Solution Manual

to accompany the textbook

Fixed Income Securities:
Valuation, Risk, and Risk Management

by Pietro Veronesi

Chapters 14 - 22

Version 1

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Solutions to Chapter 14

Exercise 1. Results should be similar to the figures presented in the book.
Exercise 2. Results for both cases should be the same as figures in the book. Changes may appear from different values in the random number generator.
Exercise 3. Figure 1 shows how (14.24) is the solution to (14.23), given a small enough dt.

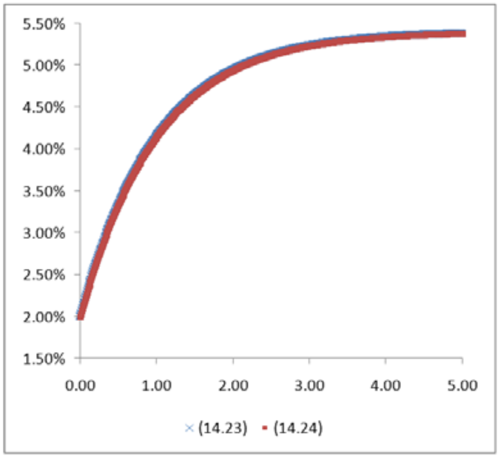

Figure 1: Equation (14.23) and (14.24) for $dt = 0.01$

Note that for $dt = 0.5$ the difference increases (see Figure 2).

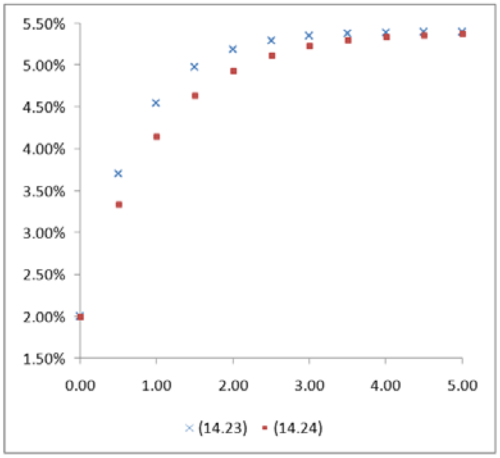

Figure 2: Equation (14.23) and (14.24) for $dt = 0.5$

So, if dt becomes larger the gap between both equations increases.

Exercise 4.

a. Consider the base case with the following parameters: $\gamma = 1$, $\bar{r} = 5.40\%$, $\sigma = 1\%$, $r(0) = 0.17\%$ and $dt = 1/252$; we have:

   i. For γ: As this parameter increases, the 'speed' by which the rate converges to the long-term rate, $\bar{r}$, increases. Inversely as this parameter decreases, the rate converges slower to $\bar{r}$ (see Figure 3).

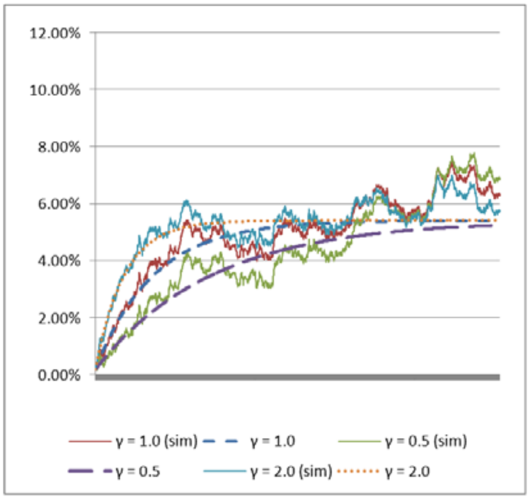

Figure 3: Vasicek simulations and expected value for different values of γ

   ii. For σ: As this parameter increases the volatility of the process goes up. Inversely as it goes down so does the volatility of the process (see Figure 4).

b. The effects of having different values for r(0) can be seen in Figure 5. We choose values that are higher, lower and the same as $\bar{r}$

Exercise 5.

a. We have that: $\beta = -\gamma dt$ and $\alpha = \gamma \bar{r} dt$. For σ we can simply estimate the standard error of the regression and annualize it by dividing the standard error by $\sqrt{dt}$. Using daily data on Treasury interest rates from 1/1/2008 to 3/5/2010 we obtain the following regression estimates: $\alpha = 0.004466$; $\beta = -0.01439$; and $\sigma_{SE} = 0.113332$. This leads to the following parameters: $\gamma = 3.625184$; $\bar{r} = 0.310414$; and $\sigma = 1.799097$. In this particular case, this leads to a highly volatile process, which easily gives negative rates. The reason for this shortcoming is twofold: on the one hand we are using a period of particular interest rate volatility to estimate the model (the range of rates during this 2 year period was between 3.37% and 0.00%); on the other hand, r_0 is 0.11% which is already very close to zero.

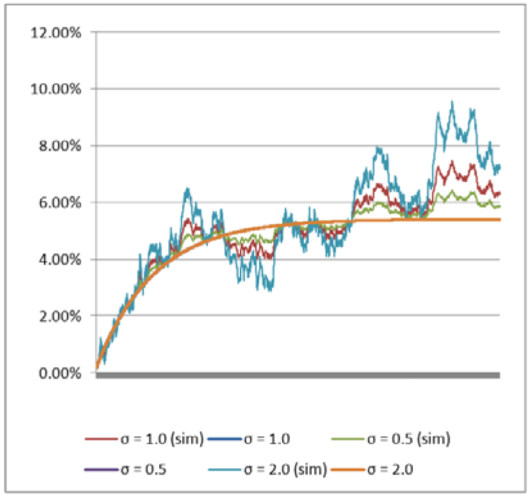

Figure 4: Vasicek simulations and expected value for different values of σ

b. Figure 6 plots the forecast with a simulation which shows the problems with the estimated model mentioned above.

c. Figure 7 shows the histograms for the forecasted values.

Exercise 6. Recall a definition of Ito's lemma for $P_t = F(X_t)$

$$dP_t = \frac{1}{2}\left(\frac{d^2F}{dX^2}\right)dt + \left(\frac{dF}{dX}\right)dX_t$$

a. For $F(X) = A + BX$:

$$\frac{dF}{dX} = B; \frac{d^2F}{dX^2} = 0$$
$$dP_t = BdX_t$$

b. For $F(X) = e^{A+BX}$:

$$\frac{dF}{dX} = Be^{A+BX}; \frac{d^2F}{dX^2} = B^2e^{A+BX}$$

$$dP_t = \frac{1}{2}e^{A+BX}dt + Be^{A+BX}dX$$

$$dP = P\left[\frac{1}{2}B^2dt + BdX\right]$$

Exercise 7. Given the process:

$$dr_t = \gamma(\bar{r} - r_t)dt + \sigma dX_t$$

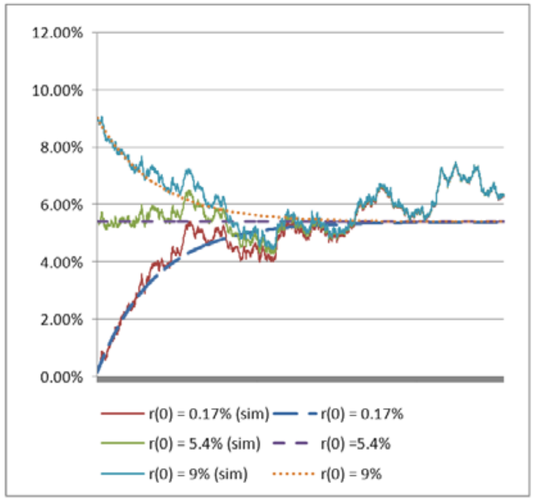

Figure 5: Vasicek simulations and expected value for different values of r_0

We have that $m(r_t, t) = \gamma(\bar{r} - r_t)dt$; and $s(r_t, t) = \sigma$. Using Ito's lemma:

$$dP_t = \left[\left(\frac{\partial F}{\partial t}\right) + \left(\frac{\partial F}{\partial r}\right) m(r_t, t) + \frac{1}{2}\left(\frac{\partial^2 F}{\partial r^2}\right) s(r_t, t)^2\right] dt + \left(\frac{\partial F}{\partial r}\right) s(r_t, t) dX_t$$

a. For $F(r) = A + Br$:

$$\frac{\partial F}{\partial t} = 0; \frac{\partial F}{\partial r} = B; \frac{\partial^2 F}{\partial r^2} = 0$$

So:

$$dP_t = B\gamma(\bar{r} - r_t)dt + B\sigma dX_t = Bdr_t$$

b. For $F(r) = e^{A-Br}$

$$\frac{\partial F}{\partial t} = 0; \frac{\partial F}{\partial r} = -Be^{A-Br}; \frac{\partial^2 F}{\partial r^2} = B^2 e^{A-Br}$$

So:

$$dP_t = \left[-BP\gamma(\bar{r} - r_t) + \frac{1}{2}B^2 P\sigma^2\right] dt + BP\sigma dX_t$$

$$dP_t = P\left[\frac{1}{2}B^2\sigma^2 dt - Bdr_t\right]$$

Exercise 8.

a. The expected capital gain return and its components for the asset prices defined in the previous exercise are:

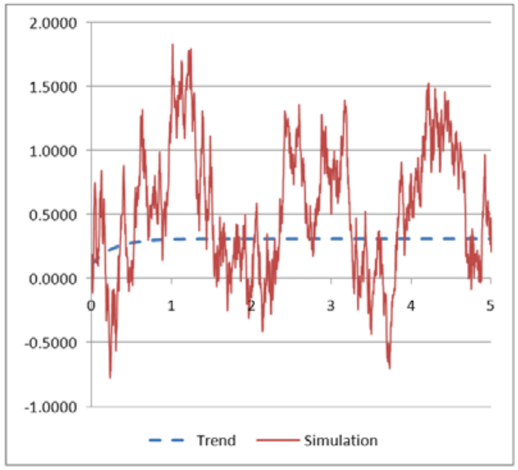

Figure 6: Vasicek model expected and simulated rates

7a. For $dP_t = B[\gamma(\bar{r} - r_t)dt + \sigma dX_t]$ we obtain:

$$E[dP] = B\gamma(\bar{r} - r_t)dt$$

In this process there are only capital gains due to the variation of r.

7b. For $dP_t = -BP[\gamma(\bar{r} - r_t)dt - \frac{1}{2}B\sigma^2 dt + \sigma dX_t]$ we obtain:

$$E[dP] = -BP\left[\gamma(\bar{r} - r_t)dt - \frac{1}{2}B\sigma^2 dt\right]$$

In this case there are two components:

i. Capital gains due to the variation of r: $-BP\gamma(\bar{r} - r_t)dt$.

ii. Captial gains due to the convexity effect: $\frac{1}{2}B\sigma^2 dt$.

b. Recall that variance (squared diffusion) is defined as:

$$Var[dP] = E[dP^2] - E^2[dP]$$

7a. In this case we have that:

$$E[dP^2] = B^2\sigma^2 dt; E^2[dP] = 0$$

So diffusion is given by: $B\sigma\sqrt{dt}$

7b. In this case we have that:

$$E[dP^2] = B^2P^2\sigma^2 dt; E^2[dP] = 0$$

So diffusion is given by: $BP\sigma\sqrt{dt}$

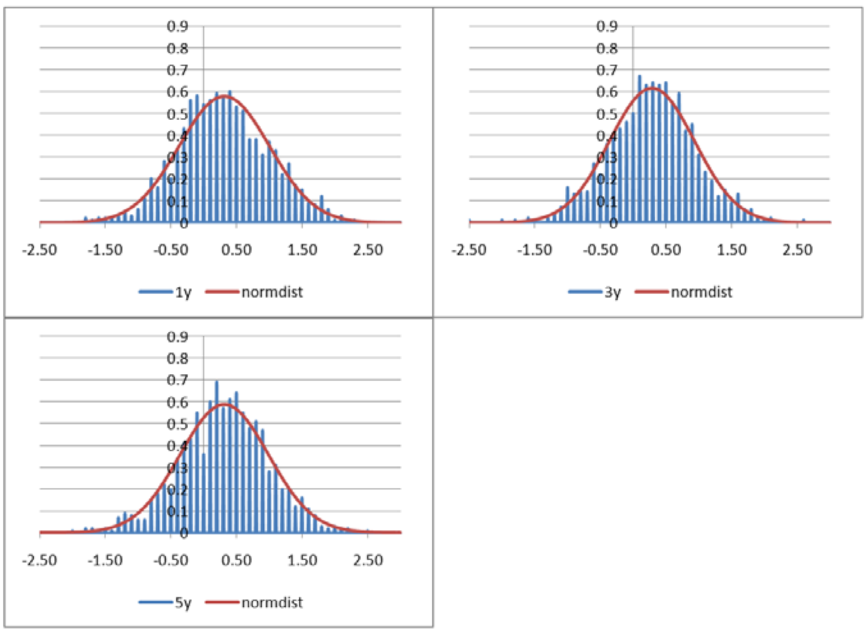

Figure 7: Histogram of rates at 1 year, 3 years and 5 years

Solutions to Chapter 15

Exercise 1.

a. In order to value the zero coupons we use the risk neutral parameters.

τ	Z(t,T)	τ	Z(t,T)
0.25	0.9944	5.25	0.7827
0.50	0.9877	5.50	0.7713
0.75	0.9802	5.75	0.7599
1.00	0.9718	6.00	0.7487
1.25	0.9627	6.25	0.7376
1.50	0.9531	6.50	0.7266
1.75	0.9429	6.75	0.7157
2.00	0.9324	7.00	0.7049
2.25	0.9216	7.25	0.6943
2.50	0.9105	7.50	0.6838
2.75	0.8992	7.75	0.6734
3.00	0.8877	8.00	0.6632
3.25	0.8761	8.25	0.6531
3.50	0.8644	8.50	0.6432
3.75	0.8527	8.75	0.6333
4.00	0.8409	9.00	0.6237
4.25	0.8292	9.25	0.6141
4.50	0.8175	9.50	0.6047
4.75	0.8058	9.75	0.5954
5.00	0.7942	10.00	0.5863

b. Figure 1 presents the yield curve for the bonds.

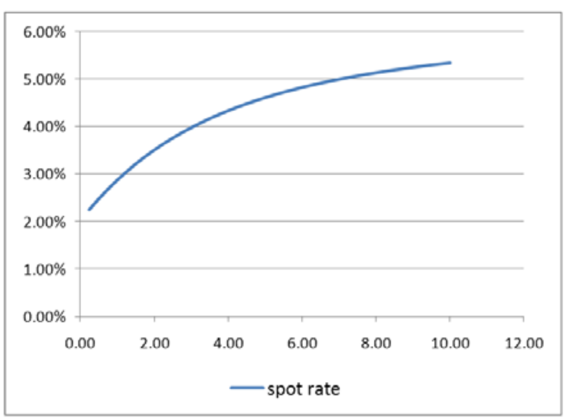

Figure 1: Spot rate of bonds up to 10 years

c. Spot rate duration is:

τ	B(t;T)	τ	B(t;T)
0.25	0.2360	5.25	1.9624
0.50	0.4461	5.50	1.9829
0.75	0.6331	5.75	2.0011
1.00	0.7996	6.00	2.0174
1.25	0.9478	6.25	2.0319
1.50	1.0797	6.50	2.0447
1.75	1.1972	6.75	2.0562
2.00	1.3017	7.00	2.0664
2.25	1.3948	7.25	2.0755
2.50	1.4776	7.50	2.0836
2.75	1.5514	7.75	2.0908
3.00	1.6170	8.00	2.0972
3.25	1.6754	8.25	2.1029
3.50	1.7275	8.50	2.1080
3.75	1.7738	8.75	2.1125
4.00	1.8150	9.00	2.1165
4.25	1.8517	9.25	2.1201
4.50	1.8844	9.50	2.1233
4.75	1.9134	9.75	2.1261
5.00	1.9393	10.00	2.1287

Exercise 2.

a. Figure 2 presents the effect on the term structure of interest rates due to changes in γ^*.

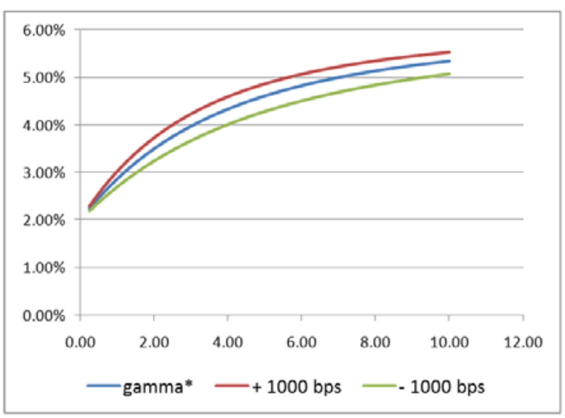

Figure 2: Term structure of interest rates for three choices of γ^* (Vasicek)

b. Figure 3 presents the effect on the term structure of interest rates due to changes in $\bar{r}^*$.

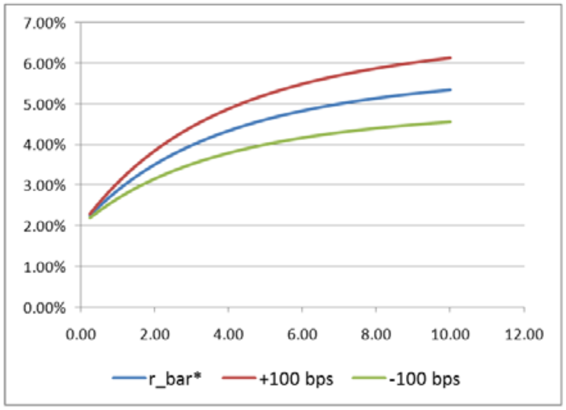

Figure 3: Term structure of interest rates for three choices of $\bar{r}^*$ (Vasicek)

c. Figure 4 presents the effect on the term structure of interest rates due to changes in σ.

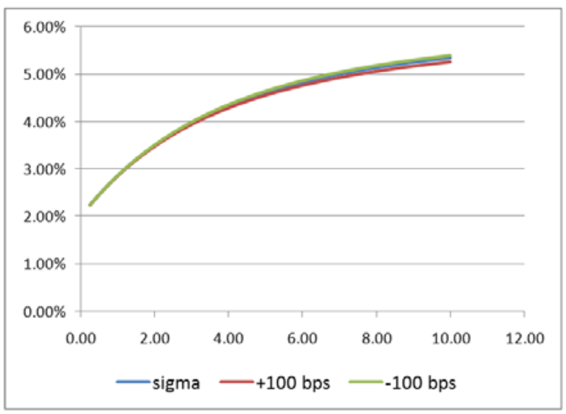

Figure 4: Term structure of interest rates for three choices of σ (Vasicek)

Exercise 3.

a. Figure 5 presents the effect on the term structure of interest rates due to changes in γ^*.

b. Figure 6 presents the effect on the term structure of interest rates due to changes in $\bar{r}^*$.

c. Figure 7 presents the effect on the term structure of interest rates due to changes in σ.

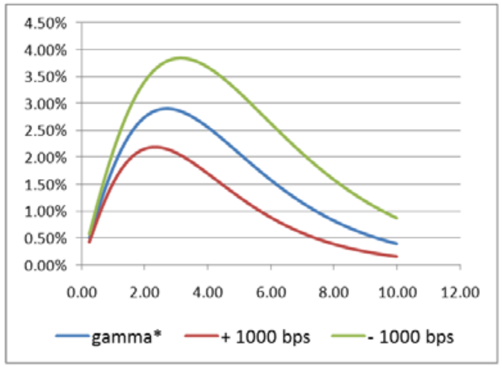

Figure 5: Term structure of interest rates for three choices of γ^* (CIR)

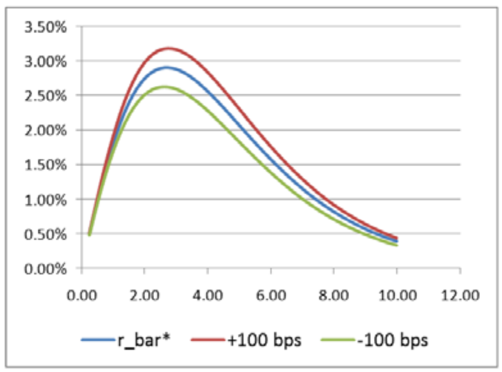

Figure 6: Term structure of interest rates for three choices of $\bar{r}^*$ (CIR)

Exercise 4. On average, the distribution of risk neutral rates is higher than the risk natural one. This is due to the additional compensation that is needed to make agents risk neutral (see Figure 8).
Exercise 5.

a. Figure 9 presents how spot rate duration depends on τ, given $\gamma = 0.3262$.

b. Figure 10 presents how spot rate duration is affected by changes in γ^*.

c. Figure 11 presents what happens to spot rate duration when $\gamma^* < 0$.

