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Stability :

1. External Stability : For a bounded input we get a bounded output then system is BIBO stable

Location of the poles

poles in LHS — Stable
poles in RHS — Unstable
poles on $j\omega$ axis - Marginally stable

$$G(s) = \frac{20(s+1)}{(s-1)(s^2+2s+2)}$$ Unstable $s=+1$

$$G(s) = \frac{20(s-1)}{(s+2)(s^2+4)}$$ Marginally stable $s=\pm j2$

$$G(s) = \frac{20(s-1)}{(s+2)(s^2+4)^2}$$ Stable ? or Unstable ? Marginally stable ?

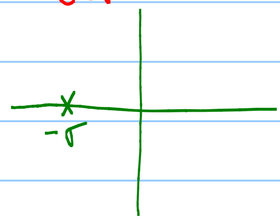
$$Y(s) = G(s)R(s) = G(s) = \frac{1}{(s^2+4)^2}$$

$$Y(s) = \underbrace{\frac{As+B}{s^2+4}}_{\downarrow ILT} + \underbrace{\frac{Cs+D}{(s^2+4)^2}}_{\downarrow ILT}$$

$$y(t) = \cos(2t) + t\cos(2t)$$

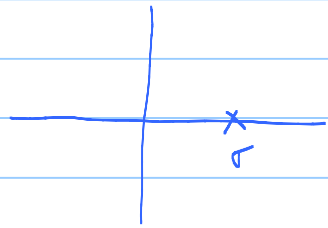
Type of roots | Nature of Impulse response | Impulse response

① Single root at $s = -\sigma$

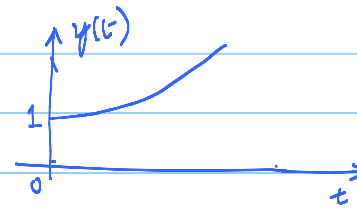


$G(s) = \frac{1}{s+\sigma}$

$y(t) = e^{-\sigma t}, t \geq 0$

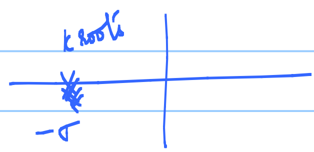


$y(t) = e^{\sigma t}, t \geq 0$



2. Multiple roots at σ (k roots)

$G(s) = \frac{1}{(s+\sigma)^k}$

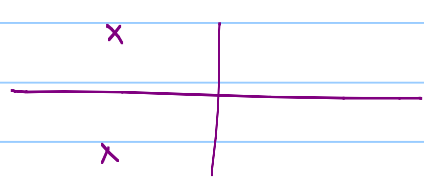


$y(t) = A_1 e^{-\sigma t} + A_2 t e^{-\sigma t} + \dots + A_k t^{k-1} e^{-\sigma t}, t \geq 0$

