Electromagnetic Field Theory (UE20EC303)

- A review of Vector Calculus
- Unit 1 Electrostatics

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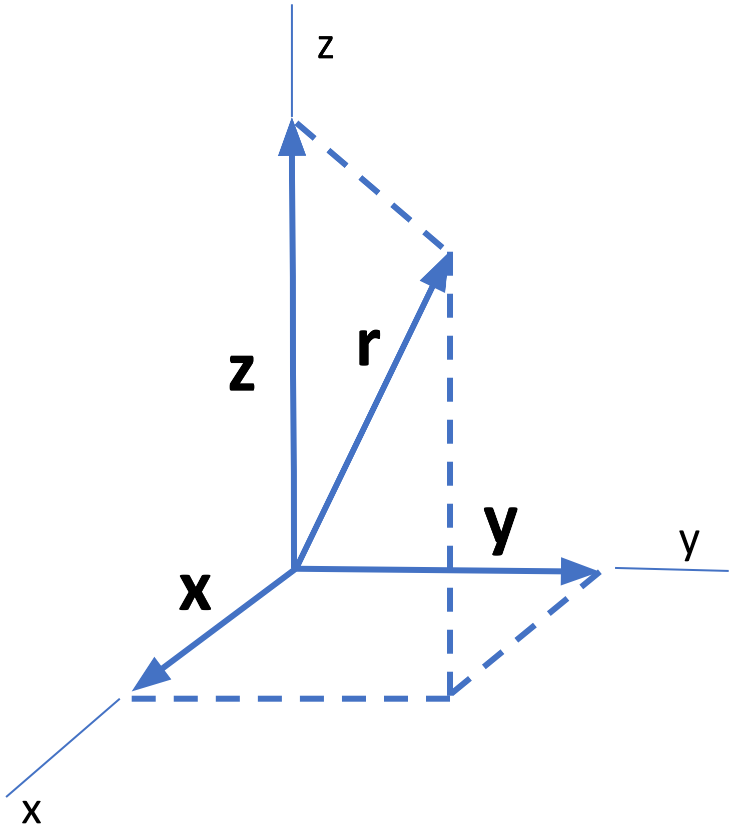
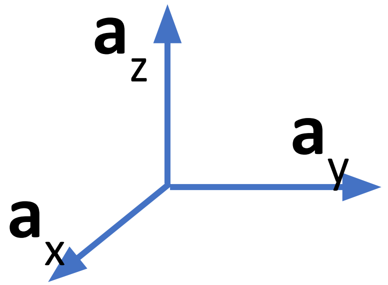




Position vectors and unit vectors in Cartesian coordinates

$\mathbf{r} = \mathbf{x} + \mathbf{y} + \mathbf{z}$,

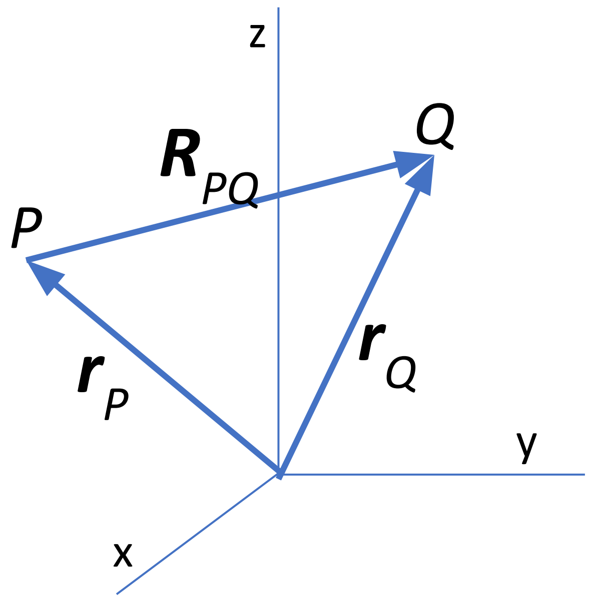
$\mathbf{r} = x\,\mathbf{a}_x + y\,\mathbf{a}_y + z\,\mathbf{a}_z$,

where $\mathbf{x} = x\,\mathbf{a}_x$, $\mathbf{y} = y\,\mathbf{a}_y$, $\mathbf{z} = z\,\mathbf{a}_z$,
(Vector quantities are in bold letters)

$\mathbf{r}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are position vectors, which originate from origin;

$\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$ are unit vectors in x, y, z directions

