Unit - II - Fast Fourier Transform

* for the efficient implementation of DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad k = 0 \cdots N-1$$

Complex Multiplications : N^2
for N-pt DFT

Complex Additions : $N(N-1)$
for N-pt DFT

when $x(n)$ ~~is complex~~ and $X(k)$ are complex valued.

$$X_R(k) = \sum_{n=0}^{N-1} \left[x_R(n) \cos \frac{2\pi kn}{N} + x_I(n) \sin \frac{2\pi kn}{N} \right]$$

$$X_I(k) = -\sum_{n=0}^{N-1} \left[x_R(n) \sin \frac{2\pi kn}{N} - x_I(n) \cos \frac{2\pi kn}{N} \right]$$

for: N-pt DFT
# of trignometric functions $- 2N^2$
# of real multiplications $- 4N^2$
# of real additions $- 4N(N-1)$

Based on Divide-and-Conquer, there are two approaches

① Decimation in time (DIT)
② Decimation in frequency (DIF)

} Fast fourier Transform

I Decimation in time fast fourier transform

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad k = 0 \cdots N-1 \quad \text{—— (1)}$$

$$= \sum_{\text{even}} x(n) W_N^{kn} + \sum_{\text{odd}} x(n) W_N^{kn}$$

$$= \sum_{m=0}^{N/2-1} x(2m) W_N^{2km} + \sum_{m=0}^{N/2-1} x(2m+1) W_N^{(2m+1)k}$$

$$= \sum_{m=0}^{N/2-1} f_1(m) W_{N/2}^{km} + W_N^k \sum_{m=0}^{N/2-1} f_2(m) W_{N/2}^{km}$$

$$X(k) = F_1(k) + W_N^k F_2(k) \quad \text{—— (2)} \quad k = 0 \cdots N/2 - 1$$

Periodic with $N/2$

$$X(k+N/2) = F_1(k+N/2) + W_N^{(k+N/2)} F_2(k+N/2)$$

$$= F_1(k) - W_N^k F_2(k) \quad \text{—— (3)}$$

$k = 0 \cdots N/2 - 1$

$|||^{ly}$

$$F_1(k) = V_{11}(k) + W_{N/2}^k V_{12}(k) \quad \text{—— (4)}$$

$$F_1(k+N/4) = V_{11}(k) - W_{N/2}^k V_{12}(k) \quad \text{—— (5)}$$

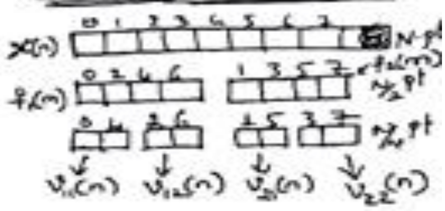
$k = 0 \cdots N/4 - 1$

$$F_2(k) = V_{21}(k) + W_{N/2}^{k} V_{22}(k)$$
$$F_2(k+N/4) = V_{21}(k) - W_{N/2}^{k} V_{22}(k)$$
$$k = 0 \cdots \tfrac{N}{4}-1$$

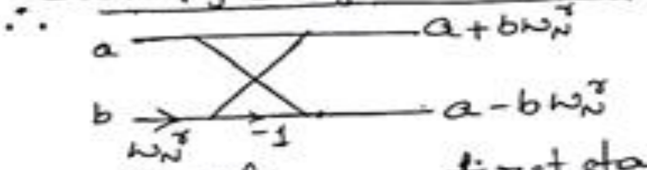
where
$$V_{11}(k) = DFT\{v_{11}(n)\} = DFT\{f_1(2n)\}$$
$$V_{12}(k) = DFT\{v_{12}(n)\} = DFT\{f_1(2n+1)\}$$
$$V_{21}(k) = DFT\{v_{21}(n)\} = DFT\{f_2(2n)\}$$
$$V_{22}(k) = DFT\{v_{22}(n)\} = DFT\{f_2(2n+1)\}$$
$$k = 0 \cdots N/4 - 1$$

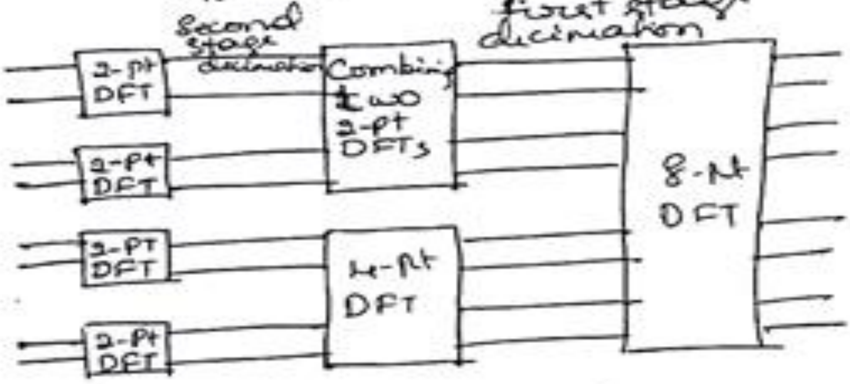
& are periodic with $N/4$

Example for DIT (N=8)



∴ Butterfly diagram-radix-2





To compute DFT,

$N = 2^{v}$, $v = \log_2 N$ times of decimation can be done.

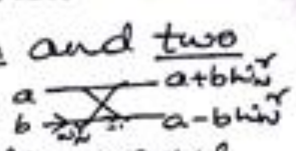
To compute an N-pt DFT using FFT we require

$\log_2 N$ # of stages

$N/2 \rightarrow$ butterfly diagrams per stage

there in total $(N/2 \log_2 N)$ # of butterfly diagrams are needed.

for one butterfly diagram, we have to perform one complex multiplication and two complex additions.



$\therefore$ for $(N/2 \log_2 N)$ # of butterfly diagrams we need $(N/2 \log_2 N)$ complex multiplications and $(N \log_2 N)$ complex additions.

Bit - reversed I/P & O/P

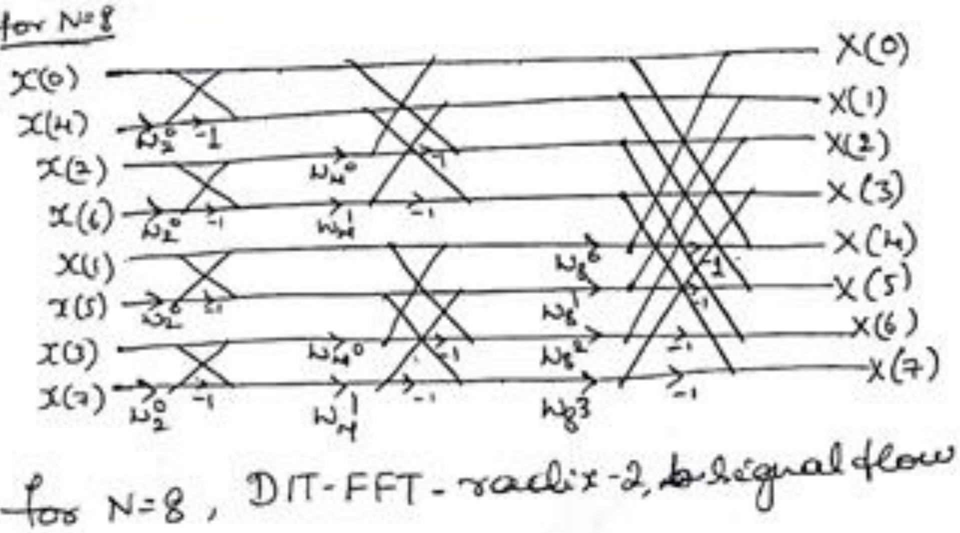
Index	$n_2 n_1 n_0$	$n_1 n_0 n_2$	$n_0 n_1 n_2$ (Bit reversed order)	
0 →	0 0 0	0 0 0	0 0 0	→ 0
1 →	0 0 1	0 1 0	1 0 0	→ 4
2 →	0 1 0	1 0 0	0 1 0	→ 2
3 →	0 1 1	1 1 0	1 1 0	→ 6
4 →	1 0 0	0 0 1	0 0 1	→ 1
5 →	1 0 1	0 1 1	1 0 1	→ 5
6 →	1 1 0	1 0 1	0 1 1	→ 3
7 →	1 1 1	1 1 1	1 1 1	→ 7

for N-pt DFT, if the input is given in order we get output in bit-reversed order.

& when the input is given in bit-reversed order output will be in order.

∗ for, Decimation-in-time, input is given in bit-reversed order and output will be in order.

Eg:- for N=8



for N=8, DIT-FFT-radix-2, signal flow graph

The above diagram for $N=8$ has

$$\log_2 N = \log_2 8 = 3 \Rightarrow \text{Stages}$$

$$N/2 = 8/2 = 4 \Rightarrow \text{Butterfly diagrams per stage.}$$

Total # of complex multiplications:

$$1\left(\frac{N}{2}\log_2 N\right) = \frac{N}{2}\log_2 N = \underline{12}$$

Total # of complex additions

$$2\left(\frac{N}{2}\log_2 N\right) = N\log_2 N = \underline{24}.$$

(Q) What is in-place computation?

For computing N-point DFT using FFT algorithms, 2N registers are enough to hold the values, ~~as~~ As output of the previous stage ~~need not to~~ need not be stored.

<u>Examples:</u>

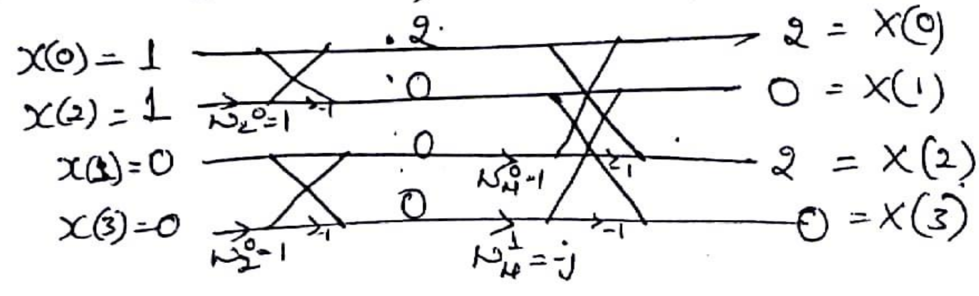
Using radix-2, DIT-FFT algorithm, determine the DFT of the following sequences

(a) $x(n) = (1, 0, 1, 0)$

(b) $x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$

(c) $x(n) = (1, 2, 3, 4, 4, 3, 2, 1)$

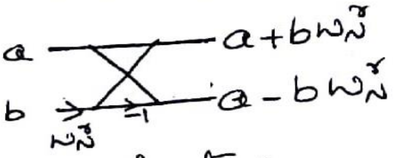
(a) $x(n) = (1, 0, 1, 0)$



$W_N^0 = \left(e^{-j\frac{2\pi}{N}}\right)^0 = 1$, $W_N^1 = \left(e^{-j\frac{2\pi}{N}}\right)^1 = -j$ (32)

$W_N^2 = \left(e^{-j\frac{2\pi}{N}}\right)^2 = -1$, $W_N^3 = \left(e^{-j\frac{2\pi}{N}}\right)^3 = +j$

