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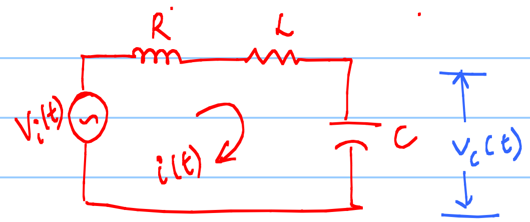
Mathematical Modelling

Series RLC



$$V_i(t) = R\,i(t) + L\frac{di(t)}{dt} + \underbrace{\frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau}_{V_c(t)}$$ Integro diff equation

Apply LT

$$V_i(s) = R\,I(s) + L s\,I(s) - L\,i(0^-) + \frac{1}{Cs}I(s)$$

Assume initial conditions $i(0^-) = 0$

O/p of interest $\rightarrow$ $i(t)$

$$\frac{I(s)}{V_i(s)} = \frac{1}{R + Ls + 1/Cs} = \frac{Cs}{LCs^2 + RCs + 1}$$

O/p $V_c(t)$ $\rightarrow$

$$V_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau$$

$$V_c(s) = \frac{1}{Cs}I(s) \Rightarrow I(s) = Cs\,V_c(s)$$

$$\boxed{\frac{\cancel{Cs}\,V_c(s)}{V_i(s)} = \frac{\cancel{Cs}}{LCs^2 + RCs + 1}}$$

O/p — Charge $q(t)$

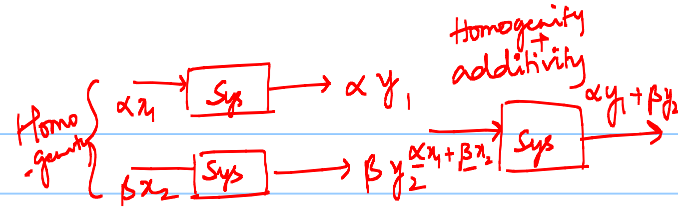
$$i(t) = \frac{dq(t)}{dt}$$

$$I(s) = s\,Q(s)$$

$$\boxed{\frac{Q(s)}{V_i(s)} = \frac{C}{LCs^2 + RCs + 1}}$$

- $\rightarrow$ Dynamics of the system is defined by denominator & hence stability is defined wrt to pole locations
- $\rightarrow$ Order of the system is the highest power of denominator polynomial ($n=2$)
- $\rightarrow$ The number of independent energy storing elements gives the order of the System
- $\rightarrow$ TF is a rational function, # of poles $\geq$ # of zeros (finite)

Idealizing Assumption:



1. Linear:

$$\alpha x_1(t) + \beta x_2(t) \rightarrow \alpha y_1(t) + \beta y_2(t) = y(t)$$

2. Time invariance

$$x(t) \rightarrow y(t) \qquad x(t-\tau) \rightarrow y(t-\tau)$$

Component value remains constant over time

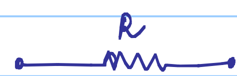
3. Lumped parameter: Value of the component is assumed to be concentrated at a point

The wavelength of operation of the component is greater than the length of the component

$f = 50\,Hz$

$\lambda = \frac{3\times10^8}{50} =$

Resistor:



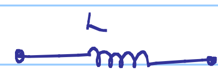
Ohm's Law

$$V_R(t) = R\, i(t) \qquad i(t) = V_R(t)/R$$

Units

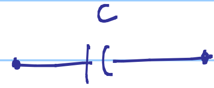
$V(t)$	V
$i(t)$	A
R	Ω
L	H
C	F

Inductor:



$$V_L(t) = L\frac{di(t)}{dt} \qquad i(t) = \frac{1}{L}\int_{-\infty}^{t} V_L(\tau)\, d\tau$$

Capacitor:



$$V_C(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau \qquad i(t) = C\frac{dV_C(t)}{dt}$$

Mechanical System

→ Translational

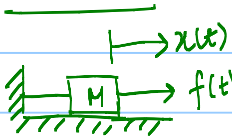
→ Rotational.

Translation: linear motion

$x(t)$ — displacement

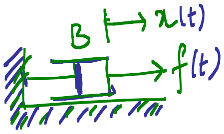
$v(t)$ — Velocity

① Mass (Kinetic Energy).



$$f(t) = M\frac{d^2x(t)}{dt^2}$$

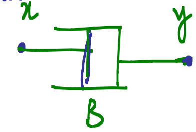
M – Mass (kg)

(2) Damper/ Dashpot 

$f(t) \propto \frac{dx(t)}{dt}$

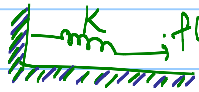
$$\boxed{f(t) = B\,\frac{dx(t)}{dt}}$$

Relative



$$f(t) = B\left(\frac{dx(t)}{dt} - \frac{dy(t)}{dt}\right)$$

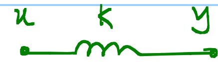
B - coefficient of viscous friction (N-s/m)

(3) Spring (Potential Energy) 

$f \propto x(t)$

$f(t) = K\,x(t)$

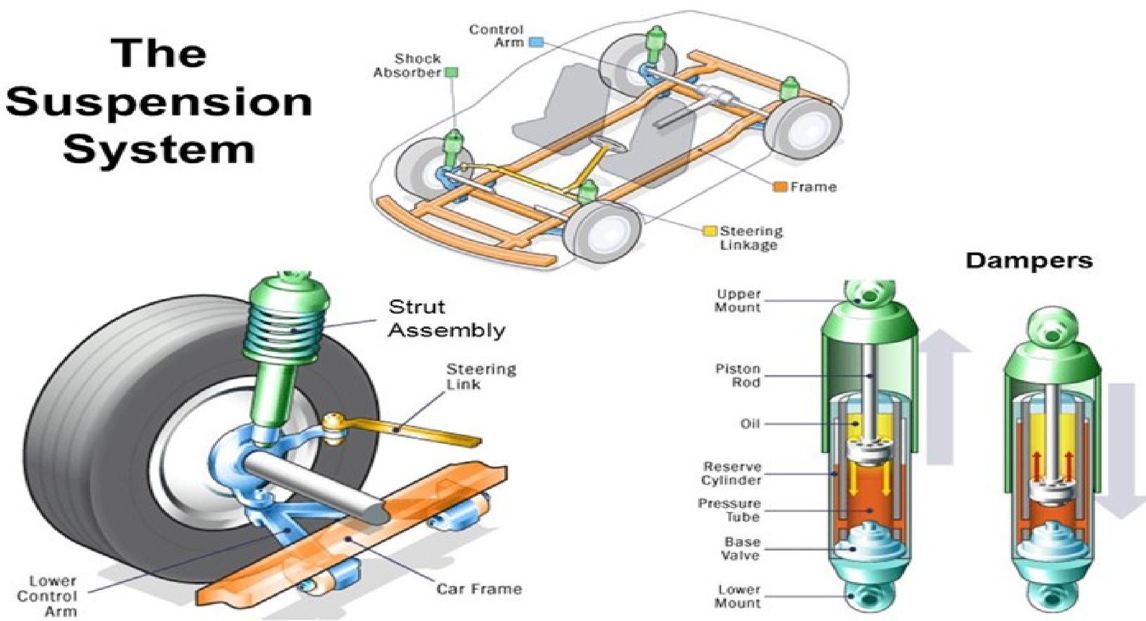
Hooks Law

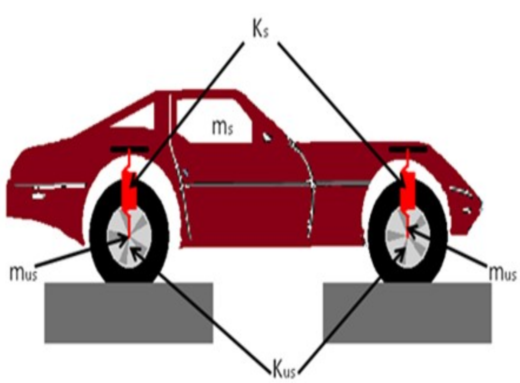


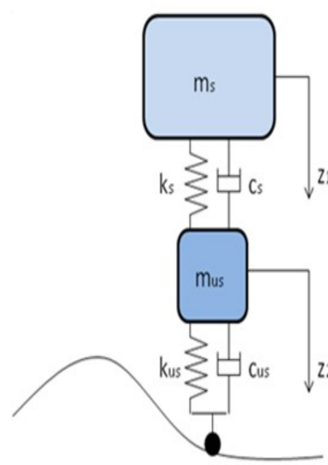
$f(t) = K(x(t) - y(t))$

K - Spring constant (N/m)

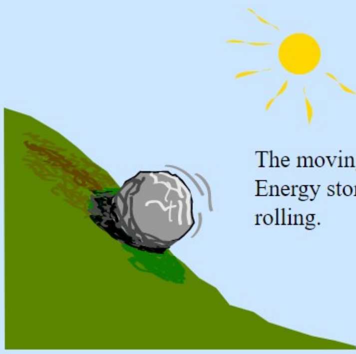
$f(t) = \mathrm{N}$ (newtons), $x(t) = \mathrm{m}$ (meters), $v(t) = \mathrm{m/s}$ (meters/second), $K = \mathrm{N/m}$ (newtons/meter), $B = \mathrm{N\text{-}s/m}$(newton-seconds/meter), $M = \mathrm{kg}$ (kilograms = newton-seconds^2/meter).







A **Kinetic store** - energy is stored in a moving **object**, for as long as the object moves.



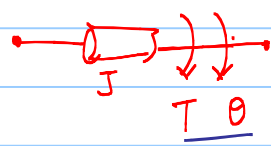
The moving boulder is a Kinetic Energy store as long as it keeps rolling.

An **Elastic Potential store** - energy is stored in a stretched elastic band or spring (**objects**), for as long as these objects are stretched (or compressed).

