DIGITAL SIGNAL PROCESSING
Unit - 3 - Analog Filter Design - Lecture Notes

June 28, 2021

Contents

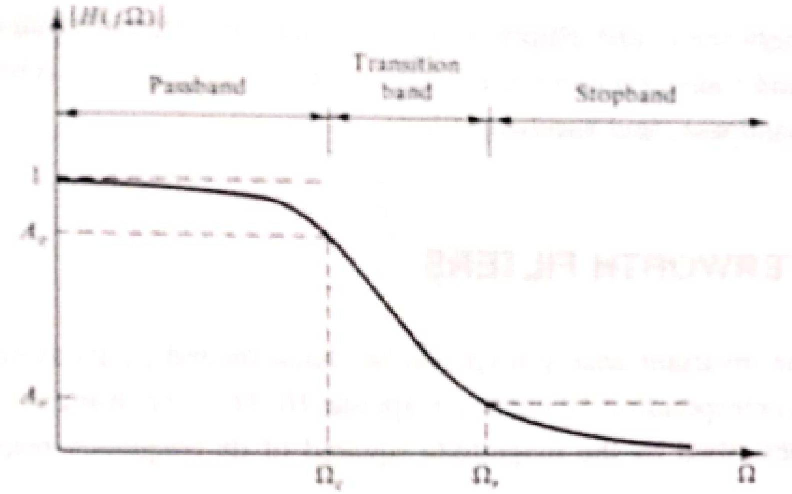

Figure 1: A typical required frequency response for a low-pass filter design

1 Introduction

- A very important approach to the design of digital filters is to apply a transformation to an existing analog filter.
- For this method of designing a digital filter,it is necessary to have a base or catalog of analog filters that can serve as the prototypes for the transformation.
- These analog procedures normally begin with a specification of the frequency response for the filter describing how the filter reacts in the steady state to sinusoidal inputs.
- If an input sinusoid is not attenuated or attenuated less than a specified tolerance as it goes through the system,it is said to be in a **pass band** of the filter.
- If an input sinusoid is attenuated more than a specified value,it is said to be stopped and within the **stop band** of the filter.
- Input sinusoids with neither a little nor a large amount of attenuation are said to be in the **transition band**.
- A typical frequency response is shown in Figure.1. showing the pass band,transition band,and stop band.
- The filter with this type of frequency response is called a low-pass filter as it passes all frequencies less than a certain value Ω_c,called the **cutoff frequency**.
- Other important basic types of filters are the **high-pass(HP),band pass(BP),and Band stop filters**,whose frequency responses are shown in Figure.2
- Also shown are the frequency responses for the ideal LP,HP,BP and BS filters which exhibit no transition.
- More complex filters can be obtained by placing four basic types in various **parallel and cascade** configurations.
- It is known that the low-pass,high-pass,band-pass,and band-stop filters can be obtained from a **normalized low pass filter** via specific transformations in the s-plane.
- Therefore,prime consideration will be given to **low-pass filter design**.
- In particular,the properties and design procedures for the **analog butterworth,chebyshev,and elliptic low-pass filters** are presented along with the procedures and transformations necessary to transform them into low-pass,high-pass,band-pass and band-stop filters.

