To Brake or Not to Brake

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Background:

The motivation behind this model is to determine the "dilemma zone" in which a driver can neither brake safely before reaching the intersection nor safely clear the intersection going at the same speed.

In the model, the driver has a reaction time $\delta = .8$ seconds, the traffic light has a yellow $\rightarrow$ red change time $\tau = 3$ seconds, an intersection width $W = 45$ meters, a braking decceleration $a = -3$ m/s^2, a variable initial position x_0, and an initial velocity $v_0 = 55$ km/hr.[1]

The geometry of the problem is such that $x_0 \leq 0$, $v_0 \geq 0$, $a < 0$, $x = 0$ is the start of the intersection, and $x = W$ is the end of the intersection. In other words, the car is initially heading towards the intersection with negative position x_0 and positive velocity v_0 when the light turns yellow at $t = 0$. After δ seconds, the driver either applies brakes with deceleration a, or continues through the intersection at constant velocity v_0. The purpose of this model is to show when the car will be able to safely stop before the intersection or safely pass the intersection maintaining constant velocity and to calculate the conditions for a "dilemma zone" where there is no safe option.

A useful equation used to calculate the piecewise equations of motion is the Heaviside function, which is shown by Equation (1),

$$\Theta(x) = \frac{1}{2}(np.sign(x) + 1), \tag{1}$$

where $np.sign(x)$ returns -1 for $x < 0$, 0 for $x = 0$, and 1 for $x > 0$. In order to avoid $\Theta(x) = \frac{1}{2}$, a small value ($\epsilon = .01$ seconds) is sometimes used to make sure the argument always falls either above or below 0.

The position as a function of time if the driver does not brake, is given by Equation (2),

$$x_{go}(t) = x_0 + v_0 t; \tag{2}$$

if the driver decides to brake, the position is given by Equation (3),

$$\begin{aligned} x_{react} &= x_0 + v_0\delta \\ x_{slow}(t) &= x_{react} + v_0(t-\delta) + \frac{1}{2}a(t-\delta)^2 \\ x_{stop} &= x_{slow}(\sigma) \\ \therefore x_{brake}(t) &= \Theta(\delta - t + \epsilon)x_{react} + \Theta(t - \delta - epsilon)x_{slow}(t)\Theta(\sigma - t - \epsilon) + \Theta(t - \sigma + \epsilon)x_{stop}(t), \end{aligned} \tag{3}$$

where σ is the time it takes for the car to come to a complete stop. σ is derived in Equation (6).

If the driver decides to pass the intersection, the velocity is found by Equation (4),[2]

$$v_{g0} = v_0; \tag{4}$$

[1]this is multiplied by a conversion factor of $\frac{5}{18}$ to convert it to m/s

[2]this is a challenging one...

if the driver decides to brake, the velocity is found by Equation (5)

$$
\begin{aligned}
v_{react} &= \Theta(\delta - t + \epsilon)v_0 \\
v_{slow}(t) &= \Theta(t - \delta - \epsilon)\Big(v_0 + a(t-\delta)\Big)\Theta(\sigma - t - \epsilon) \\
\therefore v_{brake}(t) &= v_{react} + v_{slow}(t).
\end{aligned} \tag{5}
$$

The stopping time, σ, is found by setting $v_f = 0$ in the basic kinematic equation,

$$
\begin{aligned}
v_f &= v_0 + a(t-\delta) \\
\rightarrow 0 &= v_0 + a(\sigma - \delta) \\
\therefore \sigma &= -\frac{v_0}{a} + \delta.
\end{aligned} \tag{6}
$$

This problem is also solved analytically and compared against the numerical results.

First, lets start with the min breaking distance, $|x_0^B|$, which is found by taking the condition $x_{brake}(t=\sigma) = 0$ meters,[3]

$$
\begin{aligned}
0 &= (x_0^B + v_0\delta) + v_0(\sigma - \delta) + \frac{1}{2}a(\sigma-\delta)^2 \\
&= x_0^B + v_0\sigma + \frac{1}{2}a(\sigma-\delta)^2 \\
&= x_0^B + v_0\Big(-\frac{v_0}{a} + \delta\Big) + \frac{1}{2}a\Big[\Big(-\frac{v_0}{a} + \delta\Big) - \delta\Big]^2 \\
&= x_0^B - \frac{v_0^2}{a} + v_0\delta + \frac{1}{2}\frac{v_0^2}{a} \\
&= x_0^B - \frac{v_0^2}{2a} + v_0\delta \\
\therefore x_0^B &= \frac{v_0^2}{2a} - v_0\delta.
\end{aligned} \tag{7}
$$

The minimum passing position, x_0^A, is a little easier; it is found by setting the condition $x_{go}(t=\tau) = W$ and is shown in Equation (8),[4]

$$
\begin{aligned}
W &= x_0^A + v_0\tau \\
\therefore x_0^A &= W - v_0\tau
\end{aligned} \tag{8}
$$

