Cavendish

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Author: Jacob Cluff

Partners: Jason Pickering, Philip Rybak

Abstract:
The universal constant of gravity is measured using both a static and driven method. Both methods equate the torque on the boom from gravity and from the tungsten fiber; however, the methods differ in determining the displacement angle, θ_D. The static method shows $G_s = 5.4\times10^{-11} \pm 3\times10^{-12}\ N\ m^2\ kg^{-2}$; the driven method gives $G_d = 9.4\times10^{-11} \pm 4\times10^{-12}\ N\ m^2\ kg^{-2}$. These results do not agree - within the stated uncertainty - with the presently accepted value of G.

Introduction:

The force of gravity is the weakest of all the known fundamental forces and yet is the one we experience most on a day to day basis and has had a dramatic affect on the expansion history of the universe. The same gravity that hurls galaxies through space is also the same gravity that causes a boom's equilibrium oscillation angle to be deflected in the presence of a few kilograms of lead a few centimeters away. The force of gravity is proportional to G and the product of both bodies' masses and is inversely proportional to the inverse square of the distance between both bodies' centers of mass.

In 1687, Newton introduced his theory of gravity in *Principia*. His formalism of force and gravity gave a profound theoretical explanation to the experimental results found by Kepler and Brahe. Using his three famous laws on force, concluding a central force acting orbiting bodies, and Kepler's third law - describing the orbital distance and period of a body, Newton was able to figure out that the force of gravity is inversely proportional the square of the distance.[1]

Cavendish's original experiment was in terms of Earth's density; in a correspondance, he allegedly described his experiment as "weighing the world." His result of Earth's density, 5.448 $g\ cm^{-3}$, when converted to G, gives $6.74 \times 10^{-11}\ N\ m^2\ kg^{-2}$, which is off by 1% of the currently accepted value.[2]

Theory:

The tungsten wire exerts a restoring torque on the boom, causing it to oscillate about an equilibrium angle. The gravitational attraction of of lead balls also exert a torque that deflects the equilibrium oscillation angle of the boom. These two torques are given by Equation (1),

$$\begin{aligned}
\tau_{wire} &= \tau_{grav} \\
k\theta_D &= \frac{2Gm_M m_m d}{R^2} \\
\Rightarrow G &= \frac{k\theta_D R^2}{2m_M m_m d} \\
G_{corrected} &= \frac{k\theta_D R^2}{2m_M m_c d}, \\
m_c &= (m_m - m_h)(1 - f_d) + (m_b * f_b) \\
m_h &= \text{mass of hole missing from boom} = 0.34g \\
f_b &= 0.19 \\
f_d &= 3.5/1000
\end{aligned} \tag{1}$$

which is then solved for G, the desired gravitational constant. The other variables are then measured directly or found experimentally. The full list of variables and their associated uncertainty is shown in Table (1), located in the Appendix.

The first parameter needed is k, the torsion constant. This is found by Equation (2),

$$\begin{aligned}
\omega^2 &= \omega_0^2 - \gamma^2,\ \omega_0^2 = \frac{k}{I} \\
\Rightarrow k &= \left(\omega^2 + \gamma^2\right) I \Leftrightarrow \left[\left(\frac{2\pi}{T}\right)^2 + \gamma^2\right] I,
\end{aligned} \tag{2}$$

where ω and γ are fitting parameters; T and I are found by Equations (3) and (4), respectively.

T, the period, is found by

$$T_{1,n} = \frac{1}{n}(t_n - t_1), \tag{3}$$

where t_1 is the time at some maxima or minima and t_n is the time some n complete cycles away. The period of each damped wave in Figure (3), along with their standard deviation, is used to find T and ΔT.

I, the moment of inertia, is found using

$$\begin{aligned}
I &= I_s + I_b \\
&= 2m_m(d^2 + \frac{2}{5}r_m^2) + \frac{m_b}{12}(l_b^2 + w_b^2),
\end{aligned} \tag{4}$$

where I_s and r_m are the moment of inertia and radius, respectively, of the 2 spheres m and m'.

