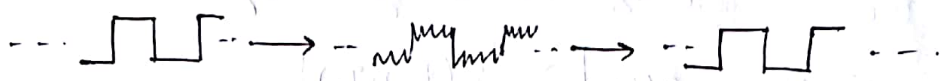
Sampling

Why digital communication ?

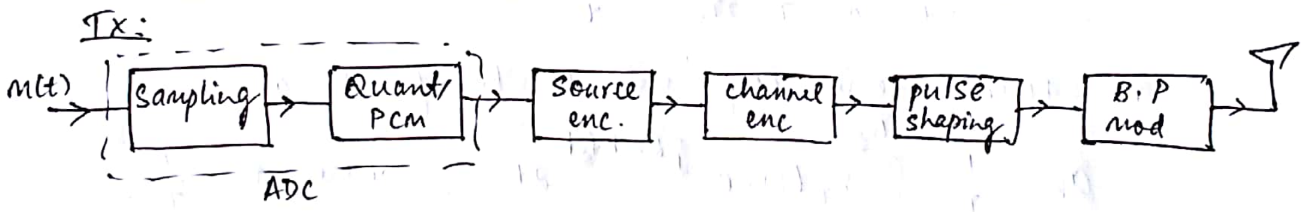
* In analog communication, distortion due to channel effects & noise cannot be undone. In digital communication, it is possible to recover the signal from distorted version.



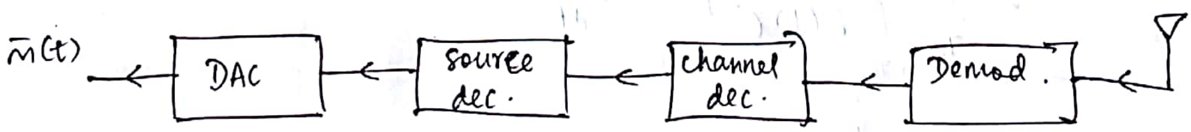
* compression, error correction coding & encryption can be performed
* common format / protocol for storage / communication of different types of signals : voice, image, video, biomedical etc
* processor / algorithm in place of components / ~~ckt~~ circuits.
* without digital communication, "NO INTERNET".

Typical digital communication system

Tx:



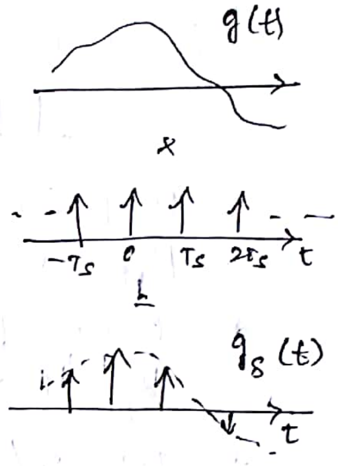
Rx



Sampling :

The sampling process converts a continuous time signal to a discrete-time signal. [Quantization & PCM convert it to a digital signal]

Consider a continuous time signal $g(t)$ that has finite energy. Suppose it is uniformly sampled with a sampling period of T_s sec. The sampling process is accomplished by multiplying $g(t)$ with a sequence of dirac impulses spaced apart by T_s sec.



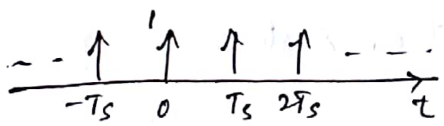
∴ The sampled signal is given by

$$g_\delta(t) = g(t)\left[\sum_{n=-\infty}^{\infty} \delta(t-nT_s)\right]$$

$$\therefore\ g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\,\delta(t-nT_s)$$

Spectrum of $g_\delta(t)$

First consider the impulse train $\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$



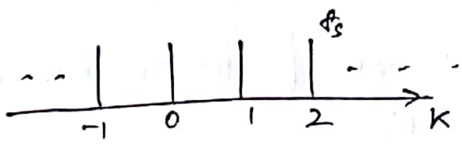
Its Fourier series coefficients can be found as

$$a_K = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta_{T_s}(t)\cdot e^{-j2\pi K f_s t}\,dt \quad , \text{ where } f_s = \frac{1}{T_s}.$$

$$= \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)\, e^{-j2\pi K f_s t}\,dt$$

$$= \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)\cdot 1\cdot dt = \frac{1}{T_s} = f_s.$$

∴ The Fourier series is



∴ The Fourier transform of $\delta_{T_s}(t)$ is

$$F\{\delta_{T_s}(t)\} = f_s \sum_{K=-\infty}^{\infty} \delta(f-Kf_s)$$

[Recall : If a continuous time periodic signal $x(t)$ with fundamental period $T_s = \frac{1}{f_s}$ has the F.S coefficients a_k, then its fourier transform is given by

$$X(f) = \sum_{k=-\infty}^{\infty} a_k \, \delta(f - kf_s)$$]

Now consider

$$g_\delta(t) = g(t) \cdot \delta_{T_s}(t)$$

Applying F.T, we have

$$G_\delta(f) = G(f) * F\{\delta_{T_s}(t)\} \quad \text{[multiplication property]}$$

