Questions and Answers

Find the Null space and column space of the following matrices

1. $A=\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}$

Now consider $Ax=0$

$$\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=0\sim\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=0;$$

$$x=0, y=0$$

Thus Null space is zero vector or (0, 0). Geometrically N(A) is an origin in 2 dimensional vector space.

The column space is R^2

2. $A=\begin{bmatrix}-1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}$

Ans: $A=\begin{bmatrix}-1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}=U$

C(A) is R^2.

To find the Null space of A , consider $Ax=0$

$$Ax=\begin{bmatrix}-1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=0$$

$$-x+y=0; \quad x=y$$

$$y+z=0 \;\; ; \quad y=-z$$

N(A) is line in R^3

3. $A=\begin{bmatrix}1 & 2\end{bmatrix}$

C(A) is R^1 and N(A) is line in R^2

Ax=0 has infinitely many solutions (-2k, 0 ,k) all of which lie on a line that obviously passes through the origin.

4. $A=\begin{bmatrix}1\\2\\2\end{bmatrix}$

C(A) is a line in R^3 and N(A) is a zero vector.

1. Decide the dependence or independence of the vectors

i) (1,3,2), (2,1,3), (0,0,0)

Ans: This set is linearly dependent, because of zero vector.

ii) (4,2,2), (2,4,2), (4,8,2)

Ans: $A=\begin{bmatrix}4 & 2 & 4\\2 & 4 & 8\\2 & 2 & 2\end{bmatrix}\sim\begin{bmatrix}4 & 2 & 4\\0 & 1 & 6\\0 & 1 & 0\end{bmatrix}\sim\begin{bmatrix}4 & 2 & 4\\0 & 1 & 6\\0 & 0 & -6\end{bmatrix}$

This set is independent.

iii) (1,2), (1,5),(3,4)

Ans: $A=\begin{bmatrix}1 & 1 & 3\\2 & 5 & 4\end{bmatrix}$

This set is linearly dependent, because n>m

Find the rank of the following matrices

1.

$$A=\begin{bmatrix}1 & 2 & 3 & 1\\ 2 & 3 & 1 & -1\\ 1 & 2 & 3 & 1\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & 3 & 1\\ 0 & -1 & -5 & -3\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Rank =2

2. $$A=\begin{bmatrix}1 & 0 & 2\\ 2 & 3 & 4\\ 1 & 6 & 2\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 2\\ 0 & 3 & 0\\ 0 & 6 & 0\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 2\\ 0 & 3 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Rank=2

3. $$A=\begin{bmatrix}1 & 2 & 2\\ -1 & -2 & 2\\ 2 & 0 & 2\end{bmatrix}\sim\begin{bmatrix}1 & 2 & 2\\ 0 & 0 & 4\\ 0 & -4 & -2\end{bmatrix}\sim\begin{bmatrix}1 & 2 & 2\\ 0 & -4 & -2\\ 0 & 0 & 4\end{bmatrix}$$

Rank=3

Find the bases and dimension of the four fundamental subspaces of the following matrices

1. $A=\begin{bmatrix}1 & 2 & 3\\ 1 & 3 & 1\\ 1 & 2 & 4\end{bmatrix}$

Ans: $A=\begin{bmatrix}1 & 2 & 3\\ 1 & 3 & 1\\ 1 & 2 & 4\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix}\sim\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}a\\ b-a\\ c-a\end{bmatrix}$

Basis of C(A)={ (1,1,1), (1,3,1),(3,1,4)}; Dim=3

Basis of C(A^T)={ (1,2,3),(1,3,1) ,(1,2,4)}; Dim=3

Basis of N(A^T)={ 0} in R^3 ; Dim=0

Basis of N(A)={ 0} in R^3 ; Dim=0

2.

$$A=\begin{bmatrix}1 & 0 & 2\\ 2 & 3 & 4\\ 1 & 6 & 2\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 2\\ 0 & 3 & 0\\ 0 & 6 & 0\end{bmatrix}\begin{bmatrix}a\\ b-2a\\ c-a\end{bmatrix}\sim$$

$$\begin{bmatrix}1 & 0 & 2\\ 0 & 3 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}a\\ b-2a\\ 3a-2b+c\end{bmatrix}$$

Basis of C(A)={ (1,2,1), (0,3,6)}; Dim=2

Basis of $C(A^T)$={ (1,0,2),(2,3,4)}; Dim=2

Basis of $N(A^T)$={ (3,-2,1)} in R^3 ; Dim=1

Basis of N(A)={ (-2,0,1)} in R^3 ; Dim=1

2. $A=\begin{bmatrix}1 & 2 & 1 & 2\\ 1 & 2 & 1 & 3\\ 3 & 6 & 3 & 7\end{bmatrix}$

Ans: $A=\begin{bmatrix}1 & 2 & 1 & 2\\ 1 & 2 & 1 & 3\\ 3 & 6 & 3 & 7\end{bmatrix};\begin{bmatrix}a\\ b\\ c\end{bmatrix}$

$$=\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\end{bmatrix};\begin{bmatrix}a\\ b-a\\ c-3a\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix};\begin{bmatrix}a\\ b-a\\ c-2a-b\end{bmatrix}$$

Basis of C(A)={ (1,1,3), (2,3,7)}; Dim=2

Basis of $C(A^T)$={ (1,2,1,2), (1,2,1,3)}; Dim=2

Basis of $N(A^T)$={ (-2,-1,1)}; Dim=2

To find the basis of N(A):

Consider UX=0 or AX=0

i.e

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 0;$$

$$x + 2y + z + 2t = 0; x = -(2y + z + 2t)$$
$$y = 0;$$

Here y and z are free variables

Basis for N(A)={(-2,1,0,0),(-1,0,1,0)} ; Dim=2

Find the left/right inverses of the following.

1. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3}$

Ans: Rank r=2=m(m<n)

$$C = A^T\left(AA^T\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

2. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}_{2\times 3}$

Rank r=2=m(m<n)

$$C = A^T\left(AA^T\right)^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/4 & 0 \\ 0 & 1/16 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}$$

3. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}_{3\times 2}$

Rank r=2=n(n<m)

$$B = \left(A^T A\right)^{-1} A^T = \begin{bmatrix} 3/2 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$