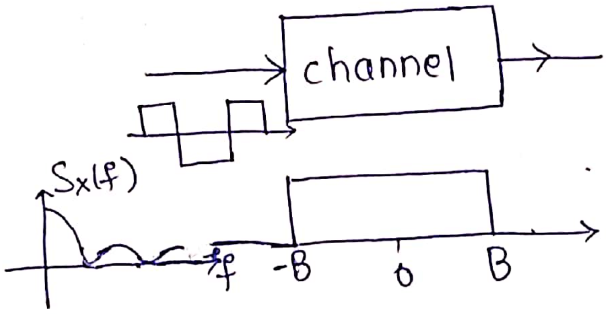
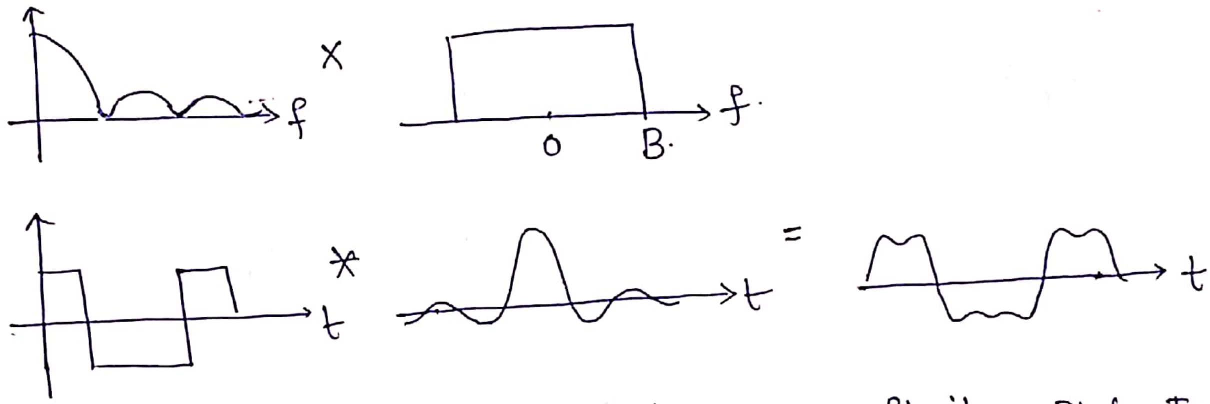
# ISI and Nyquist Criterion:

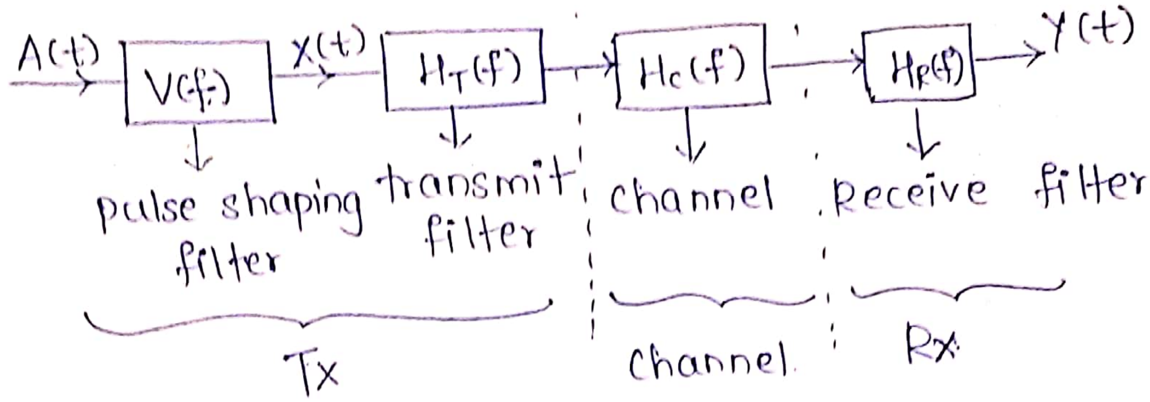
** ISI → Inter-Symbol Interference.




* Any physical channel has a finite BW. In Baseband communication we model the channel as an ideal low pass filter with bandwidth B Hz. When the discrete PAM signal is passed through such a channel it is convolved with the impulse response of the ideal Low pass filter which is the sinc function. Hence the symbols will speard in time and interfere with the other symbols. This is known as Inter-Symbol Interference (ISI).

* Consider the model for Baseband communication that has a transmit filter, a channel and a receiving filter.




$$y(t) = \underbrace{h_R(t) * h_c(t) * h_T(t) * V(t)}_{p(t)} * A(t).$$

let $\boxed{p(t) = h_R(t) * h_c(t) * h_T(t) * V(t).}$

* $\therefore$ $\boxed{Y(t) = p(t) * A(t).}$ $\rightarrow$ ①.

Equivalently we have.

* $\boxed{P(f) = H_R(f) H_c(f) H_T(f) V(f).}$

* eq ① is similar to the expression for the discrete PAM signal $X(t) = V(t) * A(t)$.

Hence we have

* $$\boxed{S_Y(f) = \frac{|P(f)|^2}{T_b} S_A(f).}$$

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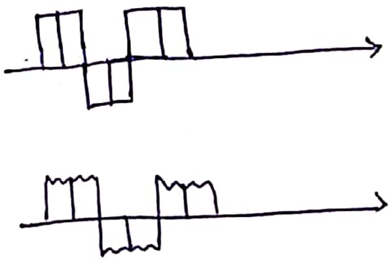
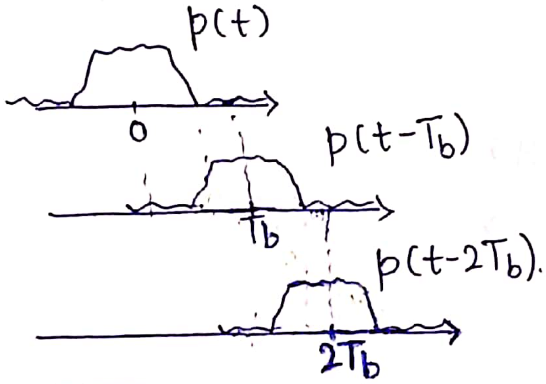


* In order to perform detection without errors at the receiver in the presence of ISI we need one sample in every symbol duration that has no interference. Therefore, we have

$$Y(t) = p(t) * A(t) = p(t) * \sum_{k=-\infty}^{\infty} A_k \delta(t - kT_b).$$

$$\boxed{Y(t) = \sum_{k=-\infty}^{\infty} A_k\, p(t - kT_b)}$$

let $p(t)$ be




$$\boxed{\begin{aligned} &p(0) = 1 \\ &p(nT_b) = 0 \quad n \neq 0. \end{aligned}}$$

→ To avoid ISI at the mid point of every pulse.

* To avoid ISI at the mid-point of every pulse we require that $\boxed{\begin{aligned} &p(0) = 1 \\ \& \ &p(nT_b) = 0,\ n \neq 0 \end{aligned}}$

(we assume that $p(t)$ is normalized to have a maximum value of 1)

# Nyquist Criterion:



$$A(t) \rightarrow \boxed{V(f)} \xrightarrow{X(t)} \boxed{H_T(f)} \rightarrow \boxed{H_c(f)} \rightarrow \boxed{H_R(f)} \rightarrow Y(t).$$

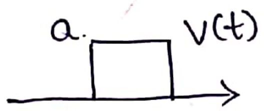
$$A(t) \rightarrow \boxed{P(f)} \rightarrow Y(t).$$

where $P(f) = H_R(f)\, H_c(f)\, H_T(f)\, V(f)$

$$p(t) = h_R(t) * h_c(t) * h_T(t) * V(t).$$

$$Y(t) = p(t) * A(t) = p(t) * \sum_{k=-\infty}^{\infty} A_k \delta(t - kT_b).$$

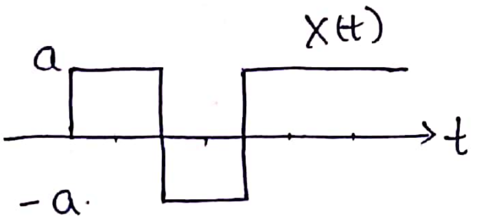
$$\boxed{Y(t) = \sum_{k=-\infty}^{\infty} A_k\, p(t - kT_b)}$$

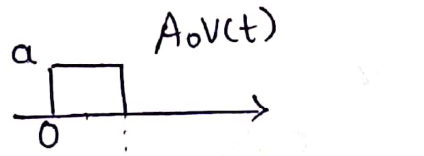


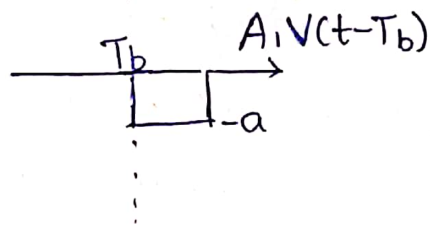

| k= | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| | 1 | 0 | 1 | 1 |

w.k.t. $X(t) = \sum_{k=-\infty}^{\infty} A_k V(t - kT_b).$

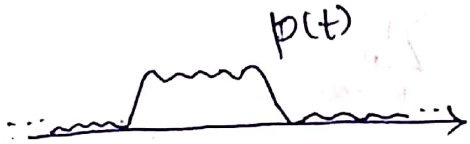
$$X(t) = A_0 V(t) + A_1 V(t - T_b) + A_2 V(t - 2T_b) + A_3 V(t - 3T_b)$$

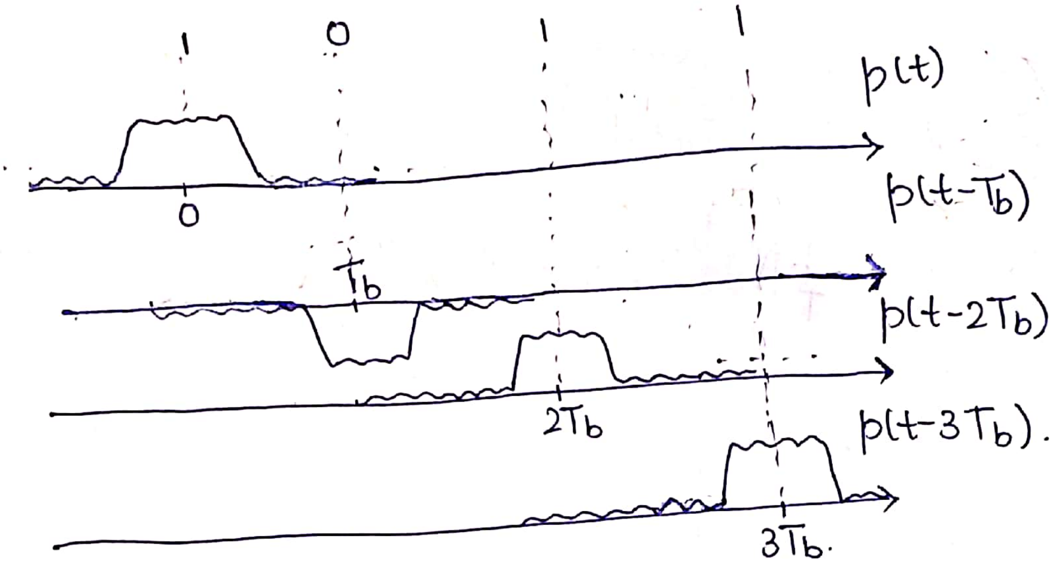


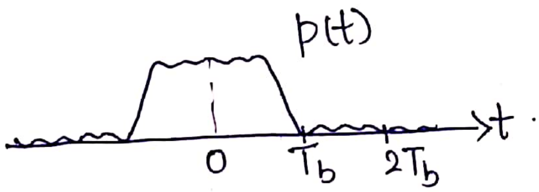
* Since $H_c(f)$ is assumed to be the ideal low pass response $P(f)$ is also limited in frequency and hence $p(t)$ is unlimited (extends till infinity) in time.



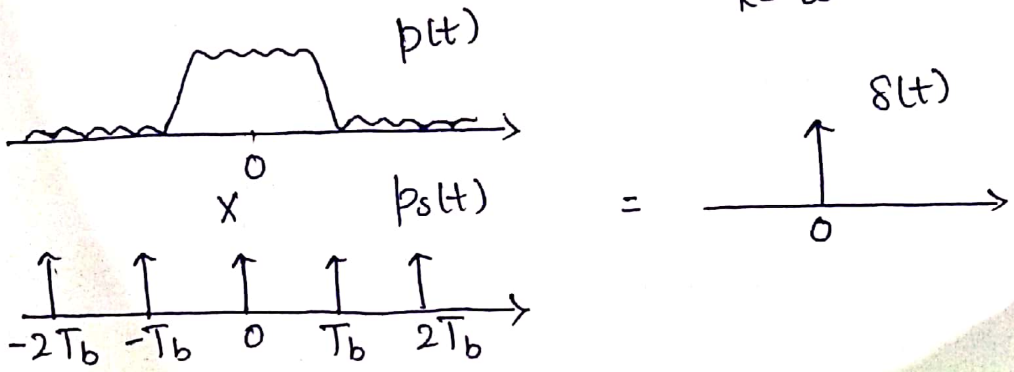




* To avoid ISI we require that

* $$\boxed{\begin{aligned} p(0) &= 1 \\ p(nT_b) &= 0, \quad n \neq 0 \end{aligned}}$$




* This implies that $p_s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_b)$



Taking Fourier Transform we have

$$P(f) * \frac{1}{T_b} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T_b}\right) = 1.$$

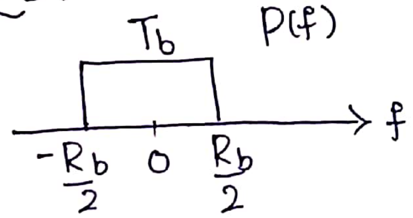
$$\because \boxed{\sum_{k=-\infty}^{\infty} \delta(t - kT_b) \xleftrightarrow{F} \frac{1}{T_b} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T_b}\right)}$$

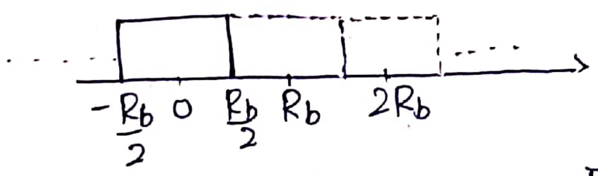
$$\frac{1}{T_b} \sum_{k=-\infty}^{\infty} P\left(f - \frac{k}{T_b}\right) = 1$$

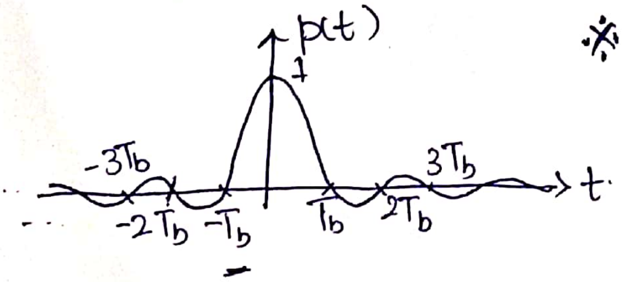
$$\boxed{\sum_{k=-\infty}^{\infty} P(f - kR_b) = T_b}$$ → Nyquist criterion for distortionless baseband transmission.

※ This is the Nyquist criterion for distortionless baseband Transmission.