Rotational Systems: rotation about a axis

$\theta(t)$ — angular displacement
$\omega(t)$ — angular velocity

1. Moment of Inertia: Kinetic energy

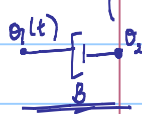


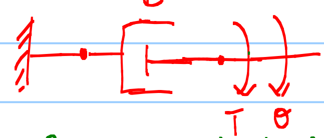
$$T = J\frac{d^2\theta(t)}{dt^2} = J\frac{d\omega(t)}{dt}$$

J – Moment of inertia (kg-m² or N-m-s²/rad)

2. Dashpot / Viscous damper

$$T = B\left(\frac{d\theta_1(t)}{dt} - \frac{d\theta_2(t)}{dt}\right)$$

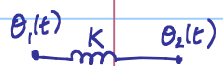




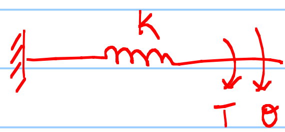
$$T = B\frac{d\theta(t)}{dt} = B\,\omega(t)$$

B - coefficient of viscous friction (N-m-s/rad)

3. Torsional Spring: potential energy

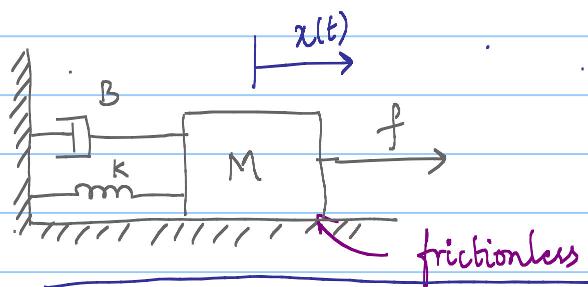


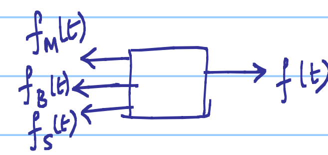
$$T = K(\theta_1(t) - \theta_2(t))$$



$$T = K\,\theta(t) = K\int_{-\infty}^{t}\omega(\tau)\,d\tau$$

K - spring constant (N-m/rad)





$$f(t) = f_M(t) + f_B(t) + f_S(t)$$

$v(t)$ — Velocity

$$\boxed{f(t) = M\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + Kx(t)} = M\frac{dv(t)}{dt} + Bv(t) + K\int_{-\infty}^{t}v(\tau)\,d\tau$$

LT $\longrightarrow$

$$F(s) = Ms^2X(s) + BsX(s) + KX(s)$$

(1) $$\boxed{\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}}$$

$$\frac{I(s)}{V_i(s)} = \frac{1}{R + Ls + \frac{1}{Cs}}$$

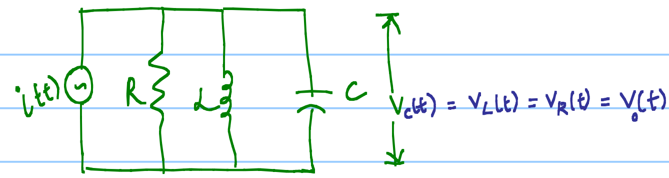
$$\frac{I(s)}{V_i(s)} = \frac{s}{Rs + Ls^2 + 1/C}$$

(2) $$\boxed{\frac{Q(s)}{V_i(s)} = \frac{1}{Rs + Ls^2 + 1/C}}$$

Force - Voltage Analogy

Electrical		Mechanical
R	$\longrightarrow$	B
L	$\longrightarrow$	M
1/C	$\longrightarrow$	K
(I/p voltage) V	$\longrightarrow$	f
q	$\longrightarrow$	x
i	$\longrightarrow$	v (velocity)

Parallel RLC



$$i(t) = i_R(t) + i_L(t) + i_c(t)$$

$$= \frac{V_o(t)}{R} + \frac{1}{L}\int_{-\infty}^{t} V_o(\tau)\,d\tau + C\frac{dV_o(t)}{dt}$$

$$I(s) = V_o(s)\left(1/R + \frac{1}{Ls} + Cs\right)$$

$$\frac{V_o(s)}{I(s)} = \frac{s}{\frac{1}{R}s + \frac{1}{L} + Cs^2}$$

$$\frac{L\,I_L(s)}{I(s)} = \frac{1}{Cs^2 + \frac{1}{R}s + \frac{1}{L}}$$

Flux linkage $\phi = iL$

$\Rightarrow \phi(s) = L\,I(s)$

$$\boxed{\frac{\phi(s)}{I(s)} = \frac{1}{Cs^2 + \frac{1}{R}s + \frac{1}{L}}}$$

Force - Current Analogy :

Mechanical → Electrical

$f \longrightarrow i$

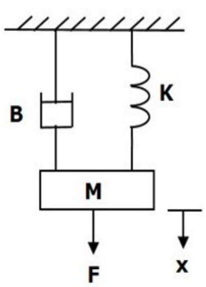
$M \longrightarrow C$

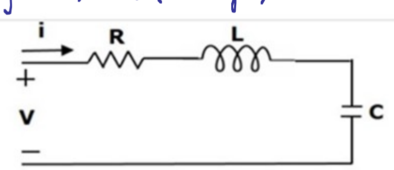
$B \longrightarrow 1/R$ (G - conductance)

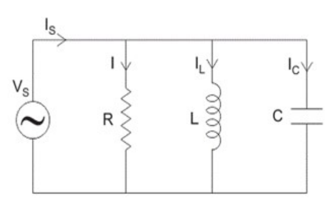
$1/K$ (compliance) $\longrightarrow L$

$x \longrightarrow \phi$

Velocity $\longrightarrow$ v (voltage)







$$F = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx$$

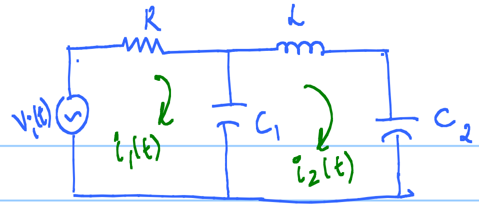
$$V = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q$$

F − V

$$i = C\frac{d^2\emptyset}{dt^2} + \frac{1}{R}\frac{d\emptyset}{dt} + \frac{\emptyset}{L}, \text{where } \phi = \psi$$

F − I

Translational Mechanical System	Electrical System	Electrical System
Force(F)	Voltage(V)	Current(I)
Mass(M)	Inductance(L)	Capacitance(C)
Frictional Coefficient(B)	Resistance(R)	Reciprocal of Resistance($\frac{1}{R}$)
Spring Constant(K)	Reciprocal of Capacitance ($\frac{1}{c}$)	Reciprocal of Inductance($\frac{1}{L}$)
Displacement(x)	Charge(q)	Magnetic Flux(ψ)
Velocity(v)	Current(i)	Voltage(V)



$$\frac{I_2(s)}{V_i(s)}$$

$$\begin{cases} \text{Loop 1}: \; v_i(t) = R i_1(t) + \dfrac{1}{C_1}\displaystyle\int_{-\infty}^{t} (i_1(\tau) - i_2(\tau))\, d\tau \\ \text{Loop 2}: \; 0 = L\dfrac{di_2(t)}{dt} + \dfrac{1}{C_1}\displaystyle\int_{-\infty}^{t} (i_2(\tau) - i_1(\tau))\, d\tau + \dfrac{1}{C_2}\displaystyle\int_{-\infty}^{t} i_2(\tau)\, d\tau \end{cases}$$

Under zero IC

$$\begin{cases} V_i(s) = R I_1(s) + \dfrac{1}{Cs}(I_1(s) - I_2(s)) & \text{— ①} \\ 0 = Ls\, I_2(s) + \dfrac{1}{Cs}(I_2(s) - I_1(s)) + \dfrac{1}{C_2 s} I_2(s) & \text{— ②} \end{cases}$$

$$② \Rightarrow \begin{cases} \dfrac{1}{C_1 s}(I_1(s) - I_2(s)) = \left(Ls + \dfrac{1}{C_2 s}\right) I_2(s) & \text{— (*)} \\ I_1(s) = C_1 s\left(Ls + \dfrac{1}{C_2 s} + \dfrac{1}{C_1 s}\right) I_2(s) & \text{— (**)} \end{cases}$$

$$① \Rightarrow \quad V_i(s) = RC_1 s\left(Ls + \frac{1}{C_2 s} + \frac{1}{C_1 s}\right) I_2(s) + \left(Ls + \frac{1}{C_2 s}\right) I_2(s)$$

CRAMER'S RULE

$$\frac{I_2(s)}{V_i(s)} = \frac{1}{\left(LRC_1 s^2 + \frac{RC_1}{C_2} + R\right) + \left(Ls + \frac{1}{C_2 s}\right)}$$

$$v_{C_2}(t) = \frac{1}{C_2}\int i_2(\tau)\, d\tau$$

$$V_{C_2}(s) = \frac{1}{C_2 s} I_2(s)$$

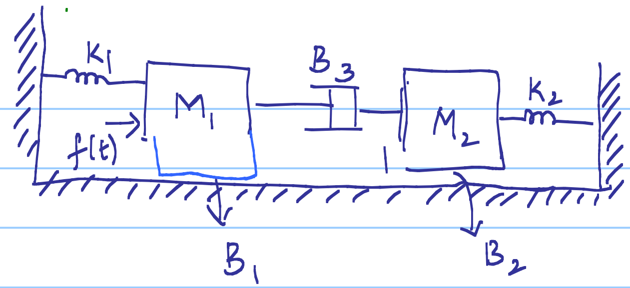
$$\boxed{\frac{I_2(s)}{V_i(s)} = \frac{C_2 s}{LRC_1C_2 s^3 + RC_1 s + RC_2 s + LC_2 s^2 + 1}}$$

$$\boxed{\frac{I_2(s)}{V_i(s)} = \frac{C_2 s}{LRC_1C_2 s^3 + LC_2 s^2 + (R(C_1+C_2))s + 1}}$$

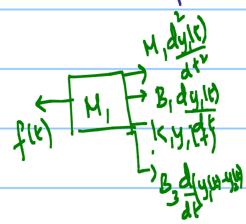
$$② \Rightarrow \begin{cases} \dfrac{1}{C_1 s}(I_1(s) - I_2(s)) = \left(Ls + \dfrac{1}{C_2 s}\right) I_2(s) \\ I_2(s) = \dfrac{\frac{1}{C_1 s} \cdot I_1(s)}{\left(Ls + \frac{1}{C_2 s} + \frac{1}{C_1 s}\right)} \end{cases}$$

$$V_i(s) = R\, I_1(s) + \frac{\left(Ls + \frac{1}{C_2 s}\right) \cdot \frac{1}{C_2 s}\, I_1(s)}{\left(Ls + \frac{1}{C_2 s} + \frac{1}{C_1 s}\right)}$$

$$\frac{I_1(s)}{V_i(s)} = \frac{\left(C_2 C_1 L s^2 + (C_1 + C_2)\right) s}{LRC_1C_2 s^3 + R(C_1+C_2)\, s + LC_2 s^2 + 1}$$



$$\frac{Y_2(s)}{F(s)}$$



wrt M_1 :

$$f(t) = M_1 \frac{d^2 y_1(t)}{dt^2} + B_1 \frac{dy_1(t)}{dt} + K_1 y_1(t) + B_3 \left(\frac{d}{dt}(y_1(t) - y_2(t)) \right)$$

wrt M_2 :

$$0 = M_2 \frac{d^2 y_2(t)}{dt^2} + B_3 \frac{d}{dt}\left(y_2(t) - y_1(t)\right) + K_2 y_2(t) + B_2 \frac{dy_2(t)}{dt}$$

LT

$$F(s) = (M_1 s^2 + B_1 s + K_1) Y_1(s) + B_3 s (Y_1(s) - Y_2(s)) \quad \text{—} \; 1$$

$$0 = (M_2 s^2 + K_2 + B_2 s) Y_2(s) + B_3 s (Y_2(s) - Y_1(s)) \quad \text{—} \; 2$$

2 $\Rightarrow$

$$\begin{cases} B_3 s (Y_1(s) - Y_2(s)) = (M_2 s^2 + K_2 + B_2 s) Y_2(s) \\ Y_1(s) = \dfrac{(M_2 s^2 + B_2 s + K_2 + B_3 s)}{B_3 s} Y_2(s) \end{cases}$$