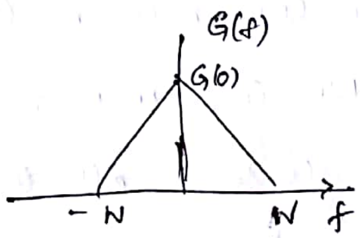
$$= G(f) * \left[f_s \sum_{k=-\infty}^{\infty} \delta(f - kf_s) \right]$$

or $$G_\delta(f) = f_s \cdot \sum_{k=-\infty}^{\infty} G(f - kf_s)$$

$\Rightarrow$ The spectrum of the sampled signal is a periodic extension of the original spectrum $G(f)$, with fundamental period equal to the sampling rate.

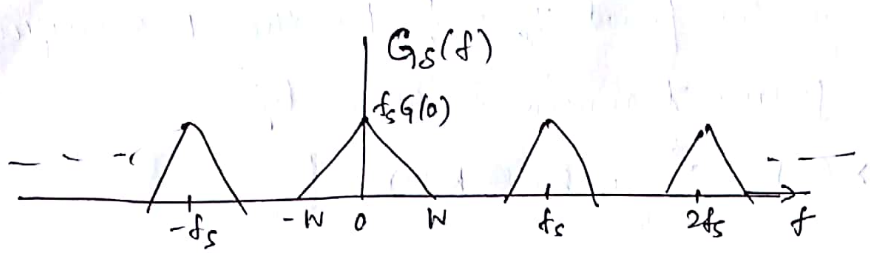
[Recall : discrete in time $\Rightarrow$ periodic in frequency]

Now, suppose $g(t)$ is "strictly bandlimited" to W Hz.
i.e, $G(f) = 0$ for $|f| > W$.



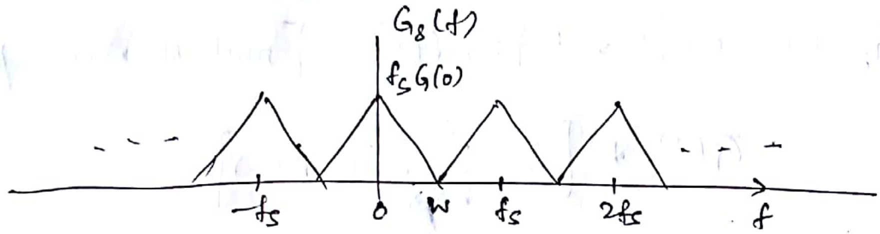
[NOTE: No practical signal can be "strictly bandlimited", since ~~their~~ bandlimited $\Rightarrow$ time unlimited]

Case 1: Consider $T_s < \frac{1}{2W}$ (or $f_s > 2W$)



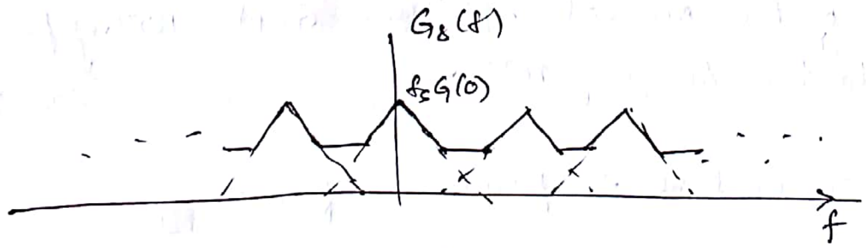
This is "over sampling".

Case 2: $T_s = \frac{1}{2W}$ ($f_s = 2W$)



This is "critical sampling / Nyquist sampling".

Case 3: $T_s > \frac{1}{2W}$ ($f_s < 2W$)



This is "undersampling" or "aliasing". We cannot recover $G(f)$ from $G_\delta(f)$.

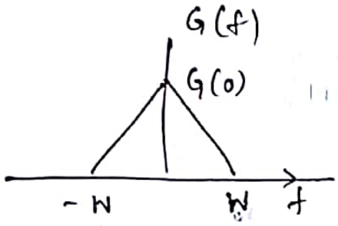
In the first two cases, we can recover $G(f)$ from $G_\delta(f)$, or equivalently, we can recover $g(t)$ from its samples. The samples contain all the information about $g(t)$.

