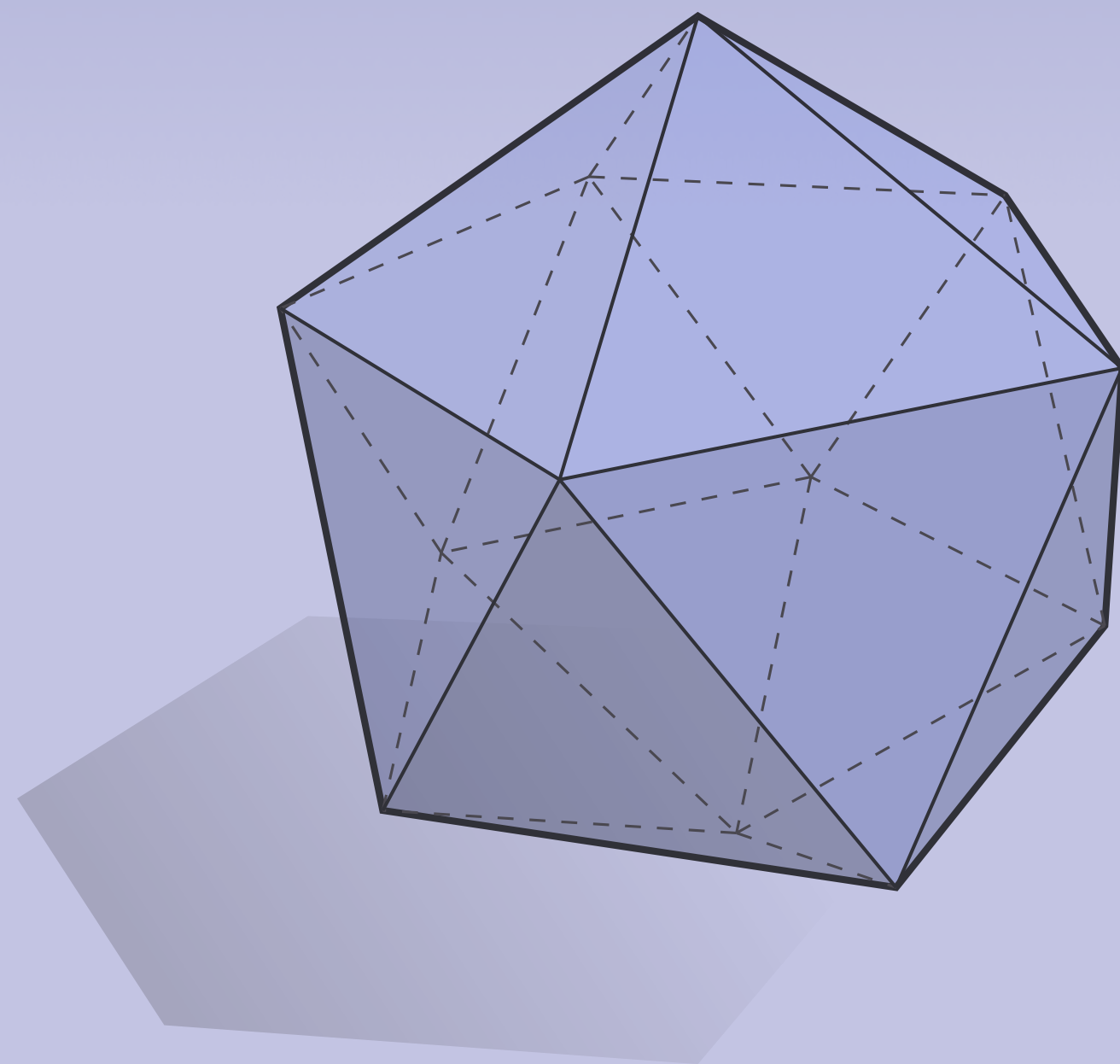


DISCRETE DIFFERENTIAL
GEOMETRY:
AN APPLIED INTRODUCTION
Keenan Crane • CMU 15-458/858

MESHES

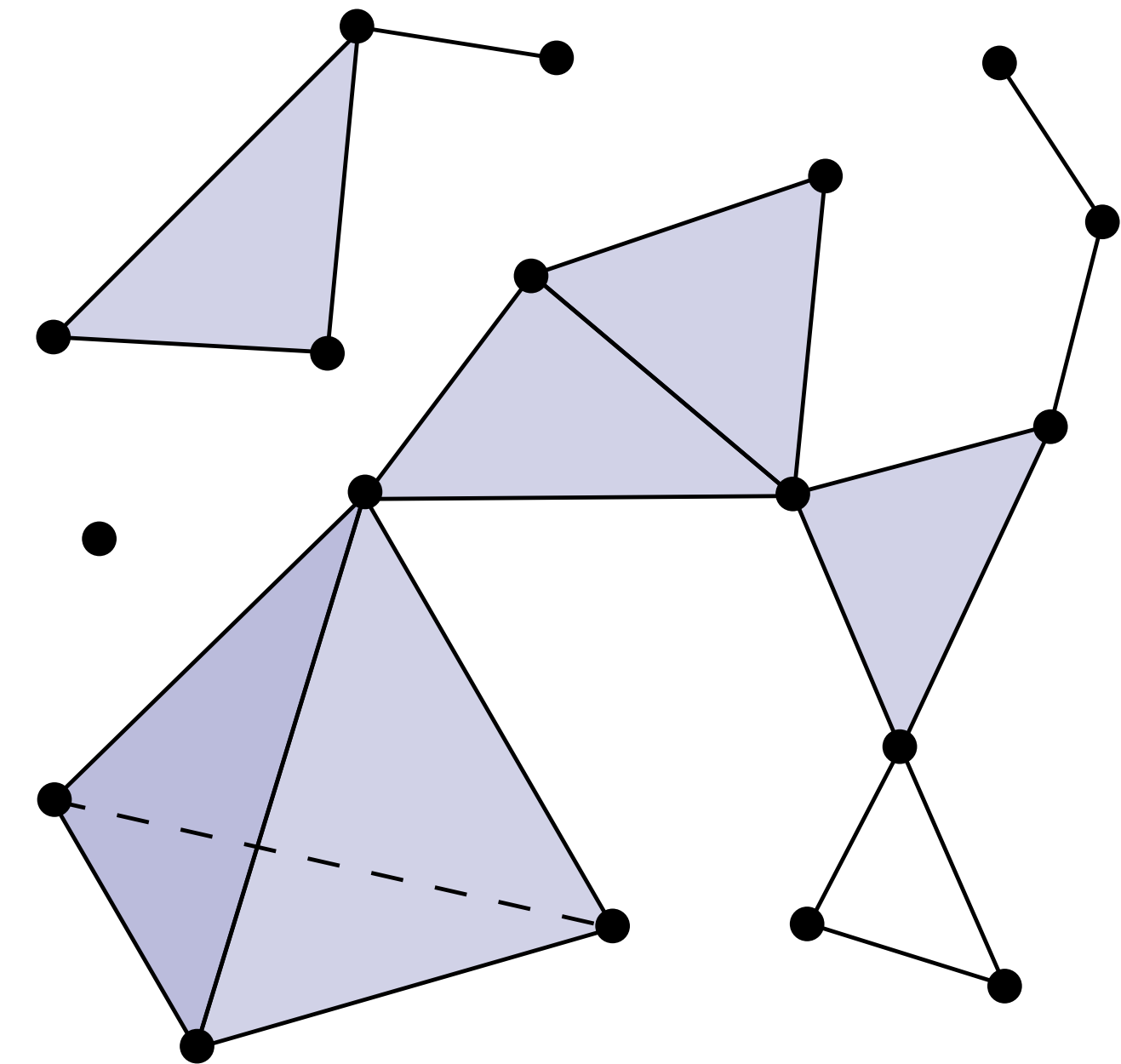
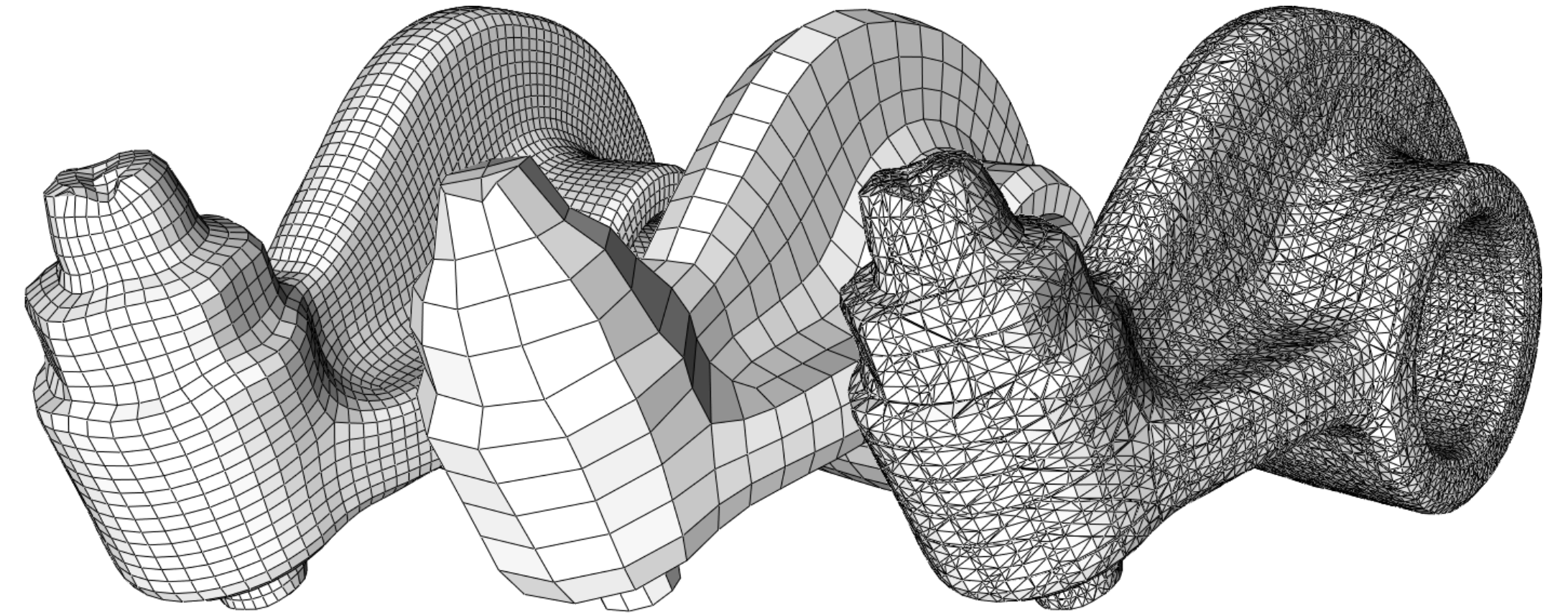


DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

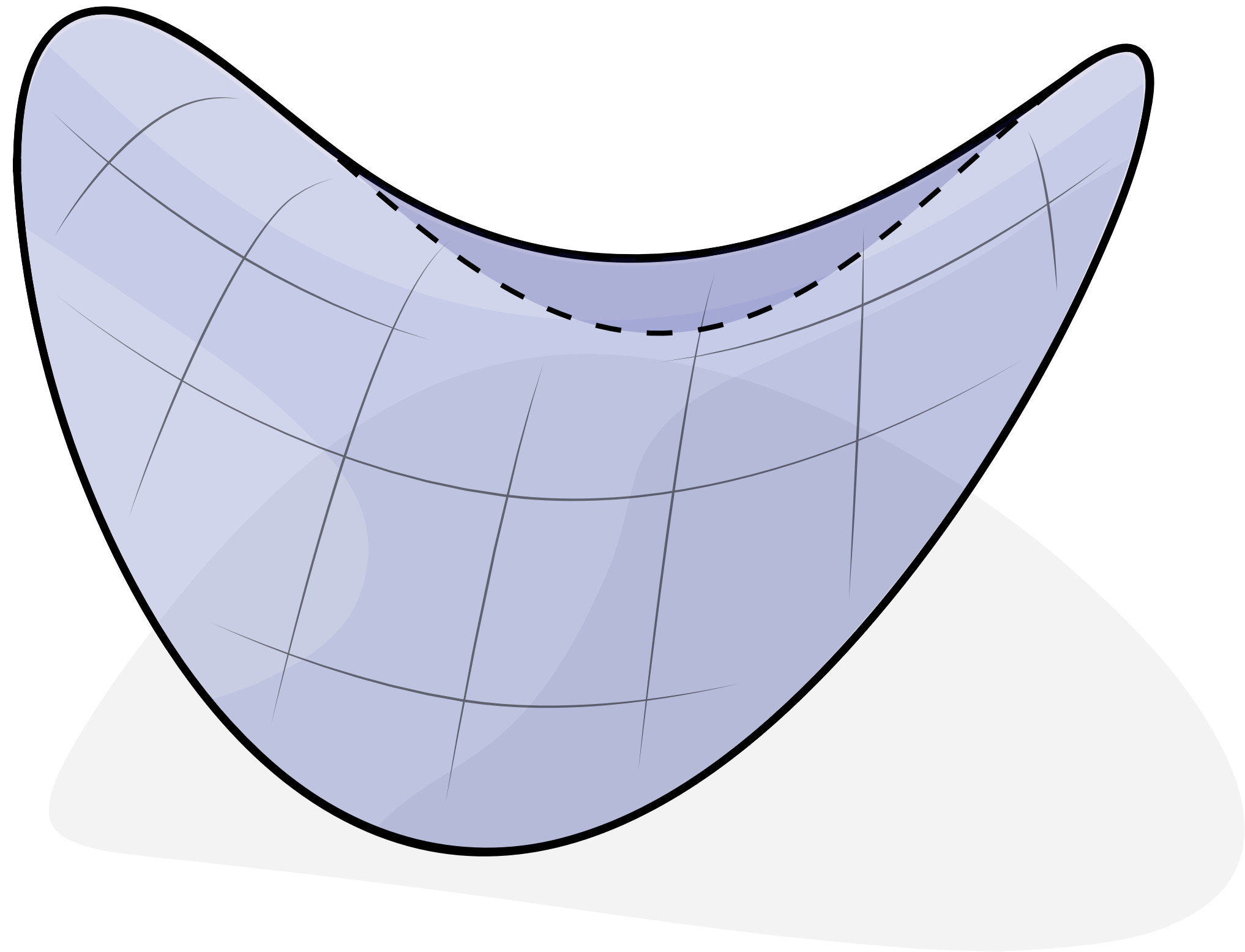
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Today: What is a “Mesh?”

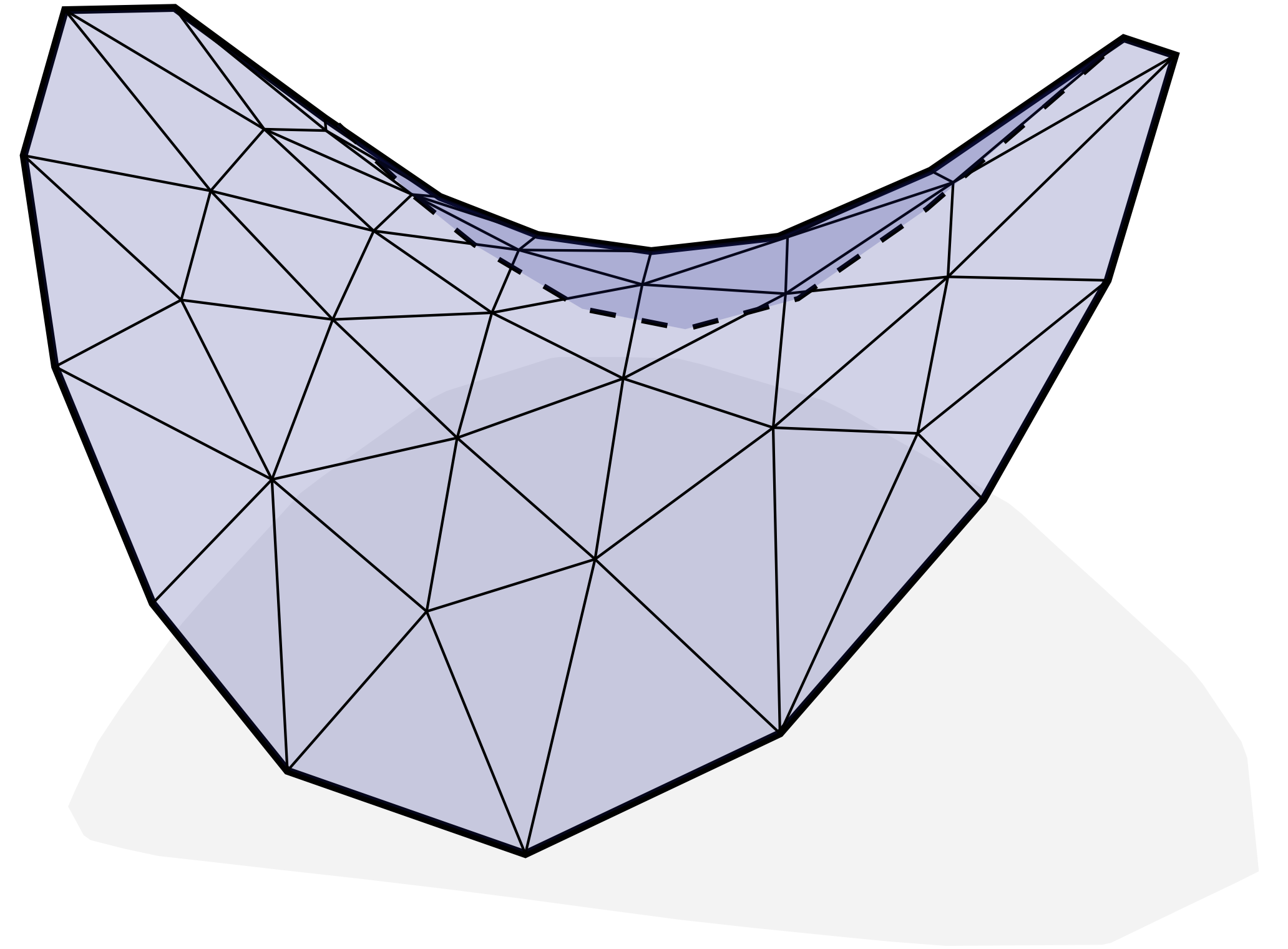
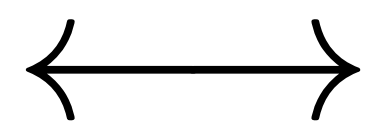
- Many possibilities...
- **Simplicial complex**
 - Abstract vs. geometric simplicial complex
 - Oriented, manifold simplicial complex
 - Application: *topological data analysis*
- **Cell complex**
 - Poincaré dual, discrete exterior calculus
- Data structures:
 - adjacency list, incidence matrix, halfedge



Connection to Differential Geometry?



topological space



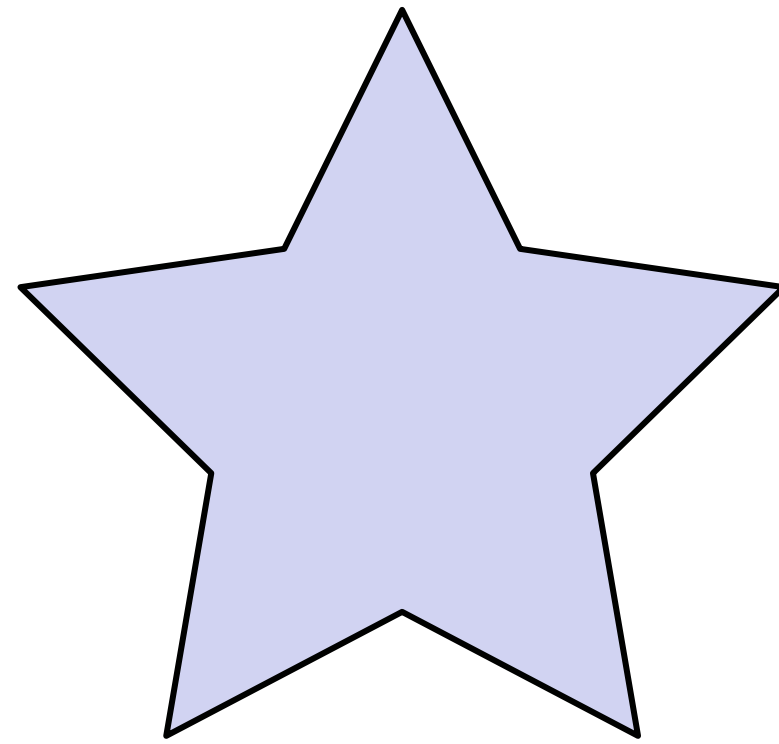
abstract simplicial complex



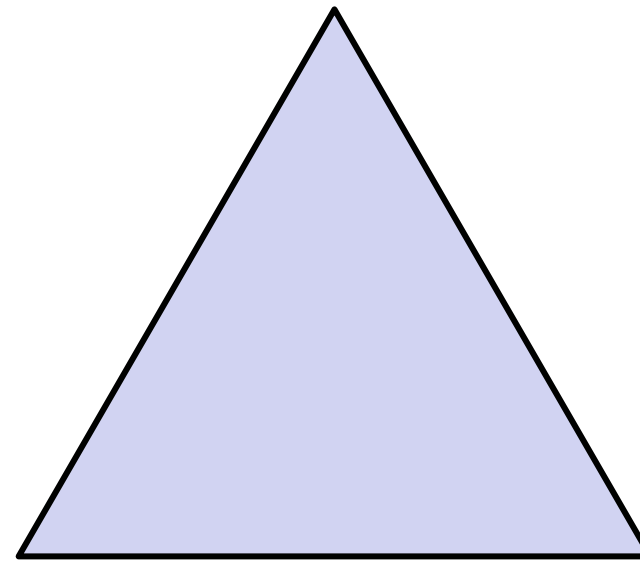
Convex Set

Convex Set—Examples

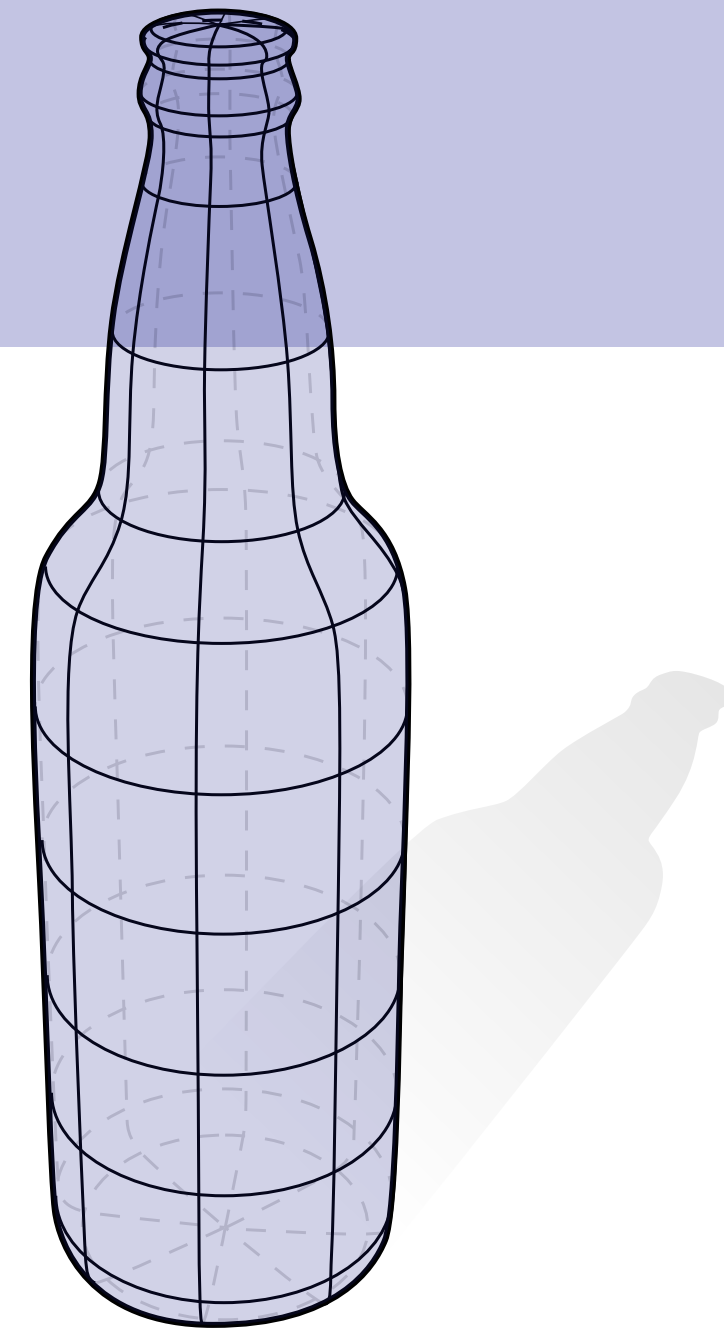
Which of the following sets are *convex*?



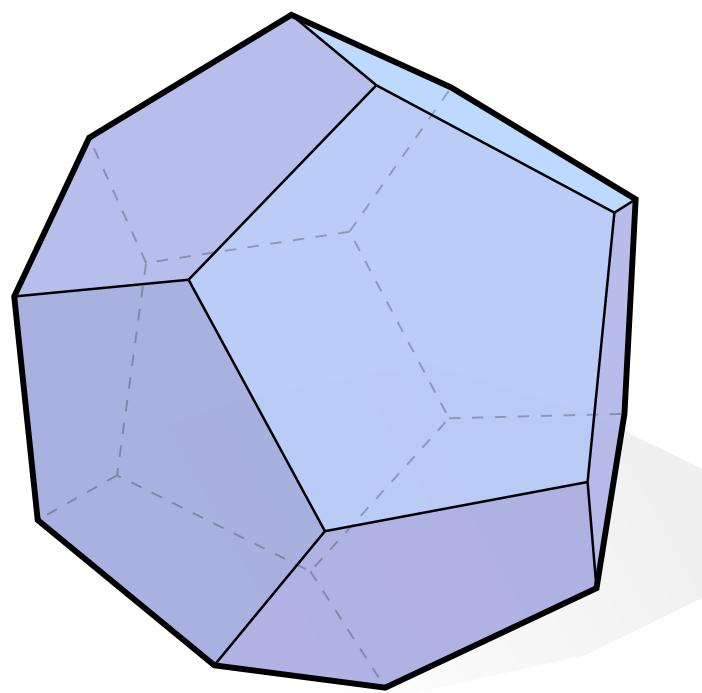
(A)



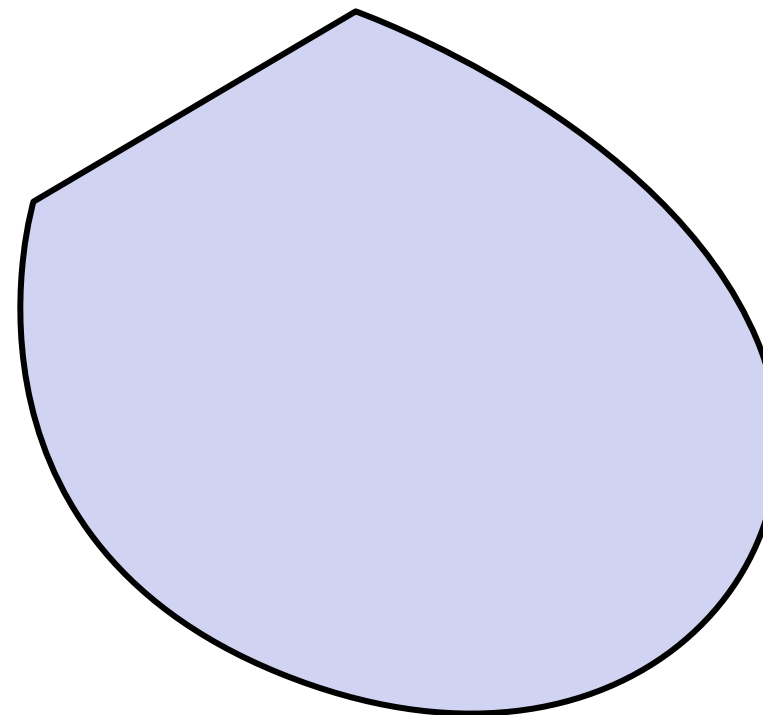
(B)



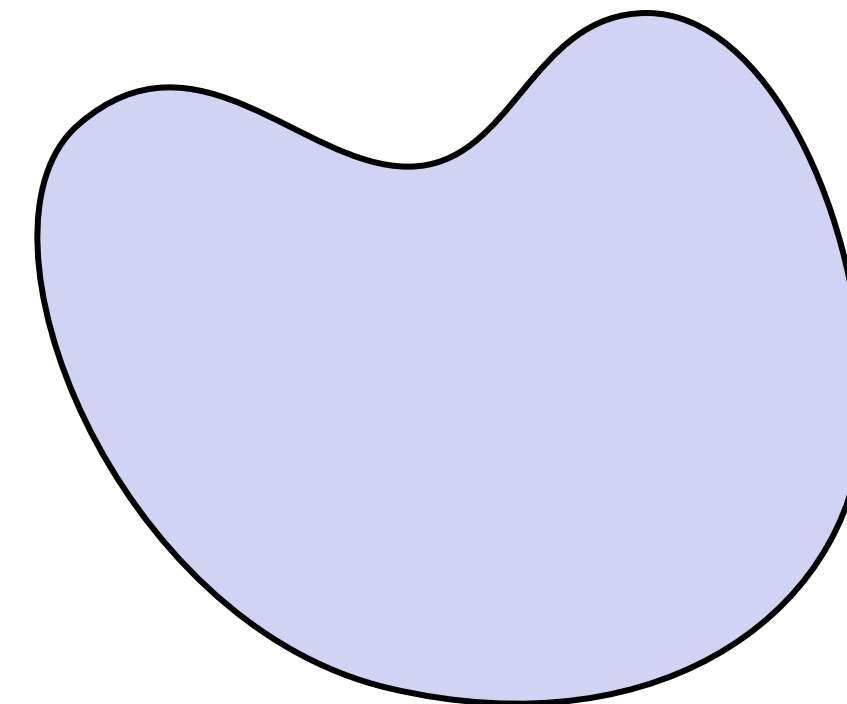
(C)



(D)