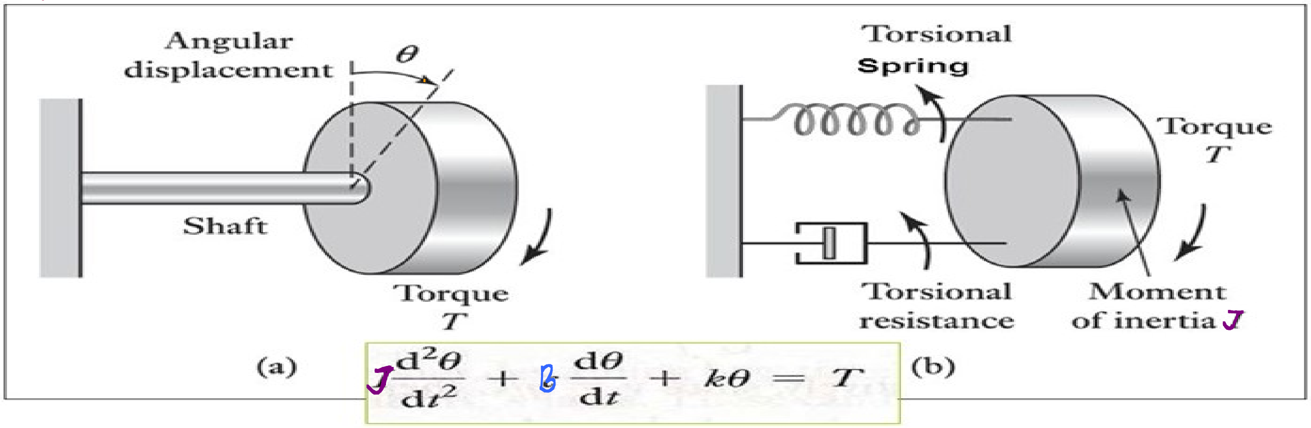
1 $\Rightarrow$

$$F(s) = \left[\frac{(M_1 s^2 + B_1 s + K_1)(M_2 s^2 + B_2 s + K_2 + B_3 s)}{B_3 s} + (M_2 s^2 + K_2 + B_2 s) \right] Y_2(s)$$

$$\frac{Y_2(s)}{F(s)} = \frac{B_3 s}{M_2 B_3 s^3 + B_3 B_2 s^2 + K_2 B_3 s + (M_1 s^2 + B_1 s + K_1)(M_2 s^2 + B_2 s + B_3 s + K_2)}$$

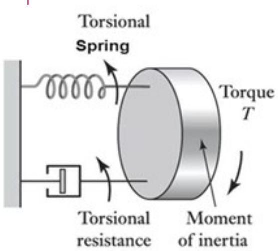
Rotational Systems:

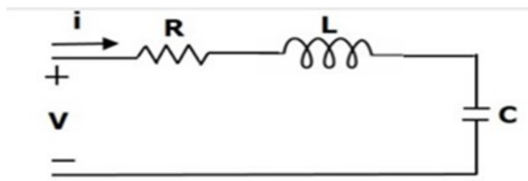
$$J\frac{d\omega(t)}{dt} + B\omega(t) + k\int_{-\infty}^{t}\omega(\tau)d\tau = \Gamma$$

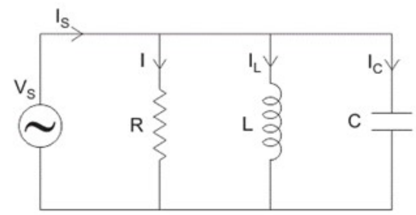


$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + k\theta = T$$

Rotating a mass on the end of a shaft: (a) physical situation, (b) building block model







$$T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta$$

$$V = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q$$

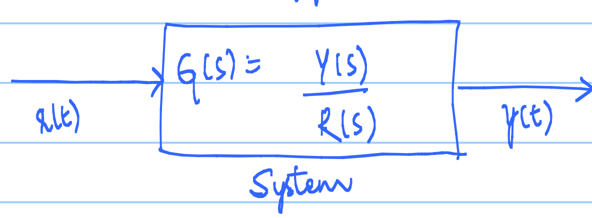
$$i = C\frac{d^2\phi}{dt^2} + \frac{1}{R}\frac{d\phi}{dt} + \frac{\phi}{L}$$

Rotational Mechanical System	Electrical System
Torque(T)	Voltage(V)
Moment of inertia(J)	Inductance(L)
Rotational friction coefficient(B)	Resistance(R)
Torsional spring constant(K)	Reciprocal of Capacitance ($\frac{1}{c}$)
Angular displacement(θ)	Charge(q)
Angular velocity(ω)	Current(i)

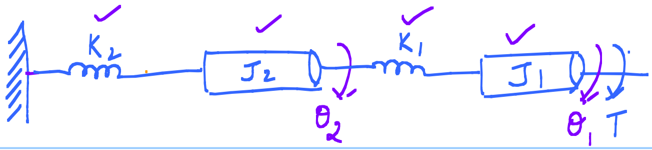
Electrical System
Current(i)
Capacitance(C)
Reciprocal of Resistance($\frac{1}{R}$)
Reciprocal of Inductance($\frac{1}{L}$)
Magnetic Flux(ψ)
Voltage(V)

The transfer function $= \left.\dfrac{LT\{y(t)\}}{LT\{x(t)\}}\right|_{\text{zero IC}}$

$y(t)$ – output / effect $x(t)$ – input / cause



⇒ Transfer function relates output of the system to input



Cause – T

Effect – θ_2

Find $\dfrac{\theta_2(s)}{T(s)}$

Wrt J_1 : $T(t) = J_1 \dfrac{d^2\theta_1(t)}{dt^2} + K_1\left(\theta_1(t) - \theta_2(t)\right)$ — ①

Wrt J_2 : $0 = J_2 \dfrac{d^2\theta_2(t)}{dt^2} + K_2\,\theta_2(t) + K_1\left(\theta_2(t) - \theta_1(t)\right)$ — ②

LT →

$T(s) = J_1 s^2 \boxed{\theta_1(s)}^{**} + \underbrace{K_1\left(\theta_1(s) - \theta_2(s)\right)}^{*}$ — ③

$0 = {}_2\,{}^2\theta_2(s) + K_2\,\theta_2(s) + K_1\left(\theta_2(s) - \theta_1(s)\right)$ — ④

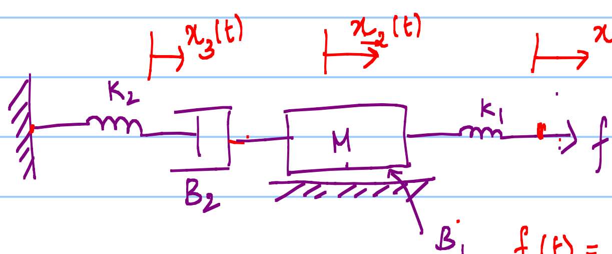
④ ⇒ $\begin{cases} K_1\left(\theta_1(s) - \theta_2(s)\right) = \left(J_2 s^2 + K_2\right)\theta_2(s) & \text{— } (*) \\ \theta_1(s) = \dfrac{\left(J_2 s^2 + K_1 + K_2\right)}{K_1}\,\theta_2(s) & \text{— } (**) \end{cases}$

$$T(s) = J_1 s^2 \left(\frac{J_2 s^2 + K_1 + K_2}{K_1}\right)\theta_2(s) + \left(J_2 s^2 + K_2\right)\theta_2(s)$$

$$= \left(\frac{J_1 s^2\left(J_2 s^2 + K_1 + K_2\right) + \left(J_2 s^2 + K_2\right)K_1}{K_1}\right)\theta_2(s)$$

$$\frac{\theta_2(s)}{T(s)} = \frac{K_1}{J_1 s^2\left[J_2 s^2 + K_1 + K_2\right) + J_2 K_1 s^2 + K_1 K_2}$$

→ Order of the system $n = 4$ (2 springs (K_1 & K_2), 2 Moment of Inertia (J_1 & J_2))



Draw the equivalent electrical circuit

→ F-V analogy

→ F-I analogy

$f_{B_1}(t) = B_1 \dfrac{dx_2(t)}{dt}$

① → $f(t) = K_1\left(x_1(t) - x_2(t)\right)$

② → $0 = M\dfrac{d^2x_2(t)}{dt^2} + B_1\dfrac{dx_2(t)}{dt} + B_2\dfrac{d}{dt}\left(x_2(t) - x_3(t)\right) + K_1\left(x_2(t) - x_1(t)\right)$

③ → $0 = B_2\dfrac{d}{dt}\left(x_3(t) - x_2(t)\right) + K_2\,x_3(t)$

<u>Force - Voltage</u> : M – L , B – R , K – 1/C , f – v(t) , x(t) – q(t)

↑ voltage

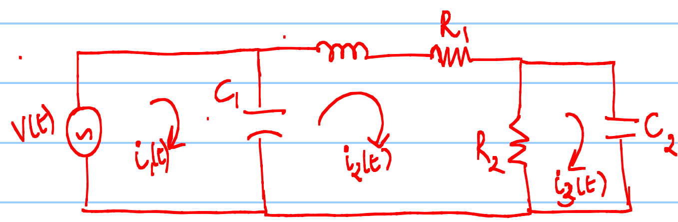
$\dfrac{d^2q(t)}{dt^2} = \dfrac{d}{dt}\dfrac{dq}{dt} = \dfrac{d}{dt}i(t)$

$$q(t) = \int_{-\infty}^{t} i(\tau)\,d\tau \qquad V(t) = \frac{1}{C_1}\left(q_1(t) - q_2(t)\right) = \frac{1}{C_1}\int_{-\infty}^{t}\left(i_1(\tau) - i_2(\tau)\right)d\tau$$

$$\frac{dq(t)}{dt} = i(t) \qquad 0 = L\frac{d^2q_2(t)}{dt^2} + R_1\frac{dq_2(t)}{dt} + R_2\frac{d}{dt}\left(q_2(t) - q_3(t)\right) + \frac{1}{C_1}\left(q_2(t) - q_1(t)\right)$$

$$\frac{d^2q(t)}{dt^2} = \frac{di(t)}{dt} \qquad 0 = L\frac{di_2(t)}{dt} + R\,i_2(t) + R_2\left(i_2(t) - i_3(t)\right) + \frac{1}{C_1}\int_{-\infty}^{t}\left(i_2(t) - i_1(t)\right)d\tau$$

$$0 = R_2\left(i_3(t) - i_2(t)\right) + \frac{1}{C_2}\int_{-\infty}^{t} i_3(\tau)\,d\tau$$



Force - Current Analogy: $f(t) - i(t)$, $M - C$, $B - 1/R$, $K - 1/L$, $x(t) - \phi(t)$

$$f(t) = K_1\left(x_1(t) - x_2(t)\right)$$

$$0 = M\frac{d^2x_2(t)}{dt^2} + B_1\frac{dx_2(t)}{dt} + B_2\frac{d}{dt}\left(x_2(t) - x_3(t)\right) + K_1\left(x_2(t) - x_1(t)\right)$$

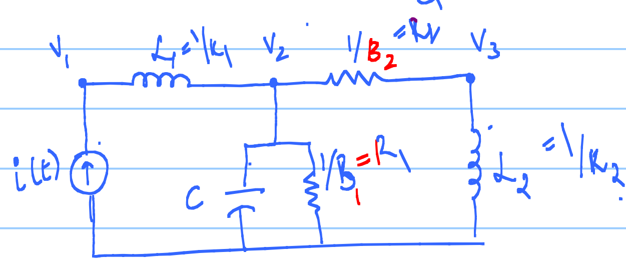
$$0 = B_2\frac{d}{dt}\left(x_3(t) - x_2(t)\right) + K_2\,x_3(t)$$