[1] http://physics.ucsc.edu/~michael/newtonreception6.pdf

[2] https://en.wikipedia.org/wiki/Cavendish_experiment

The next variable in G is θ_D, the displacement angle from equilibrium. In the static measurement, this is found by

$$\theta_D = \theta_s = \frac{1}{2}(\theta_n - \theta_{n+1}), \tag{5}$$

where θ_{n+1} and θ_n are sequential angle displacements on either side of equilibrium. This is shown in Figure (3). Several measurements are taken and the uncertainty is found by the standard deviation of the measurements.

Next is R, the distance between the centers of mass of the large and small lead balls. This is found by

$$R = \frac{W}{2} + w_t + \frac{d_M}{2}. \tag{6}$$

The next variable in G needed is d, the separation between the mass centers on the boom to the axis of rotation, which is found by,

$$d = \frac{1}{2}(l_b - d_m). \tag{7}$$

For the driven experiment, manually fitted parameters are used to model Equation (8). A "least squares" algorithm is then used to find the best fit parameters. The uncertainty is found by varying each parameter, one at a time, and recording the smallest variation that causes the fit to visually deviate from the data. The displacement angle, θ_d, is then found using Equation (9) These results are shown in Figure (4), in the Results, and Table (1), in the Appendix.

$$\theta(t) = \theta_e + Ae^{-\gamma t}\cos\omega t. \tag{8}$$

$$\begin{aligned} \theta_d(t) &= \theta_e + \frac{4Q}{\pi}\theta_s(1 - e^{-\gamma t})\cos\omega t \\ Q &= \frac{\omega}{2\gamma} \end{aligned} \tag{9}$$

Equation (9) is rewritten like so,[3]

$$\begin{aligned} \theta_n &= \theta_e + (\theta_1 - \theta_e)(-x)^{n-1} \\ \theta_1 - \theta_e &= A \\ x &= e^{-\gamma T/2}. \end{aligned} \tag{10}$$

In this scheme, t_n is the time at the n^{th} turning point and θ_n is the angular displacement at the n^{th} turning point, which makes $\theta_1 - \theta_e$ the initial amplitude, A, of the decaying wave.

x can be directly calculated from experimental values of γ and T, but the most precise way of determining x is found by averaging Equation (11) over an odd number of turning points to get Equation (12). Thirteen turning points are used in the experiment.

$$x = -\frac{\theta_{n+2} - \theta_{n+1}}{\theta_{n+1} - \theta_n} \tag{11}$$

$$< x >= 1 - \frac{\theta_1 - \theta_N}{\theta_1 - \theta_2 + \theta_3 - \theta_4 + \cdots - \theta_{N-1}} \tag{12}$$

The angular displacement for the driven experiment is then found by averaging Equation (13) over the odd number turning points, thirteen in this case, which results in Equation (14).

$$\theta_d = (-1)^n \frac{x\theta_n + (1-x)\theta_{n+1} - \theta_{n+2}}{2(1+x)} \tag{13}$$

$$< \theta_d >= \frac{(1-x)(\theta_1 - \theta_2 + \theta_3 - \theta_4 + \cdots + \theta_N) - \theta_1 + x\theta_N}{(N-1)(1+x)} \tag{14}$$

[3] the complete derivation for θ_d is found in the TEL-RP2111 Cavendish Balance Manual - http://www.telatomic.com/mechanics/docs/TEL-RP2111_Cavendish_Balance_manual.pdf

Procedure:

The static measurement starts when the boom's oscillation angle is sufficiently small - less than 5 mRad. The two massive lead balls on the exterior are swung to either side of the glass wall, which deflects the equilibrium oscillation angle. After a few cycles, the external pendulum is swung so the massive balls gently touch the opposite side of their respective walls, which deflects the equilibrium angle to the opposite side of its natural position. Several such consecutive measurements are used to get a statistical average and uncertainty for the deflection angle.

The driven measurement starts when the boom's oscillation angle is very small - less than 1 mRad. The external pendulum is then swung about at each turning point so that the smaller lead balls on the internal boom are constantly moving towards the larger exterior lead balls. After about six or seven cycles, the excitation of the oscillation starts to approach a saturation level where the torque from the square wave driving force becomes approximately equal to restoring torque of the tungsten fiber. After sufficient saturation, the external pendulum is swung to a neutral position and the decaying oscillation is used to find the decay parameter that is needed in order to find the displacement angles for the driven wave.

Regardless of method used to obtain a statistical displacement angle, θ_D is plugged into Equation (1) to find the value of G.