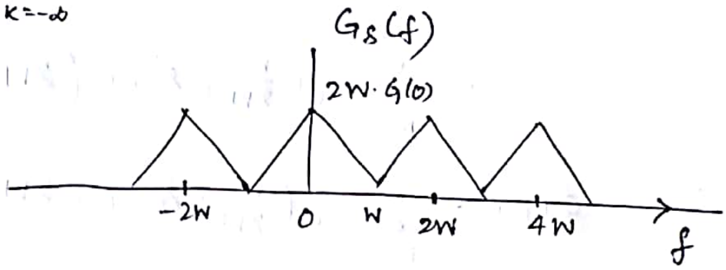
Reconstruction of $g(t)$

Let $T_s = \frac{1}{2W}$. $\therefore$ $g_\delta(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \delta\left(t - \frac{n}{2W}\right)$

Then $G_\delta(f) = f_s \sum_{k=-\infty}^{\infty} G(f - kf_s)$ becomes

$$= 2W \sum_{k=-\infty}^{\infty} G(f - k \cdot 2W)$$

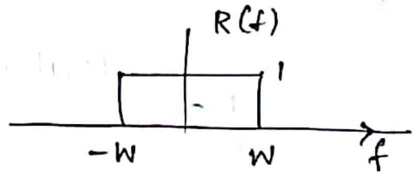




$\therefore$ We have $G(f) = \frac{1}{2W} G_\delta(f) \quad -W \le f \le W,$

or $G(f) = \frac{1}{2W} G_\delta(f) \cdot R(f),$

where $R(f) = \begin{cases} 1 & |f| \le W \\ 0 & \text{elsewhere} \end{cases}$



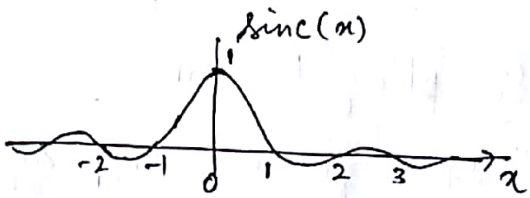
$\therefore$ we have $g(t) = \frac{1}{2W}\left[g_\delta(t) * r(t)\right]$ [convolution property]

$$r(t) = \int_{-\infty}^{\infty} R(f)\, e^{j2\pi f t}\, df$$

$$= \int_{-W}^{W} e^{j2\pi f t}\, df = \frac{1}{j2\pi t} e^{j2\pi f t}\Big|_{-W}^{W} = \frac{\sin 2\pi W t}{\pi t}$$

$$= 2W \cdot \frac{\sin \pi \cdot 2Wt}{\pi \cdot 2Wt} = 2W \operatorname{sinc}(2Wt).$$

Recall, $\operatorname{sinc}(x) = \frac{\sin \pi x}{\pi x}$



we have $\operatorname{sinc} x = \begin{cases} 1 & x = 0 \\ 0 & x = \pm 1, \pm 2, \ldots \end{cases}$

This is known as "interpolatory property".

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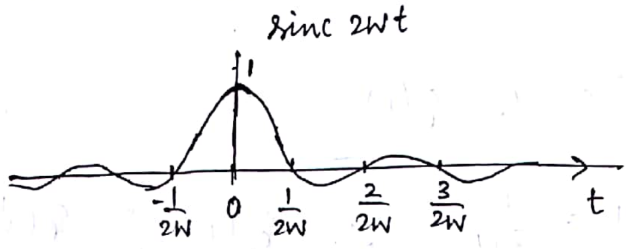
∴ we have



consider,

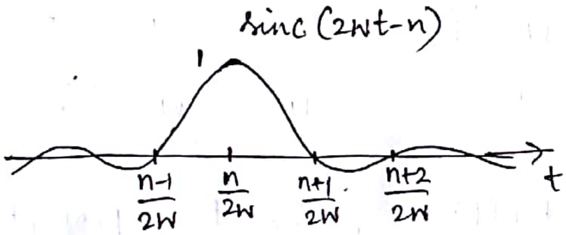
$$g(t) = \frac{1}{2W}\left[g_\delta(t) * r(t) \right]$$

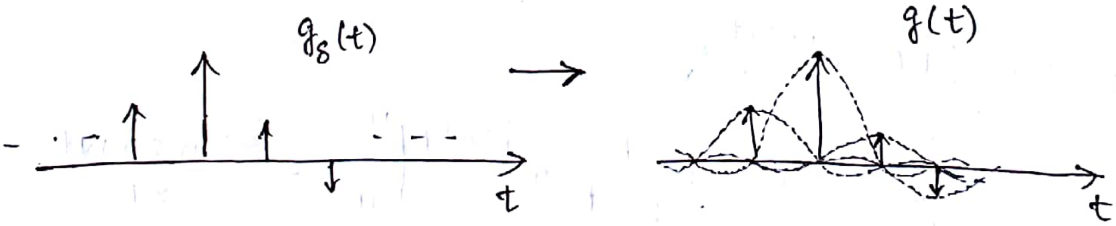
$$= \frac{1}{\cancel{2W}}\left[\sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \delta\left(t - \frac{n}{2W}\right) * \cancel{2W}\, \mathrm{sinc}(2Wt) \right]$$

$$= \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \mathrm{sinc}\left(2W\left\{t - \frac{n}{2W}\right\}\right)$$

$$g(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \mathrm{sinc}(2Wt - n)$$

This is the interpolation formula for reconstructing $g(t)$ from its critically sampled version.





* The sinc fn at every sample has its zero crossings at the locations of all other samples.

