Reminders

$$P = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \text{ or } \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$p = \begin{bmatrix} x \\ y \end{bmatrix} \text{ or } \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Equation of a line in the image plane

$$ax + by + c = 0$$

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \qquad p^T \ell = 0$$

$$\text{or } p \cdot \ell = 0$$

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if $\vec{u} \perp \vec{v} \Longrightarrow \vec{u} \cdot \vec{v} = 0$

if $\vec{u} \parallel \vec{v} \Longrightarrow \vec{u} \times \vec{v} = 0$

$$\vec{u} \times \vec{v} = [\vec{u}]_\times \vec{v}$$

$$[\vec{u}]_\times = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \longrightarrow \text{rank } 2$$

(what's the null space?)

Camera calibration matrix (intrinsic)

(extrinsic $R \; \vec{t}$)

$$K = \begin{bmatrix} f & 0 & P_x \\ 0 & f & P_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$p = K P_c = K \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} wx \\ wy \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

p: 2D pixel coordinates

P_c: 3D coordinates in camera frame

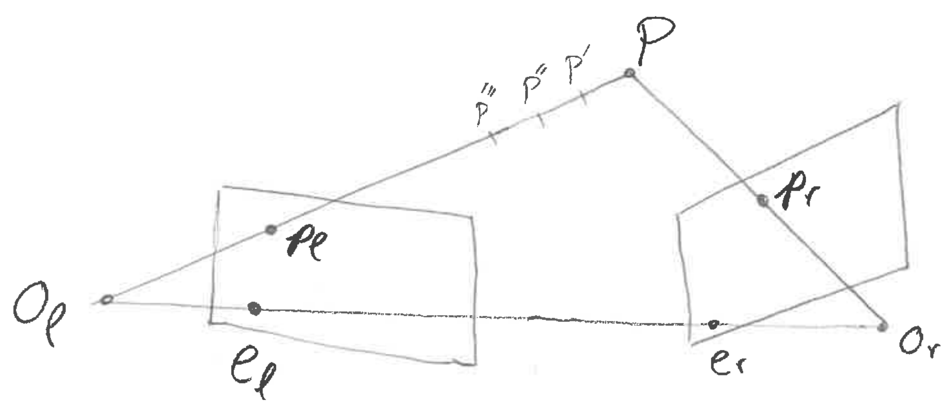
P_W : 3D coordinates, world frame

P_r : " " , Right camera frame (or X_R)

P_ℓ : " " , Left " " (or X_L)

p_r : pixel coordinates, right image

p_ℓ : " " , left image

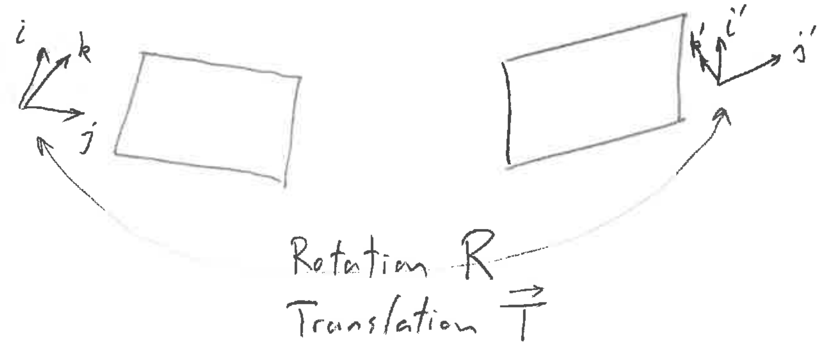


e_ℓ: left epipole

e_r: right epipole

Essential Matrix
Fundamental Matrix
Rectification

E: Essential Matrix: maps 3D points from th frame of one camera to th frame of another camera



X_L: a 3D point in left camera frame
X_R: same point, right camera frame

$$X_L = R X_R + T \xrightarrow[\text{both sides by } T]{\text{cross multiply}} T \times X_L = T \times R X_R + \underbrace{T \times T}_{\emptyset}$$

$$\xrightarrow[\text{sides by } X_L]{\text{dot Product both}} \underbrace{X_L \cdot (\underbrace{T \times X_L}_{\substack{\text{vector perpendicular} \\ \text{to } X_L \text{ and to } T}})}_{\emptyset} = X_L \cdot (T \times R X_R)$$

$$\Rightarrow \emptyset = X_L \cdot T \times R X_R$$

$$\Rightarrow \emptyset = X_L^T [T]_\times R X_R$$

R: 3×3, rank 3
$[T]_\times$: 3×3, rank 2
E: 3×3, rank 2

$$\Rightarrow X_L^T E X_R = \emptyset$$

E: Essential matrix

Fundamental Matrix (F): relation between Pixel coordinates in left & right images

$$p = \underbrace{\underset{\text{intrinsic}}{\begin{bmatrix} K \end{bmatrix}} \underbrace{\overbrace{\begin{bmatrix} R & \vec{T} \\ \emptyset & 1 \end{bmatrix}}^{\text{extrinsic}} \underbrace{\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}}_{P_W}}_{P_c}}_{\text{Pixel coordinates } p_r,\, p_e}$$