Results:

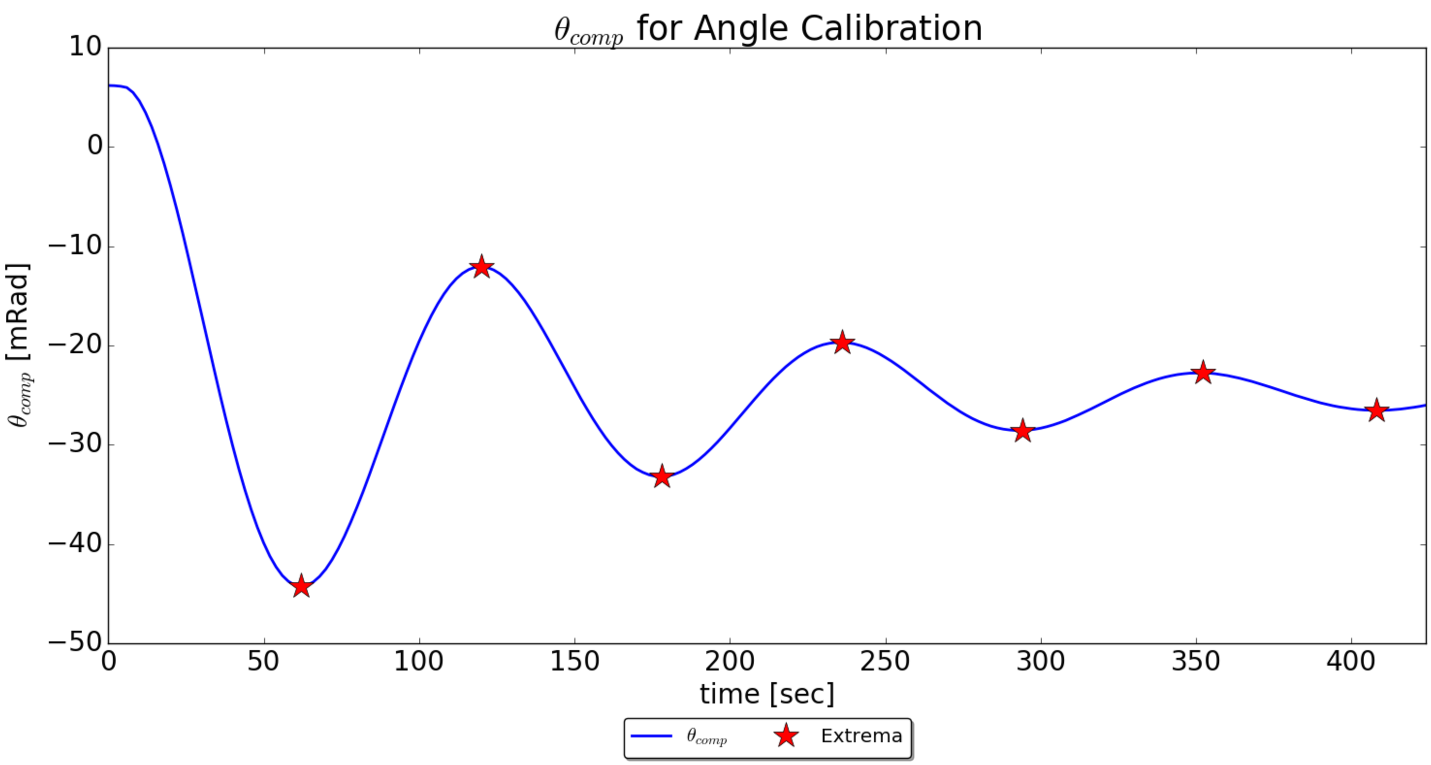

Figure 1: A laser shines on the boom as it freely oscillates without m or m'. The laser reflects off the mirror and onto the opposite wall. The turning points, $S_{i,i+1}$ are measured and compared with θ_{comp}, See Figure 2