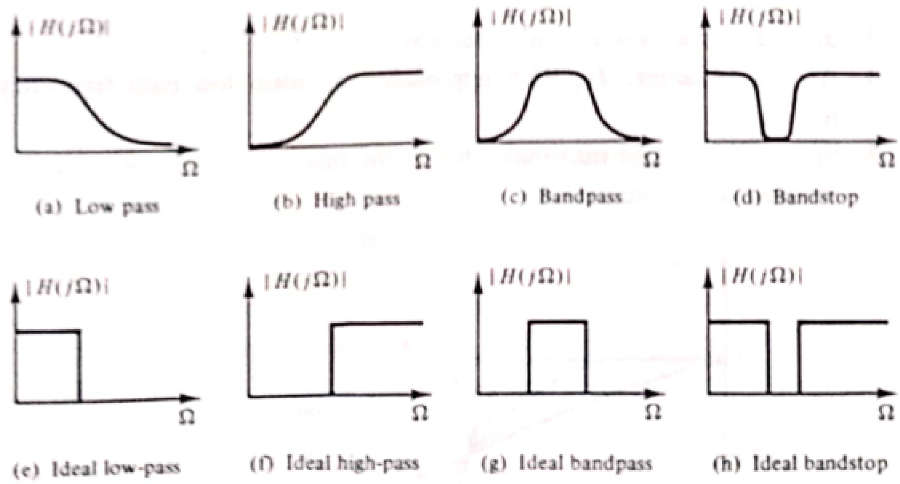

Figure 2: Basic types of frequency responses

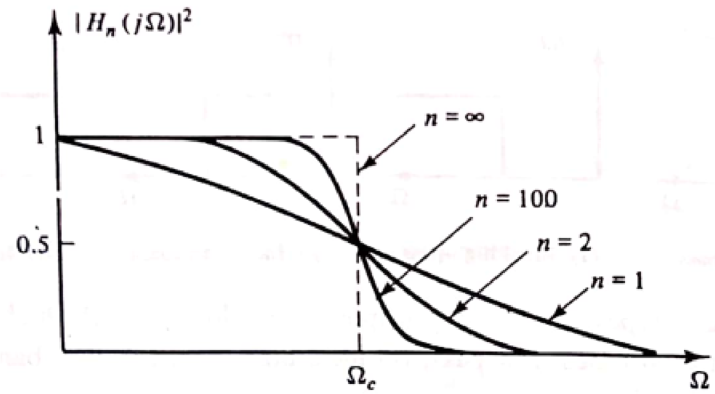

Figure 3: The magnitude squared frequency response for a Butterworth filter

2 Butterworth Filters

- A linear time invariant analog filter can be characterized by its system function $H(s)$ or its corresponding frequency response $H(j\Omega)$.
- The Butterworth filter of order n is described by the magnitude squared of its frequency response given by

$$|H_n(j\Omega)^2| = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2n}} \ldots (1)$$

- In figure 3,the magnitude squared frequency response of the butterworth filter is shown for different values of n.
- The following properties can be easily determined:

  1. $|H_n(j\Omega)^2|_{\Omega=0} = 1$ for all n
  2. $|H_n(j\Omega)^2|_{\Omega=\Omega_c} = 1/2$ for all finite n.

     This implies that $20log|H_n(j\Omega)|_{\Omega=\Omega_c} = -3.0103$

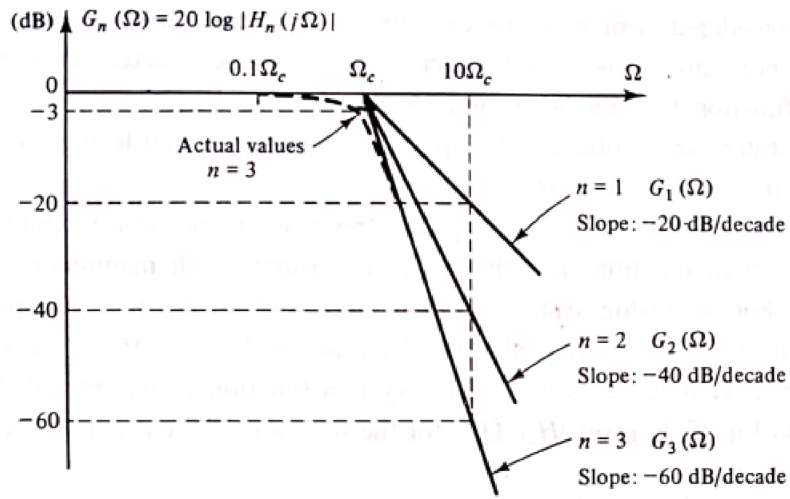

Figure 4: Filter gain plot for analog butterworth filter of various orders

3. $|H_n(j\Omega)^2|$ is a monotonically decreasing function of Ω.

4. As n gets larger,$|H_n(j\Omega)^2|$ approaches an ideal low-pass frequency response.

5. $|H_n(j\Omega)^2|$ is called maximally flat at the origin since all order derivatives exist and are zero.

- It is convenient in many cases to look at the frequency response in decibels,that is,plot $20\log|H(j\Omega)$ versus Ω.
- Figure 4 is a straight line approximation of the frequency response in decibels for the butterworth filter.
- The straight lines are determined for $\Omega << \Omega_c$ by approximating $20\log|H_n(j\Omega)$.
- We have,

$$G_n(\Omega) = 20\log|H_n(j\Omega)| \ldots (2)$$

$$G_n(\Omega) = 10\log|H_n(j\Omega)|^2 \ldots (3)$$

$$G_n(\Omega) = 10\log\left[\frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2n}}\right] \ldots (4)$$

$$G_n(\Omega) = 10\log(1) - 10\log\left[1+(\frac{\Omega}{\Omega_c})^{2n}\right] \ldots (5)$$

- Since the $\left(\frac{\Omega}{\Omega_c}\right)^{2n}$ is approximately zero for $\Omega << \Omega_c$,we have
- $$G_n(\Omega) \approx -10\log(1+0) = 0 \ldots (6)$$

.

