Finance: A Quantitative Introduction
Chapter 7 - part 2
Option Pricing Foundations

Nico van der Wijst

CAMBRIDGE UNIVERSITY PRESS

Recall general valuation formula for investments:

$$Value = \sum^{t} \frac{Exp\left[Cash\ flows_t\right]}{\left(1 + discount\ rate_t\right)^t}$$

Uncertainty can be accounted for in 3 different ways:

1. Adjust discount rate to *risk adjusted discount rate*
2. Adjust cash flows to *certainty equivalent cash flows*
3. Adjust probabilities (expectations operator) from normal to *risk neutral* or *equivalent martingale probabilities*

Introduce pricing principles in state preference theory

- old, tested modelling framework
- excellent framework to show completeness and arbitrage
- more general than binomial option pricing

Also introduce some more general concepts

- equivalent martingale measure
- state prices, pricing kernel, few more

Gives you easy entry to literature
(+ pinch of matrix algebra, just for fun, can easily be omitted)

State-preference theory

Developed in 1950's and 1960's by Nobel prize winners Arrow and Debreu:

- Time modelled as discrete points in time at which:
  - uncertainty over the previous period is resolved
  - new decisions are made
  - in periods between points 'nothing happens'
- Uncertainty in variables modelled as:
  - discrete number of states of the world
  - can occur on the future points in time
  - each state associated with different numerical value of variables under consideration.

The states of the world can be defined in different ways
Examples:

- general economic circumstances:
  - 'recession' with return on stock portfolio of -5%
  - 'expansion' with return on stock portfolio of $+16\%$
- result of a specific action, such as drilling for oil:
  - large well
  - medium-sized well
  - dry well
  - each with a cash flow attached to it

Elaborate a simple example with:

- 1 period - 2 points in time
- 3 future states of the world:

$$W = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} bust \\ normal \\ boom \end{bmatrix}$$

The states have a given probability of occurring:

$$prob(w_i) = \begin{bmatrix} 0.3 \\ 0.45 \\ 0.25 \end{bmatrix}$$

There are 3 investment opportunities, Y_1, Y_2, Y_3

Y_1 pays off:

- 4 in state 1
- 5 in state 2
- 6 in state 3, or in matrix notation:

$$Y_1(W) = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

For simplicity, the addition (W) will be omitted

Payoffs of all investments in different states in payoff matrix Ψ:

$$\Psi = \begin{bmatrix} 4 & 1 & 2 \\ 5 & 7 & 4 \\ 6 & 10 & 16 \end{bmatrix}$$

Present value of investments Y found by:

- calculating expected value of payoffs
- discounting them with an appropriate rate

The expected payoffs are e.g.:

$$E(Y_1) = 0.3 \times 4 + 0.45 \times 5 + 0.25 \times 6 = 4.95$$

or in matrix notation $prob^T \Psi$:

$$\begin{bmatrix} 0.3 \\ 0.45 \\ 0.25 \end{bmatrix}^T \begin{bmatrix} 4 & 1 & 2 \\ 5 & 7 & 4 \\ 6 & 10 & 16 \end{bmatrix} = \begin{bmatrix} 4.95 & 5.95 & 6.40 \end{bmatrix}$$

Let the required returns on the investments be:

$$\left[\begin{array}{ccc} 10\% & 13\% & 16\% \end{array}\right]$$

Gives following present values of Y_1, Y_2, Y_3:
(e.g. $4.95/1.1 = 4.5$, etc.)

$$v = \left[\begin{array}{ccc} 4.5 & 5.25 & 5.5 \end{array}\right]$$

Don't look at prices or returns as such, but:

- at mutual relations between them
- what these relations mean for capital market

Risk free and state securities

On perfect capital markets (assumed here):

- investments are costlessly and infinitely divisible
- means they can be combined in all possible ways
- to create the payoff pattern we want

An obvious candidate for a wanted pattern:

- the same payoff in all states of the world
- i.e. creating a *riskless security*.