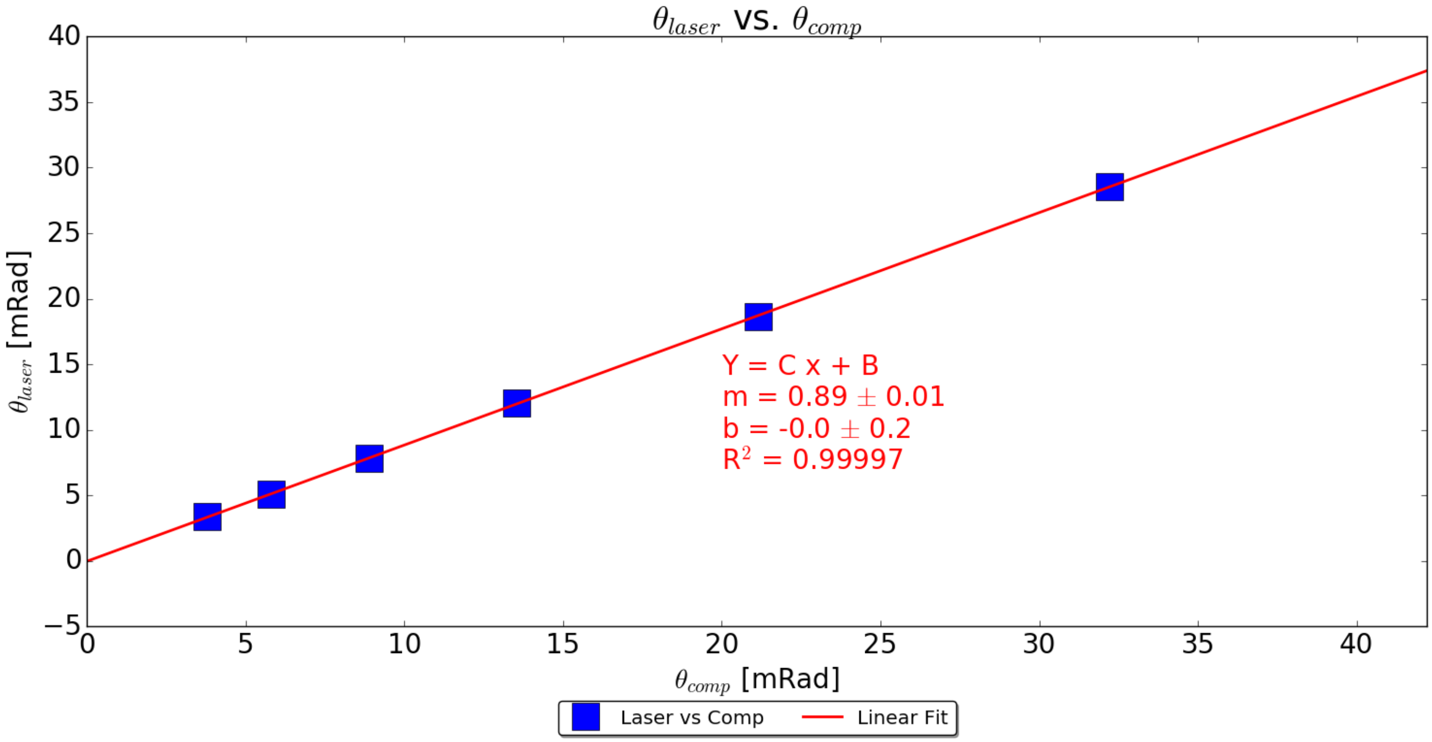

Figure 2: θ_{laser}, $1/2 \arctan(S_{ij}/L)1000$, is compared against θ_{comp}, $|\theta_j - \theta_i|$. A linear least squares fit, using the statistical equations from the fits exercise, is used to calculate the parameters. $\theta_{laser}/\theta_{comp}$, which is the slope seen above, is taken to be the calibration constant, $C = 0.89 \pm 0.01$ The data points shown above can be seen in Table 2, in the Appendix.