(b) $x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$



Scaling factors

$W_8^0 = 1$

$W_8^1 = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

$W_8^2 = -j$

$W_8^3 = -\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

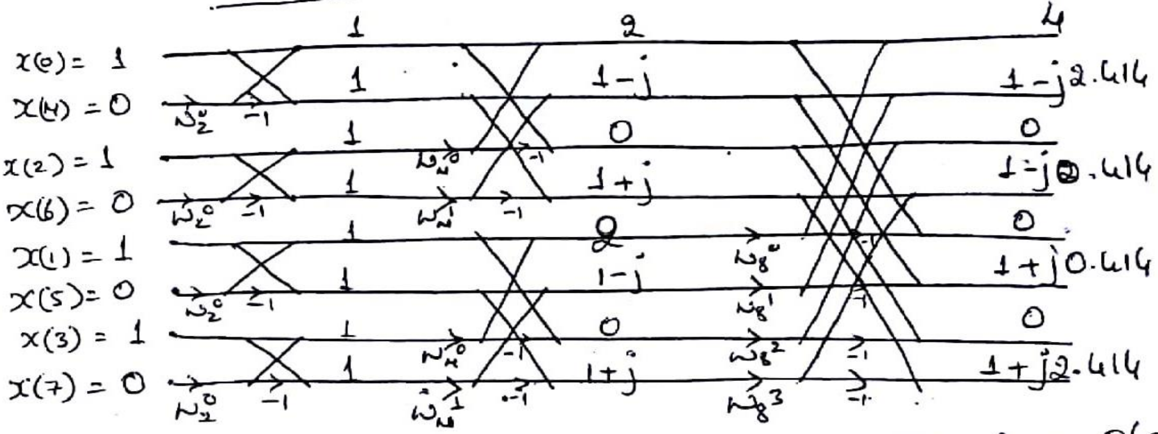
$W_8^4 = -1 = (-W_8^0)$

$W_8^5 = -\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}} = (-W_8^1)$

$W_8^6 = +j = (-W_8^2)$

$W_8^7 = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}} = (-W_8^3)$

$W_2^0 = W_4^0 = W_8^0 = 1$, $W_4^1 = W_8^2 = -j$

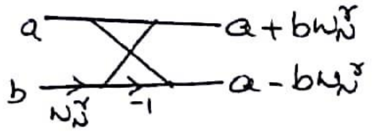


Output at first Stage	II Stage o/p	III Stage o/p
$X_1(0) = 1 + 0 \times W_2^0$	$X_2(0) = 1 + W_4^0 \times 1 = 2$	$X_3(0) = 2 + 2W_8^0 = 4$
$X_1(1) = 1 - 0 \times W_2^0$	$X_2(1) = 1 + 1 \times W_4^1 = 1-j$	$X_3(1) = 1-j + (1-j)W_8^1 = 1-j2.414$
$X_1(2) = 1 + 0 \times W_2^0$	$X_2(2) = 1 - 1 \times W_4^0 = 0$	$X_3(2) = 0 + 0 \times W_8^2 = 0$
$X_1(3) = 1 - 0 \times W_2^0$	$X_2(3) = 1 - 1 \times W_4^1 = 1+j$	$X_3(3) = (1+j) + (1+j)W_8^3 = 1-j0.414$
$X_1(4) = 1 + 0 \times W_2^0$	$X_2(4) = 1 + 1 \times W_4^0 = 2$	$X_3(4) = 2 - 2W_8^0 = 0$
$X_1(5) = 1 - 0 \times W_2^0$	$X_2(5) = 1 + 1 \times W_4^1 = 1-j$	$X_3(5) = (1-j) - (1-j)W_8^1 = 1+j0.414$
$X_1(6) = 1 + 0 \times W_2^0$	$X_2(6) = 1 - 1 \times W_4^0 = 0$	$X_3(6) = 0 - 0 \times W_8^2 = 0$
$X_1(7) = 1 - 0 \times W_2^0$	$X_2(7) = 1 - 1 \times W_4^1 = 1+j$	$X_3(7) = (1+j) - (1+j)W_8^3 = 1+j2.414$

$\therefore X(k) = [4,\ 1-j2.414,\ 0,\ 1-j0.414,\ 0,\ 1+j0.414,\ 0,\ 1+j2.414]$

Since, $x(n)$ is real, $X(k) = X^*(N-k)$

(c) $x(n) = (1, 2, 3, 4, 4, 3, 2, 1)$



$W_8^0 = W_4^0 = W_2^0 = 1$

$W_8^1 = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

$W_4^1 = W_8^2 = -j$

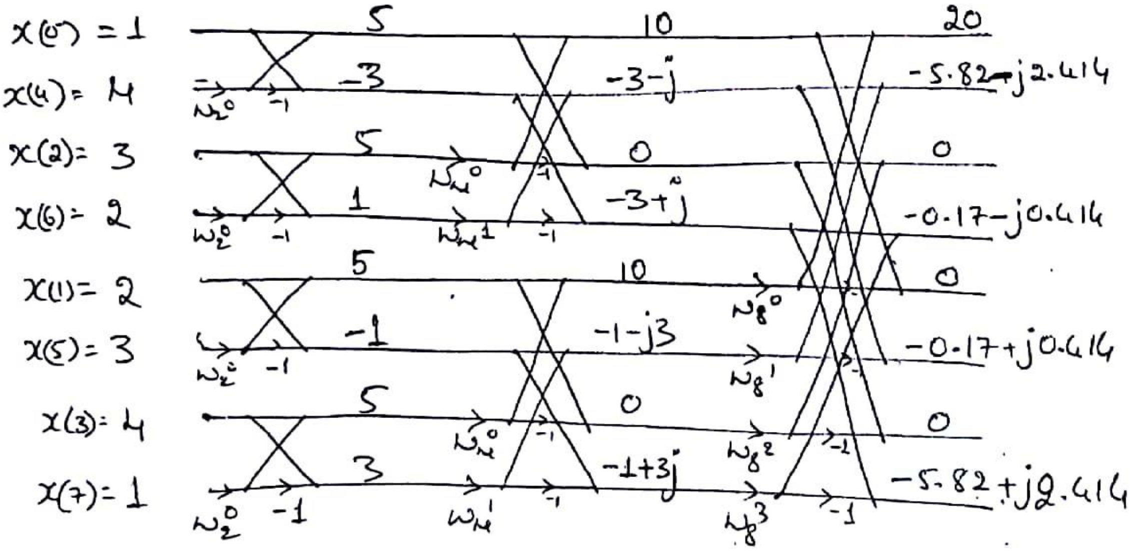
$W_8^3 = -\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

$W_8^4 = -(W_8^0)$

$W_8^5 = -(W_8^1)$

$W_8^6 = -(W_8^2)$

$W_8^7 = -(W_8^3)$



I Stage O/P	II Stage O/P	III Stage O/P
$X_1(0) = 1 + 4 \times W_2^0 = 5$	$X_2(0) = 5 + 5 \times W_4^0 = 10$	$X_3(0) = 10 + 10 \times W_8^0 = 20$
$X_1(1) = 1 - 4 \times W_2^0 = -3$	$X_2(1) = -3 + 1 \times W_4^1 = -3-j$	$X_3(1) = -3-j+(-1-3j)W_8^1 = -5.82 - j2.414$
$X_1(2) = 3 + 2 \times W_2^0 = 5$	$X_2(2) = 5 - 5 \times W_4^0 = 0$	$X_3(2) = 0 + 0 \times W_8^2 = 0$
$X_1(3) = 3 - 2 \times W_2^0 = 1$	$X_2(3) = -3 - 1 \times W_4^1 = -3+j$	$X_3(3) = -3+j+(-1+3j)W_8^3 = -0.171 - j0.414$
$X_1(4) = 2 + 3 \times W_2^0 = 5$	$X_2(4) = 5 + 5 \times W_4^0 = 10$	$X_3(4) = 10 - 10 \times W_8^0 = 0$
$X_1(5) = 2 - 3 \times W_2^0 = -1$	$X_2(5) = -1 + 3 \times W_4^1 = -1-j3$	$X_3(5) = -3-j-(-1-3j)W_8^1 = -0.171 + j0.414$
$X_1(6) = 4 + 1 \times W_2^0 = 5$	$X_2(6) = 5 - 5 \times W_4^0 = 0$	$X_3(6) = 0 - 0 \times W_8^2 = 0$
$X_1(7) = 4 - 1 \times W_2^0 = 3$	$X_2(7) = -1 - 3 \times W_8^1 = -1+3j$	$X_3(7) = -3+j-(-1+3j)W_8^3 = -5.82 + j2.414$

$\therefore X(k) = [20, -5.82-j2.414, 0, -0.17-j0.414, 0, -0.17+j0.414,$
$0, -5.82+j2.414]$

II Decimation in frequency (DIF) FFT

The output, $X(k)$ is divided into smaller and smaller subsequences.
'N' is considered to be a power of 2, $N=2^v$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad k = 0 \cdots N-1 \quad \text{—(1)}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=N/2}^{N-1} x(n) W_N^{kn}$$

$m = n - N/2$, $n = m + N/2$

$n = N/2$, $m = 0$ & $n = N-1$, $m = \frac{N}{2}-1$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{m=0}^{\frac{N}{2}-1} x(m+\tfrac{N}{2}) W_N^{k(m+\frac{N}{2})}$$

changing 'm' to 'n' ~~now~~ we get.

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + W_N^{N/2\,k} \sum_{n=0}^{\frac{N}{2}-1} x(n+\tfrac{N}{2}) W_N^{kn}$$

$$\cancel{= \sum_{n=0}^{\frac{N}{2}-1} [x(n) + x(n+\tfrac{N}{2})]}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} [x(n) + (-1)^k x(n+N/2)] W_N^{kn} \quad \text{—(2)}$$

$X(k)$ is decimated taking even and odd samples.

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} [\underbrace{x(n) + (-1)^{2k} x(n+N/2)}] W_N^{2kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} [\underbrace{x(n) + x(n+N/2)}_{e(n)}] W_{N/2}^{kn} \quad \text{—(3)}$$

$|||^{ly}$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} [x(n) + (-1)^{(2k+1)} x(n+N/2)] W_N^{(2k+1)n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \underbrace{\{[x(n) - x(n+N/2)] W_N^{n}\}}_{o(n)} \cdot W_{N/2}^{kn} \quad \text{—(4)}$$

$\therefore$ $e(n) = x(n) + x(n+N/2)$ —— (5)

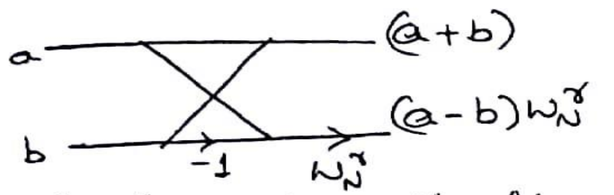
$o(n) = \left[x(n) - x(n+N/2) \right] W_N^{n}$ —— (6)

Using (5) & (6) in (3) & (4), respectively we get

$$X(2k) = \sum_{n=0}^{N/2-1} e(n)\, W_{N/2}^{kn}$$

$$X(2k+1) = \sum_{n=0}^{N/2-1} o(n)\, W_{N/2}^{kn} \qquad k = 0 \cdots \frac{N}{2}-1$$

This decimation is continued → 2-pt DFTs.