Risk free security created by:

- combining the investments Y_{1-3} in a portfolio
- choosing the portfolio weights x_n such that payoffs are equal:

$$\begin{aligned} 4x_1 + 1x_2 + 2x_3 &= 1 \\ 5x_1 + 7x_2 + 4x_3 &= 1 \\ 6x_1 + 10x_2 + 16x_3 &= 1 \end{aligned}$$

3 equations with 3 unknowns, system can be solved:

$$\begin{aligned} x_1 &= 33/124 \\ x_2 &= -5/124 \\ x_3 &= -3/248 \end{aligned}$$

These weights define:

- the riskless security
- but also the *risk free interest rate*:
- PV(investments) $\times$ weights $=$ PV(riskless security)
- $(4.5 \times 33/124) + (5.25 \times -5/124) + (5.5 \times -3/248) = 0.9194.$
- Given PV, and payoff of 1, risk free interest rate is

$$\frac{1}{0.9194} = 1.088 \text{ or } 8,8\%.$$

Notice: we use small and negative fractions of investments (short selling), use perfect market assumption

Use same procedure to create a portfolio that:

- pays off 1 if state of the world 1 occurs
- zero in all other states:

$$
\begin{aligned}
4x_{1'} + 1x_{2'} + 2x_{3'} &= 1 \\
5x_{1'} + 7x_{2'} + 4x_{3'} &= 0 \\
6x_{1'} + 10x_{2'} + 16x_{3'} &= 0
\end{aligned}
$$

System is solvable too:

$$
\begin{aligned}
x_1 &= 9/31 \\
x_2 &= -7/31 \\
x_3 &= 1/31
\end{aligned}
$$

Can repeat procedure, find portfolios that pay off 1 in state 2 or 3

Use a little matrix algebra instead, to find matrix of weights X that satisfies :

$$\Psi X = I$$

- Ψ is the payoff matrix
- X is 3×3 matrix of 3 weights in 3 equations
- I is the identity matrix:

$$\begin{bmatrix} 4 & 1 & 2 \\ 5 & 7 & 4 \\ 6 & 10 & 16 \end{bmatrix} \begin{bmatrix} x_{1,1} & x_{2,1} & x_{3,1} \\ x_{1,2} & x_{2,2} & x_{3,2} \\ x_{1,3} & x_{2,3} & x_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This system is solved by

$$X = \Psi^{-1} I = \Psi^{-1}$$

i.e. by taking the inverse of the payoff matrix :

$$\Psi^{-1} = \begin{bmatrix} \frac{9}{31} & \frac{1}{62} & -\frac{5}{124} \\ -\frac{7}{31} & \frac{13}{62} & -\frac{3}{124} \\ \frac{1}{31} & -\frac{17}{124} & \frac{23}{248} \end{bmatrix}$$

These weights give 3 portfolios, each of which:

- pays off 1 in only one state of the world
- and zero in all other states

Such securities are called:

- *state securities*, or
- *pure securities*, or
- *primitive securities*, or
- *Arrow-Debreu securities*

Prices of state securities found by multiplying:

- the weights matrix
- with present value vector (of the existing securities):

$$v\Psi^{-1} = \left[\begin{array}{ccc} 0.298 & 0.419 & 0.202 \end{array} \right]$$

Prices of state securities also known as *state prices*

In matrix algebra:

$$v = \left[\begin{array}{ccc} 4.5 & 5.25 & 5.5 \end{array}\right]$$

and

$$\Psi = \left[\begin{array}{ccc} 4 & 1 & 2 \\ 5 & 7 & 4 \\ 6 & 10 & 16 \end{array}\right]$$

so that the state prices are:

$$\left[\begin{array}{ccc} 4.5 & 5.25 & 5.5 \end{array}\right] \left[\begin{array}{ccc} 4 & 1 & 2 \\ 5 & 7 & 4 \\ 6 & 10 & 16 \end{array}\right]^{-1} = \left[\begin{array}{ccc} 0.298 & 0.419 & 0.202 \end{array}\right]$$

Market completeness defined

- State securities allow construction of *any payoff pattern*
  - simply as combination of state securities
- State securities could be constructed because:
  - the existing securities *span* all states
  - i.e. there are no states without a payoff.

A market where that is the case is said to be *complete*

It is complete because:

- no "new" securities can be constructed
- new = payoff patterns cannot be duplicated with existing securities

Can also be stated the other way around:

> *If state securities can be constructed for all states, the market has to be complete.*

On complete markets:

- all additional securities are linear combinations of original ones
- additional securities are called *redundant* securities
- in examples so far
  - risk free security
  - and state securities are redundant
  - they are formed as combinations of the existing securities

A market can only be complete if:

- number of different (i.e. not redundant) securities = number of states
- in examples: must be 3 equations with 3 unknowns