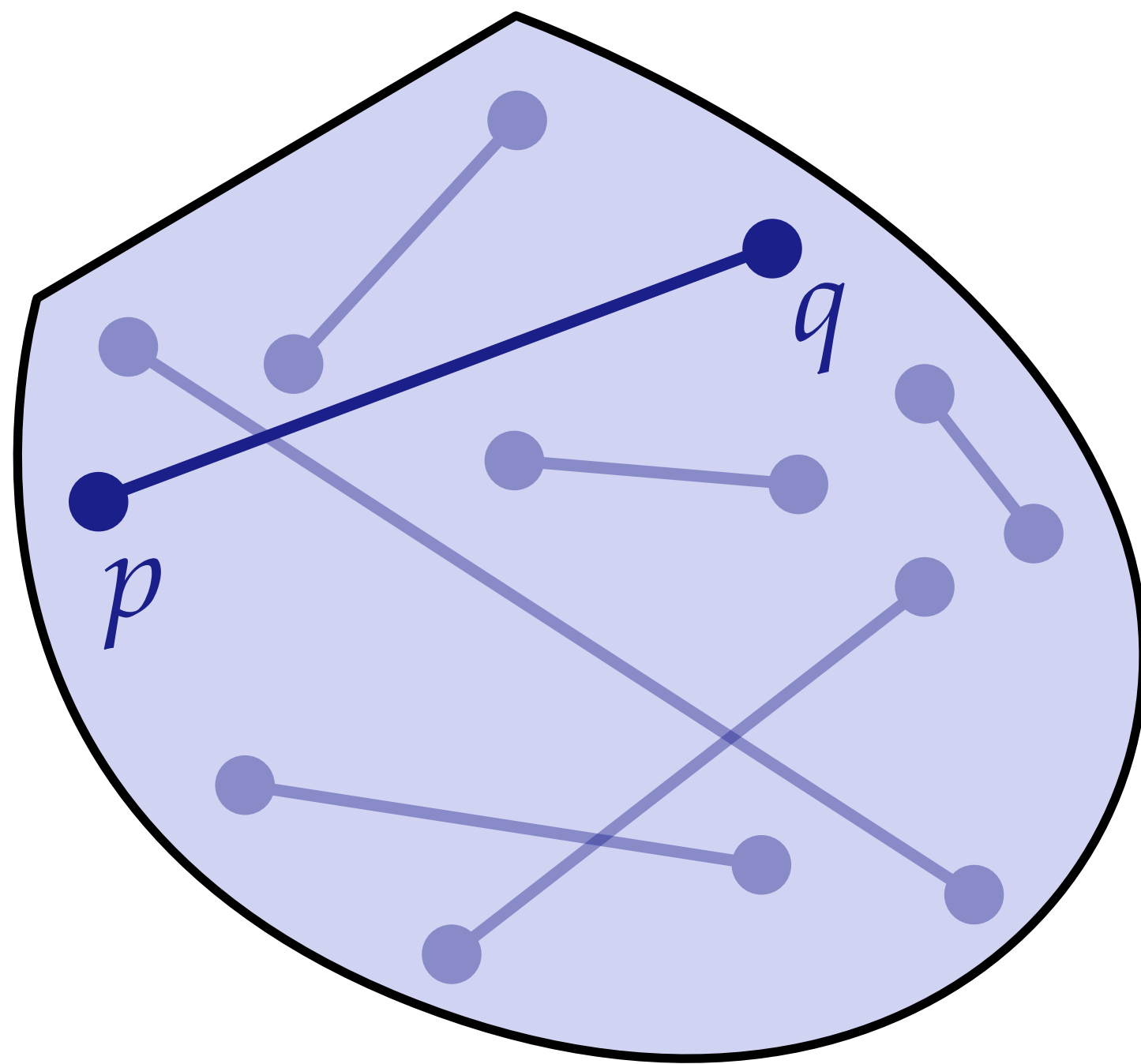
(E)



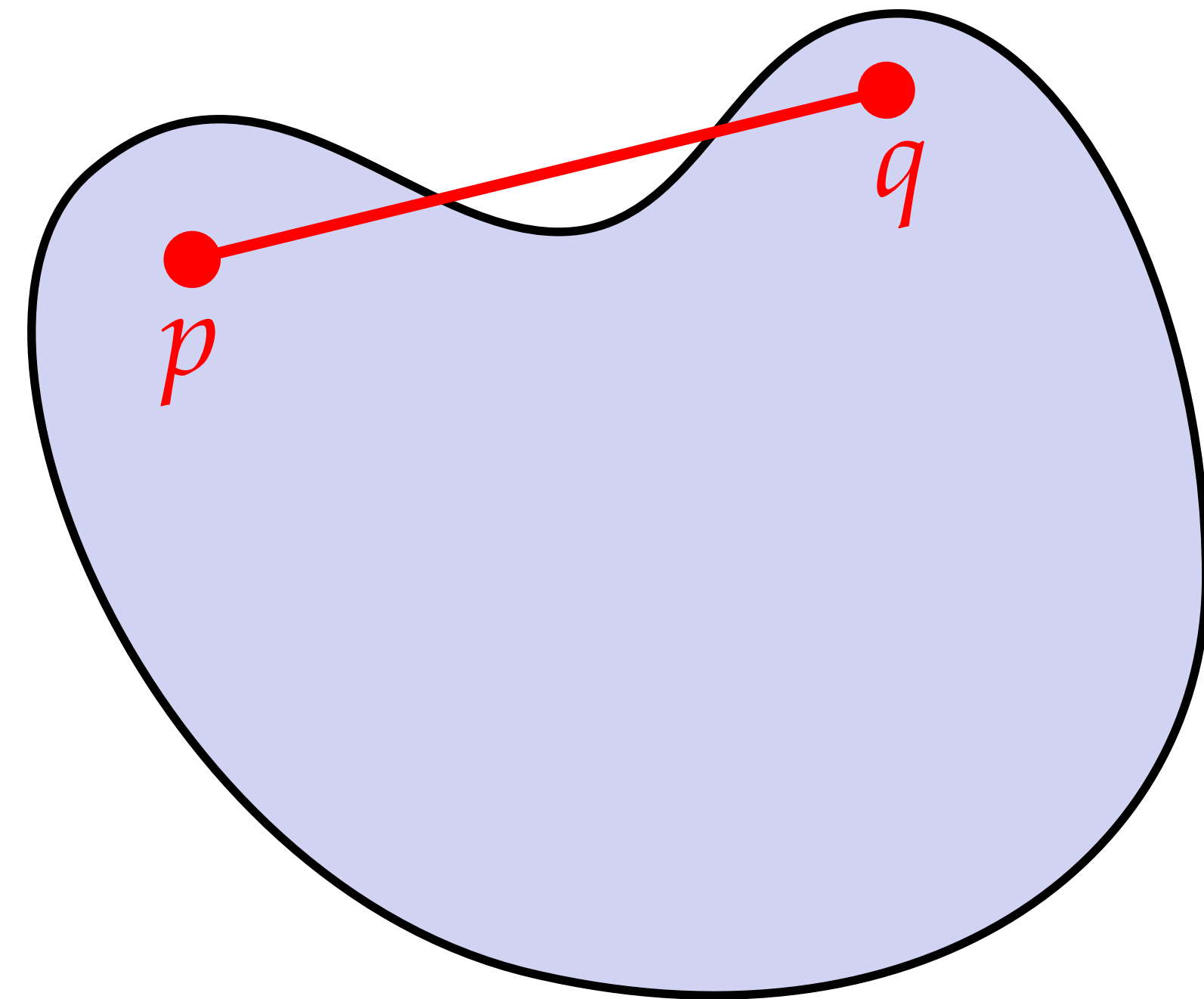
(F)

Convex Set

Definition. A subset $S \subset \mathbb{R}^n$ is *convex* if for every pair of points $p, q \in S$ the line segment between p and q is contained in S .

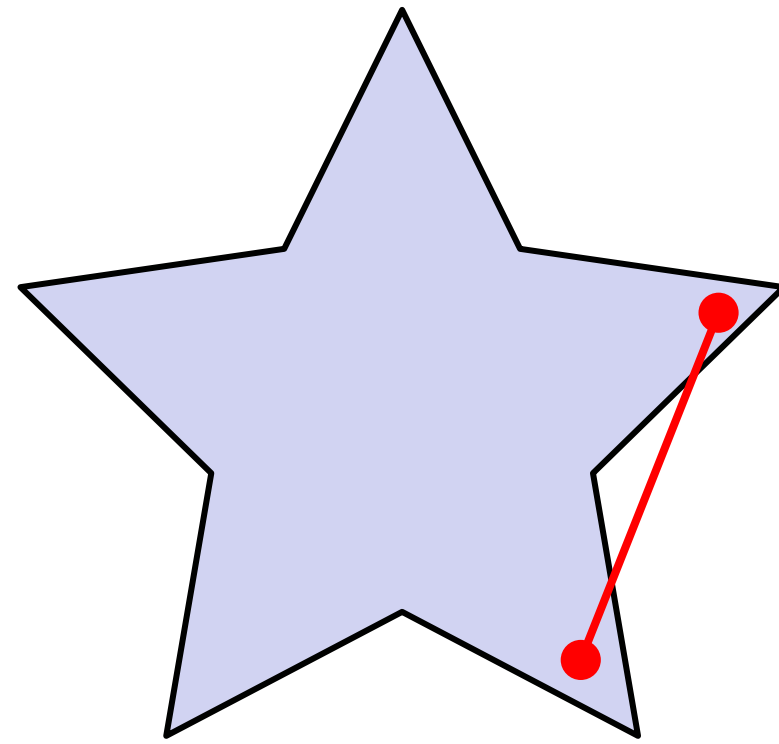


convex

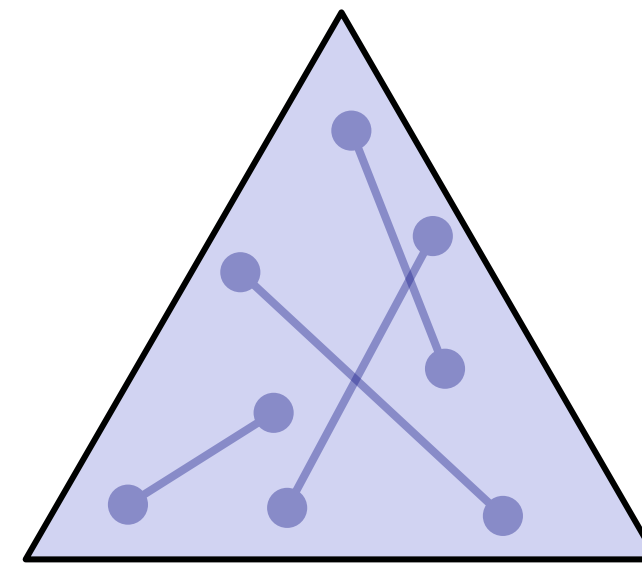


not convex

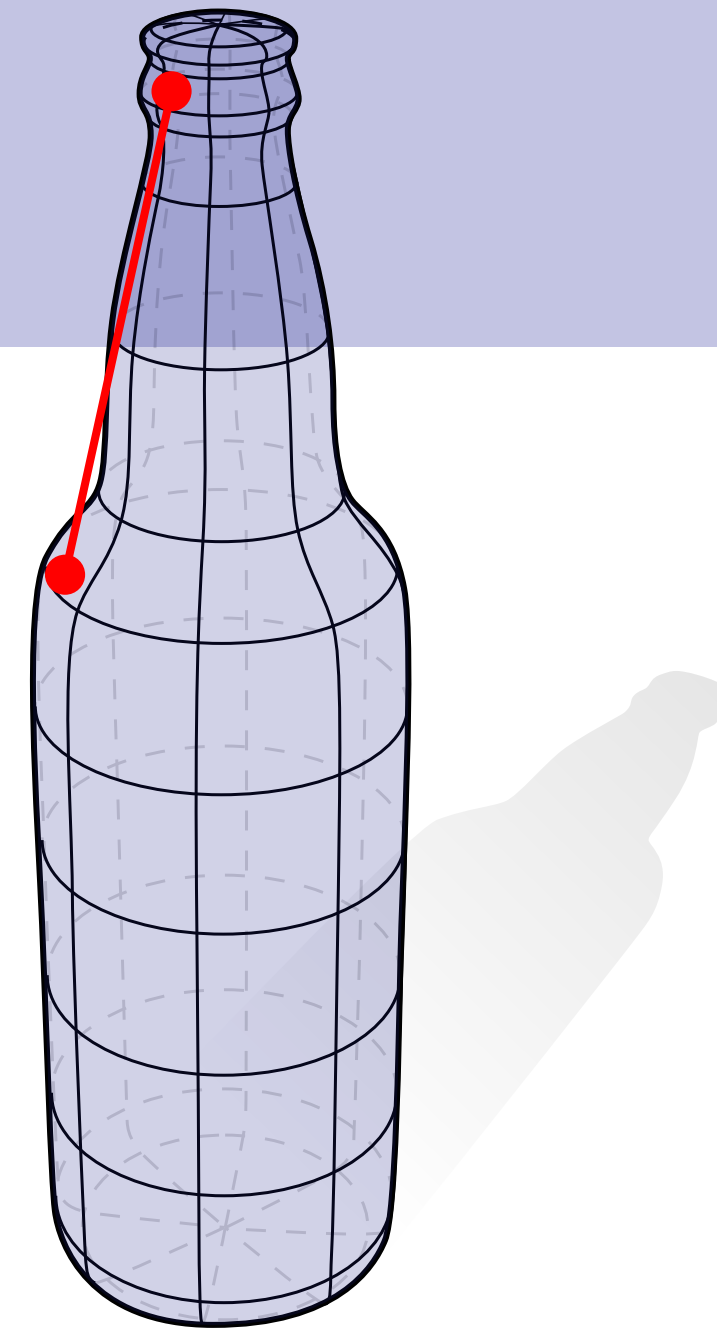
Convex Set—Examples



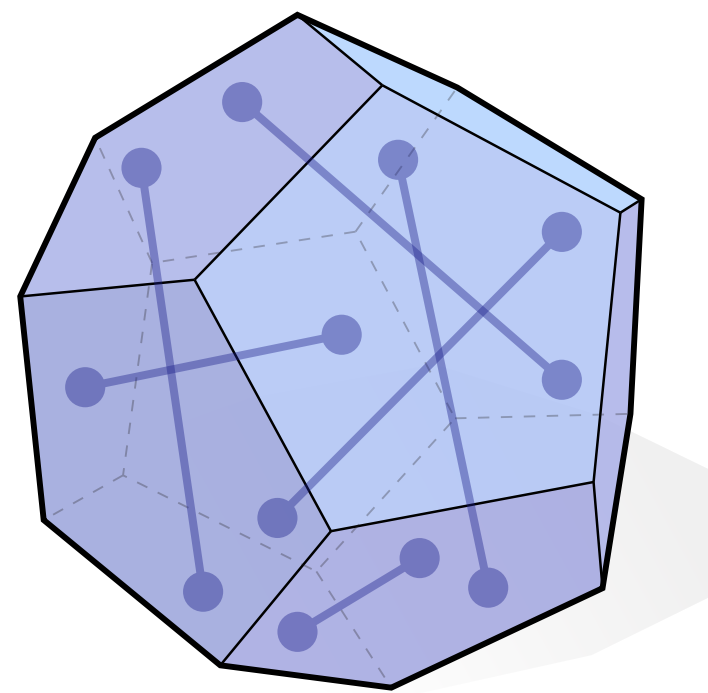
(A)



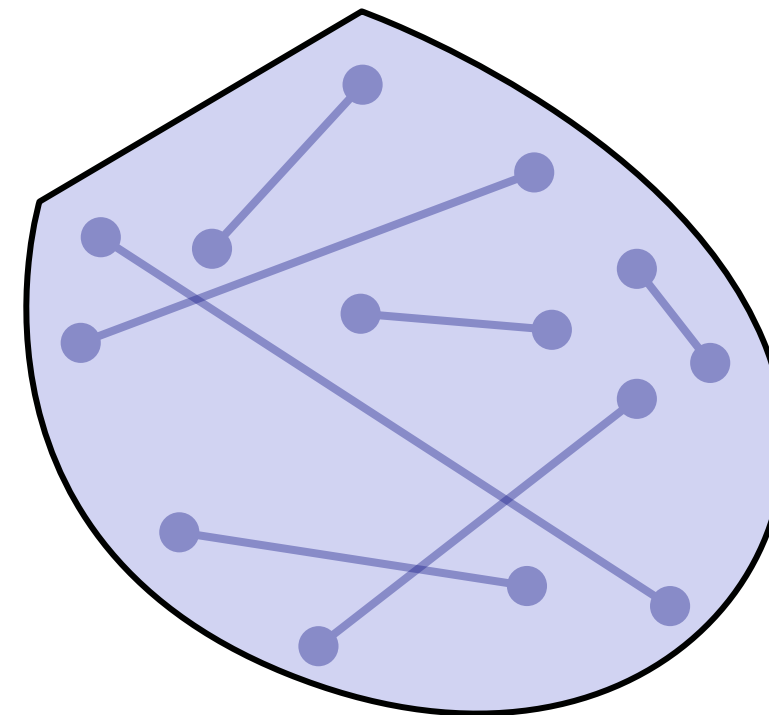
(B)



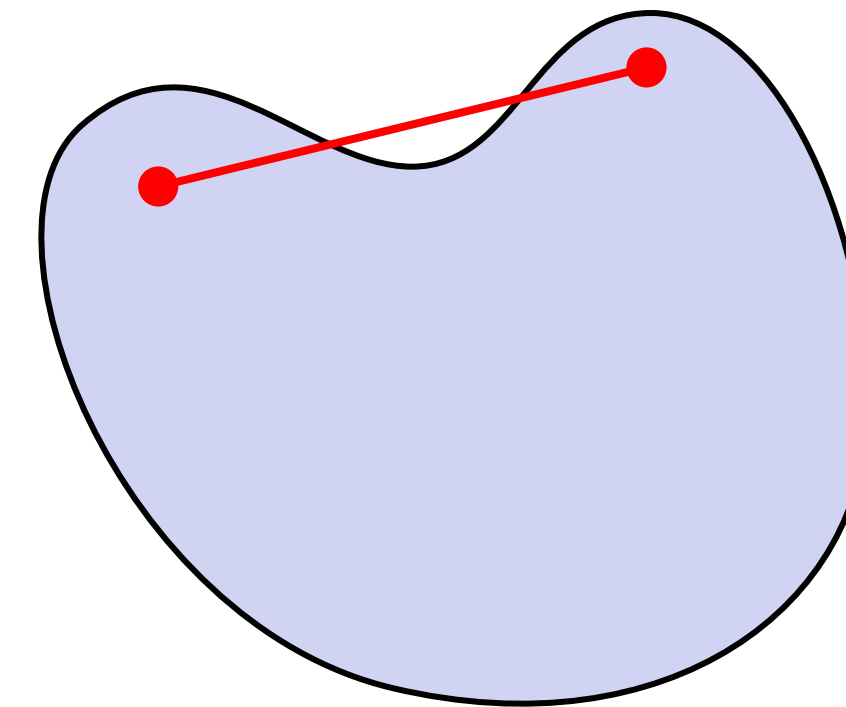
(C)



(D)



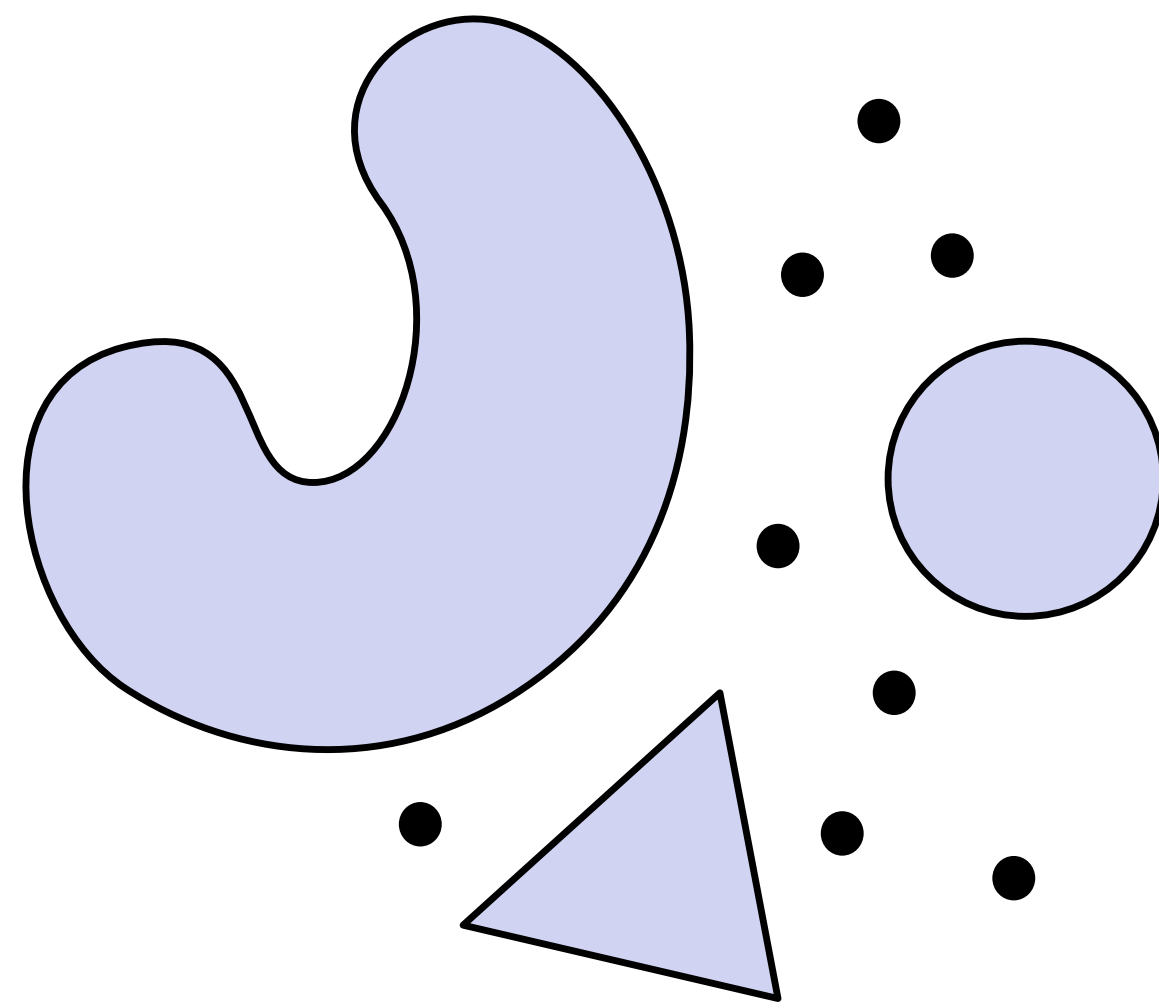
(E)



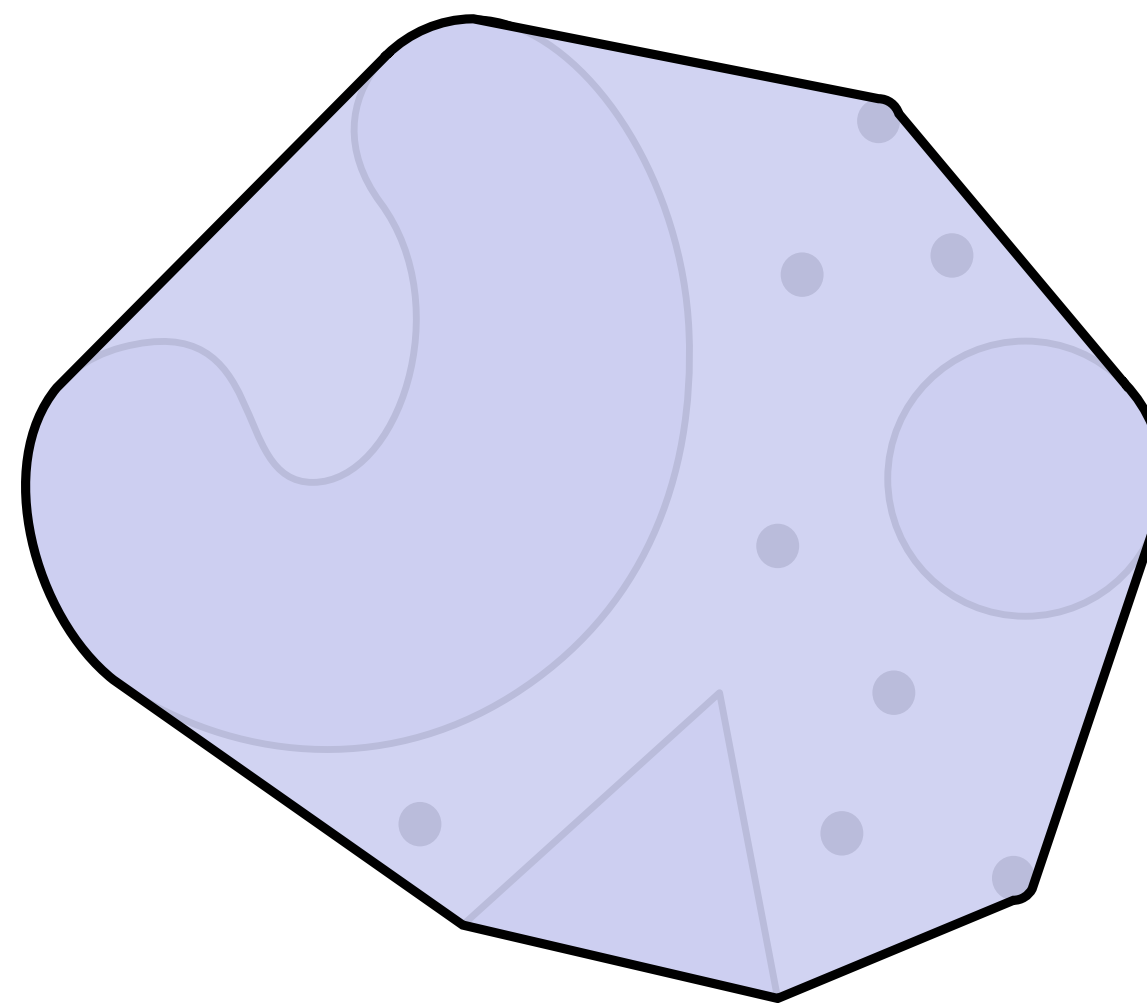
(F)

Convex Hull

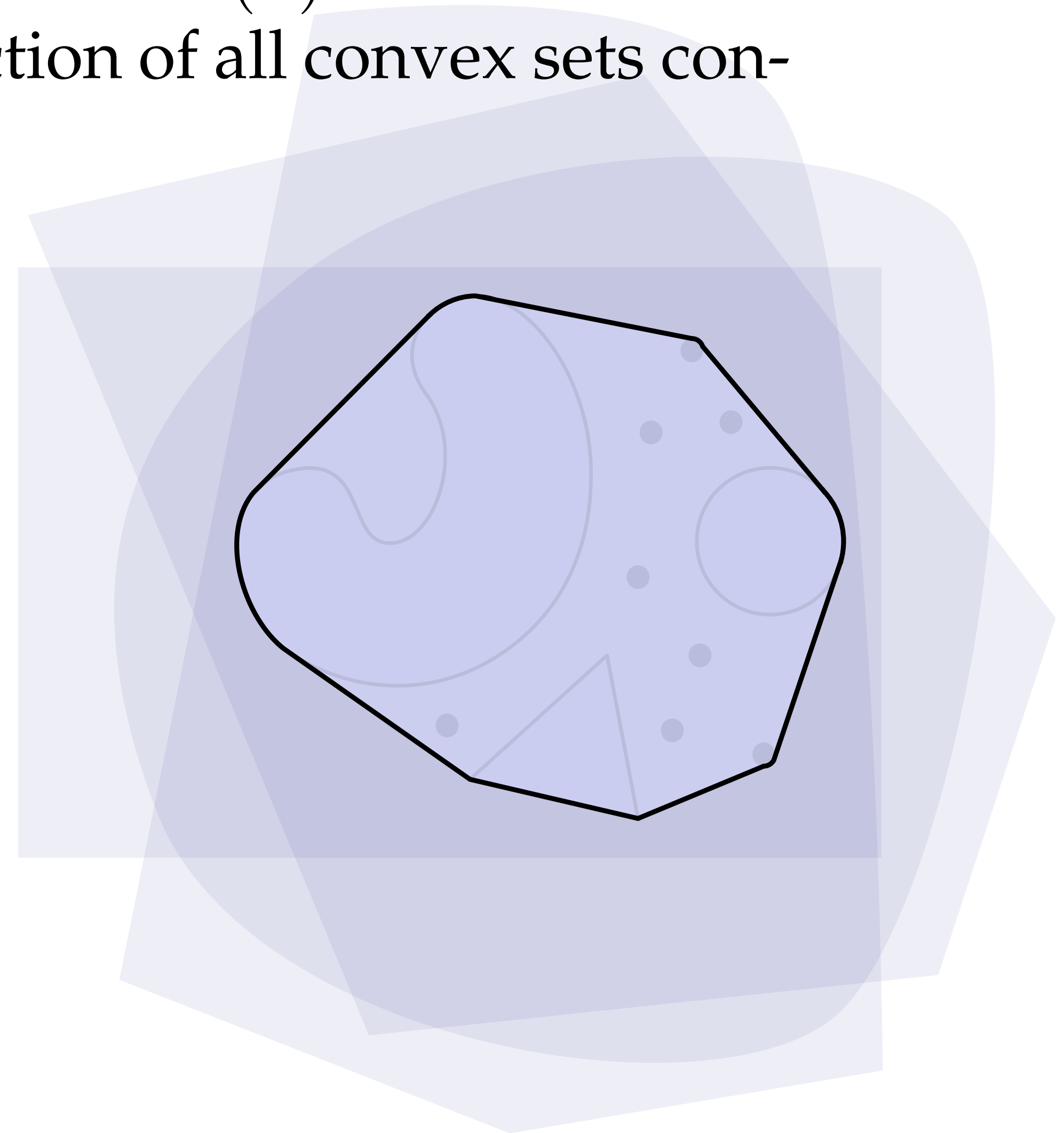
Definition. For any subset $S \subset \mathbb{R}^n$, its convex hull $\text{conv}(S)$ is the smallest convex set containing S , or equivalently, the intersection of all convex sets containing S .