Exercise 6. Short-term bonds have higher yield volatility than long-term bonds (see Figure 12). Return volatility and yield volatility are not the same since yields are a sort of "average" measure in order to make comparable different maturities. Returns will be affected by the yield volatility times the time to maturity. Return volatility is always increasing in relation to time to maturity.

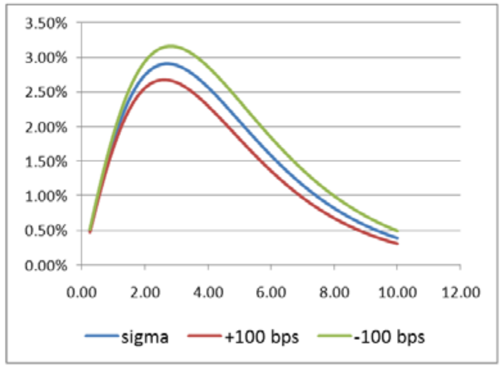

Figure 7: Term structure of interest rates for three choices of σ (CIR)

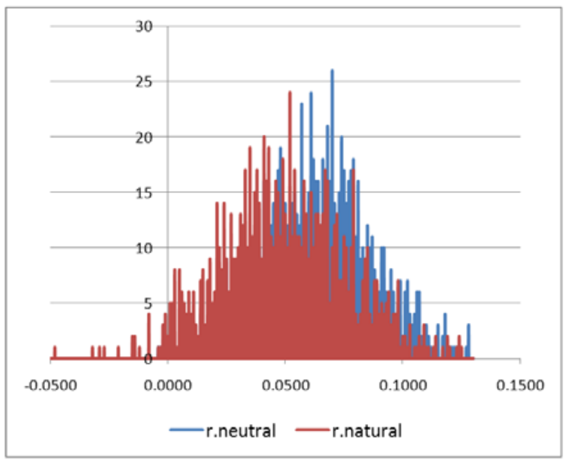

Figure 8: Distribution of r_{10} under risk neutral and natural parameters

Exercise 7. The Fundamental Pricing Equation states:

$$\frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial r} m^*(r,t) + \frac{1}{2}\frac{\partial^2 Z}{\partial r^2}\sigma^2 = rZ$$

where:

$$m^*(r,t) = \gamma^*(\bar{r}^* - r)$$

The Vasicek formula for pricing a zero coupon bond:

$$Z(r,t;T) = e^{A(t;T) - B(t;T)r}$$

where:

$$B(t;T) = \frac{1}{\gamma^*}\left(1 - e^{-\gamma^*(T-t)}\right)$$

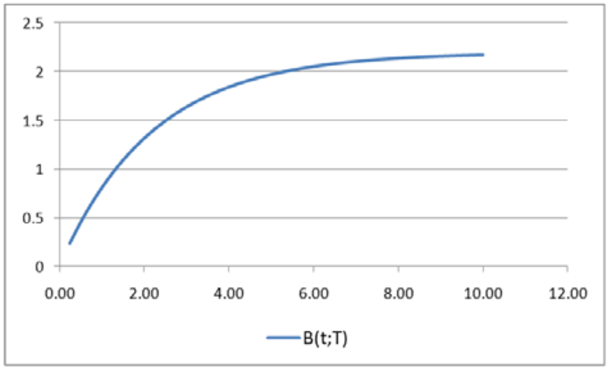

Figure 9: Spot rate duration and τ

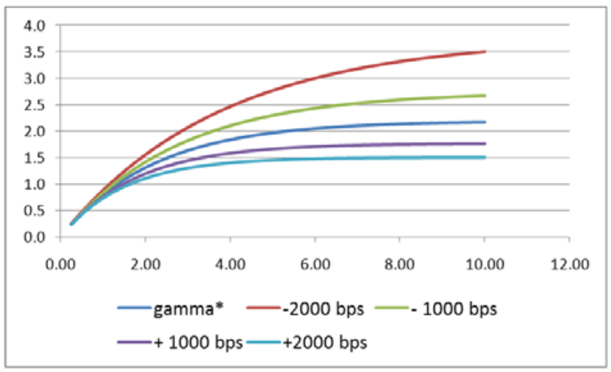

Figure 10: Spot rate duration under different values of γ^*

$$A(t;T) = (B(t;T) - (T-t))\left(\bar{r}^* - \frac{\sigma^2}{2(\gamma^*)^2}\right) - \frac{\sigma^2 B(t;T)^2}{4\gamma^*}$$

For the Left Hand Side of the Fundamental Pricing Equation we have:

$$\frac{\partial Z}{\partial t} = \left[A'(t;T) - B'(t;T)r\right] Z(r,t;T)$$

$$\frac{\partial Z}{\partial r} = -B(t;T)Z(r,t;T)$$

$$\frac{\partial^2 Z}{\partial r^2} = B(t;T)^2 Z(r,t;T)$$

Where:

$$B'(t;T) = -e^{-\gamma^*(T-t)}$$

$$A'(t;T) = (1 + B'(t;T))\left(\bar{r}^* - \frac{\sigma^2}{2(\gamma^*)^2}\right) - \frac{\sigma^2}{2\gamma^*}B'(t;T)B(t;T)$$

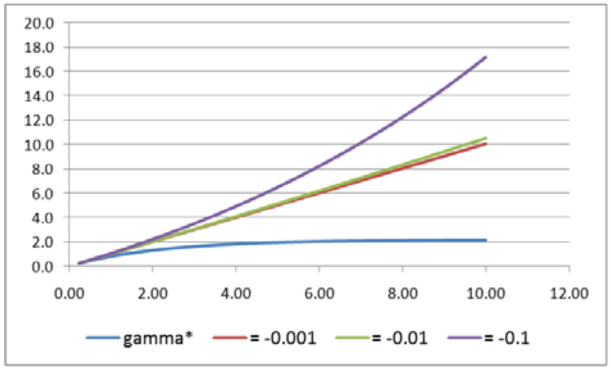

Figure 11: Spot rate duration for $\gamma^* < 0$

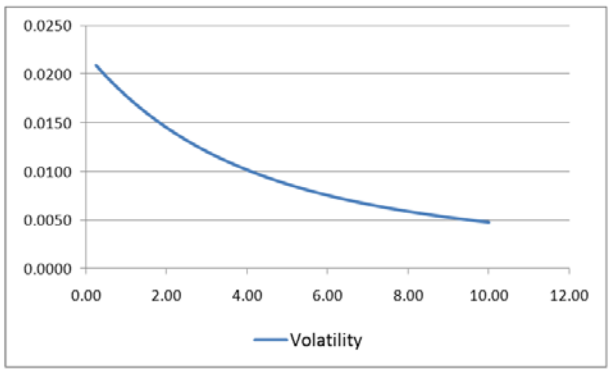

Figure 12: Yield volatility with respect to τ

Note that:

$$B(t;T) = \frac{1}{\gamma^*}(1 + B'(t;T))$$

then:

$$\gamma^* B(t;T) = 1 + B'(t;T)$$

and

$$\gamma^* B(t;T) - B'(t;T) = 1$$

So we can present $A'(t;T)$ as:

$$A'(t;T) = \gamma^* B(t;T)\left(\bar{r}^* - \frac{\sigma^2}{2(\gamma^*)^2}\right) - \frac{\sigma^2}{2\gamma^*}B'(t;T)B(t;T)$$

$$= B(t;T)\left[\gamma^*\bar{r}^* - \frac{\sigma^2}{2\gamma^*} - \frac{\sigma^2}{2\gamma^*}B'(t;T)\right] = B(t;T)\left[\gamma^*\bar{r}^* - \frac{\sigma^2}{2\gamma^*}(1 + B'(t;T))\right]$$

$$= B(t;T)\left[\gamma^*\bar{r}^* - \frac{\sigma^2}{2\gamma^*}\gamma^* B(t;T)\right] = B(t;T)\left[\gamma^*\bar{r}^* - \frac{\sigma^2}{2}B(t;T)\right]$$

So:

$$A'(t;T) = \gamma^* \bar{r}^* B(t;T) - \frac{\sigma^2}{2} B(t;T)^2$$

Which means that:

$$\frac{\partial Z}{\partial t} = \left[B(t;T)\gamma^* \bar{r}^* - \frac{\sigma^2}{2} B(t;T)^2 - B'(t;T)r \right] Z(r,t;T)$$

So we can present the Left Hand Side of the Fundamental Pricing Equation as:

$$Z(r,t;T) \left[B(t;T)\gamma^* \bar{r}^* - \frac{\sigma^2}{2} B(t;T)^2 - B'(t;T)r - B(t;T)\gamma^*(\bar{r}^* - r) + \frac{\sigma^2}{2} B(t;T)^2 \right]$$

$$= Z(r,t;T) \left[B(t;T)\gamma^* r - B'(t;T)r \right] = Z(r,t;T)r \left[B(t;T)\gamma^* - B'(t;T) \right]$$

Recall that: $B(t;T)\gamma^* - B'(t;T) = 1$, so:

$$LHS = Z(r,t;T)r$$

Exercise 8.

a. Recall Ito's lemma:

$$dZ_t = \left[\left(\frac{\partial F}{\partial t} \right) + \left(\frac{\partial F}{\partial r} \right) m(r_t,t) + \frac{1}{2} \left(\frac{\partial^2 F}{\partial r^2} \right) s(r_t,t)^2 \right] dt + \left(\frac{\partial F}{\partial r} \right) s(r_t,t) dX_t$$

The Vasicek formula for pricing a zero coupon bond states:

$$Z(r,t;T) = e^{A(t;T) - B(t;T)r}$$

where:

$$B(t;T) = \frac{1}{\gamma^*} \left(1 - e^{-\gamma^*(T-t)} \right)$$

$$A(t;T) = (B(t;T) - (T-t)) \left(\bar{r}^* - \frac{\sigma^2}{2(\gamma^*)^2} \right) - \frac{\sigma^2 B(t;T)^2}{4\gamma^*}$$

Expected capital gains are:

$$E[dZ_t] = \left[\left(\frac{\partial F}{\partial t} \right) + \left(\frac{\partial F}{\partial r} \right) m(r_t,t) + \frac{1}{2} \left(\frac{\partial^2 F}{\partial r^2} \right) s(r_t,t)^2 \right] dt$$

where:

$$\frac{\partial Z}{\partial t} = \left[A'(t;T) - B'(t;T)r \right] Z(r,t;T)$$

$$\frac{\partial Z}{\partial r} = -B(t;T) Z(r,t;T)$$

$$\frac{\partial^2 Z}{\partial r^2} = B(t;T)^2 Z(r,t;T)$$

So:

$$E[dZ_t] = Z(r,t;T)\left[A'(t;T) - B'(t;T)r - B(t;T)m(r_t,t) + \frac{1}{2}B(t;T)^2 s(t;T)^2\right]dt$$

So expected returns are:

$$E\left[\frac{dZ_t}{Z}\right] = \left[A'(t;T) - B'(t;T)r - B(t;T)m(r_t,t) + \frac{1}{2}B(t;T)^2 s(t;T)^2\right]dt$$

From the solution to Exercise 7, we know that:

$$A'(t;T) = B(t;T)\gamma^*\bar{r}^* - \frac{\sigma^2}{2}B(t;T)^2$$

Additionally, the Vasicek model states that:

$$m(r,t) = \gamma(\bar{r} - r)$$
$$s(r,t) = \sigma$$

So:

$$E\left[\frac{dZ_t}{Z}\right] = \left[B(t;T)\gamma^*\bar{r}^* - \frac{\sigma^2}{2}B(t;T)^2 - B'(t;T)r - B(t;T)\gamma(\bar{r} - r) + \frac{\sigma^2}{2}B(t;T)^2\right]dt$$

$$E\left[\frac{dZ_t}{Z}\right] = \left[B(t;T)(\gamma^*\bar{r}^* - \gamma\bar{r}) + r(\gamma B(t;T) - B'(t;T))\right]dt$$

Using this formula for 1-year and 10-year coupon bonds we have that the expected returns are 0.0111% and 0.0165%, respectively.

b. Volatility can be defined as:

$$Vol\left[\frac{dZ}{Z}\right] = \sqrt{Var\left[\frac{dZ}{Z}\right]}$$

where:

$$Var\left[\frac{dZ}{Z}\right] = E\left[\left(\frac{dZ}{Z}\right)^2\right] - E^2\left[\frac{dZ}{Z}\right]$$

since $E\left[dt^2\right] = 0$ and $E\left[dX^2\right] = dt$, we have:

$$Var\left[\frac{dZ}{Z}\right] = E\left[\left(\frac{dZ}{Z}\right)^2\right] = \frac{1}{Z^2}E\left[dZ^2\right]$$

and

$$\frac{1}{Z^2}E\left[dZ^2\right] = \frac{1}{Z^2}\left(\frac{dZ}{dr}\right)^2 s(r_t,t)^2 dt = \frac{1}{Z^2}B(t;T)^2 Z^2\sigma^2 dt$$

$$Var[dZ] = B(t;T)^2\sigma^2 dt$$
$$Vol[dZ] = B(t;T)\sigma\sqrt{dt}$$

Volatilities for the 1-year bond and 10-year bond are 0.0011 and 0.0030, respectively.

c. Risk premium for the 1-year bond and 10-year bond are 0.0032% and 0.0085%, respectively. The ratio to volatility (also known as Sharpe Ratio) is 0.0288 and 0.0288, respectively. Both have the same Sharpe Ratio, as r_0 falls the Sharpe Ratio increases, and viceversa.

d. When $\gamma^* = 0.3262 = \gamma$ we have that risk premium for the 1-year bond and 10-year bond are 0.0014% and 0.0048%, respectively. The Sharpe Ratio for both is:

$$SR = \frac{\gamma^*(\bar{r}^* - \bar{r})}{\sigma}\sqrt{dt} = 0.0116$$

Exercise 9.

a. Parameter Estimation:

   i. The risk neutral parameters are: $\gamma^* = 0.4841$ and $\bar{r}^* = 7.279\%$. Note that σ is obtained through the regression estimate.

   ii. The risk natural parameters are: $\gamma = 0.1766$, $\bar{r} = 5.97\%$ and $\sigma = 0.017431$.

   iii. The risk neutral model has a quicker convergence speed and a higher long-term rate than the risk natural model. This makes sense because we need to compensate economic agents in order for them to become risk neutral (we consider them to be risk averse).

b. Sensitivity to interest rates:

   i. Because of the assumption $r < 15\%$, we can say that:

$$P^{IF} = P_{zero} + P_{fixed} - P_{float}$$

   In other words we price the inverse floater as a portfolio of different bonds, so $\frac{\partial P^{IF}}{\partial r}$ can be presented as:

$$\frac{\partial P^{IF}}{\partial r} = \frac{\partial P_{zero}}{\partial r} + \frac{\partial P_{fixed}}{\partial r} - \frac{\partial P_{float}}{\partial r}$$

   For the Vasicek model, the price of a zero coupon bond is given by:

$$Z(r, t; T) = e^{A(t;T) - B(t;T)r}$$

   which means that:

$$\frac{\partial Z(r, t; T)}{\partial r} = -B(t; T)Z(r, t; T)$$

   We know all parameters for this so:

$$\frac{\partial P_{zero}}{\partial r} = \frac{\partial Z(r, t; 3)}{\partial r} = -1.3608$$

The fixed coupon bond is composed by several zeros:

$$\frac{\partial P_{fixed}}{\partial r} = \sum_{T=1}^{3} c\frac{\partial Z(r,t;T)}{\partial r} + \frac{\partial Z(r,t;3)}{\partial r} = -1.8547$$

From the $Z(r,0;1)$ we can obtain the annually compounded 1-year rate $r_1(0,1) = 3.99\%$ which we can use to obtain the value of a floating rate bond:

$$P_{Float} = (1 + r_1(0,1))Z(0,r;1)$$

So:

$$\frac{\partial P_{Float}}{\partial r} = (1 + r_1(0,1))\frac{\partial Z(0,r;1)}{\partial r} = -0.7927$$

Thus:

$$\frac{\partial P^{IF}}{\partial r} = -1.3608 - 1.8547 + 0.7927 = -2.4227$$

ii. For convexity we have the same idea as in the previous exercise but instead we use:

$$\frac{\partial^2 Z(r,t;T)}{\partial r^2} = B(t;T)^2 Z(r,t;T)$$

So:

$$\frac{\partial^2 P_{zero}}{\partial r^2} = \frac{\partial^2 Z(r,t;3)}{\partial r^2} = 2.1530$$

$$\frac{\partial^2 P_{fixed}}{\partial r^2} = \sum_{T=1}^{3} c\frac{\partial^2 Z(r,t;T)}{\partial r^2} + \frac{\partial^2 Z(r,t;3)}{\partial r^2} = 2.7914$$

$$\frac{\partial^2 P_{Float}}{\partial r^2} = (1 + r_1(0,1))\frac{\partial^2 Z(0,r;1)}{\partial r^2} = 0.6284$$

Thus:

$$\frac{\partial^2 P^{IF}}{\partial r^2} = 2.1530 + 2.7914 - 0.6284 = 4.3160$$

iii. For the whole portfolio we simply weigh the sensitivities of interest rate of both assets, so we get:

$$\frac{\partial \Pi}{\partial r} = -40.1588$$

$$\frac{\partial^2 \Pi}{\partial r^2} = 69.3465$$

c. The Fundamental Pricing Equation:

i. The equation that $\Pi(r,t;T)$ must satisfy is:

$$r\Pi = \frac{\partial \Pi}{\partial t} + \frac{\partial \Pi}{\partial r}\eta^* (\bar{r} - r) + \frac{1}{2}\frac{\partial^2 \Pi}{\partial r^2}\sigma^2$$

ii. In order to obtain $\frac{\partial \Pi}{\partial t}$ we can rearrange the previous equation:

$$\frac{\partial \Pi}{\partial t} = r\Pi - \frac{\partial \Pi}{\partial r}\eta^* \left(\bar{r} - r\right) - \frac{1}{2}\frac{\partial^2 \Pi}{\partial r^2}\sigma^2$$

From previous exercises we know both $\frac{\partial \Pi}{\partial r} = -40.1588$ and $\frac{\partial^2 \Pi}{\partial r^2} = 69.3465$, so:

$$\frac{\partial \Pi}{\partial t} = 1.4368$$

iii. In one day, the capital gain that can be contributed to the passage of time is:

$$0.0751 \times \frac{1}{252} = 0.0057$$

This is the partial derivative of the value of the portfolio against time. Thus, we can interpret this as the following: each day, the value of the portfolio will rise by .0057 billion when interest rate does not change. Since this portfolio is not "Delta Hedged", the Theta-Gamma relation does not hold. However, holding Delta constant, we can still see the impact that a low (or even negative) Gamma will tend to lead to high Theta.

d. The following historgram plots the possible values for the portfolio in one year. The following table summarizes the results:

VaR	Initial	1-year	Loss
@ 5%	20.5	18.81	-1.68
@ 1%	20.5	18.36	-2.13

As seen in the table the results experienced by Oragne County's portfolio where completely forseeable.

Exercise 10.

a. The price of the option is: Call (x 100) = \$0.1851.

b. The value of the option when $r_0 = 5.09\%$ is: Call (x 100) = \$0.0132 ; the value of the option when $r_0 = 10\%$ is: Call (x 100) = \$0.0000.

c. The value of the option when $T_O = 0.5$ is: Call (x 100) = \$0.0208 ; the value of the option when $T_O = 2$ is: Call (x 100) = \$1.2821.

d. The value of the option when $T_B - T_O = 1$ is: Call (x 100) = \$15.5015 ; the value of the option when $T_B - T_O = 10$ is: Call (x 100) = \$0.0000.