Distance vectors in Cartesian coordinates



Let $P\,(x_1, y_1, z_1)$ and $Q\,(x_2, y_2, z_2)$ be two points in Cartesian coordinate system. The position vectors of P and Q are

$$\boldsymbol{r}_P = x_1\,\boldsymbol{a}_x + y_1\,\boldsymbol{a}_y + z_1\,\boldsymbol{a}_z \quad \text{and} \quad \boldsymbol{r}_Q = x_2\,\boldsymbol{a}_x + y_2\,\boldsymbol{a}_y + z_2\,\boldsymbol{a}_z$$

and the distance vector starting from P to Q is $\boldsymbol{R}_{PQ}$

$$\boldsymbol{R}_{PQ} = \boldsymbol{r}_Q - \boldsymbol{r}_P = (x_2 - x_1)\boldsymbol{a}_x + (y_2 - y_1)\boldsymbol{a}_y + (z_2 - z_1)\boldsymbol{a}_z$$

Example: Find distance vector, $\boldsymbol{R}_{PQ}$, if the points are $P(1, 2, 3)$ and $Q(2, -2, 1)$

Solution: $\boldsymbol{r}_P = 1\,\boldsymbol{a}_x + 2\,\boldsymbol{a}_y + 3\,\boldsymbol{a}_z$, $\boldsymbol{r}_Q = 2\,\boldsymbol{a}_x - 2\,\boldsymbol{a}_y + 1\,\boldsymbol{a}_z$

$$\boldsymbol{R}_{PQ} = \boldsymbol{r}_Q - \boldsymbol{r}_P = \boldsymbol{a}_x - 4\,\boldsymbol{a}_y - 2\,\boldsymbol{a}_z$$

Magnitude of a vector, unit vector

Let a vector be: $\boldsymbol{B} = \boldsymbol{a}_x B_x + \boldsymbol{a}_y B_y + \boldsymbol{a}_z B_z$

The magnitude of $\boldsymbol{B}$ is:

$$|\boldsymbol{B}| = \sqrt{{B_x}^2 + {B_y}^2 + {B_z}^2}$$

The unit vector in the direction of $\boldsymbol{B}$ is: $\boldsymbol{B} / |\boldsymbol{B}|$,

$$\boldsymbol{a}_B = \frac{\boldsymbol{B}}{\sqrt{{B_x}^2 + {B_y}^2 + {B_z}^2}}$$

Example: Specify the unit vector extending from the origin toward the point *G* (2, -2, -1).

Solution: The vector extending from the origin to the point *G*,

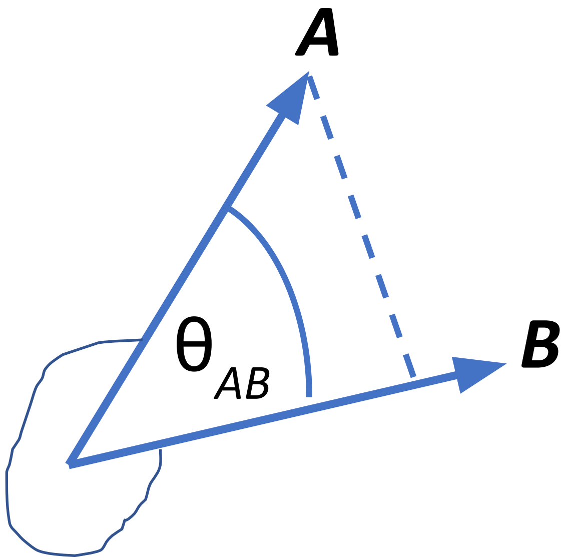
$$\boldsymbol{G} = 2\boldsymbol{a}_x - 2\boldsymbol{a}_y - \boldsymbol{a}_z$$