S



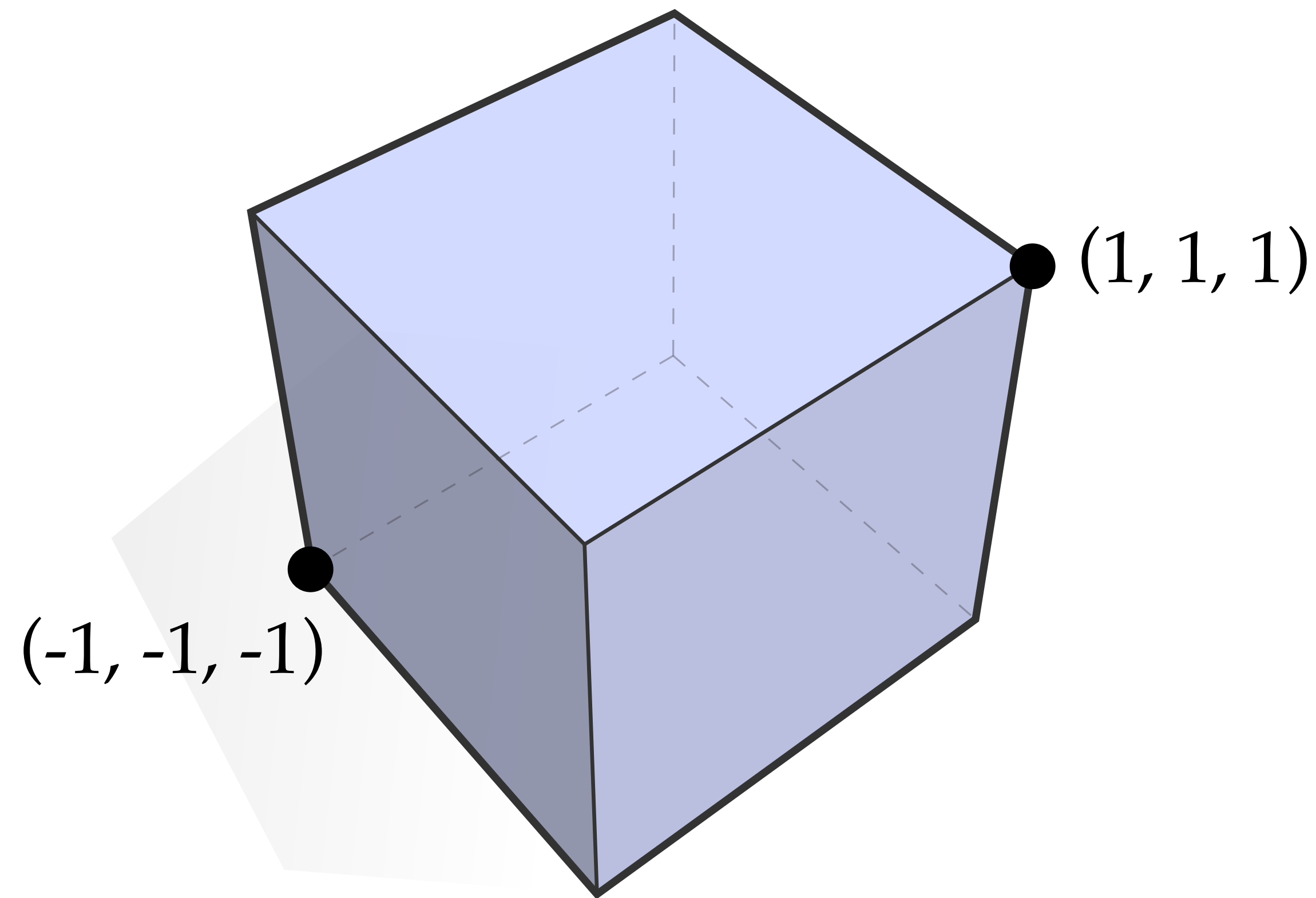
$\text{conv}(S)$

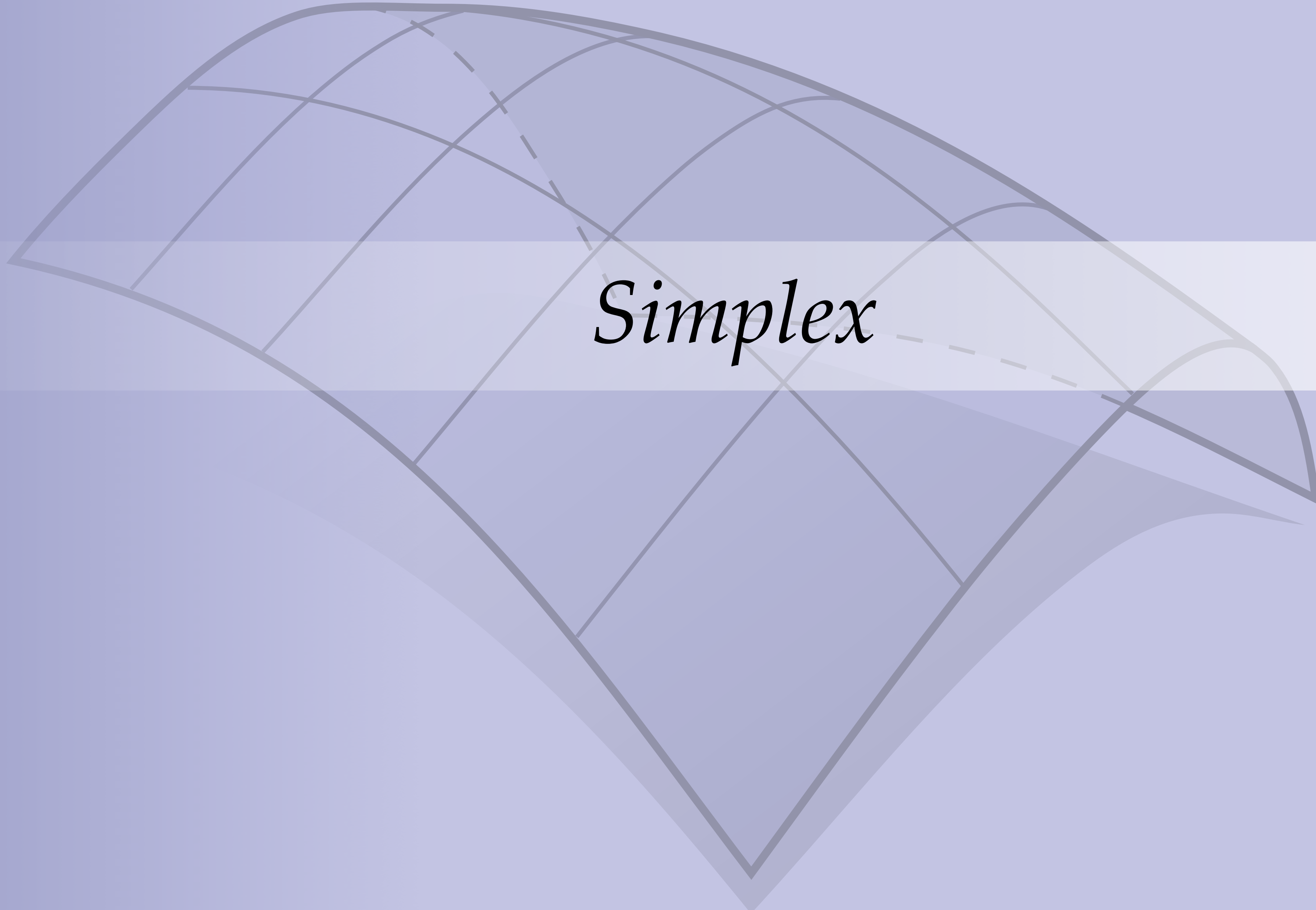


Convex Hull—Example

Q: What is the convex hull of the set $S := \{(\pm 1, \pm 1, \pm 1)\} \subset \mathbb{R}^3$?

A: A cube.

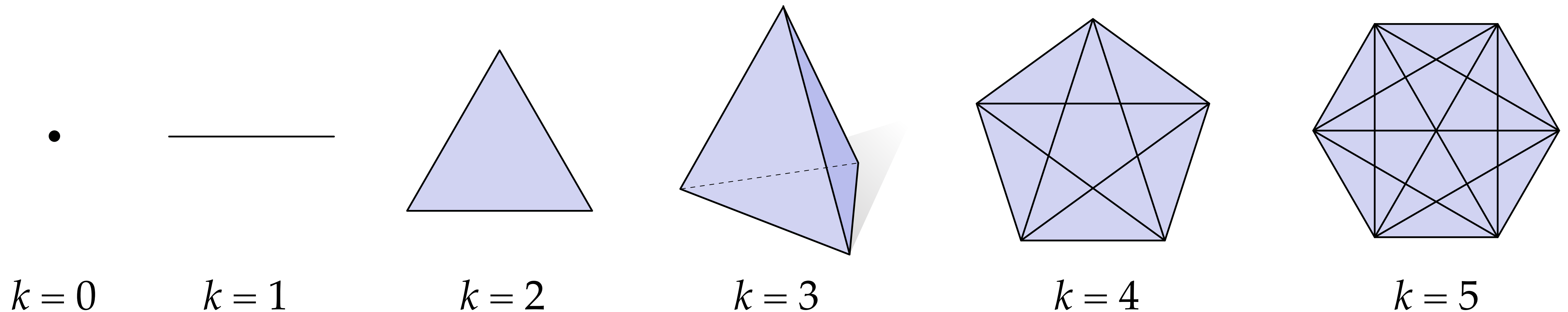




Simplex

Simplex — Basic Idea

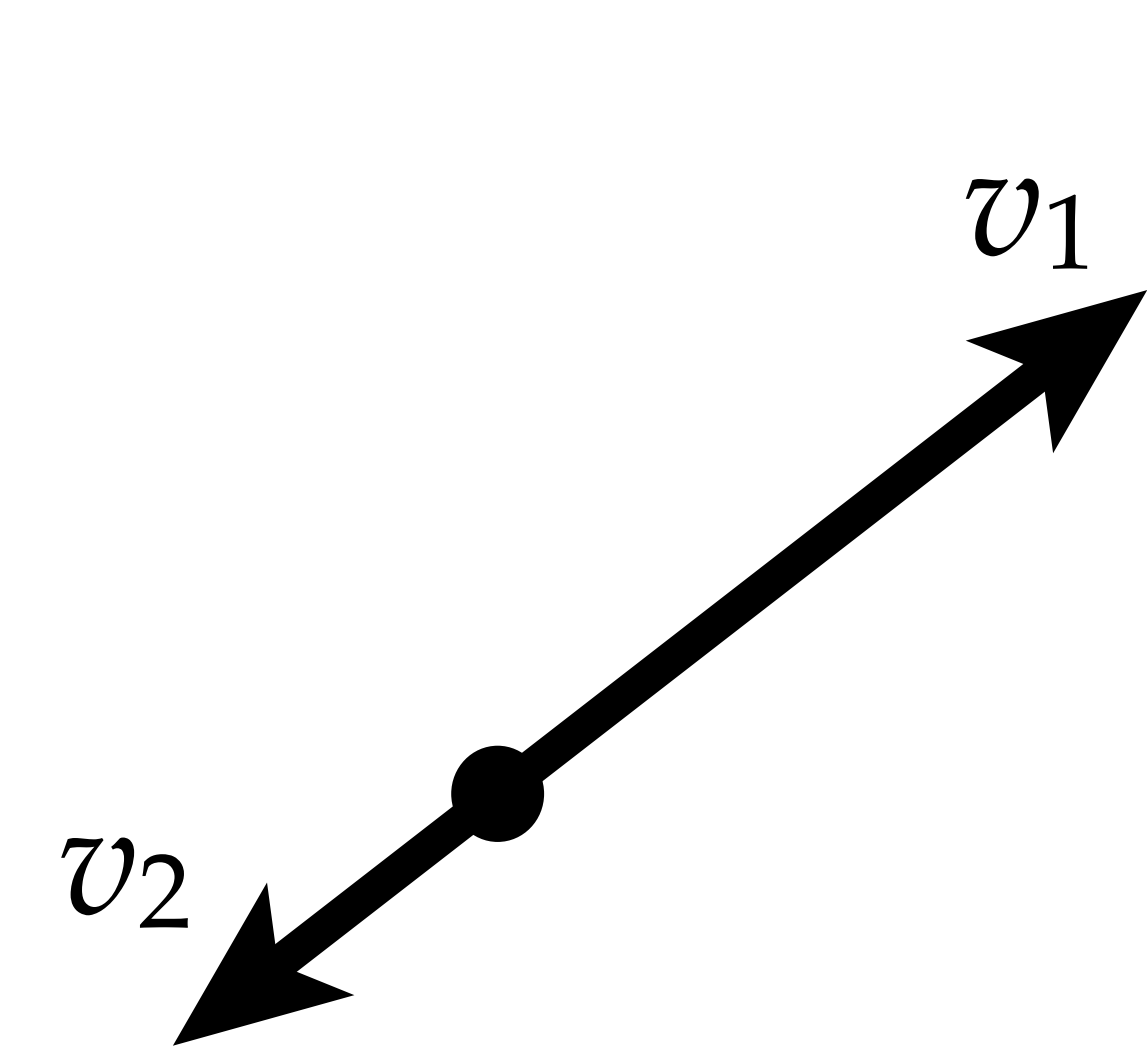
Roughly speaking, a k -simplex is a point, a line segment, a triangle, a tetrahedron...



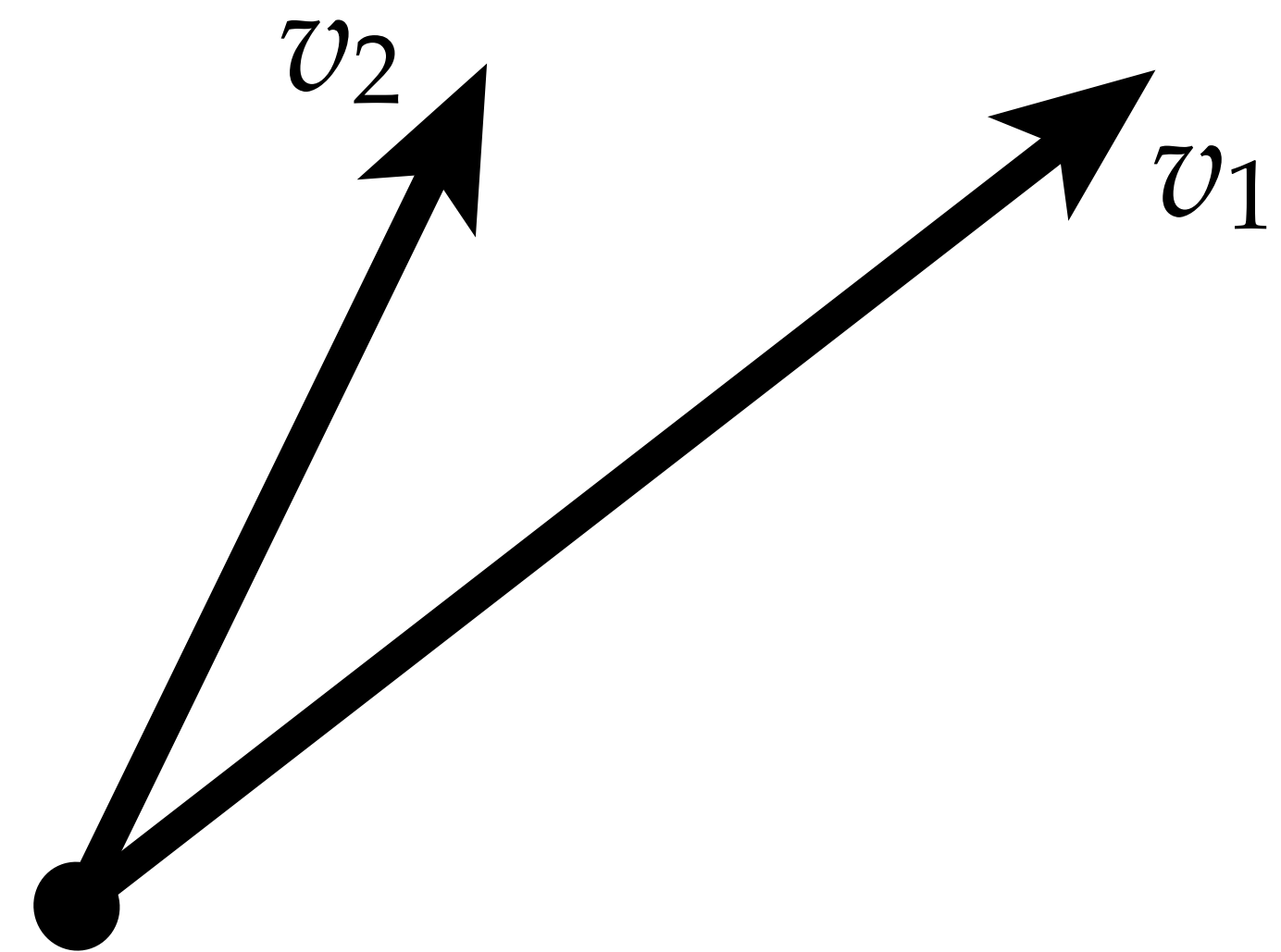
(...a bit hard to draw for $k \geq 4$!)

Linear Independence

Definition. A collection of vectors v_1, \dots, v_n is *linearly independent* if no vector can be expressed as a linear combination of the others, *i.e.*, if there is no collection of coefficients $a_1, \dots, a_n \in \mathbb{R}$ such that $v_j = \sum_{i \neq j} a_i v_i$ (for any v_j).



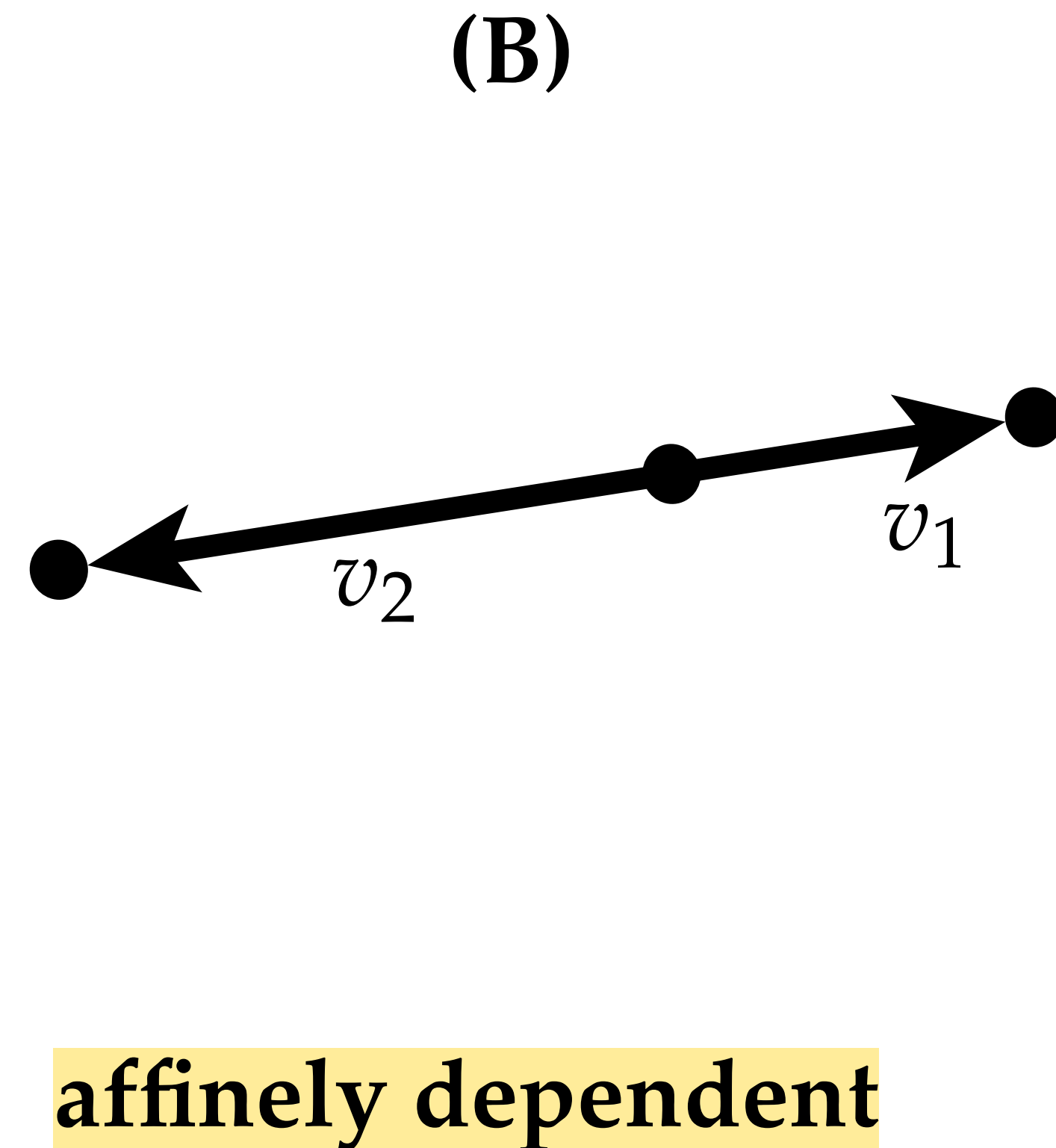
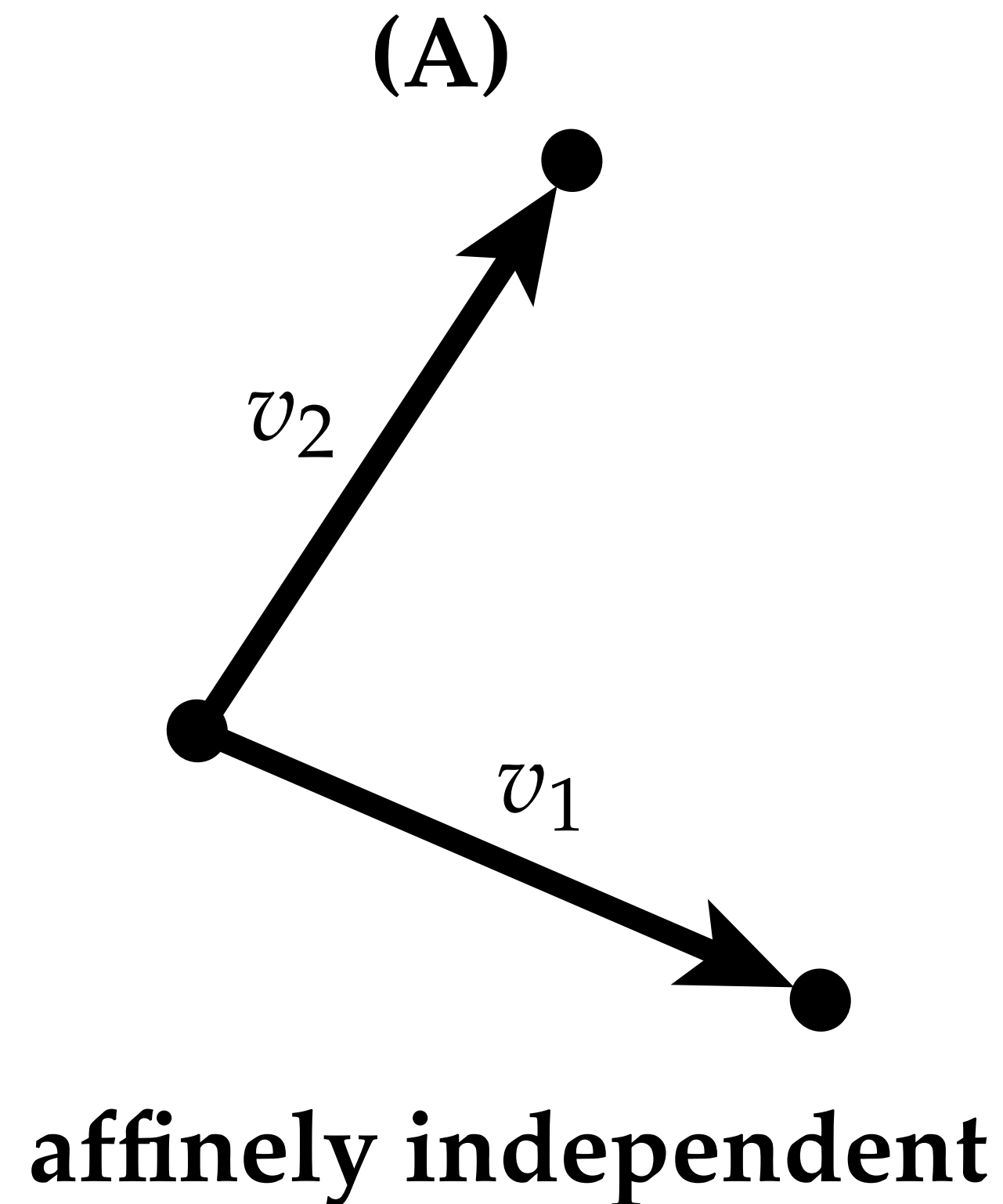
linearly dependent



