1. (a) $P(G=1 \mid B>0) = \frac{P(G=1, B>0)}{P(B>0)} = \frac{P(G=1, B=1)}{P(B>0)}$

$= \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{4}} = \frac{2}{3}$

(b). $P(C_2 = G \mid C_1 = B) = \frac{P(C_2 = G, C_1 = B)}{P(C_1 = B)}$

$= \frac{1/4}{1/2} = \frac{1}{2}$

2. (a). The prosecutor is wrong, because he assume that

$P(\text{innocent} \mid \text{blood match}) = P(\text{blood match} \mid \text{innocent})$
$= 1\%$

so he claims $P(\text{guilty} \mid \text{blood match}) = 99\%$

But his assumption itself is wrong.

the equality exists iff $P(\text{innocent}) = \frac{1}{2}$

However, it's unknown. so his argument is wrong.

(b). The defender calculated $P(\text{guilty} \mid \text{blood match})$ regardless of $P(\text{guilty})$, so he is also wrong.

3, (a) $p(X_2 = H) = p(X_2 = H \mid X_1 = H)\, p(X_1 = H) + p(X_2 = H \mid X_1 = A)\, p(X_1 = A)$

$= 0.9 \times 1 + 0.1 \times 0 = 0.9$

(b). $p(Y_2 = \text{frown})$

$= P(Y_2 = \text{frown} \mid X_2 = H)\, P(X_2 = H) + P(Y_2 = \text{frown} \mid X_2 = A)\, P(X_2 = A)$

$= 0.1 \times 0.9 + 0.6 \times 0.1$

$= 0.15$

(c). $P(X_2 = H \mid Y_2 = \text{frown})$

$$= \frac{P(Y_2 = \text{frown} \mid X_2 = H)\, P(X_2 = H)}{\sum_{x_2} P(Y_2 = \text{frown} \mid X_2 = x_2)\, P(X_2 = x_2)}$$

$$= \frac{0.1 \times 0.9}{0.1 \times 0.9 + 0.6 \times 0.1} = \frac{9}{15} = 0.6$$

(d). $P(Y_{60} = \text{yell})$

$$= P(Y_{60} = \text{yell} \mid X_{60} = A) \cdot P(X_{60} = A)$$
$$+ P(Y_{60} = \text{yell} \mid X_{60} = H)\, P(X_{60} = H)$$

For $n \geq 2$,

$$P(X_n = A) = P(X_n = A \mid X_{n-1} = A)\, P(X_{n-1} = A) + P(X_n = A \mid X_{n-1} = H)\, P(X_{n-1} = H)$$

$$P(X_n = H) = P(X_n = H \mid X_{n-1} = A)\, P(X_{n-1} = A) + P(X_n = H \mid X_{n-1} = H)\, P(X_{n-1} = H)$$

Denote $\pi_n = \begin{pmatrix} P(X_n = A) \\ P(X_n = H) \end{pmatrix}$

$\pi_n = \varphi \cdot \pi_{n-1}$

$\pi_{60} = \varphi^{59} \pi_1 = \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}^{59} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 0.50000096 \\ 0.49999904 \end{pmatrix}$

$P(Y_{60} = \text{yell}) \approx 0.1 \times 0.5 + 0.2 \times 0.5 = 0.15$

(b). In this problem, we don't assume $X_1 = H$

So we assume an initial distribution π_1,

$P(X_1 = x_1, \cdots, X_5 = x_5 \mid Y_1 = \cdots = Y_5 = \text{frown})$

$$= \frac{P(Y_1 = \cdots = Y_5 = \text{frown} \mid X_1 = x_1 \cdots X_5 = x_5)\, P(X_1 = x_1 \cdots X_5 = x_5)}{P(Y_1 = \cdots = Y_5 = \text{frown})}$$

We are optimizing on $x_1 \cdots x_5$, so.

$P(Y_1 = \cdots = Y_5 = \text{frown})$ doesn't matter.

$$P(Y_1 = \cdots = Y_5 = \text{frown} \mid X_1 = x_1 \cdots X_5 = x_5)\, P(X_1 = x_1 \cdots X_5 = x_5) = P(X_1 = x_1) \prod_{t=2}^{5} P(X_t = x_t \mid X_{t-1} = x_{t-1})$$

$$\times \prod_{t=1}^{5} P(Y_i = \text{frown} \mid X_t = x_t)$$

$$= \pi_{1, x_1} \cdot \prod_{t=2}^{5} \alpha_{x_{t-1}, x_t} \cdot \prod_{t=1}^{5} \psi_t(x_t)$$

Take log, we are maximizing:

$$\log\left(\psi_1(x_1) \cdot \pi_{1, x_1}\right) + \sum_{t=2}^{5} \log\left(\alpha_{x_{t-1}, x_t} \cdot \psi_t(x_t)\right)$$

For $t \geq 2$.

$\alpha_{x_{t-1}, x_t} \cdot \psi_t(x_t)$

(H, H) $0.9 \times 0.1 = 0.09$

(H, A) $0.1 \times 0.6 = 0.06$

(A, H) $0.1 \times 0.1 = 0.01$

(A, A) $0.9 \times 0.6 = 0.54$

so any states will goes to Angry for next step.

① If $x_1 = H$, the optimal result is:

$$\log(0.1 \cdot \pi_{1H}) + \log(0.06 \times 0.54^3)$$

② if $x_1 = A$ the optimal result is.

$$\log(0.9\, \pi_{1A}) + \log(0.54^4)$$

$$\begin{cases} \pi_{1H} + \pi_{1A} = 1 \\ 0.01\ \pi_{1H} = 0.54\ \pi_{1A} \end{cases} \qquad \begin{cases} \pi_{1A} = \dfrac{1}{55} \\ \pi_{1H} = \dfrac{54}{55} \end{cases}$$

~~therefore~~

$$\begin{cases} \text{if } P(X_1 = H) < \dfrac{54}{55} \\ \quad \text{the best path will be } (A, A, A, A, A) \\ \text{if } P(X_1 = H) > \dfrac{54}{55} \\ \quad \text{the best path will be } (H, A, A, A, A) \\ \text{if } P(X_1 = H) = \dfrac{54}{55}. \text{ both are equal.} \end{cases}$$