* case 1:



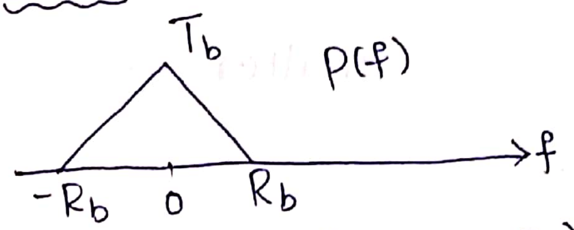



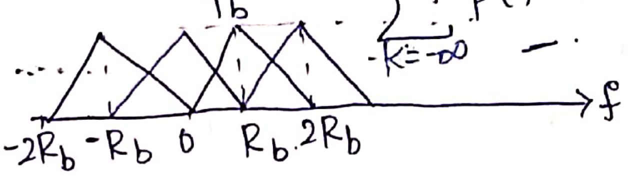




※ $\boxed{p(t) = \text{sinc}\left(\frac{t}{T_b}\right)}$

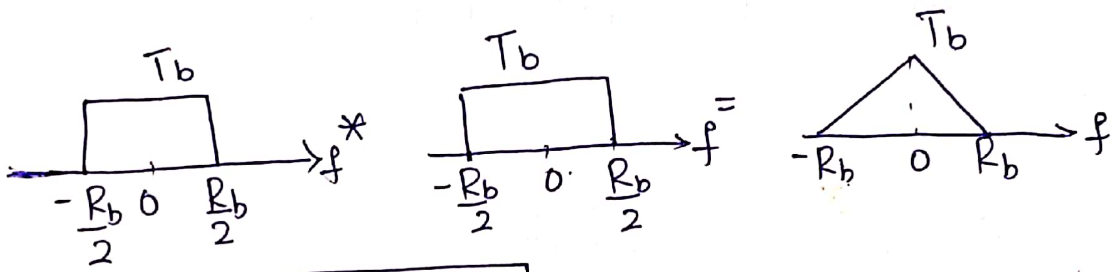
※ $\frac{R_b}{2}$ is the minimum possible Bandwidth for satisfying Nyquist criterion when the bitrate $R_b$ bits/sec.

* case 2 :

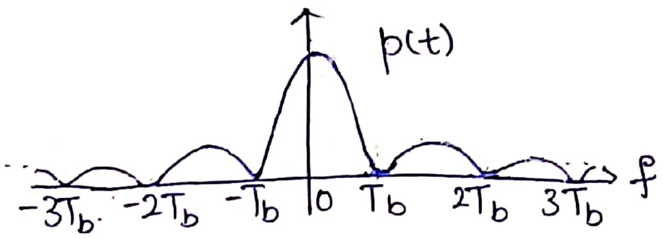



$$\sum_{k=-\infty}^{\infty} P(f-kR_b)=T_b$$




※ ∴ $\boxed{p(t) = \text{sinc}^2\left(\frac{t}{T_b}\right)}$.




※ 2 advantages when compared to $\text{sinc}\left(\frac{t}{T_b}\right)$ (practical implementation issues is easier to handle in this case)

1) sampling at multiples of $T_b$.

↳ due to timing error the error due to $\text{sinc}^2$ is very small when compared to sinc function. Because $\text{sinc}^2$ have the smaller values when compared to sinc around the zero crossing.

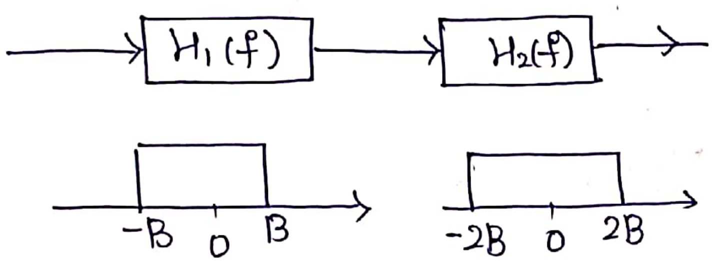
※ Hence deterioration in performance due to timing error is better in case of $sinc^2$.

2) sinc → $\frac{\sin \pi t}{\pi t}$ decreases at the rate of $t$

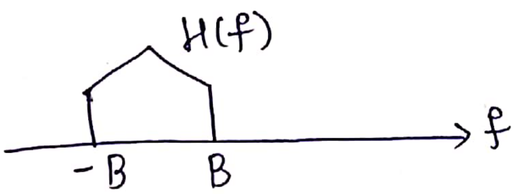
$sinc^2$ → $\frac{\sin^2 \pi t}{(\pi t)^2}$ decreases at the rate of $t^2$

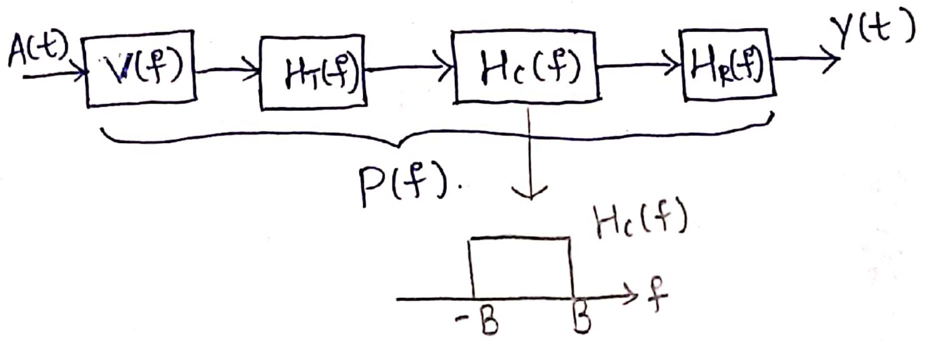
※ ∴ Better performance with compromise in Bandwidth.

※ The sinc pulse decays as $\frac{1}{t}$ but the $sinc^2$ pulse has a much faster decay as $\frac{1}{t^2}$. Truncating the pulse in time introduces distortion in the spectrum. This distortion is much smaller for $sinc^2$ than for the sinc pulse.

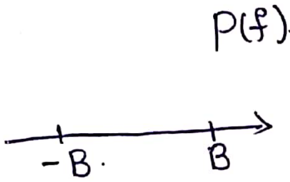


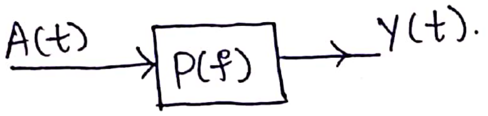

$$H(f) = H_1(f)H_2(f)$$



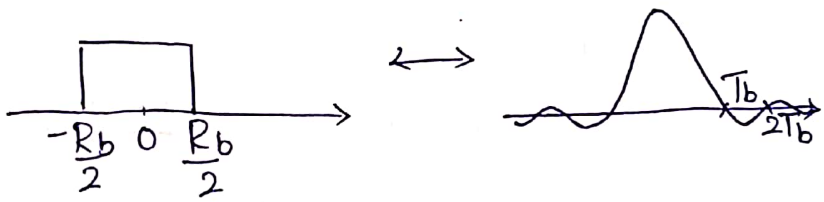



∴ P(f) is bandlimited to B. (since $H_c(f)$ is Bandlimited to B)




$$\boxed{\sum_{k=-\infty}^{\infty} P(f - kR_b) = T_b.}$$





* ↳ BW is $R_b$ Hz. It can be shown that it is the maximum BW for any P(f) that satisfies the Nyquist criterion.

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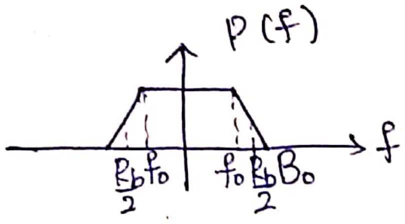
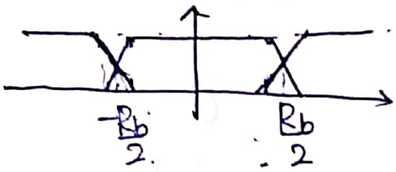
* Case 3 :




$\frac{R_b}{2} < B_o < R_b$.




* The BW in this case is $B_o = \frac{R_b}{2} + \frac{R_b}{2} - f_o$

$$B_o = \frac{R_b}{2}\left(1+1-\frac{2f_o}{R_b}\right)$$

$$\boxed{B_o = \frac{R_b}{2}(1+\alpha)}$$

where $\alpha \rightarrow \left(1-\frac{2f_o}{R_b}\right)$ which is called as the roll off factor.

* $\alpha_{min} = 0$ when $f_o = \frac{R_b}{2}$

$\Rightarrow$ Rectangular spectrum with $\underline{BW} = \frac{R_b}{2}$.

* $\alpha_{max} = 1$ when $f_o = 0$.

$\Rightarrow$ Triangular spectrum with $\underline{BW} = R_b$

* $\alpha$ indicates the excess BW beyond the minimum possible bandwidth of $\frac{R_b}{2}$.

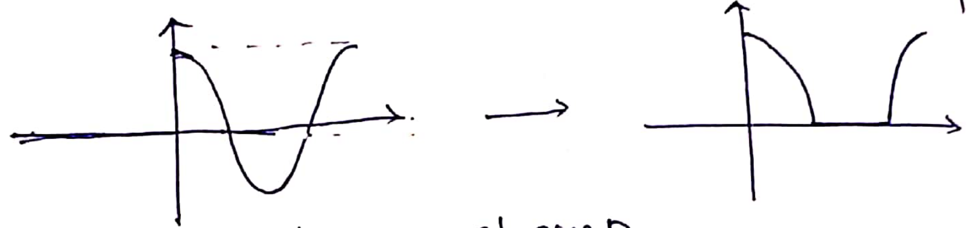
↳ as $\alpha$ varies from $0 \rightarrow 1$ the BW varies from $\frac{R_b}{2} \rightarrow R_b$.

- convolution of 2 unequal length square wave gives a trapezoid.

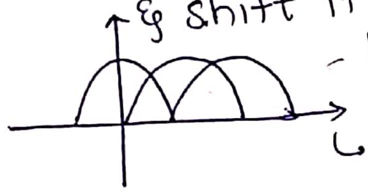
* However smoother edges in the spectrum is desired so that the signal decays faster in time & hence truncating the signal in time do not have any considerable effects in its spectrum (for practical applications). Hence we go for cosine function as P(f).

* case 4 :

take only the +ve part.



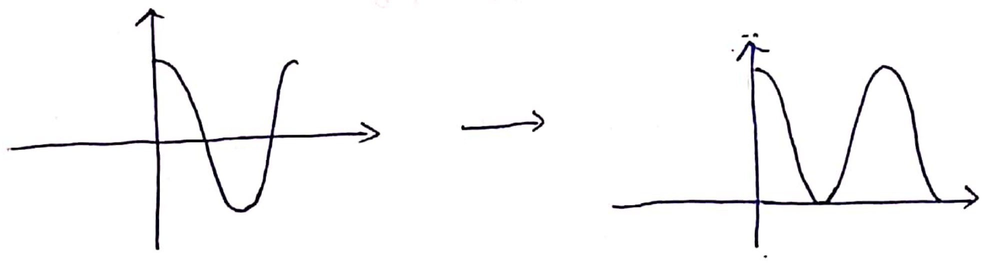
& shift it as shown below



this is not possible as

$$\sum_{k=-\infty}^{\infty} P(f-kR_b) \neq T_b$$ i.e.. the

Nyquist criterion is not satisfied.



full cycle

half cycle.

The slopes are exactly equal & opposite ∴ the sum will be constant.

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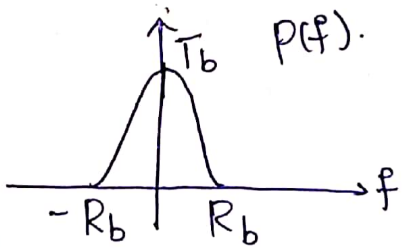
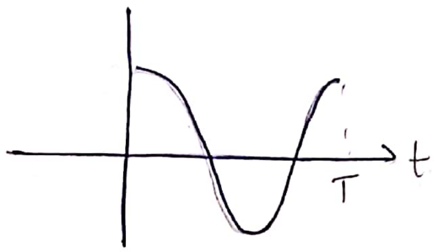
Case 4 :

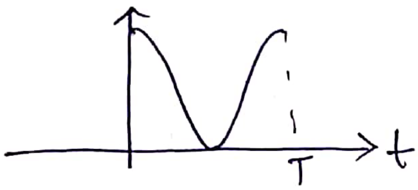
**Raised cosine**




$P(f) = ?$




$\cos\left(\frac{2\pi}{T}t\right)$



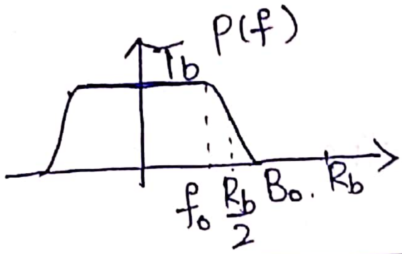

$1 + \cos\left(\frac{2\pi}{T}t\right)$.

$$P(f) = \begin{cases} \frac{T_b}{2}\left[1 + \cos\left(\frac{\pi f}{R_b}\right)\right] & |f| < R_b. \\ 0 & |f| > R_b \end{cases}$$

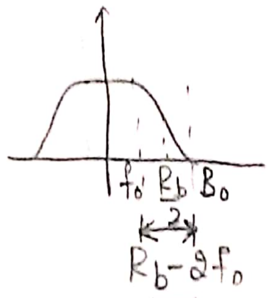
* Here the bandwidth is $R_b$.

(39)

*case 5 : Raised cosine with roll-off :




$$P(f) = \begin{cases} T_b & , |f| < f_o. \\ \frac{T_b}{2}\left(1+\cos\left(\frac{\pi(|f|-f_o)}{R_b-2f_o}\right)\right) & , f_o \le |f| \le B_o \\ 0 & , |f| > B_o. \end{cases}$$




* The corresponding p(t) is given by

$$\boxed{p(t) = \text{sinc}(R_b t) \frac{\cos \pi \alpha R_b t}{1-4\alpha^2 R_b^2 t^2}}$$

where $\alpha$ - Roll off factor

& $\boxed{\alpha = 1 - \frac{2f_o}{R_b}}$.

* In this case the $\boxed{BW = \frac{R_b}{2}(1+\alpha)}$

i.e.. same as that of the trapezoidal case.

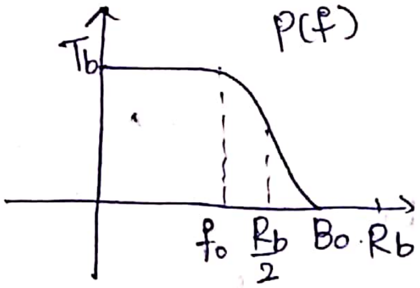
↳ $\alpha_{min} = 0 \Rightarrow f_o = \frac{R_b}{2}$

⇒ Rectangular spectrum with BW $= \frac{R_b}{2}$.

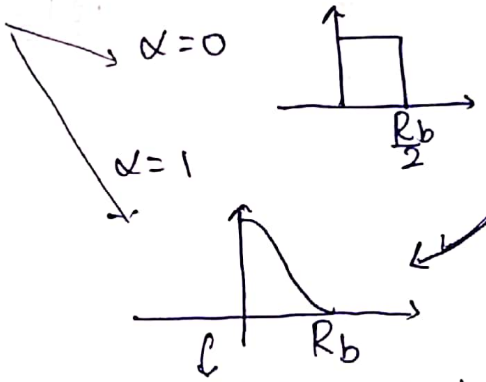
↳ $\alpha_{max} = 1 \Rightarrow f_o = R_b$

⇒ Raised cosine spectrum with BW $= R_b$

# Raised cosine Spectrum (with roll-off)




$$\boxed{p(t) = \text{sinc}(R_b t)\,\frac{\cos \pi \alpha R_b t}{1-4\alpha^2 R_b^2 t^2}}$$



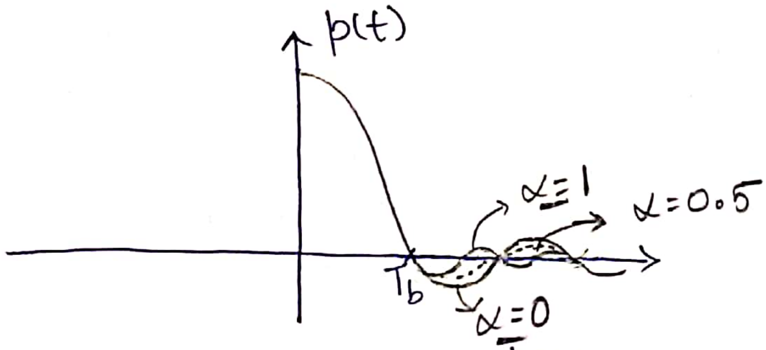

BW ↑ses from $\frac{R_b}{2}$ to $R_b$

complete raised cosine.