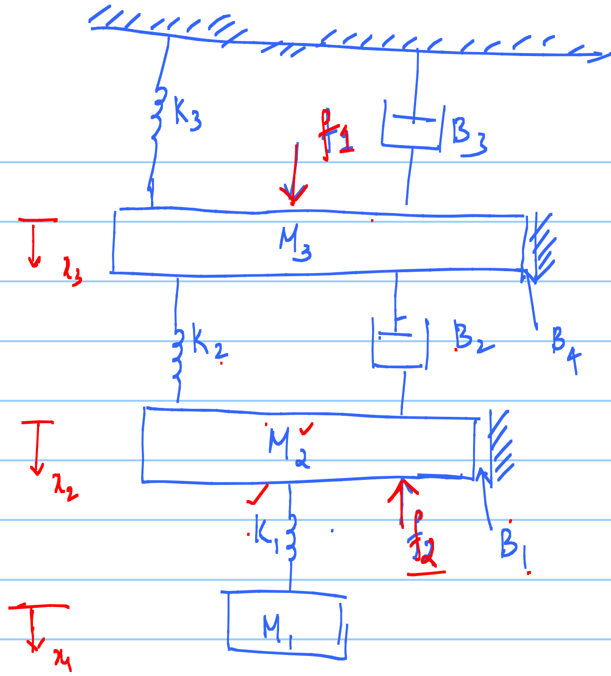
voltage

$$V(t) = \frac{d\phi(t)}{dt} \qquad i(t) = \frac{1}{L_1}\left(\phi_1(t) - \phi_2(t)\right) = \frac{1}{L_1}\int\left(V_1(t) - V_2(t)\right)d\tau$$

$$0 = C\frac{dV_2(t)}{dt} + \frac{1}{R_1}V_2(t) + \frac{1}{R_2}\left(V_2(t) - V_3(t)\right) + \frac{1}{L_1}\int\left(V_2(\tau) - V_1(t)\right)d\tau$$

$$0 = \frac{1}{R_2}\left(V_3(t) - V_2(t)\right) + \frac{1}{L_2}\int V_3(\tau)\,d\tau$$



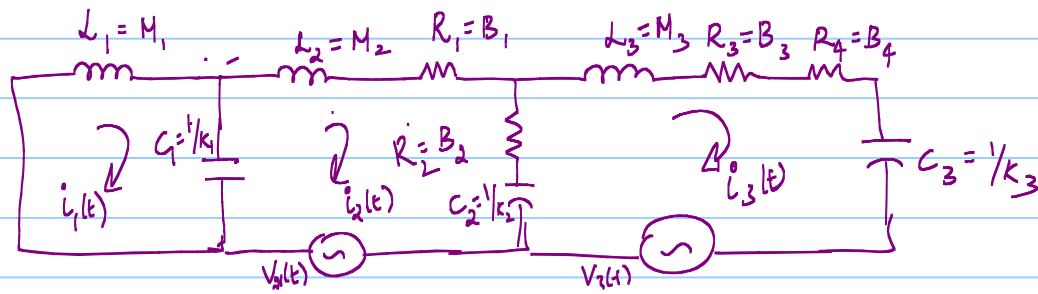


$$0 = M_1\frac{d^2x_1(t)}{dt^2} + K_1\left(x_1(t) - x_2(t)\right)$$

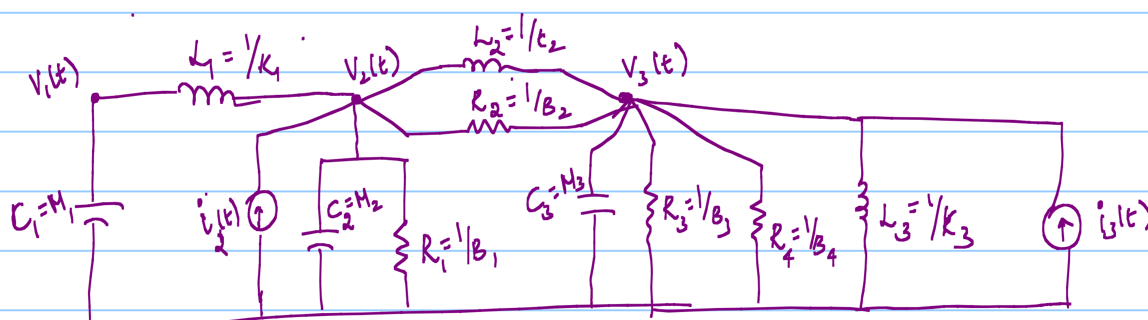
$$f_2(t) = \boxed{M_2\frac{d^2x_2(t)}{dt^2}} + K_1\left(x_2(t) - x_1(t)\right) + B_2\frac{d}{dt}\left(x_2(t) - x_3(t)\right) + K_2\left(x_2(t) - x_3(t)\right) + \boxed{B_1\frac{dx_2(t)}{dt}}$$

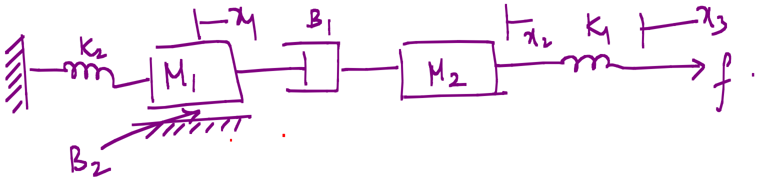
$$f_1(t) = M_3\frac{d^2x_3(t)}{dt^2} + K_3x_3(t) + B_3\frac{dx_3(t)}{dt} + K_2\left(x_3(t) - x_2(t)\right) + B_2\frac{d}{dt}\left(x_3(t) - x_2(t)\right) + B_4\frac{dx_3(t)}{dt}$$

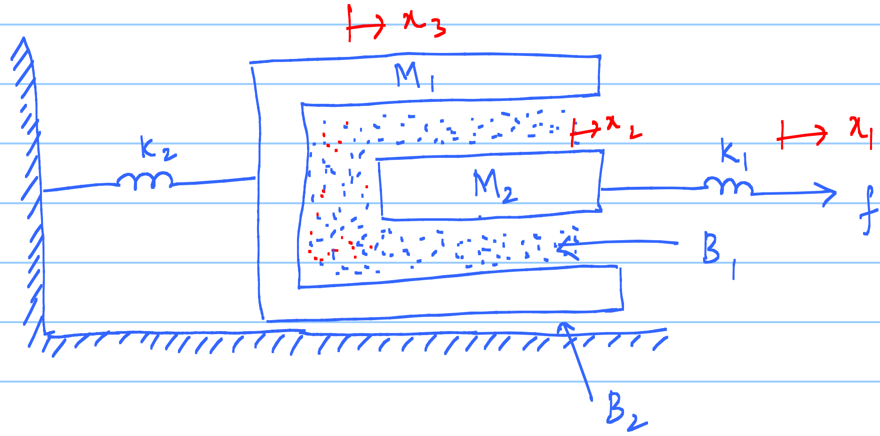
F-V



F-I







$$f(t) = K_1\,(x_3(t) - x_2(t))$$

$$0 = M_2 \frac{d^2 x_2(t)}{dt^2} + B_1 \frac{d}{dt}\left(x_2(t) - x_1(t)\right) + K_1\left(x_2(t) - x_3(t)\right)$$

$$0 = M_1 \frac{d^2 x_1(t)}{dt^2} + K_2 x_1(t) + B_2 \frac{d}{dt} x_1(t) + B_1 \frac{d}{dt}\left(x_1(t) - x_2(t)\right)$$

$\mathcal{L}T \longrightarrow$

$$F(s) = K_1\,(X_3(s) - X_2(s))$$

$$0 = M_2 s^2 X_2(s) + B_1 s\left(X_2(s) - X_1(s)\right) + K_1\left(X_2(s) - X_3(s)\right)$$

$$0 = M_1 s^2 X_1(s) + K_2 X_1(s) + B_2 s X_1(s) + B_1 s\left(X_1(s) - X_2(s)\right)$$

$$F(s) = 0\,X_1(s) - K_1 X_2(s) + K_1 X_3(s)$$

$$0 = -B_1 s X_1(s) + (M_2 s^2 + B_1 s + K_1)\,X_2(s) - K_1 X_3(s)$$

$$0 = (M_1 s^2 + B_2 s + B_1 s + K_2)\,X_1(s) - B_1 s X_2(s) + 0\,X_3(s)$$

$$\Delta = \begin{vmatrix} 0 & -K_1 & K_1 \\ -B_1 s & (M_2 s^2 + B_1 s + K_1) & -K_1 \\ (M_1 s^2 + (B_1+B_2)s + K_2) & -B_1 s & 0 \end{vmatrix}$$

$$\Delta = K_1\left(K_1\left(M_1 s^2 + (B_1+B_2)s + K_2\right)\right) + K_1\left((M_2 s^2 + B_1 s + K_1)(M_1 s^2 + (B_1+B_2)s + K_2) - B_1^2 s^2\right)$$

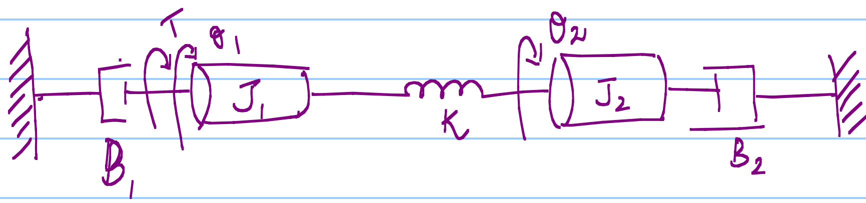
$$\Delta_3 = \begin{vmatrix} 0 & -K_1 & F(s) \\ -B_1 s & (M_2 s^2 + B_1 s + K_1) & 0 \\ (M_1 s^2 + (B_1+B_2)s + K_2) & -B_1 s & 0 \end{vmatrix}$$

$$X_2(s) = \frac{\Delta_2}{\Delta}$$

$$X_3(s) = \frac{\Delta_3}{\Delta}$$

$$\Delta_3 = F(s)\left[B_1^2 s^2 - (M_2 s^2 + B_1 s + K_1)(M_1 s^2 + (B_1+B_2)s + K_2) \right]$$

$$\frac{X_3(s)}{F(s)} = \frac{\left[B_1^2 s^2 - (M_2 s^2 + B_1 s + K_1)(M_1 s^2 + (B_1+B_2)s + K_2) \right]}{\Delta}$$



$$\therefore \frac{\theta_2(s)}{T(s)}$$

$$T = J_1 \frac{d^2\theta_1(t)}{dt^2} + B_1 \frac{d\theta_1(t)}{dt} + K(\theta_1(t) - \theta_2(t))$$

$$0 = J_2 \frac{d^2\theta_2(t)}{dt^2} + B_2 \frac{d\theta_2(t)}{dt} + K(\theta_2(t) - \theta_1(t))$$

LT →

$$T(s) = J_1 s^2 \theta_1(s) + B_1 s\theta_1(s) + K(\theta_1(s) - \theta_2(s)) \quad — (1)$$

