

# Lecture 3:

# Regularization and Optimization

# Administrative: Assignment 1

Released last week, due **Fri 4/19 at 11:59pm**

Office hours: help with high-level questions only, no code debugging. [[No Code Show Policy](#)]

# Administrative: Project proposal + Office Hours

Due **Mon 4/22**

TA expertise + Office Hours are posted on the webpage. Mix of in-person and zoom.

([http://cs231n.stanford.edu/office\\_hours.html](http://cs231n.stanford.edu/office_hours.html))

# Administrative: Ed

Please make sure to check and read all pinned Ed posts.

- Project group: fill in your information in the google form and/or look through existing responses and reach out
- SCPD: if you would like to take the midterm on-campus, send us an email: [cs231n-staff-spr24@stanford.edu](mailto:cs231n-staff-spr24@stanford.edu) + cc [scpd-exams@stanford.edu](mailto:scpd-exams@stanford.edu)

# Recap from Last Week

# Image Classification: A core task in Computer Vision



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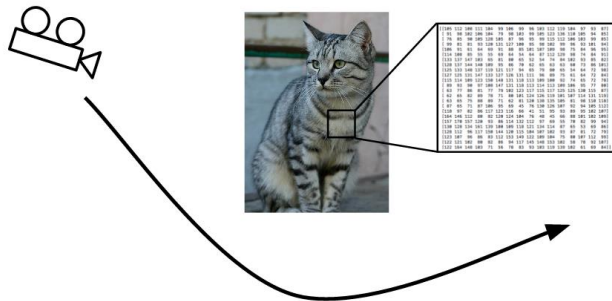
(assume given a set of labels)  
{dog, cat, truck, plane, ...}



cat  
dog  
bird  
deer  
truck

# Recall from last time: Challenges of recognition

Viewpoint



Illumination



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Deformation



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Occlusion



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Clutter



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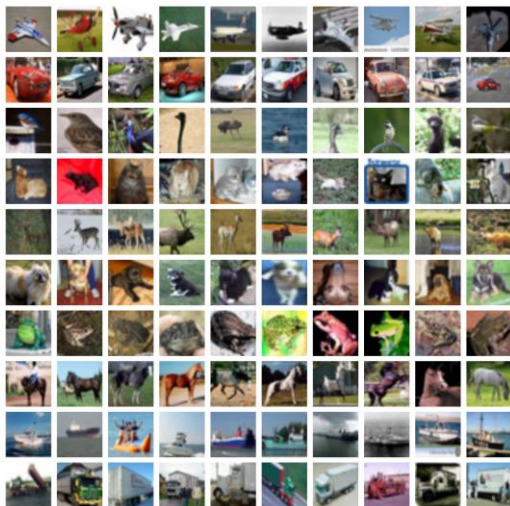
Intraclass Variation



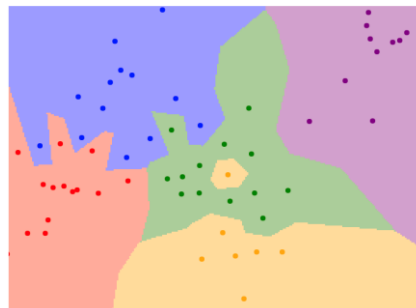
[This image](#) is [CC0 1.0](#) public domain

# Recall from last time: data-driven approach, kNN

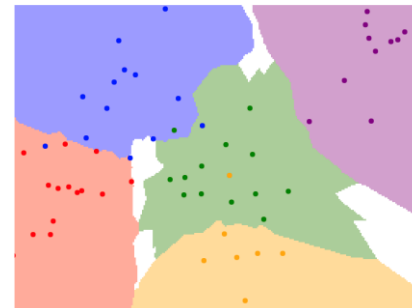
airplane  
automobile  
bird  
cat  
deer  
dog  
frog  
horse  
ship  
truck



1-NN classifier



5-NN classifier



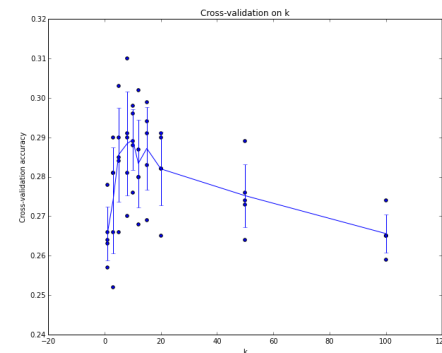
train

test

train

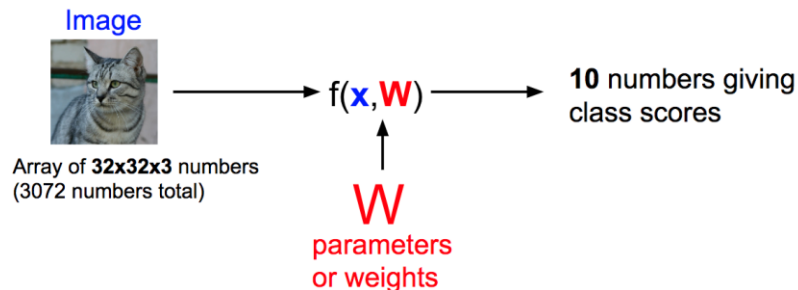
validation

test





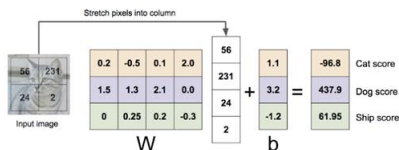
# Recall from last time: Linear Classifier



$$f(x, W) = Wx + b$$

## Algebraic Viewpoint

$$f(x, W) = Wx$$



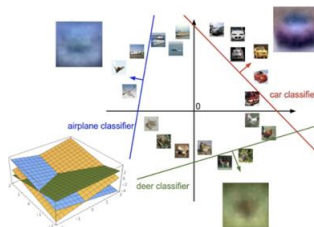
## Visual Viewpoint

One template  
per class



## Geometric Viewpoint

Hyperplanes  
cutting up space

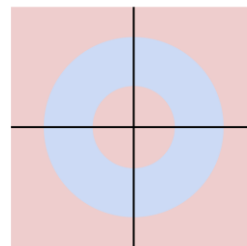


**Class 1:**

$1 \leq L2 \text{ norm} \leq 2$

**Class 2:**

Everything else

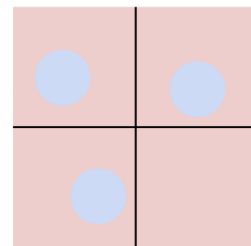


**Class 1:**

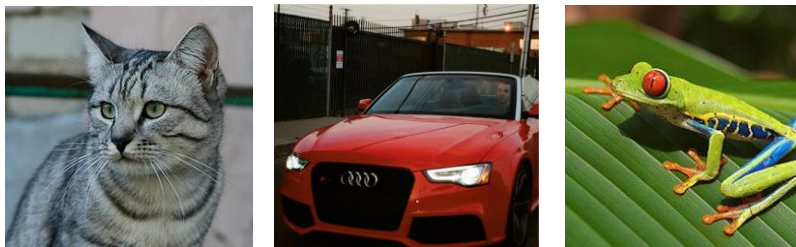
Three modes

**Class 2:**

Everything else



Suppose: 3 training examples, 3 classes.  
With some  $W$  the scores  $f(x, W) = Wx$  are:



cat	<b>3.2</b>	1.3	2.2
car	5.1	<b>4.9</b>	2.5
frog	-1.7	2.0	<b>-3.1</b>

A **loss function** tells how good our current classifier is

Given a dataset of examples

$$\{(x_i, y_i)\}_{i=1}^N$$

Where  $x_i$  is image and  
 $y_i$  is (integer) label

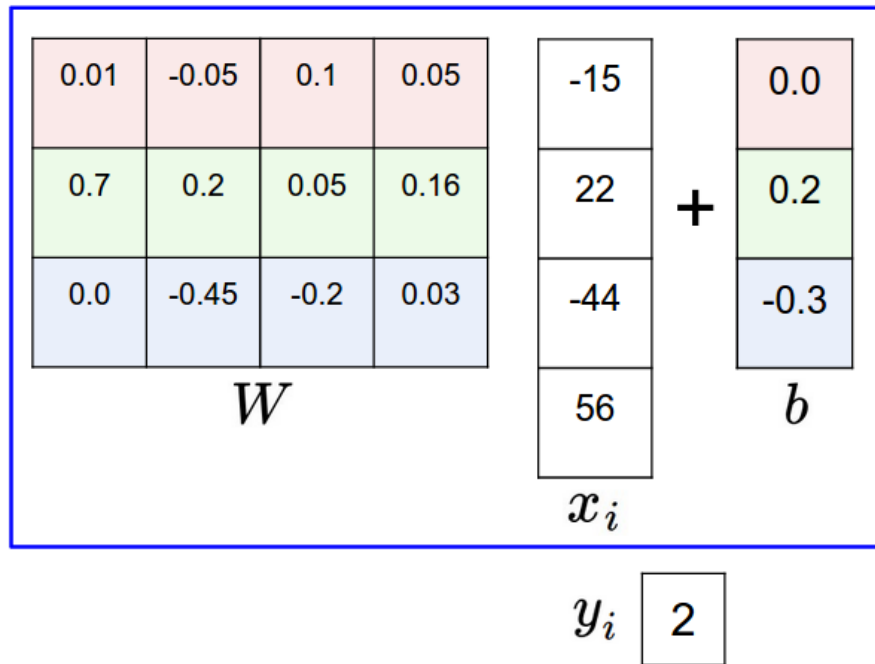
Loss over the dataset is a  
average of loss over examples:

$$L = \frac{1}{N} \sum_i L_i(f(x_i, W), y_i)$$

# Softmax vs. SVM

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right) \text{ vs. } L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

matrix multiply + bias offset

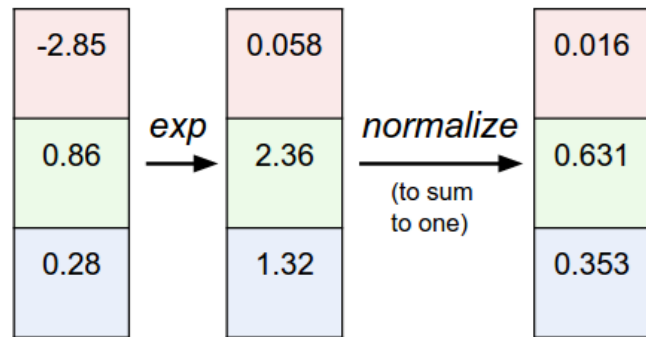


hinge loss (SVM)

-2.85
0.86
0.28

$$\begin{aligned} &\max(0, -2.85 - 0.28 + 1) + \\ &\max(0, 0.86 - 0.28 + 1) \\ &= \\ &\mathbf{1.58} \end{aligned}$$

cross-entropy loss (Softmax)



$$f(x, W) = Wx$$

$$L = \frac{1}{N} \sum_{i=1}^N \sum_{j \neq y_i} \max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + 1)$$

Q: Suppose that we found a  $W$  such that  $L = 0$ .  
Is this  $W$  unique?

$$f(x, W) = Wx$$

$$L = \frac{1}{N} \sum_{i=1}^N \sum_{j \neq y_i} \max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + 1)$$

Q: Suppose that we found a  $W$  such that  $L = 0$ .  
Is this  $W$  unique?

**No!  $2W$  is also has  $L = 0$ !**

Suppose: 3 training examples, 3 classes.  
 With some  $W$  the scores  $f(x, W) = Wx$  are:



cat	<b>3.2</b>	1.3	2.2
car	5.1	<b>4.9</b>	2.5
frog	-1.7	2.0	<b>-3.1</b>
Losses:	2.9	0	

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

**Before:**

$$\begin{aligned}
 &= \max(0, 1.3 - 4.9 + 1) \\
 &\quad + \max(0, 2.0 - 4.9 + 1) \\
 &= \max(0, -2.6) + \max(0, -1.9) \\
 &= 0 + 0 \\
 &= 0
 \end{aligned}$$

**With  $W$  twice as large:**

$$\begin{aligned}
 &= \max(0, 2.6 - 9.8 + 1) \\
 &\quad + \max(0, 4.0 - 9.8 + 1) \\
 &= \max(0, -6.2) + \max(0, -4.8) \\
 &= 0 + 0 \\
 &= 0
 \end{aligned}$$

$$f(x, W) = Wx$$


$$L = \frac{1}{N} \sum_{i=1}^N \sum_{j \neq y_i} \max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + 1)$$

E.g. Suppose that we found a  $W$  such that  $L = 0$ .  
Is this  $W$  unique?

**No!  $2W$  is also has  $L = 0$ !**

**How do we choose between  $W$  and  $2W$ ?**

# Regularization -

$$L(W) = \frac{1}{N} \sum_{i=1}^N L_i(f(x_i, W), y_i)$$


**Data loss:** Model predictions  
should match training data



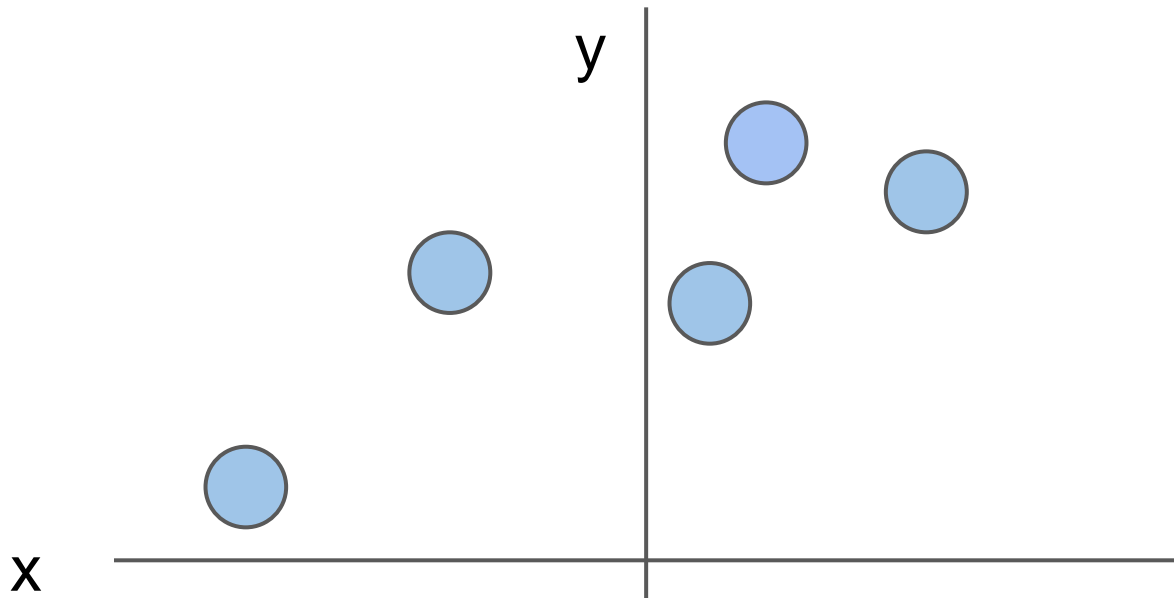
# Regularization

$$L(W) = \underbrace{\frac{1}{N} \sum_{i=1}^N L_i(f(x_i, W), y_i)}_{\text{Data loss: Model predictions should match training data}} + \underbrace{\lambda R(W)}_{\text{Regularization: Prevent the model from doing too well on training data}}$$