* when $\alpha = 1$ $\quad p(t) = \text{sinc}(R_b t)\,\frac{\cos \pi R_b t}{1-4R_b^2 t^2}$

$$p(t) = \frac{\sin \pi R_b t}{\pi R_b t}\,\frac{\cos \pi R_b t}{(1-4R_b^2 t^2)}$$

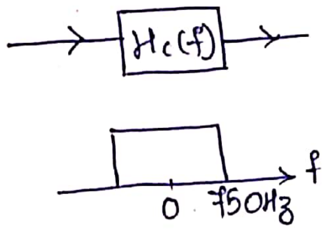
* $\boxed{p(t) = \frac{\text{sinc}(2R_b t)}{1-4R_b^2 t^2}}$ → when $\alpha = 1$.



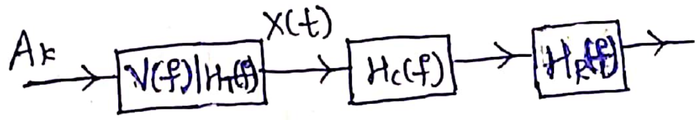

* for $\alpha = 1$ $p(t)$ has zero crossings at multiples of $\frac{T_b}{2}$ i.e.. at $t = \pm n\frac{T_b}{2}$ but at $t = \pm \frac{T_b}{2}$ it can be shown that $p(t) = 0.5$

* for $\alpha = 1$ $p(t)$ decays as $\frac{1}{t^3}$ and hence becomes negligible very quickly therefore, truncating the pulse has no significant effect on the spectrum. Hence this pulse is widely used in practice.

ex: let $R_b = 1000 bps$.

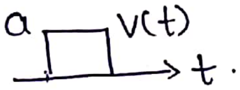


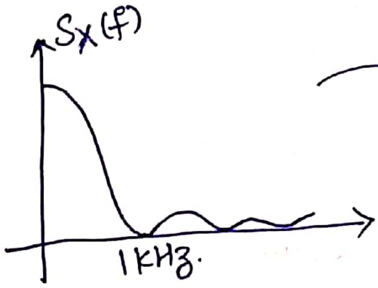

bit-stream $b_k$.
stream of impulses $A_k$




$V(f)$ & $H_T(f)$ are combined as one filter

* $v(t)$ is a pulse

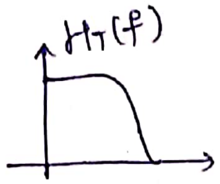



when $x(t)$ is passed through the channel whose BW is 750 Hz then this leads to distortion in the signal. Hence the signal cannot be recovered at the receiver.

* objective: No ISI at the sampling instant. should be achieved.

If $H_T(f)$ is a raised cosine function then




then $BW = \frac{R_b}{2}(1+\alpha)$.

min. BW required $= \frac{R_b}{2} = 500Hz$.

we can have roll-off upto 0.5 so when

$\alpha = 0.5$ $\quad BW = \frac{R_b}{2}(1+0.5)$.

$$\boxed{BW = 750Hz}$$

* objective/intention: use a pulse whose spectrum is limited and which will sit in the channel BW so that the channel do not cut-off any-thing. & the signal (symbol) should be recovered at receiver. which is ensured by the Nyquist criteria.

* Symbols can be recovered from the signal if its spectrum is limited & satisfies the Nyquist criteria.

* even Rectangular pulse / Trapezoidal pulse (with $\alpha \le 0.5$) can be used to achive the above conditions.

* In practice channel won't be an ideal low pass filter we also need to use a filter at the receiver.

* $\boxed{|H_T(f)| = |H_R(f)| = \sqrt{|P(f)|}}$
  ↳ Root Raised cosine filter.

* In practice the Raised cosine Response is shared b/w the Transmit & Receive filters
  * $\boxed{|H_T(f)| = |H_R(f)| = \sqrt{|P(f)|}}$

  These filters ( $H_T(f)$ & $H_R(f)$ ) are called Root Raised cosine filter.

* Same error performance ⇒ same gap b/w successive symbols.

Q) A speech signal bandlimited to 8kHz is sampled at twice the Nyquist Rate and quantized uniformly with 8 bits/sample. Find the minimum BW to transmit this signal. Also if the BW is 60kHz what is the Roll-off factor.

s/n $f_s = 12kHz$.

$\boxed{R_b = 8 \times f_s = 96kbps}$

min. BW required to transmit $= \boxed{\frac{R_b}{2} = 48kHz}$.

if BW = 60kHz $\alpha = ?$

w.k.t. $BW = \frac{R_b}{2}(1+\alpha)$

$60k = 48k(1+\alpha)$.

$\boxed{\alpha = 0.25}$

Q) In a video transmission system 24 images are transmitted per sec. Each image has a resolution of 512X 512 pixels. Each pixel is quantized with 8 bits. Find the min. & the max. BW required for transmitting this signal. Also find the BW if $\alpha = 0.25$.

S/n $* R_b = 24 \times 512^2 \times 8 \simeq 50.33$ Mbps.

$* BW_{min} = \frac{R_b}{2} = 25.165$ MHz.

$* BW_{max} = R_b = 50.33$ MHz.

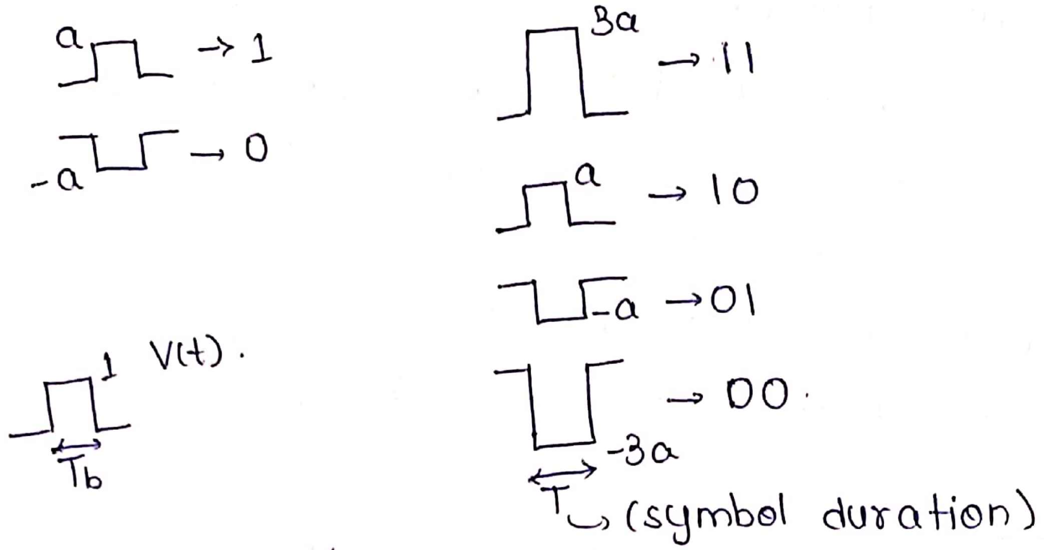
BW = ? when $\alpha = 0.25$

$* BW = \frac{R_b}{2}(1+\alpha) = 31.456$ MHz

* For colored (RGB) images the answer will be 3 times the above one. ∵ it takes 24 bits for each pixel.

## M-ary Transmission:

Binary Transmission | Quaternary Transmission




* $\boxed{\text{Symbol Rate } R_s = \frac{1}{T}}$

* M-ary Transmission is useful in increasing the bit-rate when the BW is fixed and SNR is high. Here each pulse carries a symbol that contains more than one bit of information hence we call the pulse width as symbol duration denoted by T. The symbol rate/modulation/Baud rate is the number of symbols transmitted per second & is given by $\boxed{R_s = \frac{1}{T}}$. The bit rate is $\boxed{R_b = (\log_2 M) R_s \text{ bps}}$

* The band width is given by $\boxed{BW = \frac{R_s}{2}(1+\alpha)}$

Q. A binary polar NRZ system uses rectangular pulses with a width of 1ms & $a = 0.5$.
i) Find the bit-rate, symbol rate, Band width & the avg. power.
ii). Repeat the problem for the quaternary system that uses the same pulse & has the same error performance.
iii) Repeat the problem for the quaternary case if the pulse width is doubled.

Sln

i) Bit-rate, $R_b = \frac{1}{T_b} = \frac{1}{T} = 1\text{kbps}$ | $M = 2$

Bandwidth, $BW = R_s = R_b = 1\text{kHz}$

Symbol rate, $R_s = 1000$ bauds

Avg. Power, $P_{avg} = a^2 = 0.25W$

ii) Bit-rate, $R_b = R_s \times 2 = 2\text{ kbps}$. | $R_b = \log_2 M \, R_s$, $M = 4$

Symbol rate, $R_s = 1000$ bauds

Bandwidth, $BW = \frac{1}{T} = R_s = 1\text{kHz}$.

Avg. power, $P_{avg} = 5a^2 = 1.25W$

iii). Bit-rate, $R_b = R_s \times 2 = \frac{1}{2ms} \times 2 = 1\text{kbps}$.

Symbol rate, $R_s = 500$ bauds

Bandwidth, $BW = R_s = \frac{1}{T} = 500\text{ Hz}$.

Avg Power, $P_{avg} = 5a^2 = 1.25W$.

* M-ary Transmission provides improved performance over binary communication in one of the followin-g two ways:

1) Increasing the bit-rate with the same BW. (or)
2) Achieving the same bit-rate with lesser BW.

→ but the cost we are paying for this is the increase in average power.

Q. An octal PAM system uses a raised cosine pulse shaping. If the main lobe of the pulse in time is 1ms find the symbol rate, bit-rate & the BW required. The roll-off factor is 0.5

Sln

$2T = 1ms. \Rightarrow T = 0.5ms.$

* $\boxed{R_s = 2000 \text{ Bauds}}$

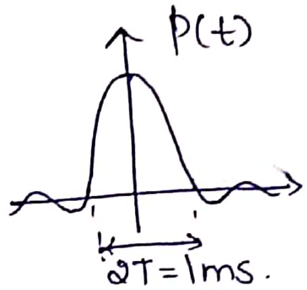
$M = 8 \Rightarrow 3$ bits/symbol.

* $\boxed{R_b = R_s \cdot \log_2 8 = 6 \text{ kbps}}$

* $BW = \frac{R_s}{2}(1+\alpha)$.

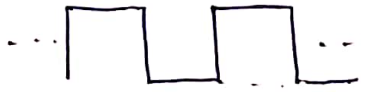
$BW = \frac{2000}{2}(1.5)$

$\boxed{BW = 1.5 kHz.}$



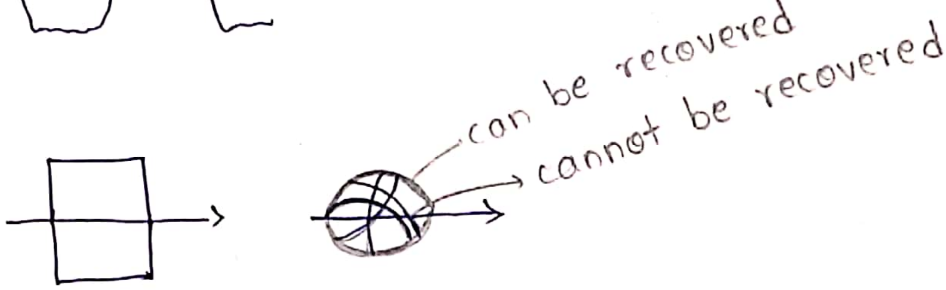

$\alpha = 0.5$

# Eye Diagram

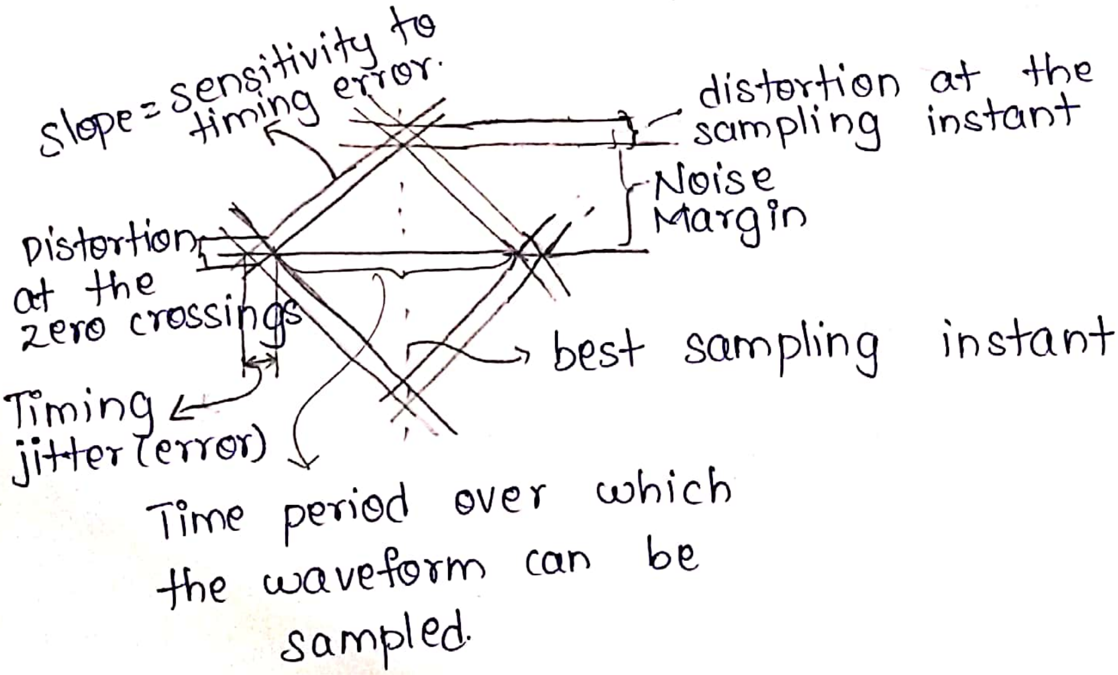


→ transmitted signal (polar form)

→ Received waveform.

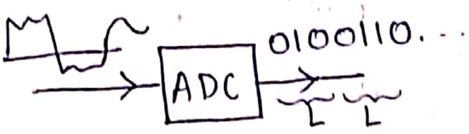



* The effect of ISI and other distortions on a PAM signal can be visualized on an oscilloscope. we apply the received waveform on the vertical deflecting plates of the oscilloscope and a saw-tooth waveform with the same period $T_b$ on the horizontal deflecting plates. This results in a superposition of different bit durations in a single frame. The resulting pattern is called Eye diagram (or) Eye pattern. If the eye is open symbols can be recovered.



* The slope indicates the sensitivity to the Timing Error.

## Signal Space Representation



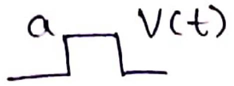

$$\left.\begin{array}{l} 00\cdots 00 \\ 00\cdots 01 \\ \vdots \\ \underbrace{11\cdots 11}_{L \text{ bits}} \end{array}\right\} M = 2^L \text{ symbols.} \qquad \left.\begin{array}{l} m_1 \\ m_2 \\ \vdots \\ m_M \end{array}\right\} M \text{ symbols}$$

let
$$m_1 \rightarrow S_1(t) = a_1 V(t)$$
$$m_2 \rightarrow S_2(t) = a_2 V(t)$$
$$\vdots \qquad\qquad \vdots$$
$$m_M \rightarrow S_M(t) = a_M V(t).$$

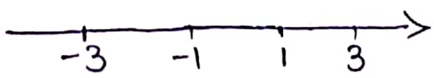
Lec-18

## Signal Space Representation:

*consider quaternary scheme

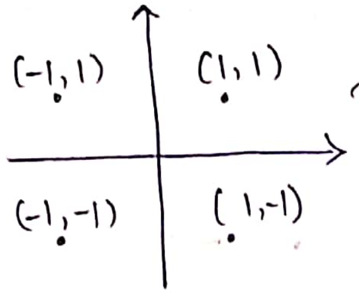



$\{-3, -1, 1, 3\}$.