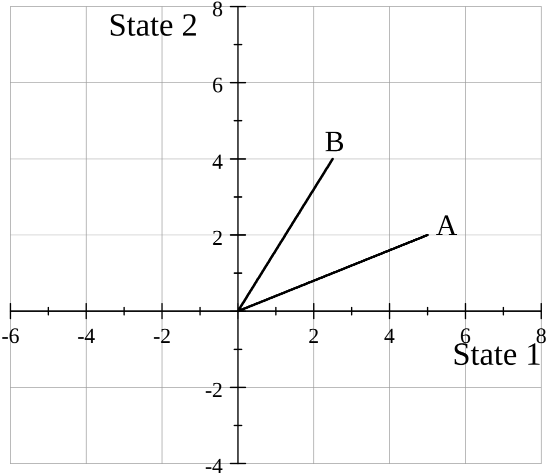
State prices offer easy way to price redundant securities:

- multiply security's payoff in each state with state prices
- sum over the states to find the security's price

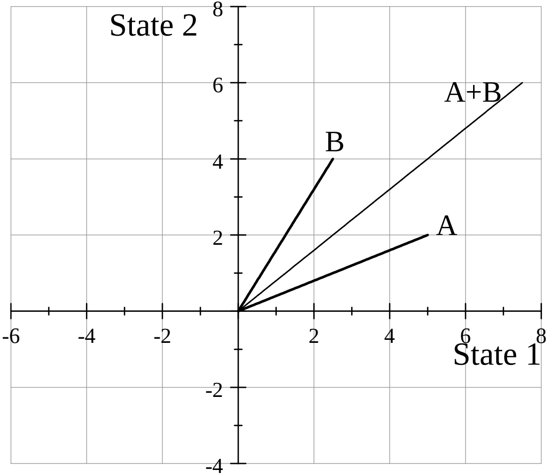
This follows directly from the definition of state prices

Completeness can be represented geometrically:

- Assume 2 securities A and B
- and 2 future states of the world
- A pays off:
  - 5 in state 1
  - 2 in state 2
- B pays off:
  - 2.5 in state 1
  - 4 in state 2
- A and B linearly independent:
- combinations of A and B span whole 2-dimensional space



Geometric representation of market completeness



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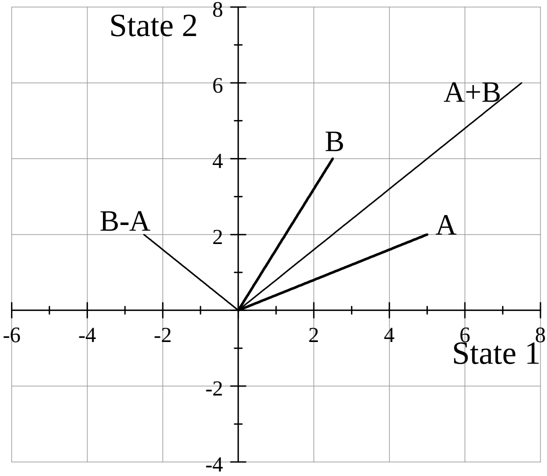
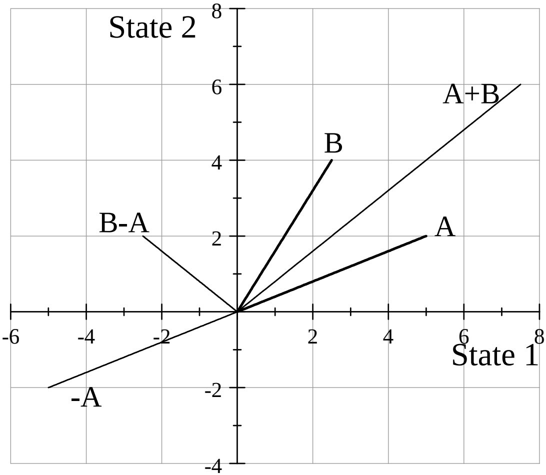


Geometric representation of market completeness



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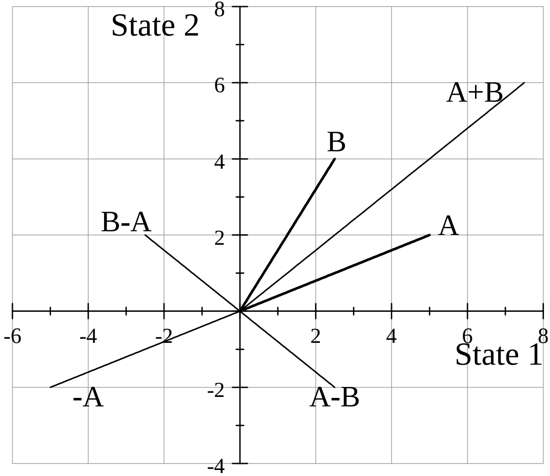

Geometric representation of market completeness

Arbitrage free markets

Complete markets imply:

- any payoff pattern can be constructed

Arbitrage free markets imply:

- patterns are properly priced

What is properly priced?