The length of the "dilemma zone" is then found by Equation (9),

$$
\begin{aligned}
s(v_0) &= x_0^A - x_0^B \\
&= \Big(W - v_0\tau\Big) - \Big(\frac{v_0^2}{2a} + v_0\delta\Big) \\
&= W + v_0(\delta - \tau) - \frac{v_0^2}{2a}
\end{aligned} \tag{9}
$$

[3]compare with Equation (3)
[4]compare against Equation (2)

Results and Discussion:

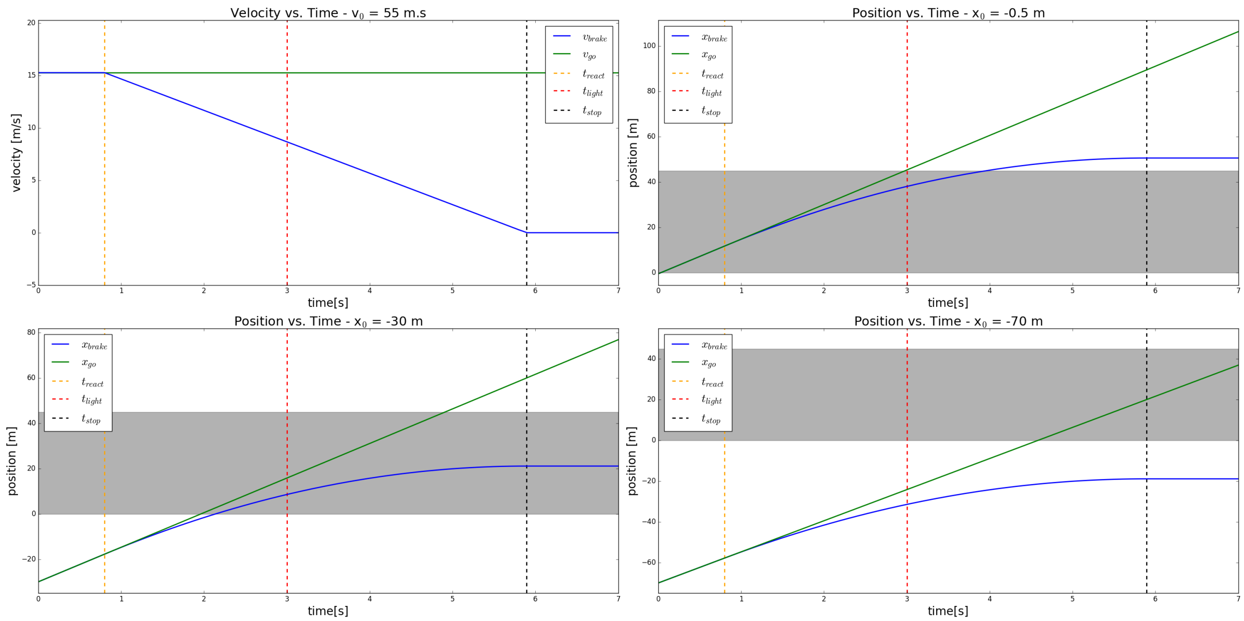

Figure 1: The time axis spans $0 < t < 7$ seconds with $\Delta t = 0.1$ seconds. All four plots use the green to signify the braking scenario and blue to signify the braking scenario. All four plots show δ (yellow), τ (red), and σ (black). The three position vs. time plots show the intersection blocked out in gray. Looking at the plot 1, it is clear to see that the driver will not be able to bring the car to a stop in time, which means the driver's initial position is critical. Plot 2, $x_0 = -.5$ m, shows that the driver can safely clear the intersection without breaking; plot 3 shows that the driver can neither safely clear the intersection nor safely come to a stop before entering intersection; plot 4 shows that the driver can safely stop before entering the intersection.