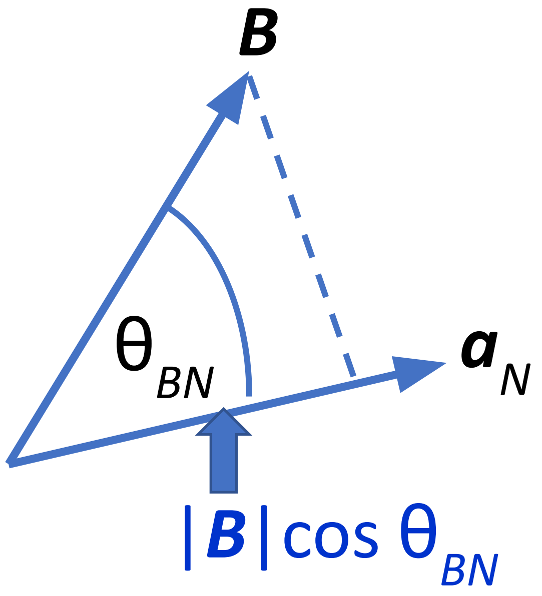
The magnitude of ***G*** is,

$$|\boldsymbol{G}| = \sqrt{4+4+1} = 3$$

The desired unit vector of **G** is,

$$\boldsymbol{a}_G = \frac{\boldsymbol{G}}{|\boldsymbol{G}|} = \frac{2}{3}\,\boldsymbol{a}_x - \frac{2}{3}\,\boldsymbol{a}_y - \frac{1}{3}\,\mathbf{a}_z$$





DOT product and its application

DOT product of two vectors: $\quad \boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}| \cos\theta_{AB}$

The angle, θ_{AB} , is the smaller angle between ***A*** and ***B***. Note that when θ_{AB} is 90˚ the DOT product vanishes.

If $\boldsymbol{A} = \boldsymbol{a}_x A_x + \boldsymbol{a}_y A_y + \boldsymbol{a}_z A_z$ and $\boldsymbol{B} = \boldsymbol{a}_x B_x + \boldsymbol{a}_y B_y + \boldsymbol{a}_z B_z$

$$\boldsymbol{A} \cdot \boldsymbol{B} = (\boldsymbol{a}_x Ax + \boldsymbol{a}_y Ay + \boldsymbol{a}_z Az) \cdot (\boldsymbol{a}_x Bx + \boldsymbol{a}_y By + \boldsymbol{a}_z Bz)$$

Since $\boldsymbol{a}_x \cdot \boldsymbol{a}_x = 1$, $\boldsymbol{a}_x \cdot \boldsymbol{a}_y = 0$, $\boldsymbol{a}_x \cdot \boldsymbol{a}_z = 0$, and so on,

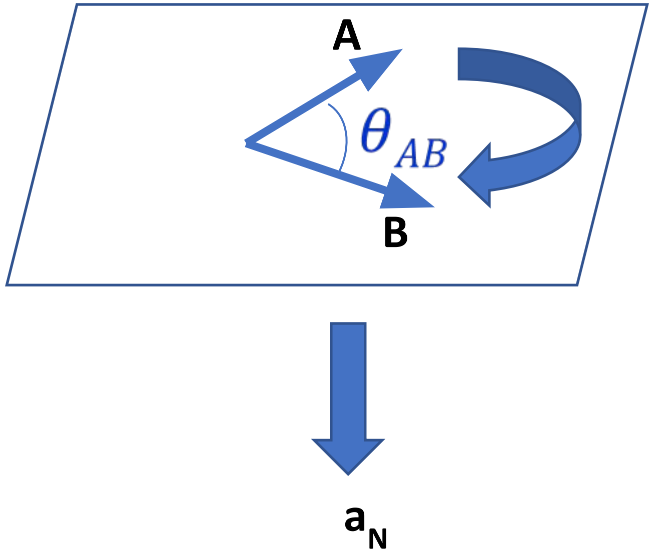
$$\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z = \text{A scalar quantity}$$

Components of a vector in given direction: The magnitude of vector ***B*** (or the scalar component) in the direction of unit vector $\boldsymbol{a}_N$ is given by the DOT product, $\boldsymbol{B} \cdot \boldsymbol{a}_N = |\boldsymbol{B}| \cos\theta_{BN}$ and the component vector of ***B*** in the direction of $\boldsymbol{a}_N$ is: $(\boldsymbol{B} \cdot \boldsymbol{a}_N)\, \boldsymbol{a}_N$

The cross product of two vectors

The cross product of two vectors, $\boldsymbol{A}$ and $\boldsymbol{B}$, is defined as,

$$\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{a}_N |\boldsymbol{A}||\boldsymbol{B}| \sin \theta_{AB}$$



where $\boldsymbol{a}_N$ is the unit vector normal to the plane containing the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, in the direction of motion of right-hand screw from $\boldsymbol{A}$ to $\boldsymbol{B}$. When θ_{AB} is zero, the cross product vanishes. So, $\boldsymbol{a}_x \times \boldsymbol{a}_x = 0$, $\boldsymbol{a}_x \times \boldsymbol{a}_y = \boldsymbol{a}_z$, $\boldsymbol{a}_x \times \boldsymbol{a}_z = -\boldsymbol{a}_y$, and so on.