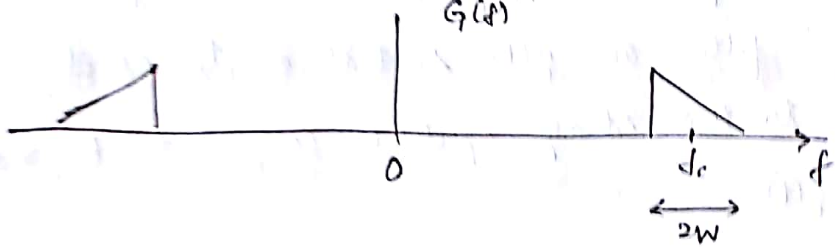
Sampling theorem

" If a finite energy signal has no frequencies higher than W Hz, then it is completely ~~specified~~ determined by specifying its samples at a sequence of samples spaced $\frac{1}{2W}$ sec. apart. The signal can be ~~xxxx~~ completely recovered from these samples."

2W is called the "Nyquist rate" or "Nyquist frequency".

Quadrature Sampling

Consider a real bandpass signal $g(t)$ whose spectrum is centered around f_c Hz, with a bandwidth of 2W Hz.

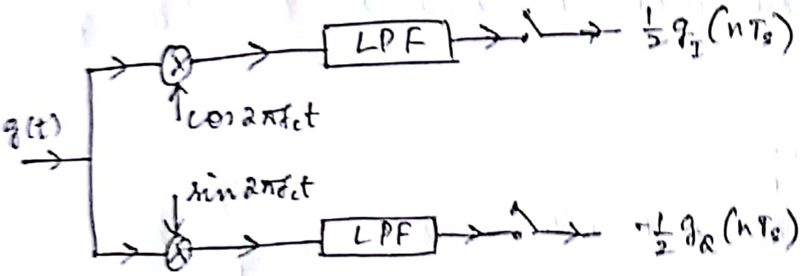


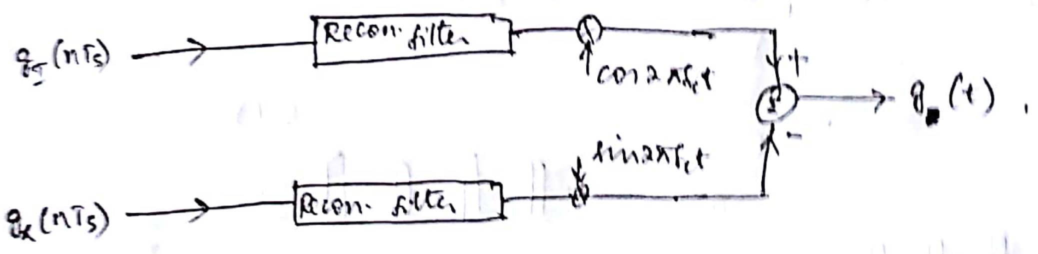
Sampling this signal with rate equal to twice the max. frequency ($f_s = 2(f_c + W)$) is clearly inefficient.
Instead, we can utilize the canonical representation of $g(t)$.

WKT, $\quad g(t) = g_I(t)\cos 2\pi f_c t - g_Q(t)\sin 2\pi f_c t \longrightarrow (1)$

where $g_I(t)$ & $g_Q(t)$ are the in-phase and quadrature components of $g(t)$ respectively. Both $g_I(t)$ & $g_Q(t)$ are baseband signals.
We can sample both at 2W samples/sec each.

To obtain $g(t)$ from samples, we can reconstruct $g_I(t)$ & $g_Q(t)$ from the samples & then use eq (1).



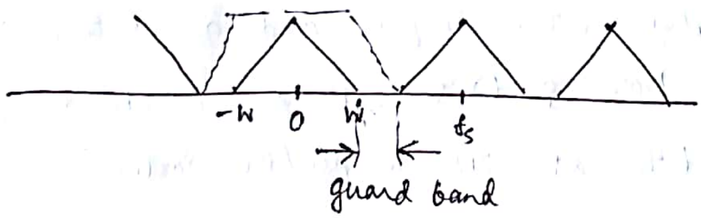


Practical Sampling

Consider the signal g(t) that has most of its energy within W Hz. (No practical signal can be strictly band limited).

1. Before sampling, we apply an LPF, called "anti-aliasing" filter, on g(t) ~~so that~~ as to avoid aliasing due to the stray spectral components beyond W Hz in g(t).

