01: Data Science Prerequisite Mathematics Exam

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1

Which point marked on the figure is equal to $f(0)$?
p

2

Consider the following expression: $\frac{f(x+h)-f(x)}{h}$. What is this equivalent to, in terms of figure 1?
Its the average slope of $f(x)$ between the interval $f(x)$ and $f(x+h)$. The function $g(x)$ happens to be equal to this average slope between the same interval.

3

The function $g(x)$ is a line, so it is of the form $mx + b$. What information about $f(x)$ can we take from this?
Its easy to compare the slope of $f(x)$ to $g(x)$, since $g(x)$ is a straight line. If the slope of $f(x)$ is steeper than the slope of $g(x)$ at a given x, then the slope of $f(x)$ is more negative than the slope of $g(x)$; if the slope of $f(x)$ is flatter than the slope of $g(x)$ at a given x, then the slope of $f(x)$ is less negative than $g(x)$.

4

The functions $g(x)$ and $u(x)$ are both lines, so let $g(x) = mx + b$ and $u(x) = nx + c$. What is $f'(s)$?
$g(x)$ and $u(x)$ are both lines and have constant slope. It seems like they are approximately equal, so $m \approxeq n$. At s, $f' \approxeq m \approx n$.

5

Which point marked on the figure represents a potential optimum?
answer: No optima are apparent looking at the graph. Its possible that the

left end point of $f(x)$ could be an optima because it could potentially start decreasing just out of range with a continous slope.

6

Is $f''(s)$ equal to 0, greater than 0, or less than 0?
The slope of $f(x)$ is negative for the entire range covered in the graph. Looking at the graph, $f'(s-\epsilon) < f'(s) < f'(s+\epsilon)$, for small ϵ. This tells us that the is $f'(x)$ is getting less negative at s, which tells us that $f''(s) > 0$.

$$A = \begin{pmatrix} 6 & 8 & 6 & 7 \\ 5 & 3 & 0 & 9 \\ 7 & 7 & 1 & 4 \\ 2 & 5 & 5 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 2 & 0 & 9 & 9 \\ 1 & 9 & 6 & 1 & 1 \\ 5 & 2 & 0 & 2 & 5 \\ 0 & 1 & 7 & 4 & 7 \end{pmatrix} \qquad u = \begin{pmatrix} 3 \\ 7 \\ -6 \\ 9 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 4 \end{pmatrix}$$

7

$\sum_{i=1}^{n} v_i = 1 + 3 + 1 + 4 = 9$

8

$\sum_{i=1}^{n} v_i^2 = 1^2 + 3^2 + 1^2 + 4^2 = 1 + 9 + 1 + 16 = 27$

9

What are the dimensions of the matrix product AB?
$A \rightarrow$ (4 x 4), $B \rightarrow$ (4 x 5). $\therefore AB \rightarrow$ (4 x 5)

10

What are the dimensions of the matrix product BA?
$B \rightarrow$ (4 x 5), $A \rightarrow$ (4 x 4), $\therefore BA$ is undefined.

11

What are the dimensions of the product Av?
$A \rightarrow$ (4 x 4), $v \rightarrow$ (4 x 1), $\therefore Av \rightarrow$ (4 x 1).

12

What are the dimensions of the product $u^T v$?
$u^T \rightarrow$ (1 x 4), $v \rightarrow$ (4 x 1), $\therefore u^T v \rightarrow$ (1 x 1), or is a scalar.

13

What are the dimensions of the product uv^T?
$u \rightarrow$ (4 x 1), $v^T \rightarrow$ (1 x 4), $\therefore uv^T \rightarrow$ (4 x 4).

14

Evaluate the following expression: $\sum_{j=1}^{4} B_{1j}$.
$\sum_{j=1}^{4} B_{1j} = 5 + 2 + 0 + 9 = 16$

15

Evaluate the following expression: $\sum_{i=1}^{m} A_{i2}$.
$\sum_{i=1}^{m} A_{i2} = 8 + 3 + 7 + 5 = 23$.

16

What is the value of $(AB)_{2,1}$?
Since we only care about element (2,1), we can just multiply the second row of A with the first column of B. We then get $(AB)_{2,1} = 5\times5+3\times1+0\times5+9\times0 = 25 + 3 + 0 + 0 = 28$.

17

What are the dimensions of B^T?
$B \rightarrow$ (4 x 5) $\therefore B^T \rightarrow$ (5 x 4).