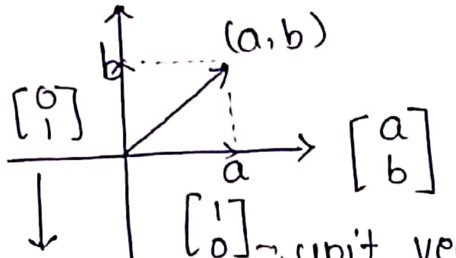

Representation of different signals/symbols on a real line.

↳ here squared length is same as power/energy that we look for signals.




→ multi-dimensional (2D) representation of signals/symbols.

* In M-ary communication we can represent the M signals corresponding to the M basic symbols as the scaled versions of a basic pulse but this representation is inefficient due to the exponentially increasing power requirement. A better representation can be obtained by including more dimensions. (↔)




$\begin{bmatrix}1\\0\end{bmatrix}$ → unit vector along horizontal direction

unit vector along vertical direction.

standard basis vectors (Euclidean basis vectors).

$$\therefore \boxed{\begin{bmatrix}a\\b\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}}$$

$$* \quad \boxed{\begin{bmatrix}a\\b\end{bmatrix} = a\phi_1 + b\phi_2}$$

*(→) Therefore, we can have an N-Dimensional Representation as follows:

$$\boxed{\begin{aligned} S_1(t) &= S_{11}\phi_1(t) + S_{12}\phi_2(t) + \dots + S_{1N}\phi_N(t) \\ S_2(t) &= S_{21}\phi_1(t) + S_{22}\phi_2(t) + \dots + S_{2N}\phi_N(t) \\ &\vdots \\ S_M(t) &= S_{N1}\phi_1(t) + S_{N2}\phi_2(t) + \dots + S_{NN}\phi_N(t). \end{aligned}}$$

w.k.t. $\langle x, y\rangle = x^T y = 0 \Rightarrow x$ & $y$ are orthogonal to each other.

$$x^T y = [x_1 \ x_2 \dots x_N]\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = x_1y_1 + x_2y_2 + \dots + x_Ny_N$$

$$= \sum_{i=1}^{N} x_i y_i = 0$$

$$\therefore \boxed{x^T y = \sum_{i=1}^{N} x_i y_i = 0.}$$

→ where $\{\phi_1(t), \phi_2(t), \dots, \phi_N(t)\}$ form an orthonormal set of signals.

* $x(t)$ & $y(t)$ are said to orthogonal if their inner product is zero.

$$\boxed{\langle x(t), y(t)\rangle = \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = 0.}$$

$$\therefore \int_0^T \phi_i(t)\,\phi_j(t)\,dt = 0 \quad \text{if } i \neq j \rightarrow \text{orthogonality}$$

$$\int_0^T |\phi_i(t)|^2 dt = 1 \rightarrow \text{orthonormal property}$$

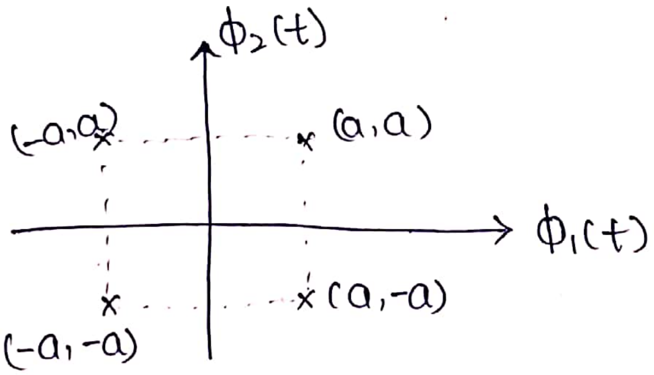
equivalently we can write

$$* \quad \boxed{\int_0^T \phi_i(t)\,\phi_j(t)\,dt = \delta(i-j)}$$

$$\Rightarrow \boxed{N \leq M}$$

Ex: Consider

$$\begin{aligned}
S_1(t) &= a\phi_1(t) + a\phi_2(t) \\
S_2(t) &= a\phi_1(t) - a\phi_2(t). \\
S_3(t) &= -a\phi_1(t) - a\phi_2(t) \\
S_4(t) &= -a\phi_1(t) + a\phi_2(t).
\end{aligned}
\qquad
\begin{aligned}
M &= 4 \\
N &= 2.
\end{aligned}$$




$$S_1(t) \rightarrow \begin{bmatrix} S_{11} \\ S_{12} \\ S_{13} \\ \vdots \\ S_{1N} \end{bmatrix}$$

* The set of co-ordinates $\begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}$ uniquely identifies the signal $s_i(t)$. Hence each signal can be represented as the point in N-dimensional space. This representation simplifies the analysis and helps in designing efficient algorithms for detection at the receiver.

$$s_i(t) \longleftrightarrow \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}$$

* Obtaining the co-ordinates Given $s_i(t)$:

consider $\boxed{\langle s_i(t), \phi_j(t)\rangle = \int_0^T s_i(t)\,\phi_j(t)\,dt}$

$$\langle s_i(t), \phi_j(t)\rangle = \int_0^T s_i(t)\,\phi_j(t)\,dt$$

$$= \int_0^T [s_{i1}\phi_1(t) + s_{i2}\phi_2(t) + \cdots + s_{iN}\phi_N(t)]\,\phi_j(t)\,dt.$$

$$\boxed{\langle s_i(t), \phi_j(t)\rangle = s_{ij}} \quad \left\{ \because \int_0^T \phi_i(t)\,\phi_j(t)\,dt = \delta(i-j) \right.$$

The co-efficient corresponding to $s_i(t)$ are obtained by taking the inner product of $s_i(t)$ with $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$.

* consider the energy of $s_i(t) = E_i$

$$E_i = \int_0^T s_i^2(t)\,dt$$

$$= \int_0^T \left( \begin{matrix} [s_{i1}\phi_1(t) + s_{i2}\phi_2(t) + \cdots + s_{iN}\phi_N(t)] \\ [s_{i1}\phi_1(t) + s_{i2}\phi_2(t) + \cdots + s_{iN}\phi_N(t)] \end{matrix} \right) dt$$

* $$\boxed{E_i = s_{i1}^2 + s_{i2}^2 + s_{i3}^2 + \cdots + s_{iN}^2} \quad \left\{ \because \int_0^T \phi_i(t)\phi_j(t)\,dt = \delta(i-j) \right.$$

⇒ Therefore, the energy of $s_i(t)$, $(E_i)$ is equal to the squared norm of the vector

$$\begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}$$

* similarly it can be shown that given two signals $s_i(t)$ & $s_k(t)$ we have.

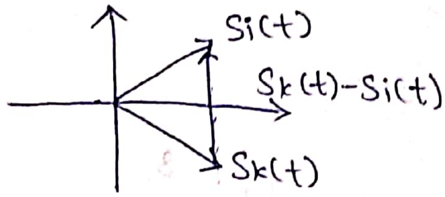
$$\boxed{\int_0^T |s_k(t) - s_i(t)|^2\,dt = \sum_{j=1}^{N} (s_{ij} - s_{kj})^2}$$

↳ (very important for detection)

# Signal Space



$$S_i(t) \rightarrow \begin{bmatrix} S_{i1} \\ S_{i2} \\ \vdots \\ S_{iN} \end{bmatrix}$$




$$\int_0^T |S_i(t)|^2 dt = S_{i1}^2 + S_{i2}^2 + \dots + S_{iN}^2$$

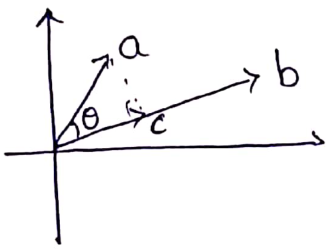
$\int_0^T |S_k(t) - S_i(t)|^2 dt$ - is the energy of the difference b/w the two signals. $S_i(t)$ & $S_k(t)$.

$\{S_1(t), S_2(t), \dots, S_M(t)\} \rightarrow$ Given.

$\{\phi_1(t), \phi_2(t), \dots, \phi_N(t)\} \rightarrow$ needs to be calculated.

& is calculated using Gram-schmidt Procedure.

## Gram-Schmidt Procedure.




$c = ?$.

$$\langle a, b \rangle = a^T b = \|a\| . \|b\| . \cos\theta$$

$$\cos\theta = \frac{\|c\|}{\cancel{\|a\|}} = \frac{a^T b}{\cancel{\|a\|}\,\|b\|}$$

$$c = \|c\| . \frac{b}{\|b\|}$$

$\frac{b}{\|b\|} \rightarrow$ unit vector along b direction.

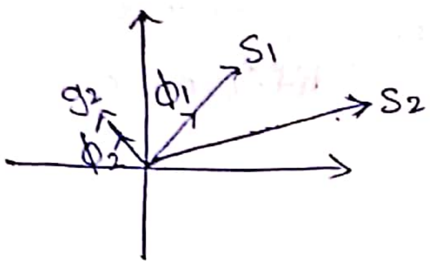
$$\therefore \boxed{c = \frac{a^T b}{\|b\|^2} . b}$$

$$※ \boxed{c = \frac{a^T b}{b^T b} . b}$$

* if $\|b\| = 1$ then $\boxed{c = [a^T b]\, b}$

* given set of vectors $\{s_1, s_2, \ldots s_M\}$.

always $\boxed{\phi_1 = \dfrac{s_1}{\|s_1\|}}$




$$g_2 = s_2 - (s_2^T \phi_1)\,\phi_1$$

$$\boxed{\phi_2 = \frac{g_2}{\|g_2\|}}$$

$$g_3 = s_3 - (s_3^T \phi_1)\,\phi_1 - (s_3^T \phi_2)\,\phi_2 .$$

$$\boxed{\phi_3 = \frac{g_3}{\|g_3\|}}$$

Q. Using the Gram-Schmidt procedure express the following vectors in terms of an ortho-normal set of vectors.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

sln:- $\phi_1 = \dfrac{s_1}{\|s_1\|} = \dfrac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\sqrt{2}}$

$$\boxed{\phi_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}}$$

$$g_2 = s_2 - (s_2^T \phi_1)\,\phi_1$$

$$= \begin{bmatrix} -3 \\ 1 \end{bmatrix} \quad \sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \rightarrow (1)$$

$$g_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

(18)

$$\phi_2 = \frac{g_2}{\|g_2\|}$$

$$\boxed{\phi_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}$$

$$g_3 = S_3 - (S_3^T\phi_1)\phi_1 - (S_3^T\phi_2)\phi_2 \rightarrow (2)$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{3}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} - \left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = 0$$

$$\therefore \boxed{\phi_3 = 0}$$

*[ Note: $g_3$ will be zero $\because$ we are looking at 2-D vectors hence there can be only 2-orthogonal vectors only & not more than that which means $S_3$ can be expressed in terms of $\phi_1$ & $\phi_2$ only].

$\therefore$ * $\boxed{S_1 = \sqrt{2}\,\phi_1}$

$S_2 = -\sqrt{2}\,\phi_1 + g_2$. $\quad\because g_2 = S_2 + \sqrt{2}\,\phi_1$ (From (1)).

* $\boxed{S_2 = -\sqrt{2}\phi_1 + 2\sqrt{2}\,\phi_2}$

* $\boxed{S_3 = \frac{3}{\sqrt{2}}\phi_1 + \frac{1}{\sqrt{2}}\phi_2.}$ $\quad\because g_3 = S_3 - (S_3^T\phi_1)\phi_1 - (S_3^T\phi_2)\phi_2 = 0$ (From (2))

hence $S_3 = (S_3^T\phi_1)\phi_1 + (S_3^T\phi_2)\phi_2$

Scanned by CamScanner

# Gram-Schmidt Procedure for Signals

Given $\{ s_1(t), s_2(t), \ldots, s_M(t) \}$.

Step-1: $\boxed{\phi_1(t) = \dfrac{s_1(t)}{\sqrt{E_{s_1}}}}$

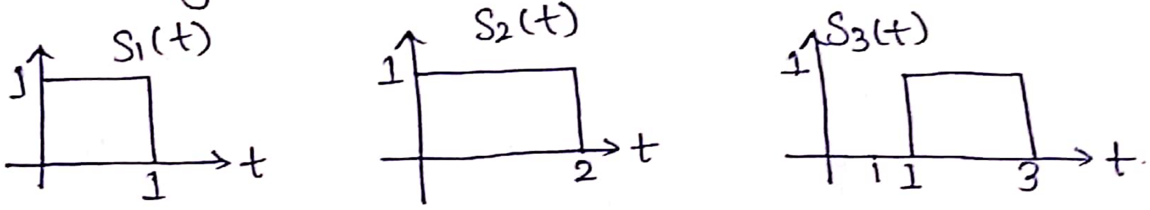
Step-2: $\boxed{g_2(t) = s_2(t) - \langle s_2(t), \phi_1(t) \rangle \phi_1(t)}$

$\boxed{\phi_2(t) = \dfrac{g_2(t)}{\sqrt{E_{g_2}}}}$

Step-3: $\boxed{g_3(t) = \left[ s_3(t) - \langle s_3(t), \phi_1(t) \rangle \phi_1(t) - \langle s_3(t), \phi_2(t) \rangle \phi_2(t) \right]}$

$\boxed{\phi_3(t) = \dfrac{g_3(t)}{\sqrt{E_{g_3}}}}$

Q. Using the Gram-Schmidt procedure find the set of orthonormal signals and express the given signals in terms of the orthonormal signals.




