Assignment 02

Instructions

1. Each assignment can contain both theoretical and practical questions.
2. Use LaTeX (preferred) or Word for theoretical question responses.
3. Practical questions are in the provided Jupyter notebook. Use Google Colab (Preferred) or Jupyter Notebook to complete questions directly in the Jupyter Notebook. Include code changes and reasoning in the Jupyter Notebook. Convert the Jupyter Notebook into an HTML file for submission.
4. Submit a PDF or Word file with reponses to theoretical questions, a Jupyter Notebook, and an HTML file (both files) with completed practical questions.
5. A 25% penalty applies to submissions on the first day after the due date, and a 50% penalty for submissions 24 to 48 hours late. No submissions will be accepted beyond 48 hours past the due date.

Theoretical Questions

Question 1

Consider a bag containing balls of three different colors: red, green and blue. Event A is drawing a red ball, Event B is drawing a ball that is either red or green and Event C is drawing a ball that is not blue. The bag contains 8 red balls, 5 green balls and 7 blue balls.

P(A) = 8/20

P(B) = drawing either red or green

(red + green)/(total) = 13/20

P(C') = Not drawing blue 13/20

(total - blue)/(total) = 13/20

Red = 8 Green = 5 Blue = 7

Total = 20

a) Calculate the conditional probability of Event A given the other two events. (without using Bayes' theorem)

P(A|B): probability of drawing a red ball given only red and green balls.

$P(A|B) = P(A \cap B) / P(B)$

P(A|B) = 8 /13 for 8 red balls out of 13 red and green balls

P(A|B): probability of drawing a red ball given all balls except blue.

$P(A|C) = P(A \cap C) / P(C)$

P(A|C) = 8 /13 for 8 red balls out of 13 balls excluding blue

b) Check if Event A is conditionally dependent on B and C.

Because P(A) != P(A|B) and P(A) != P(A|C) Event A is dependent on both events B and C

c) Prove Bayes' theorem for events A and B.

Bayes Theorem: P(A|B) = (P(A|B) * P(B)) / P(A)

(P(A|B) * P(B)) / P(A) = (8/13 * 13/20)/ (13/20) = 8/13

P(A ∩ B) /P(B) = 8/13

Because P(A ∩ B) /P(B) == (P(A|B) * P(B)) / P(A) bayes theorem stands

Show all steps in your calculation.

Question 2

a) In a Bayesian statistical framework, considering the event of tossing a coin. The probability of getting a head is 0.5 with a prior belief of 0.9 and 0.6 with a prior belief of 0.1. Find the posterior distributions for p=0.5 and p=0.6 given that you get 5 consecutive heads.

posterior probability = ∑ P(x,c) for c = 2

posterior = (prior * likelihood/evidence)

Evidence = P(5 heads|p=.5) * P(p=.5) + P(5 heads|(p=.6) * P(p=.6) = .0359

p = .5

- Prior = .9
- Likelihood = .5^5
- Posterior = .783

p = .6

- Prior = .1
- Likelihood = .6^5
- Posterior = .217

Assert(posterior(p=.5) + posterior(p=.6) == 1

i.e. We are more likely to believe that rate for heads is closer to .6 after flipping 5 heads in a row

b) In a Bayesian statistical framework, given a coin with an unknown probability of landing heads p, you initially believe (prior) that p follows a Beta(2, 2) distribution. Find the posterior distribution after observing 10 consecutive tails. (Hint: The functional form of Beta distribution is, $\beta(a,b) = x^{a-1}.(1-x)^{b-1}$. You can also consider $Posterior \propto Likelihood *$ $Prior$)

Prior = Beta(2,2) = x (1-x)

Likelihood = (1-x)^10

Prior * likelihood = (1-x)^10 * x (1-x) =x (1-x)^11

Updated beta = $\beta(a,b)$ = 2,12

i.e start with distribution of 2 heads ,2 tails and after 10 tails we now have a distribution of 2 heads, 12 tails

Question 3

Consider the Density function of Gaussian (Normal) Density.

$$f(x_i|\Theta) = \frac{1}{\sqrt{2\pi\Theta_1}} \cdot e^{-\frac{(x_i-\Theta_0)^2}{2\Theta_1}}$$

where Θ_0 represents μ (mu) and Θ_1 represent σ^2 (sigma²). Below is the Likelihood function

$$L(\Theta) = \prod_{i=1}^{n} f(X_i|\Theta)$$

Consider the given density and likelihood function, and answer the following question

1. Derive the equation for maximum likelihood estimates (MLE) of μ (Θ_0) and σ^2 (Θ_1) by maximizing the likelihood function.