If $\boldsymbol{A} = \boldsymbol{a}_x A_x + \boldsymbol{a}_y A_y + \boldsymbol{a}_z A_z$ and $\boldsymbol{B} = \boldsymbol{a}_x B_x + \boldsymbol{a}_y B_y + \boldsymbol{a}_z B_z$, then the cross product can be determined as,

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Examples using DOT product and cross product

Example 1: Consider a vector, $\boldsymbol{G} = y\,\boldsymbol{a}_x - 2.5\,x\,\boldsymbol{a}_y + 3\,\boldsymbol{a}_z$, and the point $Q(4, 5, 2)$.

- Find $\boldsymbol{G}$ at Q
- The scalar component of $\boldsymbol{G}$ at Q in the direction of $\boldsymbol{a}_N = \frac{1}{3}(2\boldsymbol{a}_x + \boldsymbol{a}_y - 2\boldsymbol{a}_z)$
- The vector component of $\boldsymbol{G}$ at Q in the direction of $\boldsymbol{a}_N$
- The angle between $\boldsymbol{G}$ at Q and $\boldsymbol{a}_N$

Solution: $\boldsymbol{G}$ at Q is: $\boldsymbol{G}(Q) = 5\,\boldsymbol{a}_x - 10\,\boldsymbol{a}_y + 3\,\boldsymbol{a}_z$,

The scalar component of $\boldsymbol{G}$ at Q in the direction of $\boldsymbol{a}_N$ is

$$\boldsymbol{G}(Q) \cdot \boldsymbol{a}_N = (5\,\boldsymbol{a}_x - 10\,\boldsymbol{a}_y + 3\,\boldsymbol{a}_z) \cdot \frac{1}{3}(2\boldsymbol{a}_x + \boldsymbol{a}_y - 2\boldsymbol{a}_z) = \frac{1}{3}(10 - 10 - 6) = -2$$

The vector component of $\boldsymbol{G}$ at Q in the direction of $\boldsymbol{a}_N$

$$[\boldsymbol{G}(Q) \cdot \boldsymbol{a}_N]\,\boldsymbol{a}_N = \frac{-2}{3}(2\boldsymbol{a}_x + \boldsymbol{a}_y - 2\boldsymbol{a}_z)$$

The angle between $\boldsymbol{G}$ at Q and $\boldsymbol{a}_N$ is obtained from $\boldsymbol{G}(\mathrm{Q}) \cdot \boldsymbol{a}_N = |\boldsymbol{G}(\mathrm{Q})| \cos\theta$,

$$\cos\theta = \frac{\boldsymbol{G}(\mathrm{Q}) \cdot \boldsymbol{a}_N}{|\boldsymbol{G}(\mathrm{Q})|}$$

The magnitude of $\boldsymbol{G}$ at Q is: $|\boldsymbol{G}(\mathrm{Q})| = \sqrt{25+100+9} = \sqrt{134}$, $\cos\theta = \frac{-2}{\sqrt{134}}$

and hence $\theta = 99.9^\circ$.

Let us use cross product also to determine the angle.

$$\boldsymbol{G}(\mathrm{Q}) \times \boldsymbol{a}_N = \frac{1}{3}(17\boldsymbol{a}_x + 16\boldsymbol{a}_y + 25\boldsymbol{a}_z)$$

The magnitude is $|\boldsymbol{G}(\mathrm{Q}) \times \boldsymbol{a}_N| = 11.402$,

$$\sin\theta = \frac{|\boldsymbol{G}(\mathrm{Q}) \times \boldsymbol{a}_N|}{|\boldsymbol{G}(\mathrm{Q})|\,|\boldsymbol{a}_N|} = 0.985,$$

The angle $\theta = 80.1^\circ, 99.9^\circ$,

so the correct angle is the common angle, $\theta = 99.9^\circ$.

Examples using DOT product and cross product

Example 2: Find the cross product of two vectors, $\boldsymbol{A} = 2\,\boldsymbol{a}_x - 3\,\boldsymbol{a}_y + \boldsymbol{a}_z$ and $\boldsymbol{B} = -4\,\boldsymbol{a}_x - 2\,\boldsymbol{a}_y + 5\,\boldsymbol{a}_z$ and determine the angle between them.