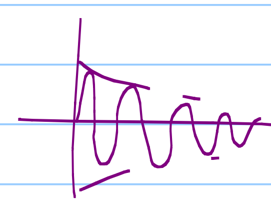
Unstable

3. Complex conjugate poles

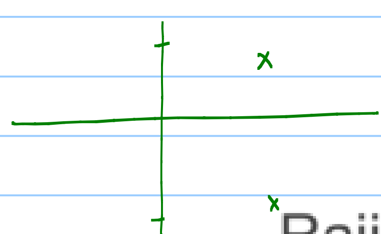
$s = -\sigma \pm j\omega$



$y(t) = e^{-\sigma t} \sin(\omega t)$

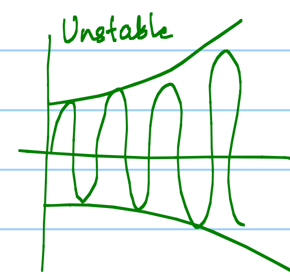


Stable



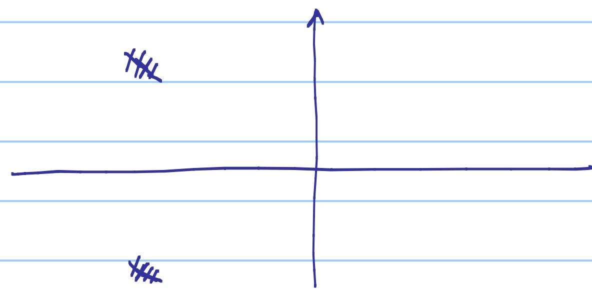
$y(t) = e^{+\sigma t} \sin \omega t$

Unstable



4. Multiple conjugate poles
(k roots)

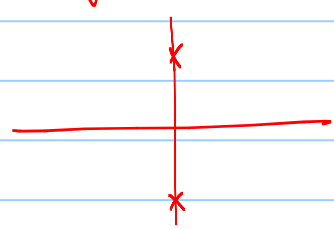
$s = -\sigma \pm j\omega$ — k times



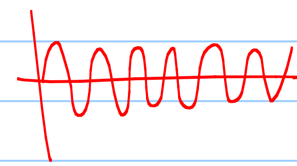
$y(t) = e^{-\sigma t} \sin \omega t + t e^{-\sigma t} \sin \omega t + \vdots + t^{k-1} e^{\sigma t} \sin \omega t$

Unstable

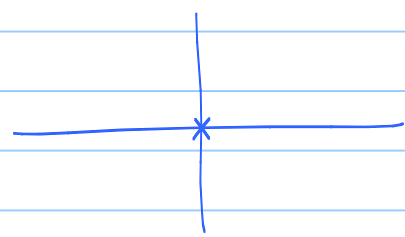
5. Single pair of roots on
jω axis.



$y(t) = \sin \omega t$

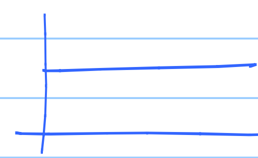


6. Single pole at origin

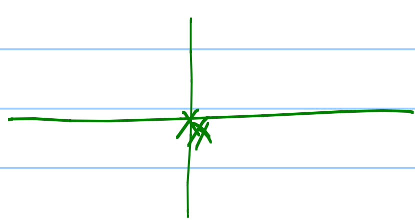


$G(s) = \frac{1}{s}$

$y(t) = 1, \; t \geq 0$



7. Multiple poles at origin
k times



$y(t) = \frac{t^{k-1}}{(k-1)!}$

Unstable

$$a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

$$\begin{array}{c|ccc}
s^3 & a_3 & a_1 & 0 \\
s^2 & a_2 & a_0 & 0 \\
s^1 & b_1 & b_2 & \\
s^0 & c_1 & &
\end{array}$$

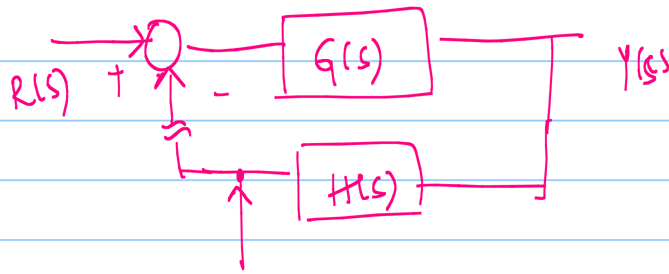
$$b_1 = \frac{a_2 a_1 - a_3 a_0}{a_2}$$

$$b_2 = \frac{a_2 \times 0 - a_3 \times 0}{a_2}$$

$$c_1 = \frac{b_1 a_0 - b_2 a_2}{b_1}$$

$$a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

$$\begin{array}{c|ccc}
s^4 & a_4 & a_2 & a_0 \\
s^3 & a_3 & a_1 & 0 \\
s^2 & \underbrace{\frac{a_3 a_2 - a_4 a_1}{a_3}}_{b_1} & \underbrace{\frac{a_3 a_0 - a_4 0}{a_3}}_{b_2} & 0 \\
s^1 & \underbrace{\frac{b_1 a_1 - b_2 a_3}{b_1}}_{c_1} & 0 & \\
s^0 & \frac{c_1 b_2 - b_1 \times 0}{c_1} & 0 &
\end{array}$$



OLTF - $G(s)H(s)$

$$Y(s) = \underbrace{G(s)H(s)}_{OLTF} R(s)$$

$$CLTF = \frac{G(s)}{1 + \underbrace{G(s)H(s)}_{OLTF}}$$

RH Criteria:

CLTF characteristic eqn = $1 + G(s)H(s)$ — apply the procedure

Root locus, Bode plots, Nyquist:

Start with $G(s)H(s)$ then apply the procedure.