linearly independent

Affine Independence

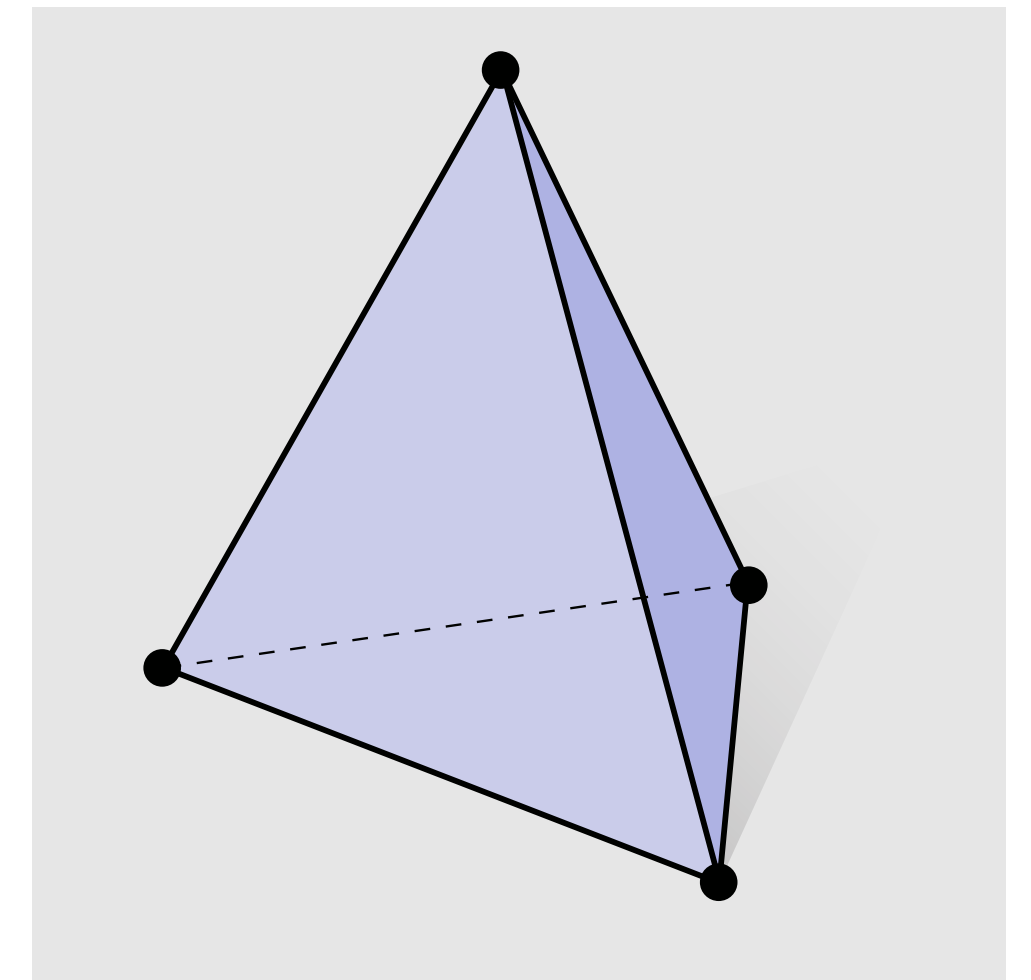
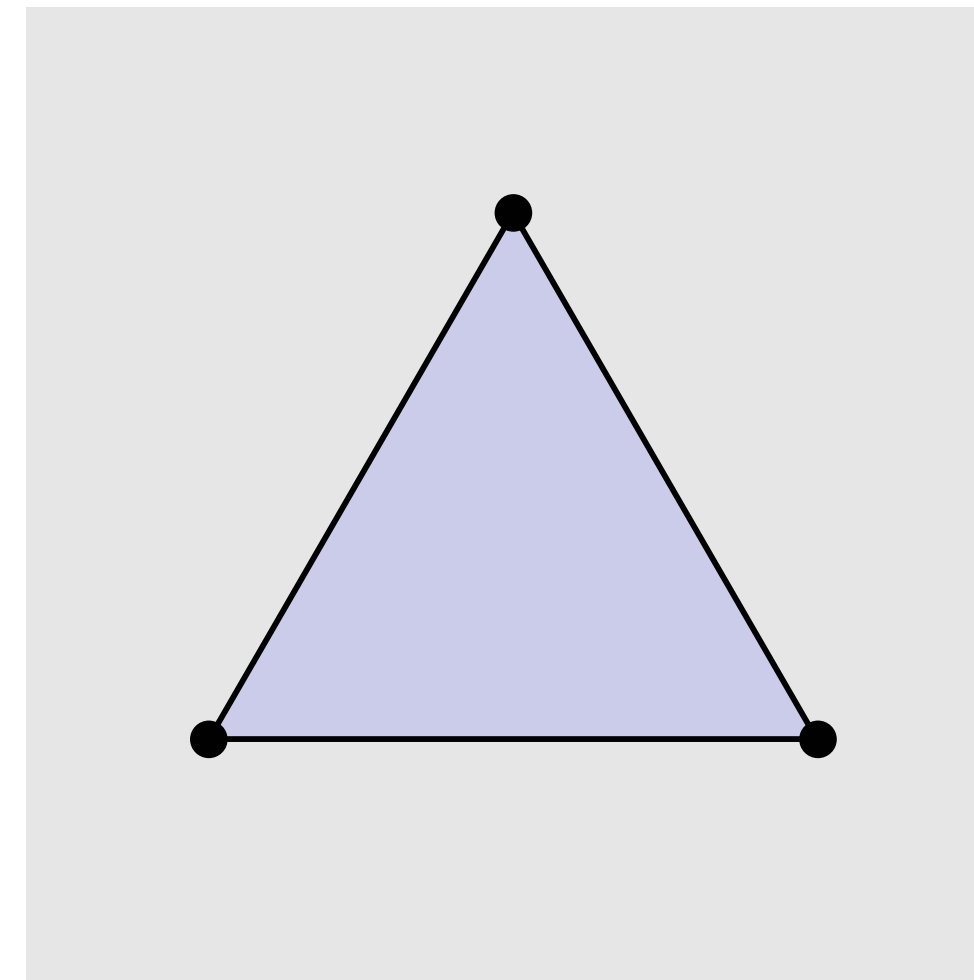
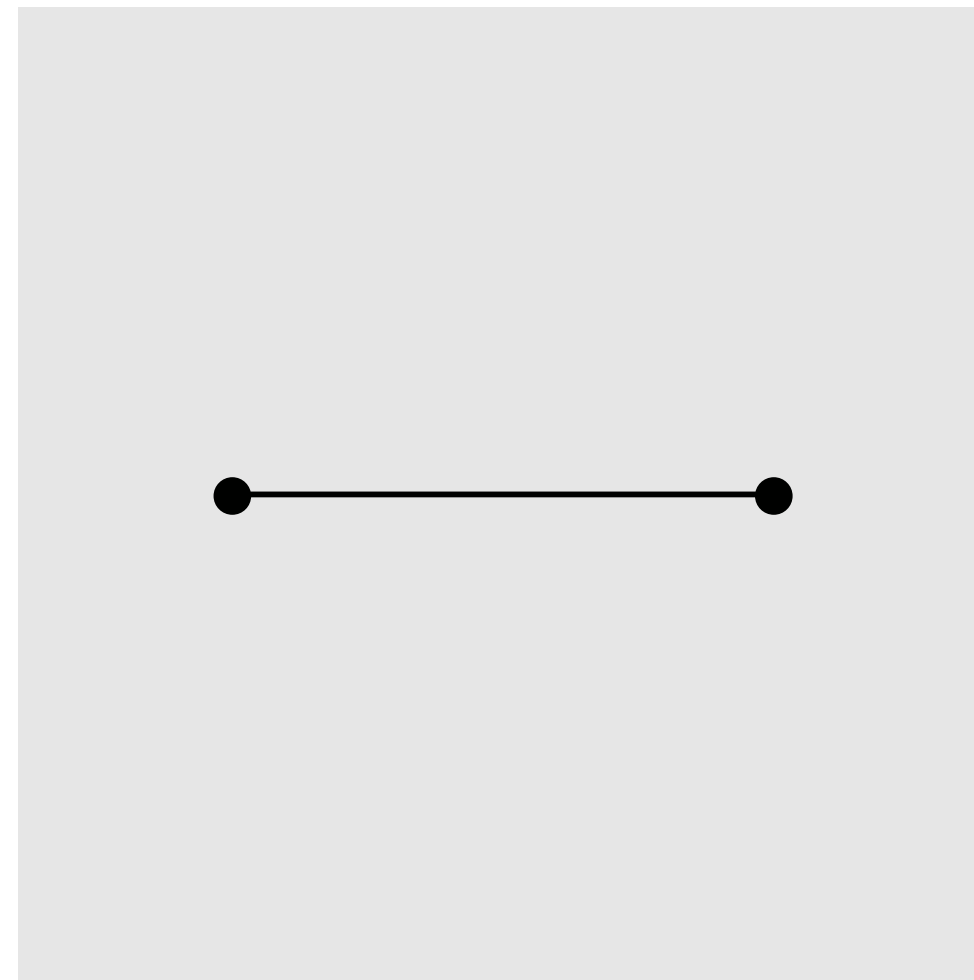
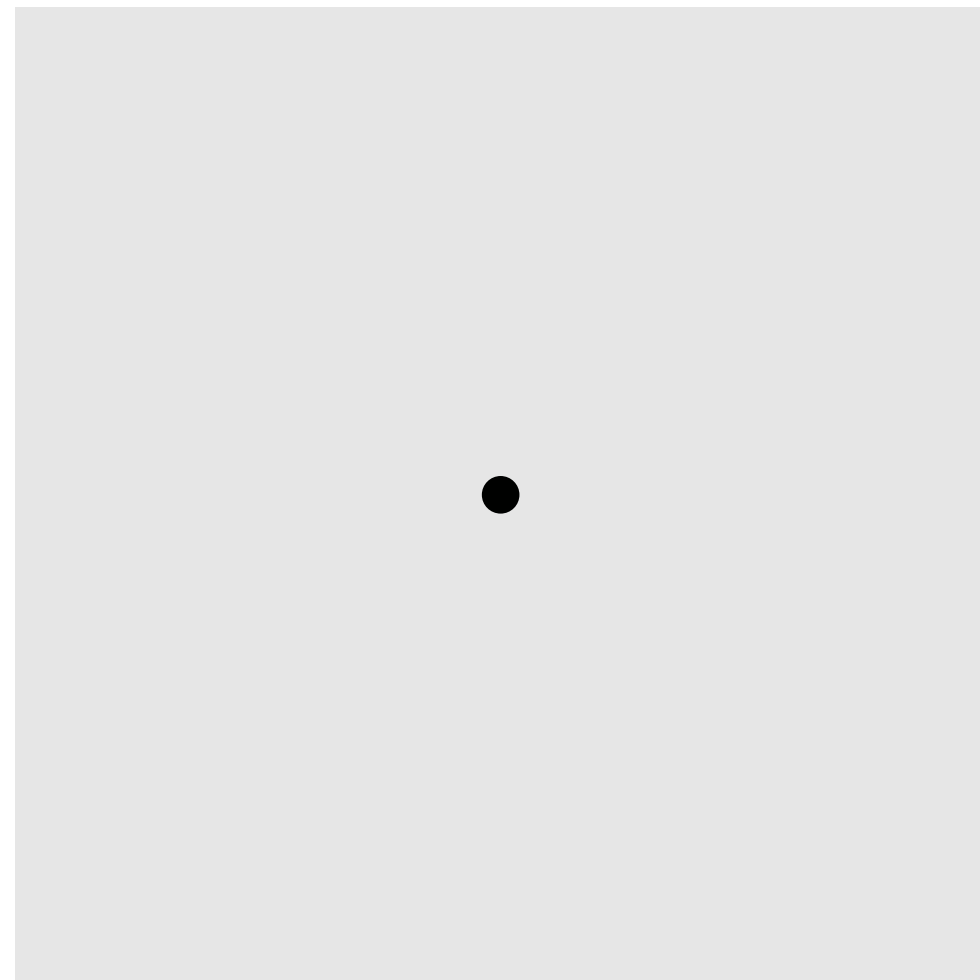
Definition. A collection of points p_0, \dots, p_k are *affinely independent* if the vectors $v_i := p_i - p_0$ are linearly independent.



(Colloquially: might say points are in “*general position*”.)

Simplex — Geometric Definition

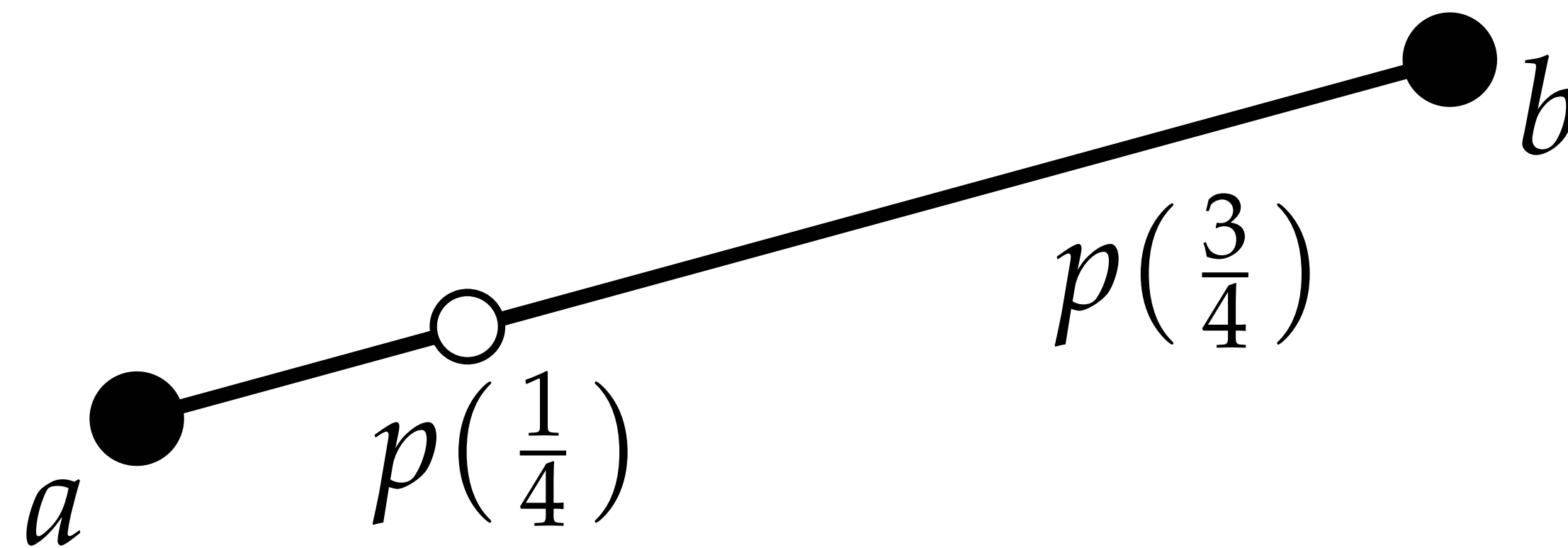
Definition. A k -simplex is the convex hull of $k + 1$ affinely-independent points, which we call its *vertices*.



Barycentric Coordinates—1-Simplex

- We can describe a simplex more explicitly using *barycentric coordinates*.
- For example, a 1-simplex is comprised of all weighted combinations of the two points where the weights sum to 1:

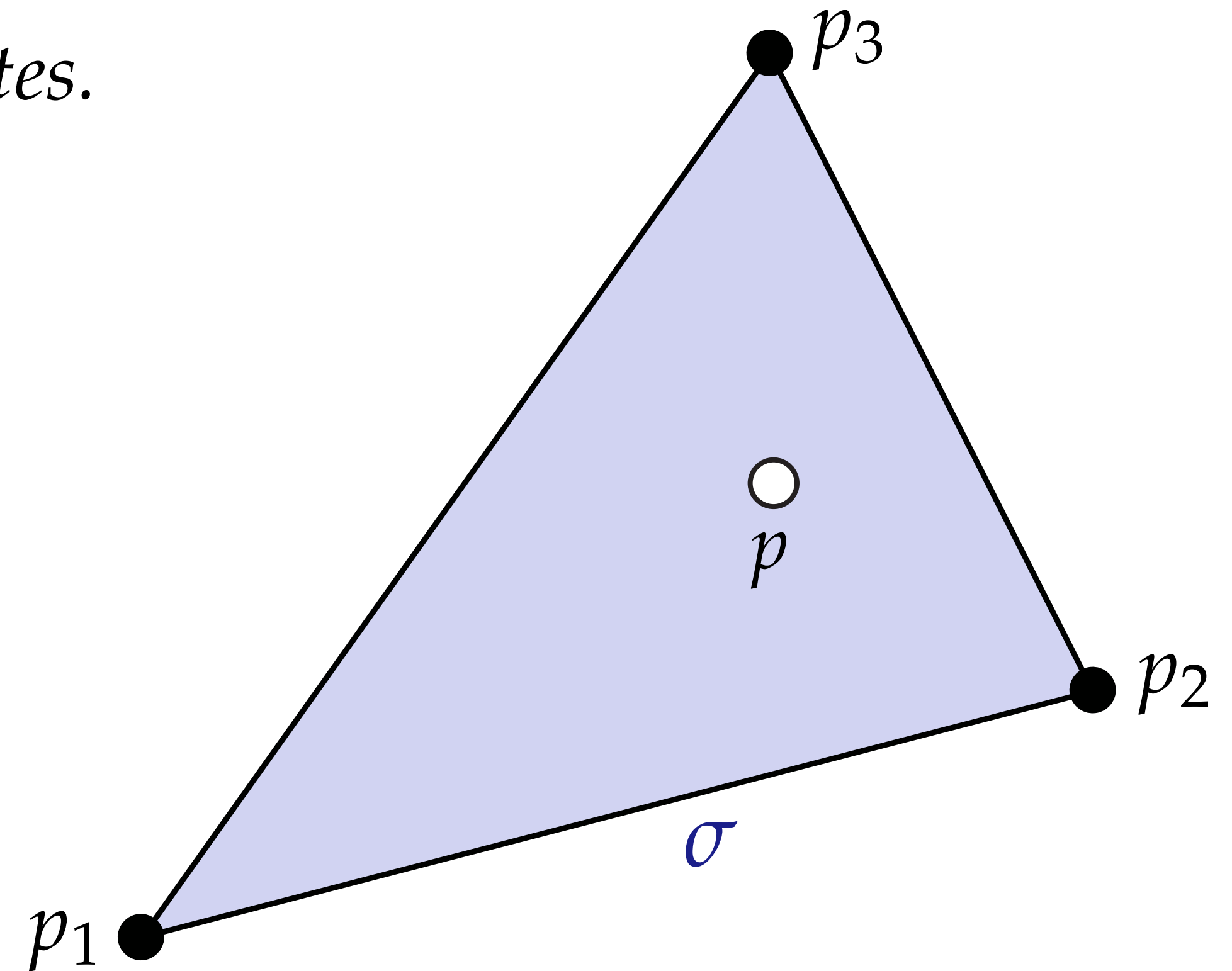
$$p(t) := (1 - t)a + tb, \quad t \in [0, 1]$$



Barycentric Coordinates— k -Simplex

- More generally, any point p in a k -simplex σ can be expressed as a (nonnegative) weighted combination of the vertices, where the weights sum to 1.
- The weights t_i are called the *barycentric coordinates*.

$$\sigma = \left\{ \sum_{i=0}^k t_i p_i \mid \sum_{i=0}^k t_i = 1, t_i \geq 0 \forall i \right\}$$



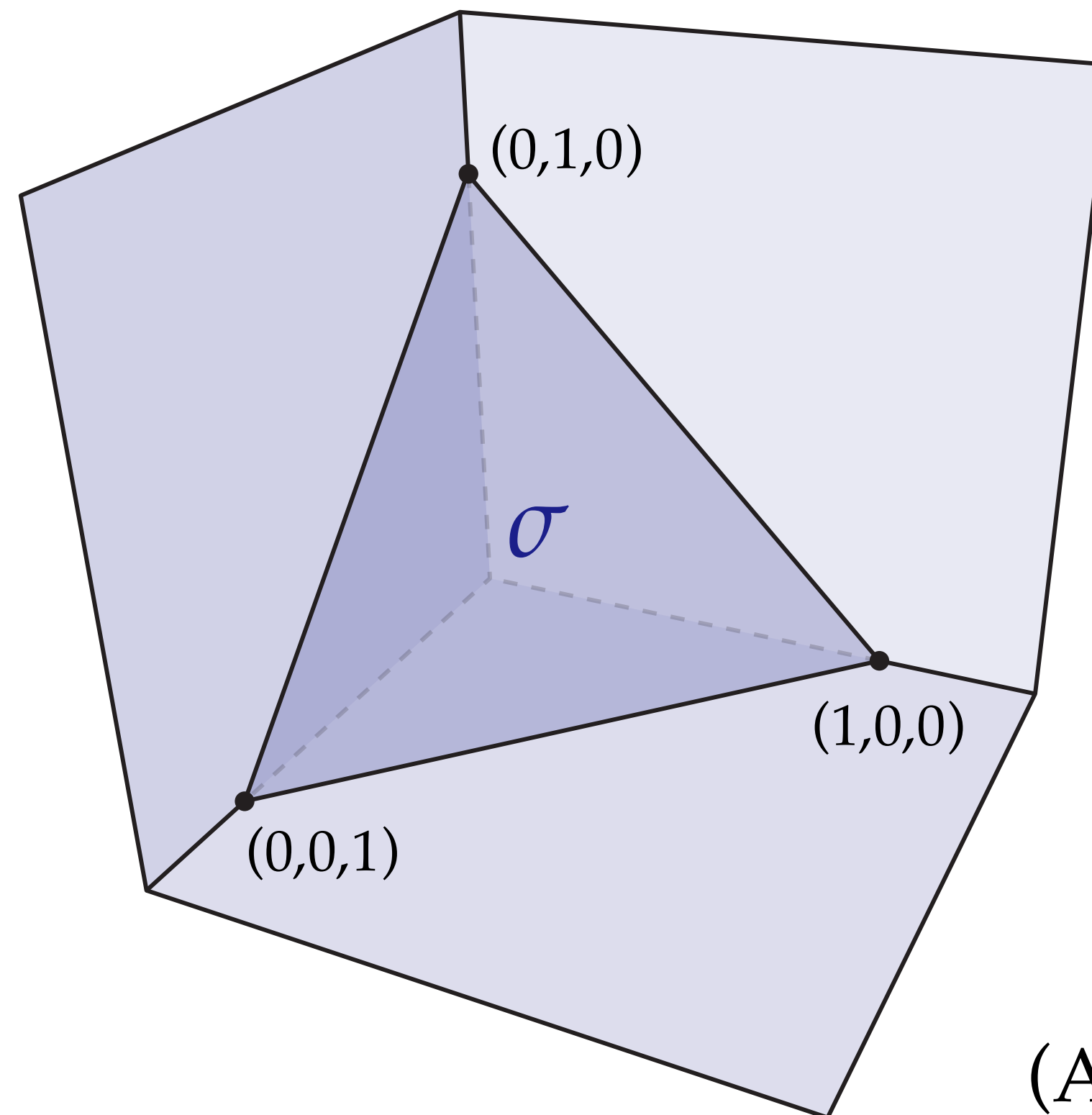
$$p = \frac{2}{10} p_1 + \frac{4}{10} p_2 + \frac{4}{10} p_3$$

(Also called a “convex combination.”)

Simplex — Example

Definition. The *standard n -simplex* is the collection of points

$$\sigma := \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i = 1, x_i \geq 0 \forall i \right\}.$$



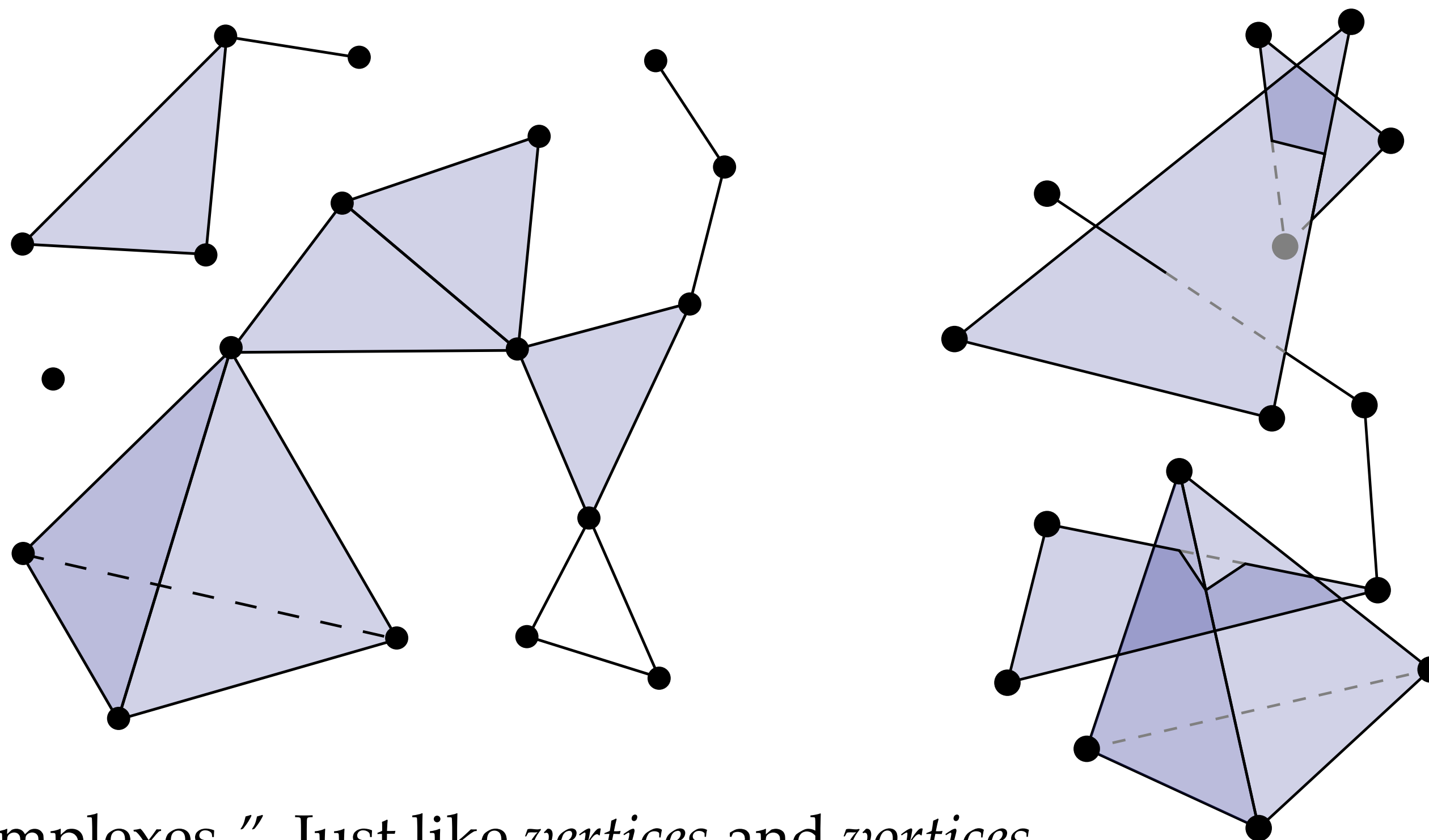
(Also known as the “probability simplex.”)



Simplicial Complex

Simplicial Complex—*Rough Idea*

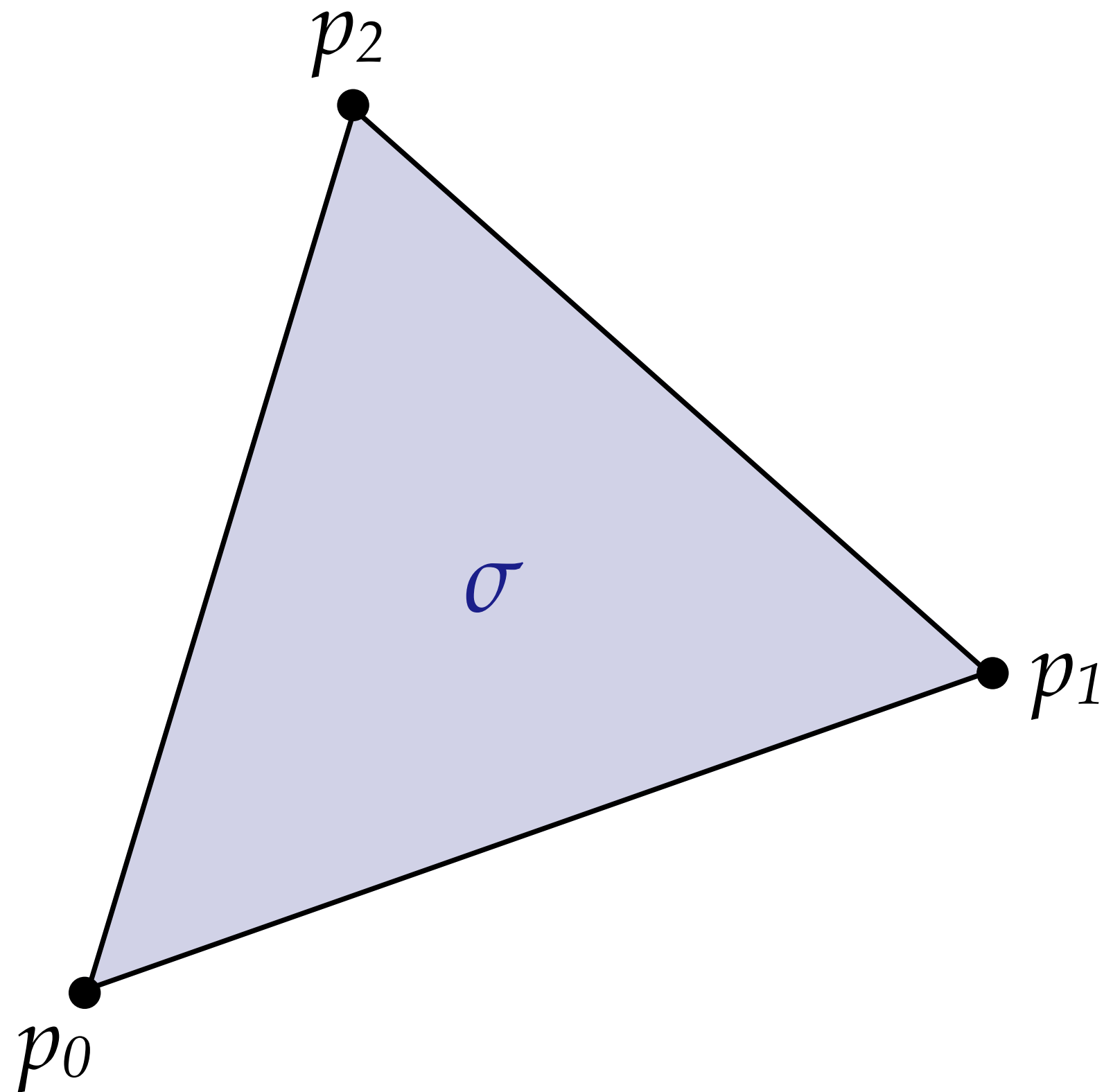
- Roughly speaking, a *simplicial complex* is “a bunch of simplices*”
 - ...but with some specific properties that make them easy to work with.
- Also have to resolve some basic questions—*e.g.*, how can simplices intersect?



*Plural of simplex; not “simplexes.” Just like *vertices* and *vortices*.

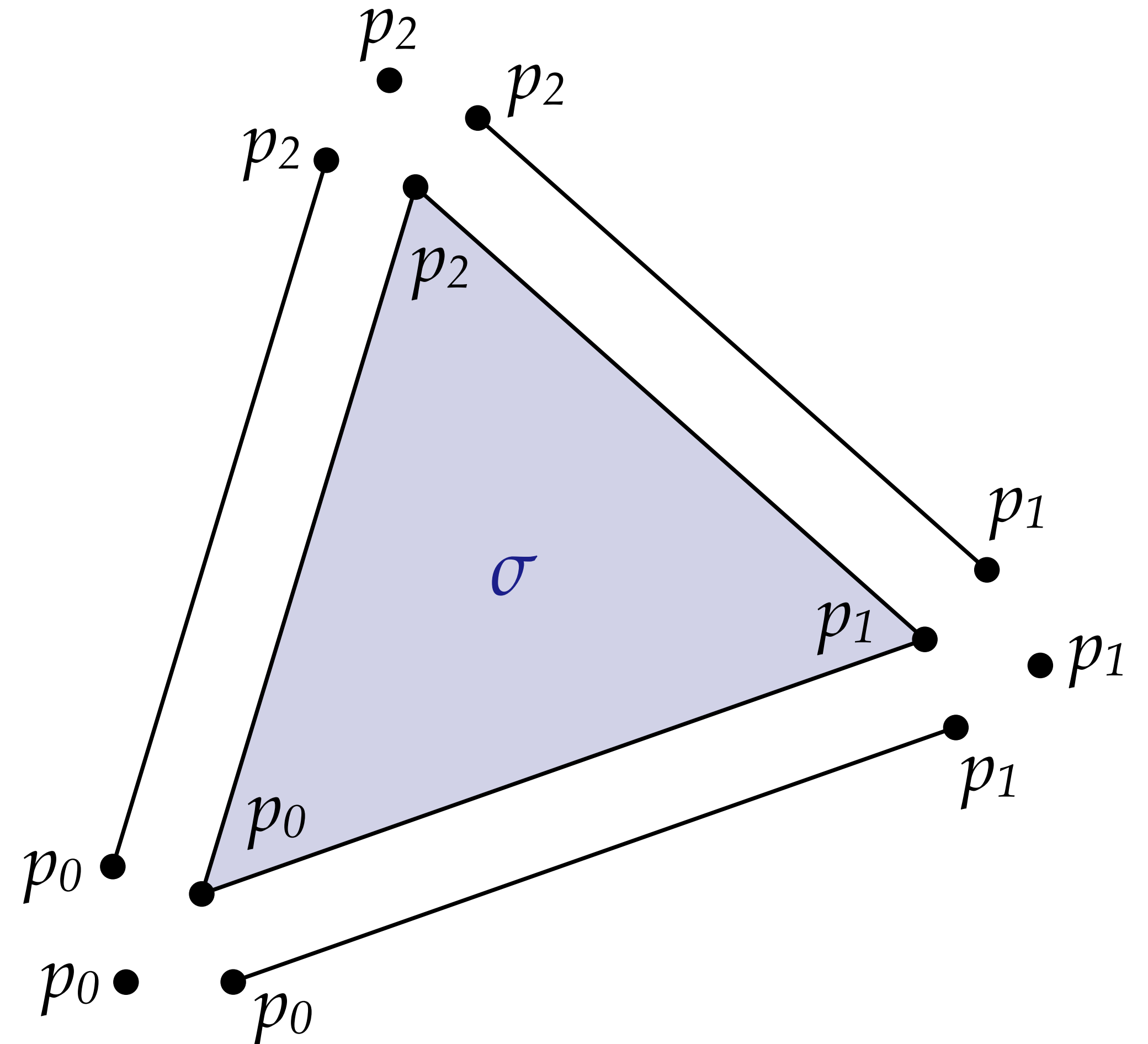
Face of a Simplex

Definition. A *face* of a simplex σ is any simplex whose vertices are a subset* of the vertices of σ .



Q: Anything missing from this picture?

A: Yes—formally, the *empty set* \emptyset .