- For $\Omega >> \Omega_c$,1 in the summation becomes insignificant compared to $\left(\frac{\Omega}{\Omega_c}\right)^{2n}$ so,

$$G_n(\Omega) = -10\log\left|\left(\frac{\Omega}{\Omega_c}\right)^{2n}\right| \ldots (7)$$

$$G_n(\Omega) = -20n\log\left|\left(\frac{\Omega}{\Omega_c}\right)\right| \ldots (8)$$

- Therefore,the appropriate gain for $\Omega << \Omega_c$ is 0dB while for $\Omega >> \Omega_c$,the slope of $G_n(\Omega) = -20ndB/decade$.
- For $\Omega = \Omega_c$,the actual value of $G_n(j\Omega)$ is approximately $-3dB$,as seen earlier,and the actual values for Ω around Ω_c can be calculated using equation(5) as shown below.

$$G_n(\Omega) = 10\log(1) - 10\log\left[1 + (\frac{\Omega}{\Omega_c})^{2n}\right] \ldots (5)$$

- In most of the work to follow,the normalized butterworth low pass filter will be considered,which is the case for $\Omega_c = 1 \ \ rad/sec$.
- It will be shown,once the normalized low-pass filter transfer function has been determined,that the transfer function for any other **butterworth low-pass,high-pass,band-pass or band-stop filter** can be obtained by applying a transformation to the normalized low-pass filter specified by $H_n(s)$.
- Starting with the magnitude squared frequency response,we would like to find the **system function** $H(s)$ that gives the Butterworth magnitude squared response.
- For an analog system,we remember that the frequency response is obtained by setting $s = j\Omega$ in the transfer function $H(s)$ for the given system.
- Therefore,if Ω is replaced by $\frac{s}{j}$,the system function is determined.
- Setting $\Omega_c = 1$ in equation below gives $|H_n(j\Omega)|^2$ for the normalized filter as follows
- We know that

$$|H_n(j\Omega)^2| = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2n}} \ldots (1)$$

- For $\Omega_c = 1$,we get

$$|H_n(j\Omega)^2| = \frac{1}{1 + \Omega^{2n}} \ldots (6)$$

$$H_n(j\Omega) * H_n(-j\Omega) = \frac{1}{1 + \Omega^{2n}} \ldots (7)$$

$$H_n(s) * H_n(-s) = \frac{1}{1 + \left(\frac{s}{j}\right)^{2n}} \ldots (8)$$

- Poles of $H_n(s) * H_n(-s)$ are given by the roots of the denominator.i.e

$$1 + \left(\frac{s}{j}\right)^{2n} = 0 \ldots (9)$$

$$1 + \left(\frac{s^{2n}}{j^{2n}}\right) = 0 \ldots (10)$$

$$\left(\frac{s^{2n}}{j^{2n}}\right) = -1 \ldots (11)$$

$$s^{2n} = -1 * (j)^{2n} \ldots (12)$$

$$s^{2n} = -1 * (j^2)^n \ldots (13)$$

$$s^{2n} = -1 * (-1)^n \ldots (14)$$

$$s^{2n} = (-1)^{n+1} \ldots (15)$$

- The roots of the above equation can be identified for the cases **when n is odd and when n is even**.
- For n odd,the poles of $H_n(s) * H_n(-s)$ become $2n^{th}$ roots of 1,while for n even,the poles are $2n^{th}$ roots of -1.
- for n odd,

$$s_k = 1\angle(\frac{k\pi}{n}), k = 0, 1, 2, \ldots, 2n - 1 \ldots (16)$$

- for n even,

$$s_k = 1\angle(\frac{\pi}{2n} + \frac{k\pi}{n}), k = 0, 1, 2, \ldots, 2n - 1 \ldots (17)$$

- These poles for n odd and n even are illustrated in figure 5.
- For n odd,we have a pole of $H_n(s)H_n(-s)$ at $s = 1$,and then poles equally spaced on the circle $\frac{\pi}{n}$ in angle,while for n even,the first pole is at $1\angle(\frac{\pi}{2n})$ and the remaining poles equally spaced around the unit circle by $\frac{\pi}{n}$.
- If we wish the filter $H_n(s)$ to be a stable and causal filter,the poles of $H_n(s)$ are selected to be those in the left half plane and $H(s)$ can be written in the following form

$$H_n(s) = \frac{1}{\Pi(s - s_k)} = \frac{1}{B_n(s)} \ldots (18)$$

- where s_k are the left half plane poles of $H_n(s) * H_n(-s)$.
- The denominator,$B_n(s)$,can be shown to be a **butterworth polynomial** of order n.