18

The L^2 norm of a vector $\vec{\mathbf{x}}$ is defined as follows: $\|x\| = \sqrt{x^T x}$. Calculate $\|v\|$

$$\begin{aligned}
\|v\| &= \sqrt{\begin{pmatrix}1 & 3 & 1 & 4\end{pmatrix}\begin{pmatrix}1\\3\\1\\4\end{pmatrix}} \\
&= \sqrt{1\times1+3\times3+1\times1+4\times4} \\
&= \sqrt{1+9+1+16} \\
&= \sqrt{27}
\end{aligned}$$

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The L^1 norm of a vector $\vec{\mathbf{x}}$ is defined as follows: $|\vec{\mathbf{x}}| = \sum_{i=1}^{n} |x_i|$. Calculate $|\vec{\mathbf{u}}|$.
$|u| = 3 + 7 + 6 + 9 = 25$.

20

Consider the columns of the matrix B to be a set of vectors in $\Re^4$. Is this set linearly independent, or linearly dependent, and why?
The are linearly dependant because the number of vectors in B ¿ 4, Therefore, at least one of the vectors must be redundant and a linear combination of the others.

21

Say we removed the fifth column of the matrix B. Is it possible for the set of the remaining columns to span the vector space $\Re^4$? Why or why not?
If the determinant of a square matrix is not equal to zero, then the matrix is linearly independent; if the determinant is equal to zero, the matrix is linearly dependent. Using numpy.linalg.det(B[:,0:3]), we find that $|B[:, 0:3]| \neq 0 \therefore B[:, 0:3]$ is linearly independent $\therefore B[:, 0:3]$ spans $\Re^4$.

22

What does the matrix I look like in the expression IA?
I is an identity matrix that is (4 x 4) because A is (4 x 4).

$$\therefore I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

23

What does the matrix I look like in the expression BI?
Because I is an identity matrix, it must be a square matrix. The only square matrix that can be defined will have a shape, (5 x 5).

$$\therefore I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

24

Say there exists some matrix C such that $AC = CA = I$. What is C?

$C \equiv A^{-1} \approx \begin{pmatrix} 3.18 & -2.03 & -0.04 & -3.81 \\ -3.01 & 1.84 & 0.25 & 3.57 \\ 1.90 & -1.15 & -0.22 & -2.03 \\ -0.76 & 0.63 & -0.06 & 0.93 \end{pmatrix}$. Which is found using numpy.linalg.inv(A).

25

Let M and N be matrices such that MN is defined. Is the product $N^T M^T$ defined?

if $M \rightarrow$ (a x b) and $N \rightarrow$ (c x d) and MN is defined, then b = c $\Rightarrow$ $N^T M^T$ is defined when (d x c) — (b x a) or (d x b) — (b x a) $\therefore$ $N^T M^T$ is defined for any d,a.

26

Tickets at Empire Cinema cost £4.00 for children and £6.00 for adults. On a given day, 3,700 tickets were sold. Total ticket revenues were £18,000. Say we wanted to determine the number of children and the number of adults that attended the cinema that day. Give the linear system we would use to find this. Next, give the equivalent coefficient matrix (A) and constant vector (b).

There are a few ways of constructing the problem. We can use a matrix multiplication,

$$\begin{pmatrix} 4 & 6 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18,000 \\ 3,700 \end{pmatrix}$$

Or, we can use an augmented matrix. I find it easier to solve using an augmented matrix.

$$\begin{aligned} \left(\begin{array}{cc|c} 4 & 6 & 18,1000 \\ 1 & 1 & 3,700 \end{array}\right) &= \left(\begin{array}{cc|c} 1 & 1 & 3,700 \\ 0 & 2 & 3,200 \end{array}\right) \\ &= \left(\begin{array}{cc|c} 1 & 1 & 3,700 \\ 0 & 1 & 1,600 \end{array}\right) \\ &= \left(\begin{array}{cc|c} 1 & 0 & 2,100 \\ 0 & 1 & 1,600 \end{array}\right) \end{aligned}$$

Where we subtract $4R_1$ from R_2, then we divide R_2 by 2, then we subtract R_2 from R_1. So, there were 2,1000 children and 1,600 adults.