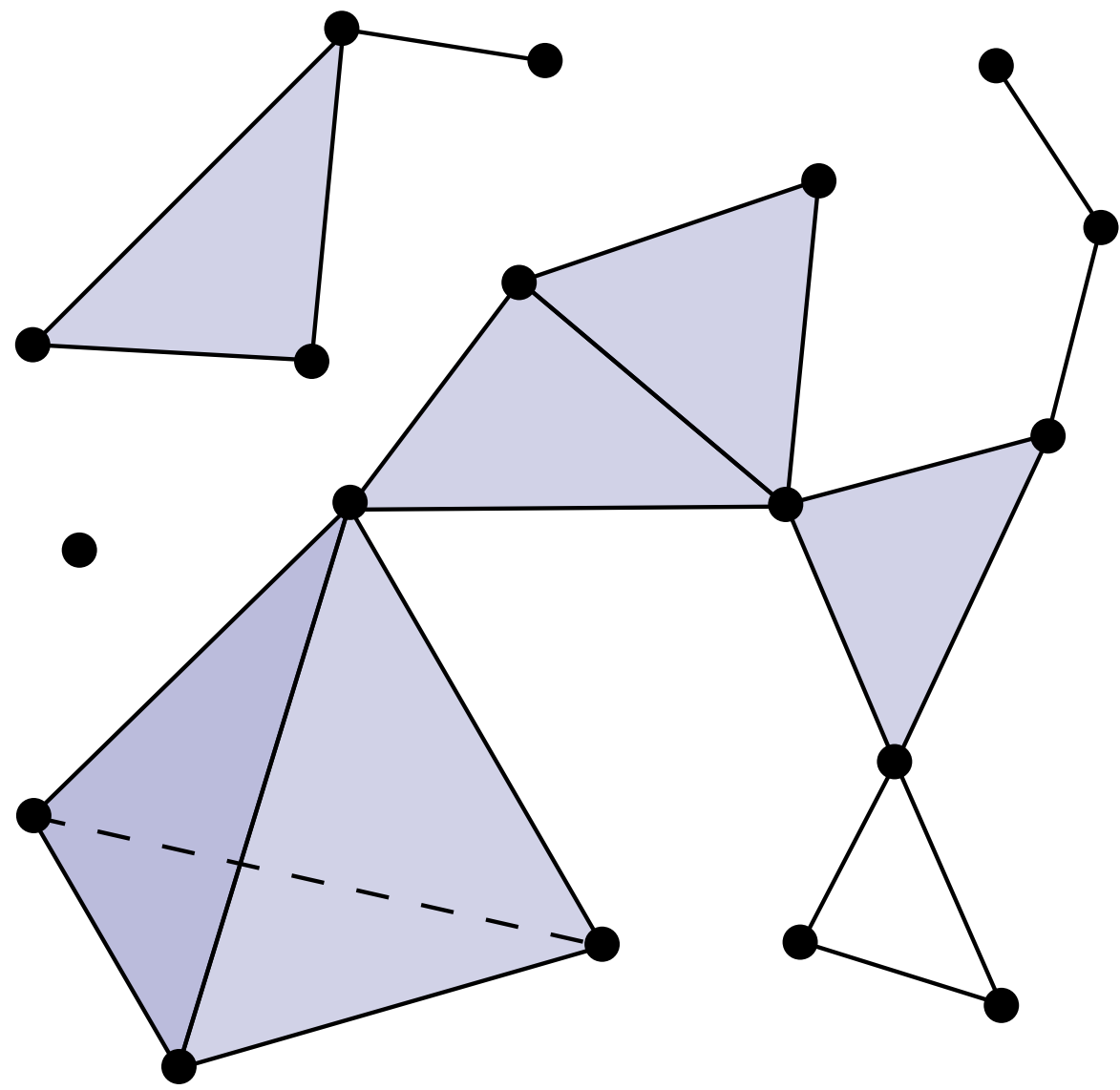


*Doesn't have to be a *proper* subset, *i.e.*, a simplex is its own face.

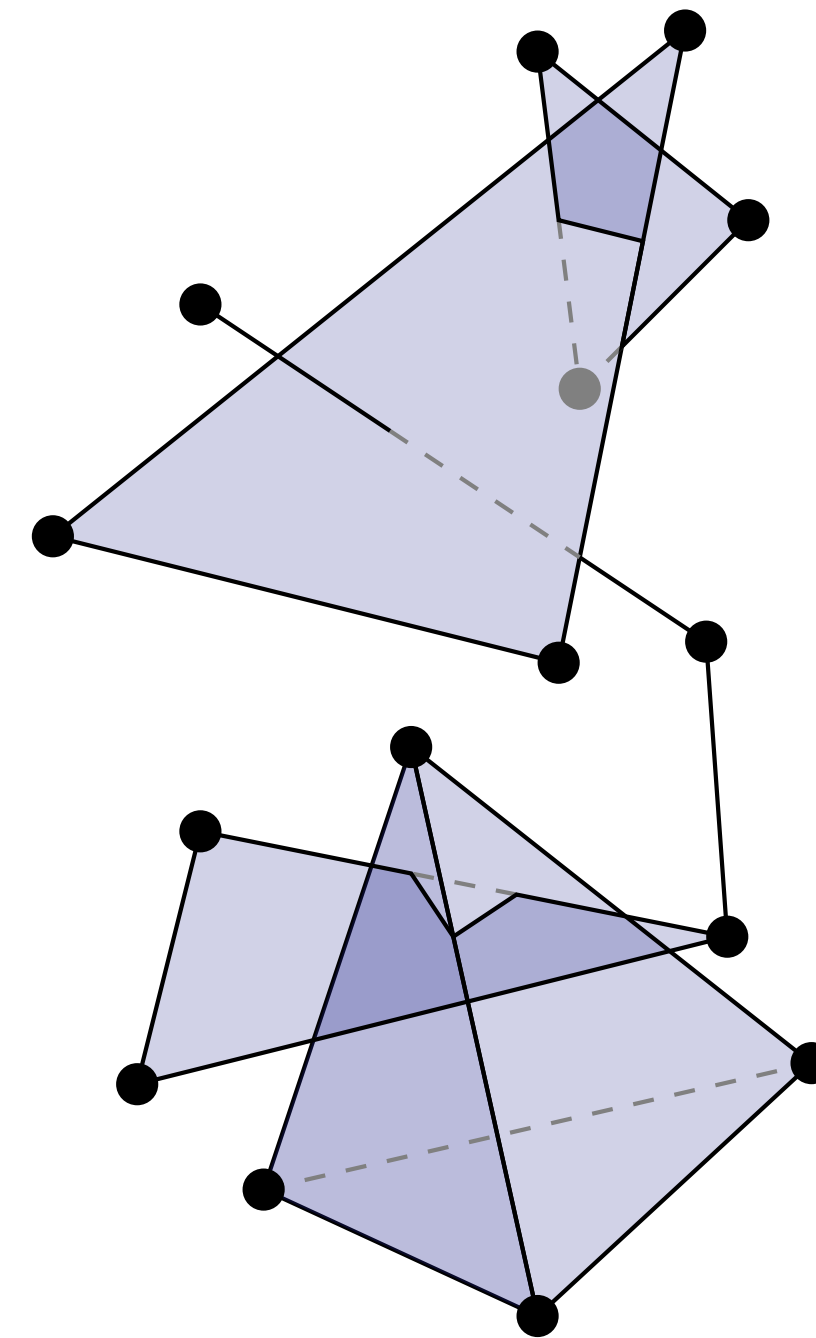
Simplicial Complex — Geometric Definition

Definition. A (geometric) simplicial complex is a collection of simplices where

- the intersection of any two simplices is a simplex in the complex, and
- every face of every simplex in the complex is also in the complex

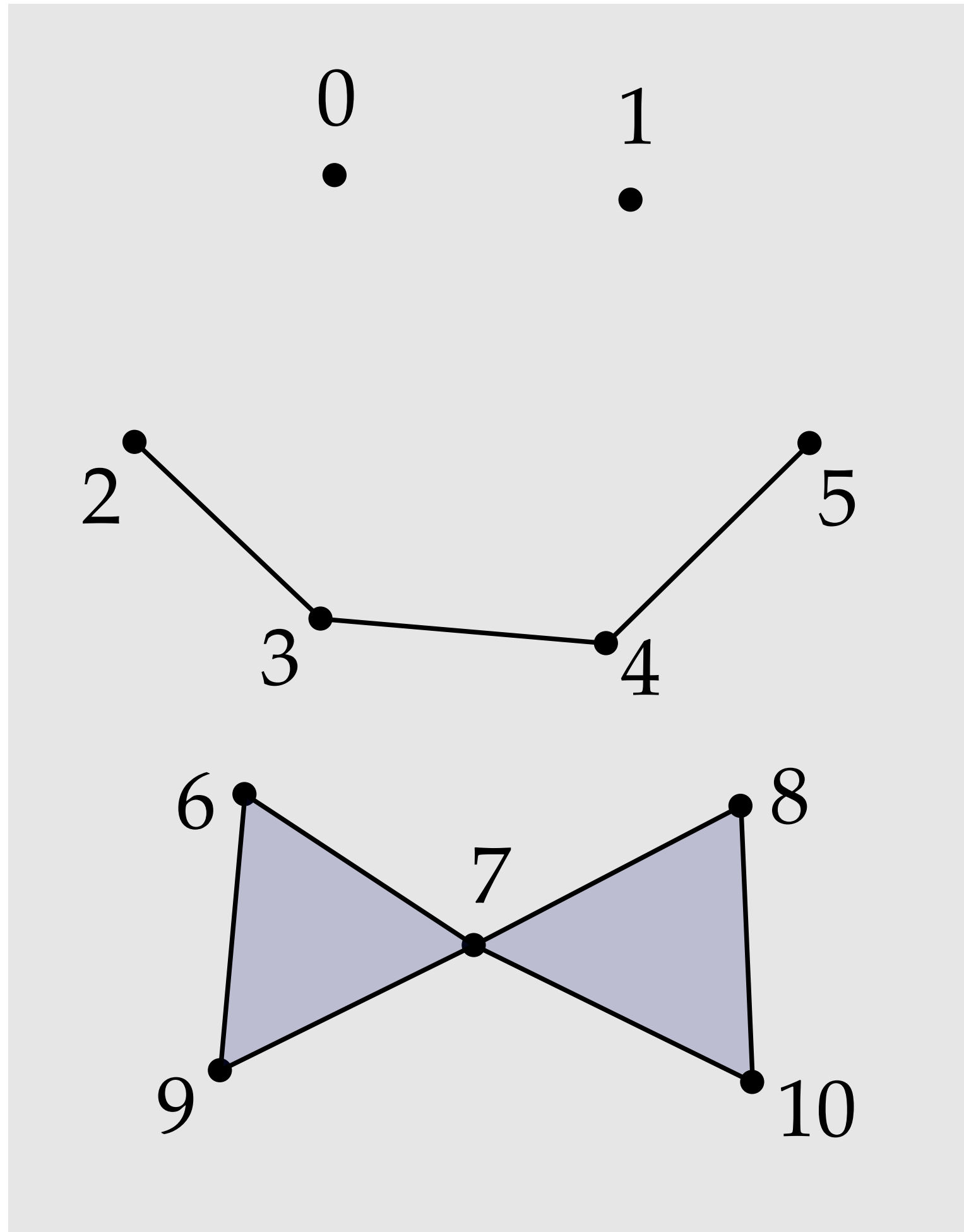


simplicial complex



not a geometric simplicial complex...

Simplicial Complex—Example



Q: What are all the simplices?

A: $\{6,7,9\}$ $\{7,10,8\}$

$\{2,3\}$ $\{3,4\}$ $\{4,5\}$ $\{6,7\}$ $\{7,9\}$ $\{9,6\}$ $\{7,8\}$ $\{8,10\}$ $\{10,7\}$

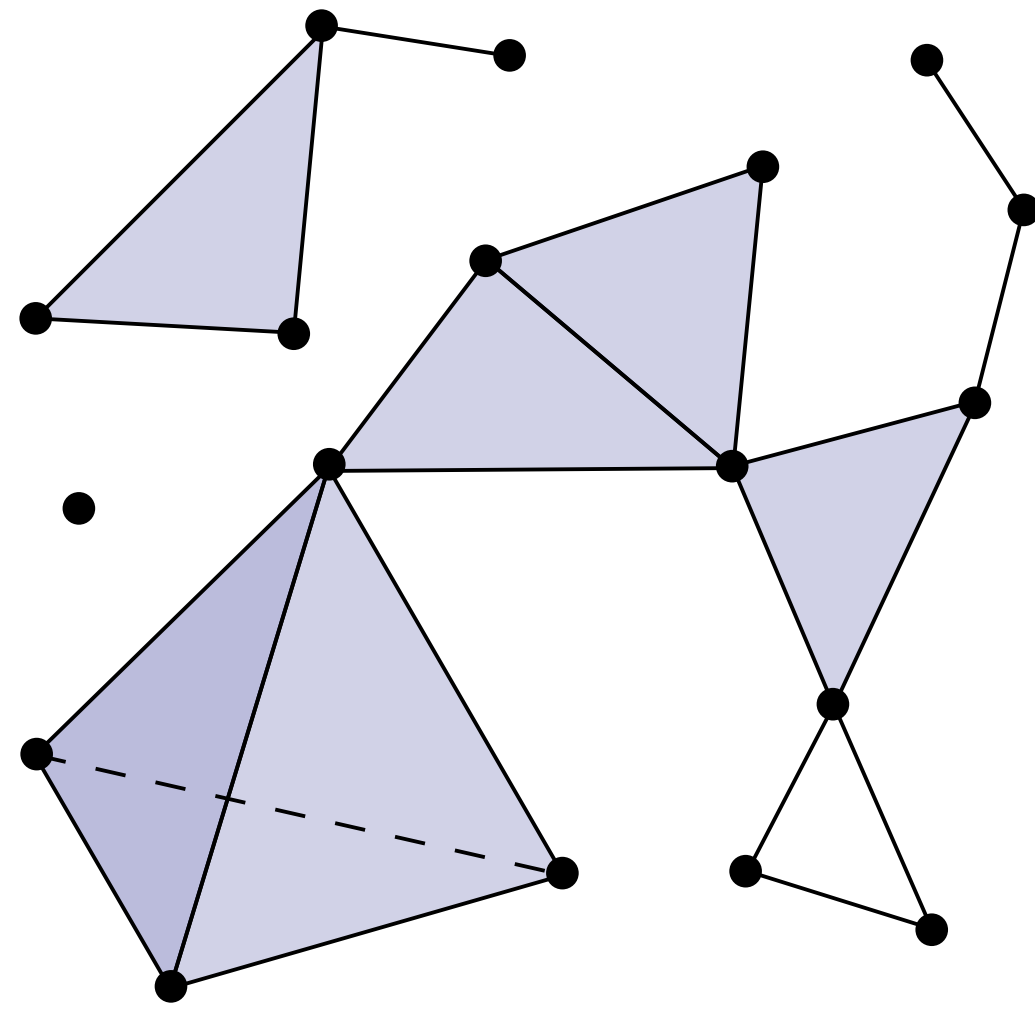
$\{0\}$ $\{1\}$ $\{2\}$ $\{3\}$ $\{4\}$ $\{5\}$ $\{6\}$ $\{7\}$ $\{8\}$ $\{9\}$ $\{10\}$

\emptyset

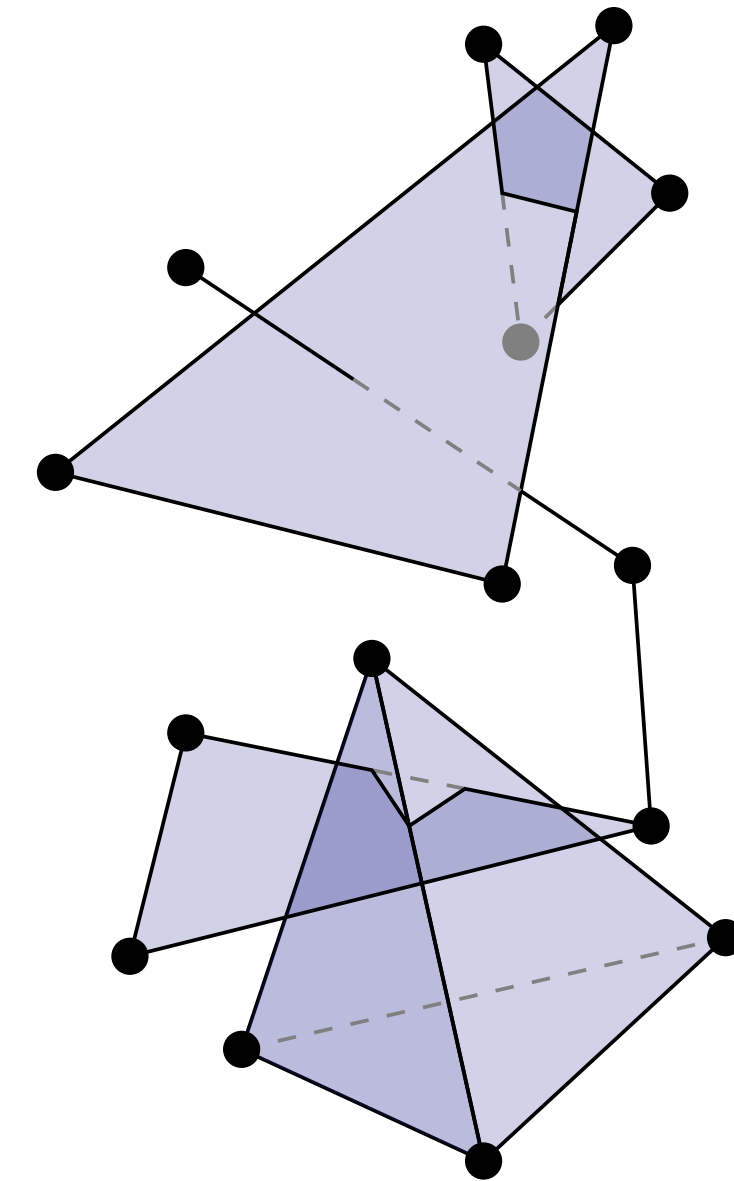
Notice: didn't really say anything about geometry here...

Abstract Simplicial Complex

Definition. Let S be a collection of sets. If for each set $\sigma \in S$ all subsets of σ are contained in S , then S is an *abstract simplicial complex*. A set $\sigma \in S$ of size $k + 1$ is an (*abstract*) *simplex*.



geometric simplicial complex



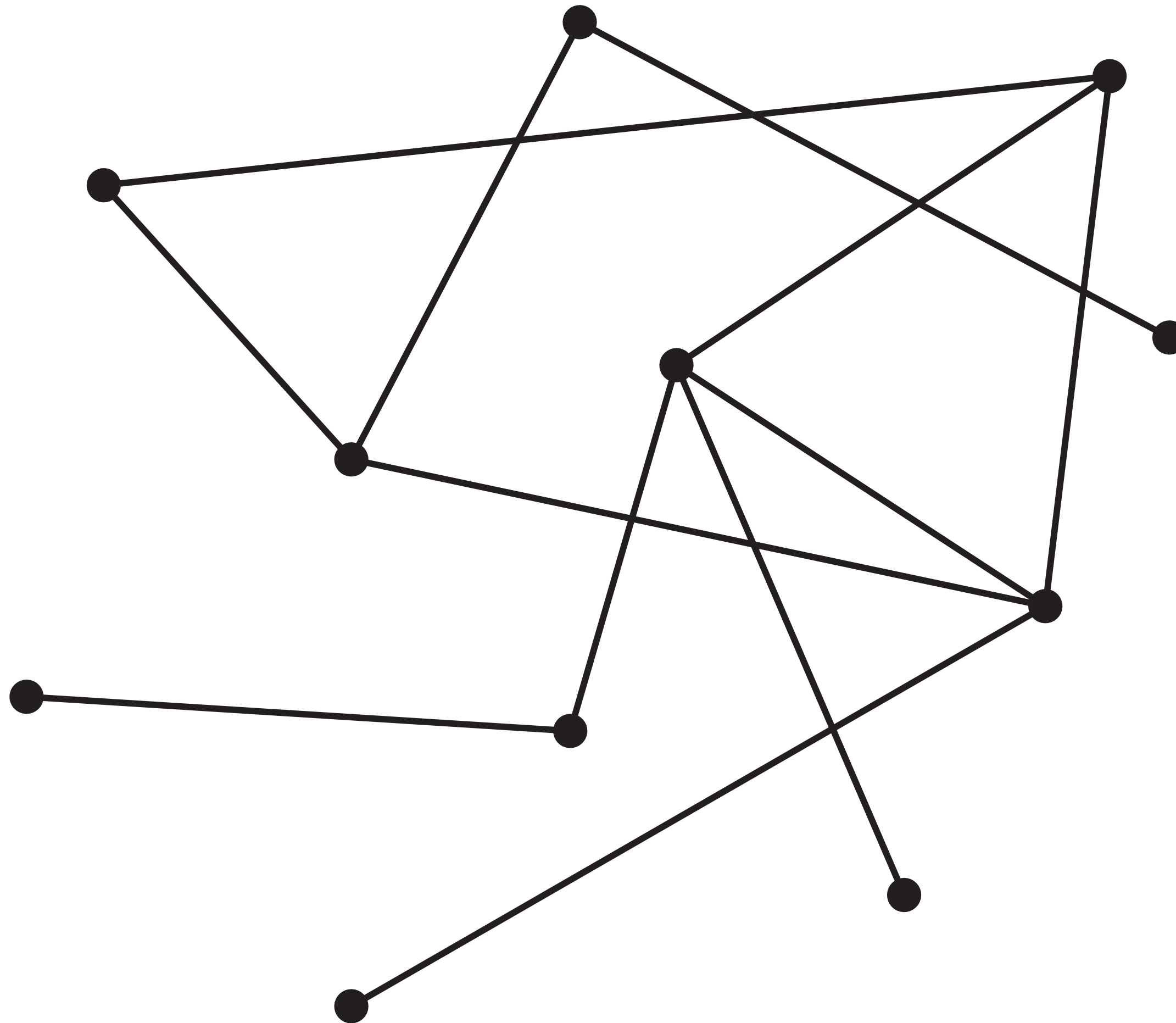
abstract simplicial complex*

- Only care about how things are *connected*, not how they are arranged geometrically.
- Provides our discrete analogue for a *topological space*

*...visualized by mapping it into R^3 .

Abstract Simplicial Complex — Graphs

- Any (*undirected*) graph $G = (V, E)$ is an abstract simplicial (1-)complex
- 0-simplices are vertices
- 1-simplices are edges



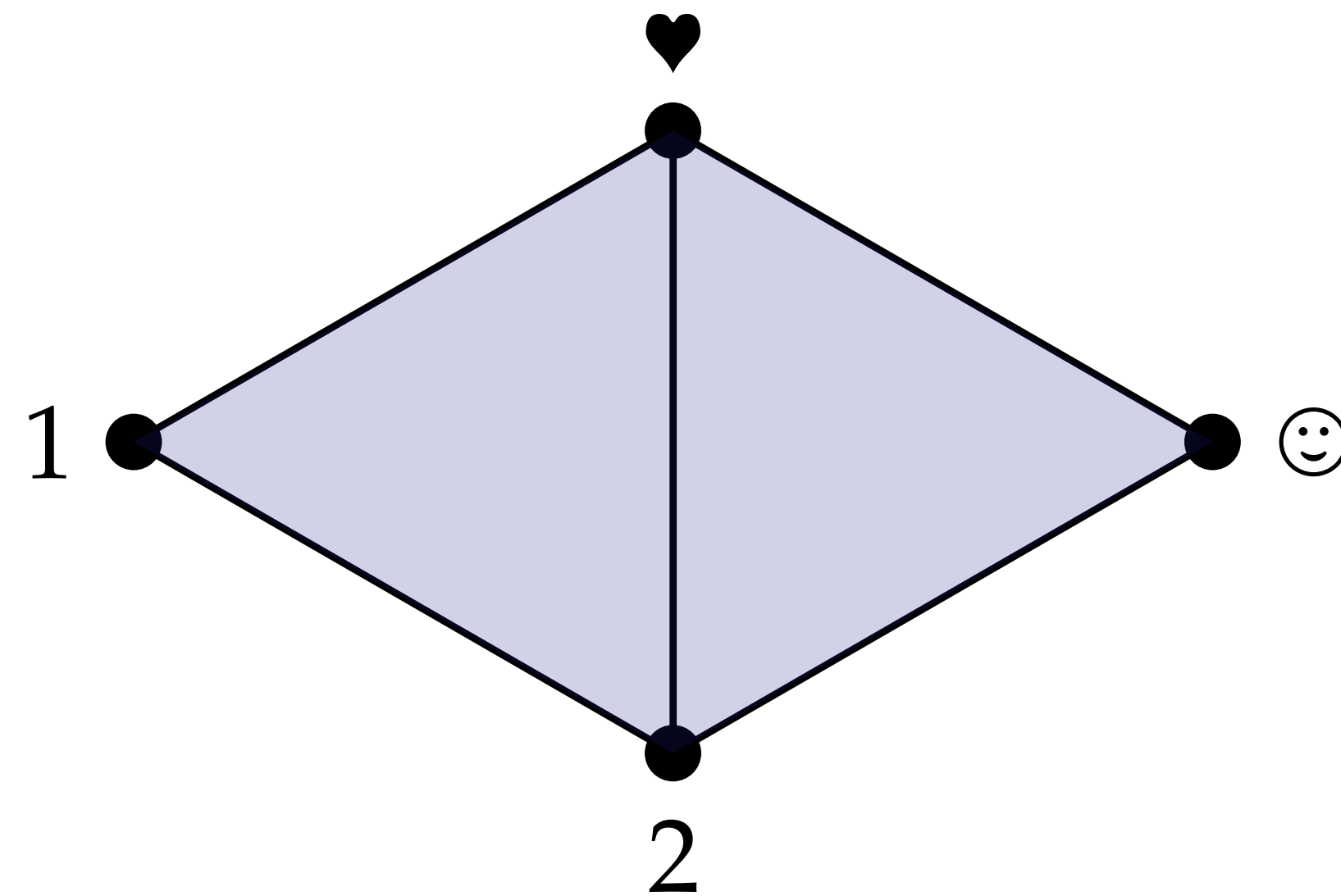
Abstract Simplicial Complex—Example

Example: Consider the set

$S := \{\{1, 2, \heartsuit\}, \{2, \heartsuit, \smile\}, \{1, 2\}, \{2, \heartsuit\}, \{\heartsuit, 1\}, \{2, \smile\}, \{\heartsuit, \smile\}, \{1\}, \{2\}, \{\heartsuit\}, \{\smile\}, \emptyset\}$

Q: Is this set an abstract simplicial complex? If so, what does it look like?

A: Yes—it's a pair of 2-simplices (triangles) sharing a single edge:



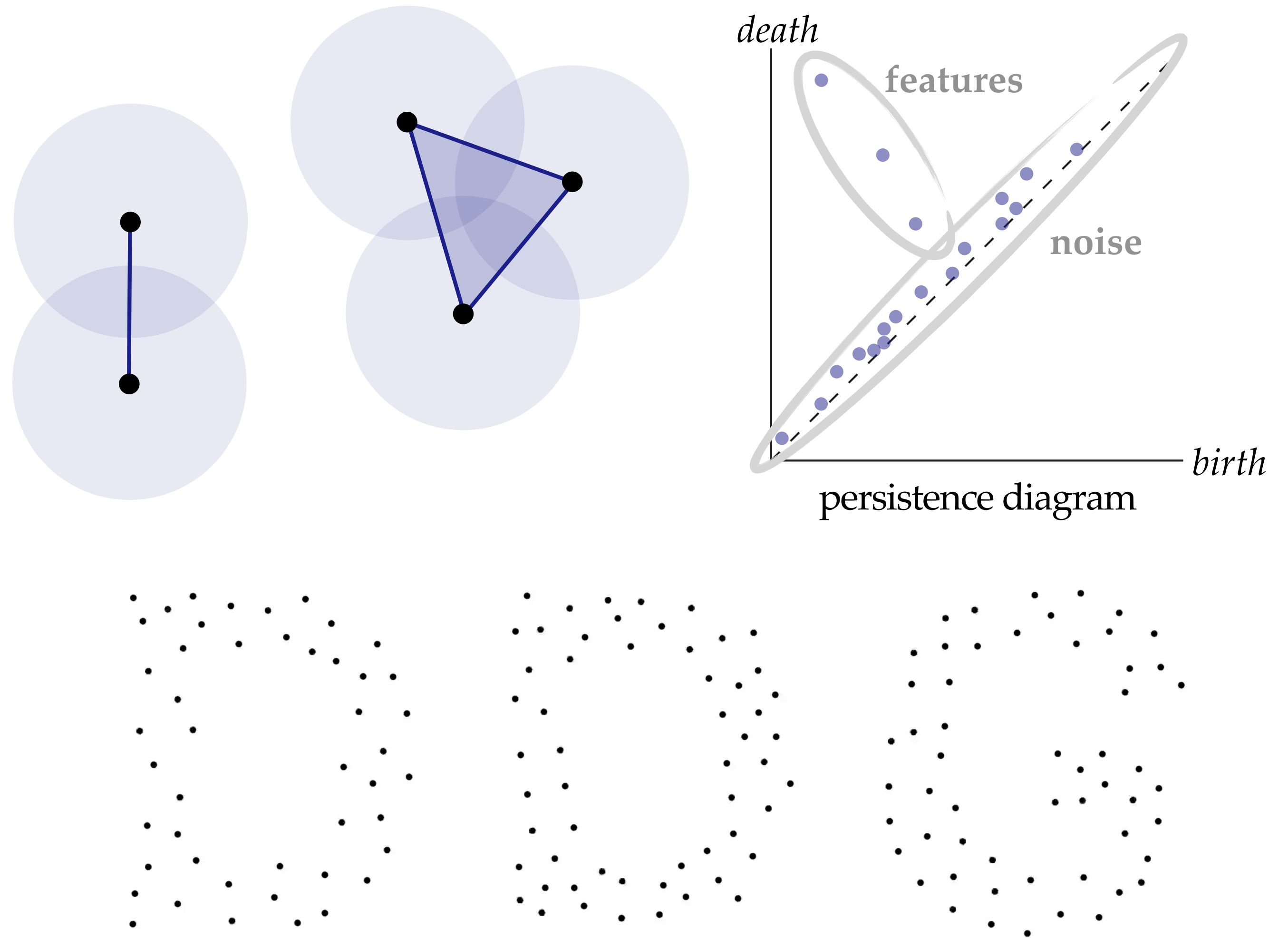
Vertices no longer have to be points in space; can represent anything at all.

Application: Topological Data Analysis

Forget (mostly) about geometry—try to understand data in terms of *connectivity*.