Ultimate goal of the above mentioned technique is stability of of CLTF

$$a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0 \qquad a_i \neq 0 \ (+ve) \qquad i = 0, 1, 2, 3, 4$$

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	b_1	b_2	0
s^1	c_1	$c_2 = 0$	
s^0	d_1		

$$b_1 = \frac{a_2 a_3 - a_1 a_4}{a_3}$$

$$b_2 = \frac{a_0 a_3 - a_4 \times 0}{a_3}$$

$$c_1 = \frac{a_1 b_1 - a_3 b_2}{b_1}$$

$$c_2 = \frac{b_1 \times 0 - a_3 \times 0}{b_1} = 0$$

$$d_1 = b_2 c_1 - c_2 b_1 / c_1$$

① $q(s) = 2s^4 + s^3 + 3s^2 + 5s + 10 = 0$

s^4	2	3	10
s^3	1	5	0
s^2	$b_1 = -7$	$b_2 = 10$	
s^1	$c_1 = 45/7$	$c_2 = 0$	
s^0	$d_1 = 10$		

$b_1 = \frac{3\times1 - 5\times2}{1} = -7$

$b_2 = \frac{10\times1 - 2\times0}{1} = 10$

$c_1 = \frac{5\times-7 - 10\times1}{-7} = \frac{45}{7}$

$d_1 = \frac{10\times\frac{45}{7} - (-7)\times0}{45/7}$

Since two sign changes in the first column of RH Table implies 2 poles in RHP

∴ Closed loop system is unstable

② $q(s) = s^4 + s^3 + 2s^2 + 2s^1 + 3 = 0$

	+	s^4	1	2	3
	+	s^3	1	2	
−	+	s^2	$\cancel{0} \to \epsilon$	3.	
+	−	s^1	$\frac{2\epsilon - 3}{\epsilon}$	0	
+	+	s^0	3.		

$\epsilon > 0$

$\lim_{\epsilon\to0} \frac{2\epsilon - 3}{\epsilon} = \lim_{\epsilon\to0} 2 - \frac{3}{\epsilon} =$

− Ve number

⇒ Two poles in RHP & system is unstable
2 poles in LHP

OR

$s = 1/z$

$$\left(\frac{1}{z}\right)^4 + \left(\frac{1}{z}\right)^3 + 2\left(\frac{1}{z}\right)^2 + 2\left(\frac{1}{z}\right) + 3 = 0$$

$1 + z + 2z^2 + 2z^3 + 3z^4 = 0$

+	z^4	3	2	1
+	z^3	2	1	
+	z^2	1/2	1	
−	z^1	−3	0	
+	z^0	1		

⇒ Two sign changes.
Hence two unstable poles

$$T(s) = \frac{20}{s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20}$$

Table 1

Table 2

s^8	1	12	39	48	20	
s^7	1	22	59	38	0	
s^6	-1	-2	1	2		(got by ÷ 10)
s^5	1	3	2	0		(got by ÷ 20)
s^4	1	3	2			
s^3	$\cancel{0} \to 4$	$\cancel{0} \to 6$	0			
s^2	$\cancel{3/2} \to 3$	$\cancel{2} \to 4$				× by 2
s^1	2/3	0				
s^0	4					

$U(s) = s^4 + 3s^2 + 2$

$\frac{dU(s)}{ds} = 4s^3 + 6s$

Table 2 → 4 roots on $j\omega$ axis (solving $U(s)$) $s^4 + 3s^2 + 2 = 0$

$s = \pm j, \pm j\sqrt{2}$

Table 1 → 2 sign changes ⇒ 2 roots in RHP

→ Remaining 2 roots in LHP.