$T = 3$

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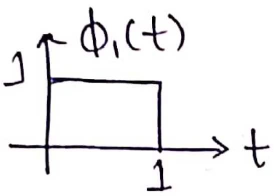
Sln $*$ $\phi_1(t) = \dfrac{S_1(t)}{\sqrt{E_{S1}}}$

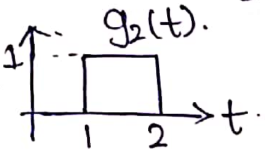
$$E_{S_1} = \int_0^T |S_1(t)|^2 dt = 1.$$

$\therefore$ $\boxed{\phi_1(t) = S_1(t).}$

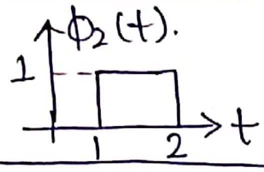


$*$ $g_2(t) = S_2(t) - \langle S_2(t), \phi_1(t)\rangle \phi_1(t).$

$$\langle S_2(t), \phi_1(t)\rangle = \int_0^T S_2(t)\phi_1(t)dt = 1$$

$g_2(t) =$ 

$g_2(t) = S_2(t) - \phi_1(t)$

$\therefore$ $\phi_2(t) =$ 

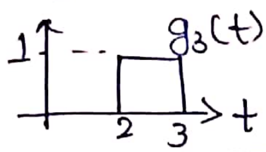
$\phi_2(t) = g_2(t).$

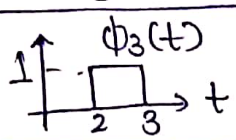
$*$ $g_3(t) = S_3(t) - \langle S_3(t), \phi_1(t)\rangle \phi_1(t) - \langle S_3(t), \phi_2(t)\rangle \phi_2(t)$

$$\langle S_3(t), \phi_1(t)\rangle = \int_0^T S_3(t)\phi_1(t)dt = 0.$$

$$\langle S_3(t), \phi_2(t)\rangle = \int_0^T S_3(t)\phi_2(t)dt = 1$$

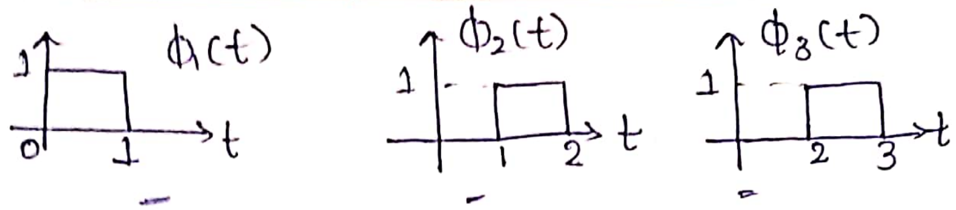
$\therefore$ $g_3(t) = S_3(t) - \phi_2(t)$

$g_3(t) =$ 

$\therefore$ $\phi_3(t) =$ 

$\phi_3(t) = g_3(t).$

$*$ $\therefore$




$\therefore$ $\boxed{\begin{aligned} S_1(t) &= \phi_1(t). \\ S_2(t) &= \phi_1(t) + \phi_2(t) \\ S_3(t) &= \phi_2(t) + \phi_3(t) \end{aligned}}$

Lec-20

Q. Using the Gram-Schmidt procedure find the orthonormal vectors corresponding to the following vectors & express the given set of vectors in terms of the orthonormal set.

$$\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

$*$ $\phi_1 = \dfrac{S_1}{\|S_1\|} = \dfrac{S_1}{2}.$ $\Rightarrow$ $\boxed{S_1 = 2\phi_1}$

$\therefore$ $\boxed{\phi_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$

$*$ $g_2 = S_2 - (S_2^T \phi_1)\phi_1$

$$g_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$g_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\phi_2 = \frac{g_2}{\|g_2\|}.$$

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$$\phi_2 = \frac{1}{\sqrt{2}} g_2 \Rightarrow \boxed{g_2 = \sqrt{2}\,\phi_2}$$

$$\therefore \boxed{\phi_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}}$$

$*$ $g_3 = S_3 - (S_3^T \phi_1)\phi_1 - (S_3^T \phi_2)\phi_2$.

$$g_3 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} - (-3)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 0\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$g_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

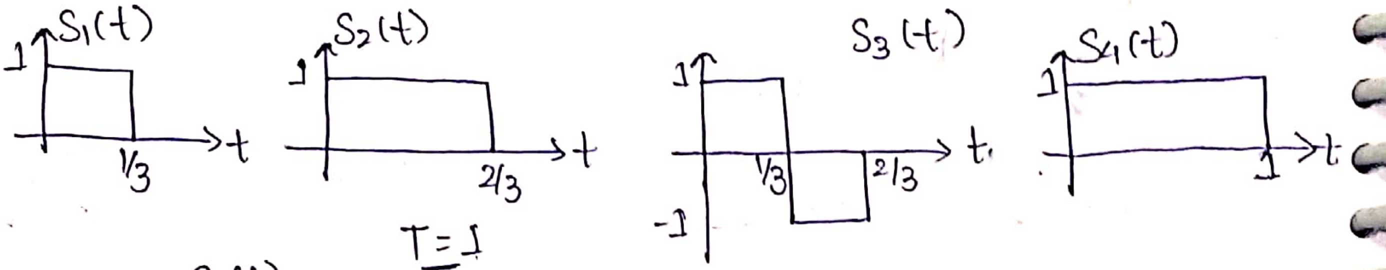
$$\phi_3 = \frac{g_3}{\|g_3\|} \Rightarrow \boxed{g_3 = \sqrt{2}\,\phi_3}$$

$$\therefore \boxed{\phi_3 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}}$$

$$※\therefore \boxed{\begin{aligned} S_1 &= 2\phi_1 \\ S_2 &= \phi_1 + \sqrt{2}\,\phi_2 \\ S_3 &= -3\phi_1 + \sqrt{2}\,\phi_3 \end{aligned}}$$

$$※ \boxed{\begin{aligned} S_2 &= (S_2^T\phi_1)\phi_1 + (S_2^T\phi_2)\phi_2 + (S_2^T\phi_3)\phi_3 \\ S_3 &= (S_3^T\phi_1)\phi_1 + (S_3^T\phi_2)\phi_2 + (S_3^T\phi_3)\phi_3. \end{aligned}}$$

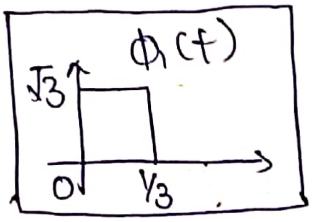
Q. Repeat the problem for the following signals.




T = 1

Sol$^n$ * $\phi_1(t) = \dfrac{S_1(t)}{\sqrt{E_{S_1}}}$

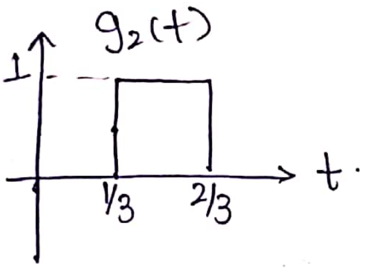
$\boxed{\phi_1 = \sqrt{3}\,S_1(t)}$

*


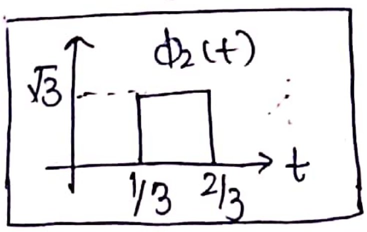

* $g_2(t) = S_2(t) - \langle S_2(t), \phi_1(t)\rangle \phi_1(t)$.

$\langle S_2(t), \phi_1(t)\rangle = \int_0^T S_2(t)\phi_1(t)\,dt = \int_0^{1/3} \sqrt{3}\,dt = \frac{1}{\sqrt{3}}$

$\therefore\ g_2(t) = S_2(t) - \frac{1}{\sqrt{3}}\phi_1(t)$.




$\phi_2(t) = \dfrac{g_2(t)}{\sqrt{E_{g_2}}} = \sqrt{3}\,g_2(t)$

*



* $g_3(t) = S_3(t) - \langle S_3(t), \phi_1(t)\rangle \phi_1(t) - \langle S_3(t), \phi_2(t)\rangle \phi_2(t)$.

$\langle S_3(t), \phi_1(t)\rangle = \int_0^T S_3(t)\phi_1(t)\,dt = \int_0^{1/3} \sqrt{3}\,dt = \frac{1}{\sqrt{3}}$.

$\langle S_3(t), \phi_2(t)\rangle = \int_0^T S_3(t)\phi_2(t)\,dt = \int^{2/3} -\sqrt{3}\,dt = -\frac{1}{\sqrt{3}}$

$$g_3(t) = S_3(t) - \frac{1}{\sqrt{3}}\phi_1(t) + \frac{1}{\sqrt{3}}\phi_2(t) = 0$$

$$\boxed{g_3(t) = 0}$$

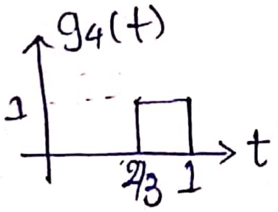
$\Rightarrow$ $S_3(t)$ can be expressed in terms of $\phi_1(t)$ & $\phi_2(t)$

* $$g_4(t) = S_4(t) - \langle S_4(t), \phi_1(t)\rangle \phi_1(t) - \langle S_4(t), \phi_2(t)\rangle \phi_2(t)$$

$$\langle S_4(t), \phi_1(t)\rangle = \int_0^T S_4(t)\phi_1(t)\,dt = \int_0^{1/3} \sqrt{3}\,dt = \frac{1}{\sqrt{3}}.$$

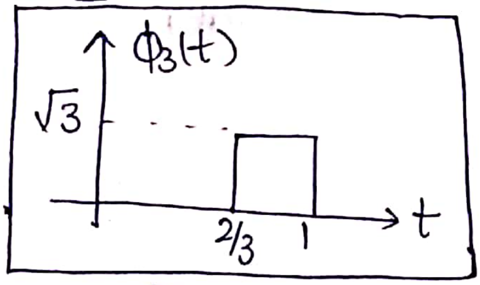
$$\langle S_4(t), \phi_2(t)\rangle = \int_0^T S_4(t)\phi_2(t)\,dt = \int_{1/3}^{2/3} \sqrt{3}\,dt = \frac{1}{\sqrt{3}}.$$

$$g_4(t) = S_4(t) - \frac{1}{\sqrt{3}}\phi_1(t) - \frac{1}{\sqrt{3}}\phi_2(t).$$




$$\phi_3(t) = \sqrt{3}\, g_4(t).$$

*



$$\boxed{\begin{aligned} s_1(t) &= \frac{1}{\sqrt{3}}\phi_1(t). \\ s_2(t) &= \frac{1}{\sqrt{3}}\phi_1(t) + \frac{1}{\sqrt{3}}\phi_2(t). \\ s_3(t) &= \frac{1}{\sqrt{3}}\phi_1(t) - \frac{1}{\sqrt{3}}\phi_2(t) \\ s_4(t) &= \frac{1}{\sqrt{3}}\phi_1(t) + \frac{1}{\sqrt{3}}\phi_2(t) + \frac{1}{\sqrt{3}}\phi_3(t). \end{aligned}}$$

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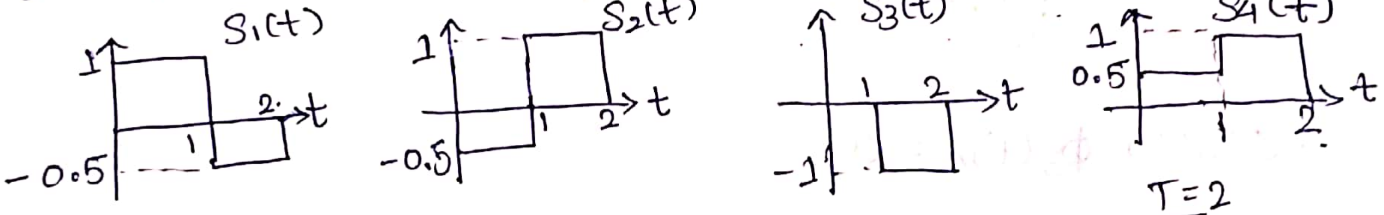
$\{s_1(t), s_2(t), \dots, s_M(t)\} \rightarrow$ M different signals/symbols

$\phi_1(t), \phi_2(t), \dots, \phi_N(t)\} \rightarrow$ N orthonormal vectors.

Q. Repeat the problem for the following signals:




T=2

Soln * $\phi_1(t) = \dfrac{s_1(t)}{\sqrt{E_{s1}}}$

$$\boxed{\phi_1(t) = \frac{2}{\sqrt{5}} s_1(t).}$$

* signals can be taken in any order & each order will give differe-nt set of orthonor-mal signal & all are valid

* Hence for simplicity we consider the following order $\{s_3(t), s_1(t), s_2(t), s_4(t)\}$

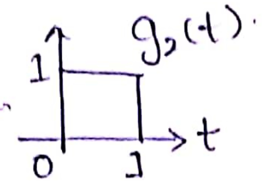
* $\therefore \phi_1(t) = \dfrac{s_3(t)}{\sqrt{E_{s3}}}$

* $\boxed{\phi_1(t) = s_3(t)}$

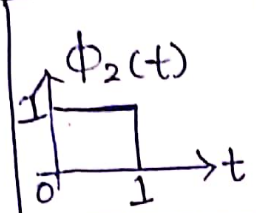
* $g_2(t) = s_1(t) - \langle s_1(t), \phi_1(t) \rangle \phi_1(t)$

$$\langle s_1(t), \phi_1(t) \rangle = \int_0^T s_1(t)\phi_1(t)\,dt = \frac{1}{2}$$

$\therefore$ $g_2(t)$ is




$\therefore$




* $g_3(t) = s_2(t) - \langle s_2(t), \phi_1(t) \rangle \phi_1(t) - \langle s_2(t), \phi_2(t) \rangle \phi_2(t)$

$\therefore \boxed{g_3(t) = 0}$

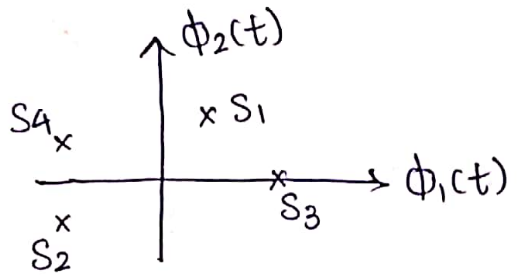
* $g_4(t) = s_4(t) - \langle s_4(t), \phi_1(t) \rangle \phi_1(t) - \langle s_4(t), \phi_2(t) \rangle \phi_2(t)$

$\therefore \boxed{g_4(t) = 0}$

* $\therefore$

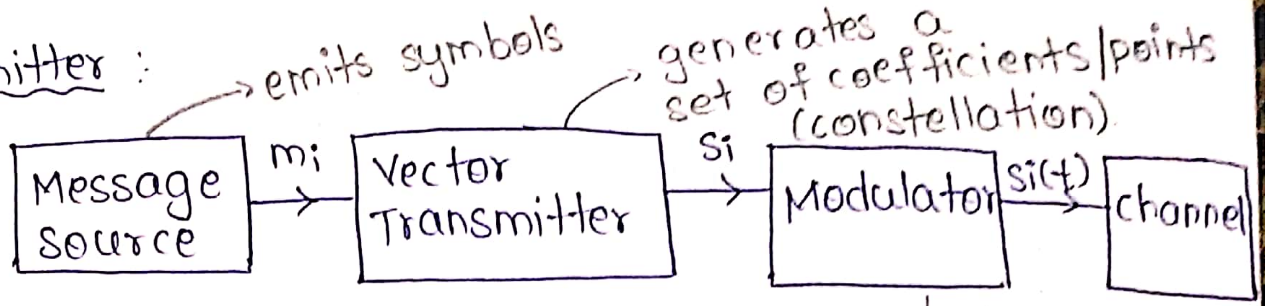
$$\boxed{\begin{aligned} s_1(t) &= \frac{1}{2}\phi_1(t) + \phi_2(t) \\ s_2(t) &= -\phi_1(t) - \frac{1}{2}\phi_2(t) \\ s_3(t) &= \phi_1(t). \\ s_4(t) &= -\phi_1(t) + \frac{1}{2}\phi_2(t). \end{aligned}}$$

$$\left.\begin{aligned} s_1(t) &\rightarrow \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \\ s_2(t) &\rightarrow \begin{bmatrix} -1 \\ -0.5 \end{bmatrix} \\ s_3(t) &\rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ s_4(t) &\rightarrow \begin{bmatrix} -1 \\ 0.5 \end{bmatrix} \end{aligned}\right\} \rightarrow \text{signal space representation}$$




* The signal $s_i(t)$ is uniquely identified with the vector $\begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}$ whose elements are the co-ordinates of $s_i(t)$ w.r.t. $\{\phi_1(t), \phi_2(t), \ldots \phi_N(t)\}$. Therefore, each signal can be represented as a point in the N-dimensional space with the axis being $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$. This is called the signal-space representation (or) the constellation Diagram.