→ 1 → complex multiplication
→ 2 → complex additions

radix-2 Butterfly diagram for DIF-FFT

X. for N-point DFT using radix-2, DIT-FFT

(1) # of stages —— $\log_2 N$

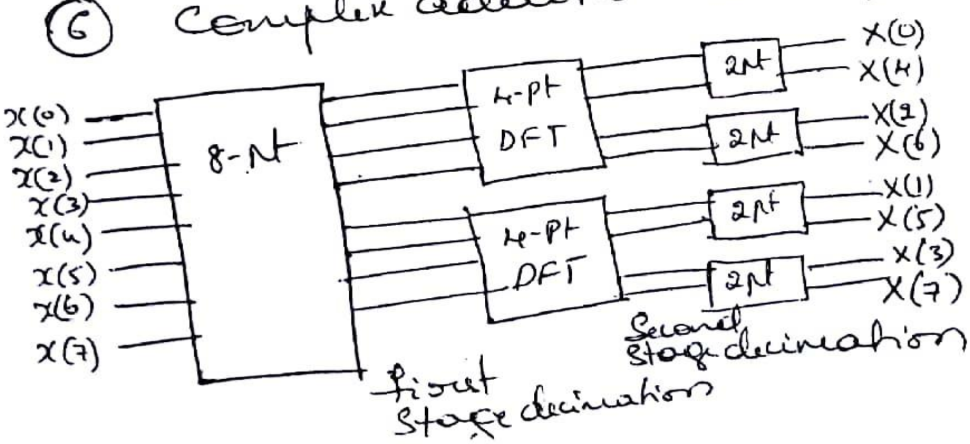
(2) # of Butterfly diagrams per stage —— $N/2$

(3) I/p is given in order

(4) O/p will be in bit-reversed order

(5) Complex multiplications → $\frac{N}{2} \log_2 N$

(6) Complex additions ——→ $N \log_2 N$



Example:

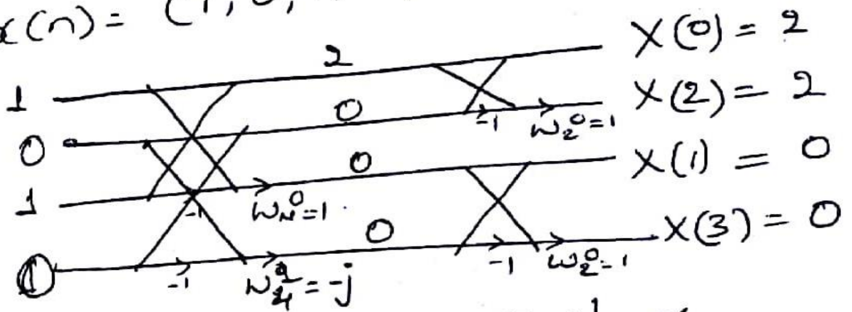
find DFT of the following sequences, using DIF-FFT ~~algorithm~~ method.

(a) $x(n) = (1, 0, 1, 0)$

(b) $x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$

(c) $x(n) = (1, 2, 3, 4, 4, 3, 2, 1)$

(a) $x(n) = (1, 0, 1, 0)$



O/P I stage

$1 + 1 = 2$

$0 + 0 = 0$

$(1 - 1)\, W_N^0 = 0$

$(0 - 0)\, {-j} = 0$

O/P II stage

$2 + 0 = 2$

$(2 - 0)\, 1 = 2$

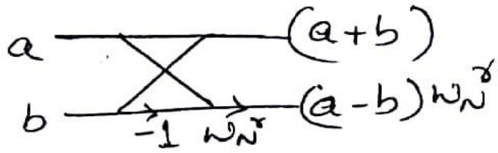
$0 + 0 = 0$

$(0 - 0)\, 1 = 0$

Since, the O/P is in bit-reversed order

$X(K) = (2, 0, 2, 0)$

(b) $x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$



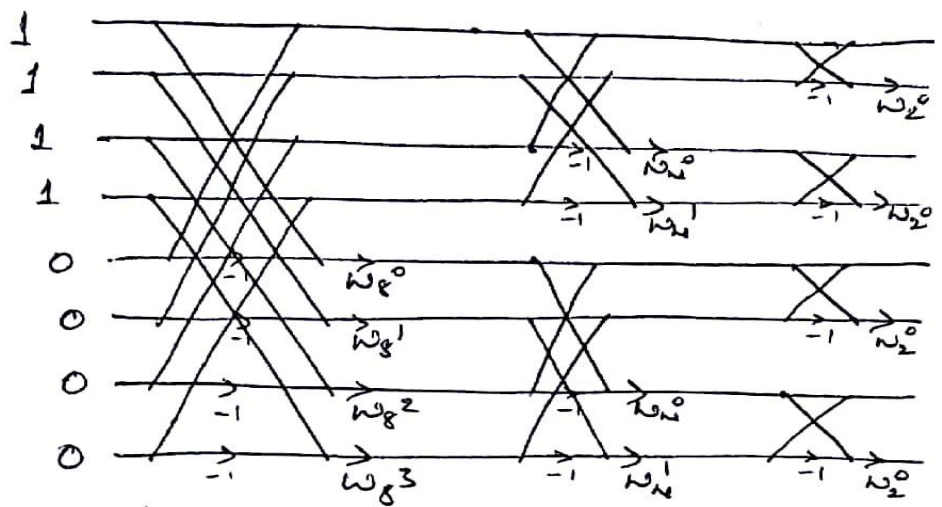
$W_8^0 = W_4^0 = W_2^0 = 1$

$W_8^1 = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

$W_8^2 = -j = W_4^1$

$W_8^3 = \frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

8



I Stage O/P

$1+0=1$

$1+0=1$

$1+0=1$

$1+0=1$

$(1-0)1 = 1 = W_8^0$

$(1-0)\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}} = 0.707-j0.707 = W_8^1$

$(1-0)-j = -j = W_8^2$

$(1-0)-\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}} = -0.707-j0.707 = W_8^3$

II Stage O/P

$1+1=2$

$1+1=2$

$(1-1)1=0$

$(1-1)-j=0$

$1+(-j)=1-j$

$W_8^1+W_8^3$

$(1-(-j))1 = 1+j$

$(W_8^1-W_8^3)W_4^1$

III Stage O/P

$X(0) = 2+2 = 4$

$X(4) = (2-2)1 = 0$

$X(2) = 0+0 = 0$

$X(6) = (0-0)1 = 0$

$X(1) = (1-j)+(W_8^1+W_8^3) = 1-j+\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}}-\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}} = 1-2.414j$

$X(5) = \left[(1-j)-(W_8^1+W_8^3)\right]1 = 1+j0.414$

$X(3) = \left[(1+j)+(W_8^1-W_8^3)W_4^1\right] = 1-j0.414$

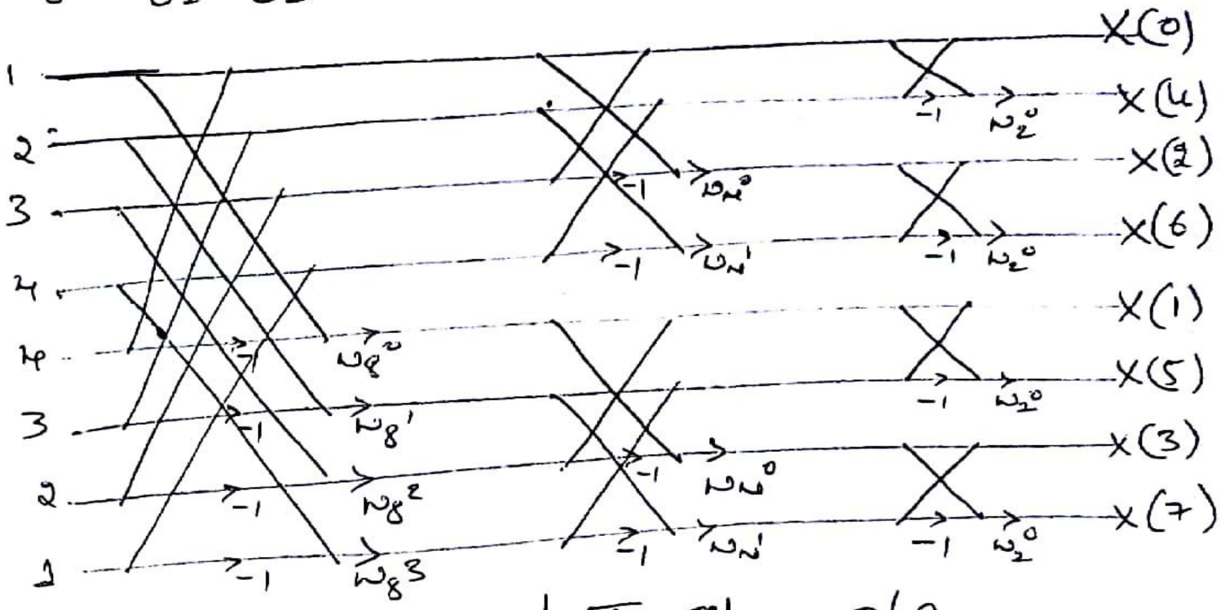
$X(7) = \left[(1+j)-(W_8^1-W_8^3)W_4^1\right]1 = 1+j2.414$

$X(K) = \{4,\ 1-j2.414,\ 0,\ 1-j0.414,\ 0,\ 1+j0.414,\ 0,\ 1+j2.414\}$

(c) $x(n) = (1, 2, 3, 4, 4, 3, 2, 1)$

$W_8^0 = W_4^0 = W_2^0 = 1$ $\qquad$ $W_8^2 = W_4^1 = -j$

$W_8^1 = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$ $\qquad$ $W_8^3 = \frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$



I Stage o/p

$1 + 4 = 5$

$2 + 3 = 5$

$3 + 2 = 5$

$4 + 1 = 5$

$(1-4)W_8^0 = -3$

$(2-3)W_8^1 = -W_8^1$

$(3-2)W_8^2 = W_8^2$

$(4-1)W_8^3 = 3W_8^3$

II Stage o/p

10

10

0

0

$(-3 + W_8^2)$

$(-W_8^1 + 3W_8^3)$

$(-3 - W_8^2)W_4^0$

$(-W_8^1 - 3W_8^3)W_4^1$

III Stage o/p

$10 + 10 = 20$

$10 - 10 = 0$

$0 + 0 = 0$

$0 - 0 = 0$

$(-3 + W_8^2) + (-W_8^1 + 3W_8^3) = -5.82 - j2.414$

$(-3 + W_8^2) - (-W_8^1 + 3W_8^3) = -0.171 + j0.414$

$(-3 - W_8^2)W_4^0 + (-W_8^1 - 3W_8^3)W_4^1 = -0.171 - j0.414$

$(-3 - W_8^2)W_4^0 - (-W_8^1 - 3W_8^3)W_4^1 = -5.82 + j2.414$

$\therefore X(K) = [20, -5.82 - j2.414, 0, -0.171 - j0.414, 0, -0.171 + j0.414, 0, -5.82 + j2.414]$