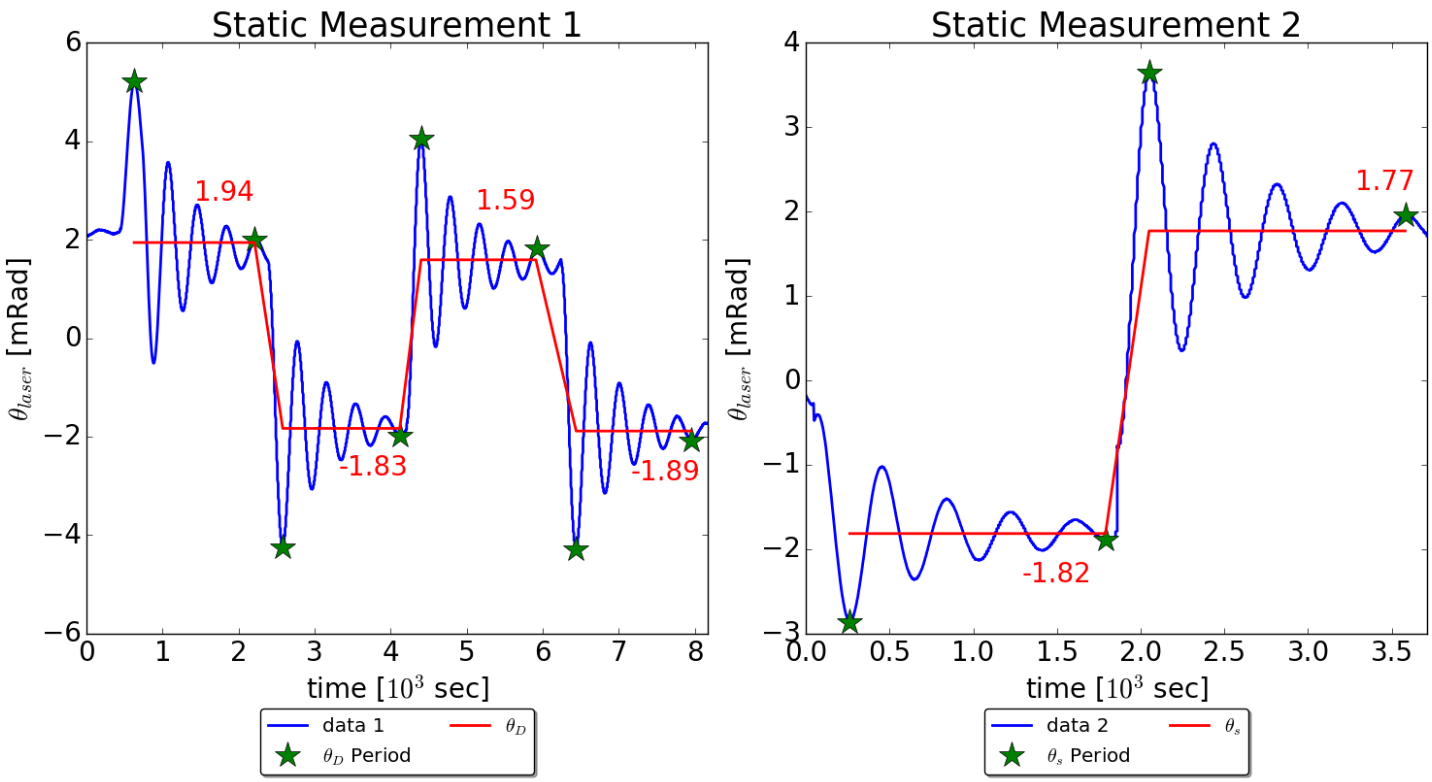

Figure 3: Due to time restraints, two different static measurements were taken; ideally, one long measurement provides better results since there are more sequential displacements, which results in a better statistical determination of G. The average of each displaced damped wave between complete cycles is θ_s. half the difference between each sequential displacement is taken as θ_s, the results of which are shown in Table 4, in the Appendix. Equation 3 is also applied to each displacement set in order to get a statistical measurement of the period, T; the results are shown in Table 3, in the Appendix.