Answer in modern finance:

Proper prices offer no arbitrage opportunities

Recall that arbitrage opportunities exist if there is investment strategy that:

- either requires
  - investment ≤ 0 today, while
  - all future payoffs ≥ 0 and
  - at least one payoff > 0
- or requires
  - investment < 0 today ($=$profit) and
  - all future payoffs ≥ 0

Less formally:

- either costs nothing today $+$ payoff later
- or payoff today without obligations later

Absence of arbitrage implies a characteristic of state prices

Can you guess what characteristic?

State prices have to be positive

Negative state price would mean:

- buying state security with negative price = receive money
- and possibly (if state occurs), a payoff of 1 later.

A negative net investment now + a non-negative profit later

Illustrate arbitrage by modifying the previous example:

- we still have the same assets Y_1, Y_2, Y_3
- with the same payoff matrix Ψ:

$$\Psi = \begin{bmatrix} 4 & 1 & 2 \\ 5 & 7 & 4 \\ 6 & 10 & 16 \end{bmatrix}$$

instead of old prices $v = \begin{matrix} 4.5 & 5.25 & 5.5 \end{matrix}$
we use price vector $u = \begin{matrix} 4.5 & 5.25 & 2 \end{matrix}$

These asset prices give following state prices:

$$u\Psi^{-1} = \begin{matrix} 0.185 & 0.899 & -0.123 \end{matrix}$$

Negative state prices cannot exist; easy to see what is wrong:

- Third security Y_3 costs less than half the second security Y_2
- but 2 times Y_3 offers a higher payoff than Y_2 in all states

This is an arbitrage opportunity:

- can sell Y_2, use the money to buy $2 \times Y_3$
- gives instantaneous profit of $5.25 - 2 \times 2 = 1.25$
- end of the period in all states, payoff of $2 \times Y_3$ is:
  - enough to pay obligations from shorting Y_2
  - and give a profit

The Arbitrage Theorem

State the no arbitrage condition more formally with example
Suppose we have 2 securities:

- risk free debt D
- stock S (D and S represent values now)

There are 2 future states:

- up, with stock return u
- down, with stock return d
- $d < u$ for normal stocks

This makes the payoff matrix Ψ (we re-use the same symbols):

$$\Psi = \begin{bmatrix} (1+r_f)D & (1+u)S \\ (1+r_f)D & (1+d)S \end{bmatrix}$$

We can represent this market as follows:

$$\left[\begin{array}{cc} D & S \end{array}\right] = \left[\begin{array}{cc} \psi_1 & \psi_2 \end{array}\right] \left[\begin{array}{cc} (1+r_f)D & (1+u)S \\ (1+r_f)D & (1+d)S \end{array}\right] \qquad (1)$$

$\psi_{1,2}$ are the state prices
Value of security = sum [payoffs in future states × state prices]

The arbitrage theorem can now be stated as follows:

Arbitrage theorem

Given the payoff matrix Ψ there are no arbitrage opportunities if and only if there is a strictly positive state price vector $\psi_{1,2}$ such that the security price vector $\left[\begin{array}{cc} D & S \end{array}\right]$ satisfies (1).

We can also formulate this the other way around:

> *if there are no arbitrage opportunities, then there is a positive state price vector $\psi_{1,2}$ such that the security price vector $\left[\begin{array}{cc} D & S \end{array}\right]$ satisfies (1).*

We analyse under which conditions this is the case
First, we write out the values of D and S :

$$\begin{aligned} D &= \psi_1(1+r_f)D + \psi_2(1+r_f)D \\ S &= \psi_1(1+u)S + \psi_2(1+d)S \end{aligned} \tag{2}$$

Then we divide first equation by D and second by S:

$$\begin{aligned} 1 &= \psi_1(1 + r_f) + \psi_2(1 + r_f) \\ 1 &= \psi_1(1 + u) + \psi_2(1 + d) \end{aligned} \tag{3}$$

Subtract second row from first and rearrange terms:

$$0 = \psi_1[(1 + r_f) - (1 + u)] + \psi_2[(1 + r_f) - (1 + d)] \tag{4}$$

With state prices $\psi_{1,2} > 0$, (4) is only zero (non-trivially) if:

