Ex 2.4

Binomial expansion

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

where $\binom{n}{i} = \dfrac{n!}{i!(n-i)!} = \dfrac{n(n-1)(n-2)\cdots(n-i+1)}{i!}$

Binomial expansion with negative exponent

$$(x+y)^{-n} = \sum_{i=0}^{\infty} \binom{-n}{i} x^{-n-i} y^i$$

where: $\binom{-n}{i} = \dfrac{(-n)(-n-1)(-n-2)\cdots(-n-i+1)}{i!} = (-1)^i \dfrac{n(n+1)(n+2)\cdots(n+i-1)}{i!}$

The Gregory-Newton backward formula is given by

$$y_{j+s} = (1-\nabla)^{-s} y_j$$

Supposing s is an integer,

$$\begin{aligned}
(1-\nabla)^{-s} &= \sum_{i=0}^{\infty} \binom{-s}{i} (1)^{-s-i} (-\nabla)^i \\
&= \sum_{i=0}^{\infty} \binom{-s}{i} (-1)^i \nabla^i \\
&= \sum_{i=0}^{\infty} (-1)^i \frac{s(s+1)(s+2)\cdots(s+i-1)}{i!} (-1)^i \nabla^i \\
&= \sum_{i=0}^{\infty} \frac{s(s+1)(s+2)\cdots(s+i-1)}{i!} \nabla^i \\
&= 1 + s\nabla + \frac{s(s+1)}{2!}\nabla^2 + \frac{s(s+1)(s+2)}{3!}\nabla^3 + \cdots + \frac{s(s+1)(s+2)\cdots(s+k-1)}{k!}\nabla^k + \cdots
\end{aligned}$$

Note: $\binom{-s}{0}(-1)^0 \nabla^0 = 1$

$$\Rightarrow y_{j+s} = (1-\nabla)^{-s} y_j$$

$$= y_j + s\nabla y_j + \frac{s(s+1)}{2!}\nabla^2 y_j + \frac{s(s+1)(s+2)}{3!}\nabla^3 y_j + \cdots + \frac{s(s+1)\cdots(s+k-1)}{k!}\nabla^k y_j + \cdots$$