Exercise 11.

a. The price of the option is: Put (x 100) = \$3.0568.

b. The value of the option when $r_0 = 5.09\%$ is: Put (x 100) = \$5.5123 ; the value of the option when $r_0 = 10\%$ is: Put (x 100) = \$9.2131.

c. The value of the option when $T_O = 0.5$ is: Call (x 100) = \$4.1708 ; the value of the option when $T_O = 2$ is: Call (x 100) = \$1.0067.

d. The value of the option when $T_B - T_O = 1$ is: Call (x 100) = \$0.0000 ; the value of the option when $T_B - T_O = 10$ is: Call (x 100) = \$22.6339.

Finally, from Put-Call parity we know:

$$Call(K) = Put(K) + Z(0, T_1) \times \left(P^{fwd}(0, T_1, T_2) - K\right)$$

This means that we can obtain the forward price from the Call and Put values:

$$P^{fwd}(0, T_1, T_2) = \frac{Call(K) - Put(K)}{Z(0, T_1)} + K$$

Using the previous values we get that $P^{fwd}(0, T_1, T_2) = \$79.77$.
Exercise 12. The value of the call option is: Call (x 100) = \$0.0008.
Exercise 13.

a. The parameters for the model are: $\gamma^* = 0.3333$, $\bar{r}^* = 0.0576$ and $\sigma = 0.0157$, with $r_0 = 0.51\%$.

b. The value of a non-callable bond is: \$111.8684.

c. The value of the option embedded in the callable bond is: \$4.0946. So the value of the callable bond is: \$107.7739.

d. If now $T_O = 1$ (with everything else staying the same) we have that the value of the non-callable bond is still \$111.8684, but the price of the call option on the bond is now \$7.7092, so the value of the callable bond is now: \$104.1592.

Solutions to Chapter 16

Exercise 1.

We follow these steps:

1. Parameter Estimation: We use the data given in Table 15.4 to fit the Vasicek model, for which we obtain $\gamma^* = 0.0290$, $\bar{r}^* = 2.09\%$ and $\sigma = 1.78\%$. The graph shows the yield curve and its fitted values according to the Vasicek model. For σ one possiblity is to estimate the value from historical data or simply include it as a free parameter in the optimization process (see Figure 1).

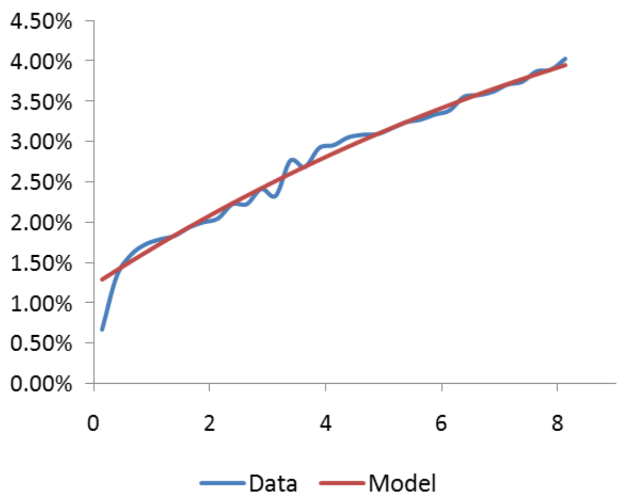

Figure 1: Yield curve and Vasicek model (September 25, 2008)

2. Relative Pricing Error Discovery: We look for the bond where there is the highest difference between the model and the data prices.

τ	Z^{Data}	Z^{Model}	Diff.
0.139	99.908	99.821	
0.389	99.488	99.457	
0.639	98.989	99.038	
0.889	98.483	98.569	
1.139	97.992	98.050	
1.389	97.497	97.485	
1.639	96.889	96.875	
1.889	96.304	96.224	
2.139	95.732	95.533	
2.389	94.840	94.805	
2.639	94.314	94.042	
2.889	93.284	93.247	
3.139	92.967	92.421	0.546
3.389	91.082	91.567	
3.639	90.715	90.688	
3.889	89.284	89.784	
4.139	88.508	88.858	
4.389	87.488	87.913	
4.639	86.694	86.950	
4.889	85.998	85.970	0.028
5.139	85.024	84.977	
5.389	84.009	83.971	
5.639	83.182	82.954	
5.889	82.195	81.928	
6.139	81.267	80.895	
6.389	79.716	79.855	
6.639	78.908	78.811	
6.889	77.982	77.764	
7.139	76.782	76.714	
7.389	75.895	75.664	
7.639	74.447	74.614	
7.889	73.603	73.566	
8.139	72.096	72.520	-0.424

Two bonds appear to be particularly promising $\tau = 3.139$ and $\tau = 8.139$, since they have the biggest gap between prices. We will use $\tau = 3.139$, which seems to be overpriced.

3. Set-Up Trading Strategy:

   i. Since $Z(3.139)$ seems to be overpriced we short sell it, making a profit of \$92.9670 per bond.

   ii. Using the bond maturing at $\tau = 4.889$ we buy a synthetic position:

$$\hat{Z}(3.139) = \Delta \times Z(4.889) + C_0 = \$92.4213$$

where:

$$\Delta = \frac{B(0, 3.139) \times Z^{Model}(r, 0; 3.139)}{B(0, 4.889) \times Z(4.889)} = 0.7074$$

$$C_0 = Z^{Model}(r, 0; 3.139) \times 100 - \Delta \times Z(4.889) \times 100 = 31.5866$$

In other words, for each $Z(3.139)$ bond we shorted, we buy 0.7074 of a $Z(4.889)$ bond and take a \$31.5866 cash position. This gives a \$0.5457 profit per bond.

Exercise 2. Note that the replication strategy gives good results, specially as we reduce the dt.

	$dt = 1/252$		$dt = 1/52$	
rdt	$Z(rdt, dt, T_1)$	Pdt	$Z(rdt, dt, T_1)$	Pdt
0.00%	95.8808	95.8930	95.9205	95.9303
0.50%	94.4544	94.4447	94.5006	94.4857
1.00%	93.0492	93.0291	93.1016	93.0735
1.22%	92.4306	92.4094	92.4858	92.4553
1.50%	91.6649	91.6453	91.7234	91.6930
2.00%	90.3011	90.2927	90.3656	90.3435
2.50%	88.9577	88.9705	89.0279	89.0243

Exercise 3. As seen in Figures 2 and 3 the replication strategy works. We can also notice an improvement as we increase n (reduce dt).

Exercise 4.

a. The price of the option is: Call (x 100) = \$13.23.

b. In order to hedge the call we pick the zero coupon bond maturing at the same time. Two methodologies were proposed to obtain the hedge ratio for the replicating portfolio:

    i. Using equation (16.31) we obtain: $\Delta = 0.6608$ and $C_0 = -0.4797$.

    ii. Through the numerical approximation we obtain: $\Delta = 0.6608$ and $C_0 = -0.4797$. Which is almost identical to the number obtained with the previous methodology.

c. Figure 4 shows that the strategy does a good job replicating the option.

Exercise 5.

a. The price of the option is: Put (x 100) = \$0.50.

b. In order to hedge the call we pick the zero coupon bond maturing at the same time. Two methodologies were proposed to obtain the hedge ratio for the replicating portfolio:

    i. Using equation (16.31) we obtain: $\Delta = -0.0617$ and $C_0 = 0.0622$.

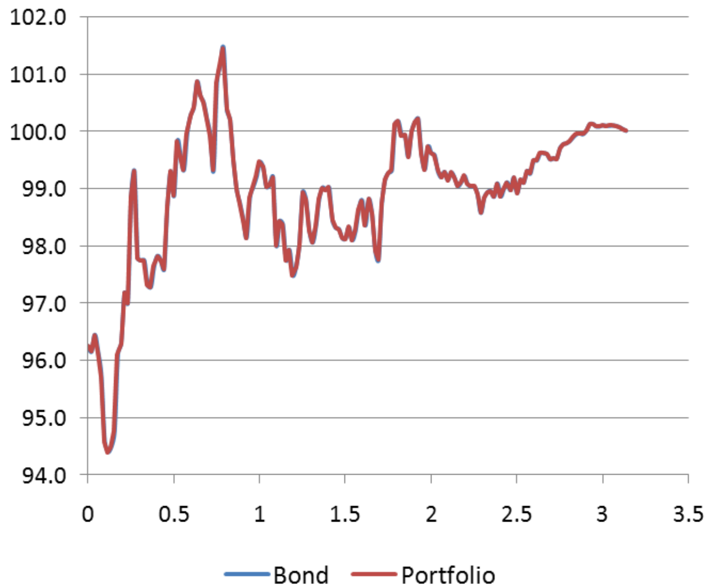

Figure 2: Vasicek model: Bond and Replicating Portfolio (n=52)

ii. Through the numerical approximation we obtain: $\Delta = -0.0617$ and $C_0 = 0.0622$. Which is almost identical to the number obtained with the previous methodology.

c. Figure 5 shows that the strategy does a good job replicating the option.

Exercise 6. The callable bond can be thought as a portfolio consisting in a long position in a non-callable bond and a short position in a call option on the bond.

a. Althought an analytical formula can be obtained, due to the coupon payments involved, using it to obtain $\frac{\partial V}{\partial r}$ would be particularly cumbersome. Instead we use the following approximation:

$$\frac{\partial V}{\partial r} \approx \frac{[V(r_{0+\delta}) - V(r_{0-\delta})]}{2 \times \delta}$$

The following table summarizes the results:

Asset	$V(r_0)$	$V(r_{0+\delta})$	$V(r_{0-\delta})$	$\frac{\partial V}{\partial r}$
Non-callable bond	111.87	111.84	111.90	-277.99
Call option	4.0946	4.0857	4.1034	-88.43
Callable Bond	107.77	107.75	107.79	-189.55

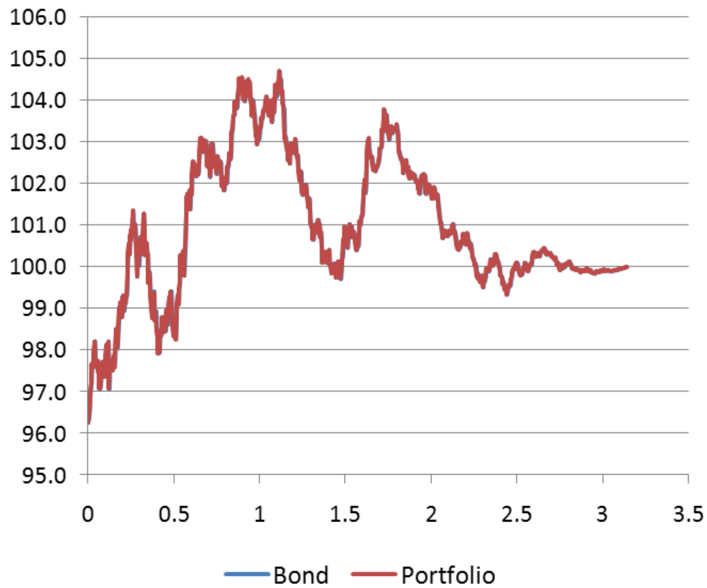

Figure 3: Vasicek model: Bond and Replicating Portfolio (n=252)

b. To hedge the position we can use the underlying bond, where $\Delta = -0.6819$ and $Cash = -31.4929$.

c. The new position will now be $\Delta = -0.3211$ and $Cash = -68.2411$.

Exercise 7. In order to perform the trade we proceed as follows:

1. Parameter Estimation:

   i. Risk Natural Parameters. Recall that the Cox-Ingersoll-Ross (CIR) process for the short rate is defined by:

$$dr = (\eta - \gamma r_t)\, dt + \sqrt{\alpha r_t} dX$$

   As in the Vasicek case we would estimate risk neutral parameters through calibrating the model to the data (using the Non-Linear Least Squares methodology), yet the volatility parameter α we would derive from the actual data. Additionally, if we want to test the model against simulated values of interest rates we will eventually need the risk natural parameters as well. For this reason, we start by obtaining the risk natural parameters and the volatility parameter, before calibrating the model to the risk neutral parameters. As a reminder, recall that in the case of the Vasicek model we had:

$$dr_t = \gamma\,(\bar{r} - r_t)\, dt + \sigma dX_t$$

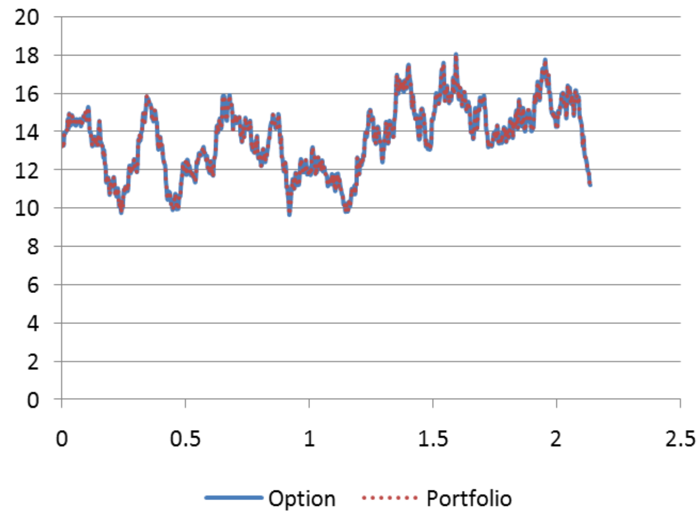

Figure 4: Value of the asset and of the replicating portfolio

and we estimated the risk natural parameters through a regression of the form (using data on the overnight repo rate from May 21, 1991 - February 17, 2004):

$$r_{t+dt} = \alpha + \beta_1 r_t + \sigma_{S.E.}\epsilon$$

where: $\alpha = \gamma\bar{r}dt$; $\beta_1 = (1-\gamma dt)$, $\sigma = \frac{\sigma_{S.E.}}{\sqrt{dt}}$ and $\epsilon \quad N(0,1)$. In the case of the CIR model this regression cannot be used because there is a $\sqrt{r_t}$ in the variance term. In order to eliminate it we must divide both sides of the equation by $\sqrt{r_t}$:

$$\frac{dr_t}{\sqrt{r_t}} = \frac{\gamma(\bar{r}-r_t)dt}{\sqrt{r_t}} + \sqrt{\alpha dt}\epsilon$$

Decomposing dr_t and rearranging a few terms, we have:

$$\frac{r_{t+dt}}{\sqrt{r_t}} = \gamma\bar{r}dt\frac{1}{\sqrt{r_t}} + (1-\gamma dt)\frac{r_t}{\sqrt{r_t}} + \sqrt{\alpha dt}\epsilon$$

Which can be summarized as the following regression:

$$\frac{r_{t+dt}}{\sqrt{r_t}} = \beta_1\frac{1}{\sqrt{r_t}} + \beta_2\frac{r_t}{\sqrt{r_t}} + \sigma_{S.E.}\epsilon$$

where: $\beta_1 = \gamma\bar{r}dt$, $\beta_2 = (1-\gamma dt)$ and $\sigma_{S.E.} = \sqrt{\alpha dt}$.
Running a simple regression gives the following results (p-values in parenthesis), as well as the following parameters for the CIR model.

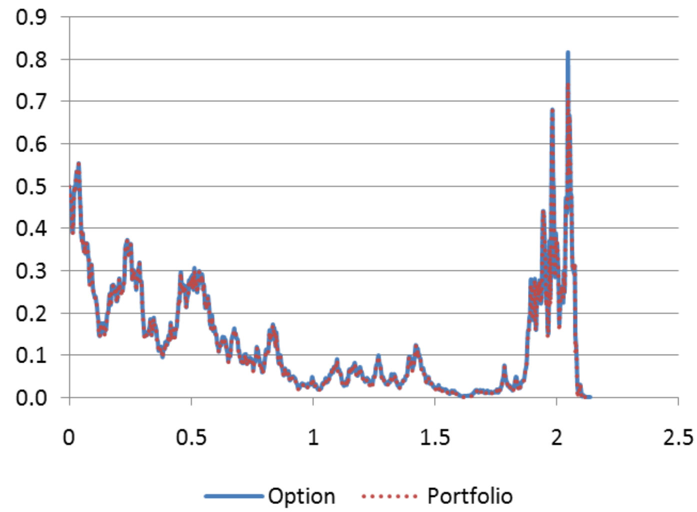

Figure 5: Value of the asset and of the replicating portfolio

α	0.000542	dt	1/252
	(0.9224)		
β_1	0.000150	γ	1.6367
	(0.7290)	$\bar{r}$	0.0231
β_2	0.993505	α	0.0265
	(0.0000)		
$\sigma_{S.E.}$	0.010258		

Note that neither the intercept nor β_1 are significant. The reason is that the regression is including the intercept, which is not present in the model above. We must run the regression without intercept in order to have a better fit with our model. The results of doing so are the following:

α	0	dt	1/252
	(N/A)		
β_1	0.000192	γ	1.2373
	(0.0026)	$\bar{r}$	0.0391
β_2	0.995090	α	0.0265
	(0.0000)		
$\sigma_{S.E.}$	0.010256		

ii. Risk Neutral Parameters. In order to price the bonds we need to find the risk neutral parameters (γ^* and $\bar{r}^*$). To do so we use the Non-Linear Least Squares approach to calibrate the parameters to the data given in Table 16.4 (we follow the same steps as in section 16.8). The minimization procedure yields $\bar{r}^* = 16.53\%$ and $\gamma = 0.0622$.

Figure 6 presents the spot rate from the CIR model.

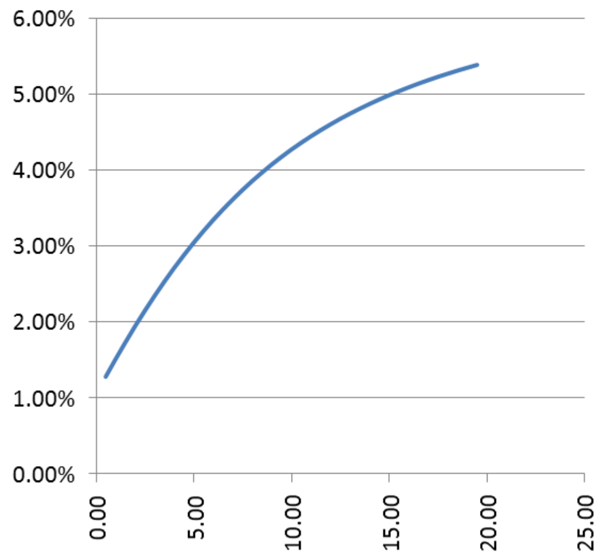

Figure 6: Spot rate

2. Relative Pricing Error Discovery. Figure 7 presents pricing errors from the CIR model. Note that these are very similar to the ones given by using the Vasicek model (see Figure 16.7 in the book), so hedging the 1.5-year bond seems like a good idea. In this case, though, it seems that the 6-year bond is a better hedging instrument than the 7.5-year bond.

3. Set Up a Trading Strategy. We decide to sell short the 1.5-coupon bond and take a long position on the 6-year bond and the cash acount (in order to replicate the shorted bond). At time zero, we have:

$$\Delta_0 = \frac{\partial P_{1.5-yr}(r,0)/\partial r}{\partial P_{6.0-yr}(r,0)/\partial r} = \frac{-147.41}{-456.45} = 0.3229$$

and:

$$C_0 = -P^{CIR}_{1.5-yr} + \Delta_0 \times P_{6.0-yr} = -107.0385 + 0.3229 \times 117.5625 = -69.0725$$

4. Simulations. We then simulate a path on interest rates to check how well the model works (see Figure 8). Note that this model is not as good as the Vasicek model in replicating the value of the bond.

