Digital Signal Processing

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Properties of DFT

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Properties of DFT
Time reversal of a sequence

If

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

Then, show that

$$x((-n))_N = x(N-n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X((-k))_N = X(N-k)$$

Proof:

$$\text{DFT}\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n) e^{-j2\pi kn/N}$$

Properties of DFT
Time reversal of a sequence

Changing the index from n to m where m=N-n

$$\mathrm{DFT}\{x(N-n)\} = \sum_{m=0}^{N-1} x(m) e^{-j2\pi k(N-m)/N}$$

$$= \sum_{m=0}^{N-1} x(m) e^{j2\pi km/N}$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j2\pi m(N-k)/N} = X(N-k)$$

$$X(N-k) = X((-k))_N, \quad 0 \le k \le N-1.$$

Properties of DFT

Circular time shift of a sequence

$$x(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k)$$

then

$$x((n-l))_N \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k)e^{-j2\pi kl/N}$$

Proof From the definition of the DFT we have

$$\text{DFT}\{x((n-l))_N\} = \sum_{n=0}^{N-1} x((n-l))_N e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{l-1} x((n-l))_N e^{-j2\pi kn/N}$$

$$+ \sum_{n=l}^{N-1} x(n-l)e^{-j\pi kn/N}$$

Properties of DFT
Circular time shift of a sequence

But $x((n-l))_N = x(N-l+n)$. Consequently,

$$\sum_{n=0}^{l-1} x((n-l))_N e^{-j2\pi kn/N} = \sum_{n=0}^{l-1} x(N-l+n) e^{-j2\pi kn/N}$$

$$= \sum_{m=N-l}^{N-1} x(m) e^{-j2\pi k(m+l)/N}$$

Furthermore,

$$\sum_{n=l}^{N-1} x(n-l) e^{-j2\pi kn/N} = \sum_{m=0}^{N-1-l} x(m) e^{-j2\pi k(m+l)/N}$$

Therefore,

$$\text{DFT}\{x((n-l))\} = \sum_{m=0}^{N-1} x(m) e^{-j2\pi k(m+l)/N}$$

$$= X(k) e^{-j2\pi kl/N}$$

Properties of DFT
Circular Frequency Shift

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

$$x(n)e^{j2\pi ln/N} \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X((k-l))_N$$

Properties of DFT
Circular Frequency Shift

Proof

$$\mathrm{DFT}\{x[n]e^{j2\pi nM/N}\} = \sum_{n=0}^{N-1} x[n]e^{j2\pi nM/N}e^{-j2\pi kn/N}$$

$$= X[(k-M)_N]$$

Properties of DFT
Complex conjugate property

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

$$x^*(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X^*((-k))_N = X^*(N-k)$$

Proof

$$\frac{1}{N}\sum_{k=0}^{N-1} X^*(k)e^{j2\pi kn/N} = \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi k(N-n)/N}\right]$$

$$x^*((-n))_N = x^*(N-n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X^*(k)$$

Properties of DFT
Circular correlation

$$x(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k)$$

$$y(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} Y(k)$$

$$\tilde{r}_{xy}(l) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} \tilde{R}_{xy}(k) = X(k)Y^*(k)$$

$$\tilde{r}xy(l) = \sum_{n=0}^{N-1} x(n)y^*((n-l))N$$

Properties of DFT
Circular correlation

Proof We can write $\tilde{r}_{xy}(l)$ as the circular convolution of $x(n)$ with $y^*(-n)$, that is,

$$\tilde{r}_{xy}(l) = x(l) \circledN y^*(-l)$$

With the help of properties seen earlier

$$\tilde{R}_{xy}(k) = X(k)Y^*(k)$$

Under special circumstances when x(n)=y(n)

$$\tilde{r}_{xx}(l) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} \tilde{R}_{xx}(k) = |X(k)|^2$$

Properties of DFT
Multiplication in time

$$x_1(n) \xleftrightarrow[N]{\text{DFT}} X_1(k)$$

$$x_2(n) \xleftrightarrow[N]{\text{DFT}} X_2(k)$$

$$x_1(n)x_2(n) \xleftrightarrow[N]{\text{DFT}} \frac{1}{N} X_1(k) \, Ⓝ \, X_2(k)$$

Properties of DFT

Multiplication in time

$$\mathrm{DFT}\{x[n]y[n]\} = \frac{1}{N}\mathrm{DFT}\left\{x[n]\sum_{\ell=0}^{N-1} Y[\ell]e^{j2\pi\ell n/N}\right\}$$

$$= \frac{1}{N}\sum_{\ell=0}^{N-1} Y[\ell]\,\mathrm{DFT}\{x[n]e^{j2\pi\ell n/N}\} = \frac{1}{N}\sum_{\ell=0}^{N-1} Y[\ell]X[(k-\ell)_N]$$

$$= \frac{1}{N}X[k] \otimes Y[k]$$

Properties of DFT

Inner Product (Parseval's Theorem)

Complex valued sequences

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

$$y(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} Y(k)$$

$$\sum_{n=0}^{N-1} x^*[n]y[n] = \sum_{n=0}^{N-1}\underbrace{\left(\frac{1}{N}\sum_{k=0}^{N-1} X^*[k]e^{-j2\pi kn/N}\right)}_{x^*[n]}y[n]$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]\underbrace{\left(\sum_{n=0}^{N-1} y[n]e^{-j2\pi kn/N}\right)}_{Y[k]}$$

Properties of DFT
Summary of properties

TABLE 7.2 Properties of the DFT

Property	Time Domain	Frequency Domain
Notation	$x(n), y(n)$	$X(k), Y(k)$
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Time reversal	$x(N-n)$	$X(N-k)$
Circular time shift	$x((n-l))_N$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l))_N$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular convolution	$x_1(n) \textcircled{N} x_2(n)$	$X_1(k)X_2(k)$
Circular correlation	$x(n) \textcircled{N} y^*(-n)$	$X(k)Y^*(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k) \textcircled{N} X_2(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

THANK YOU

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