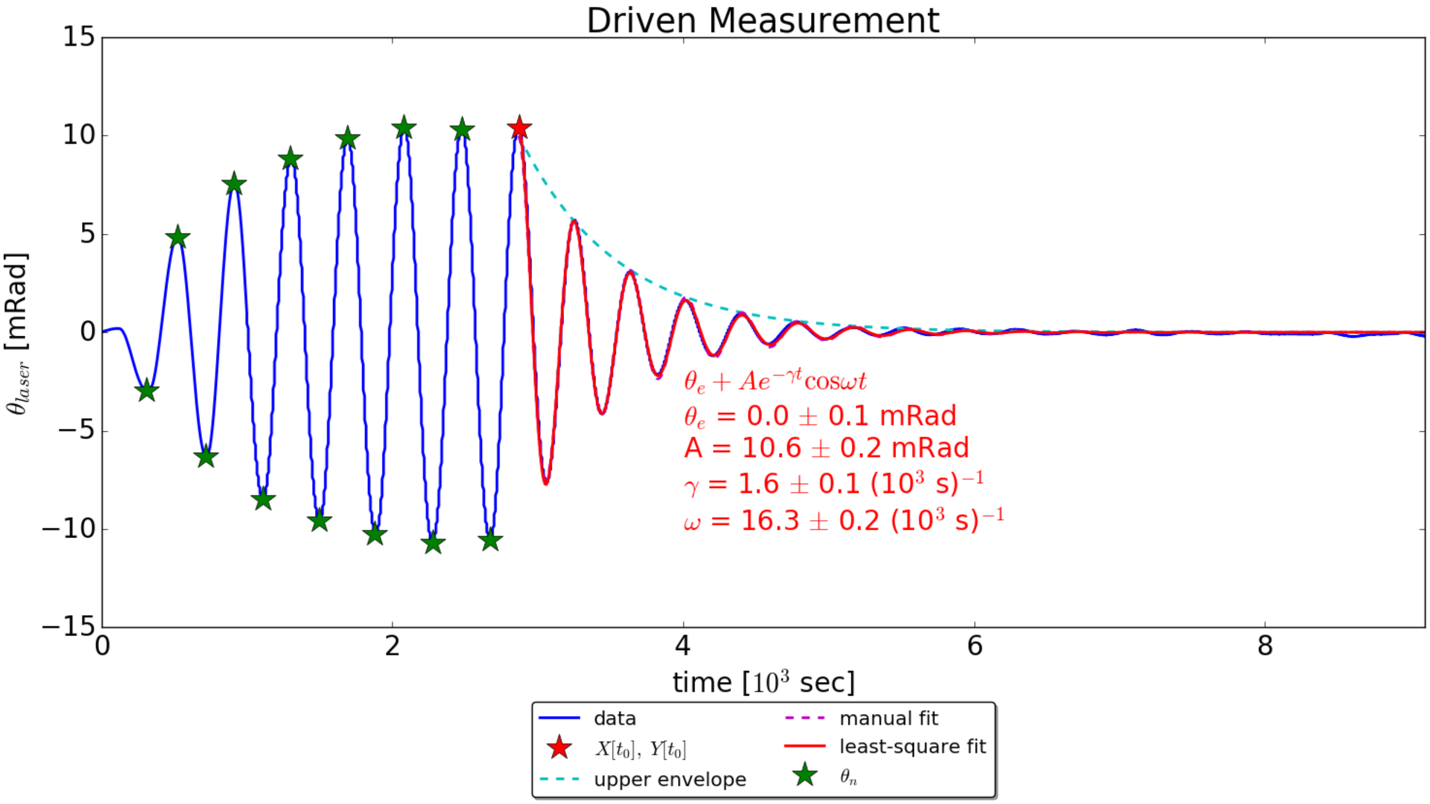

Figure 4: The decaying portion of the data is sliced and manual parameters are found to approximate $\theta(t)$. The data is shifted so that the $\theta_e \approx 0$, the initial angle is then used as A, the dashed cyan curve is used to help find γ, and ω is approximated by the average period found in Table 3. These manual parameters are then put into a least squares algorithm to find the best fit parameters.

Discussion:

The static measurement $G_s = 5.4 \times 10^{-11} \pm 3 \times 10^{-12}\ N\ m^2\ kg^{-2}$ does not agree with the accepted value of $G = 6.67 \times 10^{-11}\ N\ m^2\ kg^{-2}$. This may be because what should have been a single measurement was split up into two, due to time constraints. In addition to having fewer sequential measurements, the experimental setup was tampered with after the first measurement which resulted in a different equilibrium point (before the data is shifted so that $\theta_e = 0$).

The driven measurement of $G_d = 9.3 \times 10^{-11} \pm 4 \times 10^{-12}\ N\ m^2\ kg^{-2}$ also does not agree with the currently accepted value of G; but this time it is larger. This calculation method is extremely sensitive the the x measurement. x is calculated using Equation (10), although it could also be calculated using the turning points of the decay portion of the measurement and using Equations (11) or (12).

One unaccounted for source of error is the that the experiment was done close to a wall; when it is recommended to be done at least two meters away from a wall.