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 2 & -3 & 1 \\ -4 & -2 & 5 \end{vmatrix} = -13a_x - 14a_y - 16a_z$$

To find the angle between the vectors, the magnitudes of the two vectors, and the magnitude of their resultant cross product, have to be determined. The magnitude of ***A*** is

$$|\boldsymbol{A}| = \sqrt{4+9+1} = \sqrt{14}$$

and that of ***B*** is

$$|\boldsymbol{B}| = \sqrt{16+4+25} = \sqrt{45}\,.$$

The magnitude of ***A*** X ***B*** is

$$|\boldsymbol{A} \times \boldsymbol{B}| = \sqrt{169 + 196 + 256} = \sqrt{621}\,.\ \text{Hence,}$$

$$\sin\theta_{AB} = \frac{|\boldsymbol{A}\times\boldsymbol{B}|}{|\boldsymbol{A}||\boldsymbol{B}|} = \sqrt{\frac{621}{630}} = 0.993, \qquad \text{so,} \qquad \theta_{AB} = 83.13^\circ,\ 96.87^\circ$$

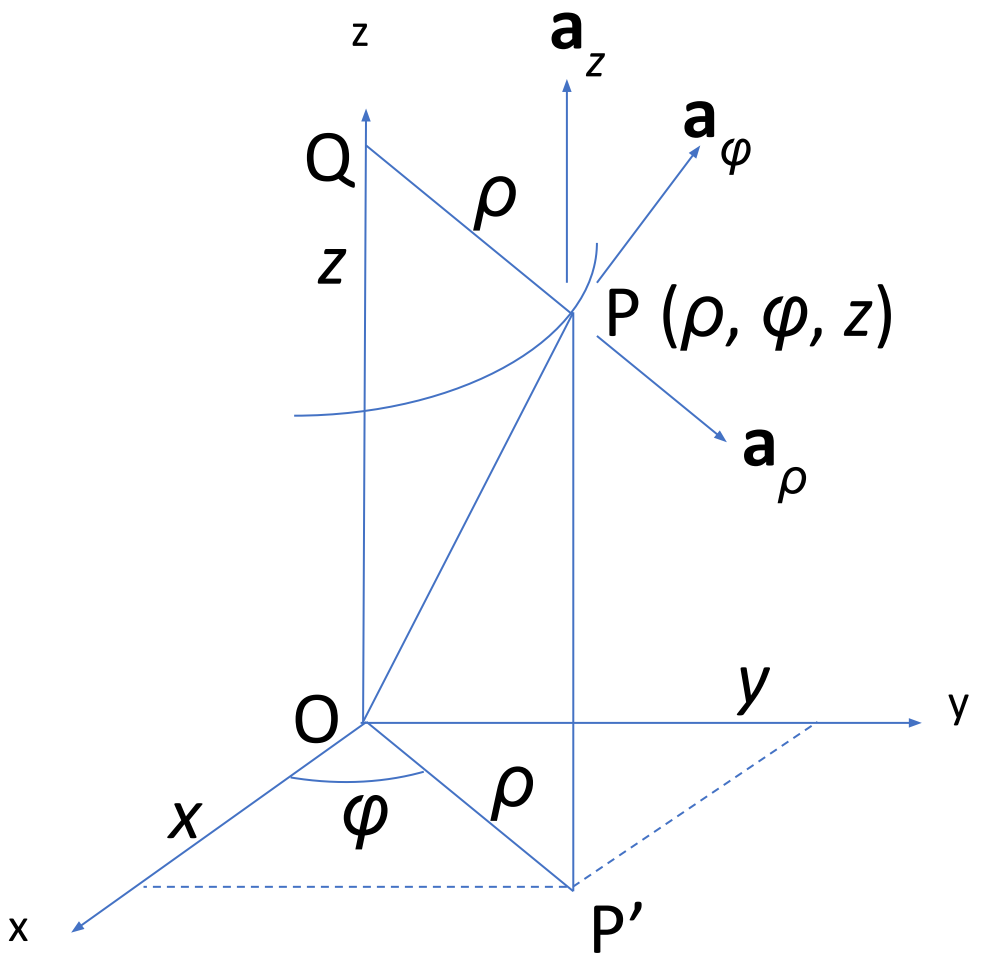
Note that two angles have emerged, and to determine the correct angle, we use the DOT product.

$$\boldsymbol{A} \cdot \boldsymbol{B} = (2\boldsymbol{a}_x - 3\boldsymbol{a}_y + \boldsymbol{a}_z) \cdot (-4\boldsymbol{a}_x - 2\boldsymbol{a}_y + 5\boldsymbol{a}_z) = 3$$

$$\theta_{AB} = \cos^{-1}\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{|\boldsymbol{A}||\boldsymbol{B}|} = \cos^{-1} 0.1195 = 83.13°$$

Hence the common angle is, 83.13°, which is the correct angle.