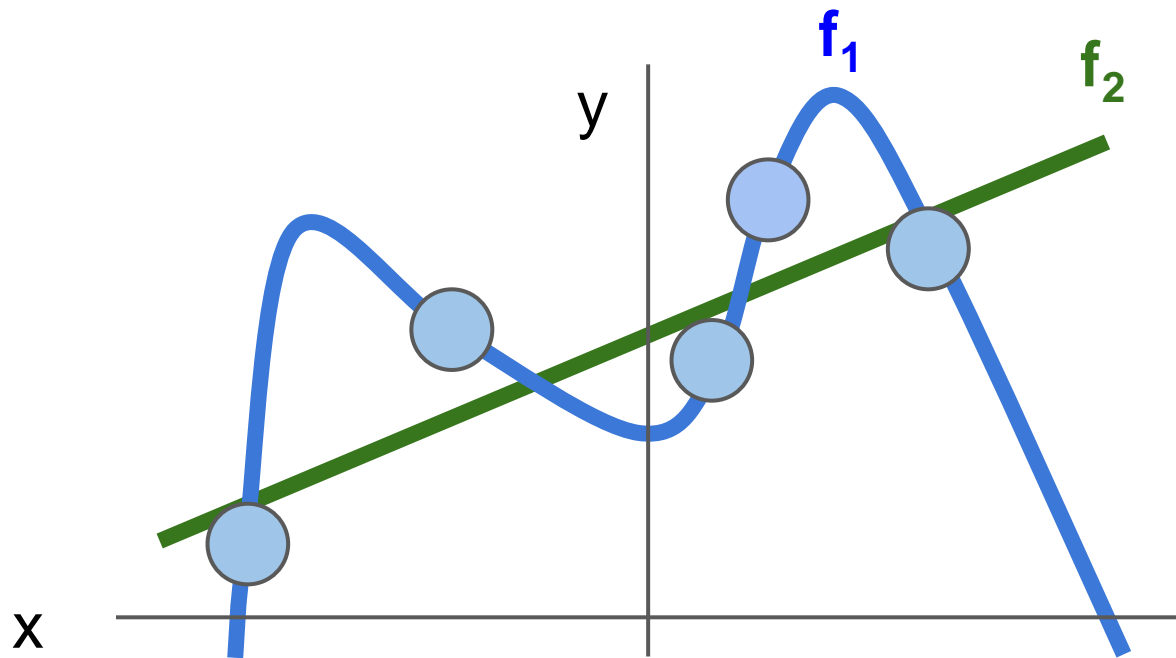
**Data loss:** Model predictions should match training data

**Regularization:** Prevent the model from doing *too* well on training data

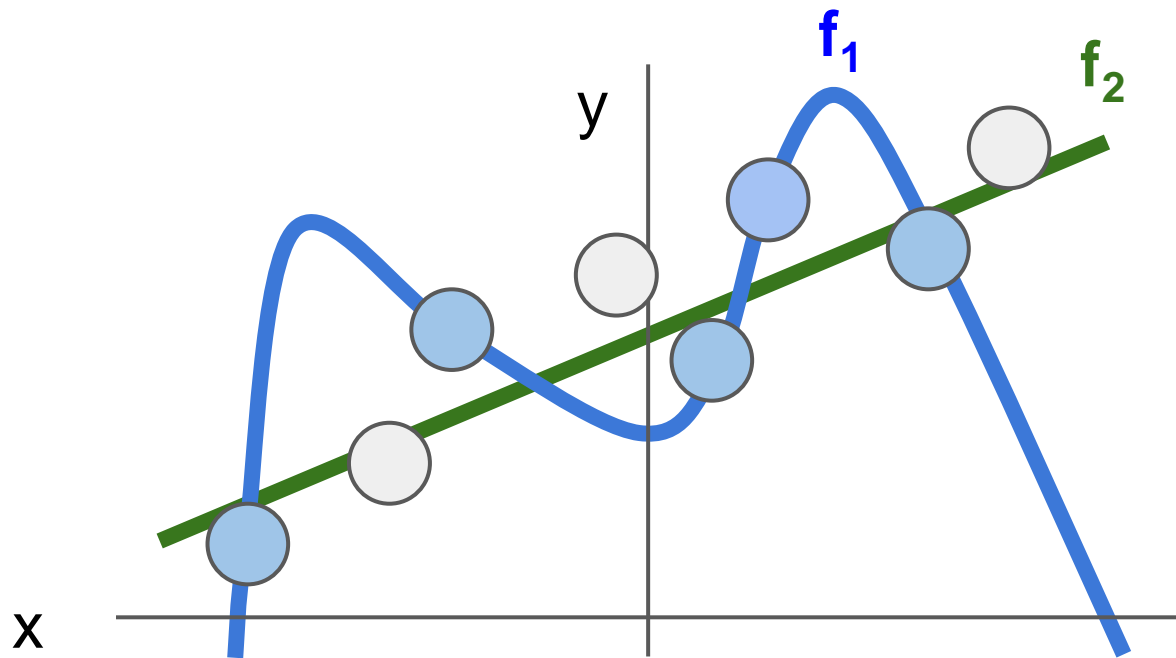
# Regularization intuition: toy example training data



# Regularization intuition: Prefer Simpler Models



# Regularization: Prefer Simpler Models



Regularization pushes against fitting the data  
*too* well so we don't fit noise in the data

# Regularization

$$L(W) = \underbrace{\frac{1}{N} \sum_{i=1}^N L_i(f(x_i, W), y_i)}_{\text{Data loss}} + \underbrace{\lambda R(W)}_{\text{Regularization}}$$

**Data loss:** Model predictions should match training data

**Regularization:** Prevent the model from doing *too well* on training data

**Occam's Razor:** Among multiple competing hypotheses, the simplest is the best, William of Ockham 1285-1347

# Regularization

$\lambda$  = regularization strength  
(hyperparameter)

$$L(W) = \underbrace{\frac{1}{N} \sum_{i=1}^N L_i(f(x_i, W), y_i)}_{\text{Data loss}} + \underbrace{\lambda R(W)}_{\text{Regularization}}$$

**Data loss:** Model predictions should match training data

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# Regularization

$\lambda$  = regularization strength  
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$$L(W) = \underbrace{\frac{1}{N} \sum_{i=1}^N L_i(f(x_i, W), y_i)}_{\text{Data loss: Model predictions should match training data}} + \underbrace{\lambda R(W)}_{\text{Regularization: Prevent the model from doing too well on training data}}$$

**Data loss:** Model predictions should match training data

**Regularization:** Prevent the model from doing *too* well on training data

## Simple examples

L2 regularization:  $R(W) = \sum_k \sum_l W_{k,l}^2$

L1 regularization:  $R(W) = \sum_k \sum_l |W_{k,l}|$

Elastic net (L1 + L2):  $R(W) = \sum_k \sum_l \beta W_{k,l}^2 + |W_{k,l}|$

# Regularization

$\lambda$  = regularization strength  
(hyperparameter)

$$L(W) = \underbrace{\frac{1}{N} \sum_{i=1}^N L_i(f(x_i, W), y_i)}_{\text{Data loss: Model predictions should match training data}} + \underbrace{\lambda R(W)}_{\text{Regularization: Prevent the model from doing too well on training data}}$$

**Data loss:** Model predictions should match training data

**Regularization:** Prevent the model from doing *too well* on training data

## Simple examples

L2 regularization:  $R(W) = \sum_k \sum_l W_{k,l}^2$

L1 regularization:  $R(W) = \sum_k \sum_l |W_{k,l}|$

Elastic net (L1 + L2):  $R(W) = \sum_k \sum_l \beta W_{k,l}^2 + |W_{k,l}|$

## More complex:

Dropout

Batch normalization

Stochastic depth, fractional pooling, etc



# Regularization

$\lambda$  = regularization strength  
(hyperparameter)

$$L(W) = \underbrace{\frac{1}{N} \sum_{i=1}^N L_i(f(x_i, W), y_i)}_{\text{Data loss}} + \underbrace{\lambda R(W)}_{\text{Regularization}}$$

**Data loss:** Model predictions should match training data

**Regularization:** Prevent the model from doing *too well* on training data

Why regularize?

- Express preferences over weights
- Make the model *simple* so it works on test data
- Improve optimization by adding curvature

# Regularization: Expressing Preferences

L2 Regularization

$$R(W) = \sum_k \sum_l W_{k,l}^2$$

Which of  $w_1$  or  $w_2$  will  
the L2 regularizer prefer?

$$x = [1, 1, 1, 1]$$

$$w_1 = [1, 0, 0, 0]$$

$$w_2 = [0.25, 0.25, 0.25, 0.25]$$

$$w_1^T x = w_2^T x = 1$$

# Regularization: Expressing Preferences

$$x = [1, 1, 1, 1]$$

$$w_1 = [1, 0, 0, 0]$$

$$w_2 = [0.25, 0.25, 0.25, 0.25]$$

$$w_1^T x = w_2^T x = 1$$

L2 Regularization

$$R(W) = \sum_k \sum_l W_{k,l}^2$$

Which of  $w_1$  or  $w_2$  will  
the L2 regularizer prefer?

L2 regularization likes to  
“spread out” the weights

# Regularization: Expressing Preferences

$$x = [1, 1, 1, 1]$$

$$w_1 = [1, 0, 0, 0]$$

$$w_2 = [0.25, 0.25, 0.25, 0.25]$$

$$w_1^T x = w_2^T x = 1$$

L2 Regularization

$$R(W) = \sum_k \sum_l W_{k,l}^2$$

Which of  $w_1$  or  $w_2$  will  
the L2 regularizer prefer?

L2 regularization likes to  
“spread out” the weights

Which one would L1  
regularization prefer?

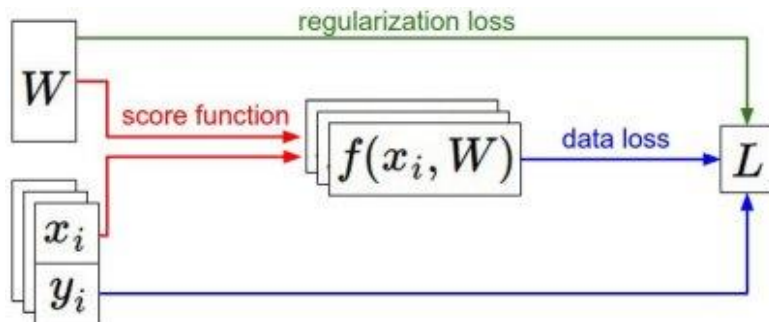
# Recap

- We have some dataset of (x,y)
- We have a **score function**:  $s = f(x; W) \stackrel{\text{e.g.}}{=} Wx$
- We have a **loss function**:

$$L_i = -\log\left(\frac{e^{sy_i}}{\sum_j e^{sj}}\right) \quad \text{Softmax}$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1) \quad \text{SVM}$$

$$L = \frac{1}{N} \sum_{i=1}^N L_i + R(W) \quad \text{Full loss}$$



# Recap

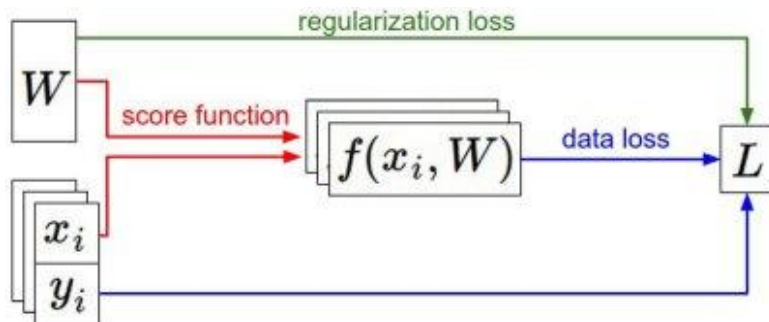
## How do we find the best $W$ ?

- We have some dataset of  $(x, y)$
- We have a **score function**:  $s = f(x; W) \stackrel{\text{e.g.}}{=} Wx$
- We have a **loss function**:

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right) \quad \text{Softmax}$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1) \quad \text{SVM}$$

$$L = \frac{1}{N} \sum_{i=1}^N L_i + R(W) \quad \text{Full loss}$$



# Optimization





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[Walking man image](#) is [CC0 1.0](#) public domain

# Strategy #1: A first very bad idea solution: Random search

```
# assume X_train is the data where each column is an example (e.g. 3073 x 50,000)
# assume Y_train are the labels (e.g. 1D array of 50,000)
# assume the function L evaluates the loss function

bestloss = float("inf") # Python assigns the highest possible float value
for num in xrange(1000):
    W = np.random.randn(10, 3073) * 0.0001 # generate random parameters
    loss = L(X_train, Y_train, W) # get the loss over the entire training set
    if loss < bestloss: # keep track of the best solution
        bestloss = loss
        bestW = W
    print 'in attempt %d the loss was %f, best %f' % (num, loss, bestloss)

# prints:
# in attempt 0 the loss was 9.401632, best 9.401632
# in attempt 1 the loss was 8.959668, best 8.959668
# in attempt 2 the loss was 9.044034, best 8.959668
# in attempt 3 the loss was 9.278948, best 8.959668
# in attempt 4 the loss was 8.857370, best 8.857370
# in attempt 5 the loss was 8.943151, best 8.857370
# in attempt 6 the loss was 8.605604, best 8.605604
# ... (truncated: continues for 1000 lines)
```

Lets see how well this works on the test set...

```
# Assume X_test is [3073 x 10000], Y_test [10000 x 1]  
scores = Wbest.dot(Xte_cols) # 10 x 10000, the class scores for all test examples  
# find the index with max score in each column (the predicted class)  
Yte_predict = np.argmax(scores, axis = 0)  
# and calculate accuracy (fraction of predictions that are correct)  
np.mean(Yte_predict == Yte)  
# returns 0.1555
```

15.5% accuracy! not bad!  
(SOTA is ~99.7%)



## Strategy #2: Follow the slope



## Strategy #2: **Follow the slope**

In 1-dimension, the derivative of a function:

$$\frac{df(x)}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

In multiple dimensions, the **gradient** is the vector of (partial derivatives) along each dimension

The slope in any direction is the **dot product** of the direction with the gradient  
The direction of steepest descent is the **negative gradient**

**current W:**

[0.34,  
-1.11,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,...]

**loss 1.25347**

**gradient dW:**

[?,  
?,  
?,  
?,  
?,  
?,  
?,  
?,  
?,...]

**current W:**

[0.34,  
-1.11,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,...]

**loss 1.25347**

**W + h (first dim):**

[0.34 + **0.0001**,  
-1.11,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,...]

**loss 1.25322**

**gradient dW:**

[?,  
?,  
?,  
?,  
?,  
?,  
?,  
?,  
?,...]

current **W**:

[0.34,  
-1.11,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,...]

**loss 1.25347**

**W + h** (first dim):

[0.34 + **0.0001**,  
-1.11,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,...]

**loss 1.25322**

gradient **dW**:

**[-2.5,**  
?,  
?,


$$(1.25322 - 1.25347)/0.0001 = -2.5$$

$$\frac{df(x)}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

?,  
?,...]



**current W:**

[0.34,  
-1.11,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,...]