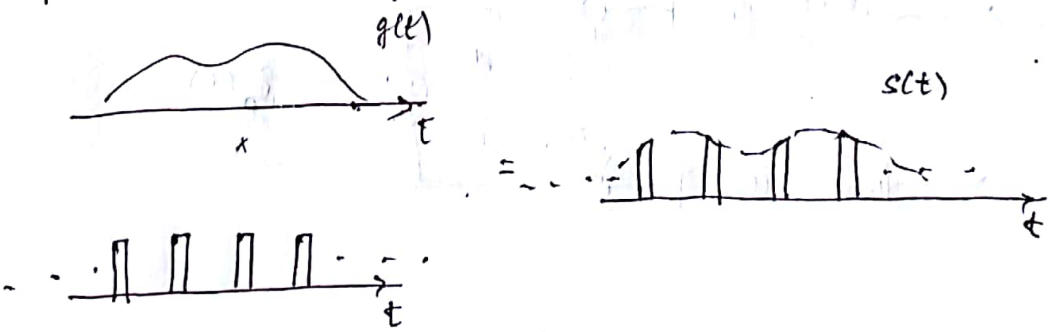
2. We select f_s to be slightly more than 2W, so that a practical filter with a ~~finite~~ non-zero transition band can be used for reconstruction. $f_s - 2W$ is called the "guard band".



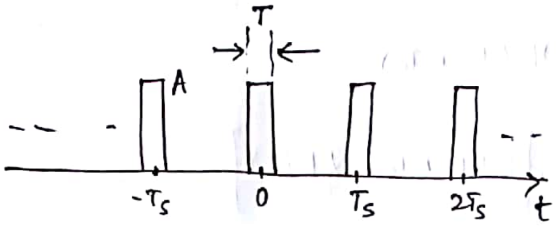
In practice, we cannot generate an ideal impulse train. Instead, short pulses are used. This leads to two types of practical sampling.

1. Natural sampling

Here, g(t) is multiplied with a sequence of short pulses, instead of the impulse train.



Let $c(t)$ denote the sequence of short pulses.



Recall the F.S for the periodic rect. signal

$$a_K = Ad \operatorname{sinc} Kd$$

$$= \frac{AT}{T_s} \operatorname{sinc} \frac{KT}{T_s}$$

$\therefore$ the F.T of $c(t)$ is

$$C(f) = \frac{AT}{T_s} \sum_{K=-\infty}^{\infty} \operatorname{sinc} \frac{KT}{T_s} \delta(f - \frac{K}{T_s})$$

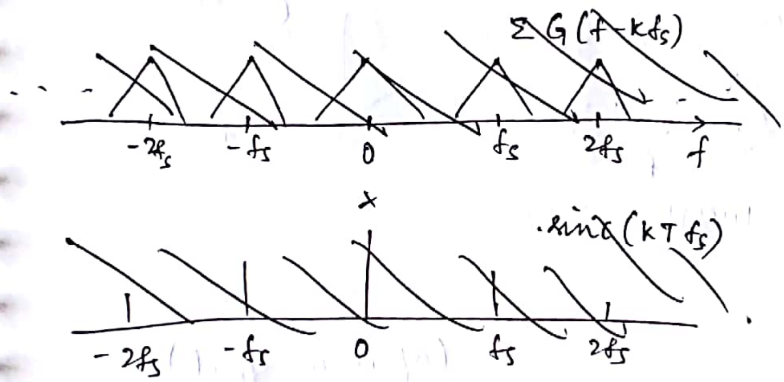
$$= f_s AT \sum_{K=-\infty}^{\infty} \operatorname{sinc}(KTf_s)\, \delta(f - Kf_s)$$

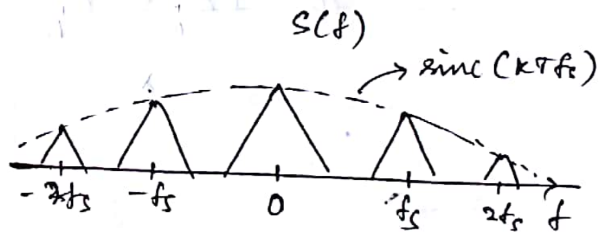
We have $\quad s(t) = g(t) \cdot c(t)$

$\therefore \quad S(f) = G(f) * C(f)$

$$= G(f) * \left[f_s AT \sum_{K=-\infty}^{\infty} \operatorname{sinc}(KTf_s)\, \delta(f - Kf_s) \right]$$

$$= f_s AT \sum_{K=-\infty}^{\infty} \operatorname{sinc}(KTf_s) . G(f - Kf_s)$$



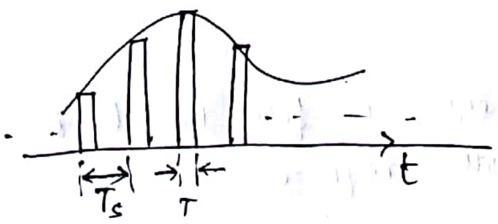


$g(t)$ $G(f)$ can be recovered without distortion from $S(f)$.

Suppose $AT = 1$. Then, as $T \to 0$, $S(f) \to G_\delta(f)$ (the ideal sampling case).

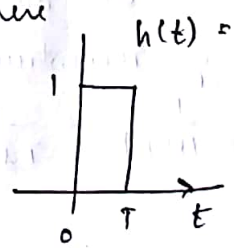
Flat-top sampling

Here, each sample has a duration of T sec.