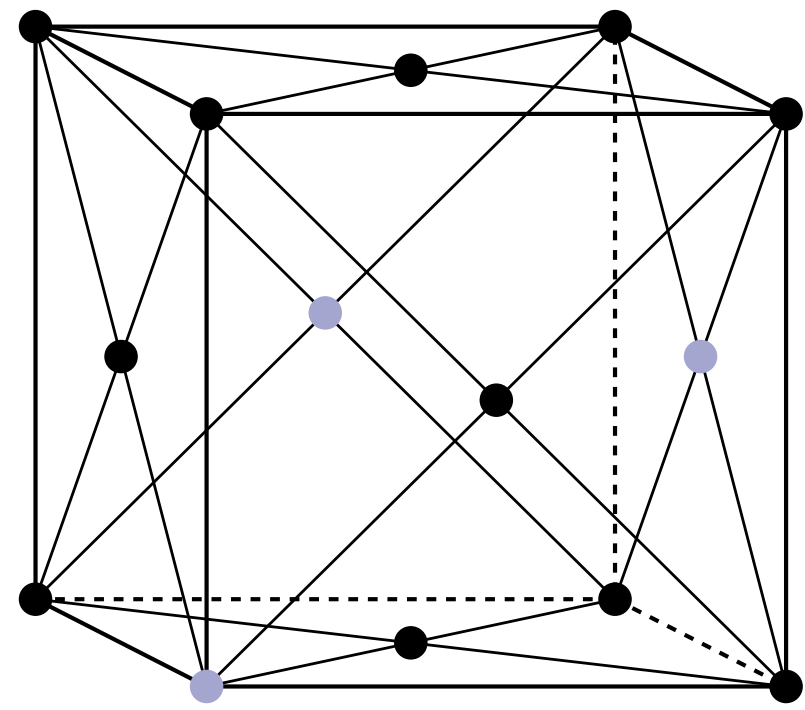
E.g., *persistent homology*:

- “grow” balls around each point
- connect $(k+1)$ overlapping balls into k -simplex
- track “birth” and “death” of features like connected components, holes, etc.
- features that persist for a long time are likely “real”
- features that quickly appear, then disappear are likely noise

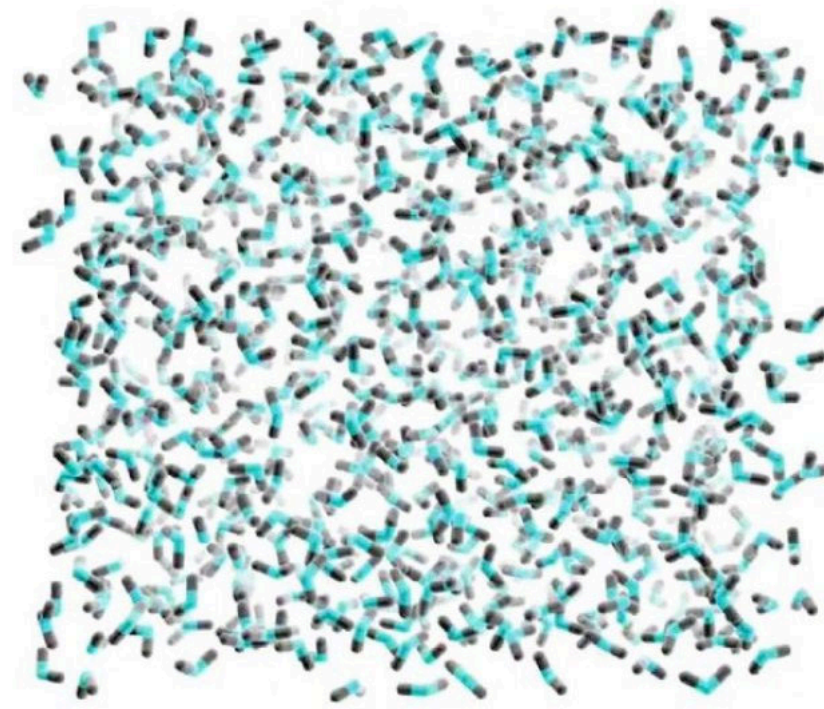


Example: Material Characterization via Persistence

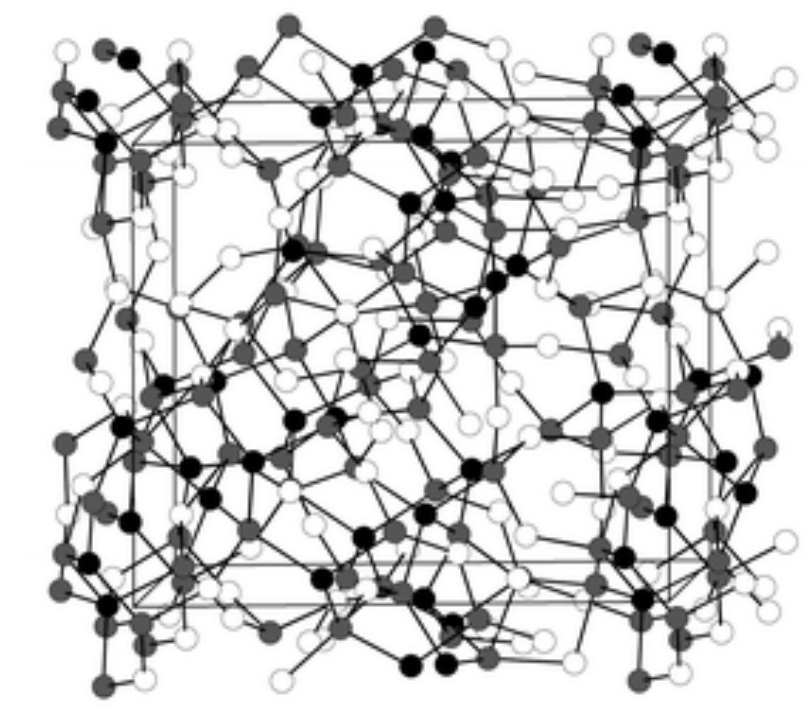
Regular



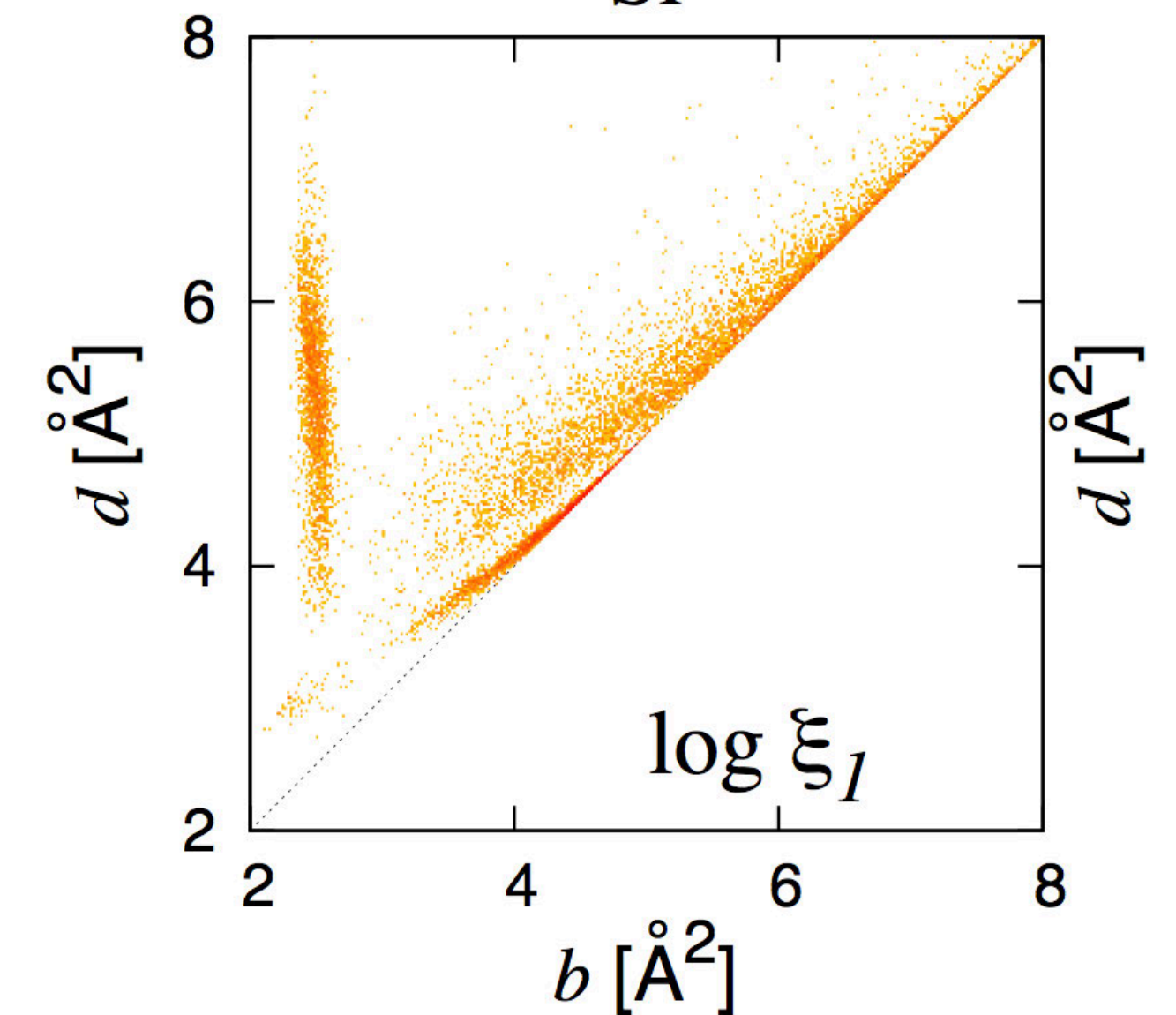
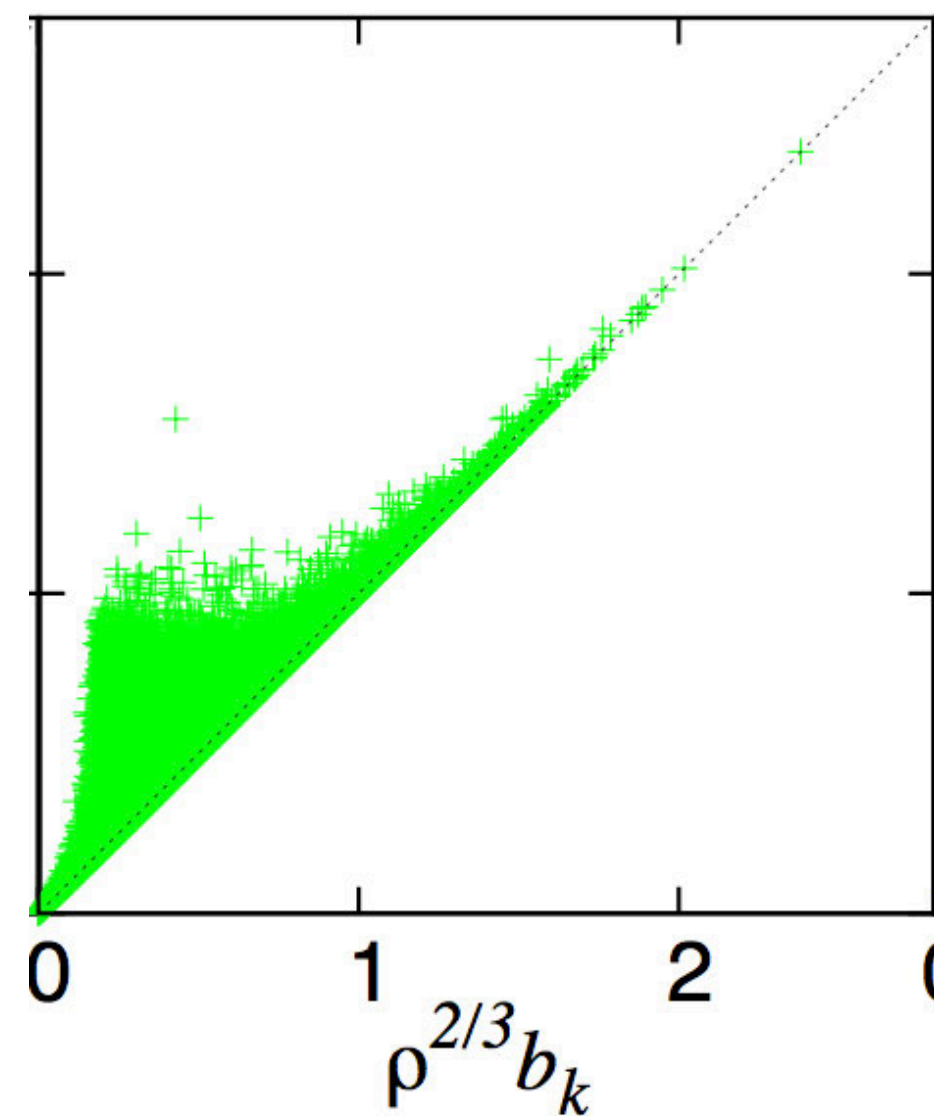
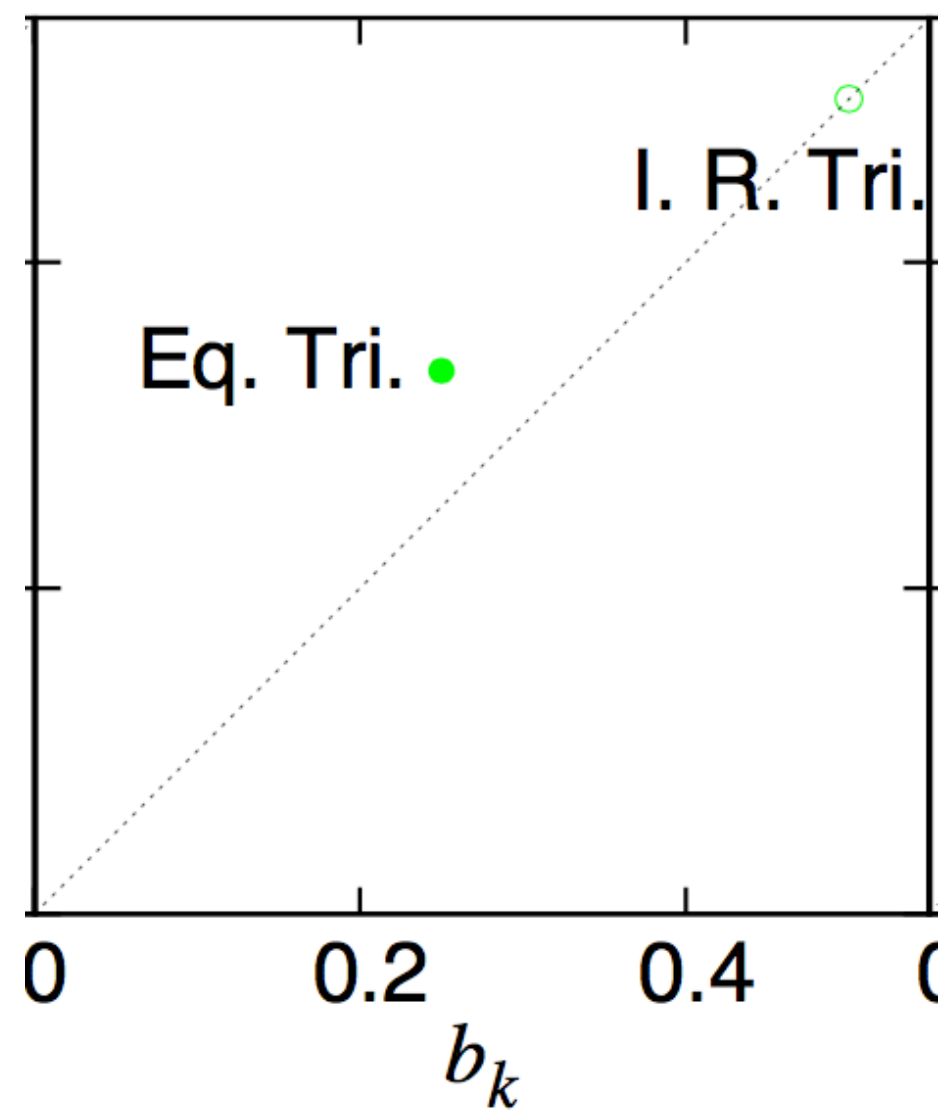
Random



Glass

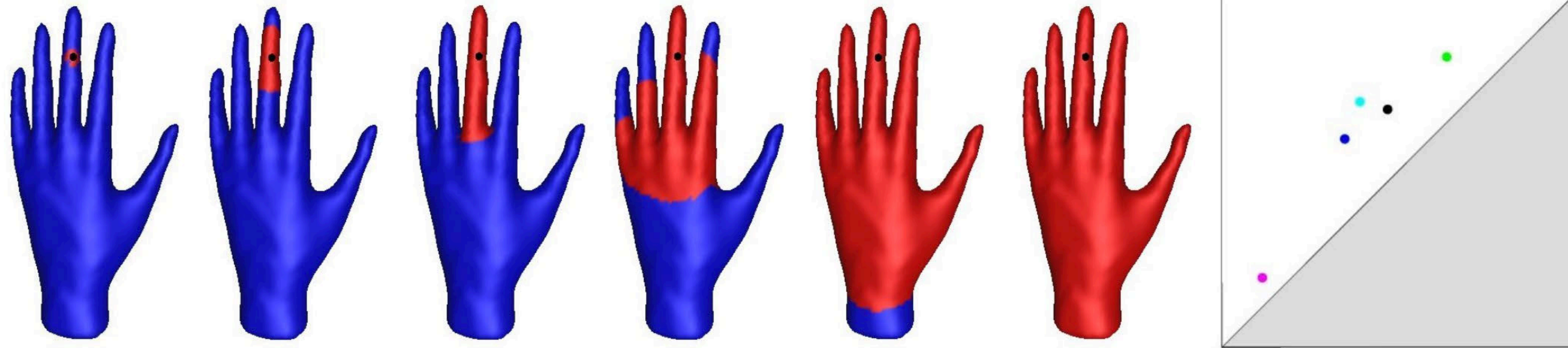


Si

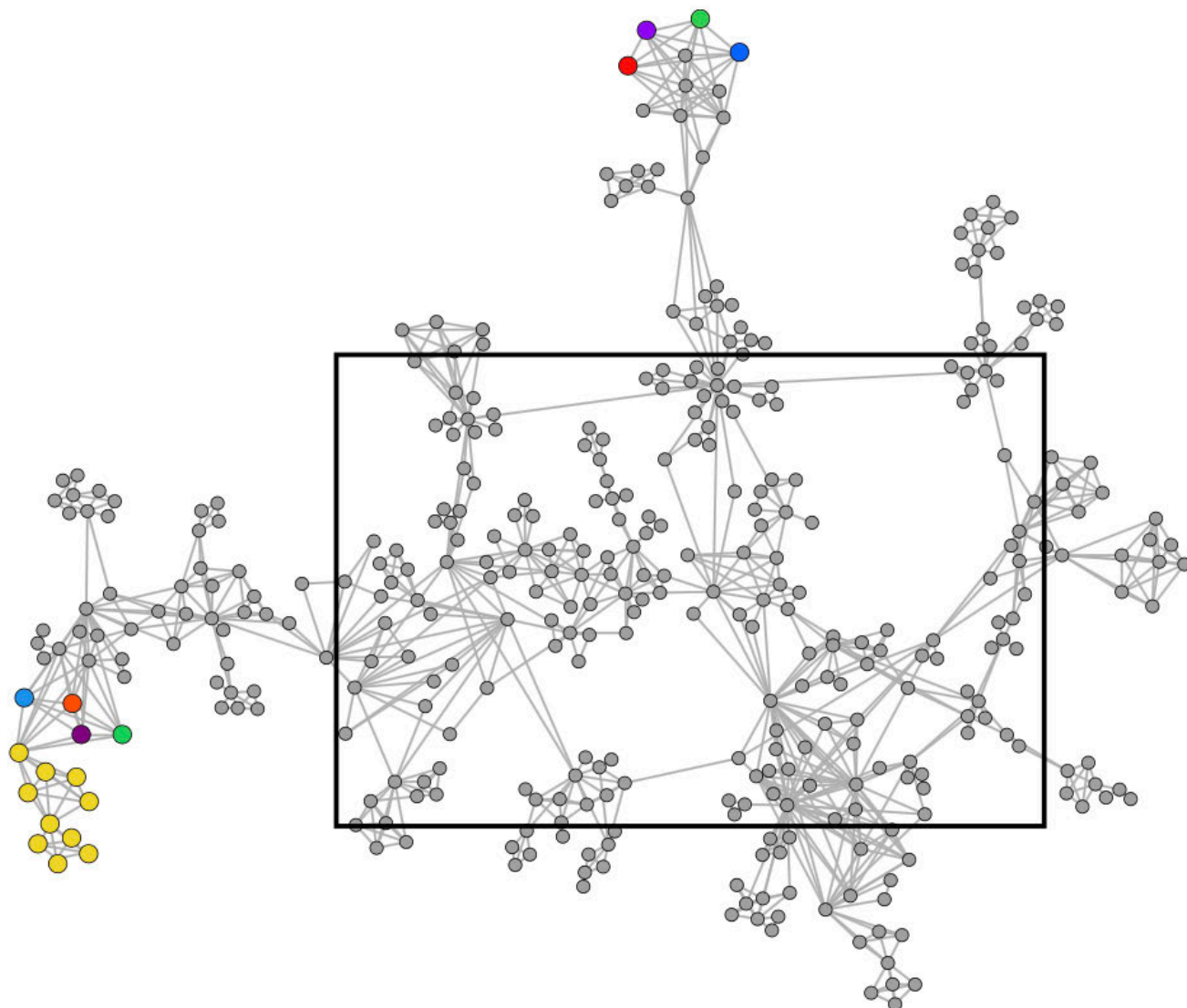


Persistent Homology—More Applications

M. Carrière, S. Oudot, M. Ovsjanikov, “Stable Topological Signatures for Points on 3D Shapes”



C. Carstens, K. Horadam,
“Persistent Homology of Collaboration Networks ”



H. Lee, M. Chung, H. Kang, B. Kim, D. Lee
“Discriminative Persistent Homology of Brain Networks”

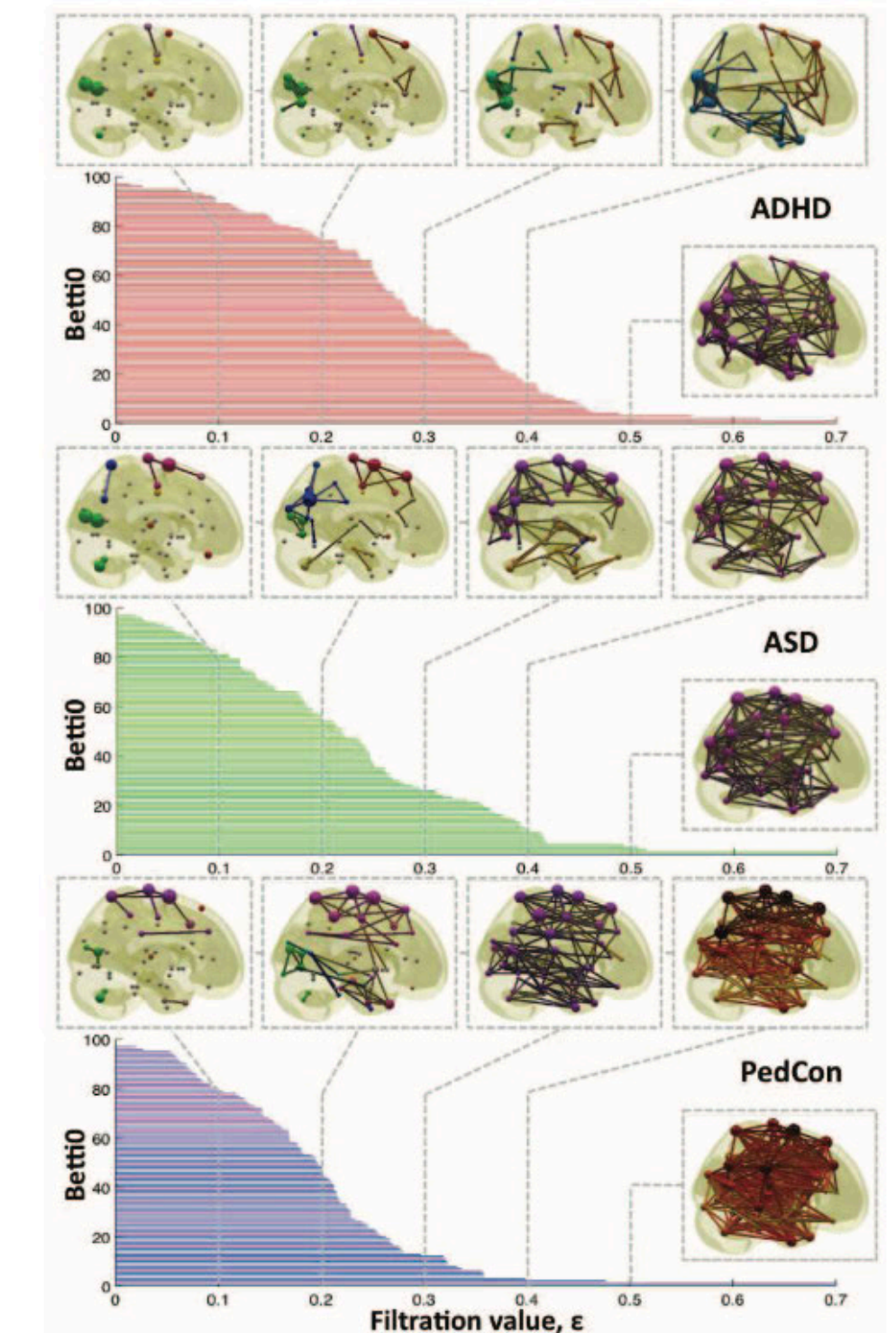
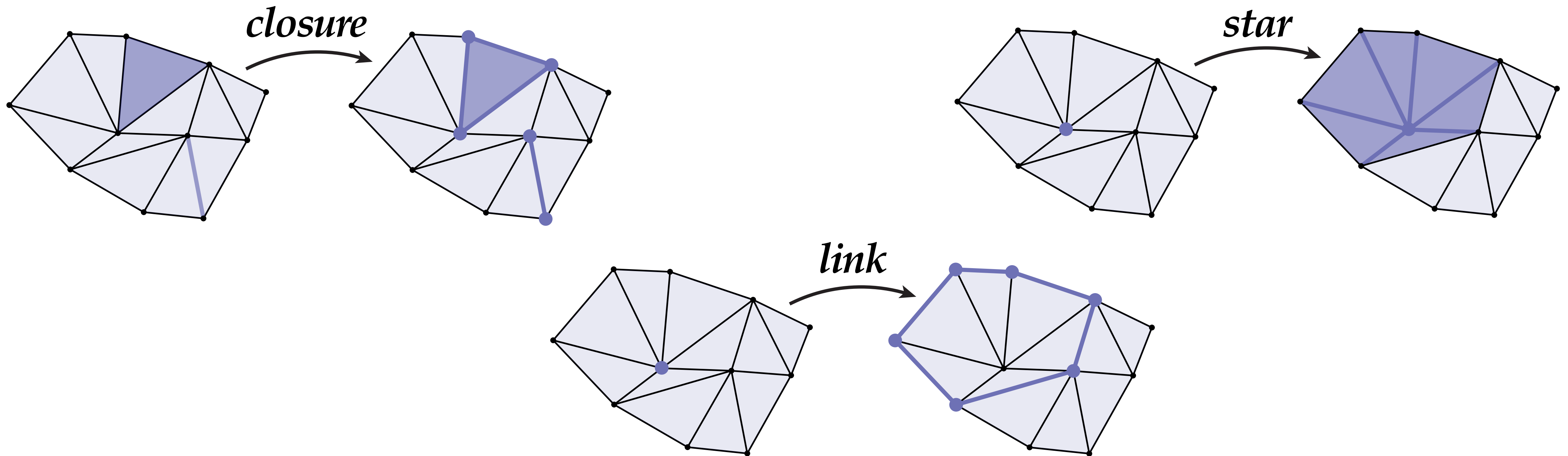


Fig. 4. Barcode of the 0-th Betti number.

...and much more (identifying patients with breast cancer, classifying players in basketball, new ways to compress images, etc.)