Inverse Decimation in-time (IDIT-FFT)

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \quad \text{——— (1)}$$

$$= \frac{1}{N}\left[\sum_{k=0}^{N/2-1} X(k)\, W_N^{-kn} + \sum_{k=N/2}^{N-1} X(k)\, W_N^{-kn}\right]$$

changing limits

$l = k - N/2 \qquad k = N/2 \quad l = 0$

$k = l + N/2 \qquad k = N-1 \quad l = \frac{N}{2} - 1$

$$= \frac{1}{N}\left[\sum_{k=0}^{N/2-1} X(k)\, W_N^{-kn} + \sum_{l=0}^{N/2-1} X(l+N/2)\, W_N^{-(N/2+l)n}\right]$$

$$= \frac{1}{N}\left[\sum_{k=0}^{N/2-1} X(k)\, W_N^{-kn} + \sum_{k=0}^{N/2-1} X(k+N/2)\, W_N^{-(k+N/2)n}\right]$$

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N/2-1}\left(X(k) + (-1)^n X(k+N/2)\right) W_N^{-kn}\right] \quad \text{——— (2)}$$

for $n = 2r$

$$x(2r) = \frac{1}{2}\left\{\frac{1}{N/2}\left\{\sum_{k=0}^{N/2-1}\underbrace{\left[X(k) + X(k+N/2)\right]}\, W_{N/2}^{-kr}\right\}\right\} \quad \text{——— (3)}$$

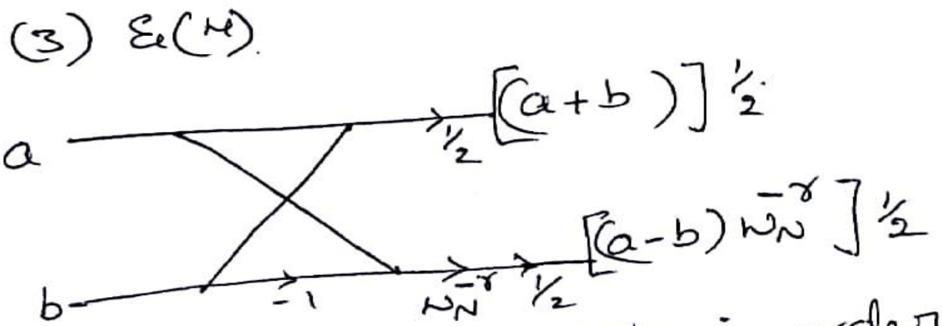
for $n = 2r+1$

$$x(2r+1) = \frac{1}{2}\left\{\frac{1}{N/2}\left\{\sum_{k=0}^{N/2-1}\left[X(k) - X(k+N/2)\right] W_N^{-k(2r+1)}\right\}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{N/2}\left\{\sum_{k=0}^{N/2-1}\left\{X(k) - X(k+N/2)\right] W_{N/2}^{-kr}\, W_N^{-k}\right\}\right]$$

$$= \frac{1}{2}\left[1/N/2\left\{\sum_{k=0}^{N/2-1}\underbrace{\left[\left[X(k) - X(k+N/2)\right] W_N^{-k}\right]}\, W_{N/2}^{-kr}\right\}\right] \quad \text{——— (4)}$$

from (3) & (4)

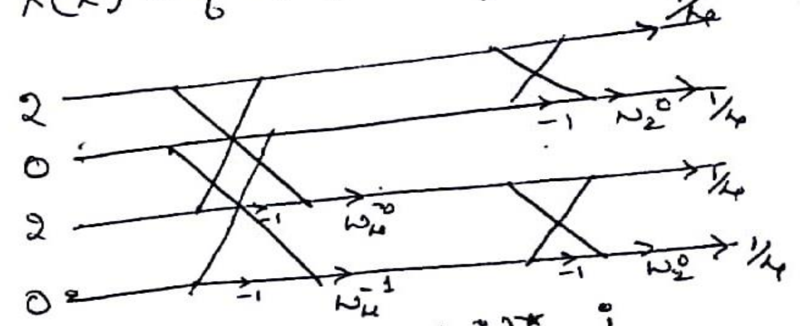


I/P in order
O/P in bit-reversed order

Examples

1] find inverse DFT of the following $X(K)$ using inverse DIT - FFT method.

(a) $X(K) = \{2, 0, 2, 0\}$



$W_4^0 = 1, \quad W_4^{-1} = (W_4^1)^* = (-j)^* = j$

I Stage O/P

$2 + 2 = 4$

$0 + 0 = 0$

$(2-2)1 = 0$

$(0-0)j = 0$

II Stage O/P

$4 + 0 = 4 \times 1/4 = 1 \quad x(0)$

$(4-0)1 = 4 \times 1/4 = 1 \quad x(2)$

$0 = 0 = 0 \quad x(1)$

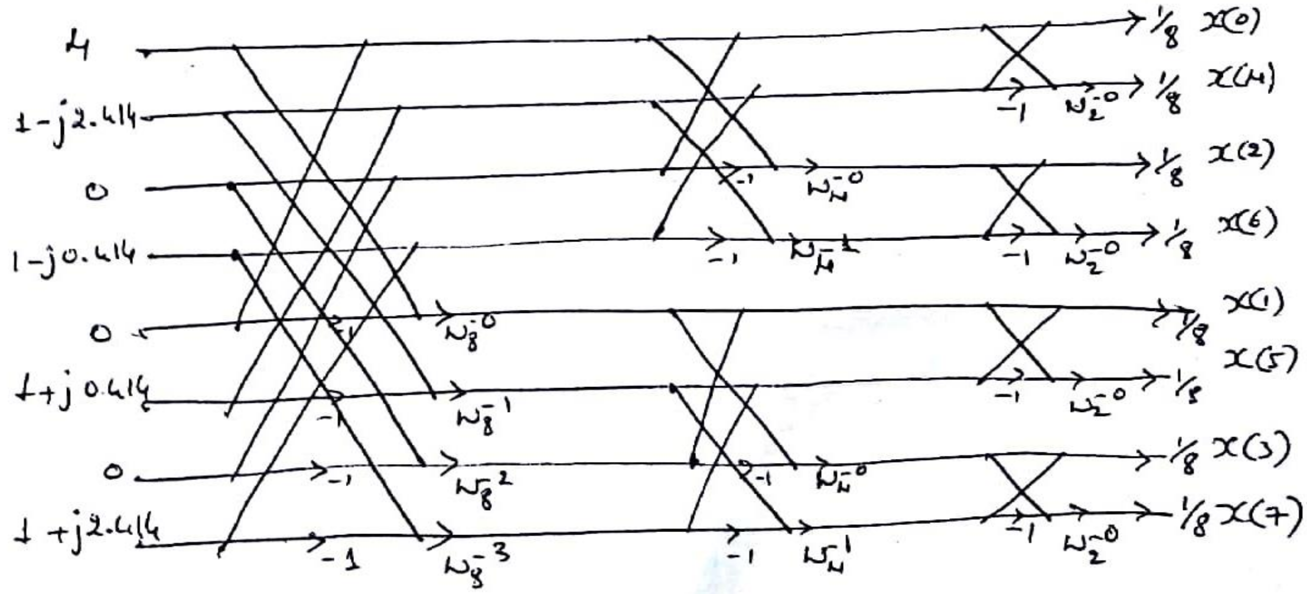
$0 = 0 = 0. \quad x(3).$

$x(n) = (1, 0, 1, 0)$

(b) $X(K) = [4, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414]$

$W_8^{-0} = W_4^{-0} = W_2^{-0} = 1 \qquad W_8^{-2} = W_4^{-1} = +j$

$W_8^{-1} = (W_8^1)^* = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}} \qquad W_8^{-3} = (W_8^3)^* = -\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$



I stage o/p

$4+0 = 4$

$1-2.414j + 1 + 0.414j = 2-2j$

$0+0 = 0$

$1 - j0.414 + 1 + j2.414 = 2+2j$

$(4-0)W_8^0 = 4$

$[(1-j2.414)-(1+j0.414)]W_8^{-1} = (-2.828j)W_8^{-1}$

$(0-0)W_8^{-2} = 0$

$[(1-j0.414)-(1+j2.414)]W_8^{-3}$
$= (-2.828j)W_8^{-3}$

II stage o/p

$4+0 = 4$

$(2-2j)+(2+2j) = 4$

$(4-0)1 = 4$

$[(2-2j)-(2+2j)]\, j = 4$

$4+0 = 4$

$(-2.828j)W_8^{-1} - (2.828j)W_8^{-3} = 4$

$(4-0)1 = 4$

$[(-2.828j)W_8^{-1} + 2.828j\, W_8^{-3}]\, j$
$= 4$

III stage o/p

$4+4 = 8 \times 1/8 = 1 = x(0)$

$(4-4)1 = 0 \times 1/8 = 0 = x(4)$

$4+4 = 8 \times 1/8 = 1 = x(2)$

$(4-4)1 = 0 \times 1/8 = 0 = x(6)$

$4+4 = 8 \times 1/8 = 1 = x(1)$

$(4-4)1 = 0 \times 1/8 = 0 = x(5)$

$4+4 = 8 \times 1/8 = 1 = x(3)$

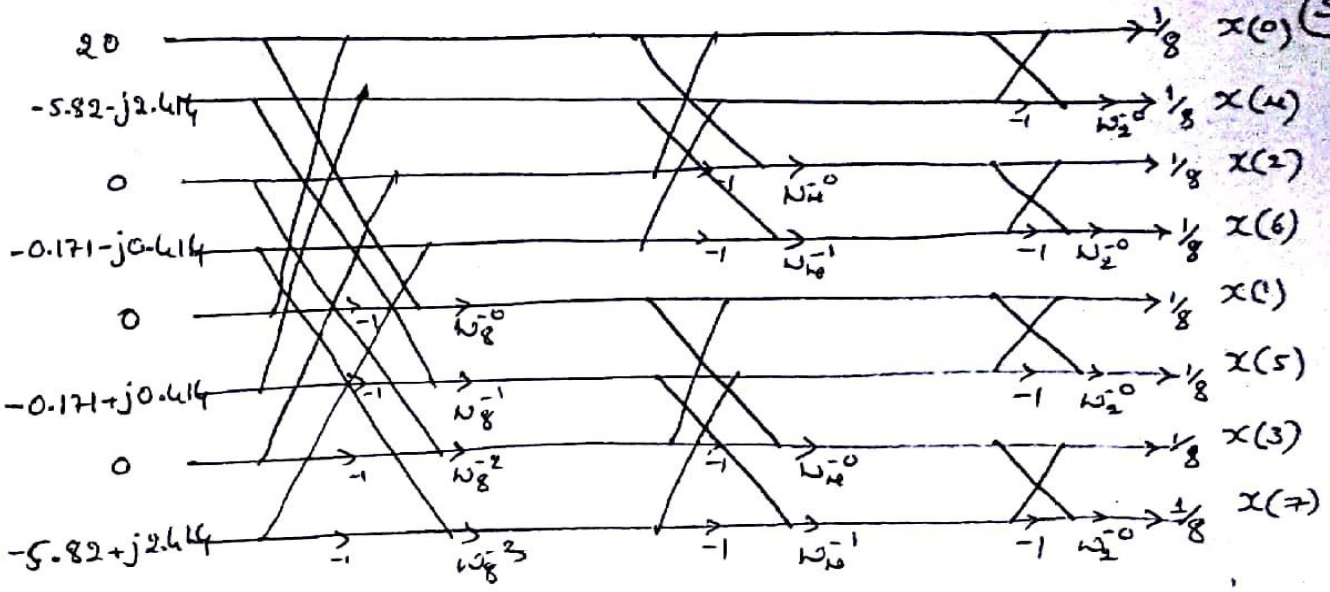
$(4-4)1 = 0 \times 1/8 = 0 = x(7)$

$x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$

(c) $X(K) = [20, -5.82 - j2.414, 0, -0.171 - j0.414, 0, -0.171 + j0.414, 0, -5.82 + j2.414]$