Ex:

$$G(s) = \frac{K(s+2)(s+1)}{(s+0.1)(s-1)}$$

$1 + G(s) = 0$ — Characteristic eqn

$$\frac{(s+0.1)(s-1) + K(s+2)(s+1)}{(s+0.1)(s-1)} = 0$$

$\Rightarrow (s+0.1)(s-1) + K(s+2)(s+1) = 0$

$(1+K)s^2 + (3K - 0.9)s + (2K - 0.1) = 0$

s^2	$(1+K)$	$(2K - 0.1)$
s^1	$3K - 0.9$	0
s^0	$2K - 0.1$	

Case (i) : No poles in RHP.

all the elements in the first column must be positive

$\Rightarrow \quad 1+K > 0 \qquad K > -1$

$\Rightarrow \quad 3K - 0.9 > 0 \qquad K > 0.3 \qquad \Rightarrow K > 0.3$

$\Rightarrow \quad 2K - 0.1 > 0 \qquad K > 0.05$

Case (ii) : Only one pole in RHP

either first two entries or last two entries of the first column of RH Table must have same sign (and other combinations also possible)

$+ \quad 1+K > 0$

$- \quad 3K - 0.9 < 0$

$- \quad 2K - 0.1 < 0$

$-1 < K < 0.05$

case (iii) : Two poles in RHP

$+ \quad 1+K > 0$

$- \quad 3K - 0.9 < 0$

$+ \quad 2K - 0.1 > 0$

$0.05 < K < 0.3$

Ex: $\ddot{x}(t) - (K+2)\dot{x} + (2K+5)x = 0$

Apply LT $\rightarrow$

$$\left(s^2 - (K+2)s + (2K+5)\right) X(s) = 0$$

$$s^2 - (K+2)s + (2K+5) = 0$$

s^2	1	$2K+5$
s^1	$-(K+2)$	0
s^0	$(2K+5)$	

case (i) : Stable.

$K+2 < 0 \rightarrow -(K+2) > 0 \qquad 2K+5 > 0$

$K < -2 \qquad K > -2.5$

$\boxed{-2.5 < K < -2}$

Case (ii) : Marginally stable

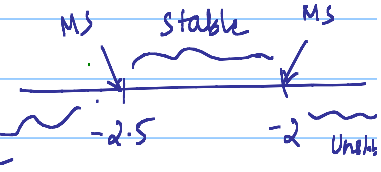
$K+2=0$ $\quad \underline{K=-2}$ or $\underline{K=-2.5}$

$Y(s)=G(s)R(s)$

When $K=-2.5$ $\quad s^2+0.5s=0 \Rightarrow s=0, -0.5$

Case (iii) : Unstable

$K>-2$ $\quad K<-2.5$



b) $-2.5<K<-2$

$$\therefore \quad s_1, s_2 = \frac{(K+2) \pm \sqrt{(K+2)^2-4\times(2K+5)}}{2}$$

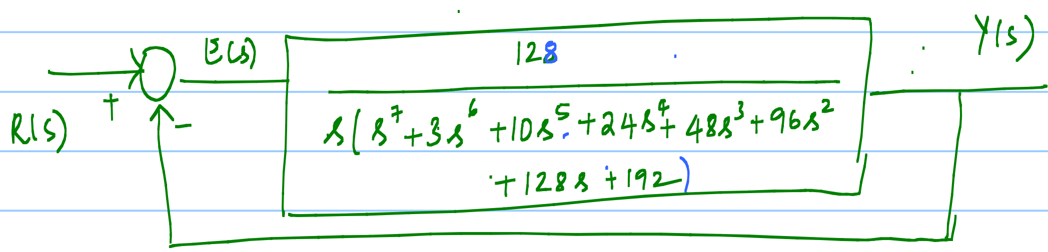
For critically damped $\qquad -2.5<K<-2$

$$(K+2)^2-4(2K+5)=0$$

$K=\underline{\underline{-2.47}}, \boxed{+6.47}$ out of stable range

Overdamped case : $-2.5<K<-2.47$ $\quad ((K+2)^2-4(2K+5))>0$

Underdamped case : $-2.47<K<-2$ $\quad ((K+2)^2-4(2K+5))<0$



Find the no of poles of closed loop system in RHP, LHP & $j\omega$ axis.