Appendix:

Table 1: Complete List of Variables

Variable	q_i	Δq_i	**Description**	**Note**
L $[m]$	7.6	0.1	distance - boom mirror to opposite wall	angle calibration
m_M $[kg]$	1.0545	5e-5	mass - M	scale
$m_{M'}$ $[m]$	1.0529	5e-5	mass - M'	scale
d_M $[m]$	5.15e-2	5e-4	diameter - M and M'	calipers
m_m $[kg]$	14.5e-3	5e-5	mass - m and m'	scale
d_m $[m]$	8.5e-3	5e-4	diameter - m and m'	calipers
l_b $[m]$	14.9e-2	1e-3	length - boom	ruler
m_b $[kg]$	7.1e-3	5e-5	mass - boom	scale
w_b $[m]$	8.0e-3	5e-4	width - boom	calipers
w_t $[m]$	1.0e-3	5e-4	thickness - window	calipers
W $[m]$	2.50e-2	5e-4	width - box	calipers
C	0.89	0.01	calibration constant	$\theta_{comp}- > \theta_{laser}$
d $[m]$	7.02e-2	6e-4	distance - axis and small spheres	equation 7
R $[m]$	3.93e-2	6e-4	distance - M and m	equation 6
I $[kg\ m^2]$	1.56e-4	2e-6	moment of inertia - boom , m, and m'	equation 4
T $[s]$	384	6	oscillation period	equation 3, figure 3, table 3
k $[N\ m]$	4.2e-8	1e-9	torsion constant - tungsten wire	equation 2
θ_e $[rad]$	0.0	1e-4	equilibrium angle	fitting parameter - equation 8, figure 4
A $[rad]$	1.05e-2	2e-4	initial amplitude of damped wave	fitting parameter - equation 8, figure 4
γ $[s^{-1}]$	1.6e-3	1e-4	damping coefficient	fitting parameter - equation 8, figure 4
ω $[s^{-1}]$	1.63e-2	2e-4	angular frequency	fitting parameter - equation 8, figure 4
θ_s $[rad]$	1.78e-3	7e-5	static deflection angle	equation 5, figure 3, table 4
θ_d $[rad]$	3.22e-3	6e-5	driven deflection angle	equation 9, figure 4
G_s $[N\ m^2\ kg^{-2}]$	5.4e-11	3e-12	gravitational constant - static	equation 1
G_d $[N\ m^2\ kg^{-2}]$	9.4e-11	4e-12	gravitational constant - driven	equation 1

Table 2: θ_{laser} vs θ_{comp}

θ_{comp} $[mRad]$	32.2	21.2	13.5	8.89	5.81	3.78
θ_{laser} $[mRad]$	28.6	18.6	12.0	7.81	5.12	3.35

Table 3: Periods of Oscillation - seconds

T_1	T_2	T_3	T_4	T_5	T_6
396	385	380	380	382	382.5

Table 4: Static Displacement Angles - mRad

θ_1	θ_2	θ_3	θ_4
1.88	1.71	1.73	1.79

Uncertainty Equations:

$$\Delta G_s = G\left[\left(\frac{\Delta k}{k}\right)^2 + \left(\frac{\Delta\theta_D}{\theta_D}\right)^2 + \left(\frac{2\Delta R}{R}\right)^2 + (m_m\Delta m_m)^2 + (m_M\Delta m_M)^2 + (d\Delta d)^2\right]^{1/2} \tag{15}$$

$$\Delta k = \left[\big((\omega^2+\gamma^2)\Delta I\big)^2 + (2I\omega\Delta\omega)^2 + (2I\gamma\Delta\gamma)^2\right]^{1/2} \tag{16}$$

$$\begin{aligned}\Delta I^2 &= \Delta I_s^2 + \Delta I_b^2\\ \Delta I_s^2 &= (2d^2\Delta m_m)^2 + (4m_m d\Delta d)^2 + \left(\frac{4}{5}r_m^2\Delta m_m\right)^2 + \left(\frac{8}{5}m_m r_m\Delta r_m\right)^2\\ \Delta I_b^2 &= \left(\frac{l_b^2}{12}\Delta m_b\right)^2 + \left(\frac{2m_b l_b}{12}\Delta l_b\right)^2 + \left(\frac{w_b^2}{12}\Delta m_b\right)^2 + \left(\frac{2m_b w_b}{12}\Delta w_b\right)^2\end{aligned} \tag{17}$$