$W_8^{-0} = W_N^{-0} = W_2^{-0} = 1$ $\quad W_8^{-2} = +j$

$W_8^{-1} = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$ $\quad W_8^{-3} = \frac{-1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$



I Stage o/p

1) $20+0 = 20$

2) $(-5.82-j2.414)+(-0.171+j0.414) = -6-2j$

3) $0+0 = 0$

4) $(-0.171-j0.414)+(-5.82+j2.414) = -6+2j$

5) $(20-0)W_8^{-0} = 20$

6) $[(-5.82-j2.414)-(-0.171+j0.414)]W_8^{-1} = -2-6j$

7) $(0-0)W_8^{-2} = 0$

8) $[(-0.171-j0.414)-(-5.82-j2.414)]W_8^{-3} = -2+6j$

II Stage o/p

1) $20+0 = 20$

2) $(-6-2j)+(-6+2j) = -12$

3) $(20-0)W_4^{-0} = 20$

4) $[(-6-2j)-(-6+2j)]W_4^{-1} = 4$

5) $(20+0) = 20$

6) $(-2-6j)+(-2+6j) = -4$

7) $(20-0)W_4^{-0} = 20$

8) $[(-2-6j)-(-2+6j)]W_4^{-1} = 12$

III Stage o/p

1) $20+(-12) = 8 \times 1/8 = 1 = x(0)$

2) $(20-(-12))1 = 32 \times 1/8 = 4 = x(4)$

3) $(20+4) = 24 \times 1/8 = 3 = x(2)$

4) $(20-4)1 = 16 \times 1/8 = 2 = x(6)$

5) $20+(-4) = 16 \times 1/8 = 2 = x(1)$

6) $(20-(-4))1 = 24 \times 1/8 = 3 = x(5)$

7) $20+12 = 32 \times 1/8 = 4 = x(3)$

8) $(20-12)1 = 8 \times 1/8 = 1 = x(7)$

$\therefore x(n) = (1, 2, 3, 4, 4, 3, 2, 1)$

Inverse Decimation in frequency (IDIF-FFT)

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \qquad n = 0 \cdots N-1 \quad \text{---} \textcircled{1}$$

$$= \frac{1}{N}\left[\sum_{l=0}^{N/2-1} X(2l)\, W_N^{-2ln} + \sum_{l=0}^{N/2-1} X(2l+1)\, W_N^{-(2l+1)n}\right]$$

$$= \frac{1}{N}\left[\sum_{l=0}^{N/2-1} X(2l)\, W_{N/2}^{-ln} + \sum_{l=0}^{N/2-1} X(2l+1)\, W_{N/2}^{-ln}\, W_N^{-n}\right]$$

$$= \frac{1}{2}\left\{\frac{1}{N/2}\left[\sum_{l=0}^{N/2-1} F_1(l)\, W_{N/2}^{-ln} + \sum_{l=0}^{N/2-1} F_2(l)\, W_{N/2}^{-ln}\, W_N^{-n}\right]\right\}$$

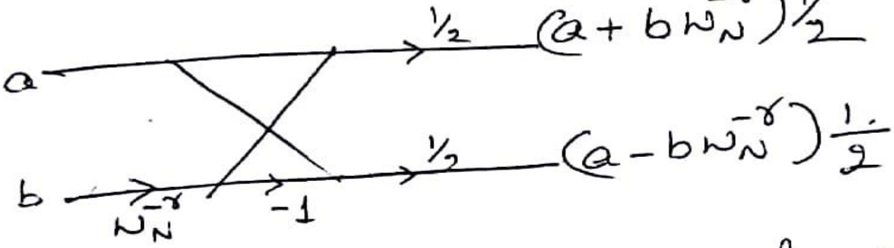
$$x(n) = \frac{1}{2}\left[f_1(n) + W_N^{-n} f_2(n)\right] \qquad n = 0 \cdots N/2 - 1$$

$$x(n) = \frac{1}{2}\left[\left(f_1(n) + W_N^{-n} f_2(n)\right)\right] \quad \text{---} \textcircled{2}$$

$$x(n + N/2) = \frac{1}{2}\left[\left(f_1(n+N/2) + W_N^{-(n+N/2)} f_2(n+N/2)\right)\right]$$

$$= \frac{1}{2}\left[\left(f_1(n) - W_N^{-n} f_2(n)\right)\right] \quad \text{---} \textcircled{3}$$

from ② & ③

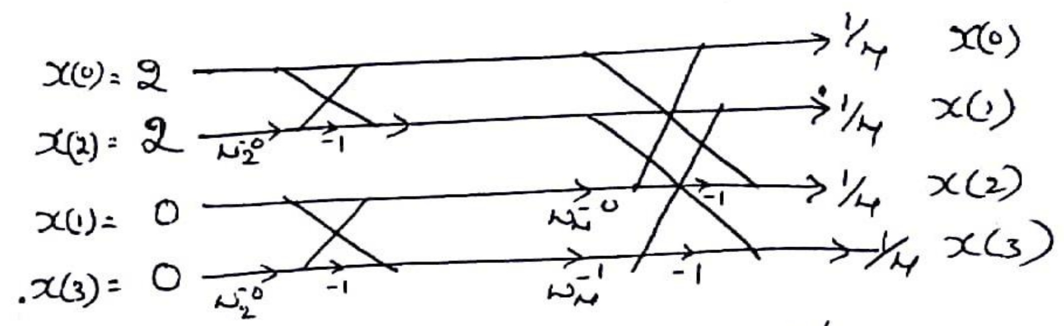


* I/p is given in bit-reversed order and o/p will be in order.

Example

find $x(n)$ for the following $X(k)$ using inverse decimation-in-frequency FFT method.

(a) $X(k) = [2, 0, 2, 0]$



I Stage o/p

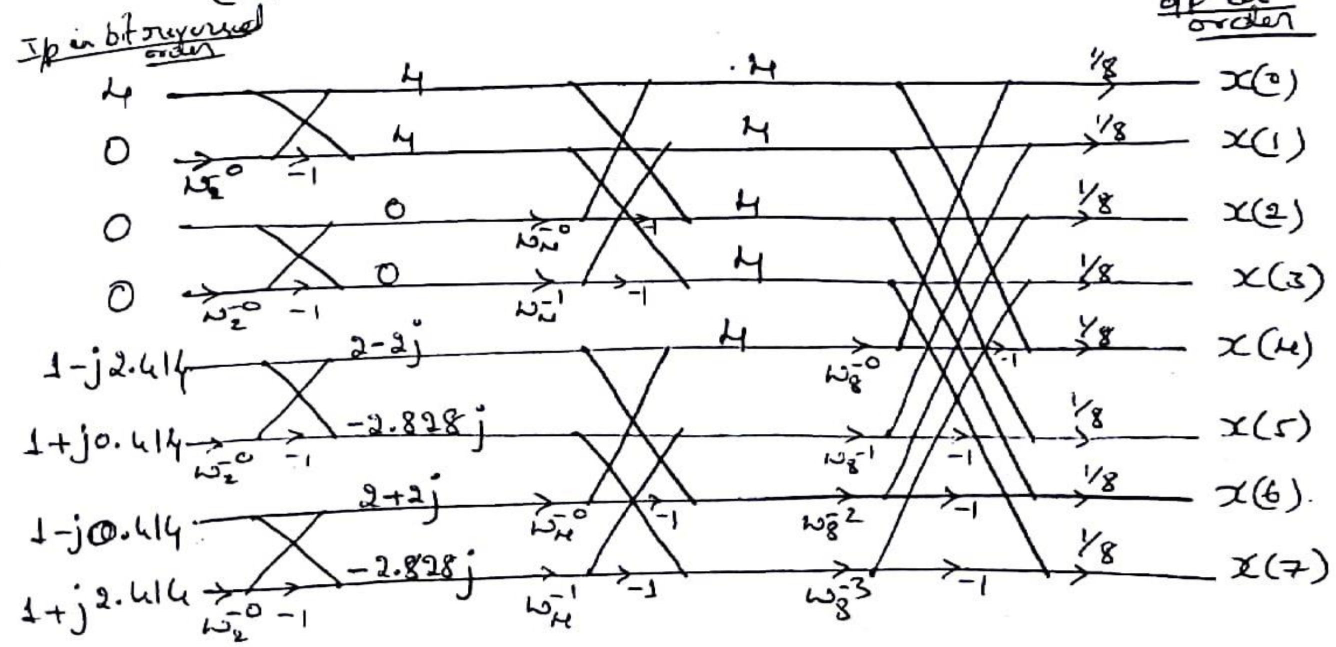
① $2 + 2(1) = 4$
② $2 - 2(1) = 0$
③ $0 = 0$
④ $0 = 0$

II Stage o/p

① $4 + 0(1) = 4 \times 1/4 = 1 \quad x(0)$
② $(0 + 0(j)) = 0 \times 1/4 = 0 \quad x(1)$
③ $4 - 0(1) = 4 \times 1/4 = 1 \quad x(2)$
④ $0 - 0(j) = 0 \times 1/4 = 0 \quad x(3)$

$\therefore x(n) = [1, 0, 1, 0]$

(b) $X(k) = [4,\ 1-j2.414,\ 0,\ 1-j0.414,\ 0,\ 1+j0.414,\ 0,\ 1+j2.414]$



I Stage o/p

① $4 + (0)(1)$
② $4 - (0)(1)$
③ $0 + 0$
④ $0 - 0$
⑤ $(1 - j2.414) + (1 + j0.414)(1)$
$= 2 - 2j$