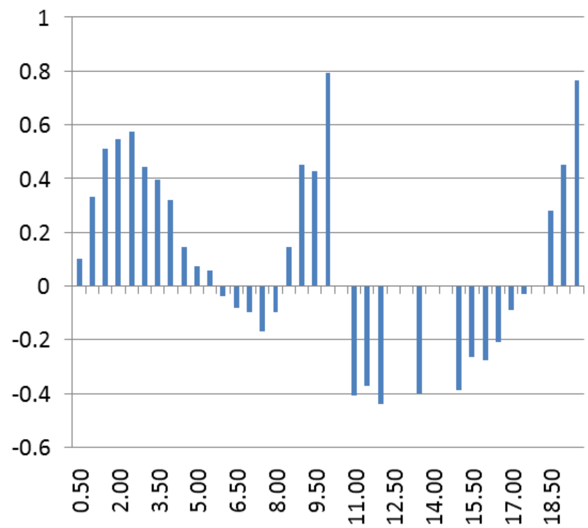

Figure 7: Pricing Errors

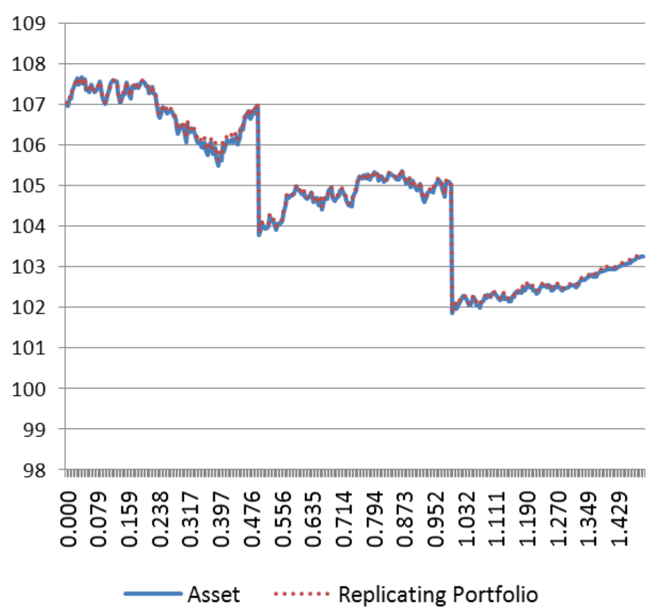

Figure 8: Asset value and replicating portfolio

Solutions to Chapter 17

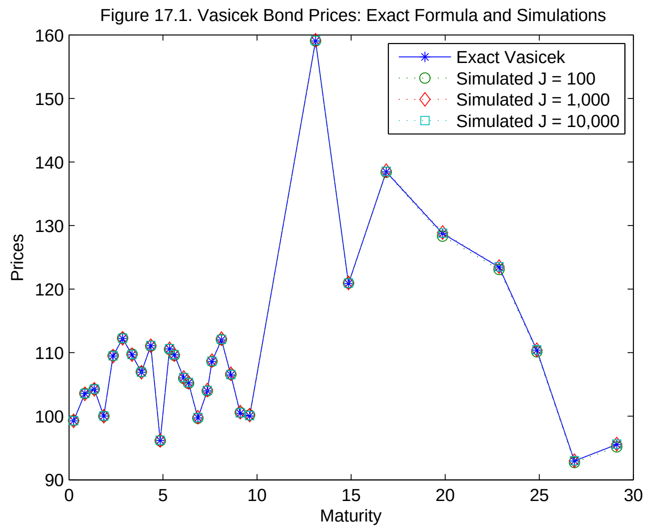

Figure 1: Vasicek Bond Prices: Exact Formula and Simulations

Exercise 1. Figure 1 presents the result of fitting the Vasicek model with the parameters given in this chapter. Additionally we present four variations:

i. Figure 2 presents the result when: γ^* + 300 bps.

ii. Figure 3 presents the result when: γ^* - 300 bps.

iii. Figure 4 presents the result when: σ + 200 bps.

iv. Figure 5 presents the result when: σ - 200 bps. All results tend to converge to the analytical formula do to the Feynman-Kac theorem. Note that decreasing γ^* and increasing σ requieres more simulations for the resutls to converge, this is intuitive since γ refers to the "speed" by which the process converges to the long-term mean ($\bar{r}$) and σ refers to the volatility of the process. Lower convergence speed and higher volatility will requiere more simulations for the methodologies to converge.

Exercise 2. Figure 6 presents the values for Delta for all maturities. Figure 7 presents the values for Gamma for all maturities.
Exercise 3. Figure 8 presents the result of fitting the CIR model with the parameters given in this chapter. Additionally we present four variations:

i. Figure 9 presents the result when: γ^* + 3000 bps.

ii. Figure 10 presents the result when: γ^* - 3000 bps.

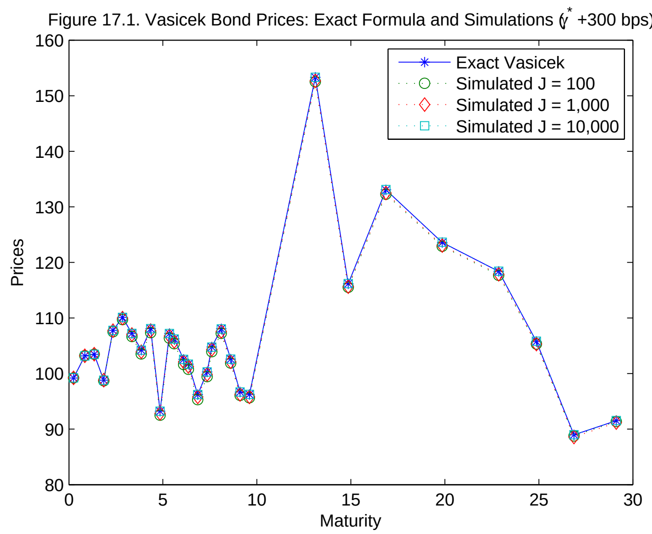

Figure 2: Vasicek Bond Prices: Exact Formula and Simulations (γ^* + 300 bps)

iii. Figure 11 presents the result when: α + 300 bps.

iv. Figure 12 presents the result when: α - 300 bps.

Exercise 4.

a. The parameters obtained by fitting the Vasicek model are: $\gamma^* = 0.1043$, $\bar{r}^*$ $= 0.0773$ and $\sigma = 0.0144$. Figure 13 presents the fit of the model with the data. Figure 14 presents the squared errors of the Vasicek model values with the data.

b. The price of the Inverse Floater is: \$1.0010.

c. The price of the Inverse Floater using simulations is: \$1.0000.

d. With the restriction the price of the Inverse Floater is: 1.0230.

e. In order to hedge the position we must have: Δ = 4.7227 and Cash = -3.3257.

Exercise 5.

a. The payoff of the option is: $max(1 \times 0.97 - V(r_{T_P}, T_P), 0)$.

b. The values of the options in each case are the following:

   i. The value of the option without the restriction is: 0.0735.

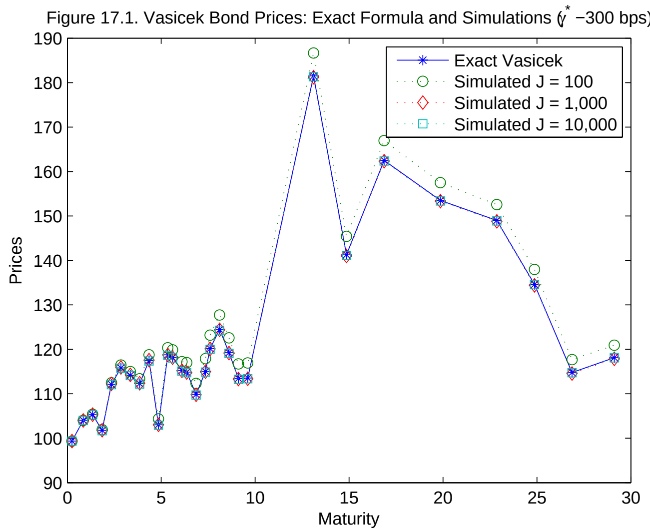

Figure 3: Vasicek Bond Prices: Exact Formula and Simulations (γ^* - 300 bps)

ii. The value of the option with the restriction is: 0.0511. This value of the option goes down because the restriction already "hedges" for scenarios where value is lost on its own.

c. The value of the Puttable Leveraged Inverse Floater is: 1.0735. With the restriction the value would be: 1.0740.

Exercise 6.

a. Fitting the Vasicek model we obtain the following parameters: $\gamma^* = 0.7738$, $\bar{r}^* = 0.0503$ and $\sigma = 8.3280\text{e-}005$. Figures 15 and 16 present the bootstrapped discount curve and yield curve, respectively.

b. Figures 17 and 18 present the comparisson of the fitted Vasicek model to the bootstrapped discount curve and yield curve, respectively.

c. The price of the Corridor Note is 1.1196.

d. These notes are essentially a bet on where the 6 month rate will stay within the next 10 years. In particular, these notes could be used as a hedge for positions which payoff if the 6 month LIBOR becomes either unusually high or low. The floating nature of the payoff for the last 5 years of the life of the note could also be used as a hedge for 3 month LIBOR swaps.

e. We have that Δ = -9.3965 and Γ = -0.0017116.

Exercise 7.

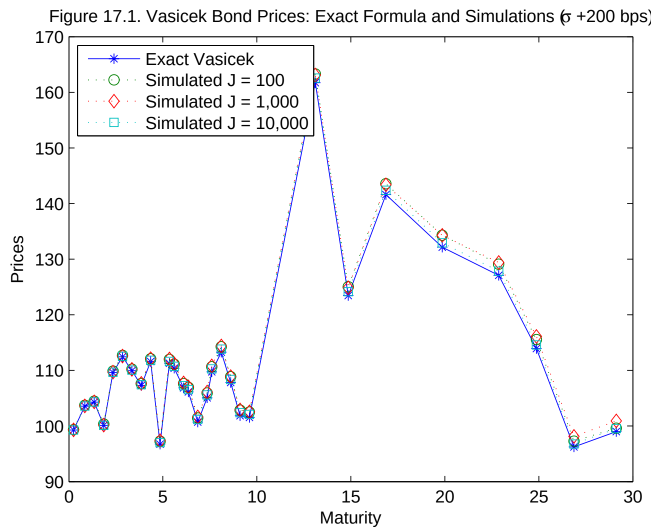

Figure 4: Vasicek Bond Prices: Exact Formula and Simulations (σ + 200 bps)

a. Fitting the Vasicek model we obtain the following parameters: $\gamma^* = 0.1345$, $\bar{r}^* = 0.0631$ and $\sigma = 0.0076$. Figures 19 and 20 present the comparisson of the fitted Vasicek model to the bootstrapped discount curve and yield curve, respectively.

b. We obtain the following:

   i. The price obtained by MCS is: \$100.6467, which is higher than the traded prices.

   ii. We need to make an adjustment given that the prepayment model is based in real values of the swap rate (not in the risk neutral world). This adjustment would increase the rates in the CPR calcualtion which would mean that more prepayments would occur, which would bring down the value of the security.

c. We have that Δ = -302.0721 and Γ = -0.0065869.

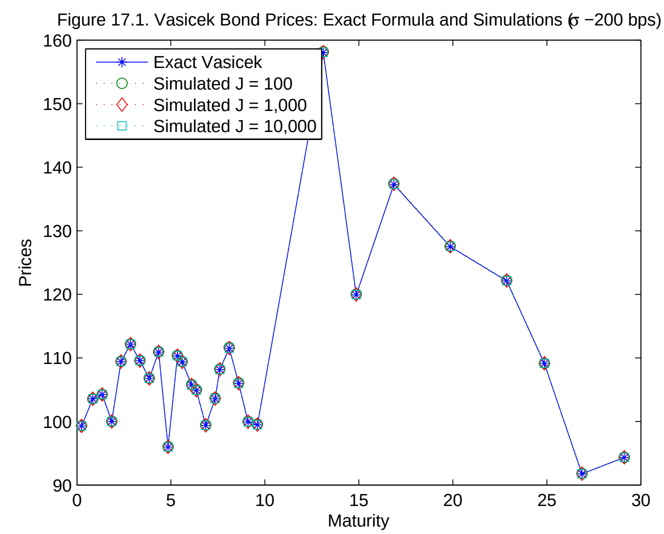

Figure 5: Vasicek Bond Prices: Exact Formula and Simulations (σ - 200 bps)

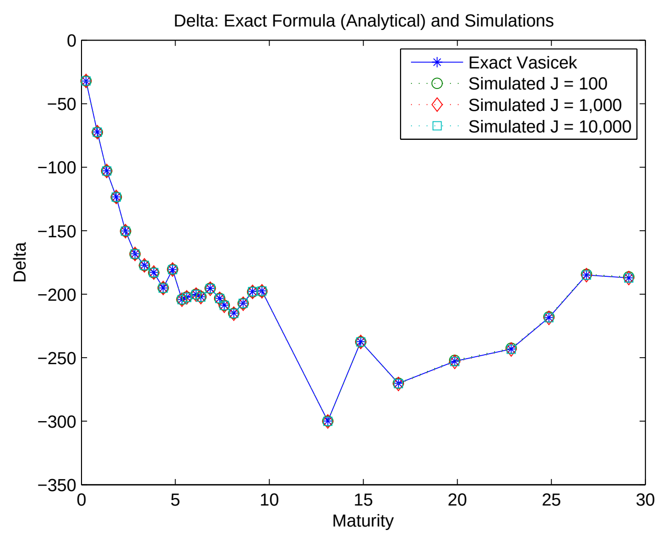

Figure 6: Delta: Exact Formula (Analytical) and Simulations

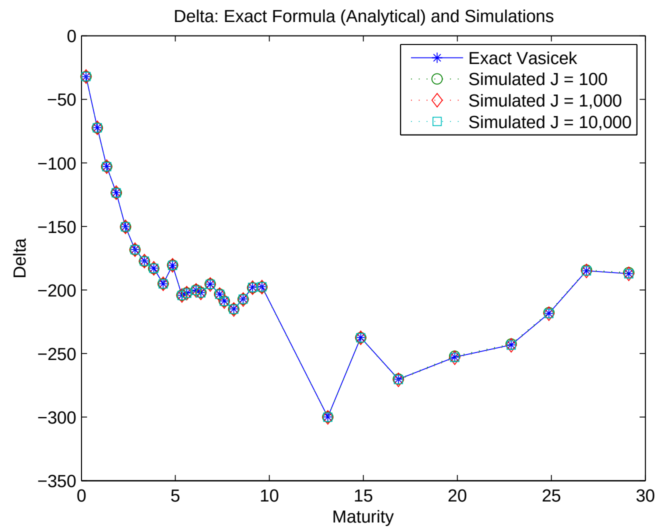

Figure 7: Gamma: Exact Formula (Analytical) and Simulations

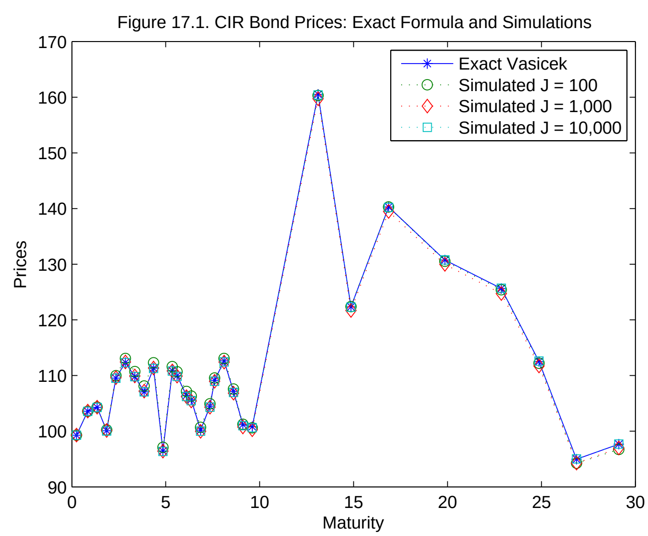

Figure 8: CIR Bond Prices: Exact Formula and Simulations

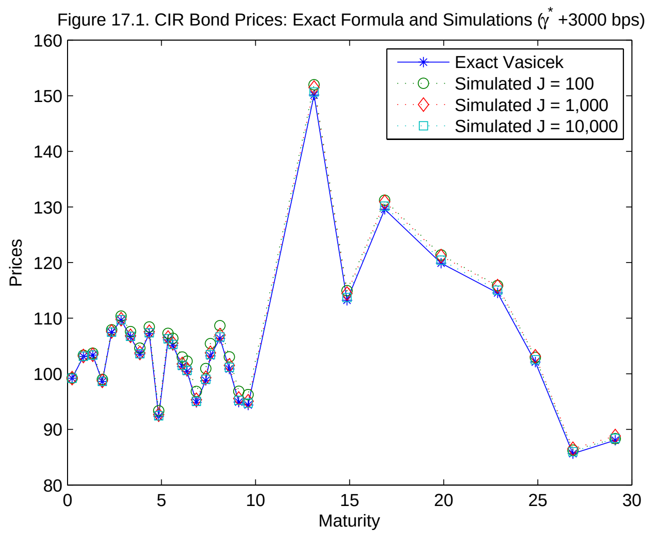

Figure 9: CIR Bond Prices: Exact Formula and Simulations (γ^* + 3000 bps)

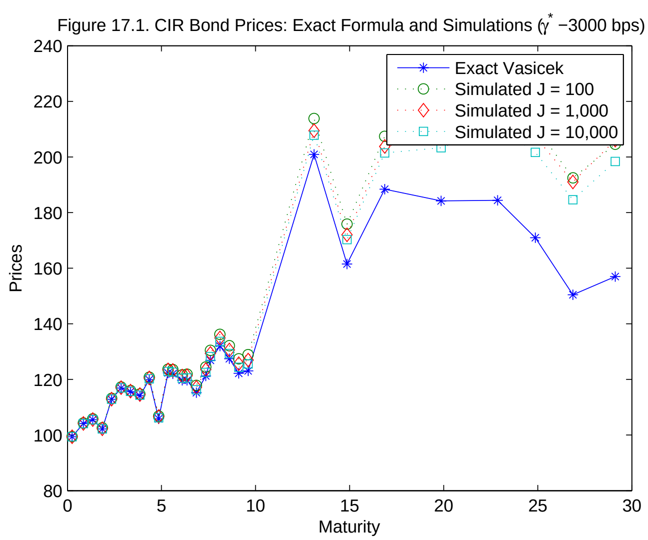

Figure 10: CIR Bond Prices: Exact Formula and Simulations (γ^* - 3000 bps)

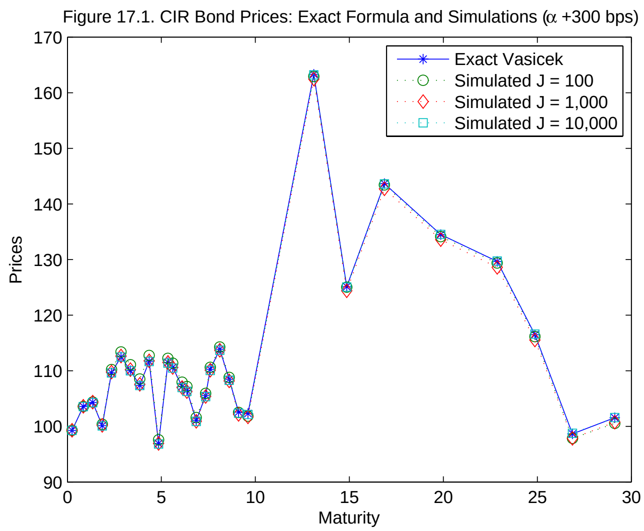

Figure 11: CIR Bond Prices: Exact Formula and Simulations (α + 300 bps)

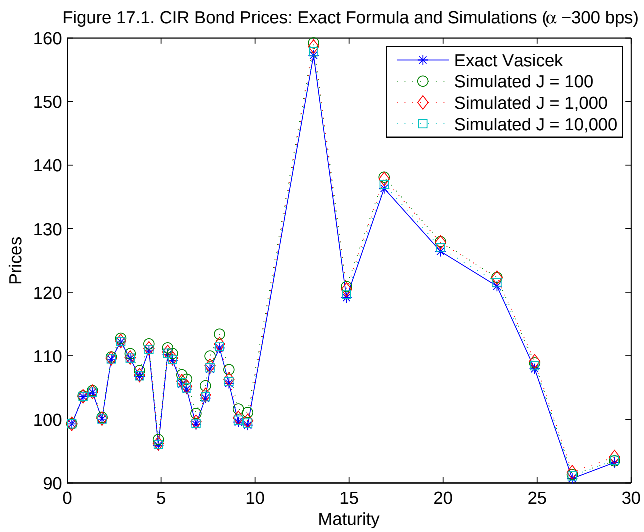

Figure 12: CIR Bond Prices: Exact Formula and Simulations (α - 300 bps)