Here, $s(t) = \sum_{n=-\infty}^{\infty} g(nT_s) h(t - nT_s)$,

where $\quad h(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases}$



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we have,

$$s(t) = \sum_{n=-\infty}^{\infty} g(nT_s)h(t-nT_s)$$

$$= \sum_{n=-\infty}^{\infty} g(nT_s).\left[\delta(t-nT_s) * h(t)\right]$$

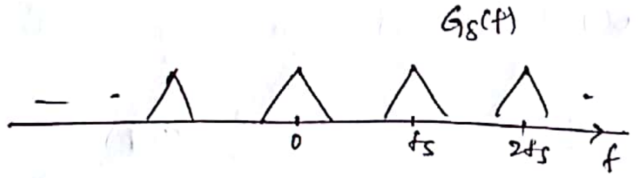
$$= \sum_{n=-\infty}^{\infty} \left[g(nT_s).\delta(t-nT_s)\right] * h(t)$$

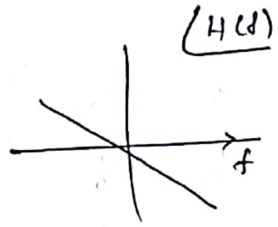
$$= g_\delta(t) * h(t).$$

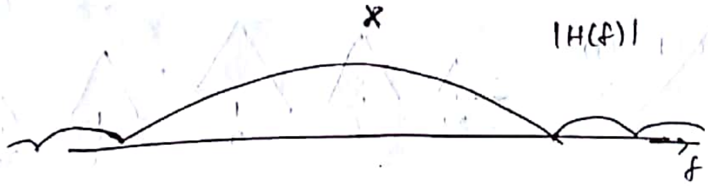
$$\boxed{\begin{array}{l} \text{WKT} \\ H(f) = T \,\text{sinc}(fT) e^{-j\pi f T} \end{array}}$$

$$\therefore S(f) = G_\delta(f) H(f).$$

$$S(f) = \frac{T}{T_s} \sum_{k=-\infty}^{\infty} G(f-kf_s)\,\text{sinc}(fT)\, e^{-j\pi f T}$$







(aperture effect)

* This leads to distortion. Higher frequencies are attenuated.
* Lower the duty cycle $\frac{T}{T_s}$, lower the distortion.

At $f = \frac{f_s}{2}$, $\text{sinc}(fT) = \text{sinc}\left(\frac{T}{2T_s}\right)$

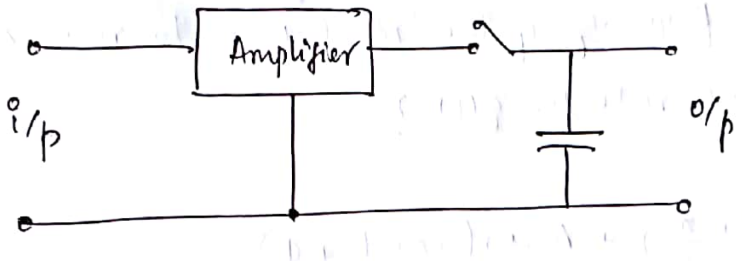
if $\frac{T}{T_s} = 0.1$, $\text{sinc}\left(\frac{T}{2T_s}\right) = \frac{1}{1.0041} \approx 1.$

$\Rightarrow$ distortion is negligible for $\frac{T_s}{T_s} \leq 0.1$.

* But making $\frac{T}{T_s}$ smaller results in attenuation, since $S(f)$ is scaled by $\frac{T}{T_s}$.
* If $\frac{T}{T_s} > 0.1$, an additional equalizer filter has to be used during reconstruction, with response

$$\frac{1}{|H(f)|} = \frac{1}{T\,\text{sinc}(fT)}$$

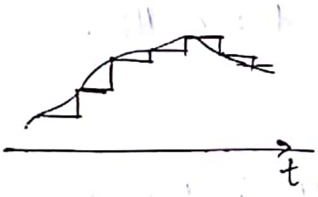
Sample and Hold circuit



* unity gain
* Low o/p impedance
* high load impedance

* when the switch is closed, the capacitor quickly charges up to the i/p voltage.
* when the switch is open, the capacitor retains the voltage level, till the switch is closed again.

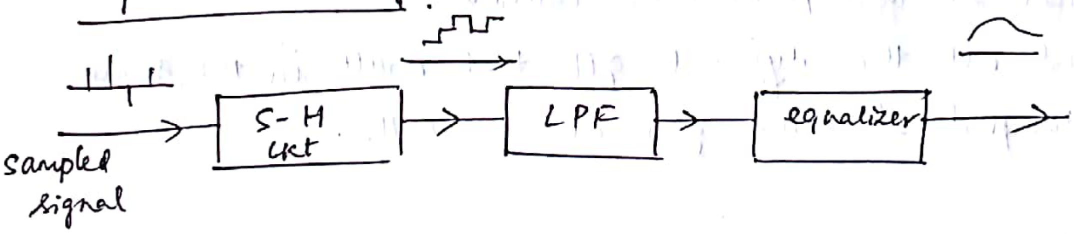
* Flat top sampling with $T = T_s$.



$$H(f) = T_s \, \text{sinc}\,(fT_s)\, e^{-j\pi f T_s}$$

$$S(f) = \sum_{k=-\infty}^{\infty} G(f - kf_s) \cdot \text{sinc}\,(fT_s)\, e^{-j\pi f T_s}.$$

Signal recovery.



problems.