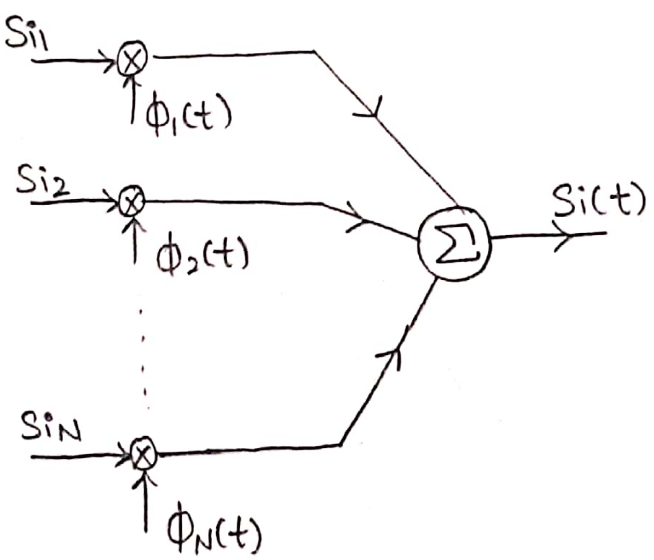
Transmitter :




generates signal for transmitting through the channel

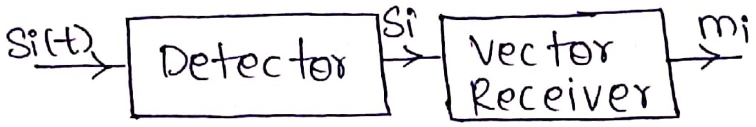
$$\left.\begin{matrix} 00\cdots 0 \\ 00\cdots 1 \\ \vdots \\ \underbrace{11\cdots 1}_{L \text{ bits}} \end{matrix}\right\} M = 2^L \qquad s_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}$$

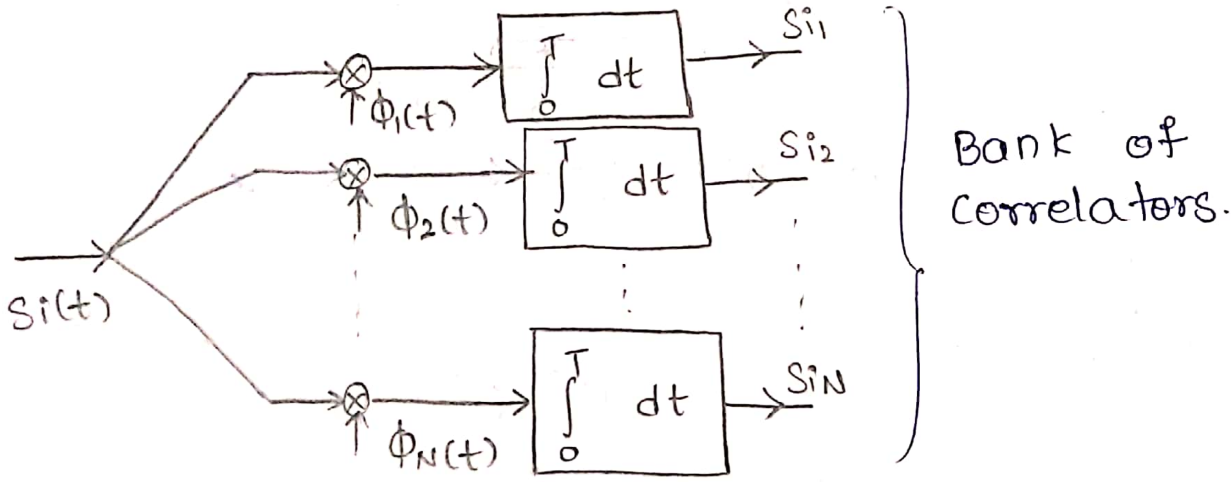
$$\boxed{s_i(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t) + \cdots + s_{iN}\phi_N(t)}$$




* Transmitter will first the set the vector and then generates the signal

Receiver :




Detector.

received signal, $x(t) = S_i(t)$ → ideal case.

practical case

$$x(t) = S_i(t) + W(t)$$

fading, $x(t) = h(t)S_i(t) + W(t)$ → wireless communication

broad band communication $x(t) = h(t) * S_i(t) + W(t)$

※ In all communication we always consider Gaussian Noise ∵ of central limit theorem (i.e.. when a large number of independent & identical random variables being added up then the sum will tend to take a Gaussian distribution'.

i.e... if $x = X_1 + X_2$ & if $X_1$ & $X_2$ are independent then

$$pdf(x) = pdf(X_1) * pdf(X_2)$$

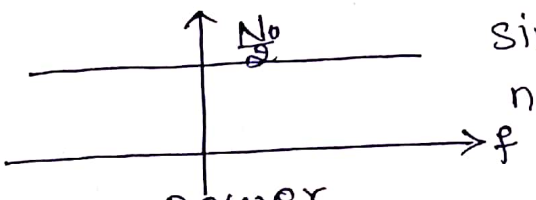
* Noise is due to collection of miniscule random/noise voltage caused due to the random movements of charge carriers. Each random movements are independent & can be considered to be identically dis-tributed.

Lec-22 & 23

* From central Limit theorem, the sum of the random movements of charge carriers with $n \to \infty$ becomes Gaussian.

## Additive White Gaussian Noise (AWGN)

* It is theoretical & not practical.




since power $= \infty$ it is not practical.

power Spectrum of AWGN

## AWGN channel :-

Here the received signal $x(t)$ is is considered as

$$\boxed{x(t) = s_i(t) + w(t)}$$

where $s_i(t)$ is, the transmitted signal corresponding to the $i^{th}$ symbol $m_i$.

$w(t)$ is a zero mean white Gaussian noise process with power spectrum $\frac{N_0}{2}$.

Note: * Unit of power spectrum = Watt / Hz.
* Unit of Fourier transform = Volt / Hz.

$$\boxed{\begin{aligned} E[w(t)] &= 0. \\ S_w(f) &= \frac{N_0}{2}. \\ R_w(t) &= \frac{N_0}{2}\delta(t). \end{aligned}}$$

∴ Area of the power spectrum gives the Average Power.

* Random Process → collection of signals.
* Random Variables → values of the collection of signals at a particular instant ( samping a random process).

(55) * If Any two random variables (different) are orthogonal then its correlation is zero.

$$\underset{=0}{C_{xy}} = \underset{=0}{R_{xy}} - \underset{=0}{\mu_x \mu_y}$$

$\therefore$ of zero-correlation & the mean $= 0$.

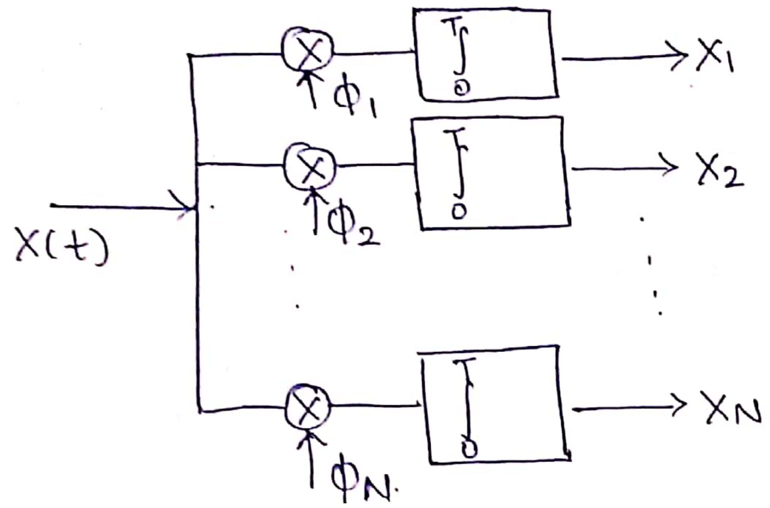
$\therefore$ covariance $= 0$

orthogonal $\Rightarrow$ correlation $= 0$.
Independence $\Rightarrow$ uncorrelated.
but Uncorrelated $\not\Rightarrow$ Independence } Except Gaussian.
Independent $\Rightarrow f_{xy}(x,y) = f_x(x) f_y(y)$.

* Independence is a property of the pdf

From moments we cannot know pdf but From pdf we can know moments.

$\therefore$ Independent $\Rightarrow$ covariance $= 0$.




w.k.t. $S_{ij}$ = $j^{th}$ component of $i^{th}$ sample.

w.k.t. $x_j = \langle x(t)\, \phi_j(t) \rangle$

$$= \langle s_i(t) + w(t), \phi_j(t) \rangle$$

$$x_j = \int_{-\infty}^{\infty} s_i(t)\,\phi_j(t)\,dt + \int_{-\infty}^{\infty} w(t)\,\phi_j(t)\,dt$$

$$\boxed{x_j = S_{ij} + W_j}$$

∴ At the output of the detector we have the vector $x$ which can be written as

$$\boxed{x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} S_{i1} \\ S_{i2} \\ \vdots \\ S_{iN} \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_N \end{bmatrix}}$$

$\boxed{x = S_i + W}$ → Gaussian random vector.

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* The vector reciever maps the vector to some decision that determines which symbol was transmitted.

* once we know the pdf of X, then we can come up with a probability distribution.

* Since W(t) is a Gaussian process the vector $\vec{W}$ is a Gaussian random vector.

Since $X = S_i + W$

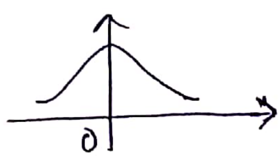
X is also a Gaussian random vector.

ex: $\underbrace{[X \quad Y]}_{\text{Random}} \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{Deterministic}} = \underbrace{X + 2Y}_{\text{Random.}}$

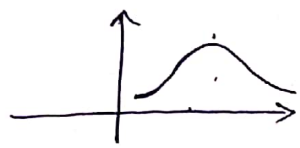
∴ The inner product of the random & deterministic vectors is random.

* $\boxed{X_1 = S_{i1} + W}$

↳ Adding a constant to the rv only changes the mean & the variance will remain the same.



+ constant =



Hence the vectors $X$ & $W$ will have the same co-variance matrix but the mean vector is different.

★ Mean of $W_j$:

$$E[W_j] = E\left[\int_0^T W(t)\,\phi_j(t)\,dt\right]$$

Since Expectation & integration are linear

$$E\left[\int_0^T W(t)\,\phi_j(t)\,dt\right] = \int_0^T E[W(t)\,\phi_j(t)]\,dt$$

since $\phi_j(t)$ is deterministic

we get $\int_0^T E[W(t)\,\phi_j(t)]\,dt = \int_0^T \phi_j(t)\,E[W(t)]\,dt$

as $E[W(t)] = 0$.

we get $E[W_j] = \int_0^T \phi_j(t)\,E[W(t)]\,dt = 0$.

$\therefore \boxed{E[W_j] = 0}$

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★ covariance of W:

w.k.t $C_W = E[(W-\mu_W)(W-\mu_W)^T]$

as mean of AWGN = 0.

$C_W = E[W W^T]$ | $WW^T \rightarrow$ outer product
$W^TW \rightarrow$ inner product

$$C_W = E\left[\begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_N \end{bmatrix} \begin{bmatrix} W_1 & W_2 & \cdots & W_N \end{bmatrix}\right]$$

w.k.t. the expectation of matrix. is the expectation of individual elements of the matrix.

$$[C_W]_{jk} = E[W_j W_k]$$

$$= E\left[\int_0^T W(t)\phi_j(t)dt \cdot \int_0^T W(\lambda)\phi_k(\lambda)d\lambda\right]$$

$$[C_W]_{jk} = \int_0^T \int_0^T \phi_j(t)\phi_k(\lambda)\, E[W(t)W(\lambda)]\, dt\, d\lambda.$$

∗ Since AWGN is a wide sense stationary process $\therefore E[W(t)W(\lambda)] \rightarrow$ the autocorrelation function of lag.

$$\therefore \boxed{E[W(t)W(\lambda)] = R_W(t-\lambda)}$$

$$\therefore [C_w]_{jk} = \int_0^T \int_0^T \phi_j(t)\phi_k(\lambda)\, R_w(t-\lambda)\, dt\, d\lambda$$

$$[C_w]_{jk} = \frac{N_0}{2} \int_0^T \int_0^T \phi_j(t)\phi_k(\lambda)\, \delta(t-\lambda)\, dt\, d\lambda$$

Since $\delta(t-\lambda)$ exists only when $t=\lambda$.

we get

$$[C_w]_{jk} = \frac{N_0}{2} \int_0^T \phi_j(t)\, \phi_k(t)\, dt.$$

Since the basis functions are taken to be orthogonal basis. functions

$$\phi_j(t)\, \phi_k(t) = \begin{cases} 1 & j=k \\ 0 & j \neq k. \end{cases}$$

$$\therefore \text{covariance} = \begin{cases} \frac{N_0}{2} & j=k \\ 0 & j \neq k \end{cases}$$

$\downarrow$

Diagonal matrix.

$$\therefore \boxed{C_W = \frac{N_0}{2} I}$$

# Detection in AWGN channel

$$\boxed{\begin{aligned} &W \sim N\left(0, \tfrac{N_0}{2} I\right) \\ &\therefore X \sim N\left(S_i, \tfrac{N_0}{2} I\right) \end{aligned}}$$ → mean is shifted & covariance remains same.

* Since $C_X$ is diagonal, the elements of vector $X = \{x_1, x_2, \dots x_N\}$ are uncorrelated.

Since these are jointly Gaussian they are also independent.

∴ The joint pdf of the vector $X$ = ?

Since $X = S_i + W$ we are assuming that we have recieved $m_i$

$$\boxed{f_X(x|m_i) = \frac{1}{(2\pi)^{N/2} |C_X|^{1/2}} \exp\left\{-\frac{1}{2}(x - s_i)^T C_X^{-1} (x - s_i)\right\}}$$

Since the elements of X are indepen-dent we can re-write the pdf has.

$$f_X(x|m_i) = \prod_{j=1}^{N} f(x_j|m_i)$$

$$f_X(x|m_i) = \prod_{j=1}^{N} \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} \exp\left\{-\frac{(x_j - s_{ij})^2}{2 \times \frac{N_0}{2}}\right\}$$

$$f_X(x|m_i) = \prod_{j=1}^{N} (\pi N_0)^{-1/2} \cdot \exp\left\{-\frac{(x_j - s_{ij})^2}{N_0}\right\}$$

$$\boxed{f_X(x|m_i) = (\pi N_0)^{-\frac{N}{2}} \exp\left\{-\frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2\right\}}$$

$\downarrow$

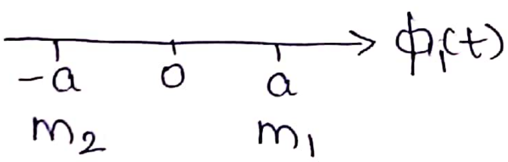
likelihood function

$\downarrow$

Eg $f_X(x_j|m_i)$ , $j = 1, 2, 3, \ldots N$.

Ex: $M = 2, \ N = 1$

$S_1(t) = a\phi_1(t)$ $\qquad$ $S_2 = -a\,\phi_1(t)$.



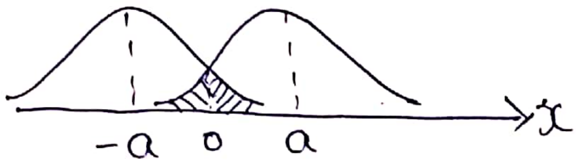

when $S_1(t)$ is transmitted

$$S_1(t) \xrightarrow{\text{channel}} X(t) = S_1(t) + w(t)$$

$\downarrow$ detector

$$\begin{pmatrix}\text{Scalar}\\ \text{as } N=1\end{pmatrix}. \leftarrow \begin{matrix} S_1 + w \\ (a + w)\end{matrix}$$

# Signal Space Representation of Recieved Signal




★ Repeating the communication several times by transmitting $m_1$. The recieved $x$ ( Gaussian noise + $m_1$) will be a Gaussian distribution with mean at $\underline{a}$.

★ Once we obtain the distribution for different message signals we should come up with a decision rule, to classify the obtained $x$ as either $m_1$ (or) $m_2$.

i.e.. the Decision Rule maps $x$ to either $m_1$ (or) $m_2$.

★ The vector reciever block implements the decision rule, that minimizes the probability of error.

## Probability of Error:

$$P_e = P\{\hat{m} \neq \text{symbol transmitted}\}$$

$$P_e = P\{T_0, R_1\} + P\{T_1, R_0\} \rightarrow \text{joint probability}$$

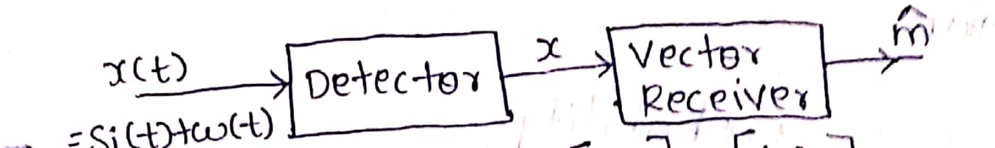
$$P_e = P(R_1|T_0)\,P(T_0) + P(R_0|T_1)\,P(T_1)$$

$$\boxed{P_e = \sum_{\substack{i \neq j \\ j, i = 1}}^{M} P(R_i|T_j)\,P(T_j)}$$

$$P_e = \sum_{i=1}^{M} P(\hat{m} \neq m_i | m_i \text{ is sent}) \cdot P(m_i \text{ is sent})$$

$$\boxed{P_e = \sum_{i=1}^{M} P\{e|m_i\}\,P(m_i)}$$

# AWGN channel




$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$$

$$\boxed{W \sim N(0, \tfrac{N_0}{2} I)}$$

co-variance matrix

$$\boxed{X \sim N(s_i, \tfrac{N_0}{2} I)}$$

## Decision Rule

↳ criteria → to minimize the probability of error.