Circular cylindrical coordinate system



In circular cylindrical coordinate system, the point P in space is defined by ρ, φ, and z.

ρ is the perpendicular distance of point P from z-axis (QP); φ is the angle between x-axis and the line segment OP' formed by projection of P on xy-plane; z is same as in Cartesian system.

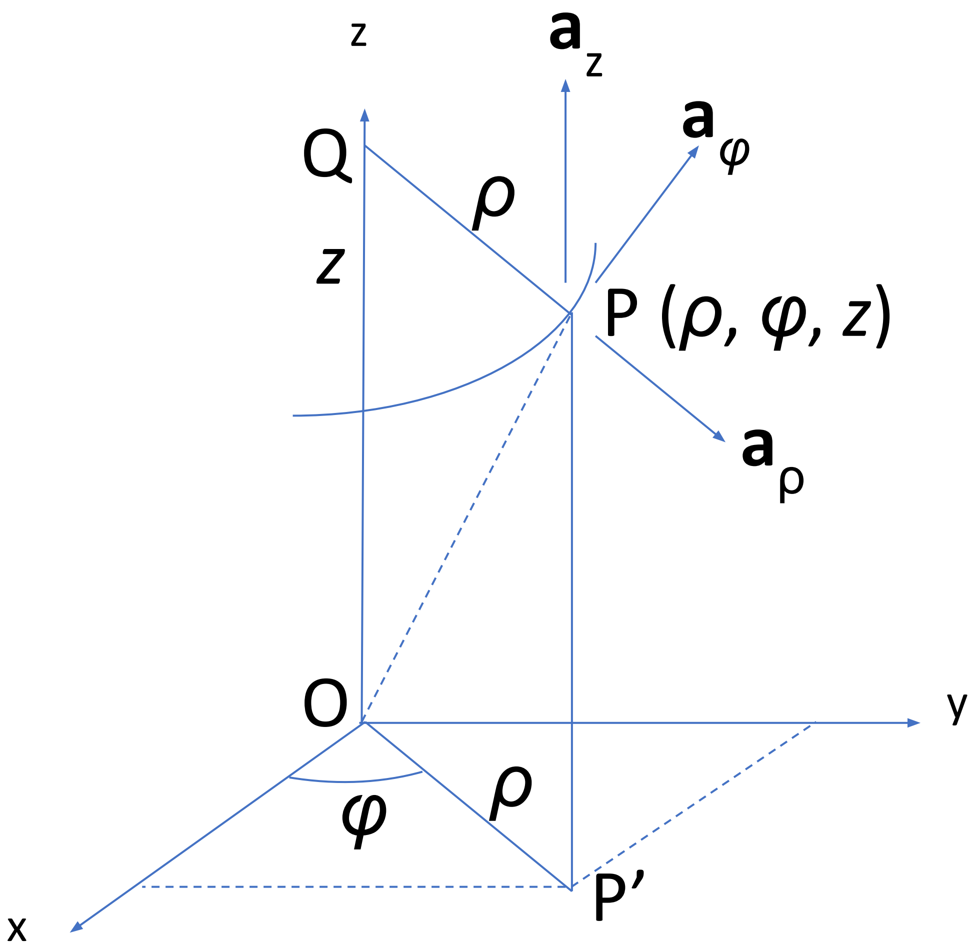
Transformation from cylindrical coordinates to Cartesian coordinates, and vice versa:

From the figure note that

$$x = \rho \cos(\varphi) \quad , \quad y = \rho \sin(\varphi)$$

$$\rho = \sqrt{x^2 + y^2} \quad , \varphi = tan^{-1}(y/x)$$

Circular cylindrical coordinate system



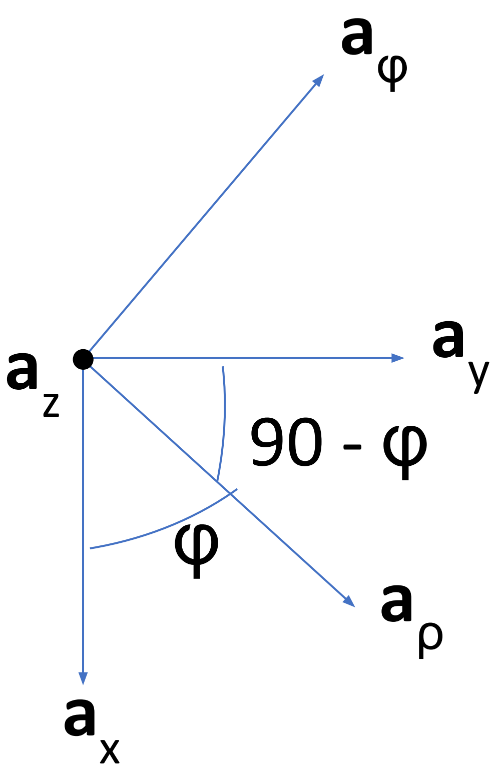
The Unit vectors in cylindrical coordinate system are: $\mathbf{a}_\rho$, $\mathbf{a}_\varphi$, $\mathbf{a}_z$ and they are mutually perpendicular, and their cross products are as below.

$\mathbf{a}_\rho \times \mathbf{a}_\varphi = \mathbf{a}_z$, $\mathbf{a}_\varphi \times \mathbf{a}_z = \mathbf{a}_\rho$, $\mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\varphi$

Note that $\mathbf{a}_\rho$ is in the direction of outward normal to the constant- ρ cylindrical surface (normal to z-axis, see figure);

$\mathbf{a}_\varphi$ is normal to the constant-φ plane (OQPP' in the figure), towards increasing φ angle.

Circular cylindrical coordinate system



Note that + z-axis is <u>coming out</u> from the screen

DOT products between unit vectors of cylindrical and Cartesian systems

$$\mathbf{a}_\rho \cdot \mathbf{a}_x = |\mathbf{a}_\rho||\mathbf{a}_x|\cos(\varphi) = \cos(\varphi)$$

$$\mathbf{a}_\rho \cdot \mathbf{a}_y = |\mathbf{a}_\rho||\mathbf{a}_y|\cos(90 - \varphi) = \sin(\varphi)$$

$$\mathbf{a}_\rho \cdot \mathbf{a}_z = |\mathbf{a}_\rho||\mathbf{a}_z|\cos(90) = 0$$

$$\mathbf{a}_\varphi \cdot \mathbf{a}_x = |\mathbf{a}_\varphi||\mathbf{a}_x|\cos(90 + \varphi) = -\sin(\varphi)$$

$$\mathbf{a}_\varphi \cdot \mathbf{a}_y = |\mathbf{a}_\varphi||\mathbf{a}_y|\cos(\varphi) = \cos(\varphi)$$

$$\mathbf{a}_\varphi \cdot \mathbf{a}_z = |\mathbf{a}_\varphi||\mathbf{a}_z|\cos(90) = 0$$

Circular cylindrical coordinate system

Example: Transform the vector $\boldsymbol{B} = y\boldsymbol{a}_x - x\boldsymbol{a}_y + z\boldsymbol{a_z}$ into cylindrical coordinates.