⑥ $(1 - j2.414) - (1 + j0.414)(1)$
$= -2.828j$

⑦ $(1 - j0.414) + (1 + j2.414)(1)$
$= 2 + 2j$

⑧ $(1 - j0.414) - (1 + j2.414)(1)$
$= -2.828j$

II Stage o/p

① $4 + 0(1) = 4$

② $4 + 0(j) = 4$

③ $4 - 0(1) = 4$

④ $4 - 0(j) = 4$

⑤ $(2-2j) + (2+2j)(1) = 4$

⑥ $(-2.828j) + (-2.828j)(j) = 2.828 - 2.828j$

⑦ $(2-2j) - (2+2j)(1) = -4j$

⑧ $(-2.828j) - (-2.828j)j = -2.828 - 2.828j$

III Stage o/p

① $4 + 4 = 8 \times 1/8 = 1$

② $4 + (2.828 - 2.828j)W_8^{-1} = 8 \times 1/8 = 1$

③ $4 + (-4j)W_8^{-2} = 8 \times 1/8 = 1$

④ $4 + (-2.828 - 2.828j)W_8^{-3} = 8 \times 1/8 = 1$

⑤ $4 - 4(1) = 0$

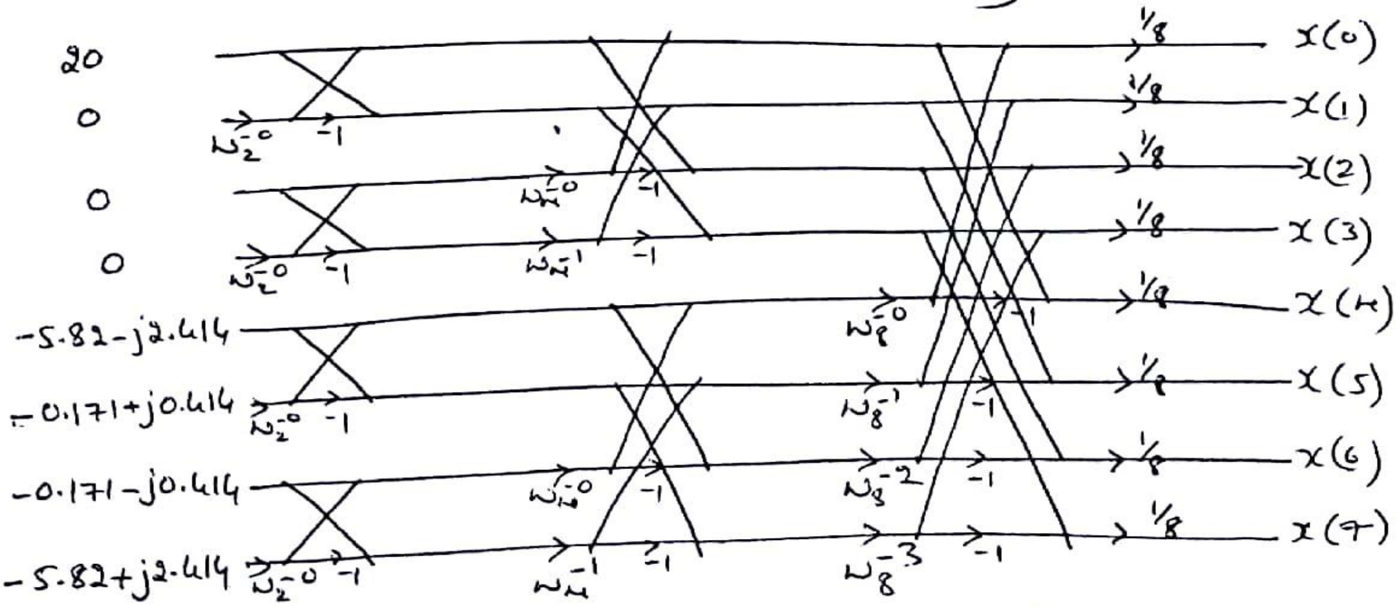
⑥ $4 - (2.828 - 2.828j)W_8^{-1} = 0$

⑦ $4 - (-4j)W_8^{-2} = 4 + 4j(j) = 0$

⑧ $4 - (-2.828 - 2.828j)W_8^{-3} = 0$

$\therefore x(n) = [1, 1, 1, 1, 0, 0, 0, 0]$

ⓒ $X(K) = [20, -5.82 - j2.414, 0, -0.171 - j0.414, 0, -0.171 + j0.414, 0, -5.82 + j2.414]$



$x(n) = (1, 2, 3, 4, 4, 3, 2, 1)$

I Stage o/p

(1) $20 + 0(1)$

(2) $20 - 0(1)$

(3) 0

(4) 0

(5) $(-5.82 - j2.414)$
$+ (-0.171 + j0.414)(1)$
$= -6 - 2j$

(6) $(-5.82 - j2.414)$
$-(-0.171 + j0.414)(1)$
$= -5.65 - 2.83j$

(7) $(-0.171 - j0.414)$
$+(-5.82 + j2.414)(1)$
$= -6 + 2j$

(8) $(-0.171 - j0.414)$
$-(-5.82 + j2.414)(1)$
$= +5.65 - 2.83j$

II Stage o/p

(1) $20 + 0(1) = 20$

(2) $20 + 0(j) = 20$

(3) $20 - 0(1) = 20$

(4) $20 - 0(j) = 20$

(5) $(-6 - 2j) + (-6 + 2j)(1)$
$= -12$

(6) $(-5.65 - j2.83) + (5.65 - 2.83j)j$
$= -2.82 + 2.82j$

(7) $(-6 - 2j) - (-6 + 2j)(1)$
$= -4j$

(8) $(-5.65 - j2.83) - (5.65 - 2.83j)j$
$= -5.65 - j2.83 - 5.65j + 2.83j^2$
$= -8.48 - 8.48j$

III Stage o/p

(1) $20 + (-12)(1) = 8 \times 1/8 = 1 = x(0)$

(2) $20 + (-2.82 + 2.82j)W_8^{-1} = 16 \times 1/8 = 2 = x(1)$

(3) $20 + (-4j)W_8^{-2} = 24 \times 1/8 = 3 = x(2)$

(4) $20 + (-8.48 - 8.48j)W_8^{-3} = 32 \times 1/8 = 4 = x(3)$

(5) $20 - (-12)(1) = 32 \times 1/8 = 4 = x(4)$

(6) $20 - (-2.82 + 2.82j)W_8^{-1} = 24 \times 1/8 = 3 = x(5)$

(7) $20 - (-4j)W_8^{-2} = 16 \times 1/8 = 2 = x(6)$

(8) $20 - (-8.48 - 8.48j)W_8^{-3} = 8 \times 1/8 = 1 = x(7)$

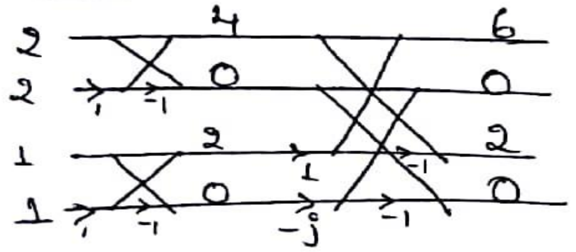
Aplications

① $x_1(n) = (2,1,2,1)$ & $x_2(n) = (1,2,3,4)$
Determine $y(n) = x_1(n) \circledN x_2(n)$ using radix 2 DIT-FFT and IDIT-FFT method.

Solution :-

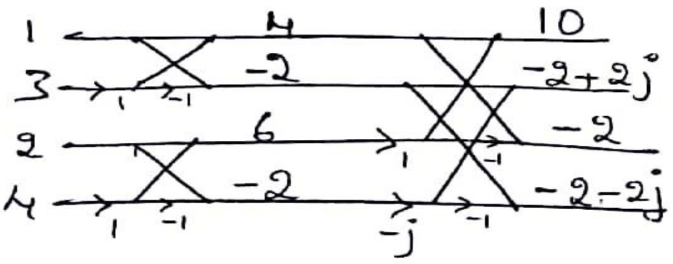
① find $X_1(k)$.
② find $X_2(k)$
③ $Y(k) = X_1(k) \cdot X_2(k)$
④ $y(n) = IDFT\{Y(k)\}$

Step 1 : $x_1(n) = (2,1,2,1)$



$\therefore X_1(k) = [6, 0, 2, 0]$

Step 2 : $x_2(n) = (1, 2, 3, 4)$



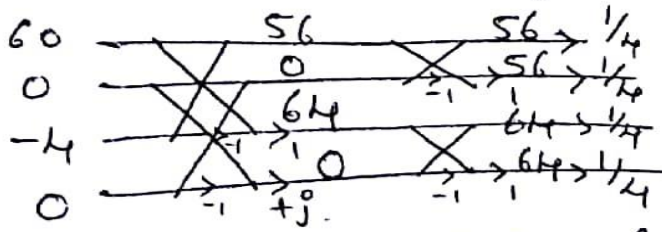
$X_2(k) = [10, -2+2j, -2, -2-2j]$

Step 3 : $Y(k) = X_1(k) \cdot X_2(k)$

$= [6, 0, 2, 0]\,[10, -2+2j, -2, -2-2j]$

$= [60, 0, -4, 0]$

Step 4 : $IDFT\{Y(k)\}$



$= 14 = y(0)$
$= 14 = y(2)$
$= 16 = y(1)$
$= 16 = y(3)$

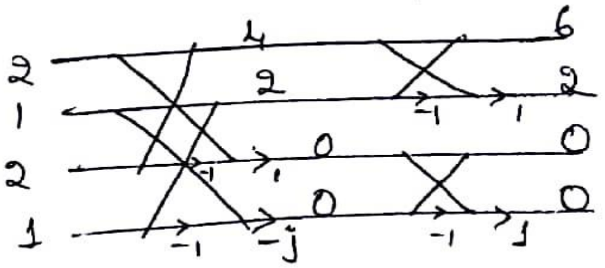
$\therefore y(n) = (14, 16, 14, 16)$ ///

② Determine $y(n) = x_1(n) \, Ⓝ \, x_2(n)$ using radix-2 DIF-FFT and IDIF-FFT methods. (4ʘ)

$x_1(n) = (2, 1, 2, 1)$ & $x_2(n) = (1, 2, 3, 4)$

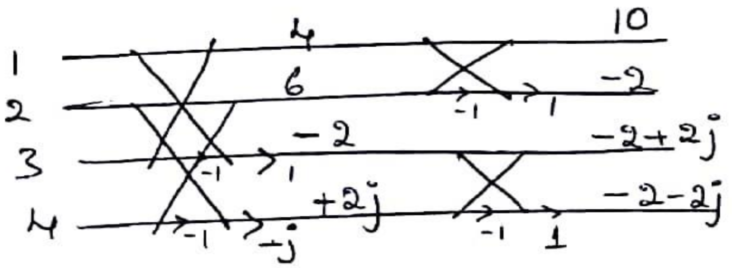
Solution

Step 1 : $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$



$X_1(k) = [6, 0, 2, 0]$

Step 2 : $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$



$X_2(k) = [10, -2+2j, -2, -2-2j]$

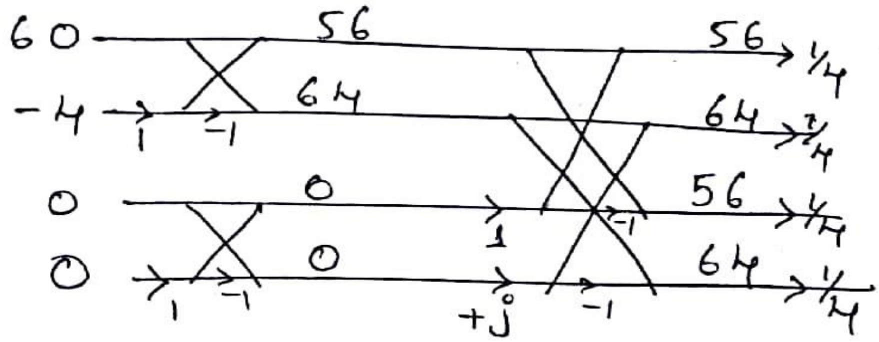
Step 3 :