**loss 1.25347**

**W + h (second dim):**

[0.34,  
-1.11 + **0.0001**,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,...]

**loss 1.25353**

**gradient dW:**

[-2.5,  
?,  
?,  
?,  
?,  
?,  
?,  
?,  
?,...]

current **W**:

[0.34,  
-1.11,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,...]

**loss 1.25347**

**W + h** (second dim):

[0.34,  
-1.11 + **0.0001**,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,...]

**loss 1.25353**

gradient **dW**:

[-2.5,  
**0.6**,  
?,  
?,


$$(1.25353 - 1.25347)/0.0001 = 0.6$$

$$\frac{df(x)}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

?,...]

**current W:**

[0.34,  
-1.11,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,...]

**loss 1.25347**

**W + h (third dim):**

[0.34,  
-1.11,  
0.78 + **0.0001**,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,...]

**loss 1.25347**

**gradient dW:**

[-2.5,  
0.6,  
?,  
?,  
?,  
?,  
?,  
?,  
?,...]

[0.34,  
-1.11,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,...]

loss 1.25347

[0.34,  
-1.11,  
0.78 + **0.0001**,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,...]

loss 1.25347

**gradient dW:**

[-2.5,  
0.6,  
**0**,  
?,  
...]

$$(1.25347 - 1.25347)/0.0001 = 0$$

$$\frac{df(x)}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

?, ...]

**current W:**

[0.34,  
-1.11,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,...]

**loss 1.25347**

**W + h (third dim):**

[0.34,  
-1.11,  
0.78 + **0.0001**,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,...]

**loss 1.25347**

**gradient dW:**

[-2.5,  
0.6,  
**0**,  
?,  
?

### Numeric Gradient

- Slow! Need to loop over all dimensions
- Approximate

?,...]

This is silly. The loss is just a function of  $W$ :

$$L = \frac{1}{N} \sum_{i=1}^N L_i + \sum_k W_k^2$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$s = f(x; W) = Wx$$

want  $\nabla_W L$

This is silly. The loss is just a function of  $W$ :

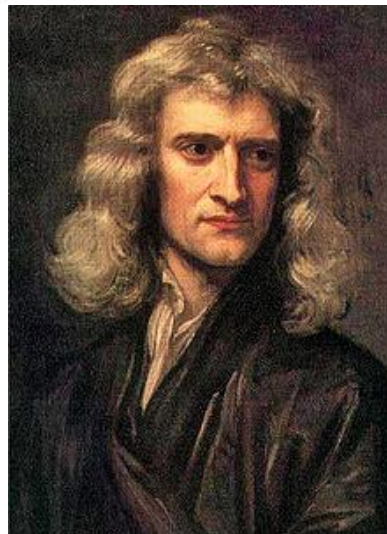
$$L = \frac{1}{N} \sum_{i=1}^N L_i + \sum_k W_k^2$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$s = f(x; W) = Wx$$

want  $\nabla_W L$

Use calculus to compute an  
**analytic gradient**



[This image](#) is in the public domain



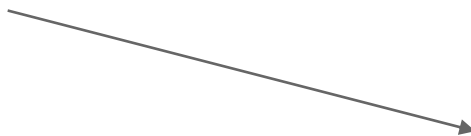
[This image](#) is in the public domain

**current W:**

[0.34,  
-1.11,  
0.78,  
0.12,  
0.55,  
2.81,  
-3.1,  
-1.5,  
0.33,...]

**loss 1.25347**

$dW = \dots$   
(some function  
data and W)



**gradient dW:**

[-2.5,  
0.6,  
0,  
0.2,  
0.7,  
-0.5,  
1.1,  
1.3,  
-2.1,...]



## In summary:

- Numerical gradient: approximate, slow, easy to write
- Analytic gradient: exact, fast, error-prone

=>

In practice: Always use analytic gradient, but check implementation with numerical gradient. This is called a **gradient check.**

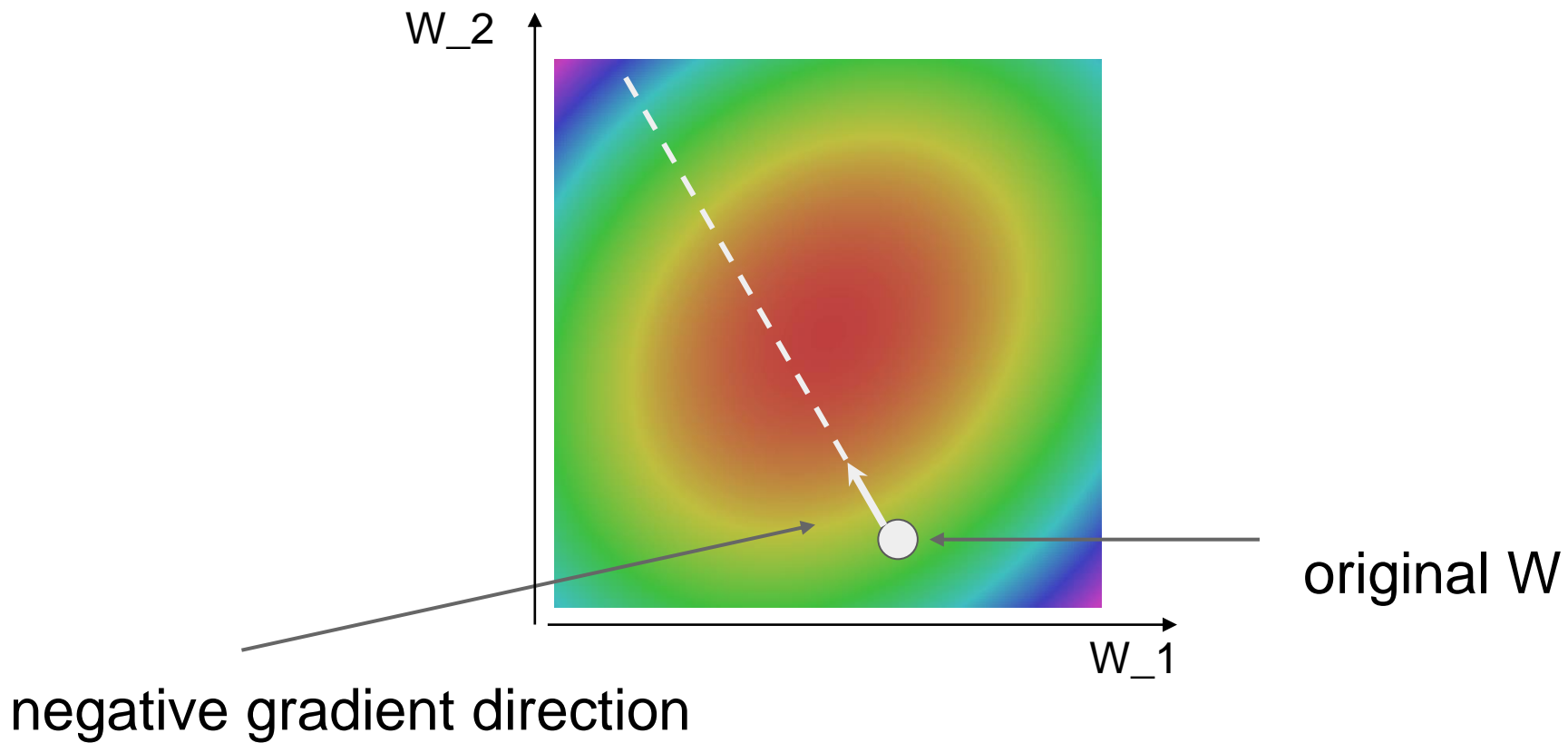
# Gradient Descent

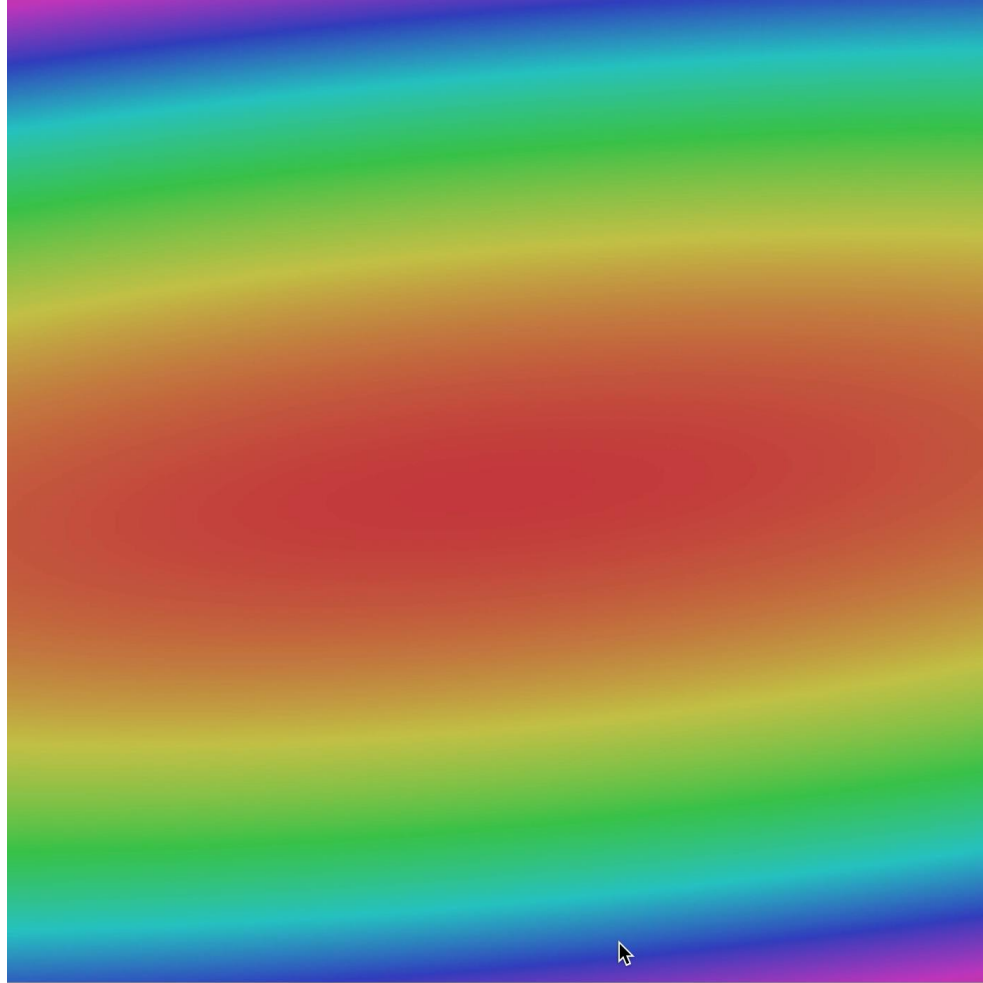
```
# Vanilla Gradient Descent
```

```
while True:
```

```
    weights_grad = evaluate_gradient(loss_fun, data, weights)
```

```
    weights += - step_size * weights_grad # perform parameter update
```





# Stochastic Gradient Descent (SGD)

$$L(W) = \frac{1}{N} \sum_{i=1}^N L_i(x_i, y_i, W) + \lambda R(W)$$

$$\nabla_W L(W) = \frac{1}{N} \sum_{i=1}^N \nabla_W L_i(x_i, y_i, W) + \lambda \nabla_W R(W)$$

Full sum expensive  
when N is large!

Approximate sum  
using a **minibatch** of  
examples  
32 / 64 / 128 common

```
# Vanilla Minibatch Gradient Descent
```

```
while True:
```

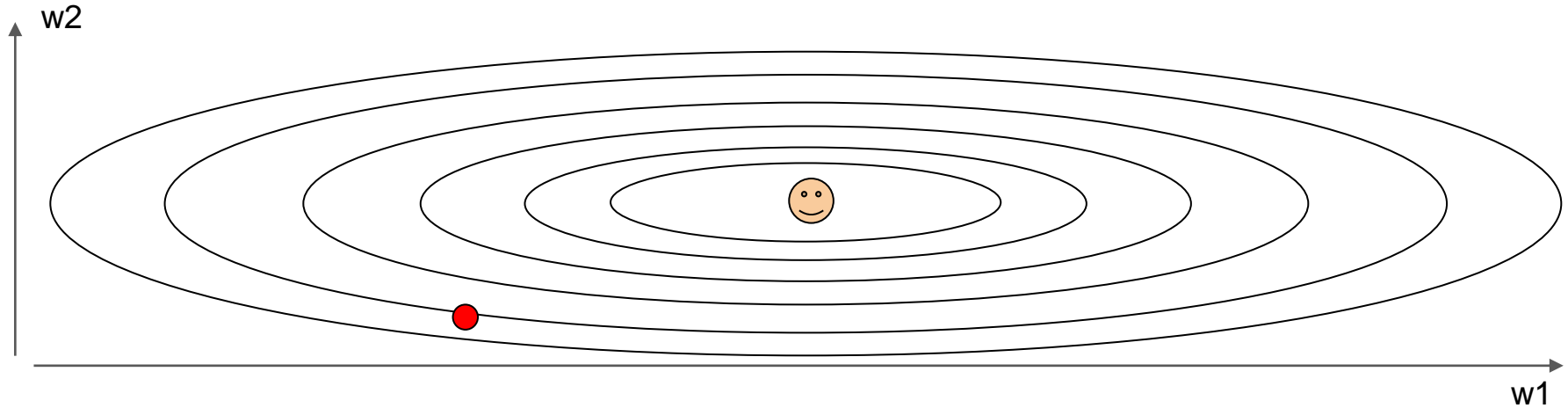
```
    data_batch = sample_training_data(data, 256) # sample 256 examples
```

```
    weights_grad = evaluate_gradient(loss_fun, data_batch, weights)
```

```
    weights += - step_size * weights_grad # perform parameter update
```

# Optimization: Problem #1 with SGD

What if loss changes quickly in one direction and slowly in another?  
What does gradient descent do?

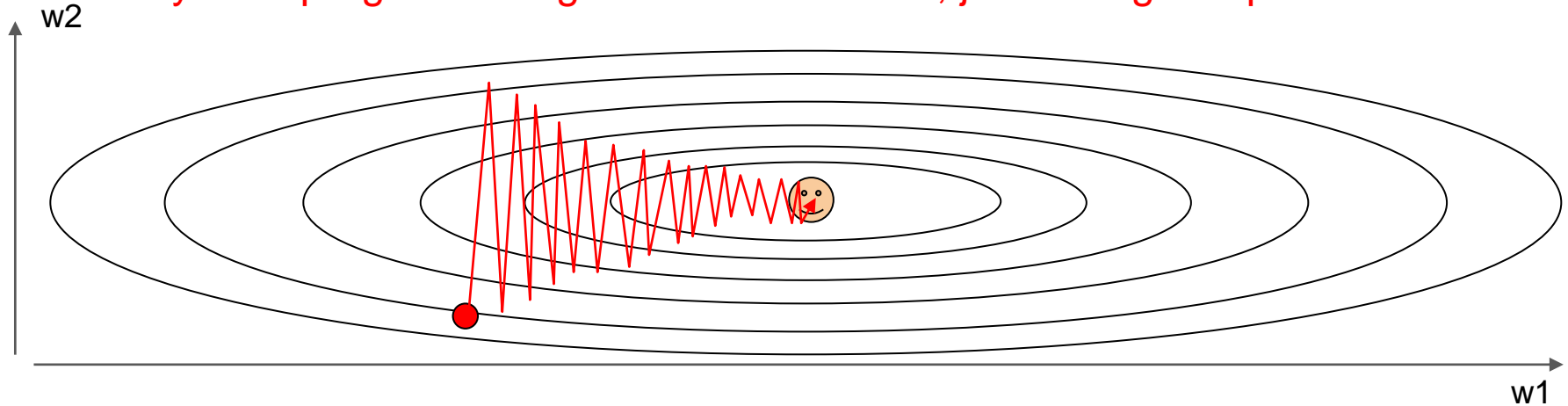


# Optimization: Problem #1 with SGD

What if loss changes quickly in one direction and slowly in another?