P_W: point in world coordinates

P_c: camera coordinates (P_r, P_e, or X_R, X_L)

Pixel coordinates p_r, p_e

$p = K P_c$ ← any camera

our left camera $p_e = K_e P_L$

our right camera $p_r = K_r P_R$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} wx \\ wy \\ w \end{bmatrix} = \begin{bmatrix} K \end{bmatrix}_{3\times3} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow P_c = K^{-1} p$$ (before getting rid of w)

$$X_L^T E X_R = \emptyset$$

$$(K_e^{-1} p_e)^T E (K^{-1} p_r) = \emptyset$$

$K_e = K_r$ ⟶ assume identical cameras

$$\Rightarrow \quad p_\ell^T \underbrace{K^{-T} E K^{-1}}_{F} p_r = \emptyset$$

$$\Rightarrow \begin{cases} p_\ell^T F p_r = \emptyset \\ \underset{\text{rank 2}}{F} = \underbrace{K^{-T}}_{\text{rank 3}} \underbrace{E}_{\text{rank 2}} \underbrace{K^{-1}}_{\text{rank 3}} \end{cases}$$

How to compute F (slides 13-14)

F is $3 \times 3 \rightarrow$ 9 elements, but 7-dof

- 1 scale (F, 2F, 3F, ... cF all the same)
  ($p_L^T F p_r = p_L^T 100 F p_r = \emptyset$)
- 1 det $= \emptyset$

$9 - 1 - 1 = 7 \rightarrow$ so 7 constraints enough to solve

- 7-point algorithm
- 8-point algorithm $\longrightarrow$ simplest, let's look at this
- normalized 8-point algorithm

solve for F without considering det $= \emptyset \longrightarrow$ then force det $= \emptyset$ later

$p_e^T F p_r = \emptyset$

each point match $\longrightarrow$ one <u>linear</u> constraint on elements of F
($f_{11}, f_{12}, f_{13}, f_{21}, \ldots$)

8 point matches $\longrightarrow$ 8 equations $\longrightarrow$ write in matrix form

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ \vdots \\ f_{33} \end{bmatrix} = \emptyset$$

$A\vec{f} = \emptyset \longrightarrow \vec{f}$: null space of A

SVD of $A \longrightarrow A = U D V^T$

set $\vec{f}$ to the column of V corresponding to the smallest singular value in D

reshape $\vec{f}_{9\times1}$ to get $F_{3\times3}$

now impose rank 2 ($\det = \emptyset$) constraint with another SVD

$F = U D V^T \longrightarrow D_{3\times3} = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}$

$d_1 > d_2 > d_3$

set $d_3 = \emptyset \longrightarrow \hat{D}$

$\hat{F} = U \hat{D} V^T$

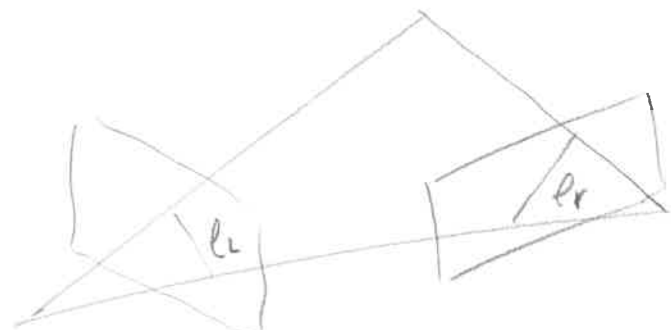
practical issues $\longrightarrow$ robust algorithms, e.g. normalized 8-point algorithm

Now we have F, how to get epipolar lines?

$$p_\ell^T (F\, p_r) = \emptyset$$

line equation: $p^T \ell_L = \emptyset$ $\Bigg\} \Rightarrow \ell_\ell = F\, p_r$

→ given a point in the right image (p_r) the epipolar line in the left image is ℓ_L



→ given a point in the left image, what's the epipolar line in the right image?

