\theta$$

$\therefore Re(Z) = 0$, $Im(Z) = 2\sin\theta$

Q1c) $Z = e^{i2\theta} \cdot e^{-i\theta}$

$$= e^{i\theta}$$

$$= \cos\theta + i\sin\theta$$

$\therefore Re(Z) = \cos\theta$, $Im(Z) = \sin\theta$

Q1d) $Z = \frac{2e^{i\theta}(e^{i\theta}-1)}{3e^{i\theta}-1} = \frac{2e^{i\theta}(e^{i\theta}-1)\cdot(3e^{-i\theta}-1)}{(3e^{i\theta}-1)\cdot(3e^{-i\theta}-1)}$

$$= \frac{2(e^{i\theta}-1)\cdot(3-e^{i\theta})}{9-3e^{i\theta}-3e^{-i\theta}+1}$$

$$= \frac{2(3e^{i\theta}-3-e^{i2\theta}+e^{i\theta})}{10-3(\cos\theta+i\sin\theta)-3(\cos\theta-i\sin\theta)}$$

$$= \frac{2[4e^{i\theta}-e^{i2\theta}-3]}{10-6\cos\theta}$$

$$= \frac{4(\cos\theta + i\sin\theta) - (\cos 2\theta + i\sin 2\theta) - 3}{5 - 3\cos\theta}$$

$$= \frac{(4\cos\theta - \cos 2\theta - 3) + i(4\sin\theta - \sin 2\theta)}{5 - 3\cos\theta}$$

$$\therefore Re(z) = \frac{4\cos\theta - \cos 2\theta - 3}{5 - 3\cos\theta}$$

$$Im(z) = \frac{4\sin\theta - \sin 2\theta}{5 - 3\cos\theta}$$

Q2a) $y_{j+1} = y_{j-1}$

$\Rightarrow \quad y_{j+1} - y_{j-1} = 0$

$\Rightarrow \quad (E^2 - 1)\, y_{j-1} = 0$

$\therefore$ The characteristic equation is

$$\xi^2 - 1 = 0$$

$\Rightarrow \quad \xi_{1,2} = \pm 1$

$\Rightarrow \quad y_j = c_1 (1)^j + c_2 (-1)^j = c_1 + c_2 (-1)^j$

Q2b) $y_{j+1} = 4y_j - 3y_{j-1}$

$\Rightarrow \quad y_{j+1} - 4y_j + 3y_{j-1} = 0$

$\Rightarrow \quad (E^2 - 4E + 3)\, y_{j-1} = 0$

$\Rightarrow \quad \xi^2 - 4\xi + 3 = 0$

$\Rightarrow \quad (\xi - 1)(\xi - 3) = 0$

$\Rightarrow \quad \xi_1 = 1, \ \xi_2 = 3$

$\Rightarrow \quad y_j = c_1 (1)^j + c_2 (3)^j$

$\Rightarrow \quad y_j = c_1 + c_2 (3)^j$

Q2c) $y_{j+1} = 2y_{j-1} - y_j$

$\Rightarrow \quad y_{j+1} + y_j - 2y_{j-1} = 0$

$\Rightarrow \quad (E^2 + E - 2)\, y_j = 0$

$\Rightarrow \quad \xi^2 + \xi - 2 = 0$

$\Rightarrow \quad (\xi + 2)(\xi - 1) = 0$

$\Rightarrow \quad \xi_1 = -2, \ \xi_2 = 1$

$\Rightarrow \quad y_j = c_1 (-2)^j + c_2 (1)^j$

$\Rightarrow \quad y_j = c_1 (-2)^j + c_2$

Q2d) $y_{j+1} + 9y_j - 9y_{j-1} - y_{j-2} = 0$

$\Rightarrow \quad (E^3 + 9E^2 - 9E - 1)\, y_{j-2} = 0$

$\Rightarrow \quad \xi^3 + 9\xi^2 - 9\xi - 1 = 0$

$\Rightarrow \quad (\xi^3 - 1) + (9\xi^2 - 9\xi) = 0$

$$\Rightarrow (\xi-1)(\xi^2+\xi+1)+9\xi(\xi-1)=0$$
$$\Rightarrow (\xi-1)(\xi^2+10\xi+1)=0$$
$$\Rightarrow \xi_1=1,\quad \xi_{2,3}=\frac{-10\pm\sqrt{10^2-4(1)(1)}}{2}=-5\pm2\sqrt{6}$$

$$\Rightarrow y_j=C_1(1)^j+C_2(-5+2\sqrt{6})^j+C_3(-5-2\sqrt{6})^j$$
$$\Rightarrow y_j=C_1+C_2(-5+2\sqrt{6})^j+C_3(-5-2\sqrt{6})^j$$