What does gradient descent do?

Very slow progress along shallow dimension, jitter along steep direction

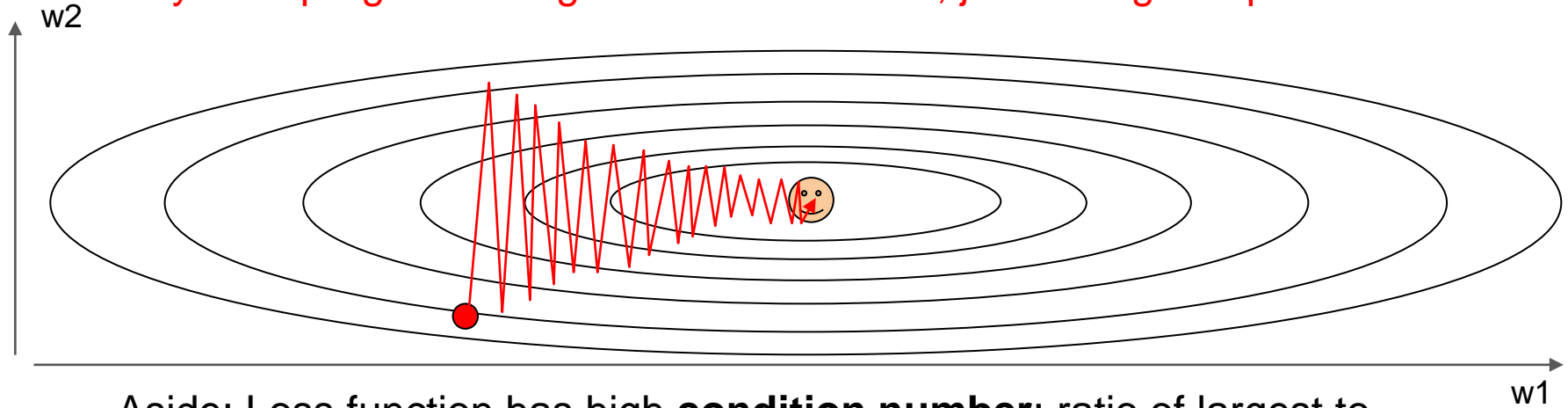


# Optimization: Problem #1 with SGD

What if loss changes quickly in one direction and slowly in another?

What does gradient descent do?

Very slow progress along shallow dimension, jitter along steep direction

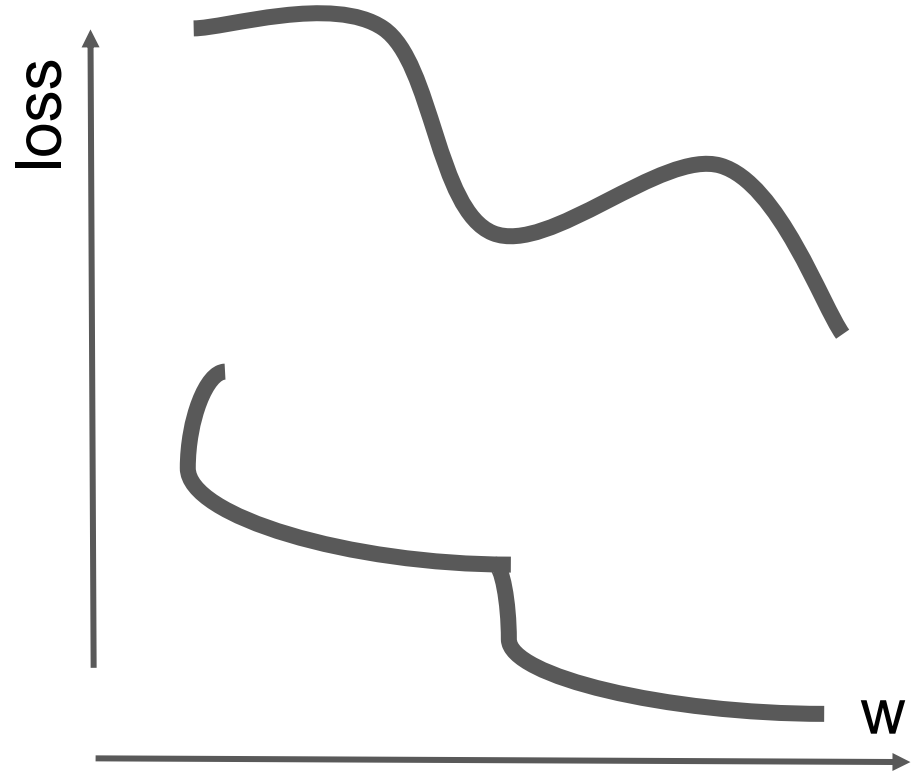


Aside: Loss function has high **condition number**: ratio of largest to smallest singular value of the Hessian matrix is large



# Optimization: Problem #2 with SGD

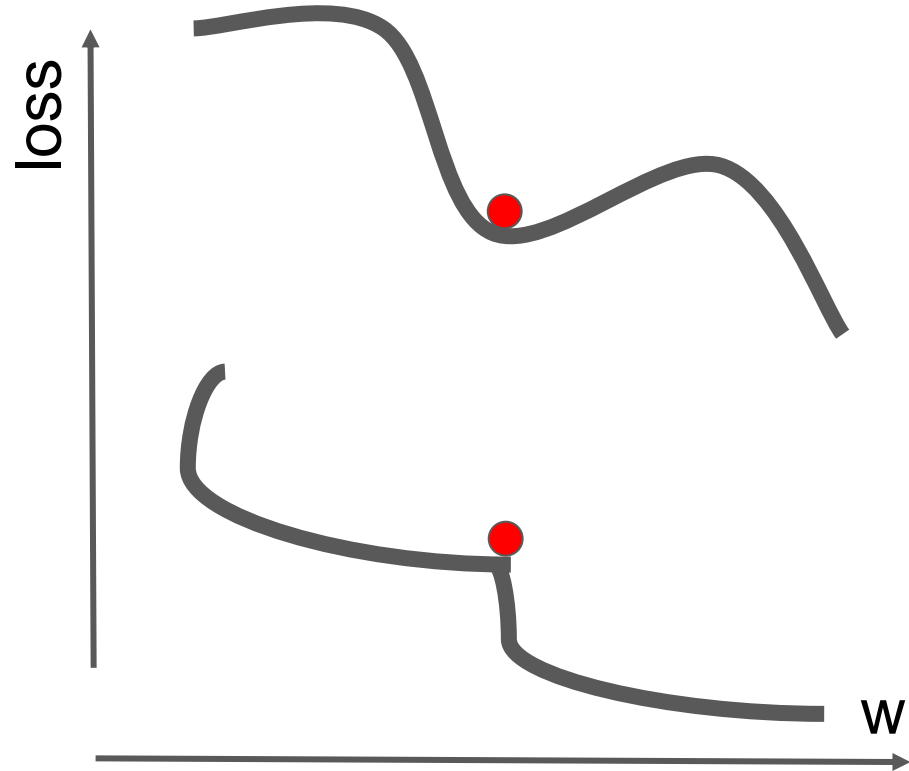
What if the loss function has a **local minima** or **saddle point**?



# Optimization: Problem #2 with SGD

What if the loss function has a **local minima** or **saddle point**?

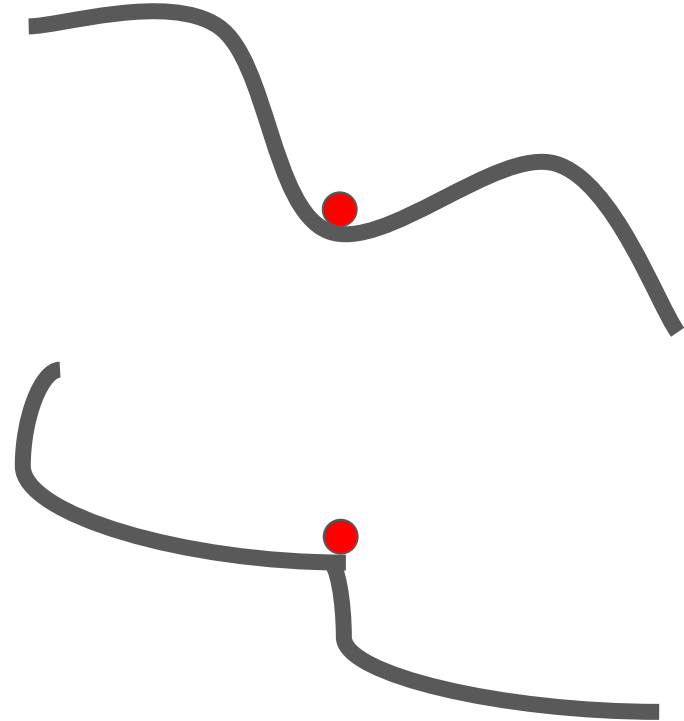
Zero gradient,  
gradient descent  
gets stuck



# Optimization: Problem #2 with SGD

What if the loss function has a **local minima** or **saddle point**?

Saddle points much more common in high dimension



Dauphin et al, "Identifying and attacking the saddle point problem in high-dimensional non-convex optimization", NIPS 2014

# Optimization: Problem #2 with SGD

**saddle point** in two dimension

$$f(x, y) = x^2 - y^2$$

$$\frac{\partial}{\partial x}(x^2 - y^2) = 2x \rightarrow 2(0) = 0$$

$$\frac{\partial}{\partial y}(x^2 - y^2) = -2y \rightarrow -2(0) = 0$$

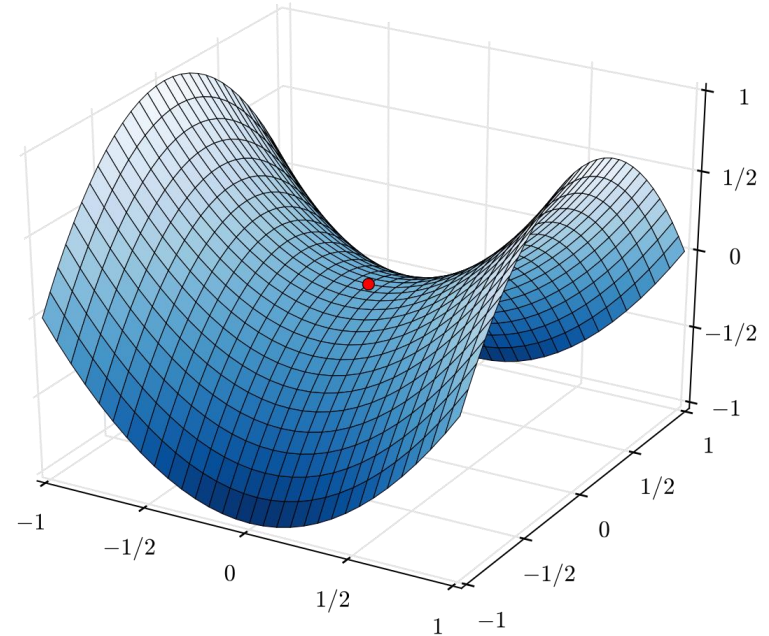


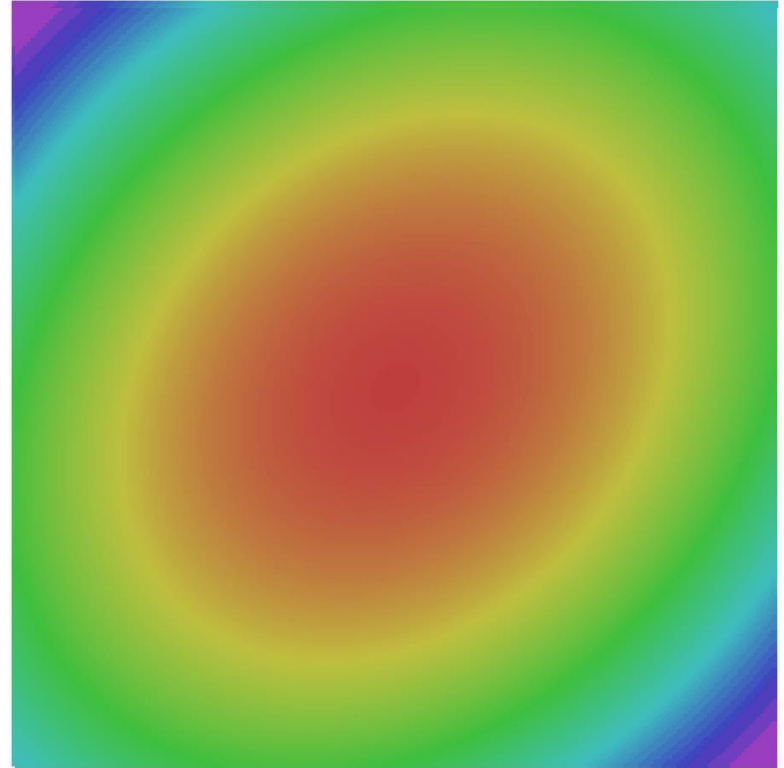
Image source: [https://en.wikipedia.org/wiki/Saddle\\_point](https://en.wikipedia.org/wiki/Saddle_point)

# Optimization: Problem #3 with SGD

Our gradients come from minibatches so they can be noisy!

