

PRELIMINARIES

CSC311 FALL 2019

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1. Some definitions.

- **Functions.** We use $f : \mathbb{R}^d \rightarrow \mathbb{R}$ to denote that f is a function, its argument is in \mathbb{R}^d and its output is real valued (or in \mathbb{R}). We denote by $\nabla f(x) \in \mathbb{R}^d$, its gradient (See Section 3 for definition).
- **ℓ_p -norms.** Since we are mostly dealing with vectors in machine learning, we will use different norms a lot. Euclidean norm, denoted as $\|\cdot\|_2$, is the most commonly used norm. But we can also use ℓ_p norms which are defined as

$$(1.1) \quad \|x\|_p = \left(\sum_{i=1}^d |x_i|^p \right)^{1/p}.$$

When we drop the subscript and use $\|\cdot\|$, this typically means the Euclidean norm $\|\cdot\|_2$.

- **Indicator function.** The indicator function is defined as $\delta(a, b) = 1$ if $a = b$, and $\delta(a, b) = 0$ if $a \neq b$. For example, $\delta(2, 2) = 1$ and $\delta(2, 2.1) = 0$.
- **argmin & argmax.** Assume that we are trying to find the point in \mathbb{R}^d that minimizes $f(x)$. This point is denoted by $x_* = \operatorname{argmin}_x f(x)$. In general, there can be many points that minimize the function $f(x)$. If this is the case, argmin function returns a set of minimizers, and notation is slightly different $x_* \in \operatorname{argmin}_x f(x)$. The function argmax is similar. For example, given a vector $a \in \mathbb{R}^d$, let $f(x) = \|x - a\|_2$ be a function. Then,

$$(1.2) \quad a = \operatorname{argmin}_x f(x).$$

Another example is that we have a binary classification problem over $\{0, 1\}$ and we are using a decision tree to solve it. We want to predict the class assignment of a region with 5 samples $t_1 = 1, t_2 = 1, t_3 = 0, t_4 = 0, t_5 = 1$. Then the majority assignment based on these samples can be found by

$$(1.3) \quad \operatorname{argmax}_{t \in \{0, 1\}} \sum_{i=1}^5 \delta(t, t_i) = 1,$$

because the summation is equal to 3 when $t = 1$, and 2 when $t = 0$.

2. Random variables and vectors. Random vectors are simply vectors with each coordinate a random variable. You can think of a d -dimensional random vector $X \in \mathbb{R}^d$ as a d random variables X_i 's stacked together to make a vector. Most probability rules we have for random variables also hold for random vectors.

- **Density.** If we have a random vector X and its each coordinate is a continuous random variable, then we can talk about probability density function $p(x) : \mathbb{R}^d \rightarrow \mathbb{R}$ associated with the random vector X . This is defined similar to the one variable case and for a set $A \subset \mathbb{R}^d$, it satisfies $\mathbb{P}(X \in A) = \int_A p(x)dx$.

For example, if we have the multivariate Gaussian random variable with mean $\mu \in \mathbb{R}^d$ and covariance $\Sigma \in \mathbb{R}^{d \times d}$, its density is given as

$$(2.1) \quad p(x|\mu, \Sigma) = \frac{1}{(2\pi)^{d/2} \det(\Sigma)^{1/2}} \exp\left\{-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu)\right\}.$$

See below for definitions of mean and variance in the multivariable setting.

If we have two random vectors $X, Y \in \mathbb{R}^d$ with joint density $p(x, y)$, the standard rules of conditional density apply.

$$(2.2) \quad X|Y \sim p(x|y) = \frac{p(x, y)}{p(y)}.$$

- **Expectation.** If $X \in \mathbb{R}^d$ is a random vector, its expectation

$$\mathbb{E}[X] = \mu \in \mathbb{R}^d$$

is also a vector and it is defined as $\mathbb{E}[X_i] = \mu_i$. Below are some properties.

- For random vectors $X, Y \in \mathbb{R}^d$, and a constant matrix $A \in \mathbb{R}^{d \times d}$, we have

$$\mathbb{E}[X + AY] = \mathbb{E}[X] + A\mathbb{E}[Y].$$

- If we have X_1, X_2, \dots, X_n random vectors with $\mathbb{E}X_i = \mu$, their sample mean has the expectation μ , i.e.,

$$\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n X_i\right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i] = \mu.$$

Note that independence is not required for the above result.