Anatomy of a Simplicial Complex

- **Closure:** smallest simplicial complex containing a given set of simplices
- **Star:** union of simplices containing a given subset of simplices
- **Link:** closure of the star minus the star of the closure



Vertices, Edges, and Faces

Some notation:

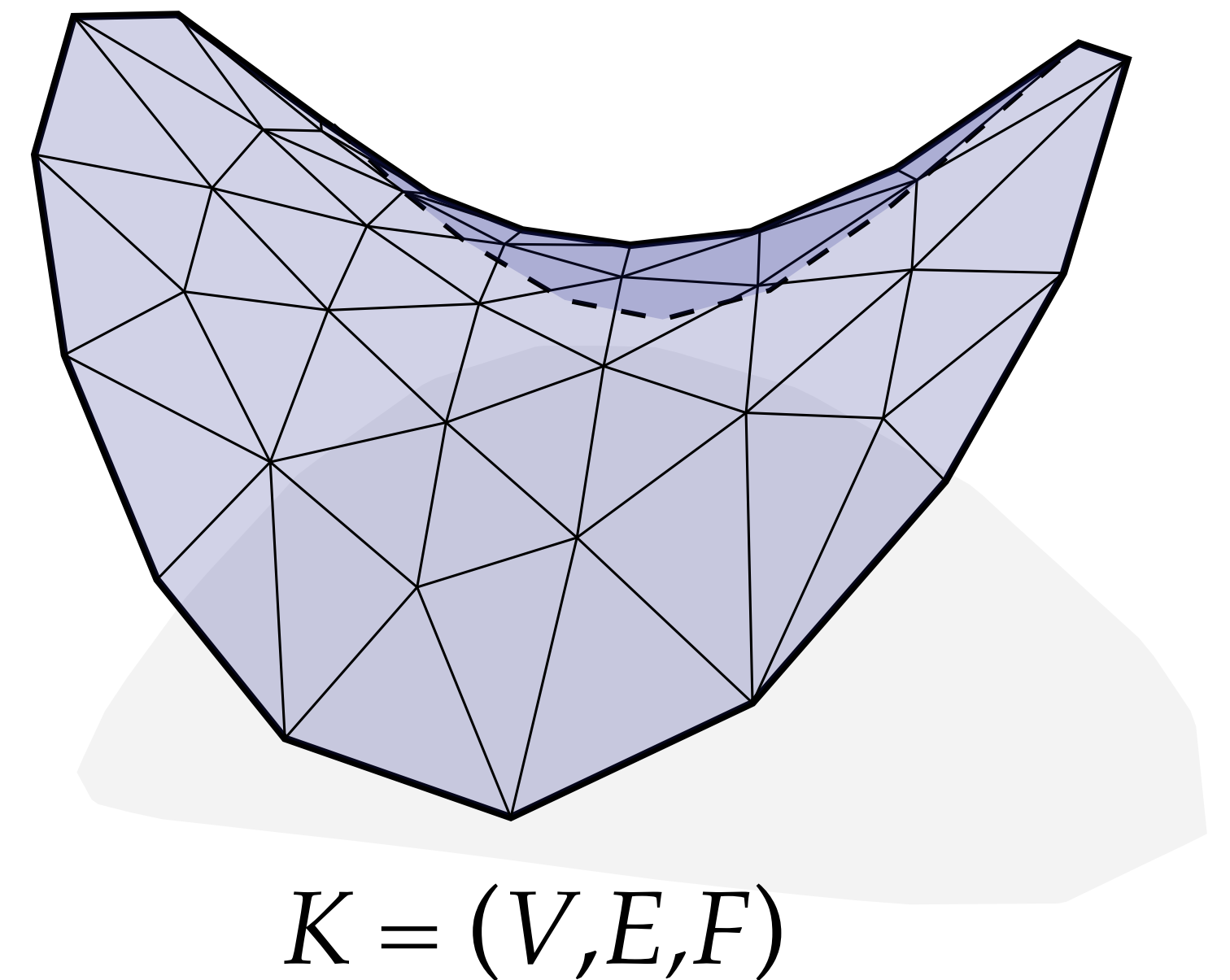
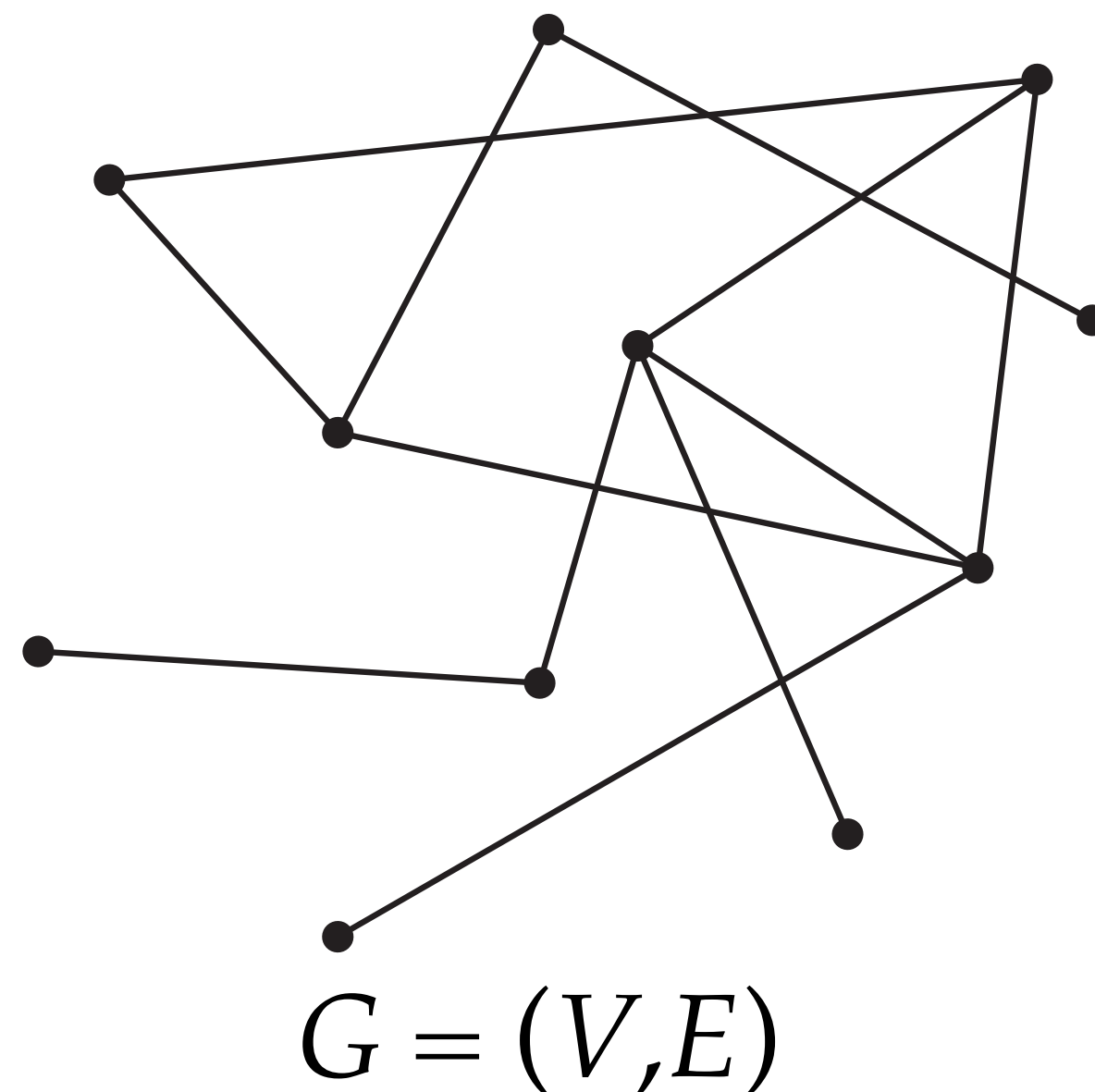
- For simplicial **1-complexes** (graphs) we often write $G = (V, E)$
- For simplicial **2-complexes** (triangle meshes) we often write $K = (V, E, F)$

– V = vertices

– E = edges

– F = faces*

– (K is for “*komplex!*”)

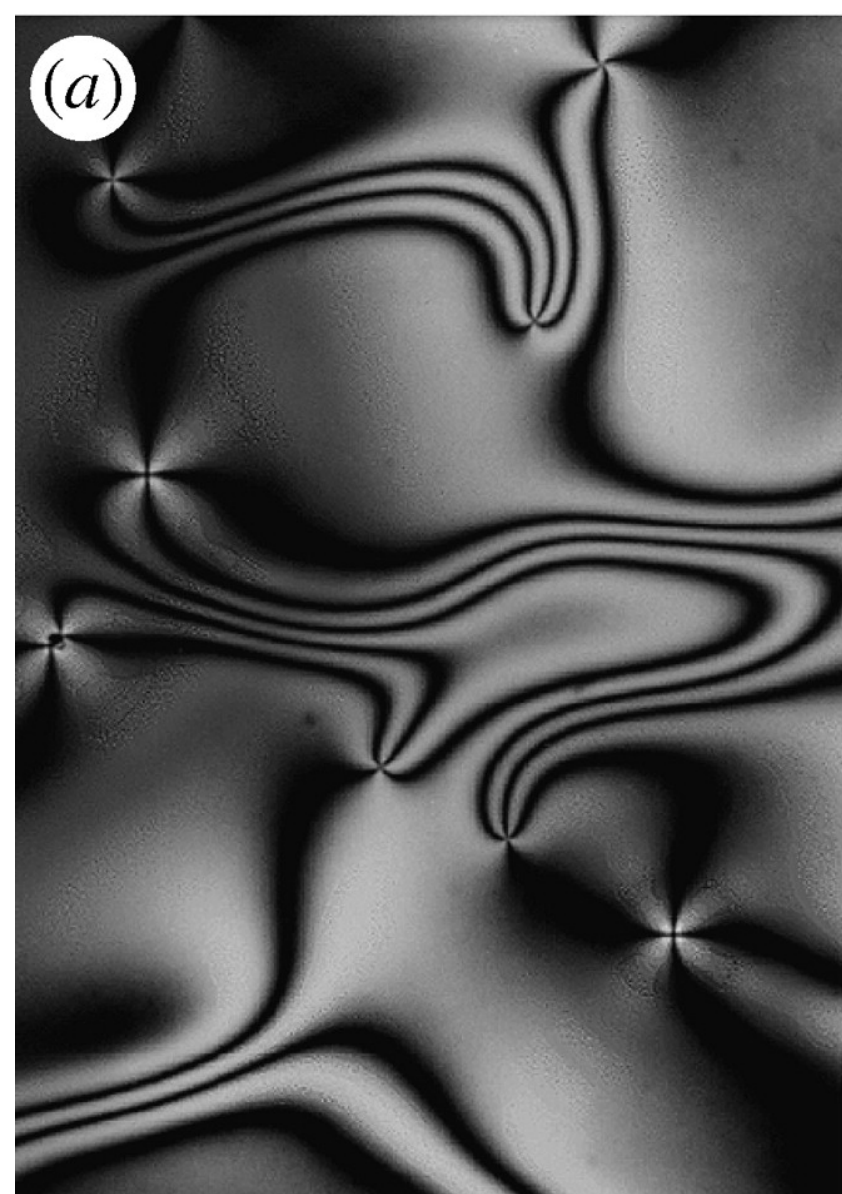
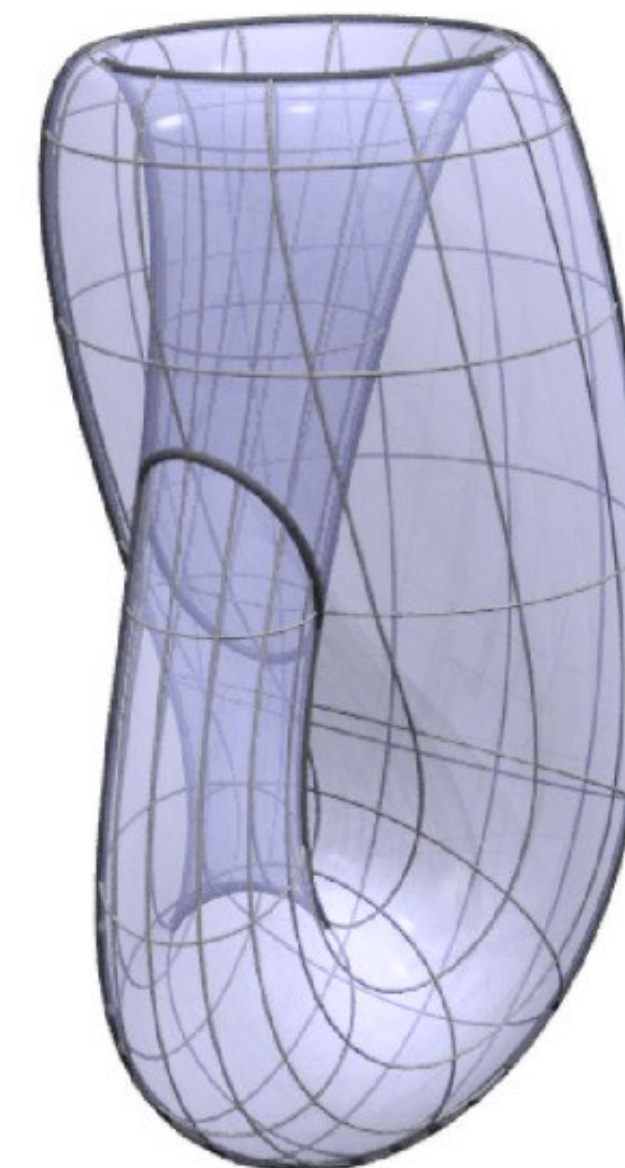
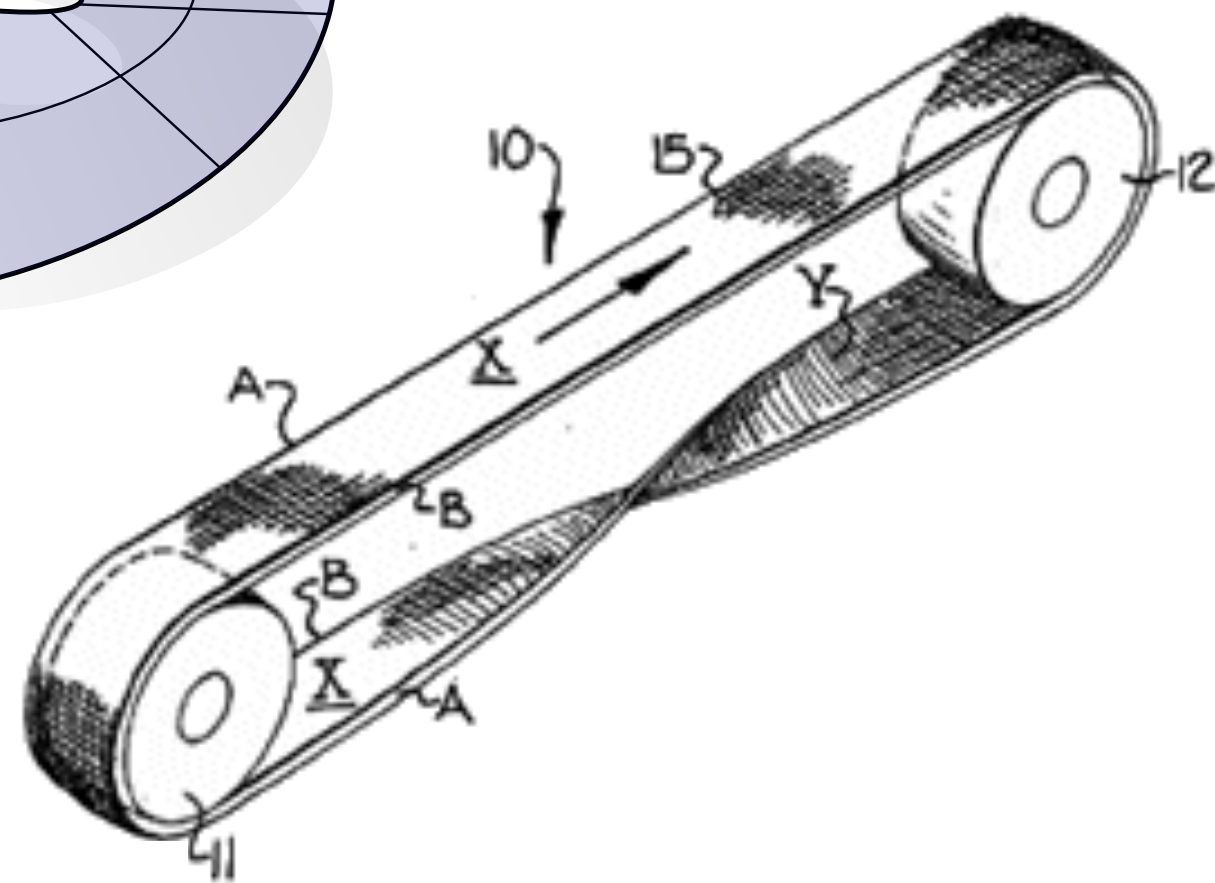
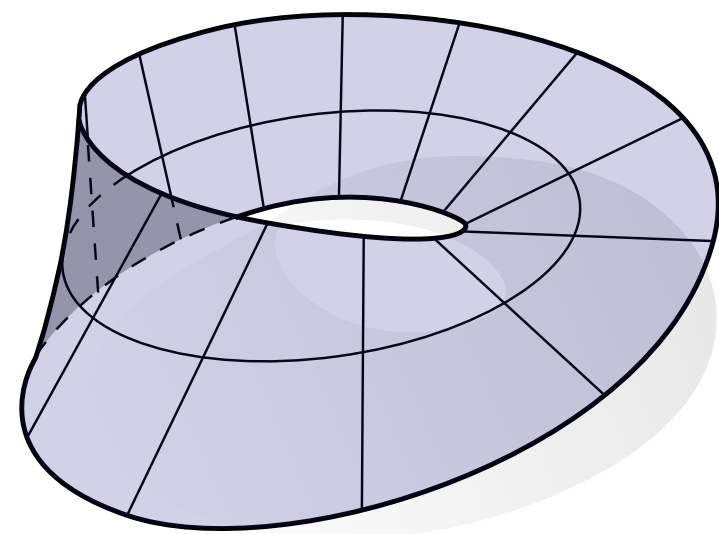
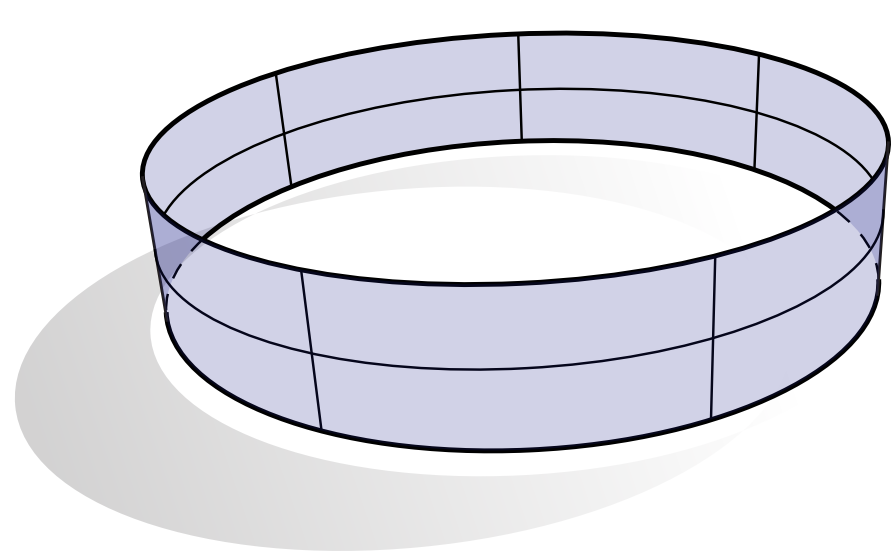


*Not to be confused with the generic *face* of a simplex...

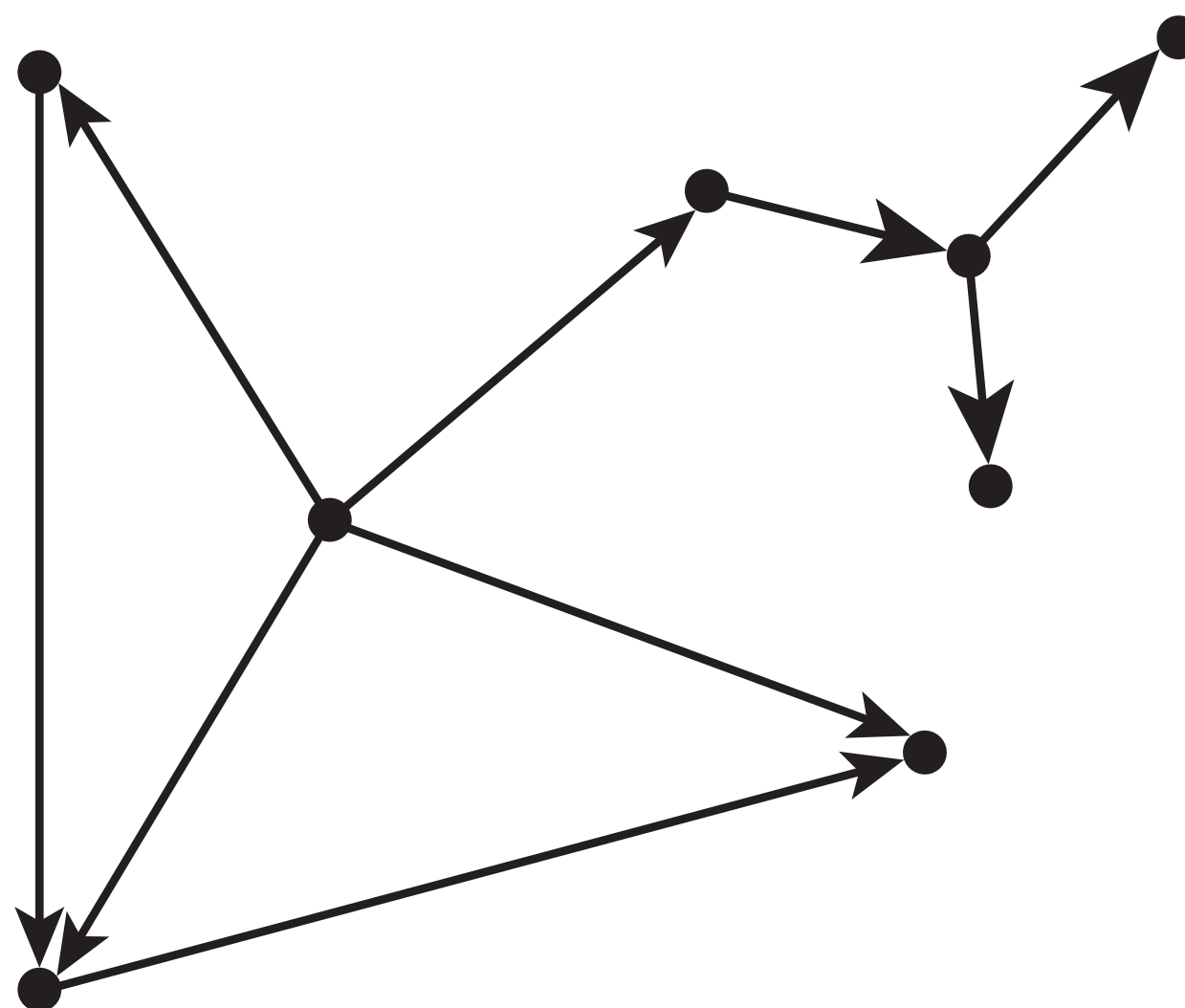


Oriented Simplicial Complex

Orientation—Visualized

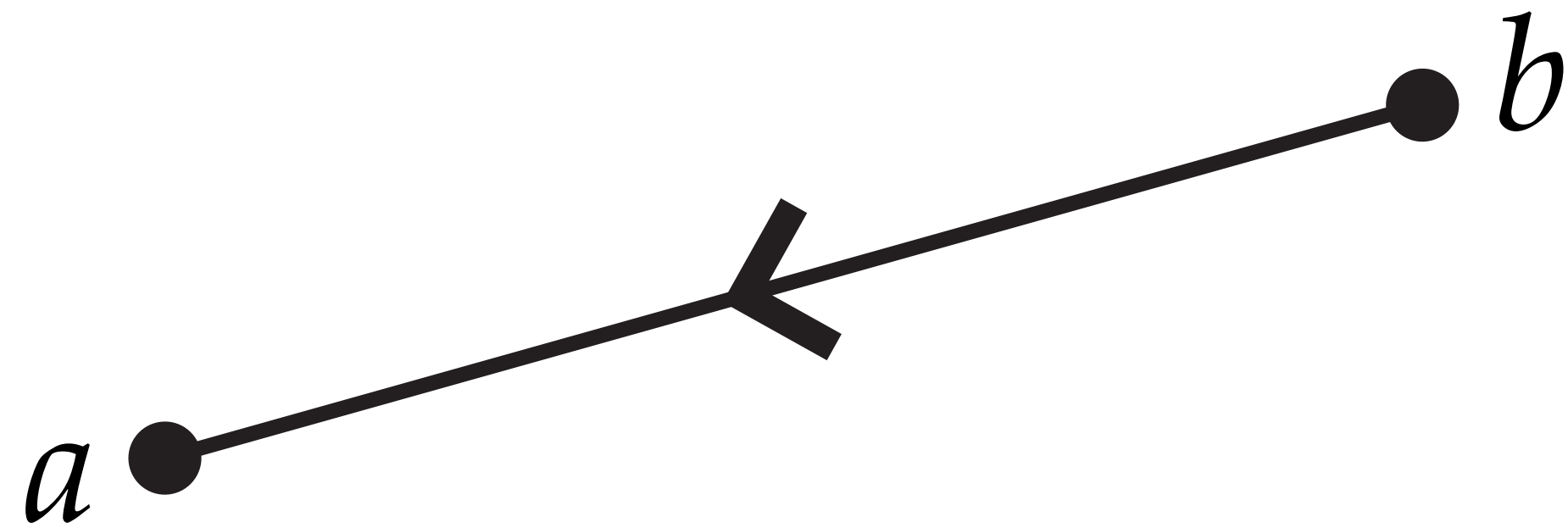
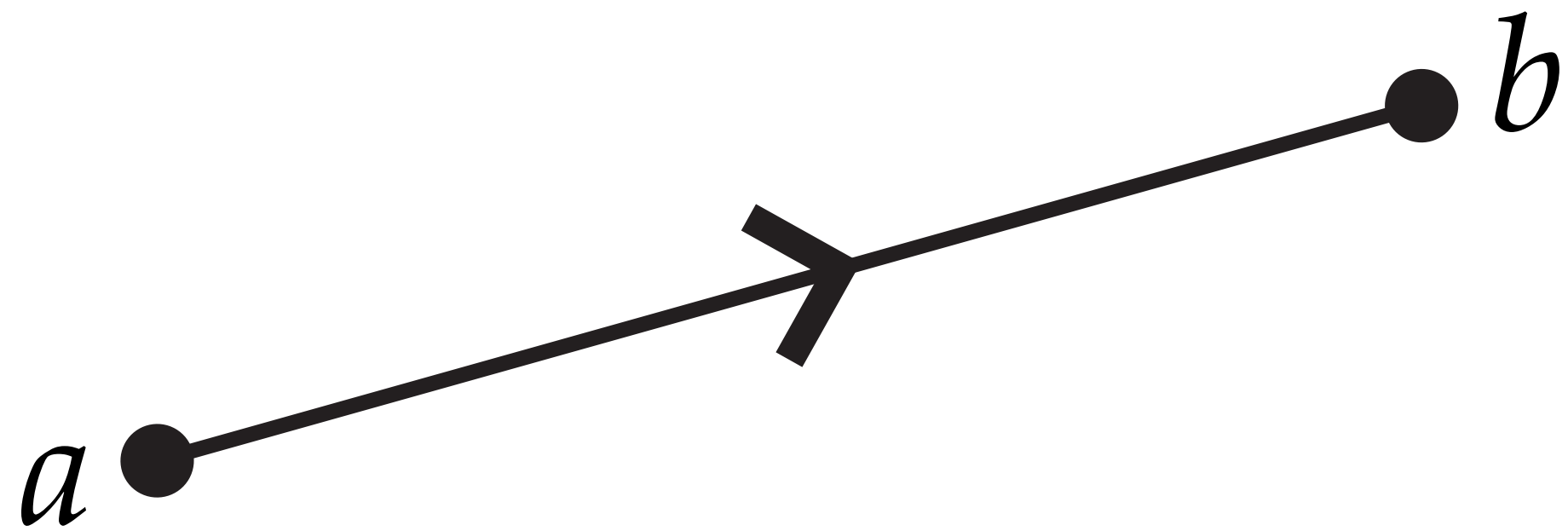


(b)



Orientation of a 1-Simplex

- Basic idea: does a 1-simplex $\{a,b\}$ go from a to b or from b to a ?
- Instead of unordered set $\{a,b\}$, now have *ordered tuple* (a,b) or (b,a)

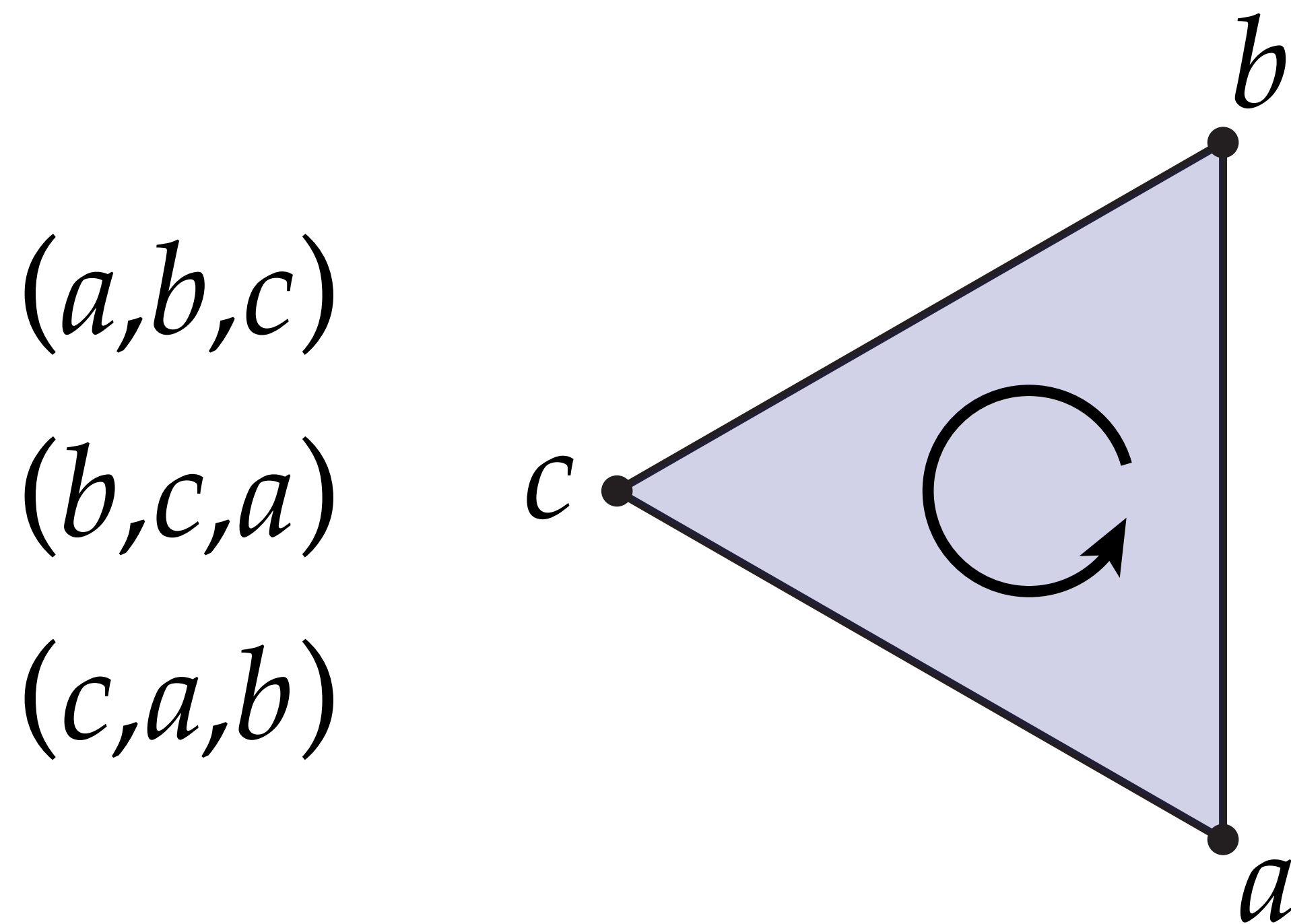


- Why do we care? *Eventually* will be useful for recording integrals...