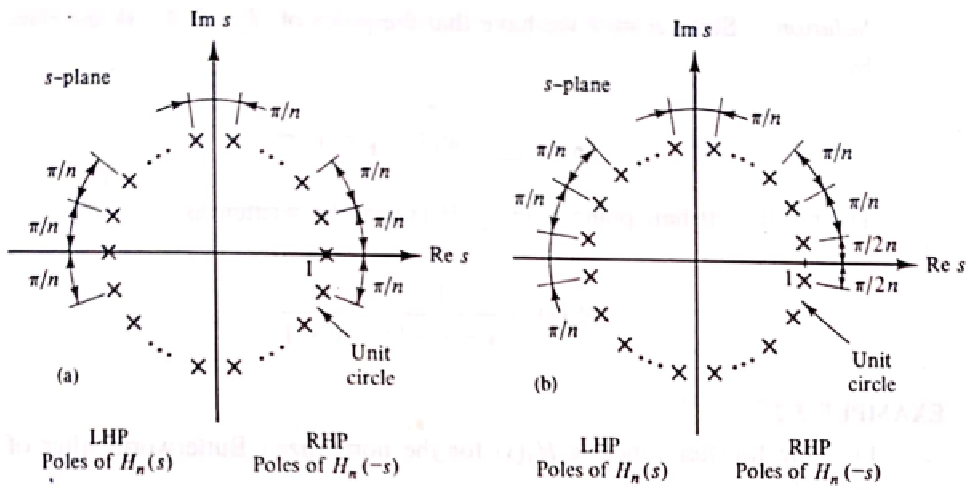

Figure 5: Poles of $H_n(s) * H_n(-s)$ for a low-pass butterworth filter with a 1-radian cut-off frequency for n even and n odd

Order n	Butterworth polynomial $B_n(s)$
1	$s + 1$
2	$s^2 + \sqrt{2}s + 1$
3	$(s^2 + s + 1)(s + 1)$
4	$(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)$
5	$(s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$

Normalized low-pass Butterworth filters

$$H_n(s) = \frac{1}{B_n(s)}$$

Figure 6: Butterworth polynomials and Normalized butterworth low-pass filters

- Table shown in Figure 6 shows the first five Butterworth polynomials in a real factored form.
- **Example 1**: Find the transfer function $H_1(s)$ for the normalized Butterworth filter of order 1.
- **Solution**: Since $n = 1$,we have that the poles of $H_1(s) * H_1(-s)$ are given by

$$s_k = 1\angle(\frac{\pi k}{n}) \ldots (1)$$

Here k=0,1

$$s_0 = 1\angle(0) \ldots (2)$$

$$s_1 = 1\angle(\pi) = -1 \ldots (3)$$

Therefore,

$$s_0 = 1, s_1 = -1 \ldots (4)$$

Taking left half plane pole s_1,$H_1(s)$ can be written as

$$H_1(s) = \frac{1}{s+1} \ldots (5)$$

- **Example 2**: Find the transfer function $H_2(s)$ for the **normalized butterworth filter** of order 2.
- **Solution**: Since $n = 2$,we have poles of $H_2(s) * H_2(-s)$ given by

$$s_k = 1\angle(\frac{\pi}{2n} + \frac{k\pi}{n}) \ldots (1)$$

Since $n = 2$,we have

$$s_k = 1\angle(\frac{\pi}{4} + \frac{k\pi}{2}) \ldots (2)$$

We know that

$$k = 0, 1, 2, \ldots, 2n - 1 \ldots (3)$$

Therefore

$$k = 0, 1, 2, 3 \ldots (4)$$

For k=0,

$$s_0 = \frac{\pi}{4}$$

For k=1,

$$s_0 = \frac{3\pi}{4}$$

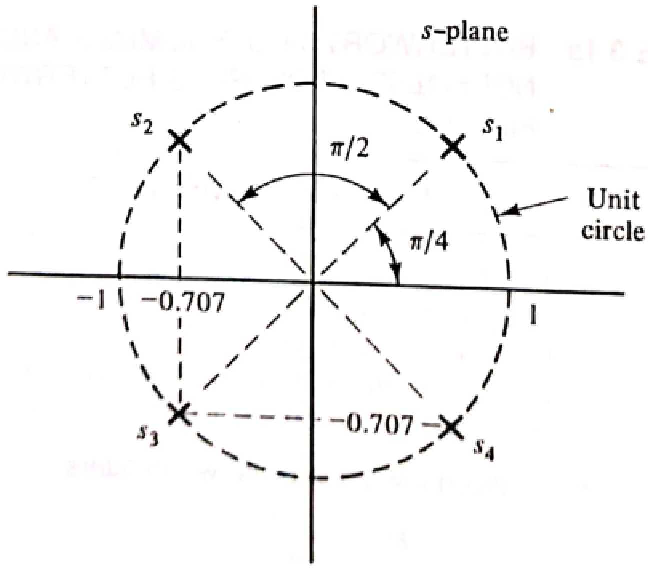

Figure 7: Pole Diagram for example 2

For k=2,

$$s_0 = \frac{5\pi}{4}$$

For k=3,

$$s_0 = \frac{7\pi}{4}$$

- These poles are shown in Figure 7.
- Considering only left half of s-plane poles, s_1 and s_2,