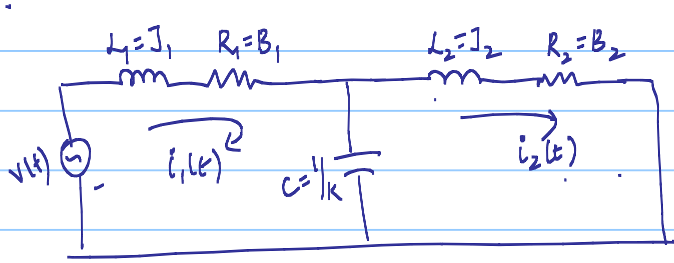
$$0 = (J_2 s^2 + B_2 s)\theta_2(s) + K(\theta_2(s) - \theta_1(s)) \quad — (2)$$

(2) ⇒
$$K(\theta_1(s) - \theta_2(s)) = (J_2 s^2 + B_2 s)\theta_2(s)$$
$$\theta_1(s) = \frac{(J_2 s^2 + B_2 s + K)}{K}\theta_2(s)$$

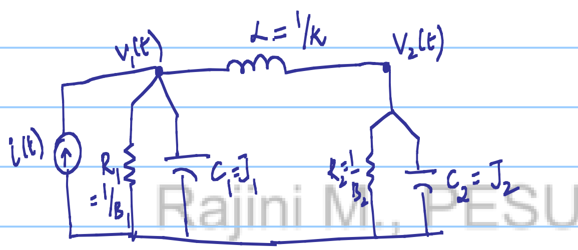
$$T(s) = (J_1 s^2 + B_1 s)\frac{(J_2 s^2 + B_2 s + K)}{K}\theta_2(s) + (J_2 s^2 + B_2 s)\theta_2(s)$$

$$\frac{\theta_2(s)}{T(s)} = \frac{K}{(J_1 s^2 + B_1 s)(J_2 s^2 + B_2 s + K) + K(J_2 s^2 + B_2 s)}$$

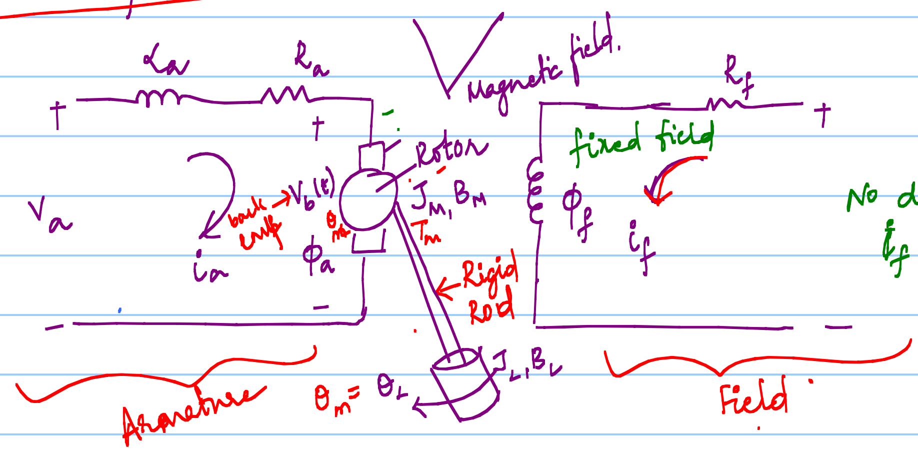
T-V



T-I



Armature Controlled DC Motor.



No dynamics
i_f is constant

→ Torque is produced due to magnetic field interaction $(\phi_a + \phi_f)$

$T_M(t) \propto i_a(t) i_f(t)$ — Non linear system

→ if i_f is constant,

$T_m(t) \propto i_a$

Armature Controlled DC Motor

→ if i_a is constant

$T_m(t) \propto i_f$

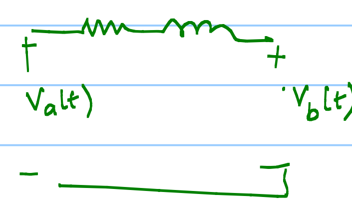
Field Controlled DC Motor

1 — $\boxed{T_m(t) = K_1 i_a(t)}$

→ Back emf $V_b(t) \propto \dfrac{d\theta_L(t)}{dt}$ $\quad$ $e_b(t) \propto$ Velocity with which the coil moves.

4 — $V_b(t) = K_2 \dfrac{d\theta_L(t)}{dt}$

→ $i_a(t)$ is produced because of $V_a(t)$

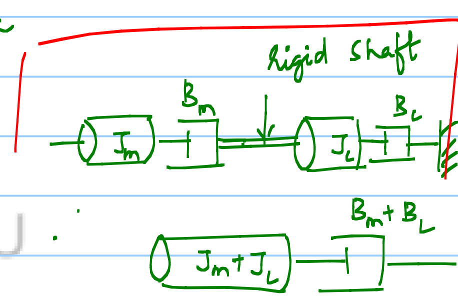


2 → $V_a(t) = R_a i_a(t) + L_a \dfrac{di_a(t)}{dt} + \underbrace{V_b(t)}_{\text{Load}}$

→ Torque applied at the coil (T_m) & Torque at the load (T_L)

$T_m(t) = J_m \dfrac{d^2\theta_L(t)}{dt^2} + B_m \dfrac{d\theta_L(t)}{dt} + T_L$

$T_L = J_L \dfrac{d^2\theta_L}{dt^2} + B_L \dfrac{d\theta_L}{dt}$



③ → $T_m(t) = J_m \frac{d^2\theta_L}{dt^2} + \underbrace{B_m \frac{d\theta_L}{dt}} + J_L \frac{d^2\theta_L}{dt^2} + B_L \frac{d\theta_L}{dt} = \overset{J_m+J_L}{J}\frac{d^2\theta_L}{dt^2} + \overset{B_m+B_L}{B}\frac{d\theta_L}{dt}$

⑤ — $T_m(s) = K_1 I_a(s)$ $\qquad V_b(s) = K_2 s\,\theta_L(s)$ — ⑥

⑦ — $V_a(s) = (L_a s + R_a) I_a(s) + V_b(s)$ ←

⑧ — $T_m(s) = \left(\underbrace{(J_m+J_L)}_{J_T} s^2 + \underbrace{(B_m+B_L)}_{B_T} s\right)\theta_L(s)$

Cause $V_a(t)$

effect $\theta(t)$.

$\frac{\theta(s)}{V_a(s)}$

from eqⁿ ⑤ $\qquad I_a(s) = \frac{T_m(s)}{K_1}$ — ⑨

Substitute eqⁿ ⑨ & ⑥ in ⑦

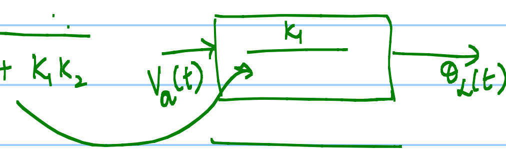
$V_a(s) = (L_a s + R_a)\frac{T_m(s)}{K_1} + K_2 s\,\theta_L(s)$ — ⑩

Substitute eqⁿ ⑧ in ⑩

$V_a(s) = \frac{(L_a s + R_a)(J_T s^2 + B_T s)\,\theta_L(s)}{K_1} + K_2 s\,\theta_L(s)$

$$\boxed{\frac{\theta_L(s)}{V_a(s)} = \frac{K_1}{(L_a s + R_a)(J_T s + B_T)s + K_1 K_2 s}}$$

Angular displacement → $\frac{\theta_L(s)}{V_a(s)} = \frac{1}{s}\,\frac{K_1}{(L_a s + R_a)(J_T s + B_T) + K_1 K_2}$

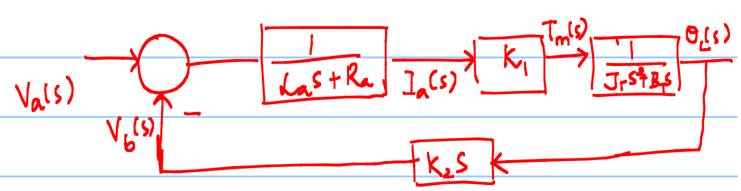


Angular velocity → $$\boxed{\frac{\omega_L(s)}{V_a(s)} = \frac{K_1}{(L_a s + R_a)(J_T s + B_T) + K_1 K_2}}$$

$T_m(s) = K_1 I_a(s)$ $\qquad V_b(s) = K_2 s\,\theta_L(s)$

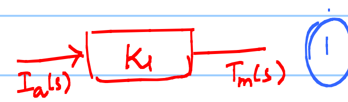
$V_a(s) = (L_a s + R_a) I_a(s) + V_b(s)$

$T_m(s) = \left(\underbrace{(J_m+J_L)}_{J_T} s^2 + \underbrace{(B_m+B_L)}_{B_T} s\right)\theta_L(s)$



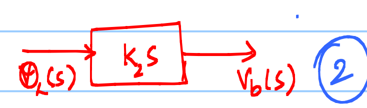
eqⁿ ⑤ → Cause $I_a(s)$ $\qquad \frac{T_m(s)}{I_a(s)} = K_1$

effect $T_m(s)$



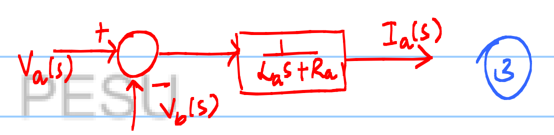
eqⁿ ⑥ → Cause $\theta_L(s)$ $\qquad \frac{V_b(s)}{\theta_L(s)} = K_2 s$

effect $V_b(s)$



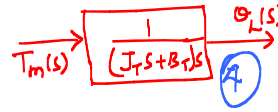
eqⁿ ⑦ → Cause $V_a(s)$ & $V_b(s)$

effect $I_a(s)$

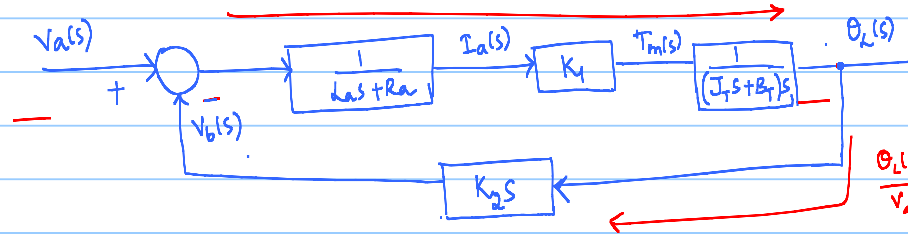


eqⁿ (8) — cause $T_m(s)$ effect $\theta_L(s)$

$$\frac{\theta_L(s)}{T_m(s)} = \frac{1}{(J_T s + B_T)s}$$



(4)



$$G(s) = \frac{K_1}{(L_a s + R_a)(J_T s^2 + B_T s)}$$

$$H(s) = K_2 s$$

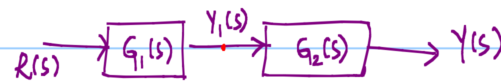
$$\frac{\theta_L(s)}{V_a(s)} = \frac{\frac{K_1}{(L_a s + R_a)(J_T s^2 + B_T s)}}{1 + \frac{K_1 \times K_2 s}{(L_a s + R_a)(J_T s^2 + B_T s)}}$$

BLOCK DIAGRAM REPRESENTATION OF DC MOTOR

$$\frac{\theta_L(s)}{V_a(s)} = \frac{K_1}{(L_a s + R_a)(J_T s^2 + B_T s) + K_1 K_2 s}$$

Block Diagram Reduction Techniques:

1. Two block in series

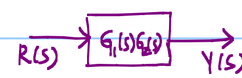


$$Y_1(s) = G_1(s)\,R(s)$$

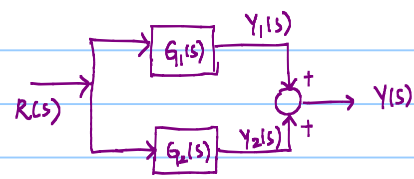
$$Y(s) = G_2(s)\,Y_1(s)$$

$$Y(s) = G_2(s)\,G_1(s)\,R(s)$$

$$\frac{Y(s)}{R(s)} = G_1(s)\,G_2(s)$$



2. Two blocks in parallel.

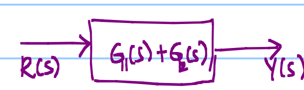


$$Y(s) = Y_1(s) + Y_2(s)$$

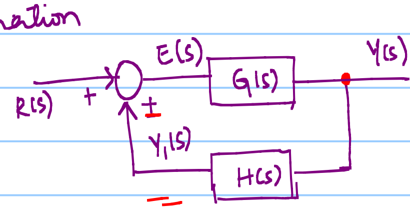
$$= G_1(s)R(s) + G_2(s)R(s)$$

$$Y(s) = (G_1(s) + G_2(s))\,R(s)$$

$$\frac{Y(s)}{R(s)} = G_1(s) + G_2(s)$$



3. Feedback Configuration

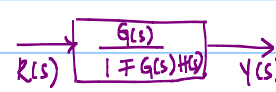


$$E(s) = R(s) \pm Y_1(s) = R(s) \pm H(s)Y(s)$$

$$Y(s) = G(s)E(s) = G(s)\left(R(s) \pm H(s)Y(s)\right)$$

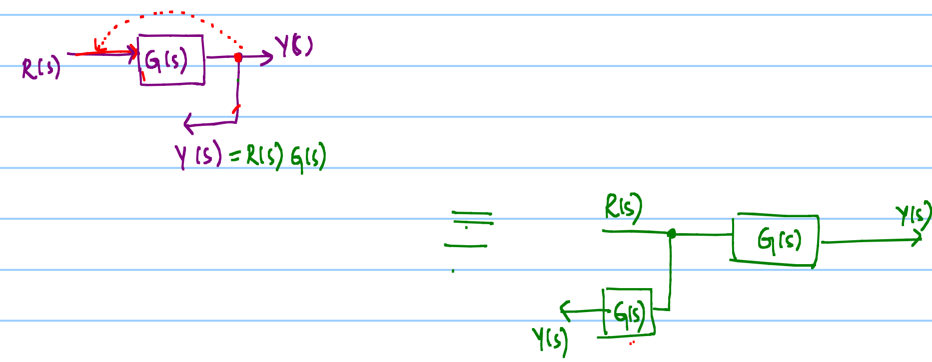
$$(1 \mp G(s)H(s))\,Y(s) = G(s)\,R(s)$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{G(s)}{1 \mp G(s)H(s)}}$$

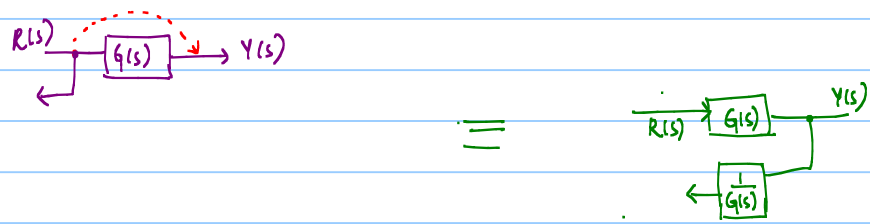


Block Diagram Reduction Rules:

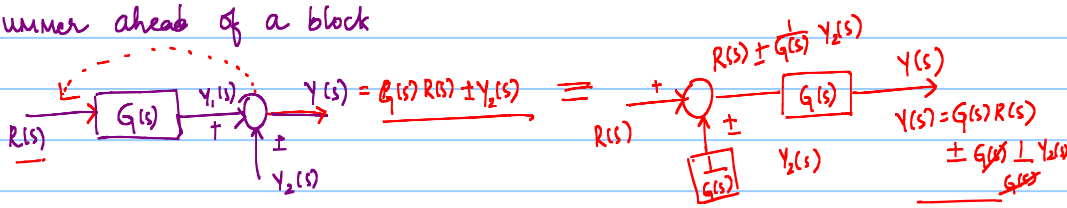
1. Moving the tap-off point ahead of the block.



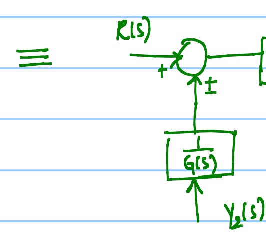
2. Moving the tap off point behind the block



3. Moving Summer ahead of a block



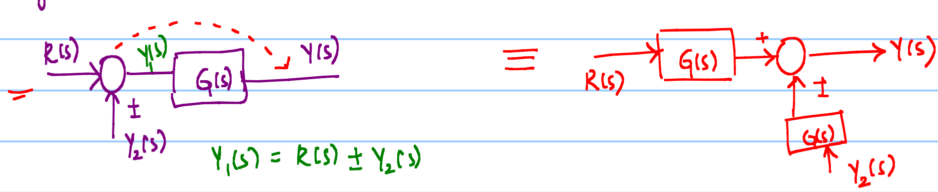
$$Y(s) = G(s) R(s) \pm Y_2(s)$$



$$Y(s) = \left(R(s) \pm \frac{Y_2(s)}{G(s)}\right) G(s)$$

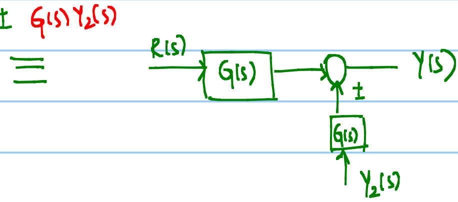
$$= R(s) G(s) \pm Y_2(s)$$

4. Moving Summer behind the block.

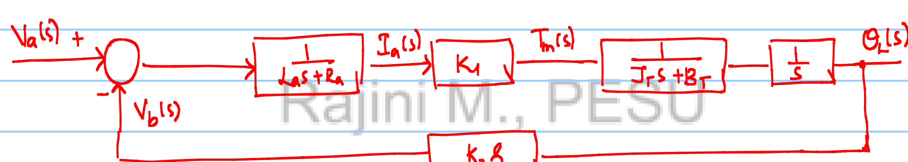


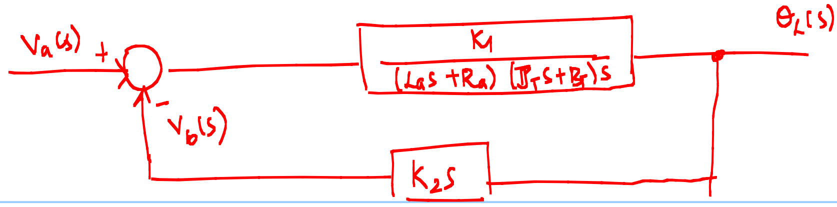
$$Y_1(s) = R(s) \pm Y_2(s)$$

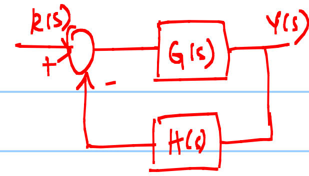
$$Y(s) = G(s) R(s) \pm G(s) Y_2(s)$$



$$Y(s) = R(s) G(s) \pm Y_2(s) G(s)$$







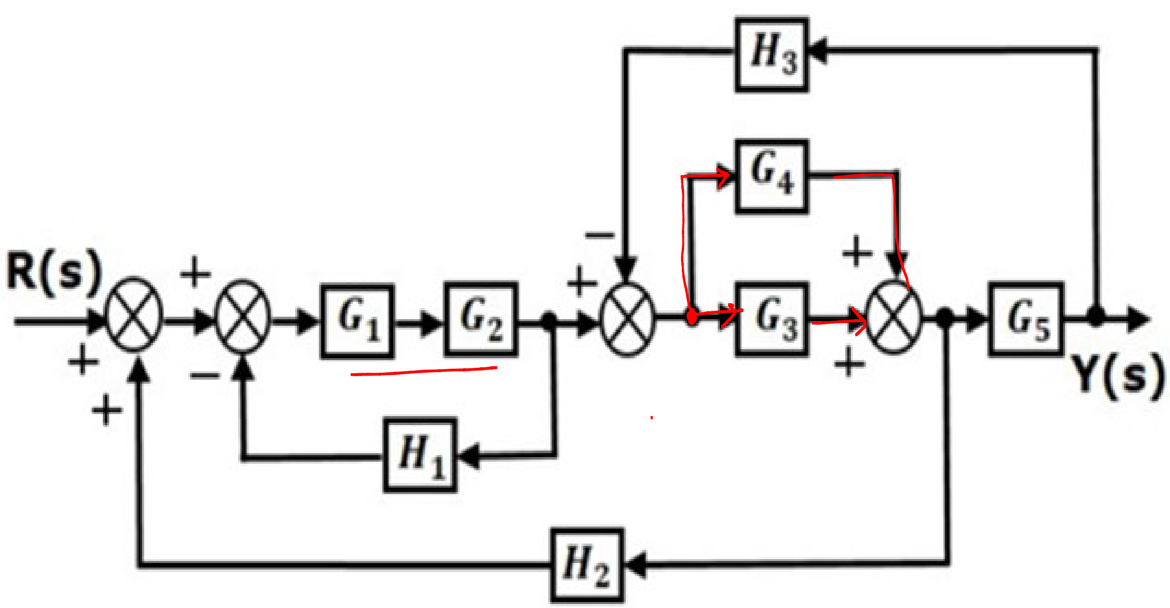
$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

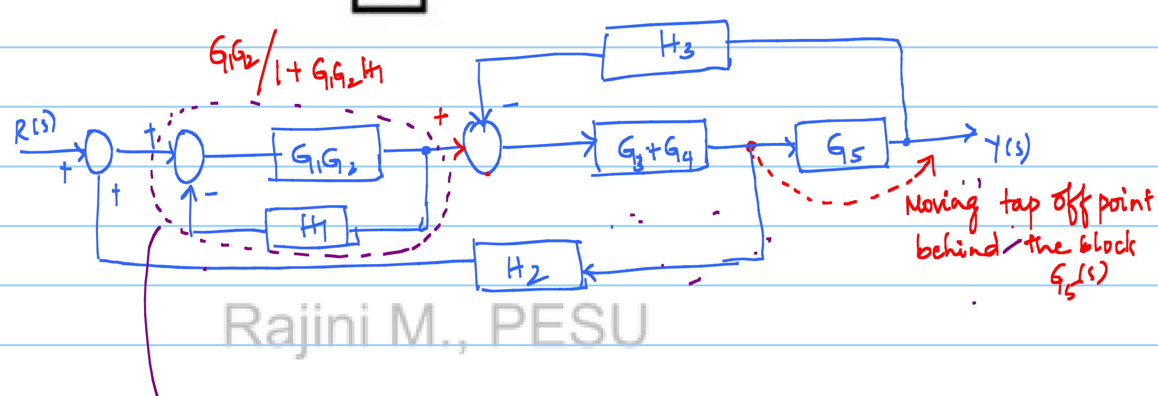
$$G(s) = \frac{K_1}{(L_as+R_a)(J_Ts+B_T)s} \qquad H(s) = K_2s$$