$$L(W) = \frac{1}{N} \sum_{i=1}^N L_i(x_i, y_i, W)$$

$$\nabla_W L(W) = \frac{1}{N} \sum_{i=1}^N \nabla_W L_i(x_i, y_i, W)$$



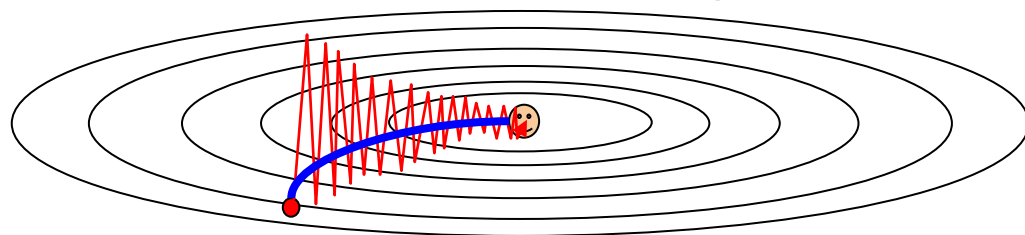
# SGD + Momentum

Local Minima

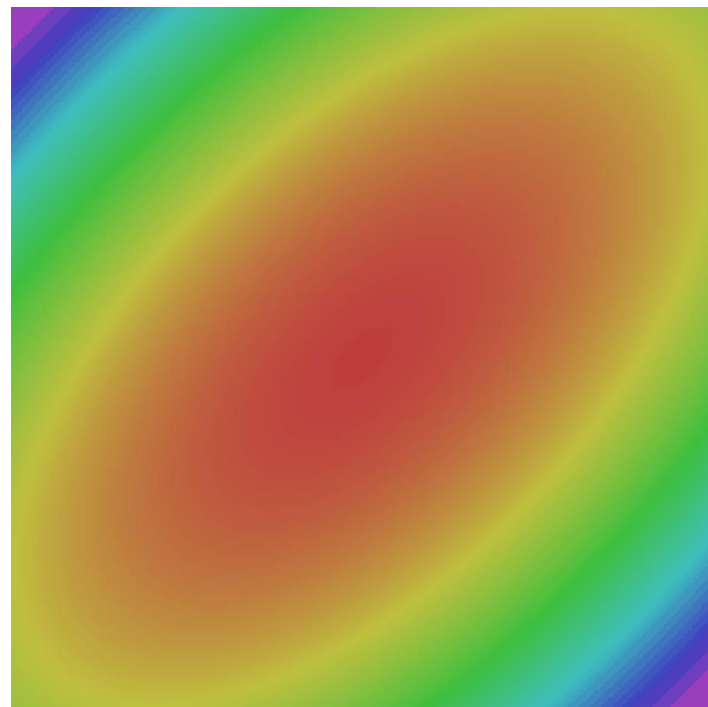
Saddle points



Poor Conditioning



Gradient Noise



— SGD

— SGD+Momentum

# SGD: the simple two line update code

## SGD

$$x_{t+1} = x_t - \alpha \nabla f(x_t)$$

```
while True:
    dx = compute_gradient(x)
    x -= learning_rate * dx
```

# SGD + Momentum:

continue moving in the general direction as the previous iterations

## SGD

$$x_{t+1} = x_t - \alpha \nabla f(x_t)$$

```
while True:
    dx = compute_gradient(x)
    x -= learning_rate * dx
```

## SGD+Momentum

$$v_{t+1} = \rho v_t + \nabla f(x_t)$$

$$x_{t+1} = x_t - \alpha v_{t+1}$$

- Build up “velocity” as a running mean of gradients
- Rho gives “friction”; typically rho=0.9 or 0.99

Sutskever et al, “On the importance of initialization and momentum in deep learning”, ICML 2013



# SGD + Momentum:

continue moving in the general direction as the previous iterations

## SGD

$$x_{t+1} = x_t - \alpha \nabla f(x_t)$$

```
while True:
    dx = compute_gradient(x)
    x -= learning_rate * dx
```

## SGD+Momentum

$$v_{t+1} = \rho v_t + \nabla f(x_t)$$

$$x_{t+1} = x_t - \alpha v_{t+1}$$

```
vx = 0
while True:
    dx = compute_gradient(x)
    vx = rho * vx + dx
    x -= learning_rate * vx
```

- Build up “velocity” as a running mean of gradients
- Rho gives “friction”; typically rho=0.9 or 0.99

Sutskever et al, “On the importance of initialization and momentum in deep learning”, ICML 2013

# SGD + Momentum:

alternative equivalent formulation

## SGD+Momentum

$$v_{t+1} = \rho v_t - \alpha \nabla f(x_t)$$

$$x_{t+1} = x_t + v_{t+1}$$

```
vx = 0
while True:
    dx = compute_gradient(x)
    vx = rho * vx - learning_rate * dx
    x += vx
```

## SGD+Momentum

$$v_{t+1} = \rho v_t + \nabla f(x_t)$$

$$x_{t+1} = x_t - \alpha v_{t+1}$$

```
vx = 0
while True:
    dx = compute_gradient(x)
    vx = rho * vx + dx
    x -= learning_rate * vx
```

You may see SGD+Momentum formulated different ways,  
but they are equivalent - give same sequence of  $x$

Sutskever et al, "On the importance of initialization and momentum in deep learning", ICML 2013

# More Complex Optimizers: RMSProp

SGD +  
Momentum

```
vx = 0
while True:
    dx = compute_gradient(x)
    vx = rho * vx + dx
    x -= learning_rate * vx
```

Adds element-wise scaling of the gradient based on the historical sum of squares in each dimension (with decay)

RMSProp

```
grad_squared = 0
while True:
    dx = compute_gradient(x)
    grad_squared = decay_rate * grad_squared + (1 - decay_rate) * dx * dx
    x -= learning_rate * dx / (np.sqrt(grad_squared) + 1e-7)
```

Tieleman and Hinton, 2012

# More Complex Optimizers: RMSProp

SGD +  
Momentum

```
vx = 0
while True:
    dx = compute_gradient(x)
    vx = rho * vx + dx
    x -= learning_rate * vx
```

“Per-parameter learning rates”  
or “adaptive learning rates”



RMSProp

```
grad_squared = 0
while True:
    dx = compute_gradient(x)
    grad_squared = decay_rate * grad_squared + (1 - decay_rate) * dx * dx
    x -= learning_rate * dx / (np.sqrt(grad_squared) + 1e-7)
```

Tieleman and Hinton, 2012

# RMSProp

## RMSProp

```
grad_squared = 0
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    x -= learning_rate * dx / (np.sqrt(grad_squared) + 1e-7)
```

Q: What happens with RMSProp?

# RMSProp

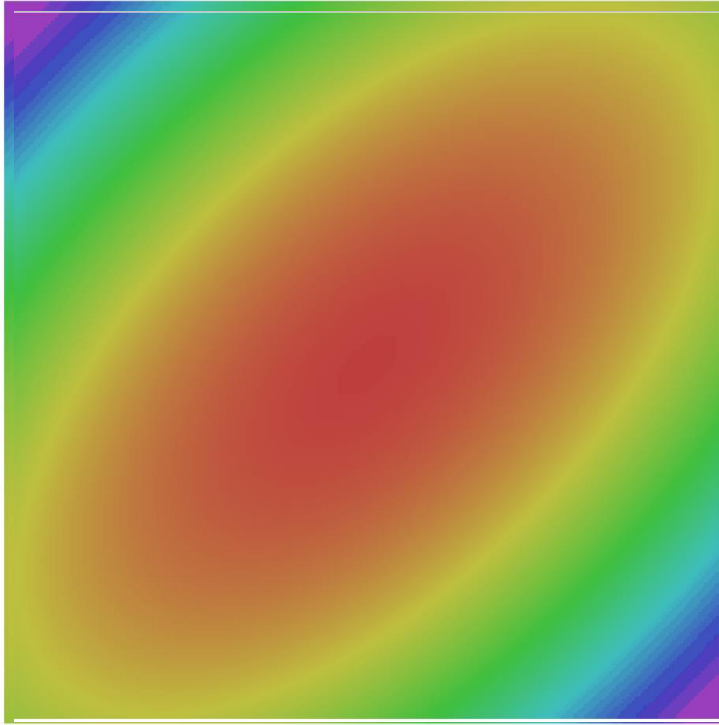
## RMSProp

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    x -= learning_rate * dx / (np.sqrt(grad_squared) + 1e-7)
```

Q: What happens with RMSProp?

Progress along “steep” directions is damped;  
progress along “flat” directions is accelerated

# RMSProp



- SGD
- SGD+Momentum
- RMSProp

# Optimizers: Adam (almost)

```
first_moment = 0
second_moment = 0
while True:
    dx = compute_gradient(x)
    first_moment = beta1 * first_moment + (1 - beta1) * dx
    second_moment = beta2 * second_moment + (1 - beta2) * dx * dx
    x -= learning_rate * first_moment / (np.sqrt(second_moment) + 1e-7))
```

Kingma and Ba, "Adam: A method for stochastic optimization", ICLR 2015



# Adam (almost)

```
first_moment = 0
second_moment = 0
while True:
    dx = compute_gradient(x)
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    second_moment = beta2 * second_moment + (1 - beta2) * dx * dx
    x -= learning_rate * first_moment / (np.sqrt(second_moment) + 1e-7))
```

Momentum

RMSProp

Sort of like RMSProp with momentum

Q: What happens at first timestep?

Kingma and Ba, "Adam: A method for stochastic optimization", ICLR 2015

# Adam (full form)

```
first_moment = 0
second_moment = 0
for t in range(1, num_iterations):
    dx = compute_gradient(x)
    first_moment = beta1 * first_moment + (1 - beta1) * dx
    second_moment = beta2 * second_moment + (1 - beta2) * dx * dx
    first_unbias = first_moment / (1 - beta1 ** t)
    second_unbias = second_moment / (1 - beta2 ** t)
    x -= learning_rate * first_unbias / (np.sqrt(second_unbias) + 1e-7))
```

Momentum

Bias correction

AdaGrad / RMSProp

Bias correction for the fact that  
first and second moment  
estimates start at zero

Kingma and Ba, "Adam: A method for stochastic optimization", ICLR 2015

# Adam (full form)

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    second_unbias = second_moment / (1 - beta2 ** t)
    x -= learning_rate * first_unbias / (np.sqrt(second_unbias) + 1e-7))
```

Momentum

Bias correction

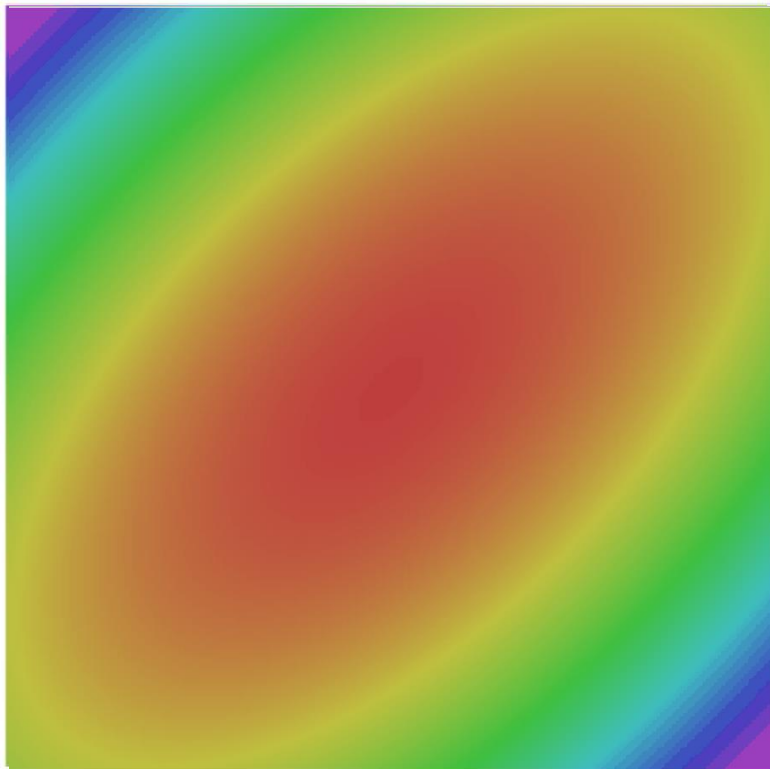
AdaGrad / RMSProp

Bias correction for the fact that first and second moment estimates start at zero

Adam with  $\text{beta1} = 0.9$ ,  $\text{beta2} = 0.999$ , and  $\text{learning\_rate} = 1\text{e-}3$  or  $5\text{e-}4$  is a great starting point for many models!

Kingma and Ba, "Adam: A method for stochastic optimization", ICLR 2015

# Adam



- SGD
- SGD+Momentum
- RMSProp
- Adam

# AdamW: Adam Variant with Weight Decay

Q: How does regularization interact with the optimizer? (e.g., L2)

```
first_moment = 0
second_moment = 0
for t in range(1, num_iterations):
    dx = compute_gradient(x)
    first_moment = beta1 * first_moment + (1 - beta1) * dx
    second_moment = beta2 * second_moment + (1 - beta2) * dx * dx
    first_unbias = first_moment / (1 - beta1 ** t)
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```

# AdamW: Adam Variant with Weight Decay

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    second_unbias = second_moment / (1 - beta2 ** t)
    x -= learning_rate * first_unbias / (np.sqrt(second_unbias) + 1e-7))
```


A: It depends!

# AdamW: Adam Variant with Weight Decay

Q: How does regularization interact with the optimizer? (e.g., L2)

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    second_unbias = second_moment / (1 - beta2 ** t)
    x -= learning_rate * first_unbias / (np.sqrt(second_unbias) + 1e-7))
```

Standard Adam computes L2 here

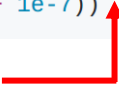


Used during moment calculations!

# AdamW: Adam Variant with Weight Decay

Q: How does regularization interact with the optimizer? (e.g., L2)

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    first_unbias = first_moment / (1 - beta1 ** t)
    second_unbias = second_moment / (1 - beta2 ** t)
    x -= learning_rate * first_unbias / (np.sqrt(second_unbias) + 1e-7)
```

AdamW (Weight Decay) adds term here 

Computed after the moments!



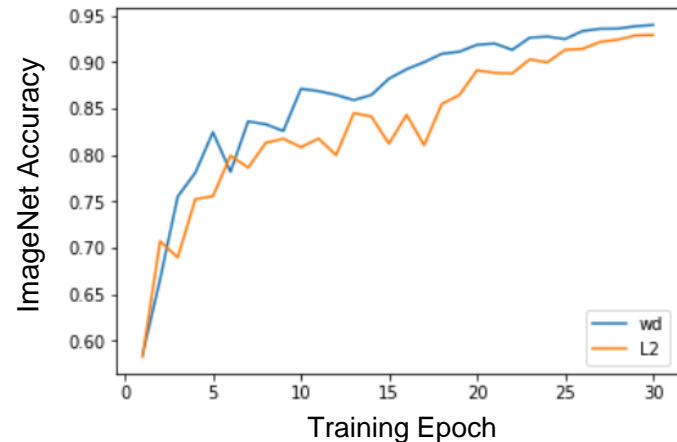
# AdamW: Adam Variant with Weight Decay

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```

Standard Adam computes L2 here

AdamW (Weight Decay) adds term here



Source: <https://www.fast.ai/posts/2018-07-02-adam-weight-decay.html>

# Learning rate schedules

```
# Vanilla Gradient Descent
```

```
while True:
```

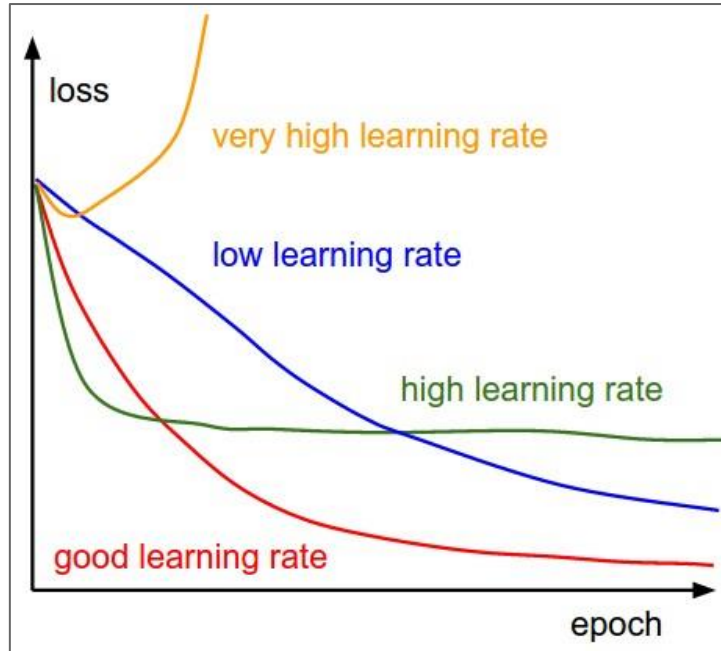
```
    weights_grad = evaluate_gradient(loss_fun, data, weights)
```

```
    weights += - step_size * weights_grad # perform parameter update
```



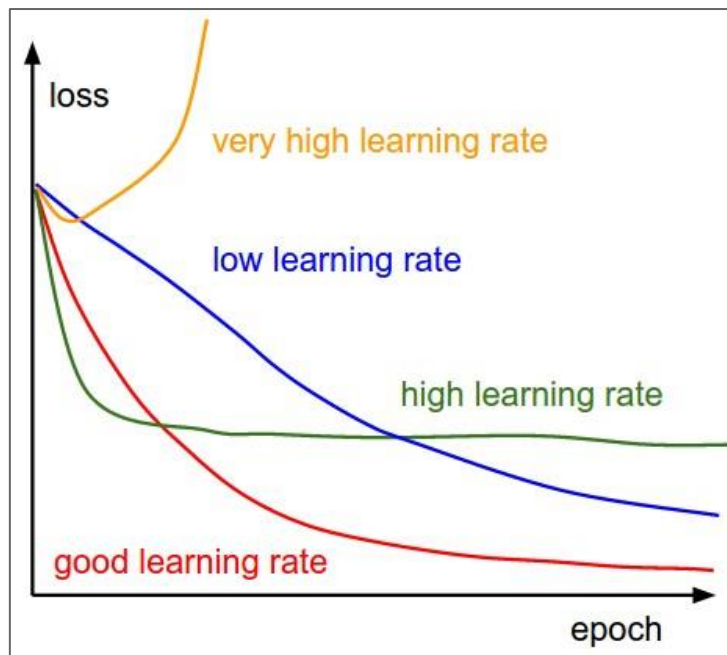
Learning rate

SGD, SGD+Momentum, RMSProp, Adam, AdamW all have **learning rate** as a hyperparameter.