Q3a) $y_{n+2}-9y_{n+1}+20y_n=0$

$$\Rightarrow (E^2-9E+20)y_n=0$$
$$\Rightarrow \xi^2-9\xi+20=0$$
$$\Rightarrow (\xi-4)(\xi-5)=0$$
$$\Rightarrow \xi_1=4,\quad \xi_2=5$$
$$\Rightarrow y_n=C_1(4)^n+C_2(5)^n$$

Q3b) $y_{n+2}+y_n+y_{n-1}=0$

$$\Rightarrow (E^2+E+1)y_{n-1}=0$$
$$\Rightarrow \xi^2+\xi+1=0$$
$$\Rightarrow \xi_{1,2}=\frac{-1\pm\sqrt{1^2-4(1)(1)}}{2}=\frac{-1\pm\sqrt{3}i}{2}$$
$$\Rightarrow y_n=C_1\left(\frac{-1+\sqrt{3}i}{2}\right)^n+C_2\left(\frac{-1-\sqrt{3}i}{2}\right)^n$$

Q3c) $y_{n+2}=\dfrac{y_{n+1}+y_{n-1}}{2}$

$$\Rightarrow 2y_{n+2}-y_{n+1}+0y_n-y_{n-1}=0$$
$$\Rightarrow (2E^3-E^2+0E-1)y_{n-1}=0$$
$$\Rightarrow 2\xi^3-\xi^2-1=0$$
$$\Rightarrow (\xi^3-\xi^2)+(\xi^3-1)=0$$
$$\Rightarrow \xi^2(\xi-1)+(\xi-1)(\xi^2+\xi+1)=0$$
$$\Rightarrow (\xi-1)(2\xi^2+\xi+1)=0$$
$$\Rightarrow \xi_1=1,\quad \xi_{2,3}=\frac{-1\pm\sqrt{1^2-4(2)(1)}}{2(2)}=\frac{-1\pm\sqrt{7}i}{4}$$

$$\Rightarrow y_n = c_1(1)^n + c_2\left(\frac{-1+\sqrt{7}i}{4}\right)^n + c_3\left(\frac{-1-\sqrt{7}i}{4}\right)^n$$

$$\Rightarrow y_n = c_1 + c_2\left(\frac{-1+\sqrt{7}i}{4}\right)^n + c_3\left(\frac{-1-\sqrt{7}i}{4}\right)^n$$

Q3d) $y_{n+2} = -\dfrac{y_{n+1}+y_{n-1}}{2}$

$\Rightarrow 2y_{n+2} + y_{n+1} + 0y_n + y_{n-1} = 0$

$\Rightarrow (2E^3 + E^2 + 0E + 1)y_{n-1} = 0$

$\Rightarrow 2\xi^3 + \xi^2 + 1 = 0$

$\Rightarrow (\xi^3 + \xi^2) + (\xi^3 + 1) = 0$

$\Rightarrow \xi^2(\xi+1) + (\xi+1)(\xi^2 - \xi + 1) = 0$

$\Rightarrow (\xi+1)(2\xi^2 - \xi + 1) = 0$

$\Rightarrow \xi_1 = -1, \quad \xi_{2,3} = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(1)}}{2(2)} = \dfrac{1 \pm \sqrt{7}i}{4}$

$$\Rightarrow y_n = c_1(-1)^n + c_2\left(\frac{1+\sqrt{7}i}{4}\right)^n + c_3\left(\frac{1-\sqrt{7}i}{4}\right)^n$$

Q3e) $y_{n+4} - 16y_n = 0$

$\Rightarrow (E^4 - 16)y_n = 0$

$\Rightarrow \xi^4 - 16 = 0$

$\Rightarrow (\xi^2 - 4)(\xi^2 + 4) = 0$

$\Rightarrow (\xi-2)(\xi+2)(\xi+2i)(\xi-2i) = 0$

$\Rightarrow \xi_{1,2} = \pm 2, \quad \xi_{3,4} = \pm 2i = 2e^{\pm i\frac{\pi}{2}}$