   For the partial derivates of μ and σ^2 given the gaussian normal density

   μ = ∑x□/n

   σ^2 = ∑(x□ -m)^2 /n

2. Taking into consideration the equations you derive just above for μ (Θ_0) and σ^2 (Θ_1), calculate the values for μ and σ^2 when the data is X = [-5, 7, 9, 12, -17, 1, 4, 24, 0, 3]

   μ = 38/10 = 3.8 or np.mean(X)

σ^2 = 104.56 or np.var(X)

Question 4

Consider the Density function of Bernoulli Distribution.

$$P(x) = p^x \cdot (1-p)^{1-x}, \qquad x \in \{0,1\}$$

Below is the Likelihood function

$$L(p|x) = \prod_{i=1}^{n} p^{x_i} \cdot (1-p)^{1-x_i}$$

Consider the given density and likelihood function, and answer the following question

1. Derive the equation for p by using Maximum Likelihood Estimation (MLE).

   $P = 1/n \sum x_i$

2. Consider an event of rolling 2 dices. So, total there are 36 outcomes. The value of x for each outcome depends on the addition of numbers on the faces of both the dices. x = 1 if the addition of both the numbers is 9 otherwise it is 0. Now based on the equation you derived above for ‘p’, calculate the value of ‘p’ based on the rolling of 2 dices. Show all steps in your calculation along with the value of x (1 or 0) for all 36 outcomes of rolling of 2 dices.

   Possible ways of rolling a sum of 9: (6,3),(3,6),(4,5),(5,4)

   Maximum likelihood = 4(possible ways of rolling sum of 9)/ 36(possible ways of two dice)

   Maximum likelihood = 4/36 = 1/9

Question 5

Consider the Bayesian Estimation.

Your parents are statistician, and you and your brother are playing in the backyard. On one fine day, they saw you on the swing and estimated that while you are on swing you come to rest at the mean rate of Θ either equal to 2 or 4. Prior collecting any data, they though the value of Θ is more likely to be 2 rather than 4. They believe that the prior probabilies are P(Θ = 2) = 0.7 and P(Θ = 4) = 0.3.

On one sunny day, your parents observed that around total 5 people including you and your brother are playing in the backyard. As per your parents observation, what is the probability of Θ = 2 and what is the probability of Θ = 4 given that 5 children are playing?

Given Information :

$P(X = 5 \mid \Theta = 2) = 0.022$

$P(X = 5 \mid \Theta = 4) = 0.105$

Show all steps in your calculation.

Evidence = $P(X = 5 \mid \Theta = 2) * P(\Theta = 2) + P(X = 5 \mid \Theta = 4) * P(\Theta = 4)$

$= .022 * .7 + .105 * .3$

$= .0469$

$P(\Theta = 2)$

Prior = .7

Likelihood = 0.022

Posterior = .7 * .022/ .0469

= .328

$P(\Theta = 4)$

Prior = .3

likelihood =.105

Posterior = .3 * .105/ .0469

= .672

Question 6

Based on your conceptual understanding answer the following question

1. Explain the concept of the Bayes' Estimator.

   The bayes estimator is a way of estimating parameters of a distribution. Using Bayes theorem, the prior estimation of the parameters are adjusted as new sample data is inputted. The estimated sample mean and standard deviation will approach the true population mean and standard deviation with more data.

2. How does the prior information influence the Bayesian estimation process?

   The prior information gives the existing knowledge of the parameter values such that we can weigh the new sample data accordingly. The prior prevents noise in the sample data from changing our parameters too far away from the true population mean and standard deviation.

Practical Questions

Please refer to and answer Questions 7, 8, and 9 in the provided Jupyter Notebook