1. Let $g(t) = A \cos 2\pi f_0 t$.

   plot the spectrum of the sampled signal, if $g(t)$ is sampled with i) $f_s = f_0$

   ii) $f_s = 2f_0$

   iii) $f_s = 3f_0$.

   Repeat for $g(t) = A \sin 2\pi f_0 t$.

   Explain the results.

2. $g(t) = 10\cos(20\pi t)\cos(200\pi t)$ is sampled with $f_s = 250$ Hz.

i) ~~plot~~ plot the spectrum of the sampled signal.

ii) specify the cut off frequency of the ideal filter to recover $g(t)$

iii) what is the Nyquist rate for $g(t)$?

3. Let $g(t) = \cos\left(2\pi f_1 t + \frac{\pi}{2}\right) + A\cos\left(2\pi f_2 t + \phi\right)$ with $f_1 = 3.9$ KHz & $f_2 = 4.1$ kHz. When $g(t)$ is sampled at $t = 0, T, 2T, \ldots$ with $T = 125\,\mu s$, the resulting signal is zero. Find A & ϕ.

4. Let $g(t) = 10\cos(50\pi t)$ be sampled with $f_s = 75$ Hz.

i) find the sampled sequence $g(n)$

ii) find another signal $g'(t)$ that results in the same sampled sequence $g(n)$, when sampled with $f_s = 75$ Hz. What is this phenomenon called?

iii) Find all the different $g(t)$ that result in the same sampled sequence $g(n)$ at $f_s = 75$ Hz.

5. Find the Nyquist frequency for the following signals.

i) $\text{sinc}(100t)$

ii) $\text{sinc}^2(100t)$

iii) $\text{sinc}(100t) + \text{sinc}(200t)$

iv) $\text{sinc}(100t) \cdot \text{sinc}(200t)$

v) $\text{sinc}(100t) * \text{sinc}(200t)$

vi) $\text{sinc}(100t)\,\text{sinc}^2(200t)$

6. The signal $g(t) = \cos 2\pi f_0 n$ is sampled at f_s Hz to obtain the discrete time sequence $g(n)$. Indicate all the sinusoids that result in the same $g(n)$ when sampled at f_s Hz.

7. Let the signal $g(t)$ be bandlimited to ~~10 Hz~~ 100 Hz. Find the Nyquist rate for the following signals:

i) $g^2(t)$

ii) $g(t-3)$

iii) $g\left(\frac{t}{3}\right)$

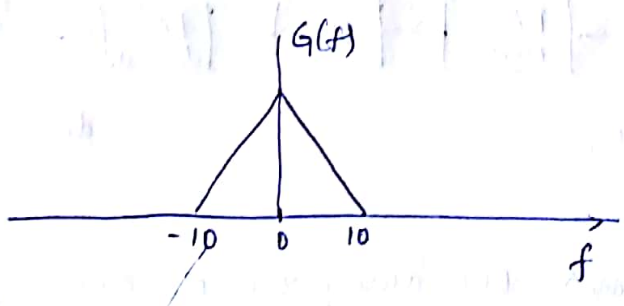
iv) ~~$g(t) * g(2t)$~~ $g(t) \cdot g(2t)$

v) ~~$f(t) = \{ g(t)$~~ $g(t) \cdot \cos 50\pi t$

8. In natural sampling, can an arbitrary periodic signal with fundamental period T_s be used instead of the rectangular pulse train? what is the condition ~~of~~ on that signal, so that the sampled signal can be recovered?

9. The spectrum of $g(t)$ is as shown in the figure. The signal is sampled with a periodic train of rectangular pulses of duration $\frac{50}{3}$ ms. plot the spectrum of the sampled signal for frequencies upto 50 Hz for the following conditions.

i) f_s = Nyquist rate   ii) $f_s = 10$ Hz.

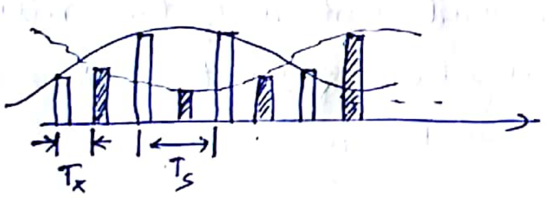


10. The signal $g(t) = 10 \sin 20\pi t + 4$ is sampled using a periodic rectangular pulse train of fundamental frequency 50 Hz. The pulses ~~are~~ are of width 10 ms. What frequencies are present in the sampled signal between 0 Hz and 200 Hz, for the following cases

i) Natural sampling
ii) flat-top sampling.