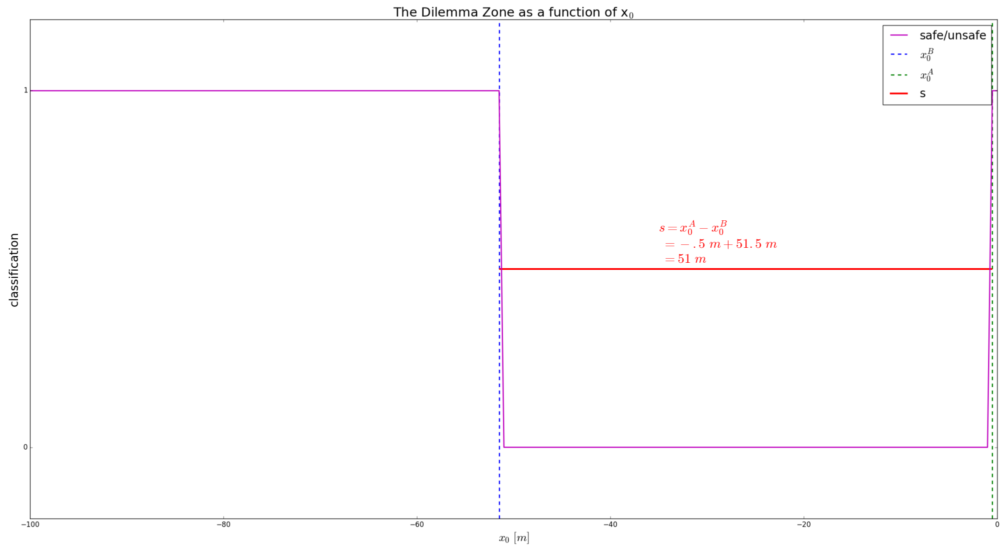

Figure 2: This plot is produced by passing a position array spanning $-100 < x_0 < 0$ meters with $\Delta x_0 = .5$ meters through the Python function "decide pos" (line 239), which returns "1" for safe and "0" for un-safe. From this plot, it is shown that $x < x_0^B = -51.5$ m is a safe initial position to brake when $v_0 = 55$ km/hr. $x > x_0^A = -.5$ m is the range to safely cross the intersection. The interval $s = 51$ meters, $x_0^B < x < x_0^A$, is the "dilemma zone" interval.

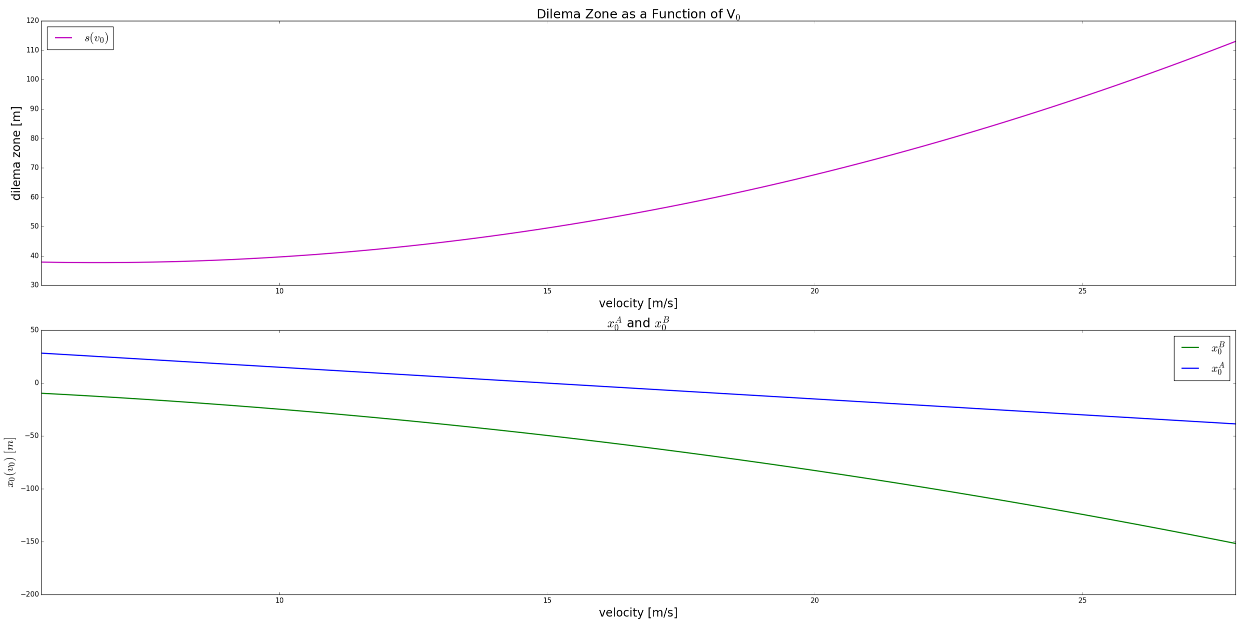

Figure 3: The top plot shows the analytical solution for $s(v_0)$ - Equation 9. The top plot does seem to agree with the numerical results. The bottom plot shows the analytical solutions to x_0^B and x_0^A - Equations 7 and 8. Mentally subtracting x_0^B from x_0^A in plot 2 also agrees with the resulting $s(v_0)$ curve in plot 1.

Summary:

The results of this model are included in Table (1)

Table 1: Results

	x_0^B	x_0^A	**s**
Numerical Value [m]	-51.5	-0.5	51
Analytical Value [m]	-51.1	-0.8	50.3
Absolute Error [m]	0.4	0.3	0.7
Percentage Error	0.7	40.0	1.4

Appendix:

Table 2: List of Variables, Definitions and Values

Description	**Variable**	**Value**
reaction time [s]	δ	0.8
yellow $\rightarrow$ red time [s]	τ	3
intersection width [m]	W	45
braking decceleration [m/s^2]	a	-3
initial position [m]	x_0	[-0.5, -30, -70]
initial velocity [km/hr]	v_0	55
stopping time [s]	σ	$-\frac{v_0}{a} + \delta$
max safe initial position to brake [m]	x_0^B	$\frac{v_0^2}{2a} - v_0\delta$
min safe initial position to clear intersection [m]	x_0^A	$W - v_0\tau$
length of "dilemma zone" interval [m]	$s(v_0)$	$W + v_0(\delta - \tau) - \frac{v_0^2}{2a}$