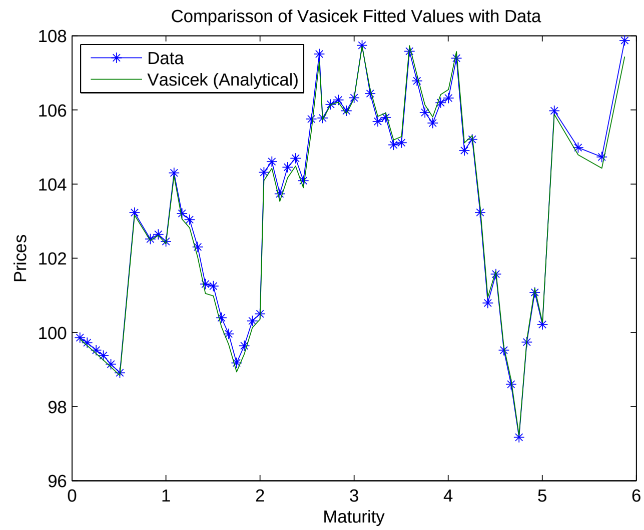

Figure 13: Comparisson of Vasicek Fitted Values with Data

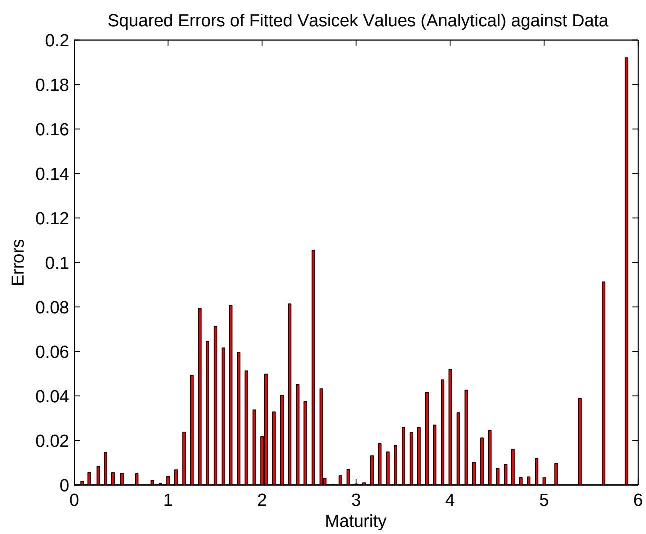

Figure 14: Squared Errors of Fitted Vasicek Values (Analytical) against Data

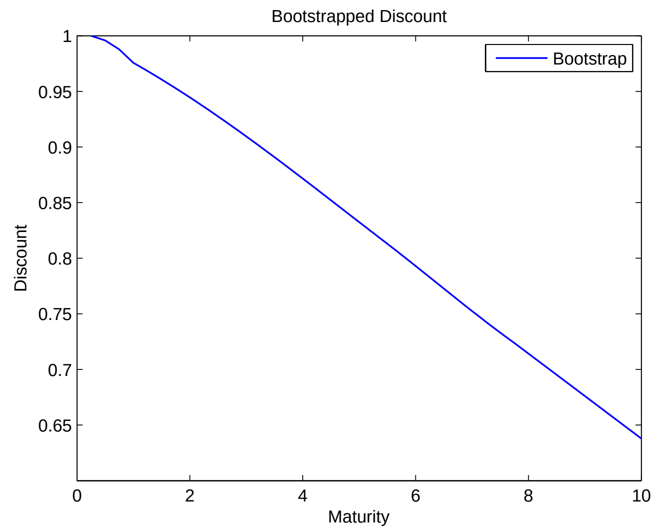

Figure 15: Bootstrapped Discount Curve

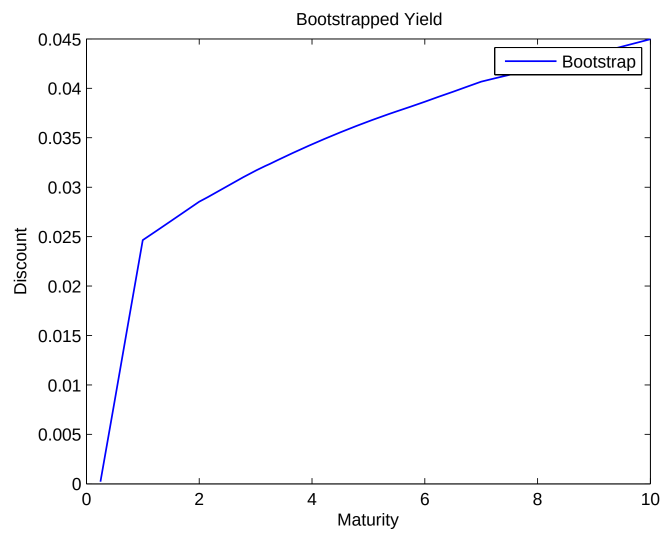

Figure 16: Bootstrapped Yield Curve

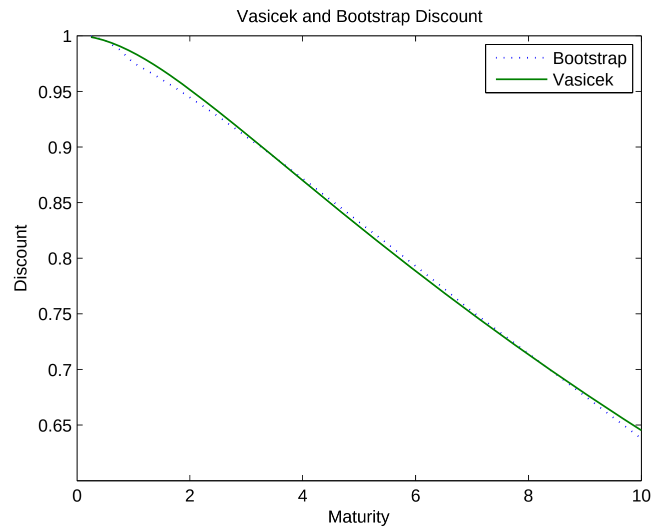

Figure 17: Vasicek and Bootstrap Discount Curve

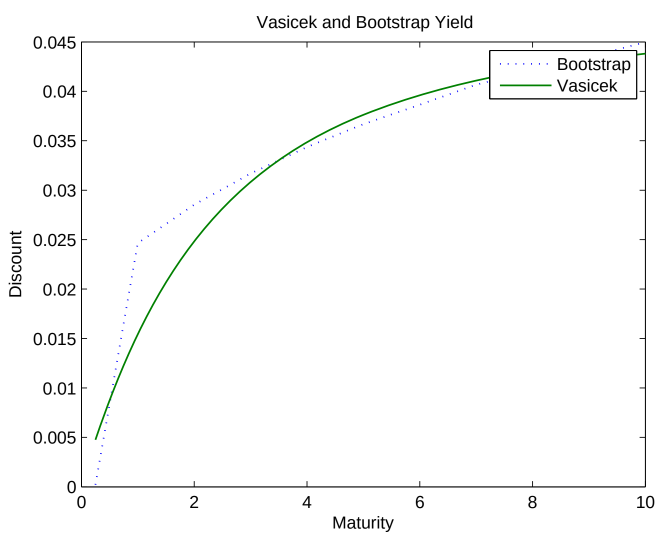

Figure 18: Vasicek and Bootstrap Yield Curve

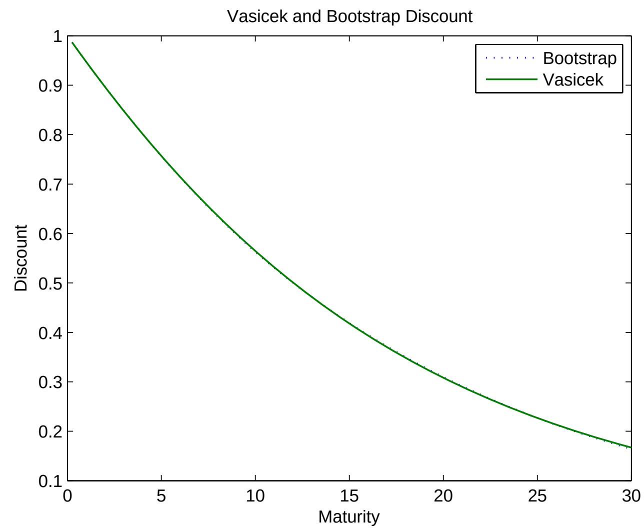

Figure 19: Vasicek and Bootstrap Discount Curve

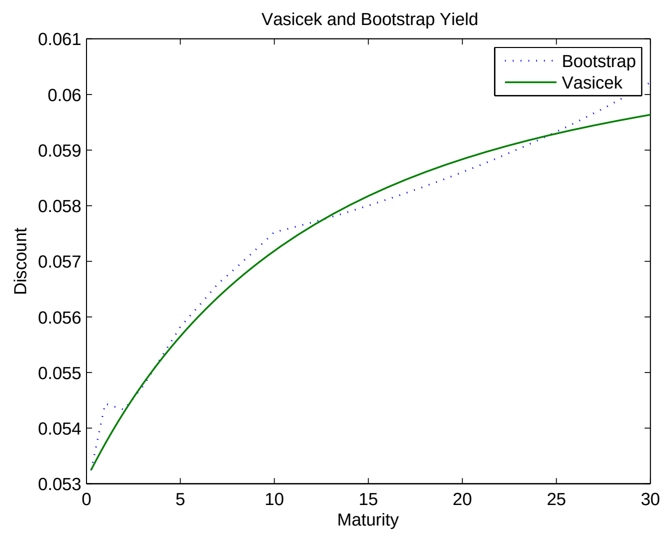

Figure 20: Vasicek and Bootstrap Yield Curve

Solutions to Chapter 18

Exercise 1.

i. For a 1-month horizon we have that the Value at Risk is (par = $100):

95% Value-at-Risk	-8.1864
99% Value-at-Risk	-11.1366

The expected shortfall is:

95% Expected Shortfall	-9.8026
99% Expected Shortfall	-11.9301

Figure 1 presents the histogram for these values.

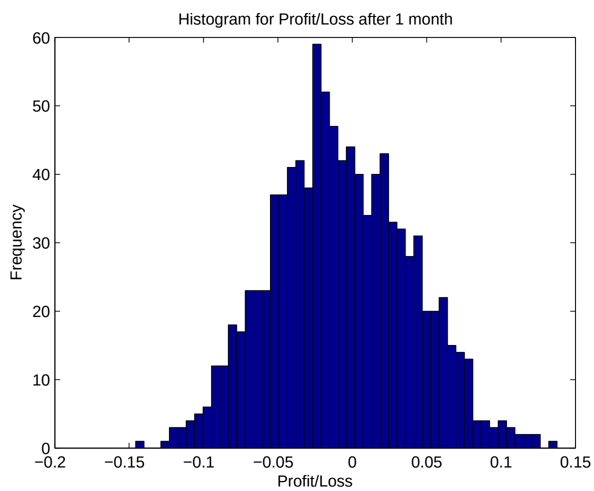

Figure 1: Histogram for Profit/Loss after 1 month

ii. For a 3-month horizon we have that the Value at Risk is (par = $100):

95% Value-at-Risk	-12.4645
99% Value-at-Risk	-18.6730

The expected shortfall is:

95% Expected Shortfall	-15.6296
99% Expected Shortfall	-20.4739

Figure 2 presents the histogram for these values.

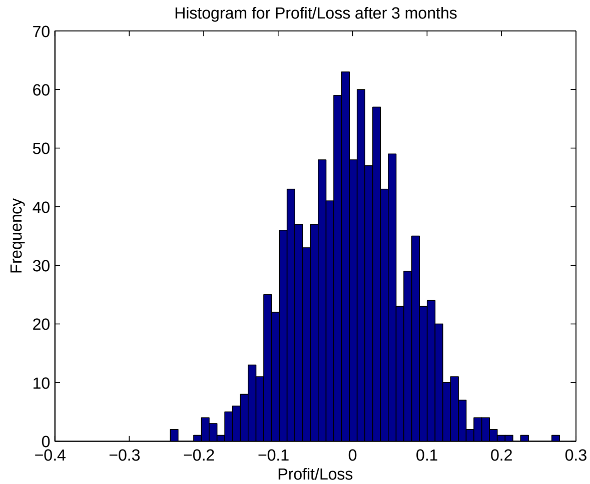

Figure 2: Histogram for Profit/Loss after 3 months

iii. For a 6-month horizon we have that the Value at Risk is (par = \$100):

95% Value-at-Risk	-16.6014
99% Value-at-Risk	-22.5328

The expected shortfall is:

95% Expected Shortfall	-19.9546
99% Expected Shortfall	-24.7113

Figure 3 presents the histogram for these values.

The methodology consists in using the risk natural model to simulate multiple paths of the short rate up to the desired horizon. At each horizon, we reprice the security by using the new value of the short rate instead of r_0 and the adjustment in timing to the cash flows. From these multiple paths we obtain the values needed for the risk analysis.

Exercise 2.

i. For a 1-month horizon we have that the Value at Risk is (par = \$100):

95% Value-at-Risk	-7.0983
99% Value-at-Risk	-9.2530

The expected shortfall is:

95% Expected Shortfall	-8.3996
99% Expected Shortfall	-10.0190

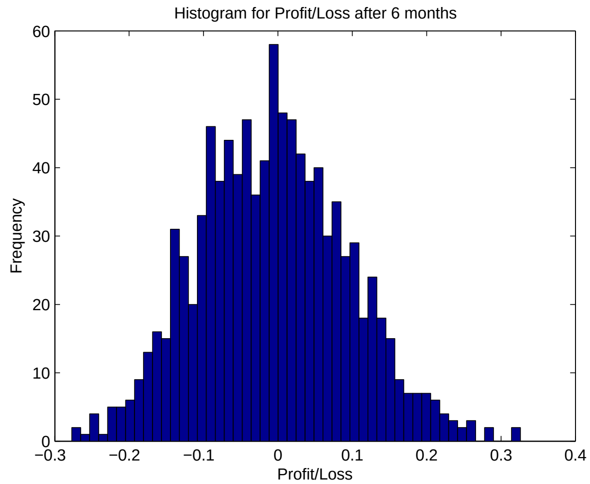

Figure 3: Histogram for Profit/Loss after 6 months

Figure 4 presents the histogram for these values.

ii. For a 3-month horizon we have that the Value at Risk is (par = \$100):

95% Value-at-Risk	-10.4219
99% Value-at-Risk	-15.0273

The expected shortfall is:

95% Expected Shortfall	-12.7550
99% Expected Shortfall	-16.2256

Figure 5 presents the histogram for these values.

iii. For a 6-month horizon we have that the Value at Risk is (par = \$100):

95% Value-at-Risk	-13.0199
99% Value-at-Risk	-17.2202

The expected shortfall is:

95% Expected Shortfall	-15.4216
99% Expected Shortfall	-18.4614

Figure 6 presents the histogram for these values.

This analysis doesn't imply any additional work other than modifying the cash flow equation for the coupon.

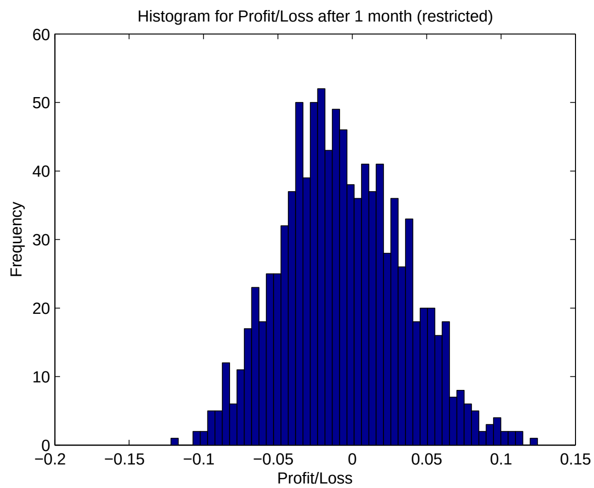

Figure 4: Histogram for Profit/Loss after 1 month (with restriction)

Exercise 3. To perform the risk analysis we have the following steps:

i. Simulate n times the short rate up to six months using the risk natural parameters.

ii. We have a cash flow occuring six months from now so we need to compute it. So we have to count the days on which the 6-month rate has been within the boundaries for the period between the previous coupon (3-months after the beginning of the contract) and the current coupon (6-month coupon after the beginning of the contract). Since this has already happened we use the rates obtained from the risk natural parameters (approximately the last half of the rates obtained in i., except the last rate). We get n cash flows occuring in 6-months, one for each risk natural process simulated in i.

iii. The last rate obtained in i. is the expected short rate in 6-months. We now use it with the risk neutral parameters to price the corridor note with 6-months less maturity. This is particularly cumbersome (i.e. takes a long time) because, in order to price, for each simulated rate in i. we have to make additional simulations until final maturity (9.5 years times 360 days), but with the risk neutral parameters. Additionally there is a lot of details to be aware of on obtaing the cash flow of the Corridor Note.

iv. We add the corresponding first cash flow (from ii.) with the value of the remaining discounted cash flows (from iii.). These is a set of random posible values of the Corridor Note in 6-months (if we want the ex-coupon value we don't include ii.).