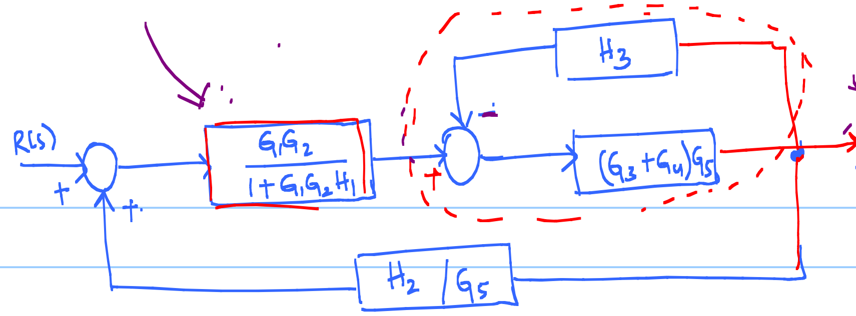
$$\frac{\theta_L(s)}{V_a(s)} = \frac{\dfrac{K_1}{(L_as+R_a)(J_Ts+B_T)s}}{1+\dfrac{K_1K_2s}{(L_as+R_a)(J_Ts+B_T)s}}$$

$$= \frac{\dfrac{K_1}{\cancel{(L_as+R_a)(J_Ts+B_T)s}}}{\dfrac{(L_as+R_a)(J_Ts+B_T)s+K_1K_2s}{\cancel{(L_as+R_a)(J_Ts+B_T)s}}}$$

$$\frac{\theta_L(s)}{V_a(s)} = \left[\frac{K_1}{(L_as+R_a)(J_Ts+B_T)+K_1K_2}\right]\frac{1}{s}$$

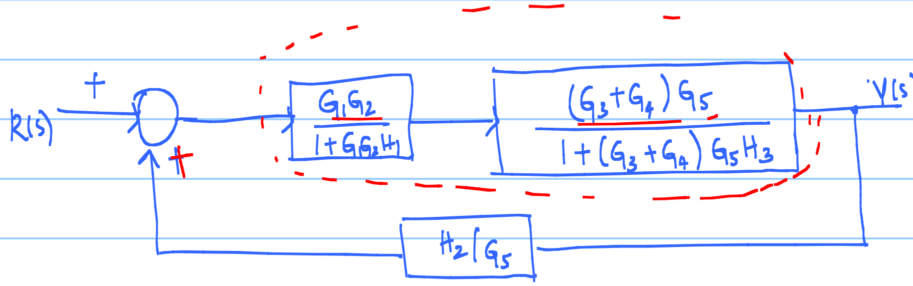


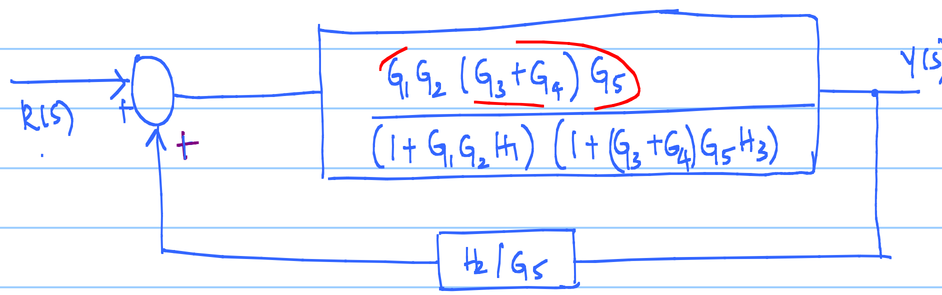




$$\frac{G}{1+GH}$$

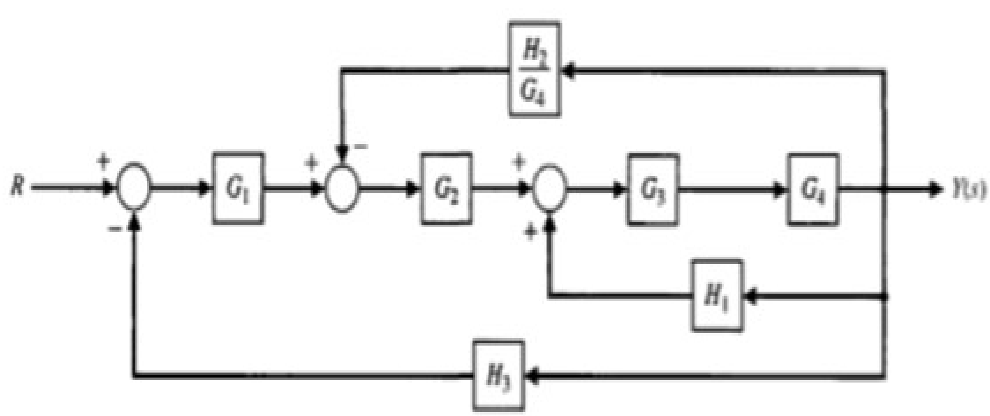
$$\frac{(G_3+G_4)G_5}{1+(G_3+G_4)G_5H_3}$$

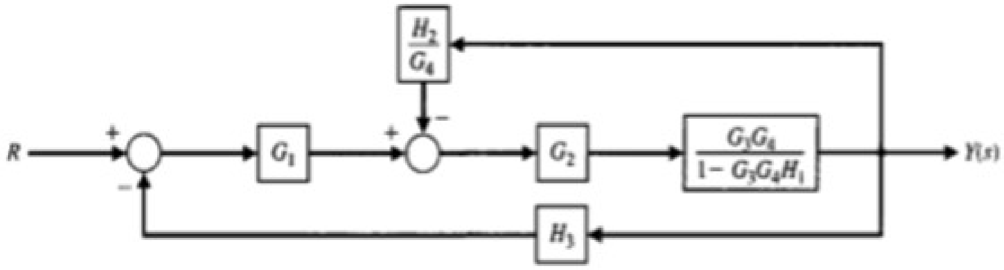


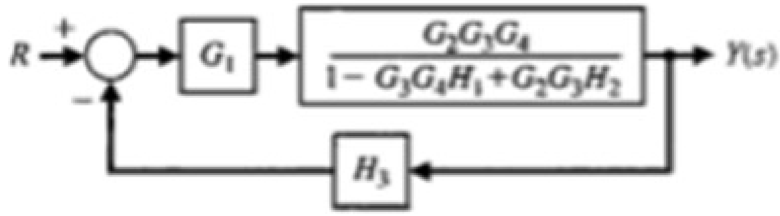


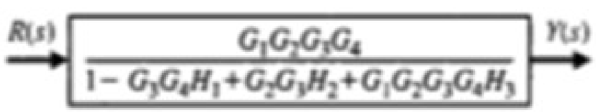
$$\frac{Y(s)}{R(s)} = \frac{\dfrac{G_1G_2(G_3+G_4)G_5}{(1+G_1G_2H_1)(1+(G_3+G_4)G_5H_3)}}{1-\dfrac{G_1G_2(G_3+G_4)\cancel{G_5}}{(1+G_1G_2H_1)(1+(G_3+G_4)G_5H_3)}\times\dfrac{H_2}{\cancel{G_5}}}$$

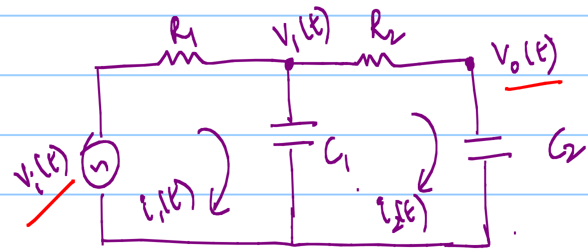
$$\boxed{\frac{Y(s)}{R(s)} = \frac{G_5G_1G_2(G_3+G_4)}{(1+G_1G_2H_1)(1+(G_3+G_4)G_5H_3)-G_1G_2(G_3+G_4)H_2}}$$





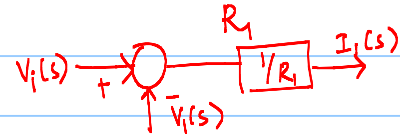






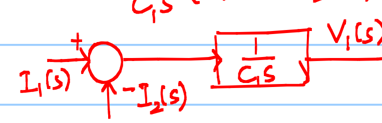
1. Cause : $v_i(t)$ & $v_1(t)$ effect : $i_1(t)$

$$i_1(t) = \frac{v_i(t) - v_1(t)}{R_1} \qquad I_1(s) = \frac{V_i(s) - V_1(s)}{R_1}$$



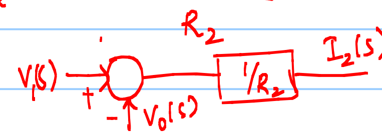
2. Cause : $i_1(t)$ & $i_2(t)$ effect $v_1(t)$

$$v_1(t) = \frac{1}{C_1}\int (i_1(t) - i_2(t))\, dt \qquad V_1(s) = \frac{1}{C_1 s}\left(I_1(s) - I_2(s)\right)$$



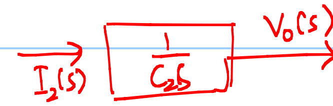
3. Cause : $v_1(t)$ & $v_o(t)$ effect : $i_2(t)$

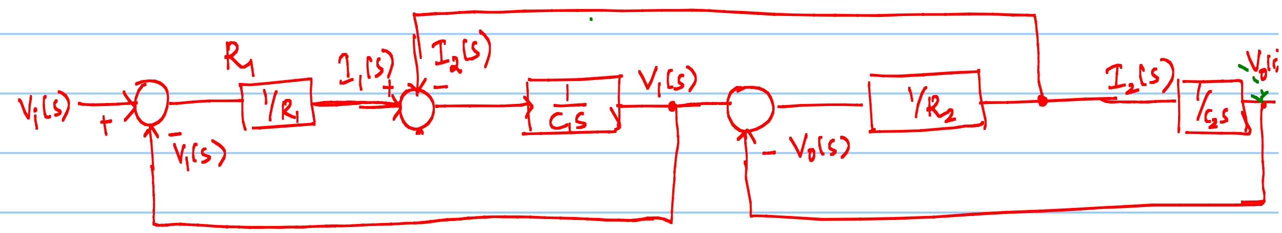
$$i_2(t) = \frac{v_1(t) - v_o(t)}{R_2} \qquad I_2(s) = \frac{V_1(s) - V_o(s)}{R_2}$$