$$S_1 = 1\angle(\frac{3\pi}{4}) \ldots (5)$$

$$S_2 = 1\angle(\frac{5\pi}{4}) \ldots (6)$$

$$s_1 = \cos(\frac{3\pi}{4}) + \sin(\frac{3\pi}{4}) \ldots (7)$$

$$s_1 = -0.707 + j0.707 \ldots (8)$$

$$s_2 = \cos(\frac{5\pi}{4}) + \sin(\frac{5\pi}{4}) \ldots (9)$$

$$s_2 = -0.707 - j0.707 \ldots (10)$$

$$H_2(s) = \frac{1}{(s - s_2)(s - s_1)} \ldots (11)$$

- Subtituting for s_1 and s_2 and simplifying,we get

$$H_2(s) = \frac{1}{s^2 + \sqrt{2}s + 1} \ldots (12)$$

3 Design of Low-Pass Butterworth Filters

- The filter requirements are normally given in terms of a set of critical frequencies,say Ω_1,Ω_2 and gains K_1 and K_2.
- A common set of conditions for the low-pass response given in Figure 8 are

$$0 \geq 20 \log |H(j\Omega)| \geq K_1 \ \ for\ all\ \ \Omega \leq \Omega_1 \ldots (1)$$

$$20 \log |H(j\Omega)| \leq K_2 \ \ for\ all\ \ \Omega \geq \Omega_2 \ldots (2)$$

- The **Butterworth LP frequency response** is characterized by only two parameters,n,the order of the filter,and Ω_c,the cut-off frequency.
- The Butterworth filter of order n is described by the magnitude squared frequency response given by

$$|H_n(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2n}} \ldots (3)$$

- If we replace $H(j\Omega)$ in equations (1) and (2) by equation (3) and consider that the equalities hold,n and Ω_c must satisfy the following.

$$10 \log_{10} \frac{1}{1 + \left(\frac{\Omega_1}{\Omega_c}\right)^{2n}} = k_1 \ldots (4)$$

$$10 \log_{10} \frac{1}{1 + \left(\frac{\Omega_2}{\Omega_c}\right)^{2n}} = k_2 \ldots (5)$$

- From equation (4),we have

$$\log_{10} \frac{1}{1 + \left(\frac{\Omega_1}{\Omega_c}\right)^{2n}} = \frac{k_1}{10} \ldots (6)$$

$$-\log_{10}\{1 + \left(\frac{\Omega_1}{\Omega_c}\right)^{2n}\} = \frac{k_1}{10} \ldots (7)$$

$$\{1 + \left(\frac{\Omega_1}{\Omega_c}\right)^{2n}\} = 10^{\frac{-k_1}{10}} \ldots (8)$$

$$\left(\frac{\Omega_1}{\Omega_c}\right)^{2n} = 10^{-\frac{k_1}{10}} - 1 \ldots (9)$$

- Similarly

$$\left(\frac{\Omega_2}{\Omega_c}\right)^{2n} = 10^{-\frac{k_2}{10}} - 1 \ldots (10)$$

- Dividing to cancel Ω_c,we have the following implicit equation relating Ω_1,Ω_2,k_1,k_2.

$$\left(\frac{\Omega_1}{\Omega_2}\right)^{2n} = \frac{10^{-\frac{k_1}{10}} - 1}{10^{-\frac{k_2}{10}} - 1} \ldots (11)$$

- Taking log on both sides

$$\log_{10}\left(\frac{\Omega_1}{\Omega_2}\right)^{2n} = \log_{10}\frac{10^{-\frac{k_1}{10}} - 1}{10^{-\frac{k_2}{10}} - 1} \ldots (12)$$

$$2n\log_{10}\left(\frac{\Omega_1}{\Omega_2}\right) = \log_{10}\frac{10^{-\frac{k_1}{10}} - 1}{10^{-\frac{k_2}{10}} - 1} \ldots (13)$$

$$n = \frac{\log_{10}\frac{10^{-\frac{k_1}{10}} - 1}{10^{-\frac{k_2}{10}} - 1}}{2\log_{10}\left(\frac{\Omega_1}{\Omega_2}\right)} \ldots (14)$$

- Using this value of n results in two different selections for Ω_c as seen from equations (9) and (10).
- If we wish to satisfy our requirement at Ω_1 exactly and do better than our requirement at Ω_2,we use

$$\Omega_c = \frac{\Omega_1}{\left(10^{\frac{-k_1}{10}} - 1\right)^{1/2n}} \ldots (15)$$

- While,if we wish to satisfy our requirement at Ω_2 and exceed our requirement at Ω_1,we use

$$\Omega_c = \frac{\Omega_2}{\left(10^{\frac{-k_2}{10}} - 1\right)^{1/2n}} \ldots (16)$$

- Choosing a value of Ω_c in between these gives an $H(j\Omega)$ that exceeds these requirements.