$$\Delta R = \frac{1}{2}\left[\Delta W^2 + 4\Delta w_t^2 + \Delta M_d^2\right]^{1/2} \tag{18}$$

$$\Delta d = \frac{1}{2}\left[\Delta l_b^2 + \Delta m_d^2\right]^{1/2} \tag{19}$$

$$\Delta\theta_{laser} = \left[(C\Delta\theta_{comp})^2 + (\Delta C\theta_{comp})^2\right]^{1/2},\ \Delta\theta_{comp} = 0.05\ mRad \tag{20}$$

$$\Delta x = \frac{\Delta\theta_{laser}}{|\theta_1-\theta_N|}(1-x)\sqrt{(N-1)(1-x)^2+2x} \tag{21}$$

$$\begin{aligned}\Delta\theta_D &= \left[\Delta\theta_{D_\theta}^2 + \Delta\theta_{D_x}^2\right]^{1/2}\\ \Delta\theta_{D_\theta} &= \Delta\theta_{laser}\frac{\sqrt{(N-1)(1-x)^2+2x}}{(N-1)(1+x)}\\ \Delta\theta_{D_x} &= \Delta x\frac{2(\theta_1-\theta_2+\theta_3-\cdots-\theta_{N-1})+(\theta_N-\theta_1)}{(N-1)(1+x)^2}\end{aligned} \tag{22}$$

Pre-Lab:

1 **Setup.**

A Measure mass and diameter: M, M', m, m'

B Measure mass and length of extra boom (not the one in the experiment)

C measure window thickness

D (TA!) Align boom: Without M, M', m, m', raise the boom to near vertical center. When the boom is oscillating in a steady damped wave, adjust the rotary so that the boom oscillates about about the desired 0 position.

E Angle Calibration: align the laser so that it will reflect off the boom mirror onto the far wall. When the boom is in damped oscillation, record both the computer angle (θ_{comp}) and mark on the wall (or a horizontally aligned paper) where the laser hits at each turning point. Save this data and plot laser vs θ_{comp} to get angle calibration. Use the period of oscillation found to get the moment of inertia for the boom alone (without masses).

F Background "noise": observe (record?) the boom with no masses at equilibrium

2 **Static Measurement of G.** When the boom is pulled towards M and M' it is displaced from its equilibrium position. When the boom is displaced from its equilibrium position, the twisted tungsten wire exerts a restoring force on the boom. This relationship causes the boom to oscillate about a displaced equilibrium angle. The displaced equilibrium angle is what is needed to find G.

A Lower the boom and carefully place m and m' on the ends.

B Place M and M' onto the carriage rotate the carriage so that it is perpendicular to the glass wall (the lead balls will be as far away from the window as they can be)

C Allow the boom with attached m and m' settle (can use RE magnet to speed this process along)

D When oscillations angle $<$ 5 mRad, start collecting data. Record data continuously for the rest of this step.

i Record the time. rotate M/M' carriage until the M and M' gently touch the glass window. Record a few cycles

ii record the time. rotate M/M' carriage so that M and M' are now gently touching the opposite side of their glass walls. Record a few more cycles.

iii repeat this set several more times for an appropriate statistical measurement of the static displacement. stop recording. The difference between the two equilibrium angles in a set of measurements is equal to $2\theta_D$.

3 **Driven (resonant) Measurement of G.** The boom is driven into an oscillating state after which it is allowed to die off in the damped system. The equilibrium angle and the damping coefficient are then used to calculate θ_D, which is then used to calculate G.

A Set M,M' to center position (far away from glass walls)

B Wait for boom oscillation $<$ 1 mR amplitude.

C Alternately swing the M/M' carriage to gently touch opposite sides of the glass walls in order to increase the amplitude of boom oscillation. This is done by positioning M/M' so that the boom is always approaching it. When the boom reaches a turning point and starts to move away from M/M', swing M/M' around to the opposite side so that the m/m' are again moving toward M/M'.

D After the boom amplitude appears to approach some asymptotic value (about 6 or 7 cycles), move M/M' to center position and record boom oscillation's natural decay (about 10 cycles).