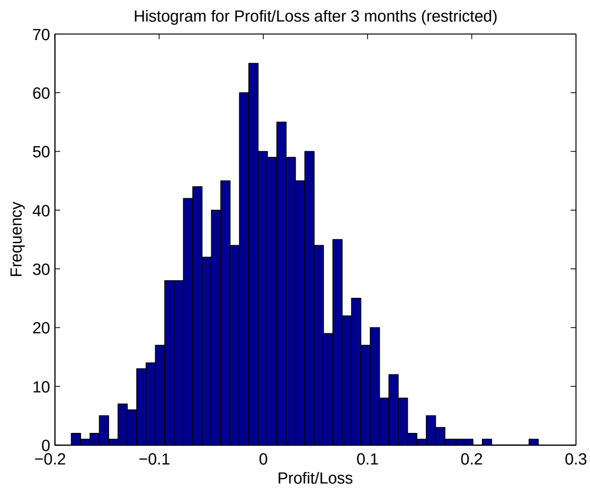

Figure 5: Histogram for Profit/Loss after 3 months (with restriction)

v. We subtract the original value of the Corridor Note (obtaining the Profit and Loss values for the Corridor Note), from which we can obtain Value-at-Risk and expected shortfall

Figure 7 presents the histogram for these values. For a 6-month horizon we have that the Value at Risk is (par = \$100):

95% Value-at-Risk	-8.4706
99% Value-at-Risk	-13.4893

The expected shortfall is:

95% Expected Shortfall	-11.3965
99% Expected Shortfall	-15.8254

Exercise 4.

a. Market price of risk is $\lambda = -0.4577$. In order to see the cause for this recall the definition of market price of risk:

$$\lambda = \frac{\mu_Z - r}{\sigma_Z}$$

where: μ_Z is the expected instantaneous return on the zero coupon bond and σ_Z is the volatility. Given that investors are bearing interest rate risk they would expect a higher rate of return than the short rate on their long-term investments, so: $\mu_Z - r > 0$. This means that the source for the negative market price of risk should be $\sigma_Z < 0$. The definition of σ_Z is:

$$\sigma_Z = \frac{1}{Z}\left(\frac{\partial Z}{\partial r}\right)\sigma$$

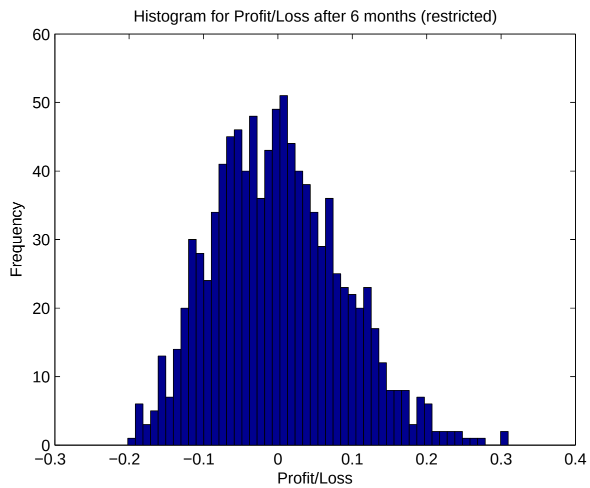

Figure 6: Histogram for Profit/Loss after 6 months (with restriction)

If we know that when interest rates go up, bond prices fall $\frac{\partial Z}{\partial r} < 0$, then it follows that it is true that $\sigma < 0$. Hence, the negative λ implies that our forecast of the interest rate in the long run will be higher for the risk neutral world.

b. Recall that λ can be decomposed in the following:

$$\lambda(r) = \lambda_0 + \lambda_1 r$$

In this particular case we have that: $\lambda_0 = -0.5835$ and $\lambda_1 = 6.2941$. This second term determines the relationship between the interest rate and the market price of risk.

c. Expected excess return is defined as:

$$E\left[\frac{dZ}{Z}\right]/dt - r = \lambda(r)\sigma_Z$$

Recall that:

$$\sigma_Z = \frac{1}{Z}\left(\frac{\partial Z}{\partial r}\right)\sigma$$

and for pricing functions of the form:

$$Z(r,t;T) = e^{A-B(t,T)r}$$

we have that σ_Z can be defined as:

$$\sigma_Z = \frac{1}{Z}\left(-B(t,T)Z\right)\sigma$$

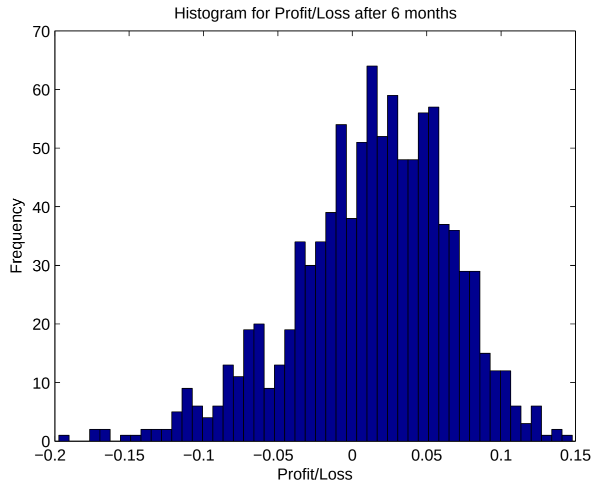

Figure 7: Histogram for Profit/Loss after 6 months

which leads to:

$$\sigma_Z = -B(t,T)\sigma$$

So we get that:

τ	$B(t;T)$	σ_Z	$E\left[\frac{dZ}{Z}\right]/dt - r$
1	0.7996	-0.0177	0.0081
3	1.6170	-0.0357	0.0164
5	1.9393	-0.0429	0.0196
10	2.1287	-0.0470	0.0215

We can find the sensitivity of expected excess returns to changes in interest rate by doing the following:

$$\frac{\partial E\left[\frac{dZ}{Z}\right]/dt - r}{\partial r} = \frac{\partial \sigma_Z \lambda(r)}{\partial r} = \frac{\partial \sigma_Z}{\partial r}\lambda(r) + \sigma_Z \frac{\partial \lambda(r)}{\partial r}$$

Note that σ_Z is not a function of r, so we have that:

$$\frac{\partial \sigma_Z \lambda(r)}{\partial r} = \sigma_Z \frac{\partial \lambda(r)}{\partial r}$$

Also, since $\lambda(r)$ can be defined as:

$$\lambda(r) = \lambda_0 + \lambda_1 r$$

it follows that:

$$\frac{\partial \sigma_Z \lambda(r)}{\partial r} = \sigma_Z \frac{\partial(\lambda_0 + \lambda_1 r)}{\partial r} = \sigma_Z \lambda_1$$

So we get that:

τ	$B(t;T)$	σ_Z	$E\left[\frac{dZ}{Z}\right]/dt - r$	Sensitivity
1	0.7996	-0.0177	0.0081	-0.1112
3	1.6170	-0.0357	0.0164	-0.2249
5	1.9393	-0.0429	0.0196	-0.2698
10	2.1287	-0.0470	0.0215	-0.2961

d.

e. Market price of risk is the same as before, since it does not depend on the asset but, as shown on the section on the macroeconomic model, on market participants' risk aversion, the volatility of GDP growth, and the correlation between GDP growth and expected inflation.

Exercise 5. The exercise uses the following basic parameters (see Table 18.1): $\bar{i} = 4.20\%$, $\gamma = 0.3805$, $g = 0.02$, $\sigma_y = 0.02$, $\sigma_q = 0.0106$, $\sigma_i = 0.0073$, $\rho_{yq} = -0.1409$, $\rho_{yi} = -0.2894$, $h = 104$, and $\rho_{iq} = 0.8360$. Additionally, we consider the case when $i_0 = 4.20\%$.

a. For variations in risk aversion (h) we pick 103.7, 104 and 104.3. As seen in Figures 8 and 9, as risk aversion increases the yield curve becomes steeper and the spread increases. Also the values for λ are -0.5835, -0.5852 and -0.5870, respectively. As explained in this chapter, as risk aversion increases so does the market price of risk (being the driving force behind the changes in the yield curve and the spread). For variations in 'impatience' (ρ) we

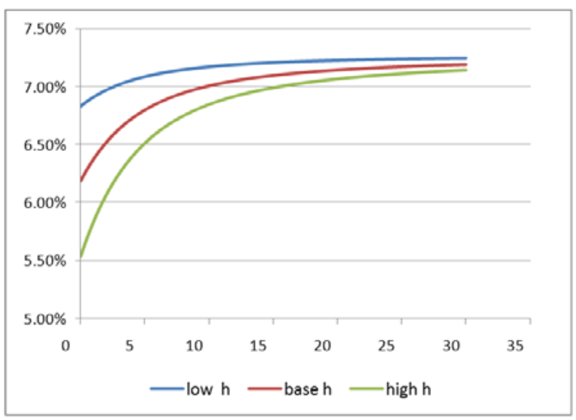

Figure 8: Yield Curve for Variations of h

pick 0.095, 0.100 and 0.105. As seen in Figures 10 and 11, as impatience increases the higher rates are required (since agents will borrow more to consume), also the spread increases. The values for λ is -0.5852 for all cases, because impatience does not affect risk aversion, volatility of GDP growth or the correlation between GDP growth and expected infaltion.

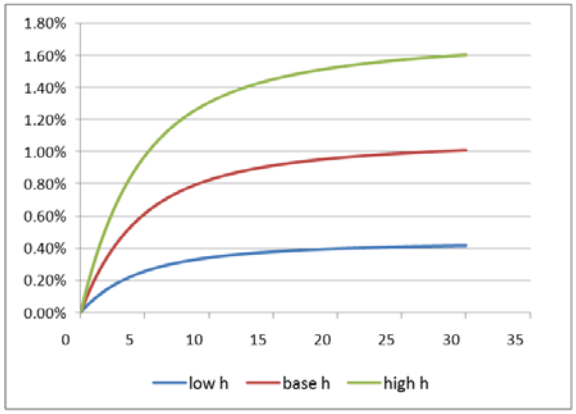

Figure 9: Term Spread for Variations of h

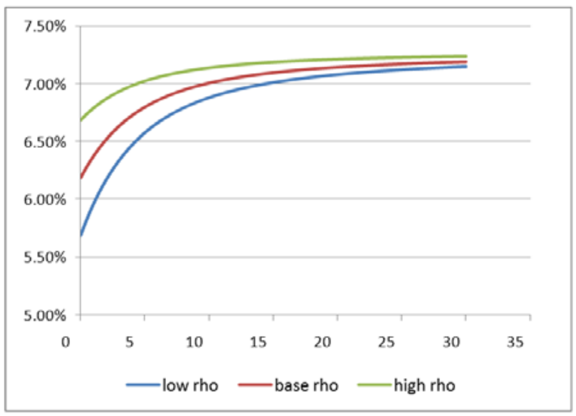

Figure 10: Yield Curve for Variations of ρ

b. For variations in the correlation between GDP growth and expected inflation (ρ_{yi}) we pick -0.1, -0.2894 and -0.5. As seen in Figure 12, as this correlation increases so does the level of interest rates demanded. Also the values for λ are -0.1913, -0.5852 and -1.0233, respectively. These results reflect the fact that this is a risk factor (investors loose money when growth is low and inflation is high), so investors will require additional compensation. For variations in the volatility of GDP (σ_y) we pick 0.01999, 0.02 and 0.02001. As volatility increases so do the level of interest rates demanded. Also the values for λ are -0.5849, -0.5852 and -0.5855, respectively. These results reflect the fact that this is a risk factor (high volatility will make investments more risky), so investors will require additional compensation (see Figure 13).

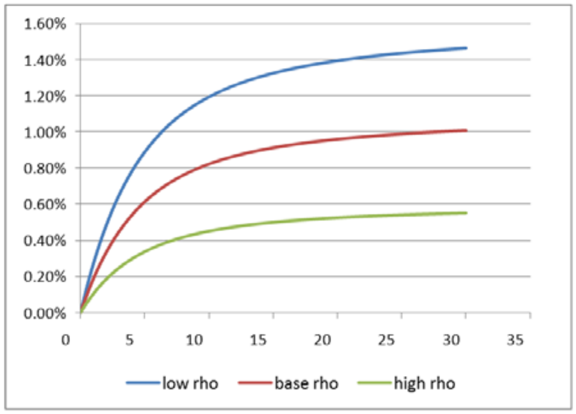

Figure 11: Term Spread for Variations of ρ

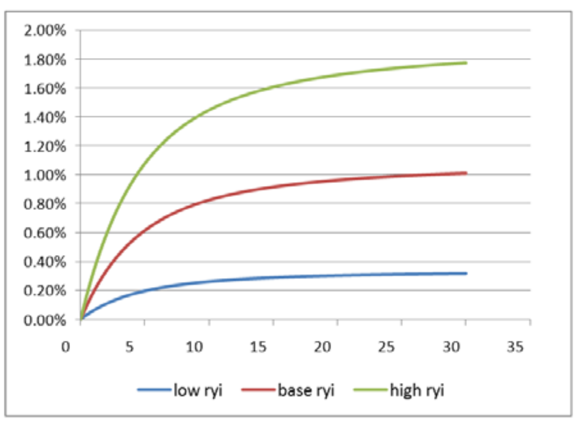

Figure 12: Yield Curve for Variations of ρ_{yi}

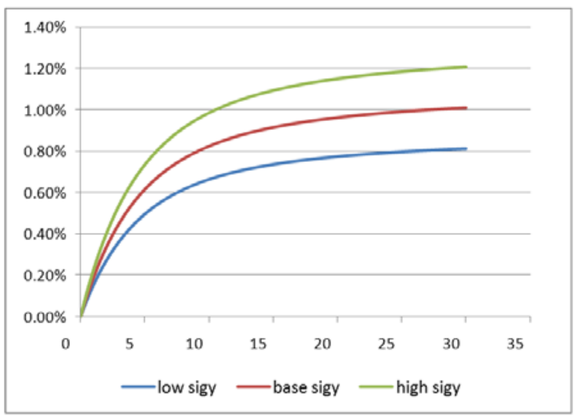

Figure 13: Yield Curve for Variations of σ_y

Solutions to Chapter 19

Exercise 1.

a. In order to obtain the LIBOR yield curve from the swap rates presented:

   i. Obtain the quarterly disocunts from the swap rates.
   ii. From the discounts obtain the continously compounded spot rates, at a quarterly frequency.
   iii. Fit a 10th order polynomial to the data to complete the gaps in the curve (see Figure 1). The parameters from the polynomial are:

parameter	value
α	2.56e-02
β_1	3.84e-02
β_2	-1.61e-01
β_3	2.69e-01
β_4	-2.47e-01
β_5	1.38e-01
β_6	-4.91e-02
β_7	1.11e-02
β_8	-1.55e-03
β_9	1.22e-04
β_{10}	-4.13e-06

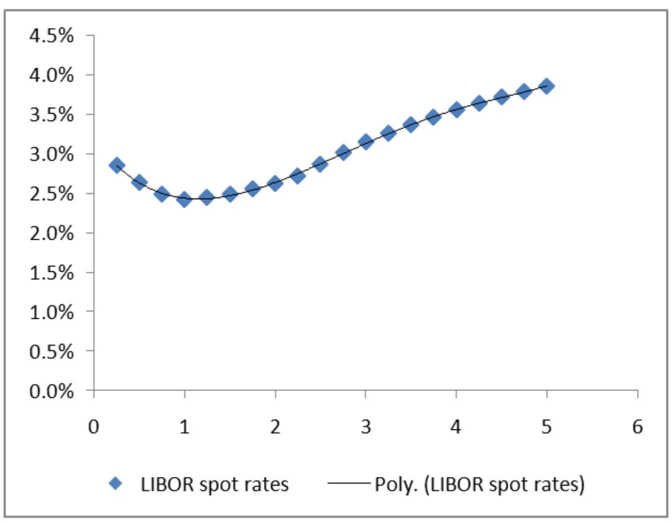

Figure 1: LIBOR yield curve

Note that, as explained in this chapter, when the forward rate is above the yield curve; the yield curve is uppward sloping. Same happens in the opposite case (see Figure 2).

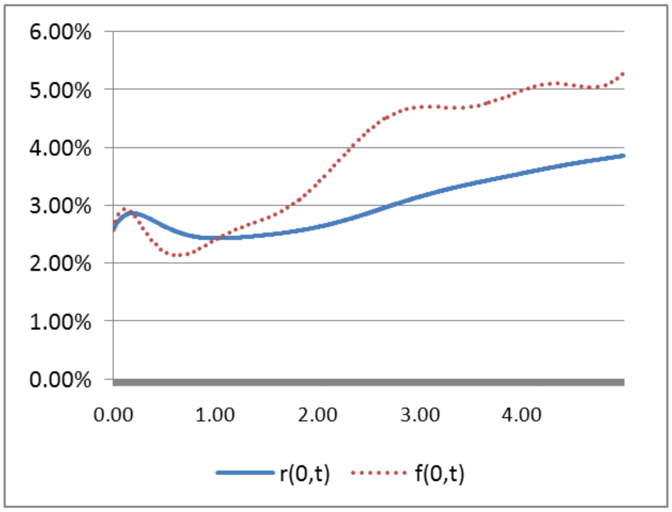

Figure 2: LIBOR yield curve and forward curve

b. The value of σ is 1.00%. The fitted values of the Ho-Lee model match exactly the yield curve.

c. The following chart presents the value of θ_t and the forward curve (See Figure 3). Note that θ_t is very close to the slope of the forward curve.

Exercise 2.

a. See previous answer.

b. For the cap with $T = 0.50$, we have $\sigma = 1.54\%$.

c. For the cap with $T = 0.75$, we have $\sigma = 1.49\%$.

d. The following table summarizes the values of σ needed for each cap maturity T:

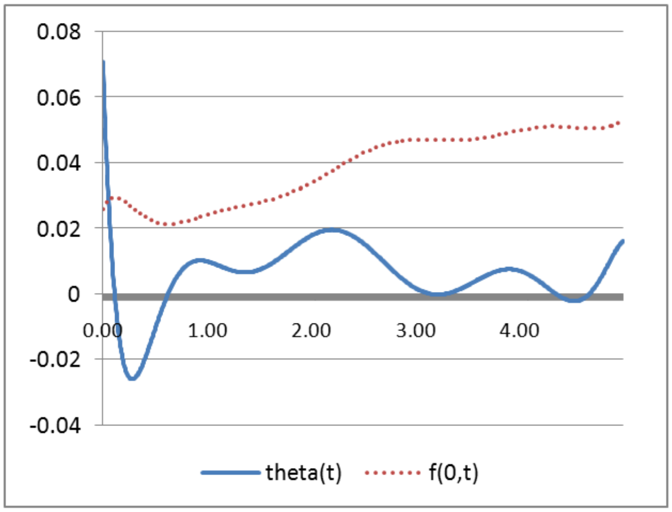

Figure 3: LIBOR yield curve

T	σ
0.25	1.00%
0.50	1.54%
0.75	1.49%
1.00	1.43%
1.25	1.43%
1.50	1.43%
1.75	1.42%
2.00	1.38%
2.25	1.36%
2.50	1.38%
2.75	1.40%
3.00	1.39%
3.25	1.37%
3.50	1.36%
3.75	1.36%
4.00	1.35%
4.25	1.34%
4.50	1.33%
4.75	1.33%
5.00	1.32%

Figure 4 plots these values.

e. The value of σ that best matches all caps is: 1.35%. Figure 5 shows the pricing errors.

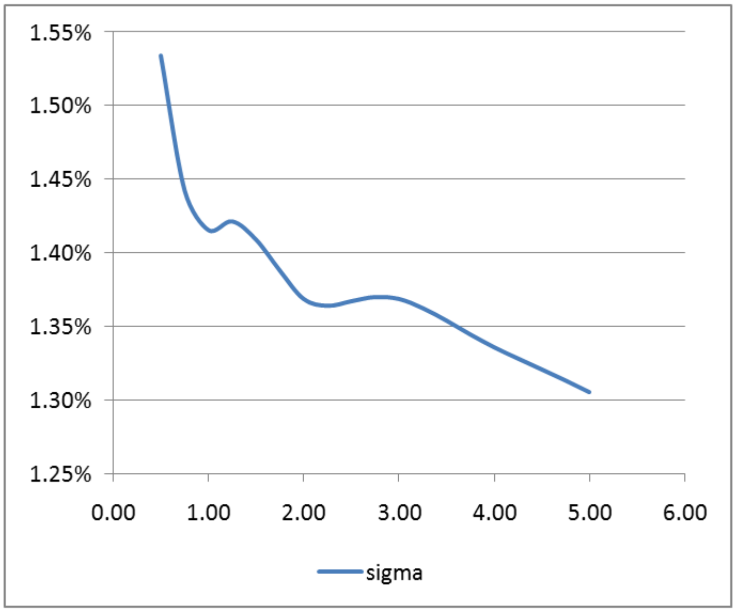

Figure 4: Value of σ that matches the quoted prices of caps

Exercise 3. The values of the parameters for the Hull-White model are: $\gamma = 0.0545$ and $\sigma = 1.49\%$. Figure 6 commpares the errors of both the Ho-Lee and the Hull-White model. Note that by adding an extra parameter the Hull-White model caputres better the term structure of volatility.
Exercise 4.

a. The value of the swaption through the Ho-Lee model is: \$0.6522.

b. The value of the swaption through the Hull-White model is: \$0.6483.

Exercise 5. Using a Ho-Lee tree the value of the American swaption is: \$0.8637, although this might be slightly overestimated since the price of the European swaption via the Ho-Lee tree is: \$0.7246. This difference comes mainly because the tree used is "cut to rough" in the sense that the time-step is: $dt = 1/4$. As dt becomes smaller the result would converge to the one obtained previously.