4 Analog to analog Transformations

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- We have placed emphasis on the one radian low-pass butterworth filters with the stipulation that other non-normalized Butterworth filters could be derived by transformational methods.
- The transformational method,however is not limited in its application to butterworth filters or normalized filters.
- In the following discussion,a normalized low-pass filter will be used as the prototype filter for the purpose of illustration.
- If we replace S of $H(s)$,the system function for a normalized low-pass filter by $\frac{s}{\Omega_u}$,we get a new transfer function $H'(s)$,given by

$$H'(s) = H(s)|_{s \to \frac{s}{\Omega_u}} = H(\frac{s}{\Omega_u}) \ldots (1)$$

- If we evaluate the magnitude of the transfer function $H'(s)$ at $s = j\Omega$ to get the frequency response,we have

$$|H'(j\Omega)| = |H\left(\frac{j\Omega}{\Omega_u}\right)| \ldots (2)$$

- At the value of $\Omega = \Omega_u$,we have

$$H'(j\Omega_u) = H\left(\frac{j\Omega_u}{\Omega_u}\right) = H(j.1) \ldots (3)$$

- That is the frequency response for the new transfer function evaluated at $\Omega = \Omega_u$ is equal to the value of the normalized filter transfer function at $\Omega = 1$.
- In a sense,we have moved the cut-off frequency from 1 rad/sec to Ω_u rad/sec and thus have a scaling of the frequency axis.
- Similar transformations can be defined for taking low-pass filter transfer functions to high-pass,band-pass and band-stop transfer functions.
- Table shown in figure gives these transformations along with design equations for both forward and backward equations.
- If the transformation $s \to \frac{s}{\Omega_u}$ is applied to the low-pass structure as shown in the table;
- The critical frequency Ω_r will be transformed in to Ω_r',which is Ω_r times Ω_u as seen under the design equation column.
- In a similar fashion,if Ω_r' is the desired critical frequency of the transformed filter,the backward equation gives the value of Ω_r that must be used such that going through the transformation $s \to \frac{s}{\Omega_u}$ results in the required Ω_r'.
- We have Ω_r equals $\frac{\Omega_r'}{\Omega_u}$.
- **Example on Analog Butterworth Low Pass Filter** :Design an analog butterworth filter that has a $-2dB$ or better cut-off frequency of 20 rad/sec and atleast 10 dB of attenuation at 30 rad/sec.
- **Solution**: The critical requirements are

  $\Omega_1 = 20rad/sec$,$K_1 = -2$,$\Omega_1 = 30rad/sec$,$K_1 = -10$

- The order n of the butterworth filter is given by

$$n = \frac{\log_{10}\left[\frac{10^{\frac{-K_1}{10}}-1}{10^{\frac{-K_2}{10}}-1}\right]}{2\log_{10}\frac{\Omega_1}{\Omega_2}} \dots (1)$$

$$n = \frac{\log_{10}\left[\frac{10^{\frac{2}{10}}-1}{10^{\frac{10}{10}}-1}\right]}{2\log_{10}\frac{20}{30}} \dots (2)$$

$$n \approx 4 \dots (3)$$

$$\Omega_c = \frac{\Omega_1}{\left(10^{\frac{-K_1}{10}} - 1\right)^{\frac{1}{2n}}} \dots (3)$$

$$\Omega_c = \frac{20}{(10^{0.2} - 1)^{\frac{1}{8}}} \dots (4)$$

$$\Omega_c = 21.3868 \ \ rad/sec \dots (5)$$

- The normalized low-pass Butterworth filter for $n = 4$ can be found from table as

$$H_4(s) = \frac{1}{(s^2 + 0.7653s + 1)(s^2 + 1.84776s + 1)} \dots (6)$$

- Applying a low-pass to low-pass transformation $s \rightarrow \frac{s}{\Omega_c}$,with $\Omega_c = 21.3868$ gives the desired transfer function as follows

$$H(s) = \frac{1}{A * B} \dots (7)$$

$$A = \frac{1}{\left[\left(\frac{s}{21.3868}\right)^2 + 0.76536\left(\frac{s}{21.3868}\right) + 1\right]} \dots (8)$$

$$B = \frac{1}{\left[\left(\frac{s}{21.3868}\right)^2 + 1.84776\left(\frac{s}{21.3868}\right) + 1\right]} \dots (9)$$

$$H(s) = \frac{0.2092 * 10^6}{A * B} \dots (10)$$

$$A = \frac{1}{s^2 + 16.3686s + 457.394} \dots (11)$$

$$B = \frac{1}{s^2 + 39.5176s + 457.394} \dots (12)$$

5 Design of Band Pass Butterworth Filters

- The Design of a band pass butterworth filter is also based on applying a transformation to a low-pass normalized butterworth filter of the proper order.
- The typical requirements are shown in Figure 8.