$$1+G(s)=0$$

$$q(s)= s^8+3s^7+10s^6+24s^5+48s^4+96s^3+128s^2+192s+128=0$$

No sign change → 2 poles in LHP

Table 1

s^8	1	10	48	128	128	
s^7	$\cancel{3}$ 1	$\cancel{24}$ 8	$\cancel{96}$ 32	$\cancel{192}$ 64		$\div 3$
s^6	$\cancel{2}$ 1	$\cancel{16}$ 8	$\cancel{64}$ 32	$\cancel{128}$ 64		$\div 2$
s^5	$\cancel{0}$ $\cancel{6}$ 3	$\cancel{0}$ $\cancel{32}$ 16	$\cancel{0}$ $\cancel{64}$ 32	0		$\div 2$

Table 2

$U(s) = s^6 + 8s^4 + 32s^2 + 64$

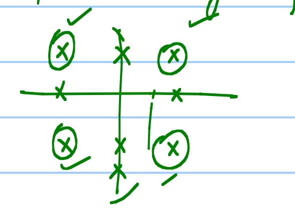
$\frac{d\,U(s)}{ds} = 6s^5 + 32s^3 + 64s + 0$

s^4	$\cancel{8/3}$ 1	$\cancel{64/3}$ 8	$\cancel{64}$ 24	$\times\ 3/8$
s^3	$\cancel{-8}$ −1	$\cancel{-40}$ −5		$\div\ 8$
s^2	$\cancel{3}$ 1	$\cancel{24}$ 8		$\div\ 3$
s^1	3	0		
s^0	8			

→ From Table 1, no sign changes ⇒ 2 poles in LHP

→ From Table 2, two sign changes ⇒ 2 poles in RHP, due to symmetry 2 poles in LHP.

→ Remaining 2 on the $j\omega$ axis.



$q(s) = s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63$

Table 1	s^5	1	4	3	+
	s^4	1	24	63	+
	s^3	$\cancel{-20}$ 1	$\cancel{-60}$ −3	0	−
Table 2	s^2	$\cancel{21}$ 1	$\cancel{63}$ 3	0	+
	s^1	$\cancel{0}$ 2	0	0	
	s^0	3			

$A(s) = 21s^2 + 63s = s^2 + 3$

$\frac{dA(s)}{ds} = 2s + 0$

→ 2 sign changes in Table 1 ⇒ 2 roots in RHP

→ Solving $A(s)$, we get roots on $j\omega$ axis.

→ Remaining roots i.e 1 root on LHP.

① $s^4 + 5s^3 + 2s + 10 = 0$

② $s^5 + 5.5s^4 + 14.5s^3 + 8s^2 - 19s - 10 = 0$

③ $2s^5 + s^4 + 6s^3 + 3s^2 + s + 1 = 0$

④ $s^5 - s^4 - 2s^3 + 2s^2 - 8s + 8 = 0$

① $s^4 + 5s^3 + 2s + 10 = 0$

→ No of roots with positive real part − 2

→ No of roots with -ve real part − 2

→ 〃 zero real part − 0

③ $2s^5 + s^4 + 6s^3 + 3s^2 + s + 1 = 0$

s^5	2	6	1
s^4	1	3	1
s^3	$\cancel{0} \rightarrow \epsilon$	-1	
s^2	$\frac{3\epsilon+1}{\epsilon}$	1	
s^1	$\frac{-3\epsilon-1-\epsilon^2}{3\epsilon+1}$	0	
s^0	1		

$+\epsilon \quad -1$

$\frac{3\epsilon+1}{\epsilon} \quad 1$

$$\frac{\frac{3\epsilon+1}{\epsilon} \times -1 - \epsilon}{\frac{3\epsilon+1}{\epsilon}}$$

$$\frac{-3\epsilon-1-\epsilon^2 / \cancel{\epsilon}}{\frac{3\epsilon+1}{\cancel{\epsilon}}}$$

→ No of roots with positive real part - 2

→ No of roots with -ve real part - 3

→ —————"— zero real part - 0

b) $s^5 + 5.5s^4 + 14.5s^3 + 8s^2 - 19s - 10 = 0$

s^5	1	14.5	-19
s^4	5.5	8	-10
s^3	13.04	-17.18	0
s^2	15.27	-10	0
s^1	-8.64	0	0
s^0	-10		

→ No of roots with positive real part - 1

→ No of roots with -ve real part - 4

→ —————"— zero real part - 0