- one of the terms in square brackets is positive
- and the other one negative

Since $d < u$ for normal stocks, this is the case if and only if:

$$(1 + d) < (1 + r_f) < (1 + u) \tag{5}$$

This is the *no arbitrage condition*:
risk free rate must be between low and high stock return

Simple market, easy to see why:

- If $(1 + r_f) < (1 + d)$:
  borrow risk free, invest in stock $\Rightarrow$ sure profit in all states
- If $(1 + u) < (1 + r_f)$:
  short sell the stock, invest risk free $\Rightarrow$ sure profit in all states

Arbitrage theorem, $\psi_{1,2} > 0$, transformed into requirements for security prices on arbitrage free market

Pricing with risk neutral probabilities

Extend analyses so far into very important pricing relation
Look again at the first row of (3):

$$1 = \psi_1(1 + r_f) + \psi_2(1 + r_f)$$

We can define:

$$p_1 = \psi_1(1 + r_f) \text{ and } p_2 = \psi_2(1 + r_f) \tag{6}$$

With this definition, $p_{1,2}$ behave as probabilities:

- $0 < p_{1,2} \leq 1$ and
- $p_1 + p_2 = 1$

$$p_1 = \psi_1(1 + r_f) \text{ and } p_2 = \psi_2(1 + r_f)$$

are different from the real probabilities, are called:

- *risk neutral* probabilities or
- *equivalent martingale* probabilities

Notice that risk neutral probabilities:

- are product of *state price* and *time value of money*
- so they contain the pricing information in this market!

Now look again at the second row of (2):

$$S = \psi_1(1+u)S + \psi_2(1+d)S$$

Multiply right hand side by $(1+r_f)/(1+r_f)$:

$$S = \frac{(1+r_f)\psi_1(1+u)S + (1+r_f)\psi_2(1+d)S}{1+r_f}$$

Using the definition

$$p_1 = \psi_1(1+r_f) \ \text{ and } \ p_2 = \psi_2(1+r_f)$$

we get:

$$S = \frac{p_1(1+u)S + p_2(1+d)S}{1+r_f} \tag{7}$$

This is a very important result. It says:

> *expected payoff of a risky asset, discounted at risk free rate, gives true asset value* **if the expected payoff is calculated with the risk neutral probabilities**

This remarkable conclusion is the essence of Black and Scholes Nobel prize winning breakthrough.

Result deserves some further attention

In *risk neutral valuation* or *arbitrage pricing:*

- we don't adjust discount rate with a risk premium
- adjust the probabilities
- Price of risk is embedded in the probability terms
- discounting done with risk free rate, easily observable
- enables pricing assets for which we cannot calculate risk adjusted discount rates, such as options

Also remarkable what does NOT appear in the formula:

- original or real probabilities of upward/downward movement
- the investors' attitudes toward risk
- reference to other securities or portfolios, e.g. market portfolio

Reasons:

- Risk neutral valuation not equilibrium model
  - no matching of demand and supply
  - but the absence of arbitrage opportunities
- Equilibrium models produce:
  - a set of market clearing equilibrium prices
  - as function of investors' preferences, demand for securities, etc.
  - equilibrium prices 'explained' by demand, supply, etc.

- Risk neutral valuation does not 'explain' prices of existing securities on a complete and arbitrage free market
- it takes them as given
- and 'translates' them into prices for additional redundant securities.
- So it is a relative, or conditional, pricing approach:
    - provides prices for additional securities
    - *given* the prices for existing securities
    - without existing securities, risk neutral valuation cannot produce prices at all.

Return equalization

Under the risk neutral probabilities:

- all securities 'earn' same expected riskless return
- all returns are equalized

Can be shown by dividing both equations in (2)

$$D = \psi_1(1 + r_f)D + \psi_2(1 + r_f)D$$
$$S = \psi_1(1 + u)S + \psi_2(1 + d)S$$

by the values of the securities now (D and S resp.):

$$1 = \psi_1 \frac{(1+r_f)D}{D} + \psi_2 \frac{(1+r_f)D}{D}$$
$$1 = \psi_1 \frac{(1+u)S}{S} + \psi_2 \frac{(1+d)S}{S}$$

Multiplying both sides by $(1+r_f)$ and using the definition of $p_{1,2}$ ($p_{1,2} = \psi_{1,2}(1+r_f)$) we get:

$$\begin{aligned}(1+r_f) &= p_1 \frac{(1+r_f)D}{D} + p_2 \frac{(1+r_f)D}{D} \\ (1+r_f) &= p_1 \frac{(1+u)S}{S} + p_2 \frac{(1+d)S}{S} \qquad (8)\end{aligned}$$

Expected return $D, S = r_f$ under risk neutral probabilities.