- **Conditional expectation.** For two random vectors X, Y with joint distribution $p(x, y)$, the conditional expectation of $X|Y$ is given by

$$\mathbb{E}[X|Y = y] = \int xp(x|y)dx.$$

This is like fixing the value of the random variable Y , and taking expectation of X after. Note that in the conditional expectation, we integrate out the variable x , but the variable we condition on is not integrated. This is why, $E[X|Y = y]$ is a function of y .

A rule that comes up in bias-variance decomposition is the law of iterated expectation:

$$\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X].$$

This can be easily shown using the properties of the density. That is,

$$\mathbb{E}[\mathbb{E}[X|Y]] = \int \left[\int xp(x|y)dx \right] p(y)dy = \int \int xp(x, y)dxdy = \mathbb{E}[X].$$

- **Variance.** Variance of a random vector $X \in \mathbb{R}^{d \times d}$ is defined as

$$\text{Var}(X) = \mathbb{E}[(X - \mu)(X - \mu)^T] \in \mathbb{R}^{d \times d}.$$

Observe that variance of a random vector is a $d \times d$ matrix and its ij -th entry is given by $\text{Var}(X)_{ij} = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)] \in \mathbb{R}$.

- **Covariance.** Let X, Y be two random vectors in \mathbb{R}^d . Then their covariance is given as

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)^T] \in \mathbb{R}^{d \times d}$$

where μ_X and μ_Y denotes the mean of X and Y respectively.

Now assume that X and Y are independent.

- (a) Their covariance is zero: $\mathbb{E}[(X - \mu_X)(Y - \mu_Y)^T] = \mathbb{E}[(X - \mu_X)]\mathbb{E}[(Y - \mu_Y)^T] = 0$.
- (b) For a constant matrix $A \in \mathbb{R}^{d \times d}$, we have

$$\text{Var}(X + AY) = \text{Var}(X) + A\text{Var}(Y)A^T.$$

Next, assume that we have X_1, X_2, \dots, X_n independent random vectors with mean μ and covariance matrix Σ . Contrary to sample mean above, in this case independence is required.

- (c) Using the above, if $X \sim \mathcal{N}(\mu, \Sigma)$, then $AX \sim \mathcal{N}(A\mu, A\Sigma A^T)$. This follows from the fact that linear transformation of a Gaussian random vector is again Gaussian – a property also used in bivariate Gaussian distributions.

2.1. Maximum likelihood estimator. Assume that we observe N i.i.d. random vectors $\mathcal{D} = \{X_1, X_2, \dots, X_N\}$ from a distribution $p(x|\theta)$. We assume that the distribution function $p(x|\theta)$ is known, but the parameter θ is not known. For example, distribution can be Gaussian, but its mean and variance is unknown.

Using the independence assumption, we write the joint density of N i.i.d. random vectors.

$$p(x_1, x_2, \dots, x_n|\theta) = \prod_{i=1}^N p(x_i|\theta).$$

The main idea behind the maximum likelihood estimation (MLE) is that we plug in our observations \mathcal{D} into the joint density and find the θ value that maximizes this likelihood function. That is,

$$\theta^{\text{MLE}} = \underset{\theta}{\text{argmax}} \prod_{i=1}^N p(X_i|\theta).$$

When finding the maximum of a function, we take its derivative and set it equal to 0. However, derivative of products is not easy to handle; therefore, we first apply the log function which doesn't change the point where the maximum is attained. This transforms the product to a summation. That is,

$$\begin{aligned} \theta^{\text{MLE}} &= \underset{\theta}{\text{argmax}} \prod_{i=1}^N p(X_i|\theta), = \underset{\theta}{\text{argmax}} \log \left(\prod_{i=1}^N p(X_i|\theta) \right), \\ &= \underset{\theta}{\text{argmax}} \sum_{i=1}^N \log p(X_i|\theta). \end{aligned}$$

Now, we can easily take derivatives and find the maximizer.