$$0 \leq 20\log|H(j\Omega)| \leq K_2 \ for \ \Omega \leq \Omega_1 \ldots (1)$$

$$0 \leq 20\log|H(j\Omega)| \leq K_1 \ for \ \Omega_l \leq \Omega \leq \Omega_u \ldots (1)$$

$$20\log|H(j\Omega)| \leq K_2 \ for \ \Omega \geq \Omega_2 \ldots (1)$$

- If $H_{LP}(s)$ represents a unit bandwidth low-pass filter with critical radian frequency Ω_r,then from table for analog-to-analog transformation,a band-pass filter with transfer function $H_{BP}(s)$ is given by

$$H_{BP}(s) = H_{LP}(s)\Big|_{s \to \frac{s^2+(\Omega_l*\Omega_u)}{s(\Omega_u-\Omega_l)}} \ldots (2)$$

- For the band-pass filter to satisfy the k_2 requirement at Ω_1,we must have the equality within the transformation,that is

$$j\Omega_r = \frac{[(j\Omega_1)^2 + (\Omega_l * \Omega_u)]}{(j\Omega_1)(\Omega_u - \Omega_l)} \ldots (3)$$

- Solving the above equation for Ω_r and a similar equation to satisfy the K_2 requirement at Ω_2 gives,

$$\Omega_r = \frac{[(\Omega_1)^2 + (\Omega_l * \Omega_u)]}{(\Omega_1)(\Omega_u - \Omega_l)} \ldots (4)$$

$$\Omega_r = \frac{[(\Omega_2)^2 + (\Omega_l * \Omega_u)]}{(\Omega_2)(\Omega_u - \Omega_l)} \ldots (5)$$

- Depending of the size of Ω_1 and Ω_2 with respect to the product $\Omega_l * \Omega_u$,the $\Omega's$ above could correspond to either positive and negative frequency values.
- Also,in most cases the Ω_r resulting from these equations will not be equal and we must choose the most restrictive value,thus giving a filter that exceeds our less restrictive requirement.
- Therefore,the selection of Ω_r becomes that given in the backward design equations for low-pass to band-pass transformation table shown previously.

$$\Omega_r = min\{|A|, |B|\} \ldots (6)$$

$$A = \frac{-(\Omega_1)^2 + \Omega_l * \Omega_u}{\Omega_1(\Omega_u - \Omega_l)} \ldots (7)$$

$$B = \frac{(\Omega_2)^2 + \Omega_l * \Omega_u}{\Omega_2(\Omega_u - \Omega_l)} \ldots (8)$$

- The procedure for the design of a band-pass filter,$H_{BP}(s)$,to satisfy the given set of specifications is composed of two steps:

  1. Design a low pass filter $H_{LP}(s)$ with the value of Ω_r
  2. Apply a low-pass to band-pass transformation using the desired Ω_l & Ω_u.

  Example on Design of Band-Pass Butterworth Filters

- Design an analog band-pass filter with the following charcteristics

  1. a -3.0103 **dB attenuation** at upper and lower cut-off frequency of $20KHz$ and $50Hz$ respectively.
  2. A **Stop-band attenuation** of atleast $20dB$ at $20Hz$ and $45KHz$.
  3. A **monotonic** frequency response.

  Solution: The monotonic requirement can be satisified with a butterworth filter.

- From the specifications,we can identify the following critical frequencies

  1. $\Omega_1 = 20 * 2\pi = 125.663 \ rad/sec$
  2. $\Omega_2 = (45k) * 2\pi = 2.82 * 10^5 \ rad/sec$
  3. $\Omega_u = (20k) * 2\pi = 1.25663 * 10^5 \ rad/sec$
  4. $\Omega_l = (50) * 2\pi = 314.159 \ rad/sec$

- Low-Pass prototype must satisfy

$$0 \geq 20 \log |H_{LP}(j.1)| \geq -3.0103dB$$

$$20 \log |H_{LP}(j.\Omega_r)| \leq -20dB$$

- We know that A is given by

$$A = \frac{-(\Omega_1)^2 + \Omega_l * \Omega_u}{\Omega_1(\Omega_u - \Omega_l)} \ldots (7)$$

- Substituting all values,we get

$$A = 2.505$$

- Similarly,

$$B = \frac{(\Omega_2)^2 + \Omega_l * \Omega_u}{\Omega_2(\Omega_u - \Omega_l)} \ldots (8)$$

- Substituting all values,we get

$$B = 2.25$$

- The most critical value,Ω_r is the minimum of two values,which is

$$\Omega_r = 2.25$$

- The low-pass butterworth filter has an order n given by the following equation

$$n = \frac{\log_{10}\left[\frac{10^{\frac{-K_1}{10}}-1}{10^{\frac{-K_2}{10}}-1}\right]}{2\log_{10}\frac{\Omega_1}{\Omega_2}} \ldots (1)$$