$p\{m_1 \text{ was sent} | x\}$

$p\{m_2 \text{ " } | x\}$

$\vdots$

$p\{m_M \text{ " } | x\}$

★ To minimize the probability of error we find all the above probabilities. The symbol for which the probability is maximum is decided as the symbol that was transmitted.

* consider $P\{m_i \text{ was sent} | x\} = ?$

$x \rightarrow$ Gaussian random vector

$m_i \rightarrow$ discrete events (discrete rv).

So we have discrete conditioned upon continuo-us.

w.k.t. $$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$p_X(x|y) = \frac{p_Y(y|x)\,p_X(x)}{p_Y(y)}$$

$$f_X(x|y) = \frac{f_Y(y|x)\,f_X(x)}{f_Y(y)}.$$

here for continuous take pdf &

for discrete take pmf

$$\therefore \boxed{P\{m_i|x\} = \frac{f_X(x|m_i)\,P\{m_i\}}{f_X(x)}}$$

$P\{m_i\} = P\{m_i \text{ was sent}\}$

↳ "prior probability"

$f_X(x|m_i)$

↳ "likelihood function

$P\{m_i|x\}$

↳ "posterior probability"

$\therefore$ to minimize the probability of error we use the following decision rule:

1) Decide $\hat{m} = m_i$ if $\boxed{P\{m_i|x\} \geq P\{m_k|x\} \quad k = 1, 2, \ldots M} \rightarrow ①$

$\therefore$ decide $\hat{m} = m_i$ if

$$\boxed{f_x(x|m_i)P\{m_i\} \geq f_x(x|m_k)P\{m_k\} \quad 1 \leq k \leq M} \rightarrow ②$$

* ① & ② are called as MAP (Maximum a posteriori) Rule.

If all the M symbols are equally likely i.e.. $\boxed{P\{m_i\} = \frac{1}{M}, 1 \leq k \leq M}$ then the decision rule becomes. i.e. (decide $\hat{m} = m_i$ if)

* $\boxed{f_x(x|m_i) \geq f_x(x|m_k), 1 \leq k \leq M}$ as $P\{m_i\} = P\{m_k\} = \frac{1}{M}$

$\downarrow$

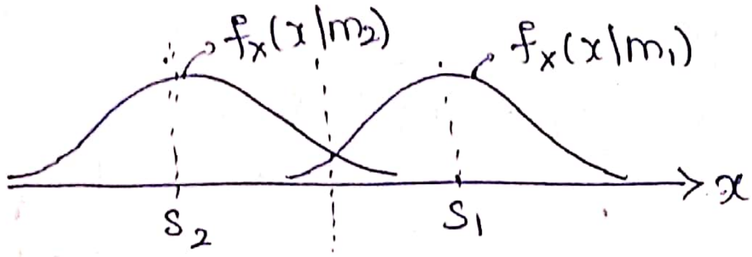
ML (maximum likelihood) Rule.

Since the log function is a monotonically increasing function we can rewrite the ML rule as

decide $\hat{m} = m_i$ if

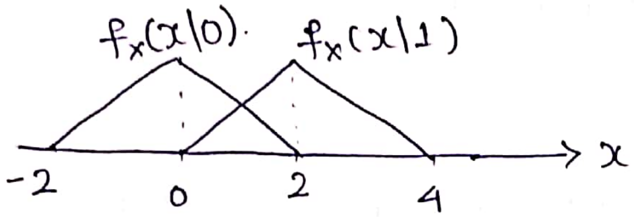
* $\boxed{\log_e\{f_x(x|m_i)\} \geq \log_e\{f_x(x|m_k)\}, 1 \leq k \leq M}$

* let $M = 2$ & $N = 1$



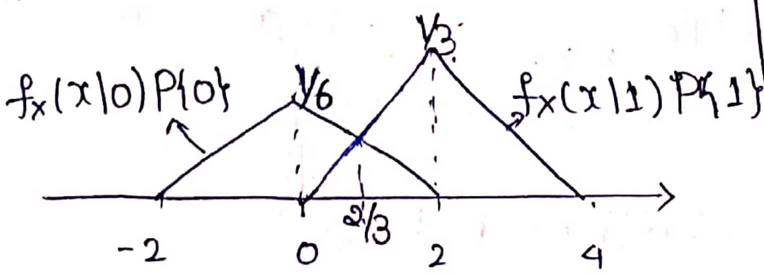

Q. For the likelihood functions shown below find

1) ML Rule. 2) MAP Rule if $P\{1\} = \frac{2}{3}$ & $P\{0\} = \frac{1}{3}$




*) ML Rule. $\begin{cases} \text{decide } \hat{m} = 1 & \text{if } x > 1 \\ \text{decide } \hat{m} = 0 & \text{if } x < 1 \end{cases}$

*) MAP Rule

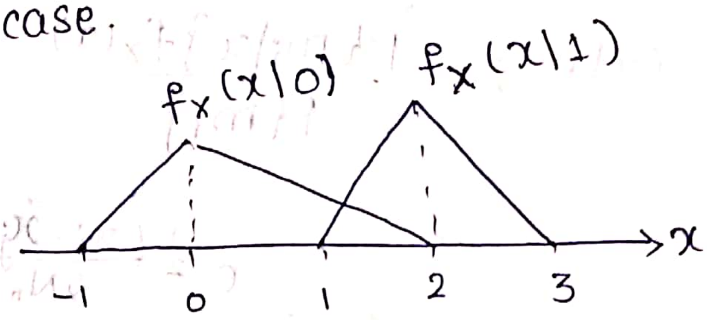



i.e. compare b/w $f_x(x|0)\,P\{0\}$ & $f_x(x|1)\,P\{1\}$

MAP Rule $\begin{cases} \text{decide } \hat{m} = 1 & \text{if } x > 2/3 \\ \text{decide } \hat{m} = 0 & \text{if } x < 2/3 \end{cases}$

(62)

Q. Repeat the problem for the following case.




Given

$P\{1\} = 3/4$

$P\{0\} = 1/4$

$$\frac{1}{2} \times h_1(1) + \frac{1}{2} \times h_1(2) = 1.$$

$$\frac{3h_1}{2} = 1 \Rightarrow h_1 = \frac{2}{3}.$$

$$\frac{1}{2} \cdot h_2(2) = 1 \Rightarrow h_2 = 1.$$

ML Rule :

We have $f_x(x|m_i) = \prod_{j=1}^{N} f_x(x_j|m_i)$

$$= \prod_{j=1}^{N} \frac{1}{\sqrt{\cancel{2}\pi \frac{N_0}{\cancel{2}}}} e^{-\frac{1}{2}\frac{(x_i - s_{ij})^2}{N_0/2}}$$

$$f_x(x|m_i) = (\pi N_0)^{-\frac{N}{2}} e^{-\frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2}$$

$$* \quad \boxed{\ln\{f_x(x|m_i)\} = -\frac{N}{2}\ln(\pi N_0) - \frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})}$$

$\therefore$ the ML Rule becomes :

$$\ln\{f_x(x|m_i)\} = -\frac{N}{2}\ln(\pi N_0) - \frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2$$

decide $\hat{m} = m_i$ if

$$-\frac{N}{2}\cancel{\ln(\pi N_0)} - \frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2$$

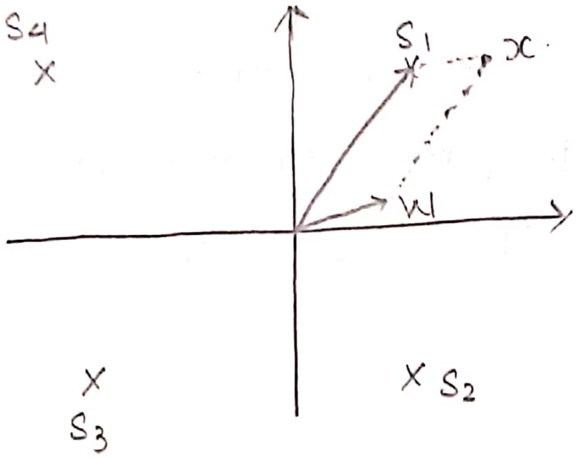
$$\geq -\frac{N}{2}\cancel{\ln(\pi N_0)} - \frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{kj})^2$$

$$* \quad \boxed{\sum_{j=1}^{N}(x_j - s_{ij})^2 \leq \sum_{j=1}^{N}(x_j - s_{kj})^2}$$

lec-26

## ML Rule :

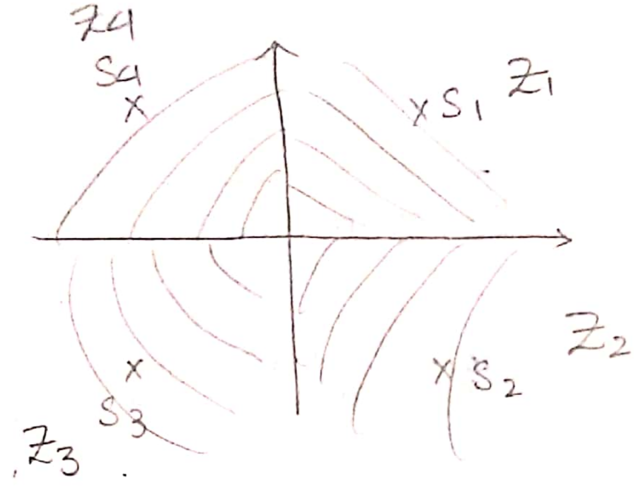
$$\boxed{\begin{gathered} f_x(x|m_i) \geq f_y(x|m_k) \\ \sum_{j=1}^{N} (x_j - s_{ij})^2 \leq \sum_{j=1}^{N} (x_j - s_{kj})^2 . \end{gathered}}$$



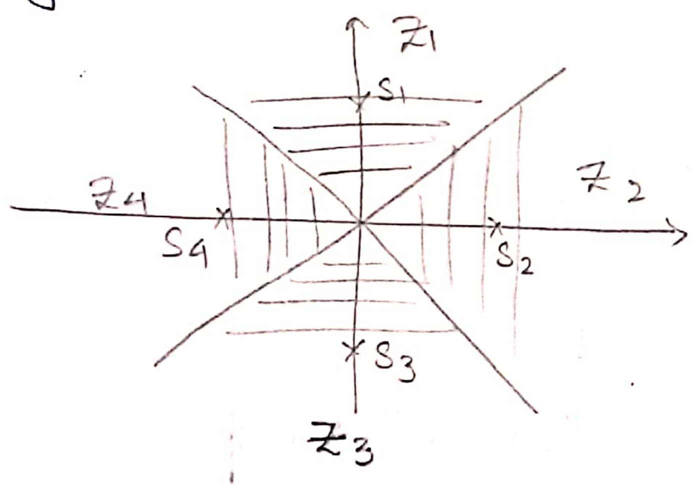

$$\underbrace{\sum_{j=1}^{N} (x_j - s_{ij}) \leq \sum_{j=1}^{N} (x_j - s_{kj}).}$$

can also be ~~written~~ as $\|x - s_j\|^2 \leq \|x - s_k\|^2, \; 1 \leq k \leq M$

* We find the distance of the recieved vector x to each of the symbol vectors $s_i$ , i from 1 to M and find the symbol that has the minimum distance. This is called as the minimum distance Rule. ( which is an alternate representation of the maximum likelihood rule) .




* Based on the minimum distance criteria we can divide the whole signal space into M non-overlapping regions called the decision regions.




$$\|x - s_i\|^2 = (x - s_i)^T (x - s_i)$$

$$= x^T x - x^T s_i - s_i^T x + s_i^T s_i$$

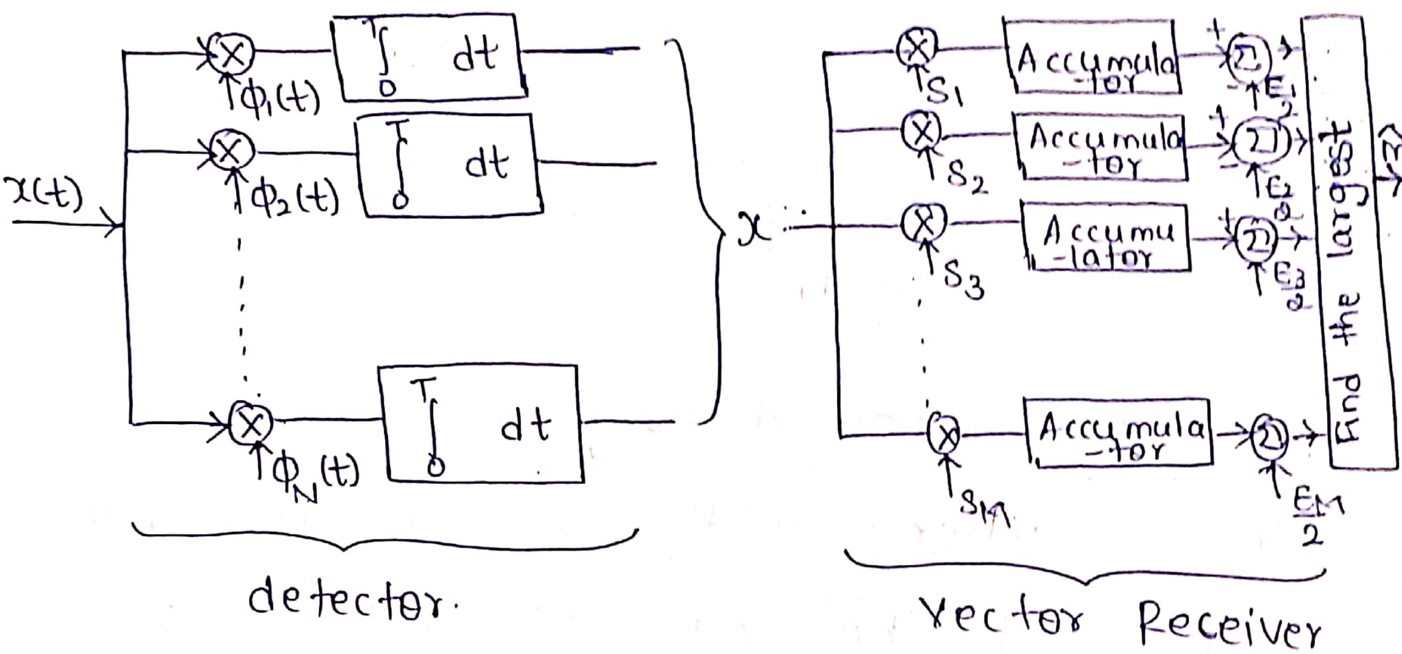
$$\|x - s_i\|^2 = x^T x - 2x^T s_i + E_i$$

$\therefore$ the minimum distance rule becomes

$$\cancel{x^T x} - 2x^T s_i + E_i \le \cancel{x^T x} - 2x^T s_k + E_k$$

$$\boxed{x^T s_i - \frac{1}{2}E_i \ge x^T s_k - \frac{1}{2}E_k}$$

The overall reciever structure is as follows



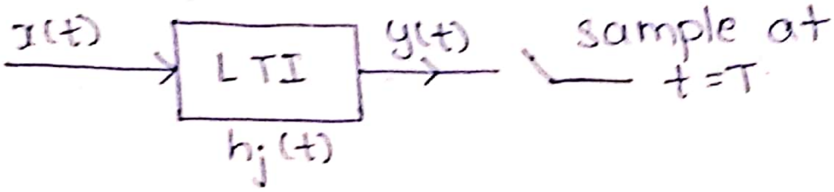

$$x^T s_i = \sum_{j=1}^{N} x_i s_{ij}$$

---

Minimum distance $(\|x - s_i\|^2)$ => inner product is maximum $(x^T s_i)$

# Matched Filter

* In the detector block of the receiver we multiply $x(t)$ with the basis signals $\phi_j(t)$ ( $j$ from 1 to N) in order find their inner product. this operation can be implemented efficiently using the matched filter.