$Y(k) = X_1(k) \, X_2(k)$

$= [6, 0, 2, 0] \, [10, -2+2j, -2, -2-2j]$

$= [60, 0, -4, 0]$

Step 4 : $IDFT\{Y(k)\} = y(n)$



$= 14 = y(0)$

$= 16 = y(1)$

$= 14 = y(2)$

$= 16 = y(3)$

$\therefore y(n) = (14, 16, 14, 16)$

Fast fourier Transform (FFT)

Radix-2

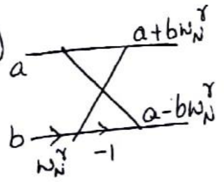
FFT → DIT, DIF

IFFT → IDIT, IDIF

DIT

① I/p in Bit-Reversed order

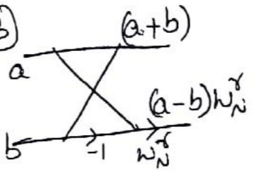
② o/p in order

③ 

DIF

① I/p in order

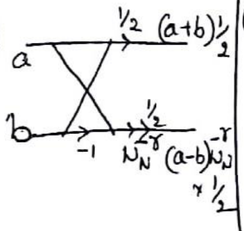
② o/p in Bit-Reversed order

③ 

IDIT

① I/p in order

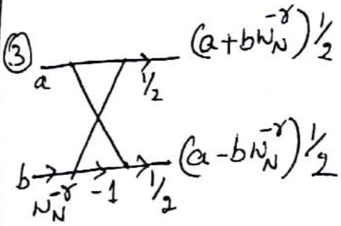
② o/p in Bit-reversed order

③ 

IDIF

① I/p in Bit-reversed order

② o/p in order

③ 

for N-pt DFT using FFT :- Complex Multiplication: $\frac{N}{2}\log_2 N$

Complex Additions: $N\log_2 N$

$N/2$ # of Butterfly diagrams per stage

$\log_2 N$ # of stages

1. Suppose that we have 1025 point data sequence (1 more than $N = 2^{10} = 1024$). Instead of discarding the last value, we append zeros to the sequence to make it a length of $N = 2^{11}$, so that we can use radix-2 FFT algorithm

a) How many multiplications and additions are required* using radix-2 FFT algorithm for computing DFT?

b) How many multiplications and additions would be required to compute a 1025 point DFT directly?

Solution:

a) $N = 2^{11}$ $\Rightarrow$ No of complex multiplications $= \frac{N}{2} \log_2 N = 11264$

No of complex additions $= N \log_2 N = 22,528$

b) $N = 1025$ $\Rightarrow$ No of complex multiplication $= N \times N = 1050625$

—— " —— additions $= N \times (N-1) = 1049600$

2. Determine the number of stages in radix-2 algorithm for

i) $N = 8$

ii) $N = 512$

iii) $N = 1024$

Solution:

i) No of stages $(N=8) = \log_2 N = \log_2 8 = 3$

ii) No of stages $(N=512) = \log_2 512 = 9$

iii) No of stages $(N=1024) = \log_2 1024 = 10$.

3. Prove that the number of multiplications for computing N-pt DFT using FFT with $N = 2^r$ is $\frac{N}{2}\log_2 N$.

<u>Solution</u>:

WKT $$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{kn}}_{N/2 \text{ pt DFT}} + W_N^k \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_{N/2}^{kn}}_{N/2 \text{ pt DFT}}$$

Let F(N) be the number of complex multiplications required to compute an N-point DFT.

$$F(N) = F\left(\frac{N}{2}\right) + F\left(\frac{N}{2}\right) + \frac{N}{2} = \frac{N}{2} + 2F\left(\frac{N}{2}\right)$$

$$= \frac{N}{2} + 2\left[\frac{\frac{N}{2}}{2} + 2F\left(\frac{N}{2^2}\right)\right]$$

$$= \frac{N}{2} + \frac{N}{2} + 2^2 F\left(\frac{N}{2^2}\right)$$

$$= \frac{N}{2} + \frac{N}{2} + 2^2\left[\frac{N}{2^3} + 2F\left(\frac{N}{2^3}\right)\right]$$

$$= \frac{N}{2} + \frac{N}{2} + \frac{N}{2} + 2^3 F\left(\frac{N}{2^3}\right)$$

$$= \underbrace{\left(\frac{N}{2} + \frac{N}{2} + \cdots + \frac{N}{2}\right)}_{r \text{ times}} + 2^r F\left(\frac{N}{2^r}\right)$$

$$= \frac{N}{2} r + 2^r F\left(\frac{N}{N}\right) = \frac{N}{2} r + 2^r \underbrace{F(1)}_{0}$$

$$F(N) = \frac{N}{2} r = \frac{N}{2}\log_2 N.$$

4 Let v(n) be a 2N length sequence and V(K) is its 2N point DFT. Given that $g(n) = v(2n)$ & $h(n) = v(2n+1)$ and $x(n) = g(n) + jh(n)$ Find V(K) in terms of G(K) & H(K) by performing only one N-point DFT.

<u>Solution</u>: WKT $$G(k) = \frac{X(k) + X^*((-k))_N}{2} \quad \left(\because g(n) = \frac{x(n) + x^*(n)}{2}\right)$$

$$H(k) = \frac{X(k) - X^*((-k))_N}{2j} \quad \left(\because h(n) = \frac{x(n) - x^*(n)}{2j}\right)$$

$$V(K) = \sum_{n=0}^{2N-1} v(n)\, W_{2N}^{kn} \qquad n = 0, 1, \ldots 2N-1$$

$$= \sum_{n=0}^{N-1} v(2n)\, W_{2N}^{2kn} + \sum_{n=0}^{N-1} v(2n+1)\, W_{2N}^{(2n+1)k}$$

$$= \sum_{n=0}^{N-1} g(n)\, W_N^{kn} + W_{2N}^{k} \sum_{n=0}^{N-1} h(n)\, W_N^{kn} \qquad \left(\because W_{2N}^{2k} = W_N^{2k/2} = W_N^{k} \right)$$

$$\boxed{V(K) = G(K) + W_{2N}^{k} H(K)}$$

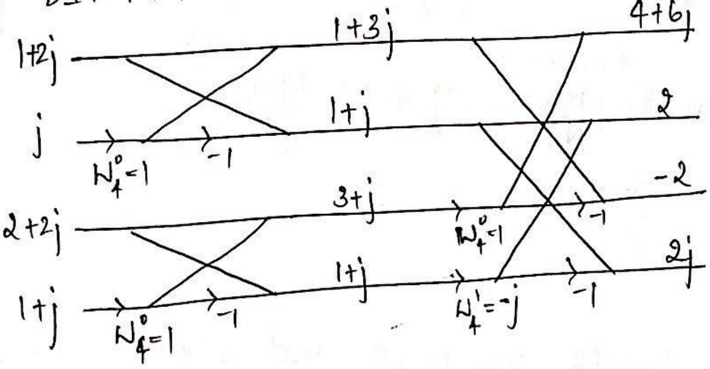
where $G(K)$ and $H(K)$ are periodic with N point

5. Find $V(K)$ given $v(n) = \{1, 2, 2, 2, 0, 1, 1, 1\}$. Find 8-pt DFT using single 4 point DFT.

Solution: Let $g(n) = \{1, 2, 0, 1\}$ $\quad h(n) = \{2, 2, 1, 1\}$.

$x(n) = g(n) + j\,h(n) = \{1+2j, 2+2j, j, 1+j\}$

Using DIT FFT



$X(K) = \{4+6j, 2, -2, 2j\}$

$$G(k) = \frac{1}{2}\left[X(k) + X^*(-k) \right]$$

$$= \frac{1}{2}\left[\{4+j6, +2, -2, 2j\} + \{4-j6, -2j, -2, +2\} \right]$$

$$= [4, 1-j, -2, 1+j]$$

$$H(k) = \frac{1}{2j}\left[X(k) - X^*(-k)\right]$$

$$= \frac{1}{2j}\left[12j\ ,\ 2+2j\ ,\ 0\ ,\ -2+2j\right]$$

$$= \left[6,\ 1-j\ ,\ 0\ ,\ 1+j\right]$$

$$\Rightarrow V(k) = G((k))_4 + W_8^{k} H((k))_4$$

$$V(0) = G(0) + W_8^0 H(0) = 10$$

$$V(1) = 1-j + W_8^1(1-j) = 1-(1+\sqrt{2})j = 1-j\,2.414$$

$$V(2) = -2 + W_8^2 \times 0 = -2$$

$$V(3) = 1+j + W_8^3(1+j) = 1+(1-\sqrt{2})j = 1-j\,0.414$$

$$V(4) = G(4) + W_8^4 H(4) = G(0) + (-1)H(0) = 4-6 = -2$$

$$V(5) = V^*(3) = 1+j\,0.414$$

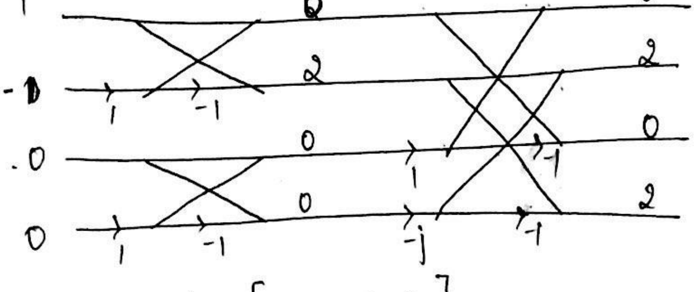
$$V(6) = V^*(2) = -2$$

$$V(7) = V^*(1) = 1+j\,2.414$$

$$\Rightarrow V(k) = \{10,\ 1-j2.414,\ -2,\ 1-j0.414,\ -2,\ 1+j2.414\}.$$

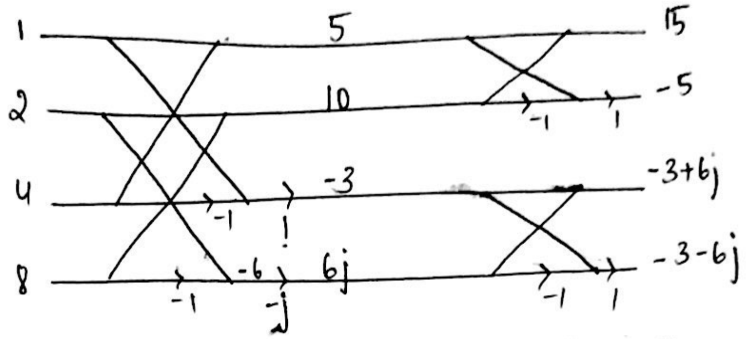
6. Find DFT of the following sequence $x(n)$ using DIT FFT $N=4$

a) $x(n) = \cos\left(\frac{2\pi n}{4}\right) = [1, 0, -1, 0]$



$$X(k) = [0, 2, 0, 2]$$

b) $x(n) = 2^n = [1, 2, 4, 8]$ DIF



$X(k) = [15, -3+6j, -5, -3-6j]$

7. Using FFT algorithm find the output of filter whose impulse response $h(n) = \{1, 2, 3, 2, 1\}$ & input $x(n) = \{1, 1, 1, 1\}$.