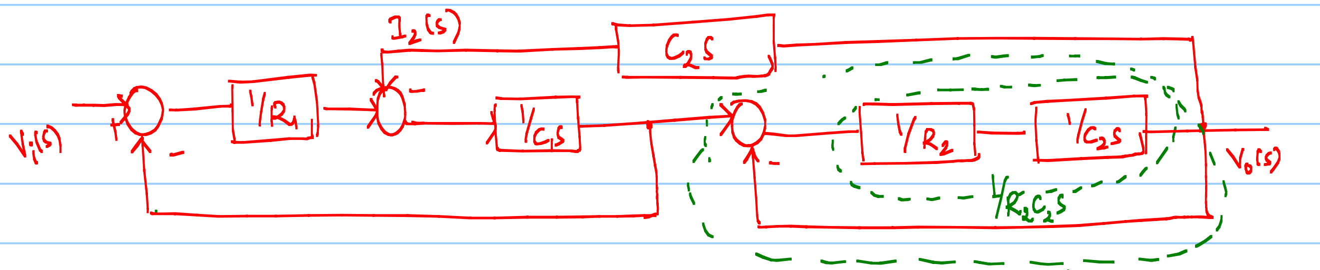


4. Cause: $i_2(t)$ effect: $V_0(t)$

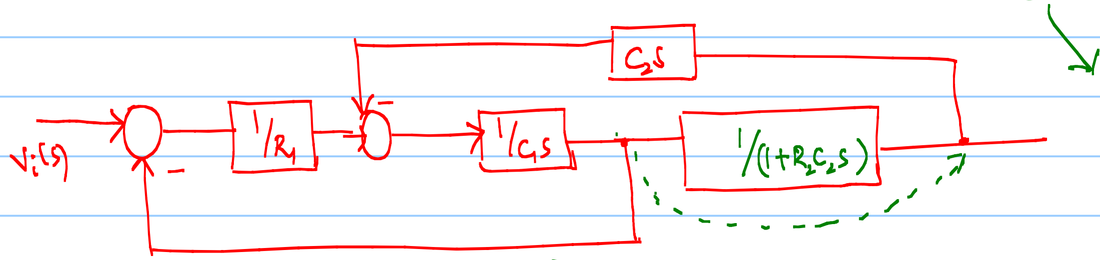
$$V_0(t) = \frac{1}{C_2}\int i_2(\tau)\,d\tau \qquad V_0(s) = \frac{1}{C_2 s} I_2(s)$$

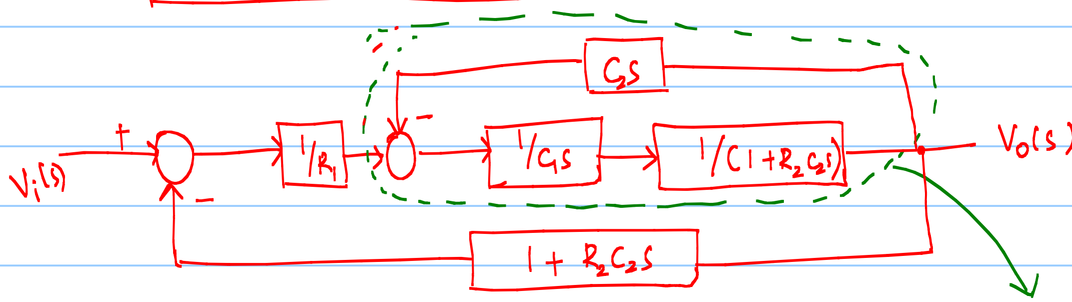




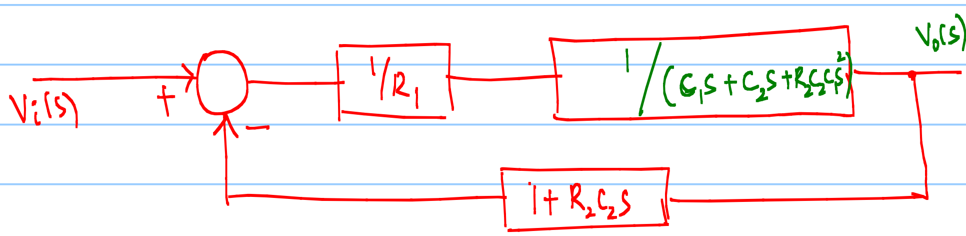


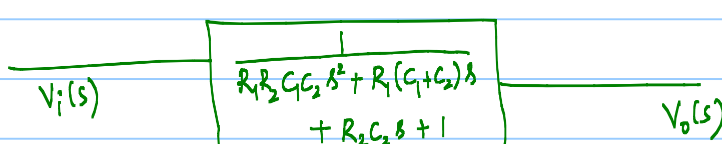
$$\frac{1/R_2C_2s}{1+\frac{1}{R_2C_2s}} = \frac{1}{1+R_2C_2s}$$

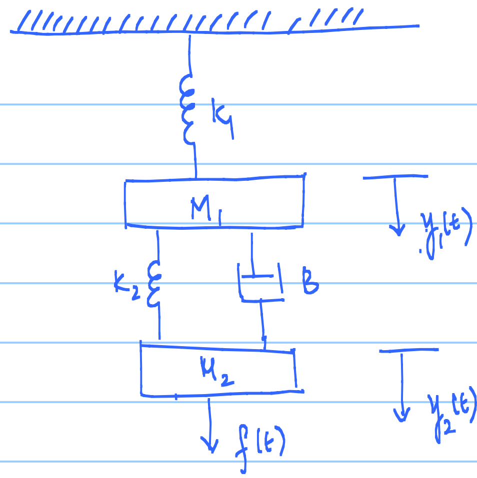




$$\frac{1/C_1s(1+R_2C_2s)}{1+\frac{C_2s}{C_1s(1+R_2C_2s)}} = \frac{1}{C_1s(1+R_2C_2s)+C_2s}$$







$$f(t) = M_2 \frac{d^2 y_2(t)}{dt^2} + K_2\left(y_2(t) - y_1(t)\right) + B\left(\frac{d}{dt} y_2(t) - \frac{d y_1(t)}{dt}\right)$$

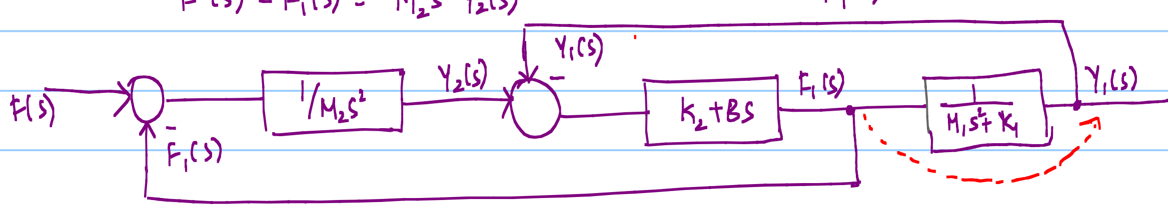
$$0 = M_1 \frac{d^2 y_1(t)}{dt^2} + K_1 y_1(t) + K_2\left(y_1(t) - y_2(t)\right) + B \frac{d}{dt}\left(y_1(t) - y_2(t)\right)$$

$(K_2 + Bs)(Y_2(s) - Y_1(s))$

$$F(s) = M_2 s^2 Y_2(s) + \overbrace{K_2\left(Y_2(s) - Y_1(s)\right) + Bs\left(Y_2(s) - Y_1(s)\right)}^{F_1(s)}$$

$$0 = M_1 s^2 Y_1(s) + K_1 Y_1(s) + \underbrace{(K_2 + Bs)\left(Y_1(s) - Y_2(s)\right)}_{-F_1(s)} \Rightarrow F_1(s) = (M_1 s^2 + K_1) Y_1(s)$$

$$F(s) - F_1(s) = M_2 s^2 Y_2(s)$$



Signal Flow graph: (SFG)

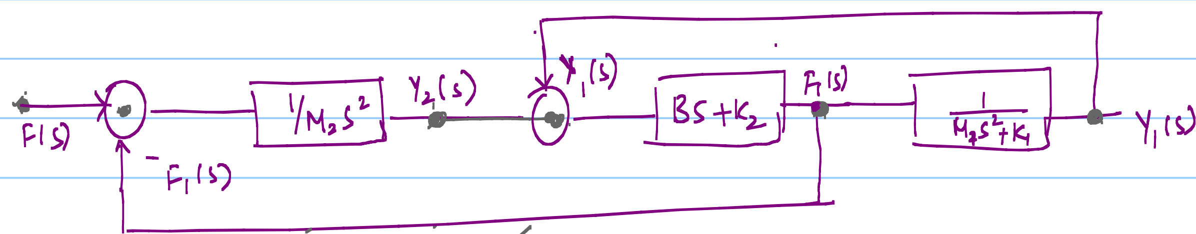
Converting Block diagram to signal flow graph

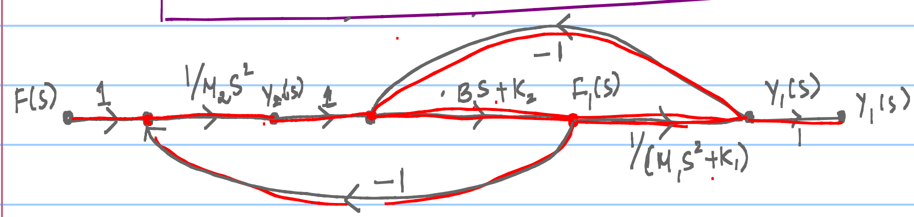
→ Represent the variables, signals, summers & tap off with 'nodes'

→ Represent the blocks of the block diagram as branches

→ Represent the transfer functions as the gains of the branches

→ Connect the nodes as per the block diagram





MASON'S GAIN FORMULA.

$$\boxed{T(s) = \frac{\sum_{i=1}^{N} P_i \Delta_i}{\Delta} = \frac{Y(s)}{R(s)}}$$

$Y(s)$ — Output Node

$R(s)$ — Input node

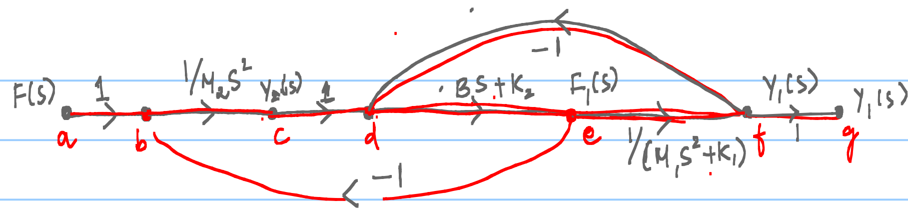
P_i — i^{th} forward path gain

Δ = 1 − [sum of all the individual loop gains] + [Sum of gain products of all possible 2 non touching loops]

− [Sum of gain products of all possible 3 non touching loops]

+ . . . . .

Δ_i — Obtained from Δ by removing the loops which are touching the i^{th} forward path.



$$P_1 = abcdefg = 1 \times \frac{1}{M_2 s^2} \times 1 \times (Bs+K_2) \times \frac{1}{M_1 s^2 + K_1} \times 1$$

$$L_1 = defd = (Bs+K_2) \times \frac{1}{(M_1 s^2 + K_1)} \times -1$$

$$L_2 = bcdeb = \frac{1}{M_2 s^2} \times 1 \times (Bs+K_2) \times -1$$

$$\Delta = 1 - (L_1 + L_2) + \cancel{(L_1 L_2)} \; X$$

$$= 1 - \left(-\frac{Bs+K_2}{M_1 s^2 + K_1} - \frac{Bs+K_2}{M_2 s^2} \right)$$

$$\Delta_1 = 1$$

$$T = \frac{Y(s)}{R(s)} = \frac{P_1 \Delta_1}{\Delta} = \frac{(Bs+K_2) / (M_1 s^2 + K_1) M_2 s^2 \times 1}{1 + \frac{(Bs+K_2)(M_2 s^2 + M_1 s^2 + K_1)}{(M_1 s^2 + K_1) M_2 s^2}}$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{Bs+K_2}{(M_1 s^2 + K_1) M_2 s^2 + (Bs+K_2)(M_2 s^2 + M_1 s^2 + K_1)}}$$