Q: Which one of these learning rates is best to use?

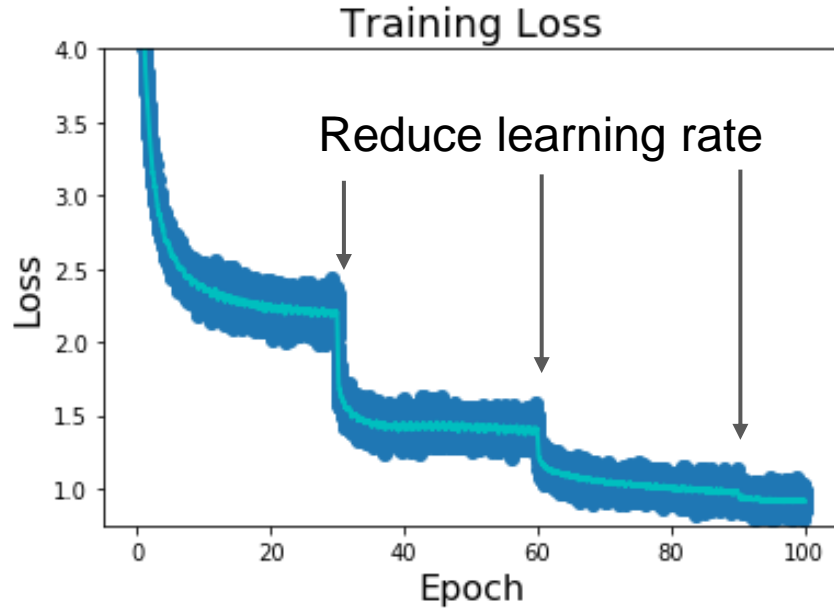
SGD, SGD+Momentum, RMSProp, Adam, AdamW all have **learning rate** as a hyperparameter.



Q: Which one of these learning rates is best to use?

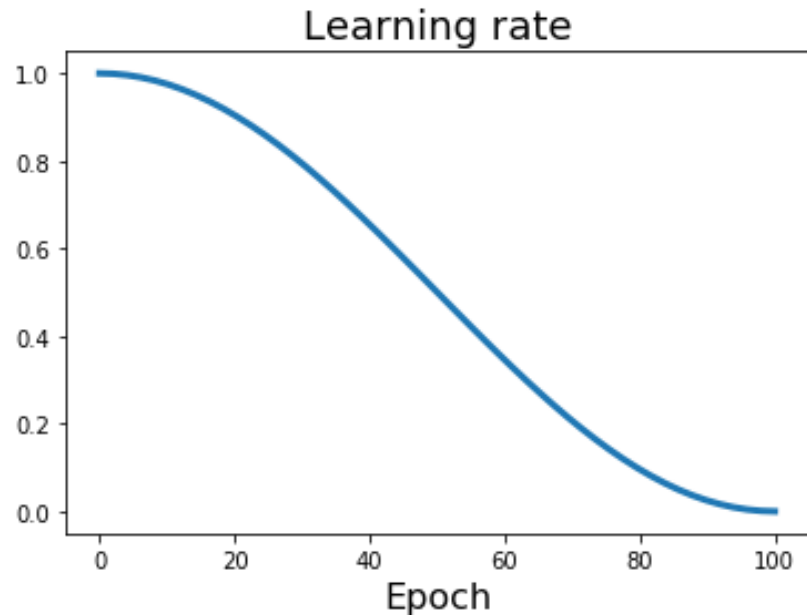
A: In reality, all of these could be good learning rates.

# Learning rate decays over time



**Step:** Reduce learning rate at a few fixed points. E.g. for ResNets, multiply LR by 0.1 after epochs 30, 60, and 90.

# Learning Rate Decay



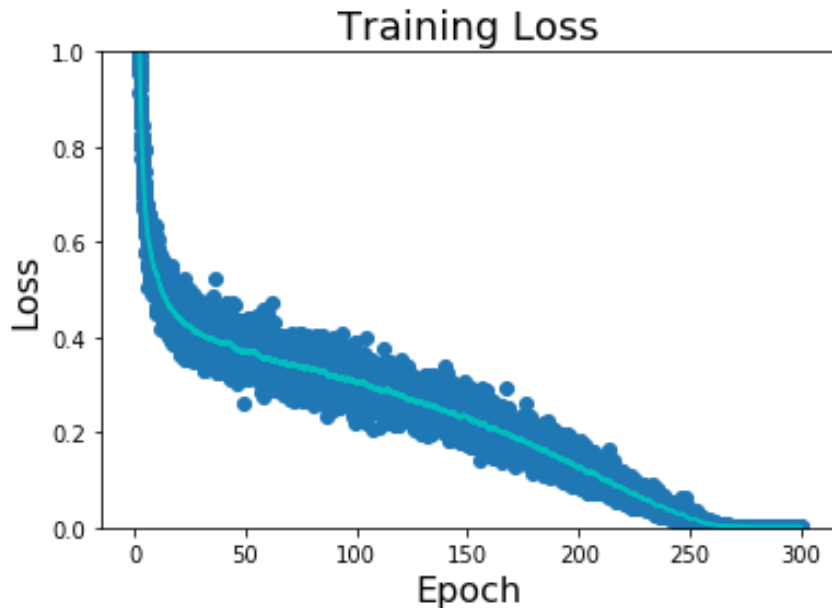
**Step:** Reduce learning rate at a few fixed points. E.g. for ResNets, multiply LR by 0.1 after epochs 30, 60, and 90.

**Cosine:**  $\alpha_t = \frac{1}{2}\alpha_0 (1 + \cos(t\pi/T))$

$\alpha_0$  : Initial learning rate  
 $\alpha_t$  : Learning rate at epoch  $t$   
 $T$  : Total number of epochs

Loshchilov and Hutter, "SGDR: Stochastic Gradient Descent with Warm Restarts", ICLR 2017  
Radford et al, "Improving Language Understanding by Generative Pre-Training", 2018  
Feichtenhofer et al, "SlowFast Networks for Video Recognition", arXiv 2018  
Child et al, "Generating Long Sequences with Sparse Transformers", arXiv 2019

# Learning Rate Decay



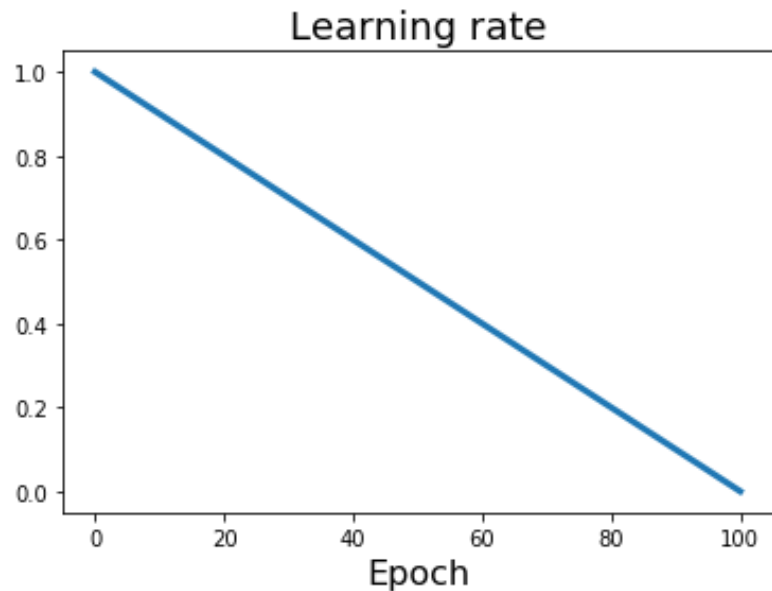
**Step:** Reduce learning rate at a few fixed points. E.g. for ResNets, multiply LR by 0.1 after epochs 30, 60, and 90.

**Cosine:**  $\alpha_t = \frac{1}{2}\alpha_0 (1 + \cos(t\pi/T))$

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# Learning Rate Decay



**Step:** Reduce learning rate at a few fixed points. E.g. for ResNets, multiply LR by 0.1 after epochs 30, 60, and 90.

**Cosine:**  $\alpha_t = \frac{1}{2}\alpha_0 (1 + \cos(t\pi/T))$

**Linear:**  $\alpha_t = \alpha_0(1 - t/T)$

$\alpha_0$  : Initial learning rate

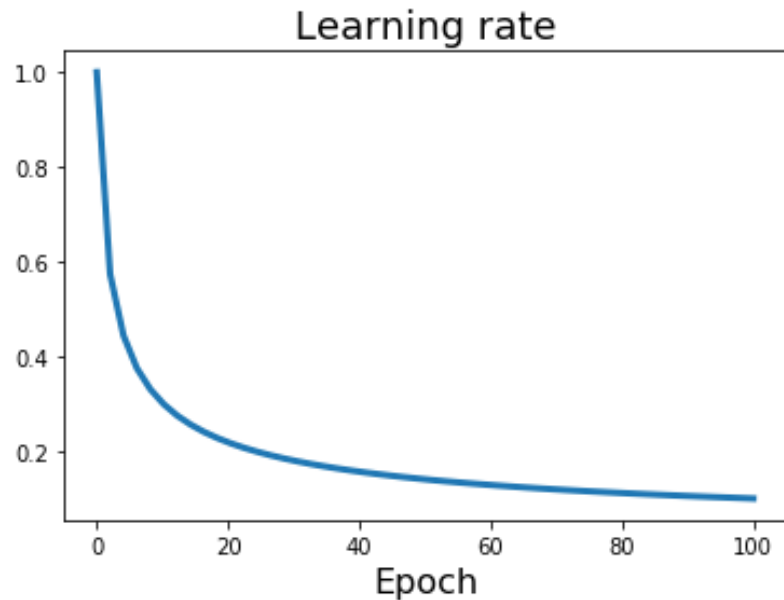
$\alpha_t$  : Learning rate at epoch  $t$

$T$  : Total number of epochs

Devlin et al, "BERT: Pre-training of Deep Bidirectional Transformers for Language Understanding", 2018



# Learning Rate Decay



**Step:** Reduce learning rate at a few fixed points. E.g. for ResNets, multiply LR by 0.1 after epochs 30, 60, and 90.

**Cosine:**  $\alpha_t = \frac{1}{2}\alpha_0 (1 + \cos(t\pi/T))$

**Linear:**  $\alpha_t = \alpha_0(1 - t/T)$

**Inverse sqrt:**  $\alpha_t = \alpha_0/\sqrt{t}$

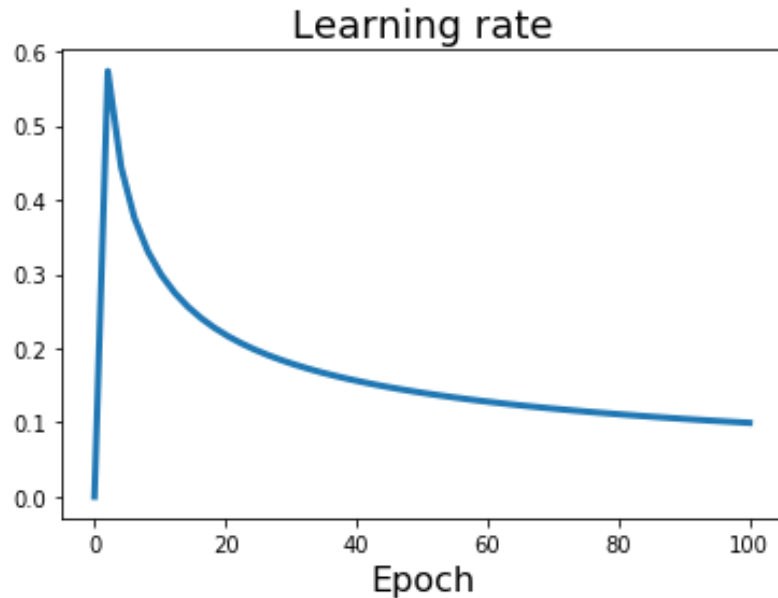
$\alpha_0$  : Initial learning rate

$\alpha_t$  : Learning rate at epoch  $t$

$T$  : Total number of epochs

Vaswani et al, "Attention is all you need", NIPS 2017

# Learning Rate Decay: Linear Warmup

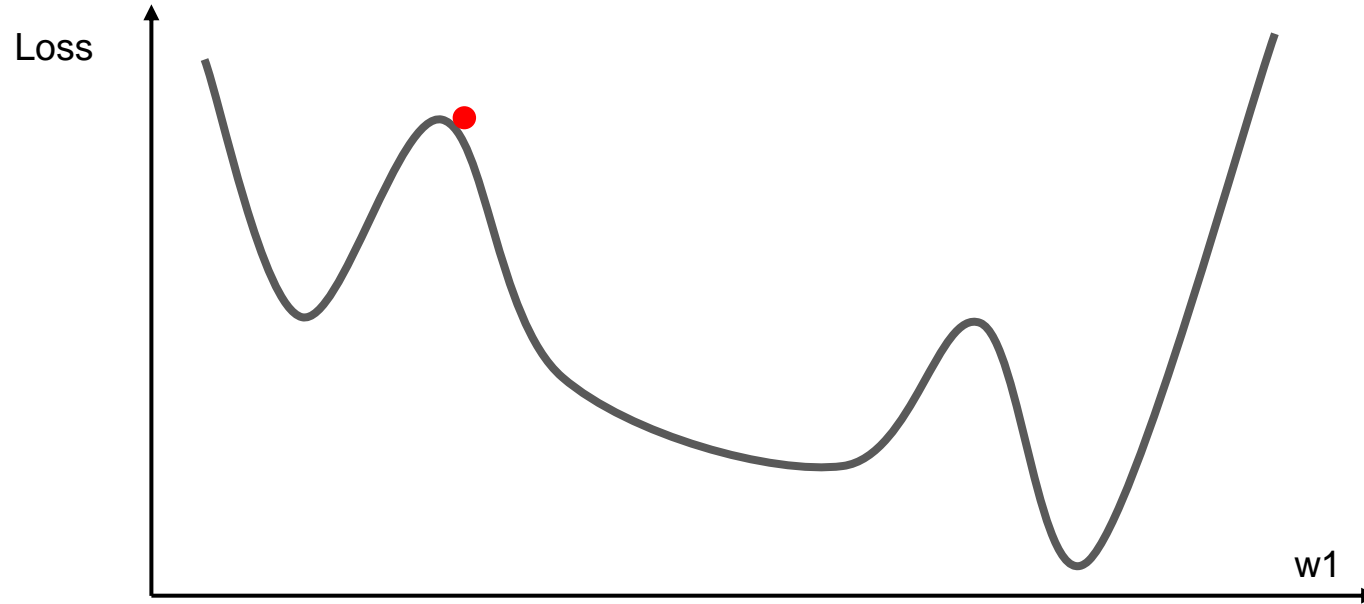


High initial learning rates can make loss explode; linearly increasing learning rate from 0 over the first ~5,000 iterations can prevent this.