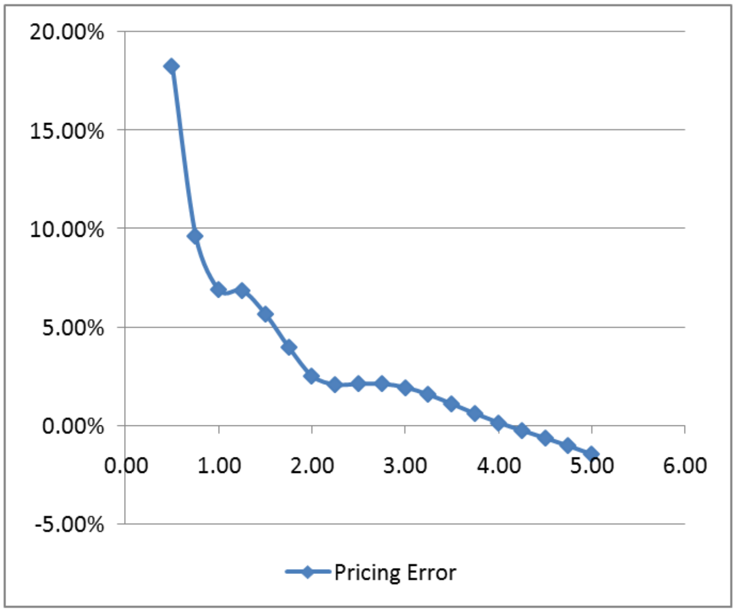

Figure 5: Pricing errors

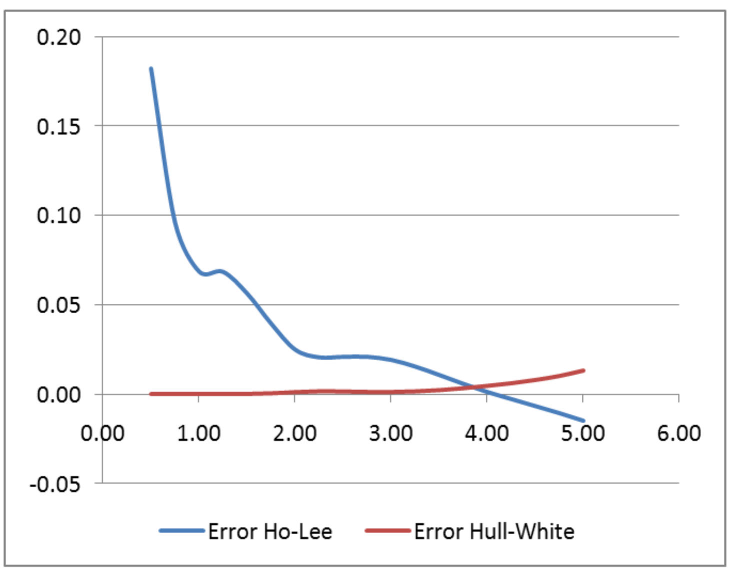

Figure 6: Pricing errors for Ho-Lee model and Hull-White

Solutions to Chapter 20

Exercise 1.

a. Figure 1 presents the results for the LIBOR curve. Note that the swap curve was interpolated using a polynomial of 6th order. This is not the ideal interpolation method (splines or cubic Hermite interpolation are better), but it is done in order to present the result in Excel.

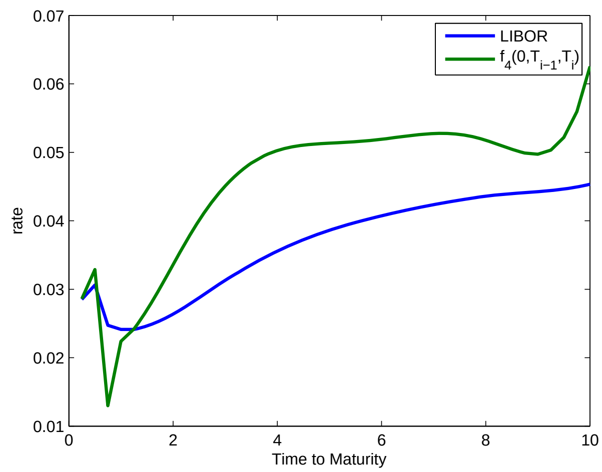

Figure 1: LIBOR yield curve and forward curve

b. The dollar value of a 1-year cap ($\times$ 100) is: \$0.3314 while the dollar value of a 2-year cap ($\times$ 100) is: \$1.0273.

Exercise 2.

a. See previous exercise.

b. The value of the caps up to 10 years maturity is:

τ	Cap	τ	Cap
0.50	0.1905	5.50	4.6871
0.75	0.2237	5.75	4.9388
1.00	0.3338	6.00	5.1861
1.25	0.4685	6.25	5.4297
1.50	0.6276	6.50	5.6699
1.75	0.8103	6.75	5.9074
2.00	1.0150	7.00	6.1424
2.25	1.2388	7.25	6.3749
2.50	1.4785	7.50	6.6048
2.75	1.7311	7.75	6.8313
3.00	1.9933	8.00	7.0533
3.25	2.2625	8.25	7.2692
3.50	2.5361	8.50	7.4779
3.75	2.8119	8.75	7.6782
4.00	3.0882	9.00	7.8706
4.25	3.3632	9.25	8.0575
4.50	3.6358	9.50	8.2460
4.75	3.9051	9.75	8.4503
5.00	4.1702	10.00	8.6965
5.25	4.4310		

Some small discrepancies with the previous results occur do to the interpolation techiniques and rounding errors.

c. Forward volatility is:

τ	Fwd.Vol.	τ	Fwd.Vol.
0.50	58.97%	5.50	24.68%
0.75	63.05%	5.75	24.06%
1.00	61.56%	6.00	23.42%
1.25	58.29%	6.25	22.83%
1.50	53.90%	6.50	22.31%
1.75	48.76%	6.75	21.90%
2.00	43.82%	7.00	21.62%
2.25	39.18%	7.25	21.43%
2.50	35.33%	7.50	21.33%
2.75	32.43%	7.75	21.28%
3.00	30.29%	8.00	21.19%
3.25	28.96%	8.25	20.99%
3.50	28.09%	8.50	20.63%
3.75	27.59%	8.75	20.05%
4.00	27.24%	9.00	19.23%
4.25	26.94%	9.25	18.31%
4.50	26.65%	9.50	17.39%
4.75	26.26%	9.75	16.81%
5.00	25.82%	10.00	16.99%
5.25	25.28%		

Figure 2 shows both forward and flat volatility.

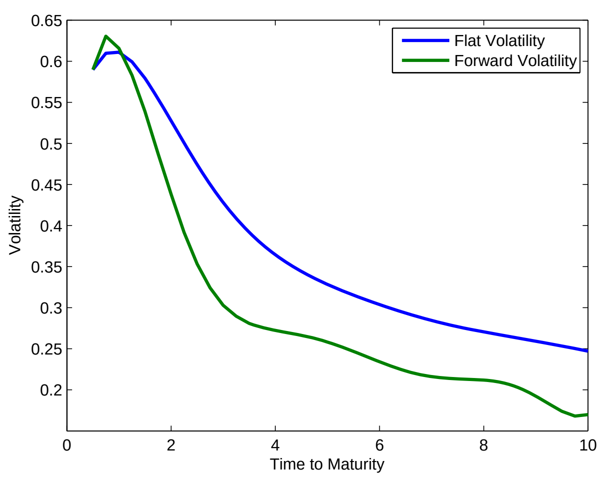

Figure 2: Forward and Flat volatilities

d. The shape of the forward volatility is not the typical humped. It is strongly decreasing. When compared to the average forward volatility between

1997-2008 (see Figure 20.3 in the book) we see that what happened was that short term volatility had increased significantly (from being within the 20% range it increased to the 60% range). This is mainly due to the subprime mortgage crisis that was at its worst in November 2008 (in the last three months the government had bailed out Fannie Mae, Freddie Mac and AIG, while Lehman Brothers had gone bankrupt). Note that the small hump around the 0.75 maturity is completely artificial since I decided to also interpolate missing volatilities before the first maturity, instead of assuming it to be constant (which would have given a plateau and then a drop).

Exercise 3. The value of the swaption is: $1.3996.

Exercise 4. The flat volatilities corresponding to Table 20.7 are:

τ	Volatility
1Y	118.77%
2Y	75.79%
3Y	49.56%
4Y	35.85%
5Y	29.85%
7Y	24.65%
10Y	21.97%

Exercise 5.

a. The BDT can be calibrated through two methodologies:

   i. Leaving σ_i free so that it is calibrated along with $r_{i,1}$.

   ii. Using the forward volatilities computed in the Black model as σ_i, only $r_{i,1}$ is free.

   Figure 3 presents the caps through each different methodology.

b. Figure 4 presents the errors of each methodology with respect to the data. Note that using the forward volatilities as σ_i is not as far from the results using the first methodology.

Exercise 6. We have two methodologies to compute the swaption price through the BDT model:

i. Through the first methodolgy we obtain a value for the swaption of: $1.2738.

ii. Through the second methodology we obtain a value for the swaption of: $1.2270.

Both values are reasonably close to each other and to the value from the Black formula.

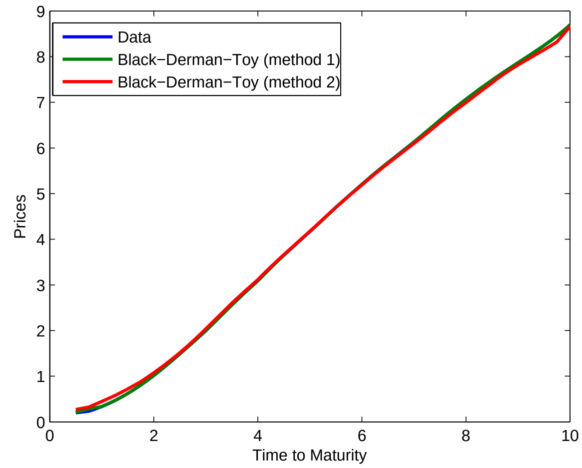

Figure 3: Cap prices under both BDT methodologies

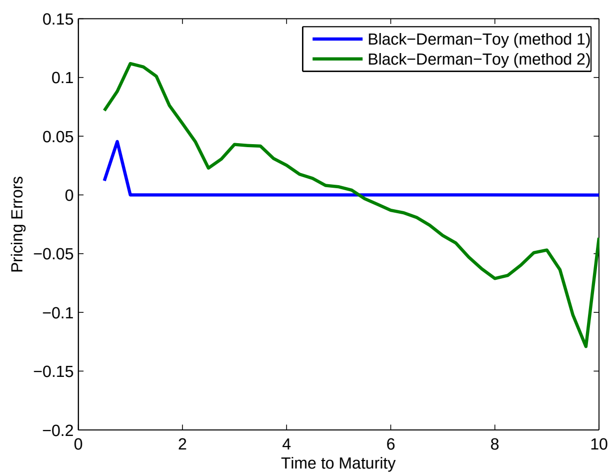

Figure 4: Pricing errors under both BDT methodologies

Solutions to Chapter 21

Exercise 1.

a. Figure 1 presents different payoff structures for the power option. The payoff is larger as α decreases.

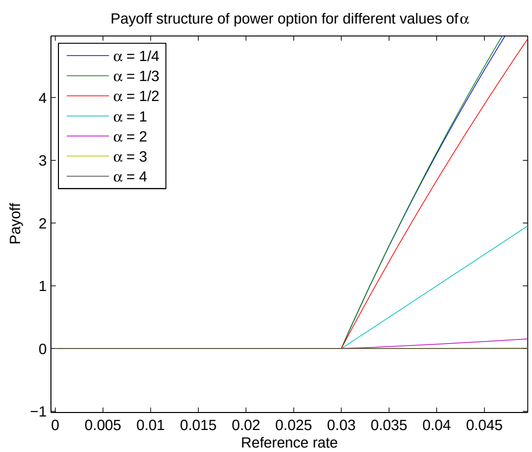

Figure 1: Payoff structure of power option for different values of α

b., c. The following table summarizes the values of different power options under the analytical formula and simulations:

α	Analytical	Simulations
0.2500	2.2951	2.2934
0.3333	2.3457	2.3439
0.5000	2.0703	2.0687
1.0000	0.8538	0.8530
2.0000	0.0774	0.0773
3.0000	0.0059	0.0059
4.0000	0.0005	0.0005

Exercise 2. Applying the analytical formulas discussed in section 21.7 we have that the value under the Unnatural Lag is: 0.5100. If we applied the convexity adjustment as an approximation we would have: 0.5083, which is lower than the actual value but higher than what we would have with the Natural Lag: 0.4873.
Exercise 3. Although not shown explicitly an analytical formula (similar to the one shown in section 21.7) can also be developed for this security. I provide the details of how it was obtained: Consider a payoff:

$$V = Z(0,T)E_f^*\left[\frac{G(r_n(\tau,T))}{Z(\tau,T)}\right]$$

$$= Z(0,T)E_f^*\left[max(r_n(\tau,T)-K,0)(1+r_n(\tau,T)\Delta)\right]$$
$$= Z(0,T)E_f^*\left[max(r_n(\tau,T)-K+r_n(\tau,T)^2\Delta-Kr_n(\tau,T)\Delta,0)\right]$$

The distribution of the rates is:

$$r_n(\tau,T)\ LN(f_n(0,\tau,T),\sigma_f^2\tau)$$

or

$$ln(r_n(\tau,T))\ N\left(ln(f_n(0,\tau,T))-\frac{1}{2}\sigma_f^2\tau,\sigma_f^2\tau\right)$$

For:

$$E_f^*[max(V_T-K,0)] =$$
$$\int_{-\infty}^{K} 0p_f^*(V_T)\,dV_T+\int_K^\infty V_T p_f^*(V_T)\,dV_T-K\int_K^\infty p_f^*(V_T)\,dV_T$$
$$+\Delta\int_K^\infty V_T^2 p_f^*(V_T)\,dV_T-\Delta K\int_K^\infty p_f^*(V_T)\,dV_T$$

Apply substitution: $v_T = ln(V_T)$, $V_T = e^{v_T}$ and $dV_T = e^{v_T}\,dv_T$.

$$= (1-K\Delta)\int_{lnK}^\infty e^{v_T}q_f^*(v_T)\,dv_T-K\int_{lnK}^\infty q_f^*(v_T)\,dv_T+\Delta\int_{lnK}^\infty e^{2v_T}q_f^*(v_T)\,dv_T$$

From section 21.10.2 we can find what the first two terms of the equation are in terms of the standard normal distribution, but we still need to derive the what the third term equals. The third term of the equation is (I use abreviated terms for mean, standard deviation and the variable):

$$\Delta\int_{lnK}^\infty \frac{1}{\sqrt{2\pi}s}e^{2x}e^{-\frac{1}{2}\left(\frac{x-b}{s}\right)^2}\,dx$$

We can make the following manipulations on the exponents:

$$-\frac{1}{2}\left(\frac{x-b}{s}\right)^2+2x = -\frac{1}{2}\left(\frac{x-b}{s}\right)^2+2x+\frac{1}{2}s^2-\frac{1}{2}s^2$$
$$= -\frac{1}{2}\left[\left(\frac{x-b}{s}\right)^2+s^2-2s\left(\frac{x-b}{s}\right)+2s\left(\frac{x-b}{s}\right)\right]+2x+\frac{1}{2}s^2$$
$$= -\frac{1}{2}\left(\frac{x-b}{s}-s\right)^2-x+b+2x+\frac{1}{2}s^2 = -\frac{1}{2}\left(\frac{x-b}{s}-s\right)^2+x+b+\frac{1}{2}s^2$$
$$= -\frac{1}{2}\left(\frac{x-b}{s}-s\right)^2+x+b+\frac{1}{2}s^2+\frac{1}{2}s^2-\frac{1}{2}s^2$$
$$= -\frac{1}{2}\left[\left(\frac{x-b-s^2}{s}\right)^2+s^2-2s\left(\frac{x-b-s^2}{s}\right)+2s\left(\frac{x-b-s^2}{s}\right)\right]+x+b+s^2$$

$$= -\frac{1}{2}\left(\frac{x-b}{s} - 2s\right)^2 + 2b + 2s^2$$

So we get that the third term is:

$$\Delta e^{2b+2s^2}\int_{\frac{lnK-b}{s}-2s}^{\infty}\frac{1}{\sqrt{2\pi}s}e^{-\frac{1}{2}y^2}\,dy = \Delta e^{2b+2s^2}N\left(2s - \frac{lnK-b}{s}\right)$$

Putting this term in the rest of the equation we get that:

$$E_f^*[...] = (1-K\Delta)e^{b+\frac{1}{2}s^2}N\left(s+\frac{b-a}{s}\right)+\Delta e^{2b+2s^2}N\left(2s+\frac{b-a}{s}\right)-KN\left(\frac{b-a}{s}\right)$$

So given that: $v_T\ N\left(ln(f_n(0,\tau,T)) - \frac{1}{2}\sigma_f^2\tau, \sigma_f^2\tau\right)$

$$E_f^*\left[\frac{G(r_n(\tau,T))}{Z(\tau,T)}\right] = (1-K\Delta f_n(0,\tau,T))N\left(\frac{1}{\sigma_f\sqrt{\tau}}ln\left(\frac{f_n(0,\tau,T)}{K}\right)+\frac{1}{2}\sigma_f\sqrt{\tau}\right)$$

$$-KN\left(\frac{1}{\sigma_f\sqrt{\tau}}ln\left(\frac{f_n(0,\tau,T)}{K}\right)+\frac{1}{2}\sigma_f\sqrt{\tau}\right)$$

$$+\Delta e^{\sigma_f^2\tau}f_n(0,\tau,T)^2N\left(\frac{1}{\sigma_f\sqrt{\tau}}ln\left(\frac{f_n(0,\tau,T)}{K}\right)+\frac{3}{2}\sigma_f\sqrt{\tau}\right)$$

Applying this analytical formula we have that under the Unnatural Lag the value is: 1.8463. We can also apply a convexity adjustment and obtain: 1.8440, which (again) is lower than the actual value but higher than what we woudl have with the Natural Lag: 1.8164.

Exercise 4. Figure 2 presents the comparisson between both types of volatilities (it is identical to Figure 21.2 from the book).

Exercise 5. The following table compares the results from the simulations versus using cap volatilities. On the vertical axis is the maturity of the option in the swaption and on the vertical axis is the maturity of the underlying swap.

	Volatility of Forwards				Forward Volatility		
	0.25	0.50	1.00		0.25	0.50	1.00
1	0.0628	0.0813	0.1248	1	0.0719	0.1198	0.2046
2	0.2187	0.2405	0.2327	2	0.2455	0.3269	0.4312
3	0.2799	0.3293	0.3756	3	0.4065	0.5345	0.6886
4	0.3715	0.4052	0.384	4	0.5776	0.7498	0.9495
5	0.3702	0.4175	0.4499	5	0.7437	0.9712	1.2268
7	0.5806	0.6745	0.7248	7	0.9934	1.3100	1.6812

Note that as we move away from the top left corner the differences between the prices becomes larger. This difference comes from the difficulties in estimating the volatility of forward rates. When reviewing Figure 2 we note that there are 'valleys' where the volatility is zero. The more a swaption is affected by these

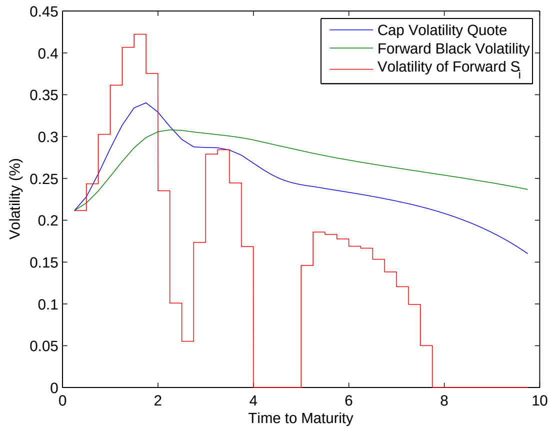

Figure 2: Forward Volatility and Volatility of Forwards

valleys (when using the volatility of forwards) the more it will diverge from the price we obtain using forward volatilities.

Exercise 6. The value of the Constant Maturity Swap is: 3.3055.

Exercise 7. Figure 3 presents the comparisson between both types of volatilities for this new date. This is in the middle of the subprime mortgage crisis with short term volatility being particularly large. This fall has a very strong impact on the volatility of forward rates which goes quickly to zero. As explained before this will significantly reduce the effectivness of pricing through simulations on the LIBOR market model. The result under simulations is: 0.0003, while

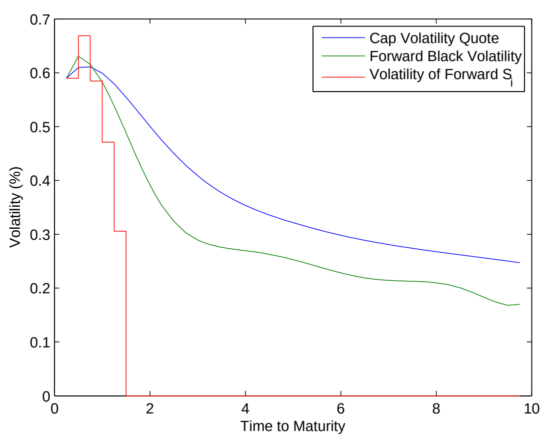

Figure 3: Forward Volatility and Volatility of Forwards

under forward volatilities it is: 0.8387. Clearly the model is not doing a good job pricing.