The received signal $x(t)$ is i/p to a filter with impulse response $h_j(t)$ and the o/p of the filter is sampled at $t=T$. Let $h_j(t) = \phi_j(T-t)$

$$\therefore \quad y(t) = \int_0^T x(\tau) h_j(t-\tau)\, d\tau$$

$$y(t) = \int_0^T x(\tau)\, \phi_j(T-t+\tau)\, d\tau$$

$$\because \quad h_j(\tau) = \phi_j(T-\tau)$$

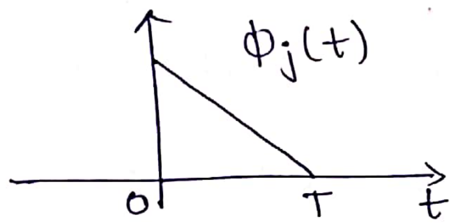
$$\Rightarrow \quad h_j(t-\tau) = \phi_j(T-(t-\tau))$$

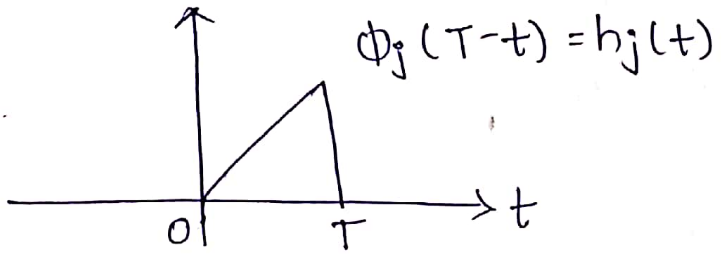
$$h_j(t-\tau) = \phi_j(T-t+\tau)$$

$$\boxed{y(T) = \int_0^T x(\tau)\,\phi_j(\tau)\,d\tau = x_j}$$

Therefore, the o/p of the co-rrelator can be obtained by filtering followed by sampling at $t=T$. The filter with impulse response $h_j(t) = \phi_j(T-t)$ is said to be matched to the signal $\phi_j(t)$

ex:




objective : to maximize the SNR at the o/p of the filter. i.e... we should increase the energy of the signal component w.r.t. the power of the noise component.

$$x(t) \rightarrow \boxed{h(t)} \xrightarrow{y(t)} \overset{t=T}{\diagdown}$$

consider $\boxed{x(t) = \phi(t) + w(t).}$

where $w(t) \rightarrow$ Additive white Gaussian Noise.

let $\boxed{y(t) = \phi_o(t) + w_o(t)}$

where
$$\boxed{\begin{aligned} \phi_o(t) &= h(t) * \phi(t) \quad \rightarrow (1) \\ w_o(t) &= h(t) * w(t) \quad \rightarrow (2). \end{aligned}}$$
&

* we would like to find the $h(t)$ that maximizes the SNR. at the o/p

* The SNR at the o/p is given by

$$\boxed{SNR = \frac{|\phi_o(T)|^2}{E[W_o^2(T)]}}$$

* Since $w(t)$ is a wss process $w_o(t)$ [o/p of an LTI system] is also a wss process.

$$\therefore \boxed{E[W_o^2(T)] = E[W_o^2]}$$

since $w_o(t)$ is a wss process & hence its mean squared value is constant

$$E[W_o^2(T)] = \int_{-\infty}^{\infty} S_{W_o}(f)\,df$$

* (since the avg. power is the area of the power spectrum)

From (2) we can write

$$\boxed{S_{W_0}(f) = |H(f)|^2\, S_W(f).}$$

$$\boxed{S_{W_0}(f) = |H(f)|^2\, \frac{N_0}{2}.}$$

$$\therefore \boxed{E[W_0^2] = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df.}$$

using the IFT expression we have.

$$\phi_0(t) = \int_{-\infty}^{\infty} \phi_0(f)\, e^{j2\pi ft}\, df$$

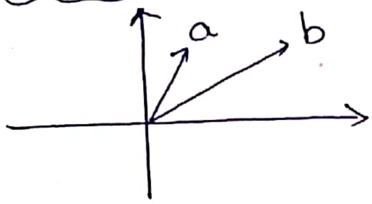
$$\boxed{\phi_0(t) = \int_{-\infty}^{\infty} H(f)\, \phi(f)\, e^{j2\pi ft}\, df.}$$

$$\therefore \boxed{\phi_0(T) = \int_{-\infty}^{\infty} H(f)\, \phi(f)\, e^{j2\pi fT}\, df.}$$

$$\therefore SNR = \frac{\left| \int_{-\infty}^{\infty} H(f)\, \phi(f)\, e^{j2\pi fT}\, df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df.}$$

## Cauchy-Schwarz inequality:

1) Vectors:




$$a^T b = \|a\| . \|b\| \cos\theta$$

$$(a^T b)^2 = \|a\|^2 . \|b\|^2 \cos^2\theta$$

$$\boxed{(a^T b)^2 \leq \|a\|^2 . \|b\|^2}$$

equality iff $a = \alpha b$.

2) Random variables

$$\rho = \frac{C_{xy}}{\sigma_x \sigma_y}$$

$$\rho^2 \leq 1$$

$$\therefore \boxed{C_{xy}^2 \leq \sigma_x^2 \sigma_y^2}$$

equality iff $X = cY$ | where $c \rightarrow$ constant.

3) Signals:

* Cauchy-Schwarz inequality for finite energy signals is given by

$$\boxed{\left| \int_{-\infty}^{\infty} x(t) y(t) \, dt \right|^2 \leq \int_{-\infty}^{\infty} |x(t)|^2 dt . \int_{-\infty}^{\infty} |y(t)|^2 dt}$$

equality iff $x(t) = c y^*(t)$.

(67)

consider
$x(t) = H(f)$
$y(t)$
$= \cancel{\ldots}$

$$\therefore \quad SNR \le \frac{\int_{-\infty}^{\infty} \cancel{|H(f)|^2 df} \cdot \int_{-\infty}^{\infty} |\phi(f) e^{j2\pi fT}|^2 df}{\frac{N_0}{2} \int_{-\infty}^{\infty} \cancel{|H(f)|^2 df}}$$

$\boxed{y(t) = \phi(t) e^{j2\pi fT}}$

$$* \quad \boxed{SNR \le \frac{\int_{-\infty}^{\infty} |\phi(f)|^2 df}{\frac{N_0}{2}}}$$

equality iff $\boxed{H(f) = \phi^*(f) e^{-j2\pi fT}}$ | $x(t) = y^*(t)$

* the constant c will effect both signal & noise equally & hence it won't effect the SNR hence the constant c is considered to be 1.

$\therefore$ with $H(f) = \phi^*(f) e^{-j2\pi ft}$ the maximum SNR is given by

$$* \quad \boxed{SNR_{max} = \frac{2 \int_{-\infty}^{\infty} |\phi(f)|^2 df}{N_0}}$$

By Parseval's theorem for Fourier Transform we have

$$E_\phi = \int_{-\infty}^{\infty} |\phi(t)|^2 dt = \int_{-\infty}^{\infty} |\phi(f)|^2 df$$

$$\therefore \quad \boxed{SNR_{max} = \frac{2E_\phi}{N_0}}$$

$\therefore$ we get $H(f) = \phi^*(f)\, e^{-j2\pi f T}$
(when SNR is maximum)

w.k.t. $\phi(t) \xleftrightarrow{F} \phi(f)$

$\phi^*(-t) \xleftrightarrow{F} \phi^*(f)$

$\phi^*(-\{t-T\}) \longleftrightarrow \phi^*(f)\, e^{-j2\pi f T}$

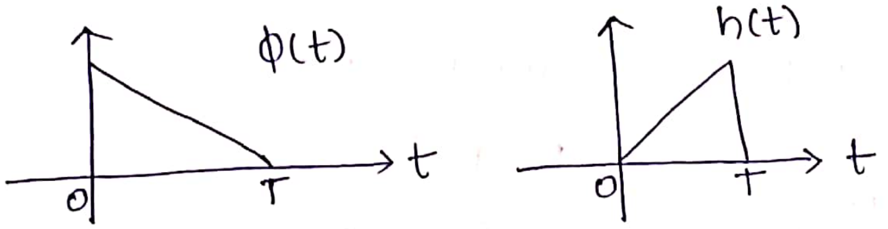
$= \phi^*(T-t)$

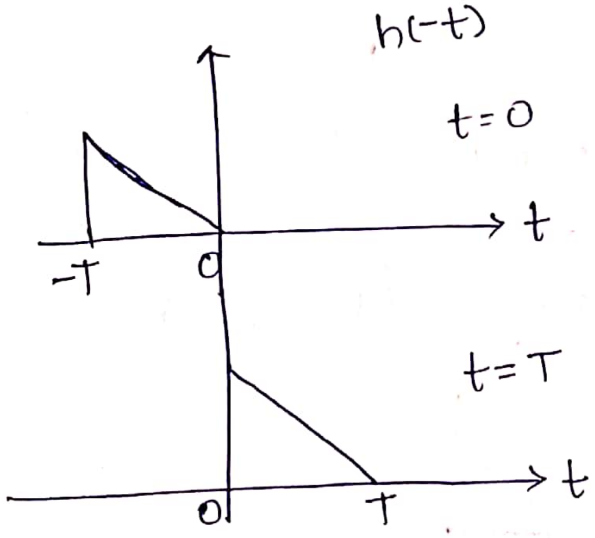
Since $\phi(t)$ is real we have

$\phi^*(T-t) = \phi(T-t)$.

$\therefore \boxed{h(t) = \phi(T-t)} \rightarrow$ to achieve maximu-m SNR.

**Result**: Therefore the Matched filter maximizes the SNR at the o/p.



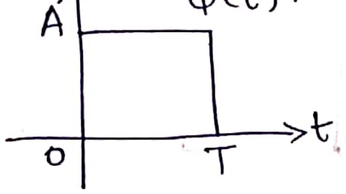



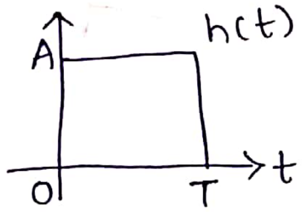
* $\phi(t) * h(t)$ is maximum at $t = T$ & its value will be $E_\phi$ $\therefore$ we sample the o/p of the matched filter at $t=T$.

---

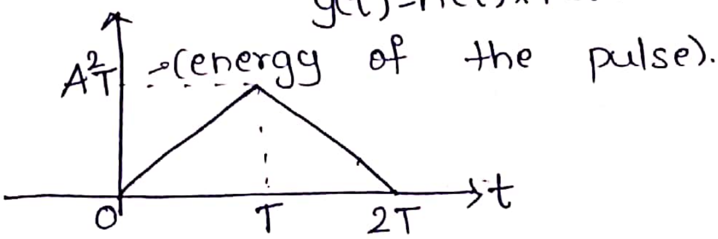
Q. For the following signals find the Matched Filter & plot its o/p when the i/p is the signal itself.

a) $\phi(t)$.




$y(t) = h(t) * \phi(t)$



b) $\phi(t) = \begin{cases} A\cos 2\pi f_c t & 0 \le t \le T \\ 0 & \text{elsewhere.} \end{cases}$ where $f_c$ is an integer multiple of $\frac{1}{T}$

Sln Since $f_c$ is an integer multiple of $\frac{1}{T}$

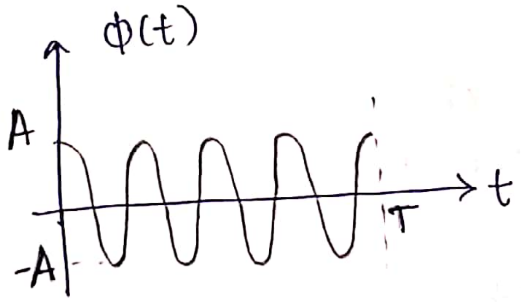
$$h(t) = A\cos 2\pi(f_c(T-t))$$

$$= A\cos 2\pi(f_c T - f_c t)$$

$$h(t) = A\cos(2\pi n - 2\pi f_c t)$$

$$\boxed{h(t) = \begin{cases} A\cos 2\pi f_c t ; & 0 \le t \le T \\ 0 & , \text{elsewhere} \end{cases}}$$

$\therefore h(t) = \phi(t)$

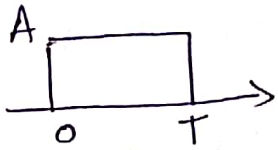



$$h(t) * \phi(t) \overset{F}{\longleftrightarrow} H(f)\,\Phi(f)$$

$$\cancel{H(f)\Phi(f) = \frac{A}{2}[\delta(f+f_c)+\delta(f-f_c)] \cdot \frac{A}{2}[\delta(f+f_c)+\delta(f-f_c)]}$$

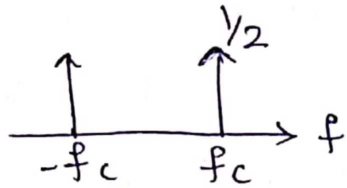
(69)

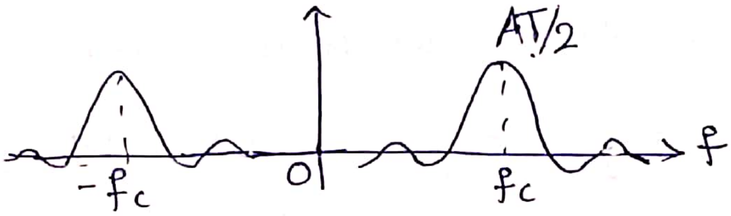
w.k.t.

$\phi(t) =$ 

$\times \cos 2\pi f_c t$

$\phi(f) =$ 

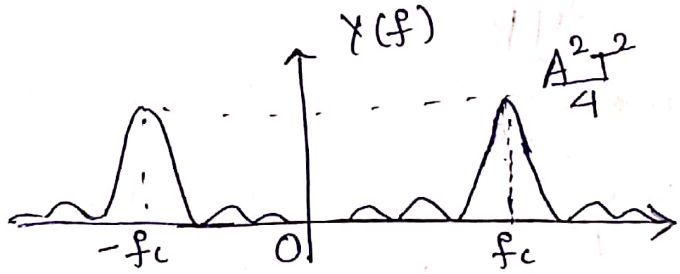
$*$ 


$\phi(f) =$ 

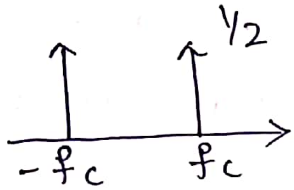

since $h(t) = \phi(t)$

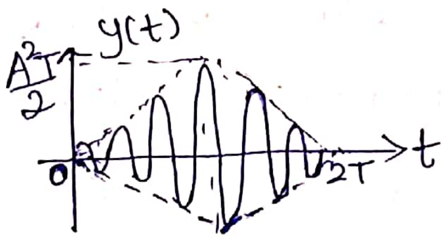
$H(f) = \phi(f)$

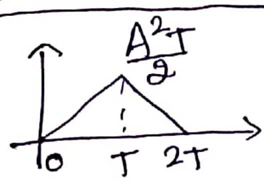
$\therefore \quad H(f)\phi(f) = [\phi(f)]^2 = Y(f)$

$\therefore$ 


$Y(f) =$ 

$*$ 


$\therefore\ y(t) =$ 


$\therefore\ y(t) =$ 

$\times \cos 2\pi f_c t$