$$\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx$$

Orientation of a 2-Simplex

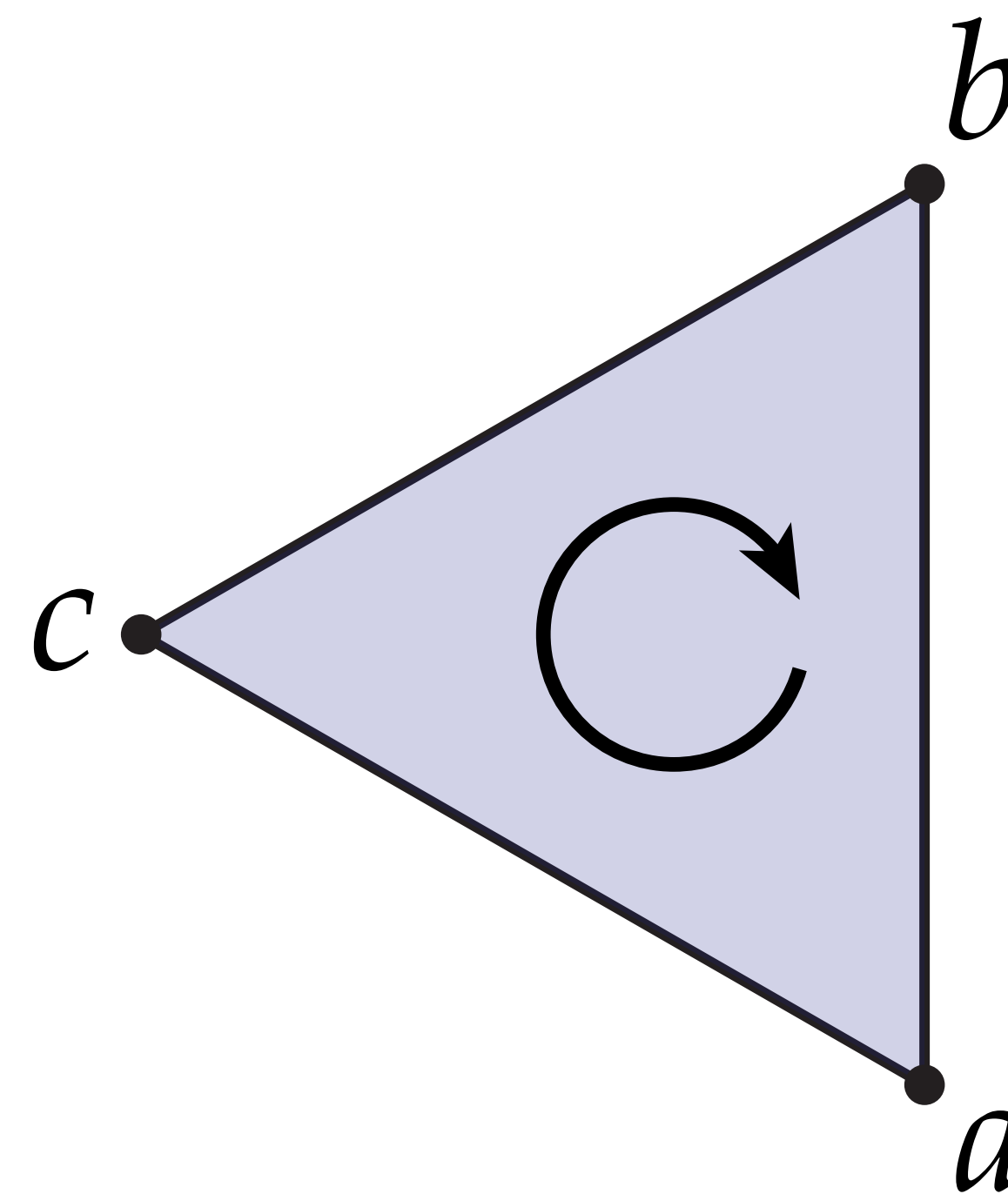
For a 2-simplex, orientation given by “winding order” of vertices:



(a,b,c)

(b,c,a)

(c,a,b)



(a,c,b)

(c,b,a)

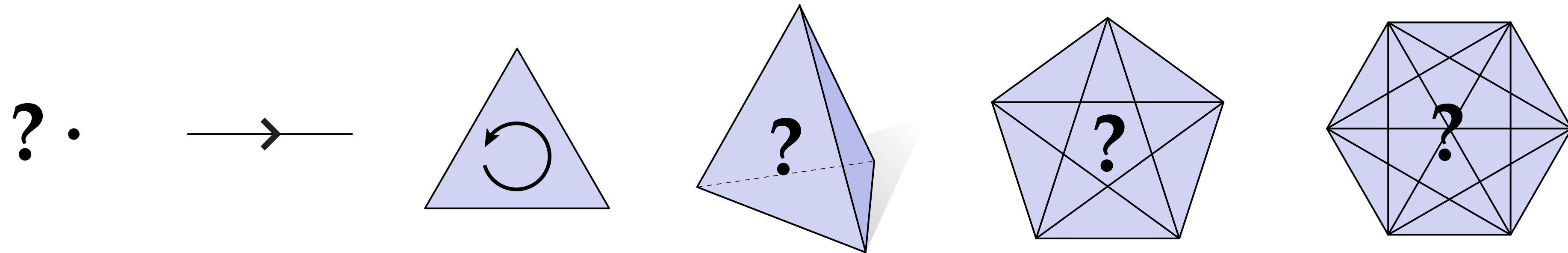
(b,a,c)

Hence, an *oriented 2-simplex* can be specified by a 3-tuple.

(Circular shifts describe the same oriented 2-simplex.)

Oriented k -Simplex

How do we define orientation in general? (Hard to draw arrows...)



Similar idea to orientation for 2-simplex:

Definition. An oriented k -simplex is an ordered tuple, up to even permutation.

Hence, always* two orientations: *even* or *odd* permutations of vertices.
Convention: even permutations of $(0, \dots, k)$ “**positive**”; otherwise “**negative**.”

Oriented 0-Simplex?

Q: What's the orientation of a single vertex?



A: Only one permutation of vertices; only one orientation! (Positive):

(a)

Oriented 3-Simplex

Hard to draw pictures as k gets large!

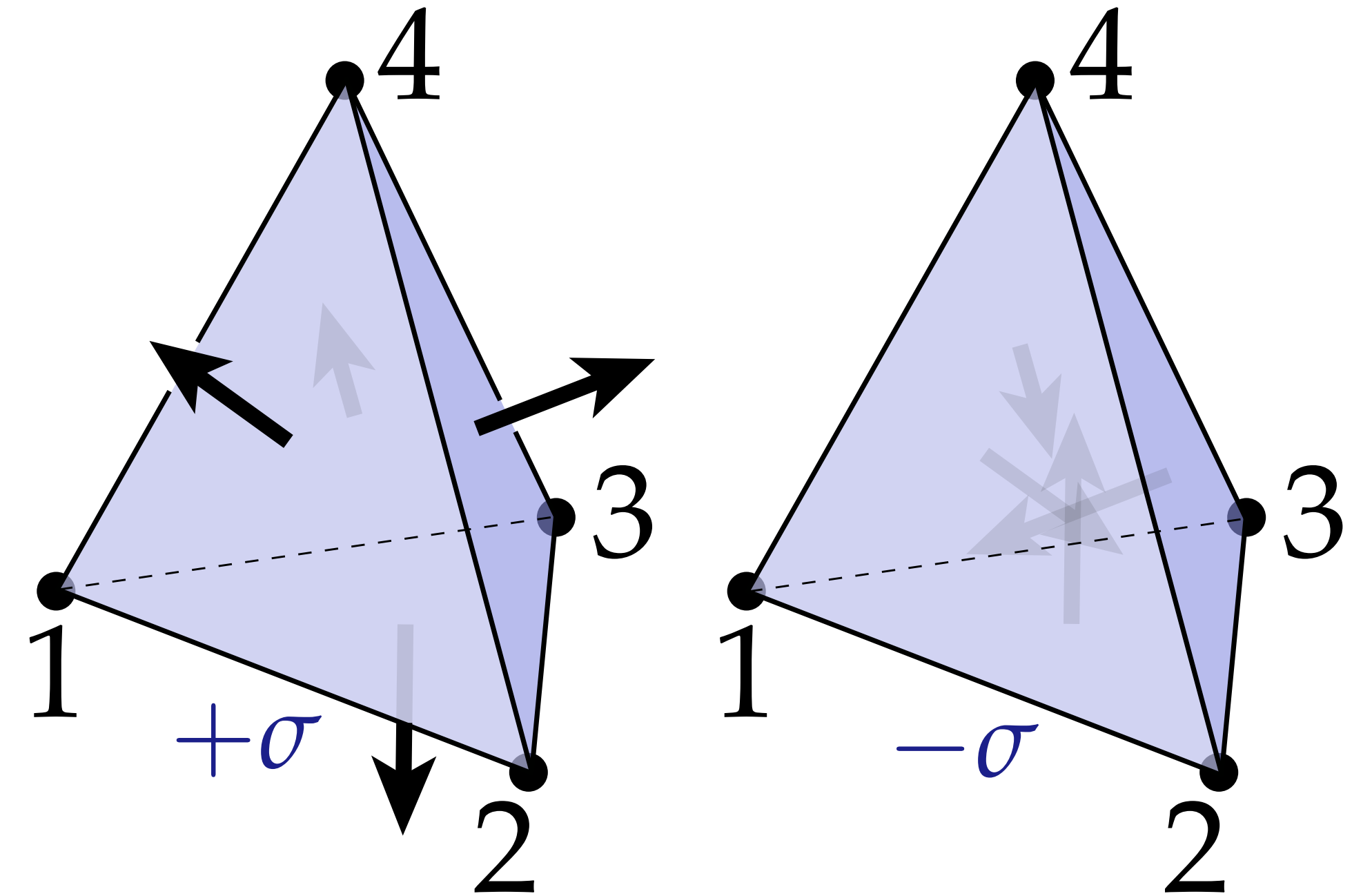
But still easy to apply definition:

even / positive

| | |
|----------------|----------------|
| $(1, 2, 3, 4)$ | $(3, 1, 2, 4)$ |
| $(1, 3, 4, 2)$ | $(3, 2, 4, 1)$ |
| $(1, 4, 2, 3)$ | $(3, 4, 1, 2)$ |
| $(2, 1, 4, 3)$ | $(4, 1, 3, 2)$ |
| $(2, 3, 1, 4)$ | $(4, 2, 1, 3)$ |
| $(2, 4, 3, 1)$ | $(4, 3, 2, 1)$ |

odd / negative

| | |
|----------------|----------------|
| $(1, 2, 4, 3)$ | $(3, 1, 4, 2)$ |
| $(1, 3, 2, 4)$ | $(3, 2, 1, 4)$ |
| $(1, 4, 3, 2)$ | $(3, 4, 2, 1)$ |
| $(2, 1, 3, 4)$ | $(4, 1, 2, 3)$ |
| $(2, 3, 4, 1)$ | $(4, 2, 3, 1)$ |
| $(2, 4, 1, 3)$ | $(4, 3, 1, 2)$ |



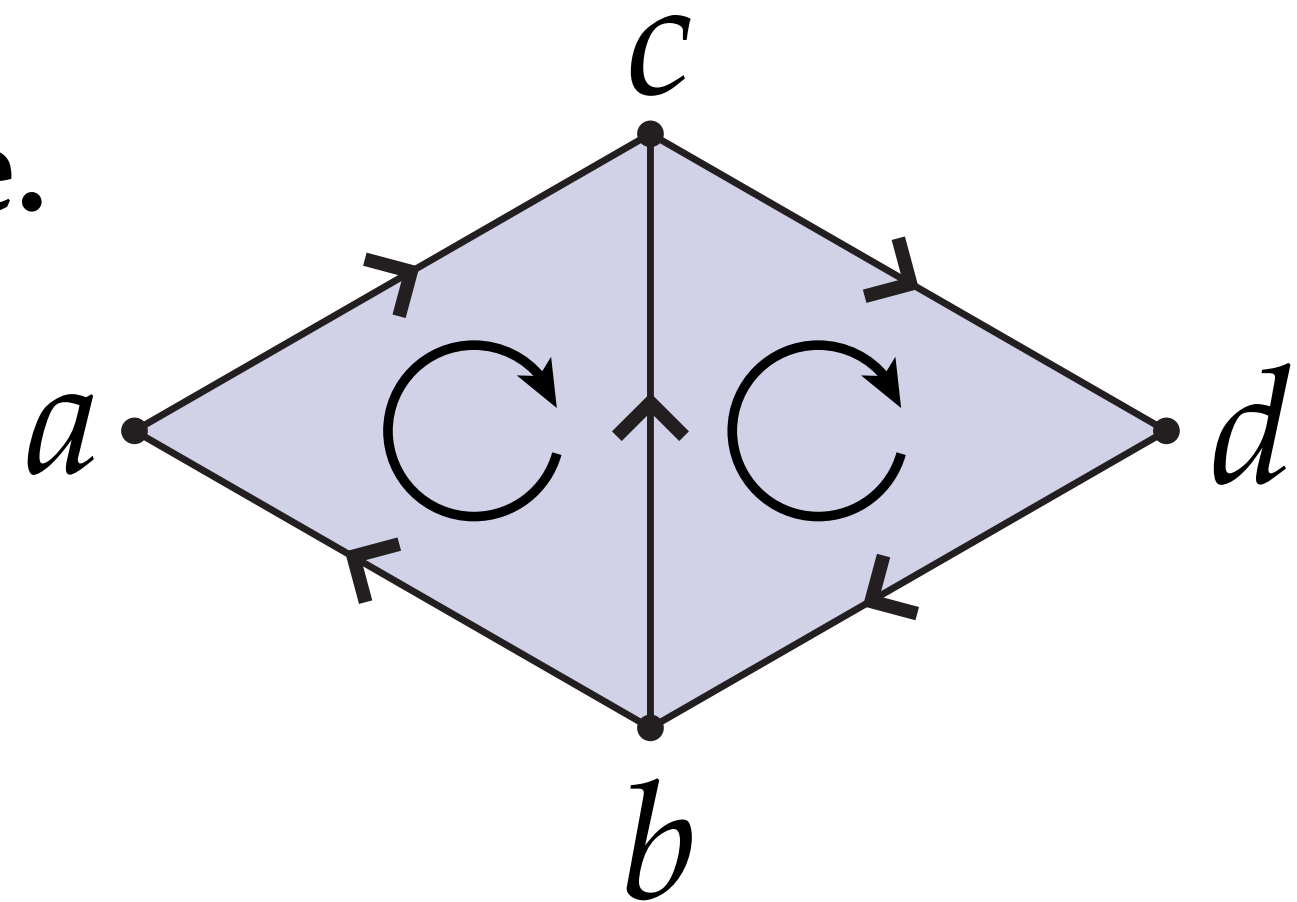
...much easier, of course, to just pick a single representative.

E.g., $+\sigma := (1, 2, 3, 4)$, and $-\sigma := (1, 2, 4, 3)$.

Oriented Simplicial Complex

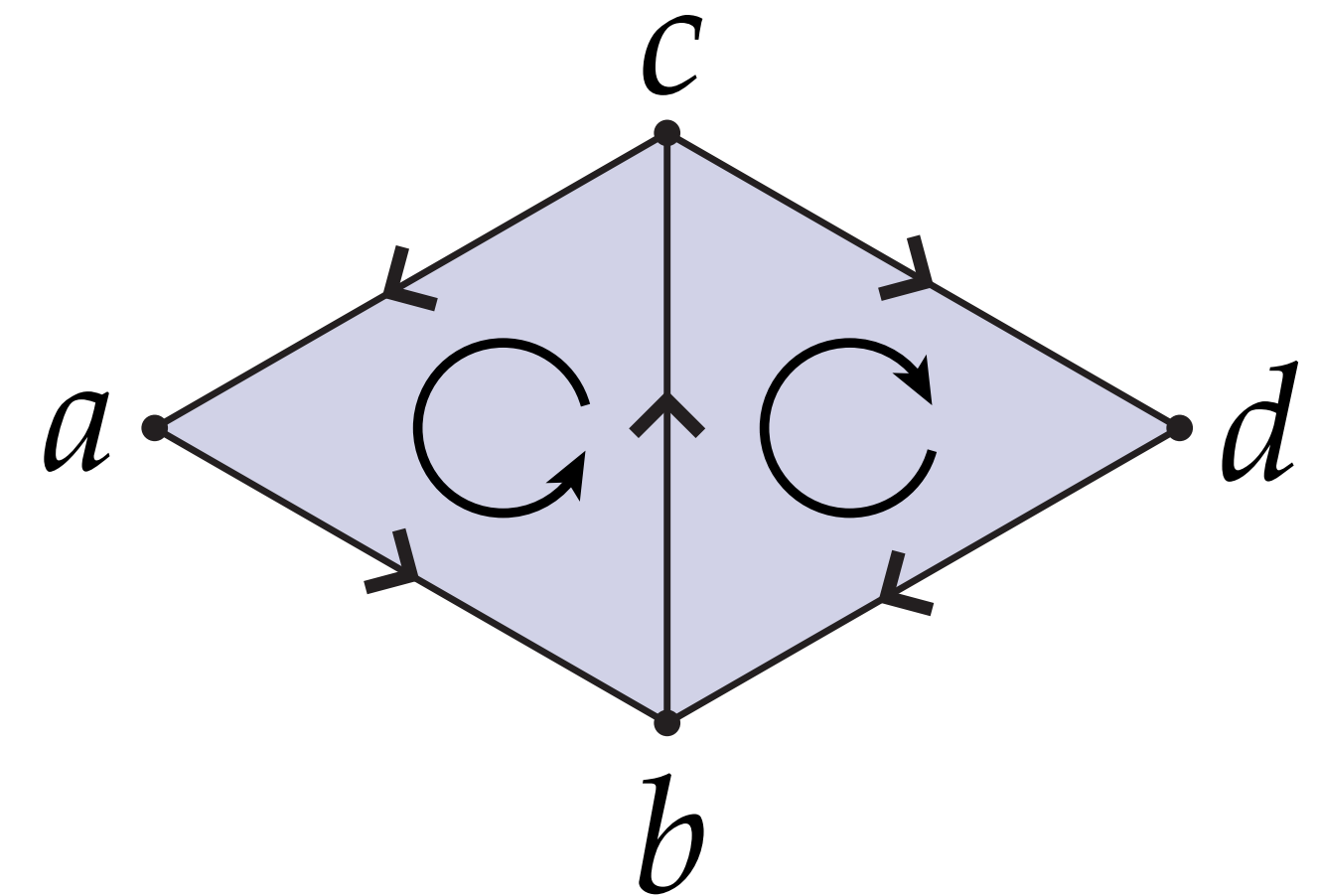
Definition. An *orientation* of a simplex is an ordering of its vertices up to even permutation; one can specify an oriented simplex via one of its representative ordered tuples. An *oriented simplicial complex* is a simplicial complex where each simplex is given an ordering.

Example.



$$\{\emptyset, (a), (b), (c), (d),$$

$$(a, c), (b, a), (b, c), (c, d), (d, b),$$

$$(a, c, b), (b, c, d)\}$$


$$\{\emptyset, (a), (b), (c), (d),$$

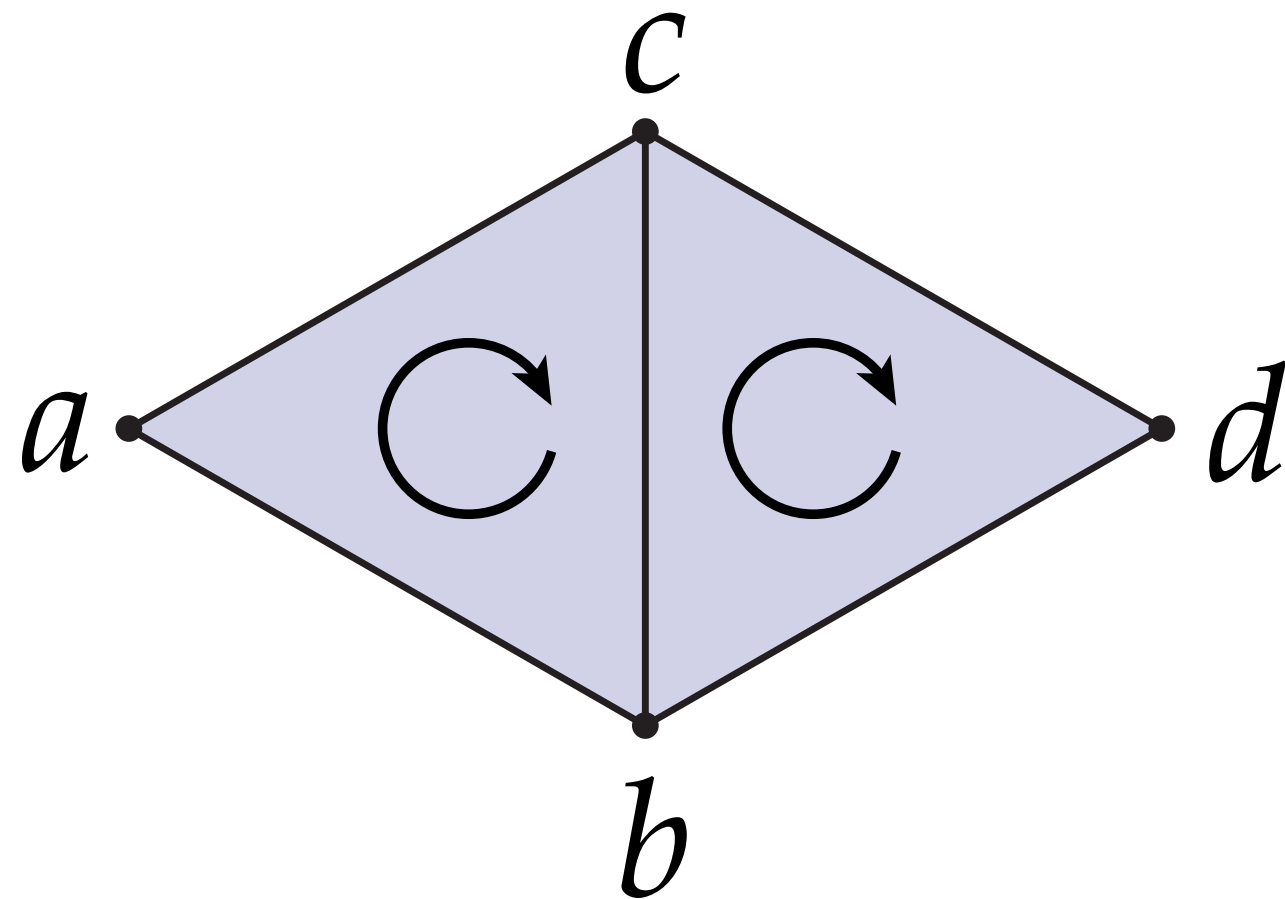
$$(c, a), (a, b), (b, c), (c, d), (d, b),$$

$$(a, b, c), (b, c, d)\}$$

Relative Orientation

Definition. Two distinct oriented simplices have the same *relative orientation* if the two (maximal) faces in their intersection have **opposite** orientation.

Example: Consider two triangles that intersect along an edge:

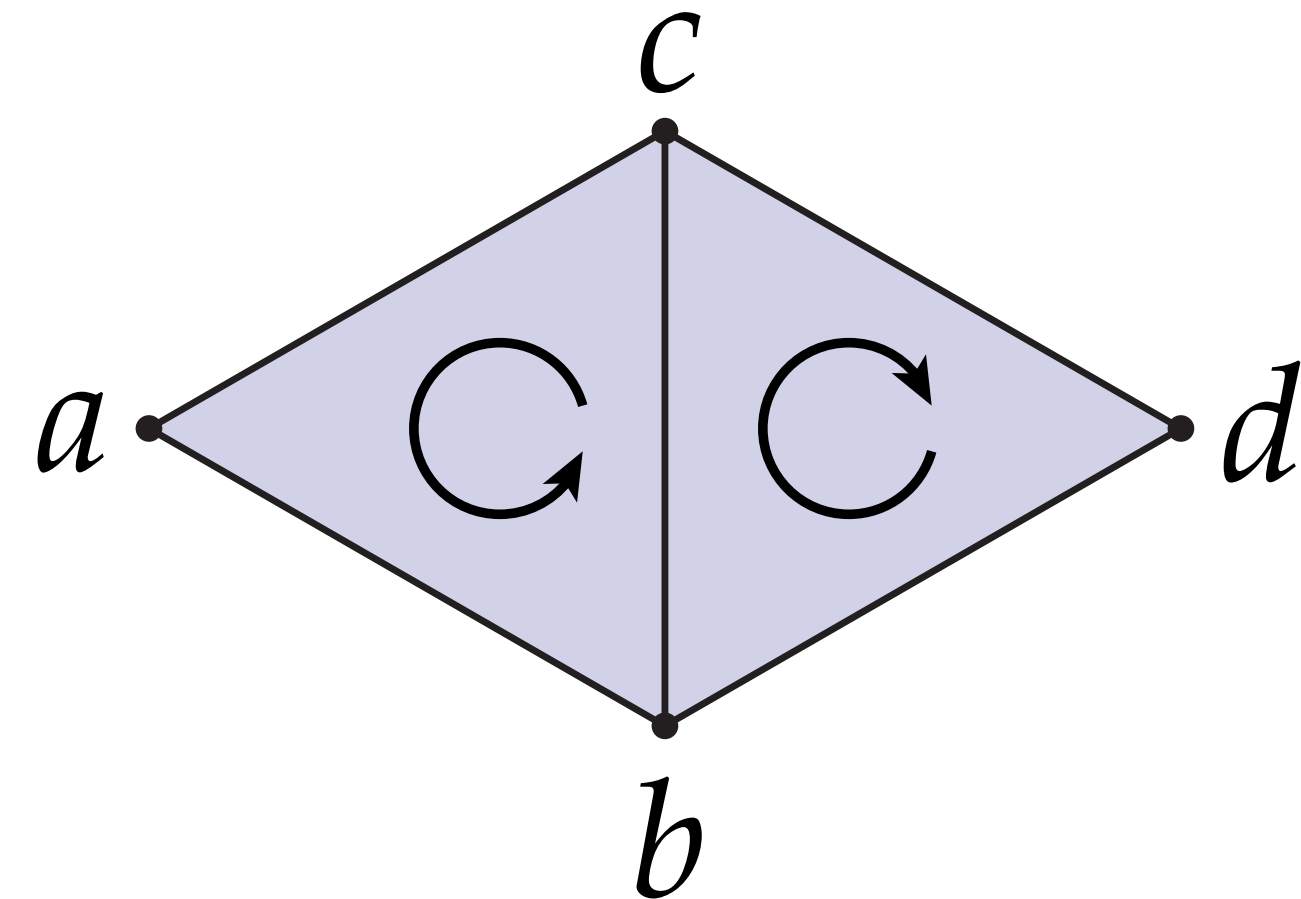


same relative orientation

$$(a, c, b) \Rightarrow (c, b)$$

$$(b, c, d) \Rightarrow (b, c)$$

$$(c, b) = -(b, c)$$



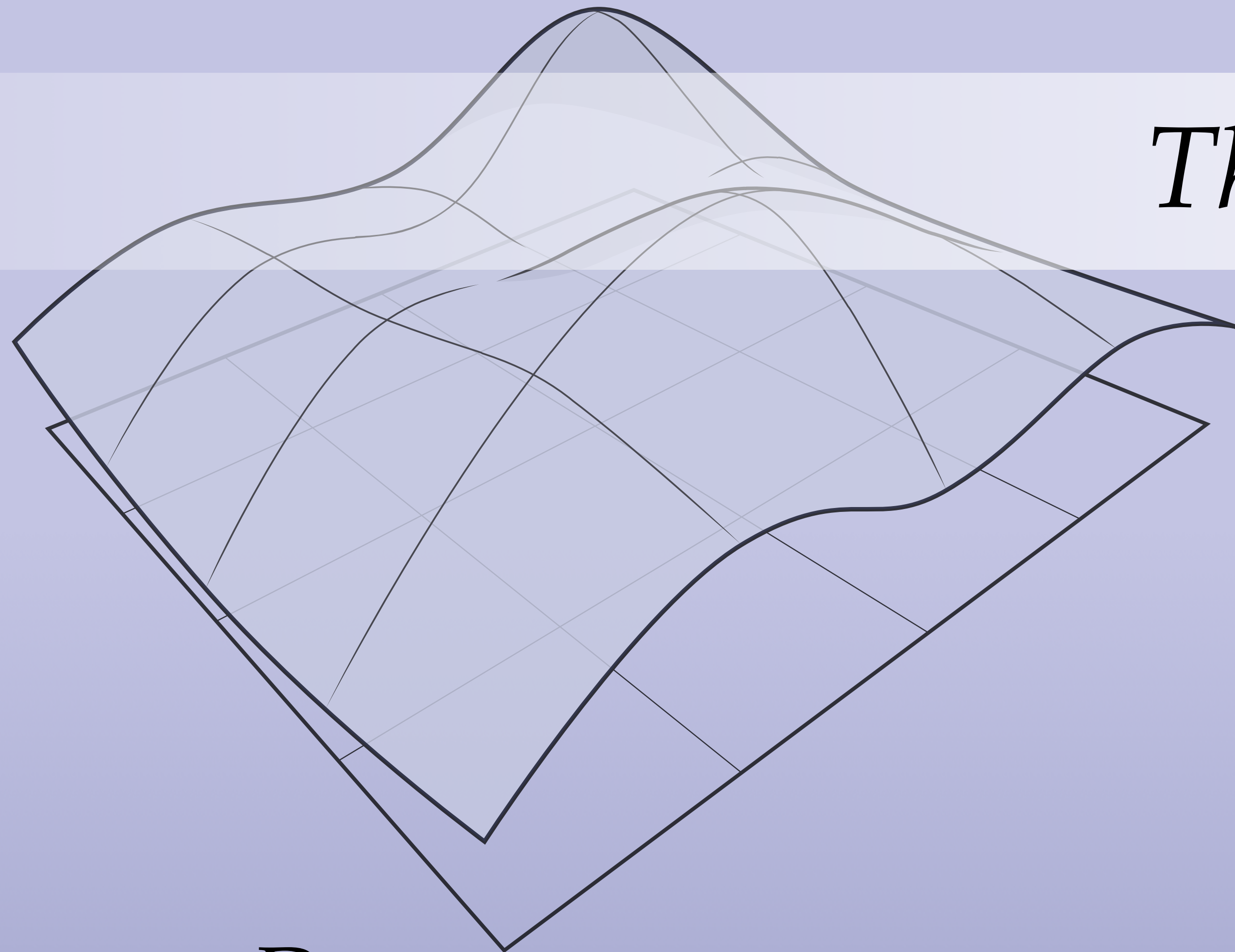
different relative orientation

$$(a, b, c) \Rightarrow (b, c)$$

$$(b, c, d) \Rightarrow (b, c)$$

$$(b, c) = +(b, c)$$

Thanks!



DISCRETE DIFFERENTIAL
GEOMETRY:
AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858