27

In a particular clinic, 10% of patients are prescribed narcotic pain killers. Overall, 5% of the clinic's patients are addicted to narcotics. Out of all the people

who are prescribed pain pills, 8% are addicts. If a patient is addicted to narcotics, what is the probability that they will be prescribed pain pills?
Let the probability of being an addict be $P(a) = \frac{5}{100} = \frac{1}{20}$, the probability of being prescribed narcotics be $P(p) = \frac{10}{100} = \frac{1}{10}$, and the probability of being an addict-given you are prescribed be $P(a|p) = \frac{8}{100} = \frac{2}{25}$. We then have,

$$\begin{aligned} P(p|a) &= \frac{P(a|p)P(p)}{P(a)} \\ &= \frac{\frac{2}{25}\frac{1}{10}}{\frac{1}{20}} \\ &= \frac{\frac{2}{25}}{\frac{1}{2}} \\ &= \frac{4}{25} = 16\% \end{aligned}$$

28

A guitar manufacturer has factories in Fresno, Nashville, and Memphis. The Fresno factory produces 32% of all the guitars, the Nashville factory produces 43%, and the Memphis factory produces 25%. 16% of all the guitars produced at the Fresno factory are acoustic, the rest are electric. At the Nashville factory 88% of the guitars produced are electric. At the Memphis factory, 73% of the guitars produced are acoustic.
Let the probability of a guitar being manufactured from the Fresno factory be $P(F) = \frac{32}{100} = \frac{8}{25}$, from the Nashville factory $P(N) = \frac{43}{100}$, and from the Memphis factory $P(M) = \frac{25}{100} = \frac{1}{4}$. Let a and e represent accoustic and ellectric. Let probability a given F is $P(a|F) = \frac{16}{100} = \frac{4}{25}$, $P(e|N) = \frac{88}{100} = \frac{22}{25}$, $P(a|M) = \frac{73}{100}$.

28.1

What is the probability that a guitar produced by this company is electric?

$$\begin{aligned} P(e) &= P(e|M)P(M) + P(e|N)P(N) + P(e|F)P(F) \\ &= \frac{27}{100}\frac{1}{4} + \frac{22}{25}\frac{43}{100} + \frac{21}{25}\frac{8}{25} \\ &= 0.7147 \end{aligned}$$

28.2

We select a guitar at random that was produced by this company. The guitar is an acoustic guitar. What is the probability it was manufactured in Memphis?

$P(a|M) = \frac{73}{100} \Rightarrow P(e|M) = \frac{27}{100}$. Using $P(e)$ from the previous problem,

$$\begin{aligned}
P(M|a) &= \frac{P(a|M)P(M)}{P(a)} \\
&= \frac{P(a|M)P(M)}{1 - P(e)} \\
&= \frac{\frac{73}{100}\frac{1}{4}}{1 - 0.7147} \\
&\approx 64.0\%
\end{aligned}$$

29

Let X and Y be independent random variables.

29.1

Say the probability of some observation $x \epsilon X$ is $P(x) = 0.81$. Now say we observe some $y \epsilon Y$. What is $P(x|y)$?
X and Y are independent, so $P(x|y) = P(x) = 0.81$.

29.2

Say the distribution of X is given by $X\ p\big(x|\mu, \sigma^2\big)$. Now, say we take a sample of size N from X. Our sample is $x_1, x_2, ..., x_N$. What is the likelihood function of our sample?

$$\begin{aligned}
P(x_i|\mu, \sigma^2) &= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi\sigma^2}^N} \prod_{i=1}^{N} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}}
\end{aligned}$$

30

Starting with the definition for the conditional probability of an event, given the observation of another event $P(A|B)$, derive the formula from Bayes' Theorem.
The joint probability is the probability of both A and B, we can write this as either $P(A|B)P(B)$ or $P(B|A)P(A)$. All we have to do is set them both equal to each other and solve for one of the conditional probabilities.

$$\begin{aligned}
&P(A \cap B) = P(A|B)P(B) = P(B \cap A) = P(B|A)P(A) \\
&\Rightarrow P(A|B) = \frac{P(B|A)P(A)}{P(B)}
\end{aligned}$$

31

A ship is transporting livestock across the sea. The ship contains 38 sheep, 12 cows, and 22 pigs. Let s, c, p, stand for sheep, cow and pig. The total number of animals is $T = s + c + p = 72$.

31.1

What it the probability that an animal on the ship is a sheep?
$P(s) = \frac{s}{T} = \frac{38}{72} = \frac{19}{36}$

31.2

What is the probability that an animal on the ship is a pig, given that it's not a sheep?
$P(p|p \cap c) = \frac{p}{p+c} = \frac{22}{22+12} = \frac{22}{34} = \frac{11}{17}$

31.3

What is the age of the ship's captain? Lets assume the age of the captain is $A_c = 40$ years. 16 years to start as a sailor, 20 to become captain, and he's relatively new in his career because he's not hauling anything more exciting.