- For a normalized butterworth low pass filter,$\Omega_1 = 1rad/sec$ and $\Omega_r = 2.25rad/sec$.
- Substituting these values in the expression for n,we have

$$n = \frac{\log_{10}\left[\frac{10^{0.30103}-1}{10^2-1}\right]}{2\log_{10}\frac{1}{2.25}} \ldots (2)$$

$$n = 2.81 \ldots (3)$$

- Equating to the next highest integer,we have

$$n = 3 \ldots (4)$$

- The Normalized butterworth filter of order $n = 3$ has a transfer function given by

$$H_{LP}(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \ldots (5)$$

- The required analog-to-analog transformation is determined from Ω_l and Ω_u as

$$s \to \frac{s^2 + (\Omega_l * \Omega_u)}{s(\Omega_u - \Omega_l)}$$

- Substituting all values,we get

$$s \rightarrow \frac{s^2 + 3.95 * 10^7}{s(1.25 * 10^5)}$$

$$H_{BP}(s) = \left. \frac{1}{s^3 + 2s^2 + 2s + 1} \right|_{s \rightarrow \frac{s^2+3.95*10^7}{s(1.25*10^5)}}$$

- This is the transfer function of the desired band-pass butterworth filter

6 Chebyshev Filters

- There are two types of Chebyshev filters,one containing a ripple in the pass band(Type-I) and the other containing a ripple in the stop band(Type-II).
- A Type-I low pass normalized (unit bandwidth) Chebyshev filter with a ripple in the pass band is characterized by the following magnitude squared frequency response.

$$|H_n(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_n^2(\Omega)} \ldots (1)$$

- Where $T_n(\Omega)$ is the nth-order Chebyshev polynomial.The Chebyshev polynomial can be generated and thus defined from the following recursive formula

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), n > 2 \ldots (2)$$

- With $T_0(x) = 1$ and $T_1(x) = x$
- A list of ten chebyshev polynomials is given in table below for reference in Figure.9.
- It can easily be seen from the list of chebyshev polynomials that $T_n(x)$ at $x = 0$ is 1 when n even and zero when n odd,resulting in $|H_n(j\Omega)|^2$ to be $\frac{1}{1+\epsilon^2}$ at $\Omega = 0$ for n even and $|H_n(j\Omega)|^2$ to be 1 at $\Omega = 0$ for n odd.
- The two general shapes of magnitude squared frequency response of the type I chebyshev filter for n even and n odd are shown in Figure.10.
- The magnitude squared frequency response oscillates between 1 and $\frac{1}{1+\epsilon^2}$ within the pass band,the so called equiripple.
- The magnitude squared frequency response has a value of $\frac{1}{1+\epsilon^2}$ at $\Omega = 1$,the so called cut-off frequency.
- The magnitude squared frequency response $|H_n(j\Omega)|^2$ is monotonic outside the pass band,including both the transition band and the stop band.
- The Stop band begins at Ω_r with magnitude squared frequency response at value $\frac{1}{A^2}$.

- To obtain the causal stable transfer function $H_n(s)$ that corresponds to the Chebyshev magnitude squared function given in equation(1) below,we must find the poles of $H_n(s) * H_n(-s)$ and select the left-half plane poles for $H_n(s)$.

$$|H_n(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_n^2(\Omega)} \ldots (1)$$

- The poles of $H_n(s) * H_n(-s)$ are obtained by finding the roots of the following equation

$$1+\epsilon^2 \ T_n^2(\frac{s}{j}) \ldots (3)$$

- It can be shown that the roots of equation(3),that is,the poles of $H_n(s) * H_n(-s)$ fall on the ellipse as shown in Figure.11.

- If $s_k = \sigma_k + j\Omega_k$ represents a pole,then σ_k and Ω_k were shown to satisfy

$$\frac{\sigma_k^2}{a^2} + \frac{\Omega_k^2}{b^2} = 1 \ldots (4)$$

- where a,b,σ_k,and Ω_k are given by

$$a = \frac{1}{2}\left\{\frac{1+(1+\epsilon^2)^{1/2}}{\epsilon}\right\}^{1/n} - \frac{1}{2}\left\{\frac{1+(1+\epsilon^2)^{1/2}}{\epsilon}\right\}^{-1/n} \ldots (5)$$

$$b = \frac{1}{2}\left\{\frac{1+(1+\epsilon^2)^{1/2}}{\epsilon}\right\}^{1/n} + \frac{1}{2}\left\{\frac{1+(1+\epsilon^2)^{1/2}}{\epsilon}\right\}^{-1/n} \ldots (6)$$