Exercise 8. The convexity adjustment states that:

$$f(0,T,T+\tau) = f^{fut}(0,T,T+\tau) - \int_0^T \frac{\sigma_Z(t,T+\tau)^2 - \sigma_Z(t,T)^2}{2\tau}\,dt$$

Recall that: $x^2 - y^2 = (x+y)(x-y)$, so:

$$\sigma_Z(t,T+\tau)^2 - \sigma_Z(t,T)^2 = \left(\sigma_Z(t,T+\tau) + \sigma_Z(t,T)\right)\left(\sigma_Z(t,T+\tau) - \sigma_Z(t,T)\right)$$

For the Hull-White model we have:

$$\sigma_Z(t,T) = -B(t,T)\sigma = -\frac{1-e^{-\gamma^*(T-t)}}{\gamma^*}\sigma$$

$$\sigma_Z(t,T+\tau) = -B(t,T+\tau)\sigma = -\frac{1-e^{-\gamma^*(T+\tau-t)}}{\gamma^*}\sigma$$

also keep in mind that:

$$\sigma_f(t,T) = \sigma e^{-\gamma^*(T-t)}$$

$$\sigma_f(t,T+\tau) = \sigma e^{-\gamma^*(T+\tau-t)}$$

So:

$$\sigma_Z(t,T+\tau) + \sigma_Z(t,T) = \frac{-2\sigma + \sigma_f(t,T+\tau) + \sigma_f(t,T)}{\gamma^*}$$

$$\sigma_Z(t,T+\tau) + \sigma_Z(t,T) = \frac{\sigma_f(t,T+\tau) - \sigma_f(t,T)}{\gamma^*}$$

We have then that:

$$\sigma_Z(t,T+\tau)^2 - \sigma_Z(t,T)^2 = \frac{1}{(\gamma^2)^2}\left[-2\sigma\sigma_f(t,T+\tau) + 2\sigma\sigma_f(t,T) + \sigma_f(t,T+\tau)^2 - \sigma_f(t,T)^2\right]$$

The integral in the adjustment can be presented as:

$$\int_0^T \frac{\sigma_Z(t,T+\tau)^2 - \sigma_Z(t,T)^2}{2\tau}\,dt =$$

$$\int_0^T \frac{1}{2\tau\,(\gamma^2)^2}\left[-2\sigma\sigma_f(t,T+\tau) + 2\sigma\sigma_f(t,T) + \sigma_f(t,T+\tau)^2 - \sigma_f(t,T)^2\right]\,dt =$$

$$\frac{1}{2\tau\,(\gamma^2)^2}\left[\underbrace{-2\sigma\int_0^T \sigma_f(t,T+\tau)\,dt}_{A} + \underbrace{2\sigma\int_0^T \sigma_f(t,T)\,dt}_{B} + \underbrace{\int_0^T \sigma_f(t,T+\tau)^2\,dt}_{C} - \underbrace{\int_0^T \sigma_f(t,T)^2\,dt}_{D}\right]$$

- For A:

$$-2\sigma \int_0^T \sigma_f(t, T+\tau)\, dt = -2\sigma \int_0^T \sigma e^{-\gamma^*(T+\tau-t)}\, dt = -2\sigma^2 \frac{e^{-\gamma^*(T+\tau-t)}}{\gamma^*}\Bigg|_0^T =$$

$$-\frac{2\sigma^2}{\gamma^*}e^{\gamma^*\tau}+\frac{2\sigma^2}{\gamma^*}e^{\gamma^*(T+\tau)}+\underbrace{\left(\frac{2\sigma^2}{\gamma^*}-\frac{2\sigma^2}{\gamma^*}\right)}_{=0} = \frac{2\sigma^2}{\gamma^*}\left(1-e^{\gamma^*\tau}\right)-\frac{2\sigma^2}{\gamma^*}\left(1-e^{\gamma^*(T+\tau)}\right)$$

Recall the definition of σ_Z, so that:

$$-2\sigma \int_0^T \sigma_f(t, T+\tau)\, dt = -2\sigma\sigma_Z(0,\tau) + 2\sigma\sigma_Z(0, T+\tau)$$

- For B:

$$2\sigma \int_0^T \sigma_f(t,T)\, dt = 2\sigma \int_0^T \sigma e^{-\gamma^*(T-t)}\, dt = 2\sigma^2 \frac{e^{-\gamma^*(T-t)}}{\gamma^*}\Bigg|_0^T =$$

$$\frac{2\sigma^2}{\gamma^*} - \frac{2\sigma^2}{\gamma^*}e^{\gamma^* T} = \frac{2\sigma^2}{\gamma^*}\left(1-e^{\gamma^* T}\right) = 2\sigma\sigma_Z(0,T)$$

- For C:

$$\int_0^T \sigma_f(t, T+\tau)^2\, dt = \sigma^2 \int_0^T e^{-2\gamma^*(T+\tau-t)}\, dt = \frac{\sigma^2}{2\gamma^*}e^{-2\gamma^*(T+\tau-t)}\Bigg|_0^T =$$

$$\frac{\sigma^2}{2\gamma^*}e^{-2\gamma^*\tau} - \frac{\sigma^2}{2\gamma^*}e^{-2\gamma^*(T+\tau)} + \underbrace{\left(\frac{\sigma^2}{2\gamma^*}\right)-\left(\frac{\sigma^2}{2\gamma^*}\right)}_{=0} =$$

$$-\frac{\sigma^2}{2\gamma^*}(1-e^{-2\gamma^*\tau})+\frac{\sigma^2}{2\gamma^*}(1-e^{-2\gamma^*(T+\tau)}) = \frac{\sigma}{2}\sigma_Z(0,2\tau)-\frac{\sigma}{2}\sigma_Z(0,2(T+\tau))$$

- For D:

$$-\int_0^T \sigma_f(t,T)^2\, dt = -\sigma^2 \int_0^T e^{-2\gamma^*(T-t)}\, dt = -\frac{\sigma^2}{2\gamma^*}e^{-2\gamma^*(T-t)}\Bigg|_0^T =$$

$$-\frac{\sigma^2}{2\gamma^*} + \frac{\sigma^2}{2\gamma^*}e^{-2\gamma^* T} = -\frac{\sigma^2}{2\gamma^*}\left(1-e^{-2\gamma^* T}\right) = \frac{\sigma}{2}\sigma_Z(0,2T)$$

So putting everything together we have that:

$$\int_0^T \frac{\sigma_Z(t,T+\tau)^2 - \sigma_Z(t,T)^2}{2\tau}\, dt =$$

$$\frac{\sigma}{2\tau\left(\gamma^*\right)^2}\left[2[\sigma_Z(0,T+\tau)-\sigma_Z(0,T)-\sigma_Z(0,\tau)]-\frac{1}{2}[\sigma_Z(0,2(T+\tau))-\sigma_Z(0,2T)-\sigma_Z(0,2\tau)]\right]$$

So we have that the convexity adjustment for the Hull-White model is:

$$f(0,T,T+\tau) = f^{fut}(0,T,T+\tau)$$

$$-\frac{\sigma}{2\tau\left(\gamma^*\right)^2}\left[2[\sigma_Z(0,T+\tau)-\sigma_Z(0,T)-\sigma_Z(0,\tau)]-\frac{1}{2}[\sigma_Z(0,2(T+\tau))-\sigma_Z(0,2T)-\sigma_Z(0,2\tau)]\right]$$

Exercise 9. The following table presents the relatioship between futures rates and forward rates for various maturities under the Ho-Lee and Hull-White models.

t	f^{fut}	Ho-Lee	Hull-White
0.112	2.641%	2.641%	2.641%
0.364	2.691%	2.690%	2.690%
0.614	2.919%	2.917%	2.916%
0.86	3.088%	3.083%	3.083%
1.112	3.286%	3.279%	3.278%
1.364	3.450%	3.439%	3.437%
1.614	3.658%	3.643%	3.640%
1.86	3.787%	3.767%	3.763%
2.112	3.931%	3.906%	3.899%
2.364	4.054%	4.023%	4.015%
2.614	4.183%	4.146%	4.133%
2.86	4.252%	4.208%	4.192%
3.112	4.331%	4.279%	4.258%
3.364	4.406%	4.345%	4.319%
3.614	4.495%	4.425%	4.392%
3.863	4.544%	4.465%	4.424%
4.115	4.603%	4.514%	4.464%
4.367	4.663%	4.562%	4.502%
4.616	4.737%	4.625%	4.553%

Solutions to Chapter 22

Exercise 1. Figure 1 and Figure 2 present the yield curve and the discount prices under both the Vasicek and the 2-Factor Vasicek models, respectively. Apparently there is no gain by using the 2-Factor model model. Yet this is mostly due to the time framed selected to present these observations (Figure 22.1 expands the chart over 30 years, where there is a larger divergence in values from the models). Additionally, when we take into account volatility we note that the 2-Factor model does a much better job. Figure 3 presents the yield volatility for both of these models and, also, the 2-Factor model with correlated factors. Note that both 2-Factor models significantly outperform the single factor Vasicek model. Figure 4 presents the correlation and how well each model captures this. The 2-Factor model with correlated factors does a much better job in capturing this.

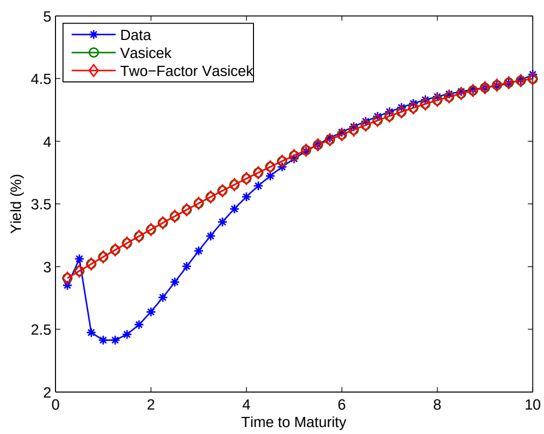

Figure 1: Yield curve under Single Factor and 2-Factor Vasicek

Exercise 2.

a. Figure 5 presents the cap prices under the Hull-White models (Single Factor and 2-Factor), as well as the Ho-Lee model. It is very difficult to tell which one does a better job pricing the caps from this chart. Figure 6 shows the pricing errors. Here we can see clearly that the 2-Factor Hull-White model does a better job than both the Ho-Lee and the Single Factor Hull-White models.

b. Figure 3 presents volatility estimates for all three models. Both Hull-White models are very close to the volatility implied by the data.

c. Unsing the 2-Factor Hull-White we price the Eurpean swaptions quoted in Table 22.3 (or 20.6). As it can be seen this swaption implied price is

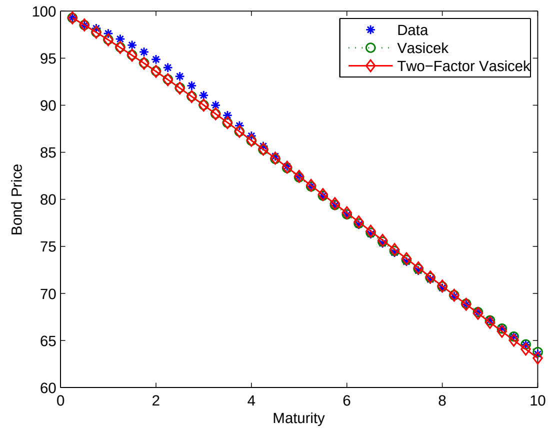

Figure 2: Discounts under Single Factor and 2-Factor Vasicek

much closer to the data than the cap implied price (see Figures 8 and 9). Also included is the volatility surface of these swaptions (Figure 10).

Exercise 3. Price of the caplet is 0.1253.
Exercise 4. Reuslts for this section are summarized in exercise 2.
Exercise 5. The value of the swap under the Single Factor model is: -0.0699, while under the 3-Factor model we have that it is: -0.1356.
Exercise 6. The value of the Constant Maturity Swap under the Single Factor model is: 2.4304, while under the 3-Factor model we have that it is: 4.1422.

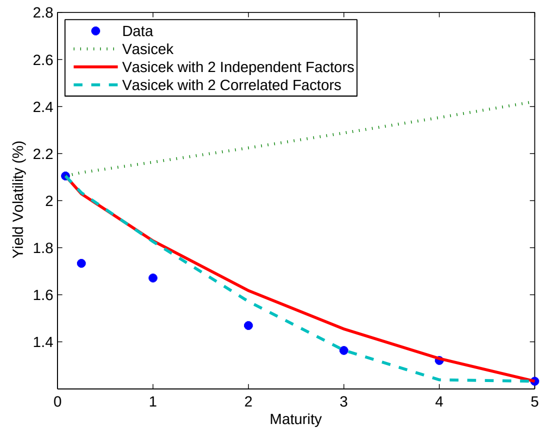

Figure 3: Yield volatility under Single Factor, 2-Factor and 2-Factor with Correlated Factors Vasicek

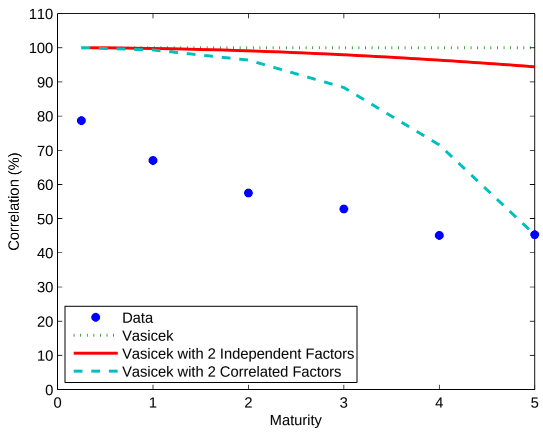

Figure 4: Correlation under Single Factor, 2-Factor and 2-Factor with Correlated Factors Vasicek

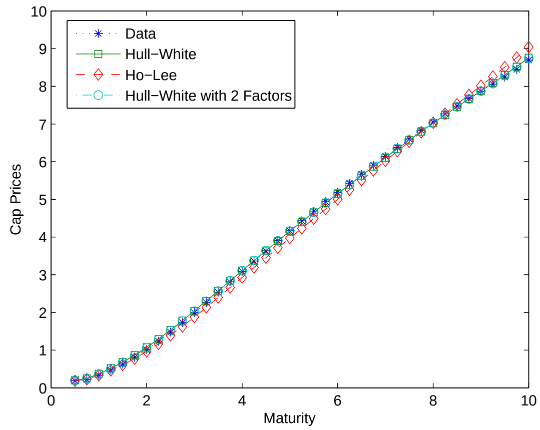

Figure 5: Cap Prices for Ho-Lee, Single Factor Hull-White and 2-Factor Hull-White models

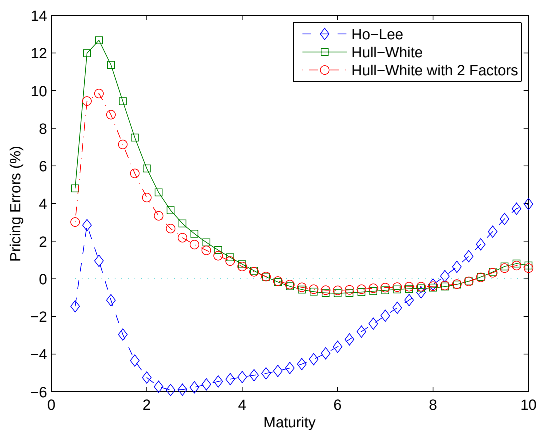

Figure 6: Pricing Errors for Ho-Lee, Single Factor Hull-White and 2-Factor Hull-White models

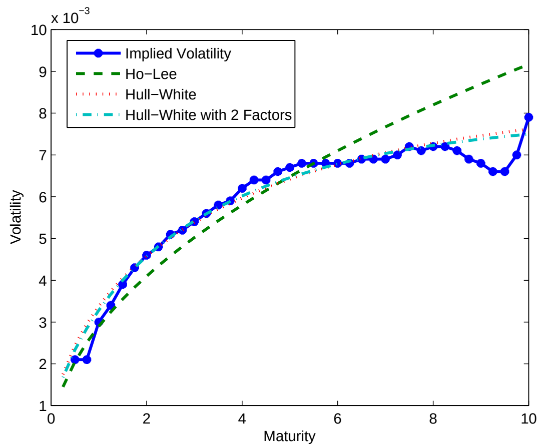

Figure 7: Volatility estimates for Ho-Lee, Single Factor Hull-White, 2-Factor Hull-White models

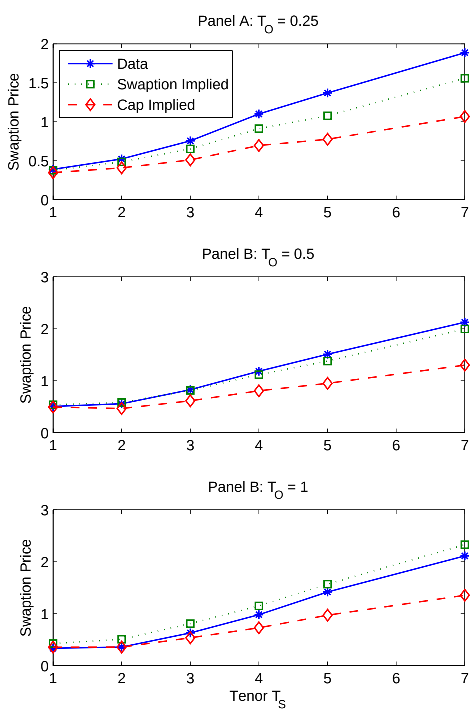

Figure 8: Prices for swaptions

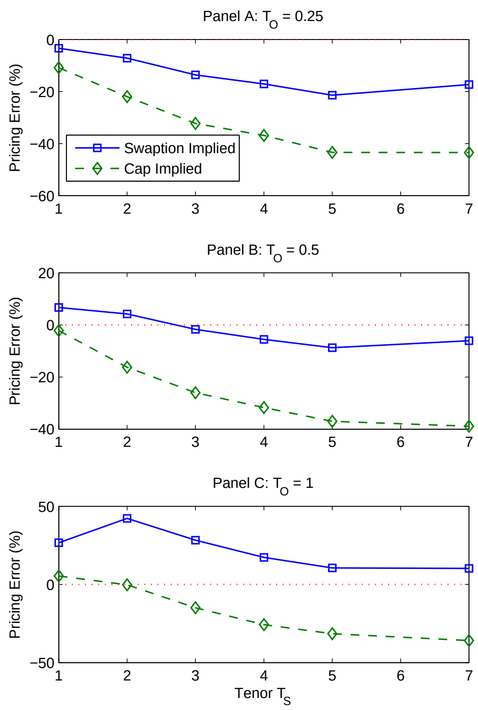

Figure 9: Pricing Errors for swaptions

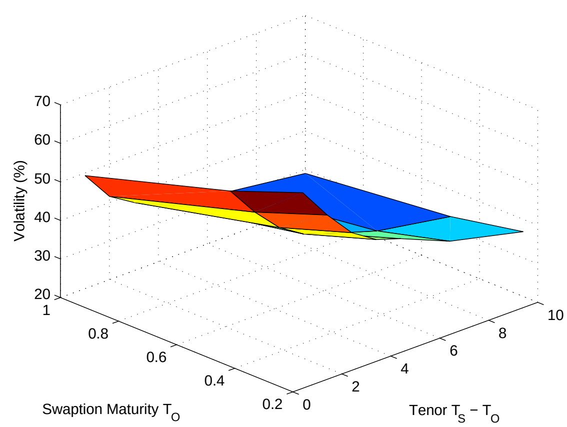

Figure 10: Volatility surface of swaptions