2.2. Maximum A posteriori Probability. Assume the previous setup that we observe N i.i.d. random vectors $\mathcal{D} = \{X_1, X_2, \dots, X_N\}$ from a distribution $p(x|\theta)$. This time, we will also assume that θ is a random vector and its prior distribution is given by $p(\theta)$. This means that instead of treating the parameter θ as a constant as in MLE, we assume some prior knowledge on θ which comes from $p(\theta)$. Maximum A posteriori Probability (MAP) estimator of θ maximizes the posterior distribution $p(\theta|\text{data})$ obtained by the Bayes rule

$$p(\theta|\text{data}) = \frac{p(\text{data}|\theta)p(\theta)}{p(\text{data})}.$$

Therefore, MAP estimator is given by

$$\theta^{\text{MAP}} = \underset{\theta}{\operatorname{argmax}} p(\theta|\text{data}) = \underset{\theta}{\operatorname{argmax}} p(\text{data}|\theta)p(\theta).$$

In the above maximization problem, we dropped the term in the denominator since it doesn't have the optimization parameter θ in it and doesn't contribute to the minimization problem.

For example in the previous setup, MAP estimator can be written as

$$\begin{aligned} \theta^{\text{MAP}} &= \underset{\theta}{\operatorname{argmax}} p(X_1, \dots, X_N|\theta)p(\theta) = \underset{\theta}{\operatorname{argmax}} p(\theta) \prod_{i=1}^N p(X_i|\theta), \\ &= \underset{\theta}{\operatorname{argmax}} \log p(\theta) + \sum_{i=1}^N \log p(X_i|\theta). \end{aligned}$$

3. Basic multivariable calculus. For a given function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, we denote its partial derivative with respect to its i -th coordinate as $\partial f(x)/\partial x_i \in \mathbb{R}$. Gradient of this function is simply a vector with i -th coordinate $\partial f(x)/\partial x_i \in \mathbb{R}$. That is,

$$(3.1) \quad [\nabla f(x)]_i = \frac{\partial f(x)}{\partial x_i}.$$

The gradient of a function points in the direction of greatest increase, and its magnitude is the rate of increase in that direction. Therefore, when you are minimizing a function, it makes sense to move in the direction opposite to its gradient.

Similarly, we can define the second derivative of the function f , which is generally referred to as the Hessian of f . It is a matrix and its i, j -th entry is given by

$$(3.2) \quad [\nabla^2 f(x)]_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}.$$

Using the above definition, for $x, y \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d \times d}$ we obtain

- (a) the gradient with respect to x of $x^T y$ is y ,
- (b) the gradient with respect to x of $x^T x$ is $2x$,
- (c) the gradient with respect to x of $x^T A x$ is $2Ax$,
- (d) the gradient with respect to x of Ax is A .

In some cases, you can see that the above gradients are transposed. This is a matter of definition. You should check the wikipedia page https://en.wikipedia.org/wiki/Matrix_calculus which contains a very detailed list of rules.

3.1. *Least squares problem.* In the least squares problem, we are given a target vector $\mathbf{t} \in \mathbb{R}^N$, a design matrix $\mathbf{X} \in \mathbb{R}^{N \times D}$. We would like to find the weights \mathbf{w} that minimizes the objective function given by the least squares problem

$$\underset{\mathbf{w}}{\text{minimize}} \mathcal{J}(\mathbf{w}) =: \frac{1}{2} \|\mathbf{t} - \mathbf{X}\mathbf{w}\|_2^2.$$

We know that a minimum occurs at a critical at which the partial derivatives are equal to 0. i.e. $\partial \mathcal{J}(\mathbf{w}) / w_j = 0$ for $j = 1, \dots, D$. This is equivalent to saying the gradient $\nabla \mathcal{J}(\mathbf{w}) = 0$. We can write

$$\mathcal{J}(\mathbf{w}) = \frac{1}{2} \|\mathbf{t}\|_2^2 + \frac{1}{2} \mathbf{w}^\top \mathbf{X}^\top \mathbf{X} \mathbf{w} - \mathbf{t}^\top \mathbf{X} \mathbf{w}.$$

Taking derivative with respect to the vector \mathbf{w} and setting it equal to 0, we obtain

$$\nabla \mathcal{J}(\mathbf{w}) = \mathbf{X}^\top \mathbf{X} \mathbf{w} - \mathbf{X}^\top \mathbf{t} = 0.$$

If $\mathbf{X}^\top \mathbf{X}$ is invertible, a solution to above linear system is given by

$$\mathbf{w}^{\text{LS}} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{t}.$$