$$(p_l^T F p_r)^T = 0$$

$$\left\{\begin{array}{l} p_r^T F^T p_l = 0 \\ p_r l_r = 0 \end{array}\right\} \longrightarrow l_r = F^T p_l$$

epipoles: all epipolar lines go through the epipoles

e.g. right epipoles

$$p_\ell^T F p_r = 0$$

e_r: right epipole
e_ℓ: left epipole

for every point in the left image this should be zero

$$\Rightarrow F p_r = 0 \longrightarrow \text{to find } e_r \text{, solve for } p_r$$

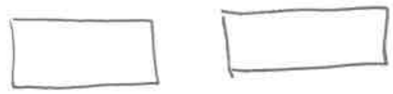
$$F e_r = 0 \longrightarrow e_r = \text{vector that is null space of } F$$

(recall that F is rank 2)

How do you find the left epipole (e_ℓ)?

given F, how to rectify ?

$$R = I_{3\times3}$$



$$T = \begin{bmatrix} t_x \\ 0 \\ 0 \end{bmatrix}$$

$$F = \underbrace{K^{-T} E K^{-1}}$$

$$K = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$E = [T]_x R$$

$$R = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$[T]_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -t_x \\ 0 & t_x & 0 \end{bmatrix}$$

$$\Rightarrow F = c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

c: some scalar , $c = 1$

$$\begin{bmatrix} x_\ell & y_\ell & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ 1 \end{bmatrix} = 0$$

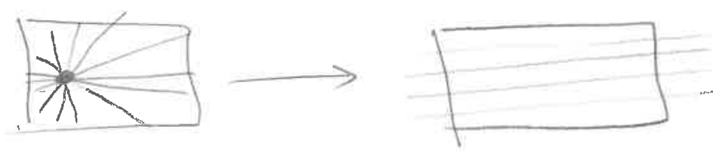
$$\Rightarrow y_\ell = y_r$$

right epipole : null space of F $\longrightarrow$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$

what does this mean? $\longleftarrow$ $e_r = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

recall that $\begin{bmatrix} wx \\ wy \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

rectified $\longrightarrow e_r = \begin{bmatrix}1\\0\\0\end{bmatrix}$



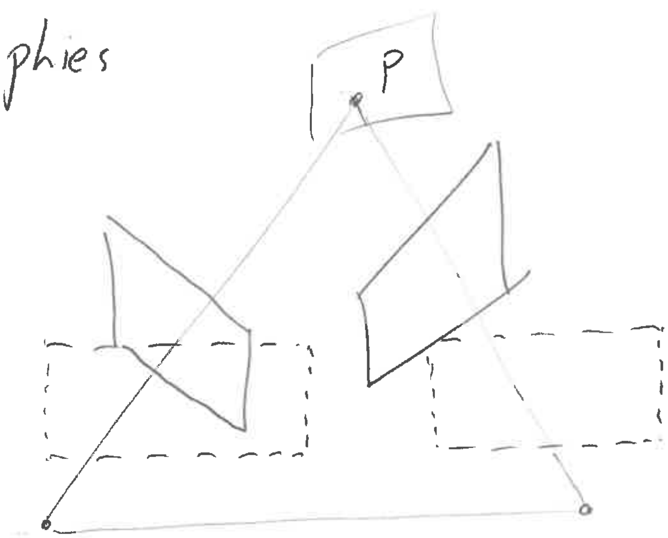
Parallel lines do not intersect

to rectify: we can warp left, or right, or both images

$\longrightarrow$ ambiguity $\longrightarrow$ we can keep one as is and warp only one

let's start with warping both

H_ℓ, H_r : homographies



assume point P is on a plane (any plane)

$\longrightarrow$ p_ℓ and p_r mapping via a homography

$$p_\ell^T F p_r = 0$$

$$p_\ell = H_r p_r \quad , \quad p_r = H_\ell p_\ell$$

$$(H_r p_r)^T F (H_\ell p_\ell) = 0$$

$$\Longrightarrow \quad p_r^T \left(H_r^T F H_\ell\right) p_\ell = \emptyset$$

transpose

$$p_\ell^T \underbrace{\left(H_\ell F H_r^T\right)^T}_{= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & \emptyset \end{bmatrix}} p_r = \emptyset$$

let's set $H_\ell = \left[I_{3\times 3} \;\middle|\; \vec{\emptyset} \right]$

and solve for H_r (slide 21)

solution $\longrightarrow$ $H_r = \left[[e_r]_\times F \;\middle|\; e_r \right]$