Time Division Multiplexing

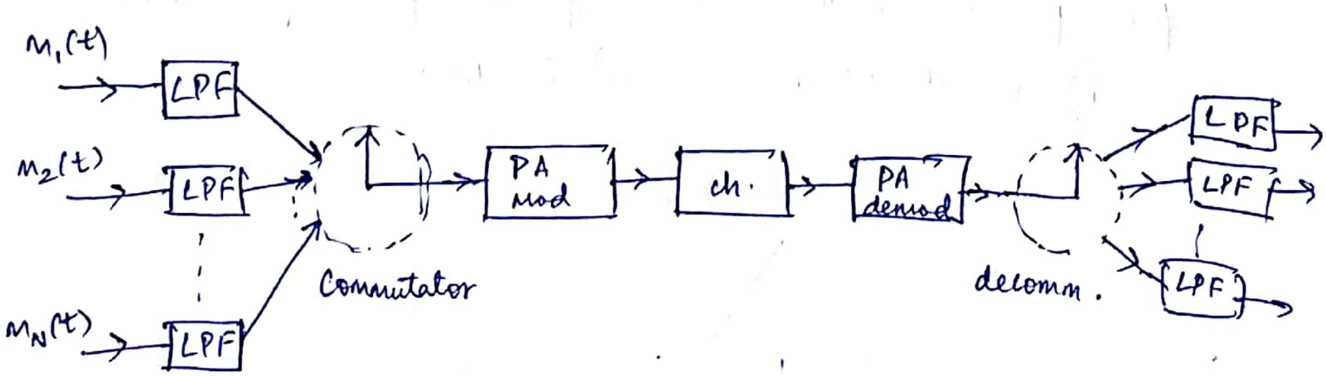
In pulse Amplitude modulation (Flat-top sampling), each pulse occupies the comm. channel for only a fraction of the sampling interval. ∴ we can accommodate more than one message signal on the same channel.



$T_x = \frac{T_s}{N}$

where N : no. of message signals.

TDM is a system in which many independent message signals are transmitted over the same comm. channel without mutual interference, on a time-sharing basis.



Let W be the max. BW among the N message signals.

The pre-alias filter have a BW of W Hz.

The commutator takes a narrow sample of each of the N i/p signals at a rate of f_s Hz ~~($f_s > 2N$)~~ (f_s is slightly more than 2N), and interleaves the N samples. The spacing b/w samples is $T_x = \frac{T_s}{N}$. $\left(T_s = \frac{1}{f_s}\right)$.

The ~~PA~~ pulse amplitude modulator performs "pulse shaping".

The commutator & decommutator need to be synchronized.

Ex: 24 voice signals are ~~time~~ sampled & then time-division multiplexed. The flat-top samples have a duration of 1 μs. An extra pulse of 1 μs width is added for synchronization in every sampling period. If the sampling rate is 8 kHz, find the spacing between successive pulses of the multiplexed signal.

Ans: $T_s = \frac{1}{8000} = 125\ \mu s$.

$T_x = \frac{T_s}{N} = \frac{125}{25} = 5\ \mu s$ (24 + 1 synch. pulse)

pulse duration = 1 μs ⇒ spacing = 4 μs.

Answers to problems.

3. $A = 1$ & $\phi = \pi/2$

or $A = -1$ & $\phi = -\pi/2$

4. i) $g(n) = \cos \frac{2\pi n}{3}$

ii) $g'(t) = \cancel{\cos 10\pi}\ 10 \cos 100\pi t$: Aliasing

iii) $10 \cos 2\pi f_0 t$, where $f_0 = 25, 100, 175, \ldots$
$50, 125, 200, \ldots$

5. i) 100 Hz iv) 300 Hz

ii) 200 Hz v) 100 Hz

iii) 200 Hz vi) 500 Hz

6. $\cos 2\pi f_1 t$, where $f_1 = f_0 \pm k f_s$

k = integer.

7. i) 400 Hz
ii) 200 Hz
iii) $\frac{200}{3}$ Hz
iv) 600 Hz
v) 250 Hz.

8. Yes. The signal must have a non-zero d.c value.

10. i) <u>Natural sampling</u>:

$$S(f) = \frac{A}{2} \sum_{k=-\infty}^{\infty} \operatorname{sinc}\left(\frac{k}{2}\right) G(f - k.50)$$

$\therefore$ frequencies present: 0, 10, 40, 50, 60, 140, 150, 160

ii) Flat-top sampling

$$\cancel{S(f) = G_\delta(f)} \times \cancel{AT \operatorname{sinc} f}$$

$$S(f) = G_\delta(f) \times T \operatorname{sinc}(fT)$$

$$= G_\delta(f) \times 10^{-2} \underbrace{\operatorname{sinc}(0.01 f)}_{\text{zero crossings at 100 Hz \& 200 Hz.}}$$

$\therefore$ frequencies present:

0, 10, 40, 50, 60, 90, 110, 140, 150, 160, 190.