Trivial for risk free debt, not for the stock

Martingale property

If all assets are expected to earn the risk free rate
then expected future prices, discounted at r_f, must be price now
Adding time subscript to last formula:

$$(1 + r_f) = p_1 \frac{(1+u)S_t}{S_t} + p_2 \frac{(1+d)S_t}{S_t}$$

By definition:

$$E^p[S_{t+1}] = p_1(1+u)S_t + p_2(1+d)S_t$$

E^p expectation operator w.r.t. risk neutral probabilities p
This means:

$$S_t = \frac{E^p[S_{t+1}]}{(1+r_f)}$$

We recognize the martingale dynamic process from market efficiency:

$$X \text{ is martingale if } E(X_{t+1} \mid X_0, ...X_t) = X_t$$

Under risk neutral probabilities:

- discounted exp. future asset prices are martingales
- hence 'martingale' in equivalent martingale measure.

Notice:

- asset prices not martingales
- but asset prices discounted at r_f
- Asset prices expected to grow with risk free rate

Probability measure:

- set of probabilities on all possible outcomes
- describing likelihood of each outcome
- e.g. sides of coin $\frac{1}{2}$, or sides of a die $1/6$
- Real prob. measures based on e.g. long term frequency
- Risk neutral probabilities based on state prices

Probability measures are *equivalent* if:

- they assign positive probability to same set of outcomes
- i.e. agree on which outcomes have zero prob.

Hence term: *equivalent martingale probabilities*

State prices and probabilities

Recall definition of risk neutral probabilities in (6):

$$p_1 = \psi_1(1 + r_f) \text{ and } p_2 = \psi_2(1 + r_f).$$

Rewrite in term of state prices:

$$\psi_1 = \frac{p_1}{(1 + r_f)} \text{ and } \psi_2 = \frac{p_2}{(1 + r_f)}$$

divide both sides by the sum of the two:

$$\frac{\psi_1}{\psi_1 + \psi_2} = \frac{p_1/(1 + r_f)}{\frac{p_1+p_2}{(1+r_f)}} \text{ and } \frac{\psi_2}{\psi_1 + \psi_2} = \frac{p_2/(1 + r_f)}{\frac{p_1+p_2}{(1+r_f)}}$$

$$\frac{\psi_1}{\psi_1 + \psi_2} = \frac{p_1}{p_1 + p_2} = p_1 \text{ and } \frac{\psi_2}{\psi_1 + \psi_2} = \frac{p_2}{p_1 + p_2} = p_2$$

Risk neutral probabilities are:

- standardized state prices
- i.e. defined to sum to 1
- makes the embedded pricing info even more explicit

From this we can conclude that:

- positive state price vector
- positive risk neutral probabilities
- equivalent martingale measure

are all same condition

So we can reformulate no arbitrage condition:

There is no arbitrage if and only if there exists an equivalent martingale measure.

The pricing kernel

Found the state price vector as:

$$v\Psi^{-1} = \left[\begin{array}{ccc} 0.298 & 0.419 & 0.202 \end{array} \right]$$

Why are prices of one money unit different across states?
Two reasons:

- Probability that state occurs:
  - higher probability $\Rightarrow$ higher state price
- Marginal utility of money:
  - market assigns different utility to different states
  - expresses the risk aversion in the market

Eliminate probability that state occurs:

- by calculating the price per unit of probability
- have to use the real probabilities for this
- not the equivalent martingale probabilities

Resulting vector of 'probability deflated' state prices is called the *pricing kernel*

State	Price	Real probability	Pricing kernel
bust	.298	.3	.9933
normal	.419	.45	.9311
boom	.202	.25	.808

Marginal utility of extra money unit is higher in a bust than in a boom:

- In a bust, good results are scarce
  $\Rightarrow$ investments that pay off just then are valuable.
- In a boom, almost all investments pays off
  $\Rightarrow$ extra money unit contributes little to total wealth

State prices and probabilities must be positive
allows yet another reformulation of no arbitrage condition:

the existence of a positive pricing kernel excludes arbitrage possibilities

We now have three equivalent ways of formulating the no arbitrage condition:

There are no arbitrage possibilities if and only if:

1. there exists a positive state price vector
2. there exists an equivalent martingale measure
3. there exists a positive pricing kernel