$\Rightarrow y_n = c_1(2)^n + c_2(-2)^n + 2^n\left(c_3 \cos\frac{n\pi}{2} + c_4 \sin\frac{n\pi}{2}\right)$

Q3f) $y_{n+4} + 16y_n = 0$

$\Rightarrow (E^4 + 16)y_n = 0$

$\Rightarrow \xi^4 + 16 = 0$

$\Rightarrow \xi^4 = -16 = 16e^{i\pi}$

Let $\xi = re^{i\theta}$, $r > 0$, $\theta \in [0, 2\pi]$

$$\Rightarrow \begin{cases} \xi^4 = r^4 e^{i4\theta} \\ \xi^4 = 16e^{i\pi} \end{cases}$$

$$\Rightarrow \begin{cases} r^4 = 16 \\ 4\theta = (2k+1)\pi \end{cases} \Rightarrow \theta = \frac{2k+1}{4}\pi \quad , k \in \mathbb{Z}$$

$$\Rightarrow \quad r = 2$$

$$\left.\begin{aligned} k=0, \quad \theta &= \tfrac{1}{4}\pi \\ k=1, \quad \theta &= \tfrac{3}{4}\pi \\ k=2, \quad \theta &= \tfrac{5}{4}\pi \\ k=3, \quad \theta &= \tfrac{7}{4}\pi \end{aligned}\right\} \begin{aligned} &\Rightarrow \xi_1 = \sqrt{2} + \sqrt{2}i \\ &\Rightarrow \xi_3 = -\sqrt{2} + \sqrt{2}i \\ &\Rightarrow \xi_4 = -\sqrt{2} - \sqrt{2}i \\ &\Rightarrow \xi_2 = \sqrt{2} - \sqrt{2}i \end{aligned} \Bigg\} \Rightarrow \begin{cases} \xi_{1,2} = \sqrt{2} \pm \sqrt{2}i \\ \xi_{3,4} = -\sqrt{2} \pm \sqrt{2}i \end{cases}$$

$$k=4, \quad \theta = \tfrac{9}{4}\pi = 2\pi + \tfrac{1}{4}\pi$$

$$\Rightarrow \xi_{1,2} = 2e^{\pm i\frac{\pi}{4}} \quad , \xi_{3,4} = 2e^{\pm i\frac{3\pi}{4}}$$

$$\Rightarrow y_n = 2^n\left(c_1 \cos\frac{n\pi}{4} + c_2 \sin\frac{n\pi}{4}\right) + 2^n\left(c_3 \cos\frac{3n\pi}{4} + c_4 \sin\frac{3n\pi}{4}\right)$$

Q4) $y_n = y_{n-1} + y_{n-2} \quad , y_0 = 0, \quad y_1 = 1$

$$\Rightarrow \quad y_n - y_{n-1} - y_{n-2} = 0$$

$$\Rightarrow \quad (E^2 - E - 1)y_{n-2} = 0$$

$$\Rightarrow \quad \xi^2 - \xi - 1 = 0$$

$$\Rightarrow \quad \xi_{1,2} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

$$\Rightarrow \quad y_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\begin{cases} y_0 = c_1 + c_2 = 0 \\ y_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{cases}$$

$$\Rightarrow \quad c_1\left(\frac{1+\sqrt{5}}{2}\right) - c_1\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$\Rightarrow \quad c_1 = \frac{1}{\sqrt{5}} \quad , c_2 = -\frac{1}{\sqrt{5}}$$

$$\Rightarrow \quad y_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Q5) $y_{n+2} - 2a y_{n+1} + a^2 y_n = 0 \qquad (\ast)$

$y_n = c_1 a^n + c_2 n a^n \qquad (\triangle)$

Proof: Substitute $(\triangle)$ into $(\ast)$

$$[c_1 a^{n+2} + c_2 (n+2) a^{n+2}] - 2a[c_1 a^{n+1} + c_2 (n+1) a^{n+1}] + a^2 [c_1 a^n + c_2 n a^n]$$

$$= c_1 [a^{n+2} - 2a \cdot a^{n+1} + a^2 \cdot a^n] + c_2 [(n+2) a^{n+2} - 2a (n+1) a^{n+1} + a^2 \cdot n a^n]$$

$$= c_1 [a^{n+2} - 2a^{n+2} + a^{n+2}] + c_2 \cdot a^{n+2} \cdot [(n+2) - 2(n+1) + n]$$

$$= 0$$

Q.E.D.