Solution: WKT $y(n) = x(n) * h(n)$

$N_y = N_x + N_y - 1 = 8$.

$\therefore N = 8$.

$x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$.

$X(k) = \{4, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$.

-from fig. 1

$h(n) = \{1, 2, 3, 2, 1, 0, 0, 0\}$

$H(k) = \{9, -j5.828, -1, j0.171, 1, -j0.171, -1, j5.828\}$.

- from fig 2.

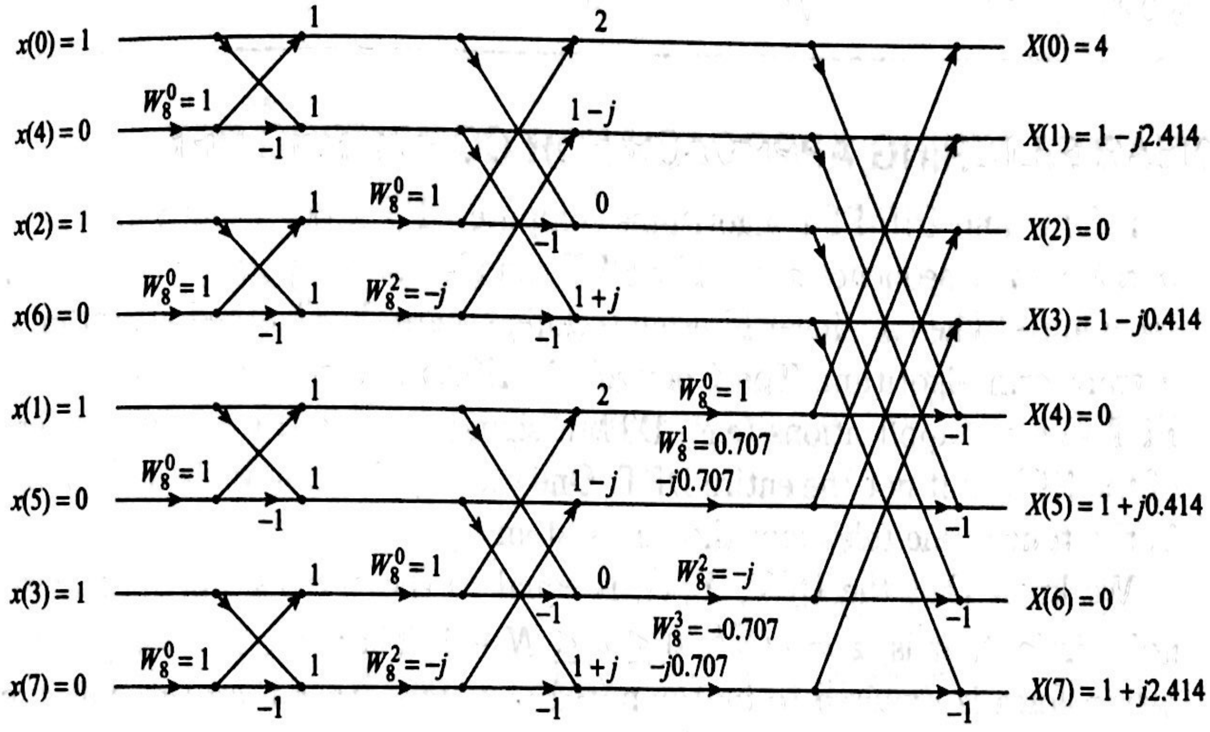
$Y(k) = H(k)X(k)$

$= \{36, -14.07, -j5.828, 0, 0.071+j0.171, 0, 0.071-j0.171,$

$0, -14.071-j5.828\}$.

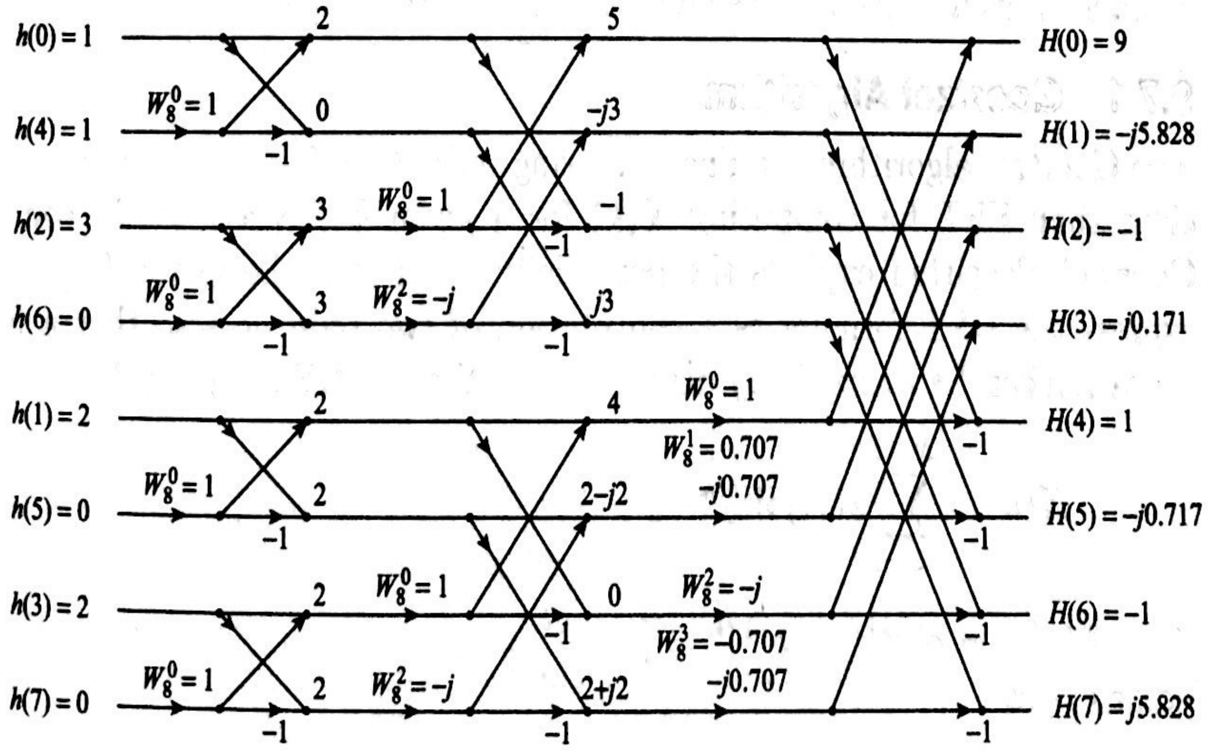
$y(n) = \{1, 3, 6, 8, 8, 6, 3, 1\}$.

- from fig 3.



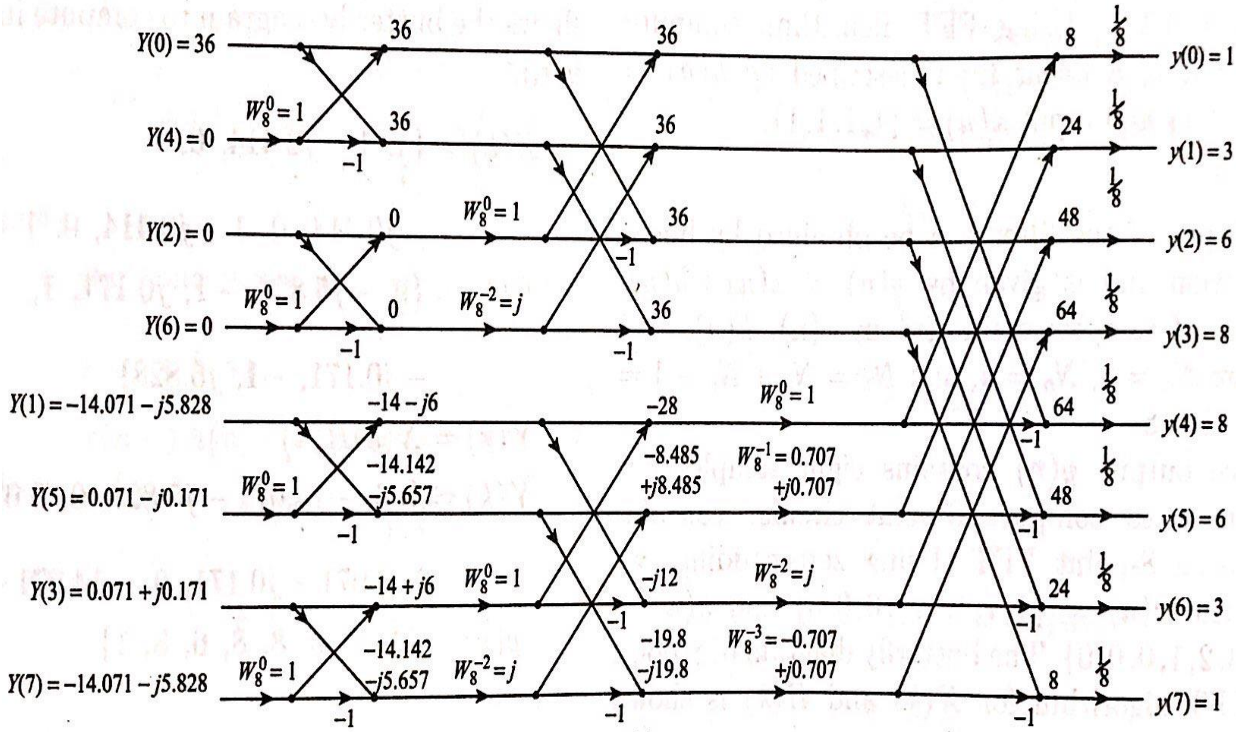
x(0) = 1
x(4) = 0
x(2) = 1
x(6) = 0
x(1) = 1
x(5) = 0
x(3) = 1
x(7) = 0
X(0) = 4
X(1) = 1 – j2.414
X(2) = 0
X(3) = 1 – j0.414
X(4) = 0
X(5) = 1 + j0.414
X(6) = 0
X(7) = 1 + j2.414

(a)



h(0) = 1
h(4) = 1
h(2) = 3
h(6) = 0
h(1) = 2
h(5) = 0
h(3) = 2
h(7) = 0
H(0) = 9
H(1) = –j5.828
H(2) = –1
H(3) = j0.171
H(4) = 1
H(5) = –j0.717
H(6) = –1
H(7) = j5.828

(b)



Y(0) = 36
Y(4) = 0
Y(2) = 0
Y(6) = 0
Y(1) = −14.071 − j5.828
Y(5) = 0.071 − j0.171
Y(3) = 0.071 + j0.171
Y(7) = −14.071 − j5.828
W80 = 1
W8−2 = j
W8−1 = 0.707 +j0.707
W8−3 = −0.707 +j0.707
−1
36
0
−14 − j6
−14.142 −j5.657
−14 + j6
−28
−8.485 +j8.485
−j12
−19.8 −j19.8
8
24
48
64
1/8
y(0) = 1
y(1) = 3
y(2) = 6
y(3) = 8
y(4) = 8
y(5) = 6
y(6) = 3
y(7) = 1