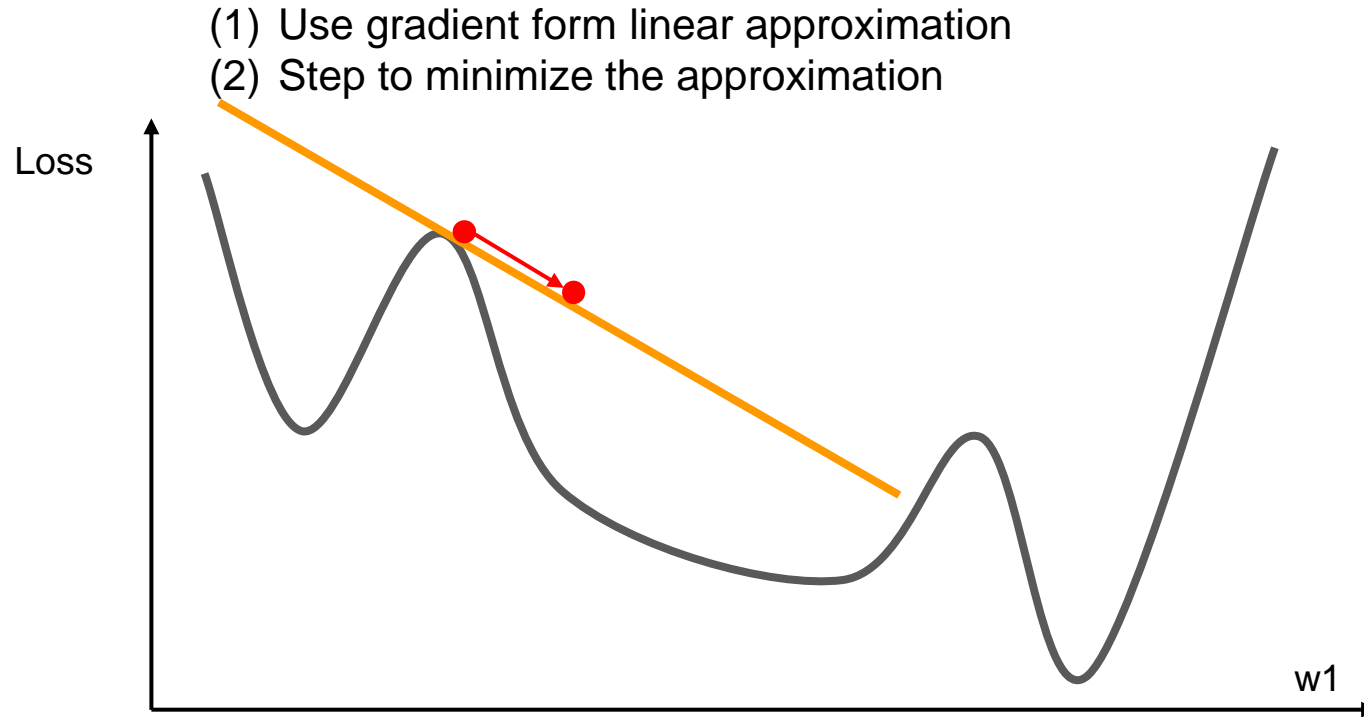
Empirical rule of thumb: If you increase the batch size by  $N$ , also scale the initial learning rate by  $N$

Goyal et al, "Accurate, Large Minibatch SGD: Training ImageNet in 1 Hour", arXiv 2017

# First-Order Optimization

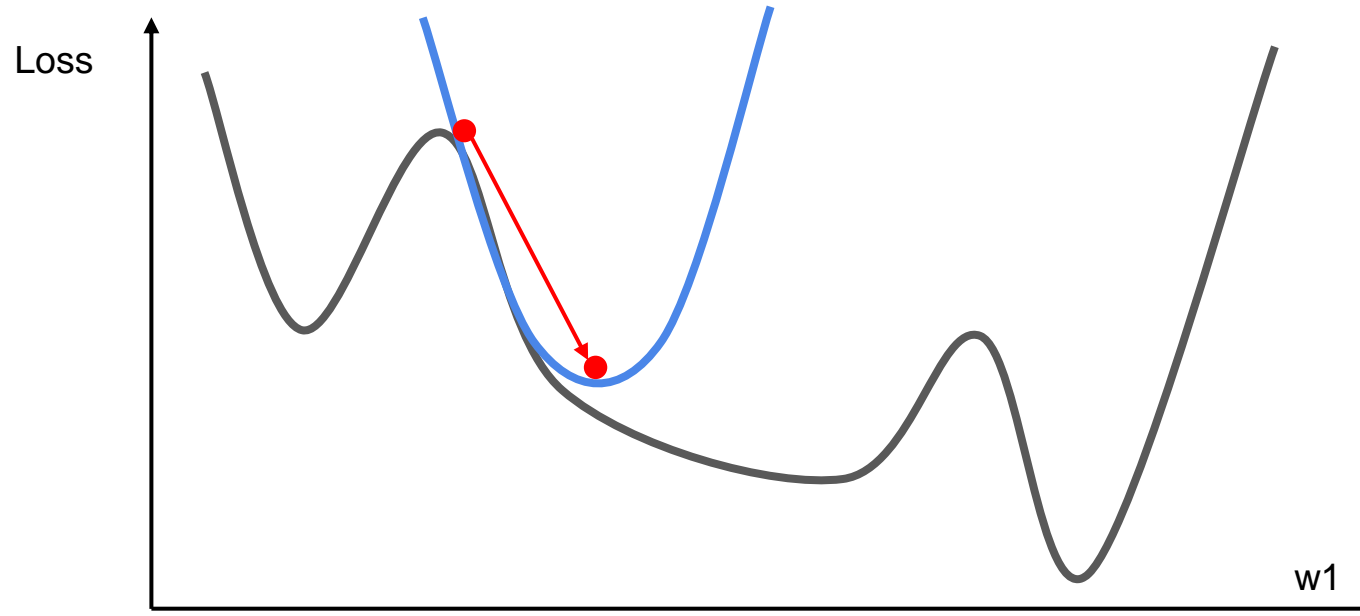


# First-Order Optimization



# Second-Order Optimization

- (1) Use gradient **and Hessian** to form **quadratic** approximation
- (2) Step to the **minima** of the approximation



# Second-Order Optimization

second-order Taylor expansion:

$$J(\boldsymbol{\theta}) \approx J(\boldsymbol{\theta}_0) + (\boldsymbol{\theta} - \boldsymbol{\theta}_0)^\top \nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}_0) + \frac{1}{2} (\boldsymbol{\theta} - \boldsymbol{\theta}_0)^\top \mathbf{H} (\boldsymbol{\theta} - \boldsymbol{\theta}_0)$$

Solving for the critical point we obtain the Newton parameter update:

$$\boldsymbol{\theta}^* = \boldsymbol{\theta}_0 - \mathbf{H}^{-1} \nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}_0)$$

Q: Why is this bad for deep learning?

# Second-Order Optimization

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Hessian has  $O(N^2)$  elements

Inverting takes  $O(N^3)$

$N$  = (Tens or Hundreds of) Millions

Q: Why is this bad for deep learning?

# In practice:

- **Adam(W)** is a good default choice in many cases; it often works ok even with constant learning rate
- **SGD+Momentum** can outperform Adam but may require more tuning of LR and schedule
- If you can afford to do full batch updates then look beyond 1<sup>st</sup> order optimization (**2<sup>nd</sup> order and beyond**)



# Looking Ahead: How to optimize more complex functions?

(Currently) Linear score function:  $f = Wx$

$$x \in \mathbb{R}^D, W \in \mathbb{R}^{C \times D}$$

# Neural networks: 2 layers

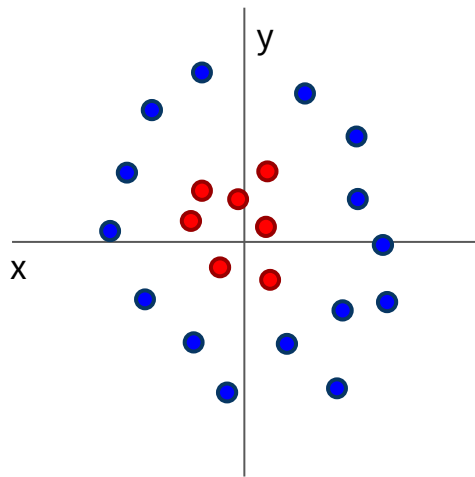
(Currently) Linear score function:  $f = Wx$

(Next Class) 2-layer Neural Network  $f = W_2 \max(0, W_1 x)$

$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

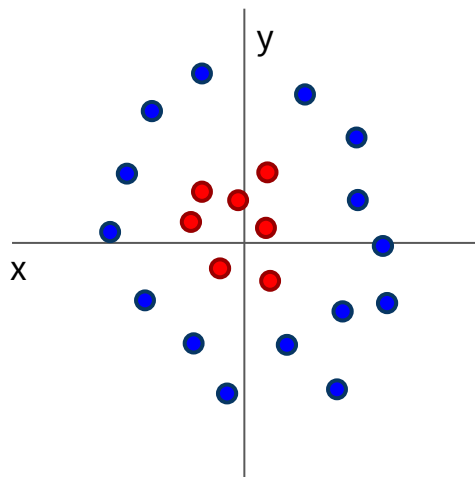
(In practice we will usually add a learnable bias at each layer as well)

# Why do we want non-linearity?



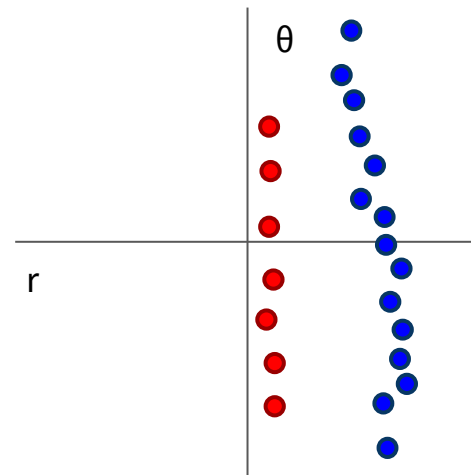
Cannot separate red and blue points with linear classifier

# Why do we want non-linearity?



Cannot separate red and blue points with linear classifier

$$f(x, y) = (r(x, y), \theta(x, y))$$



After applying feature transform, points can be separated by linear classifier

# Neural networks: also called fully connected network

(Currently) Linear score function:  $f = Wx$

(Next Class) 2-layer Neural Network  $f = W_2 \max(0, W_1 x)$

$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

“Neural Network” is a very broad term; these are more accurately called “fully-connected networks” or sometimes “multi-layer perceptrons” (MLP)

(In practice we will usually add a learnable bias at each layer as well)

# Next time:

Introduction to neural networks

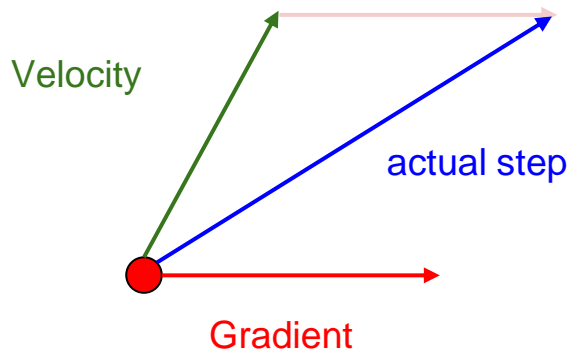
Backpropagation (How do you calculate  $dx$  for neural nets?)

# Appendix Material

Extraneous content that may be of interest

# SGD+Momentum

Momentum update:



Combine gradient at current point with velocity to get step used to update weights

Nesterov, "A method of solving a convex programming problem with convergence rate  $O(1/k^2)$ ", 1983

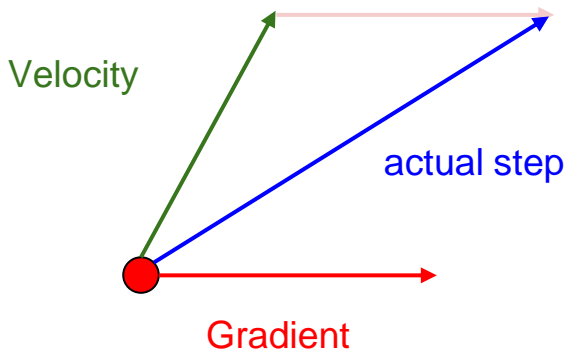
Nesterov, "Introductory lectures on convex optimization: a basic course", 2004

Sutskever et al, "On the importance of initialization and momentum in deep learning", ICML 2013



# Nesterov Momentum

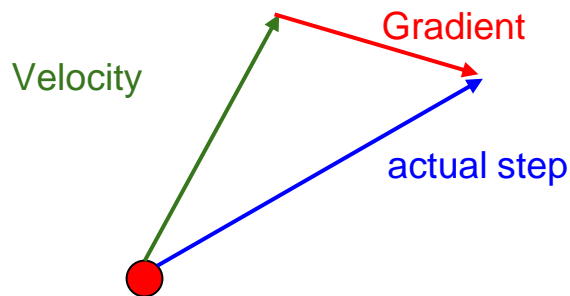
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Nesterov Momentum

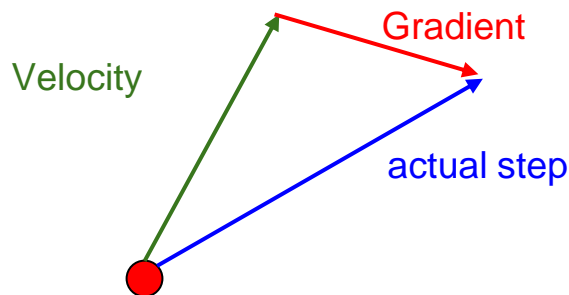


"Look ahead" to the point where updating using velocity would take us; compute gradient there and mix it with velocity to get actual update direction

# Nesterov Momentum

$$v_{t+1} = \rho v_t - \alpha \nabla f(x_t + \rho v_t)$$

$$x_{t+1} = x_t + v_{t+1}$$



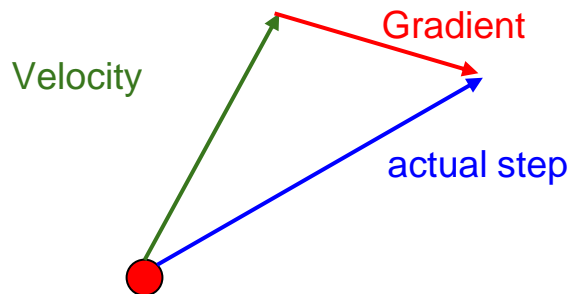
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Annoying, usually we want update in terms of  $x_t, \nabla f(x_t)$



“Look ahead” to the point where updating using velocity would take us; compute gradient there and mix it with velocity to get actual update direction

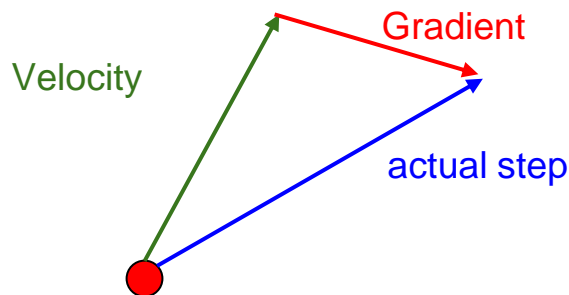
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$$v_{t+1} = \rho v_t - \alpha \nabla f(x_t + \rho v_t)$$

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Change of variables  $\tilde{x}_t = x_t + \rho v_t$  and rearrange:

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“Look ahead” to the point where updating using velocity would take us; compute gradient there and mix it with velocity to get actual update direction

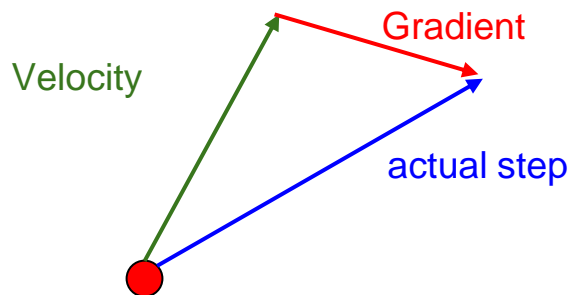
# Nesterov Momentum

$$\begin{aligned}v_{t+1} &= \rho v_t - \alpha \nabla f(x_t + \rho v_t) \\x_{t+1} &= x_t + v_{t+1}\end{aligned}$$

Change of variables  $\tilde{x}_t = x_t + \rho v_t$  and rearrange:

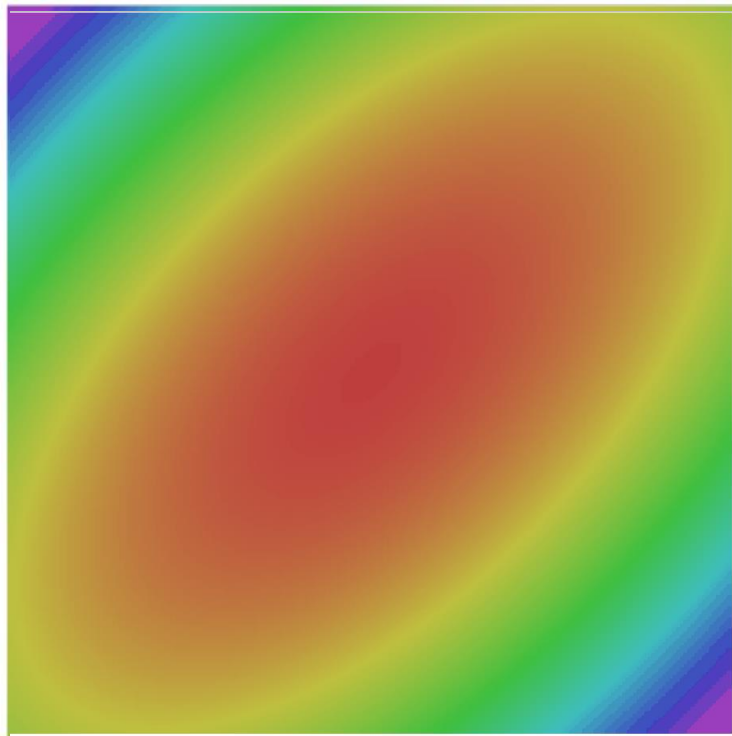
$$\begin{aligned}v_{t+1} &= \rho v_t - \alpha \nabla f(\tilde{x}_t) \\ \tilde{x}_{t+1} &= \tilde{x}_t - \rho v_t + (1 + \rho)v_{t+1} \\ &= \tilde{x}_t + v_{t+1} + \rho(v_{t+1} - v_t)\end{aligned}$$

Annoying, usually we want update in terms of  $x_t, \nabla f(x_t)$



“Look ahead” to the point where updating using velocity would take us; compute gradient there and mix it with velocity to get actual update direction

# Nesterov Momentum



- SGD
- SGD+Momentum
- Nesterov

# AdaGrad

```
grad_squared = 0
while True:
    dx = compute_gradient(x)
    grad_squared += dx * dx
    x -= learning_rate * dx / (np.sqrt(grad_squared) + 1e-7)
```

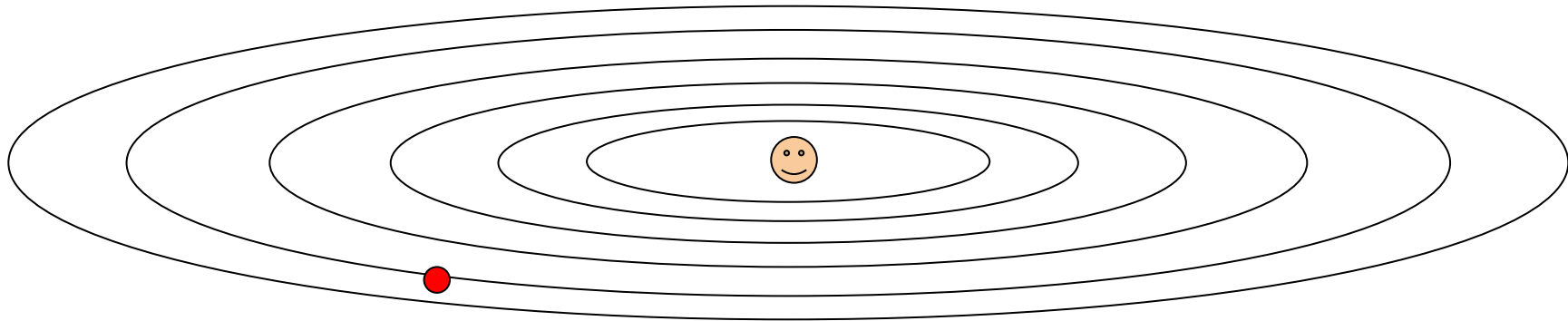
Added element-wise scaling of the gradient based on the historical sum of squares in each dimension

“Per-parameter learning rates”  
or “adaptive learning rates”

Duchi et al, “Adaptive subgradient methods for online learning and stochastic optimization”, JMLR 2011

# AdaGrad

```
grad_squared = 0
while True:
    dx = compute_gradient(x)
    grad_squared += dx * dx
    x -= learning_rate * dx / (np.sqrt(grad_squared) + 1e-7)
```

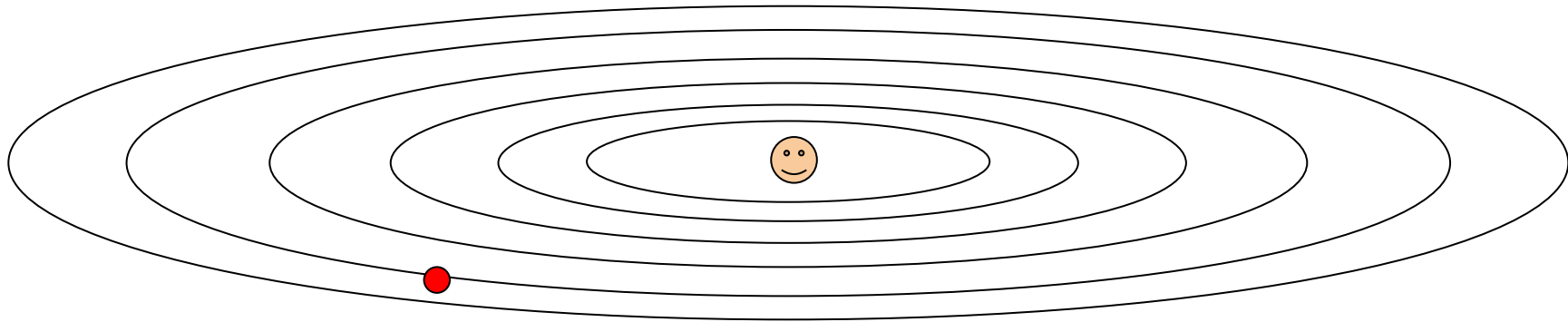


Q: What happens with AdaGrad?



# AdaGrad

```
grad_squared = 0
while True:
    dx = compute_gradient(x)
    grad_squared += dx * dx
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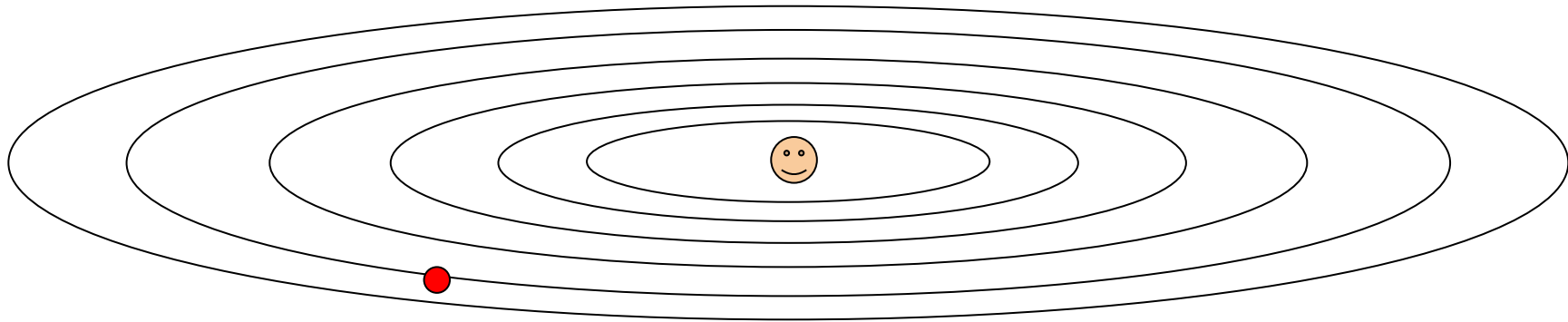


Q: What happens with AdaGrad?

Progress along “steep” directions is damped;  
progress along “flat” directions is accelerated

# AdaGrad

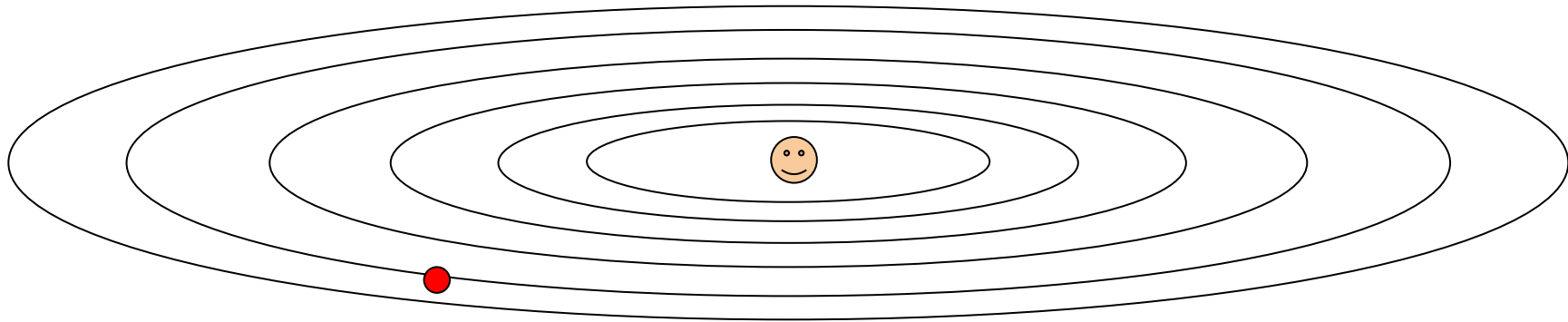
```
grad_squared = 0
while True:
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    grad_squared += dx * dx
    x -= learning_rate * dx / (np.sqrt(grad_squared) + 1e-7)
```



Q2: What happens to the step size over long time?

# AdaGrad

```
grad_squared = 0
while True:
    dx = compute_gradient(x)
    grad_squared += dx * dx
    x -= learning_rate * dx / (np.sqrt(grad_squared) + 1e-7)
```



Q2: What happens to the step size over long time? Decays to zero

# RMSProp: “Leaky AdaGrad”

AdaGrad

```
grad_squared = 0
while True:
    dx = compute_gradient(x)
    grad_squared += dx * dx
    x -= learning_rate * dx / (np.sqrt(grad_squared) + 1e-7)
```



RMSProp

```
grad_squared = 0
while True:
    dx = compute_gradient(x)
    grad_squared = decay_rate * grad_squared + (1 - decay_rate) * dx * dx
    x -= learning_rate * dx / (np.sqrt(grad_squared) + 1e-7)
```

Tieleman and Hinton, 2012

# Second-Order Optimization

$$\theta^* = \theta_0 - H^{-1} \nabla_{\theta} J(\theta_0)$$

- Quasi-Newton methods (**BGFS** most popular):  
*instead of inverting the Hessian ( $O(n^3)$ ), approximate inverse Hessian with rank 1 updates over time ( $O(n^2)$  each).*
- **L-BFGS** (Limited memory BFGS):  
*Does not form/store the full inverse Hessian.*

# L-BFGS

- **Usually works very well in full batch, deterministic mode** i.e. if you have a single, deterministic  $f(x)$  then L-BFGS will probably work very nicely
- **Does not transfer very well to mini-batch setting.** Gives bad results. Adapting second-order methods to large-scale, stochastic setting is an active area of research.

Le et al, "On optimization methods for deep learning, ICML 2011"

Ba et al, "Distributed second-order optimization using Kronecker-factored approximations", ICLR 2017

# In practice:

- **Adam** is a good default choice in many cases; it often works ok even with constant learning rate
- **SGD+Momentum** can outperform Adam but may require more tuning of LR and schedule
- If you can afford to do full batch updates then try out **L-BFGS** (and don't forget to disable all sources of noise)