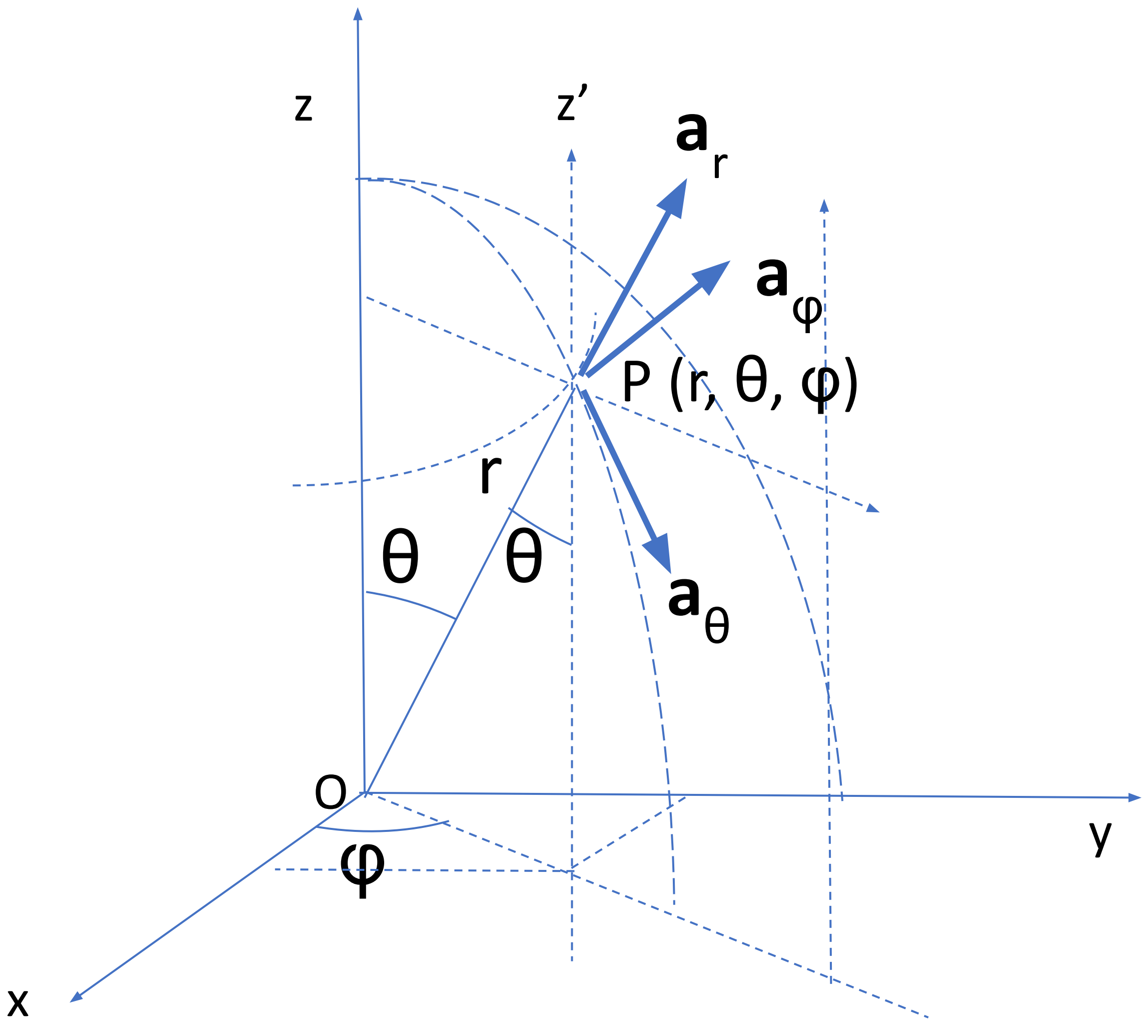
Solution: In cylindrical coordinates, $\boldsymbol{B} = B_\rho\boldsymbol{a}_\rho + B_\varphi\boldsymbol{a}_\varphi + B_z\boldsymbol{a}_z$, where B_ρ , B_φ, and B_z are scalar components of ***B*** in ρ, φ, and z directions, which are to be determined.

$$B_\rho = \boldsymbol{B} \cdot \boldsymbol{a}_\rho = \left(y\boldsymbol{a}_x - x\boldsymbol{a}_y + z\boldsymbol{a_z}\right) \cdot \boldsymbol{a}_\rho = y\left(\boldsymbol{a}_x \cdot a_\rho\right) - x\left(\boldsymbol{a}_y \cdot \boldsymbol{a}_\rho\right)$$
$$= y\cos\varphi - x\sin\varphi = \rho\sin\varphi\ \cos\varphi - \rho\cos\varphi\ \sin\varphi = 0$$

$$B_\varphi = \boldsymbol{B} \cdot \boldsymbol{a}_\varphi = \left(y\boldsymbol{a}_x - x\boldsymbol{a}_y + z\boldsymbol{a_z}\right) \cdot \boldsymbol{a}_\varphi = y\left(\boldsymbol{a}_x \cdot \boldsymbol{a}_\varphi\right) - x\left(\boldsymbol{a}_y \cdot \boldsymbol{a}_\varphi\right)$$
$$= \ -y\sin\varphi - x\cos\varphi = -\rho sin^2\varphi - \rho cos^2\varphi = -\rho$$

Hence ***B*** in cylindrical coordinates is: $\boldsymbol{B} = -\rho\boldsymbol{a}_\varphi + z\boldsymbol{a}_z$

Spherical coordinate system

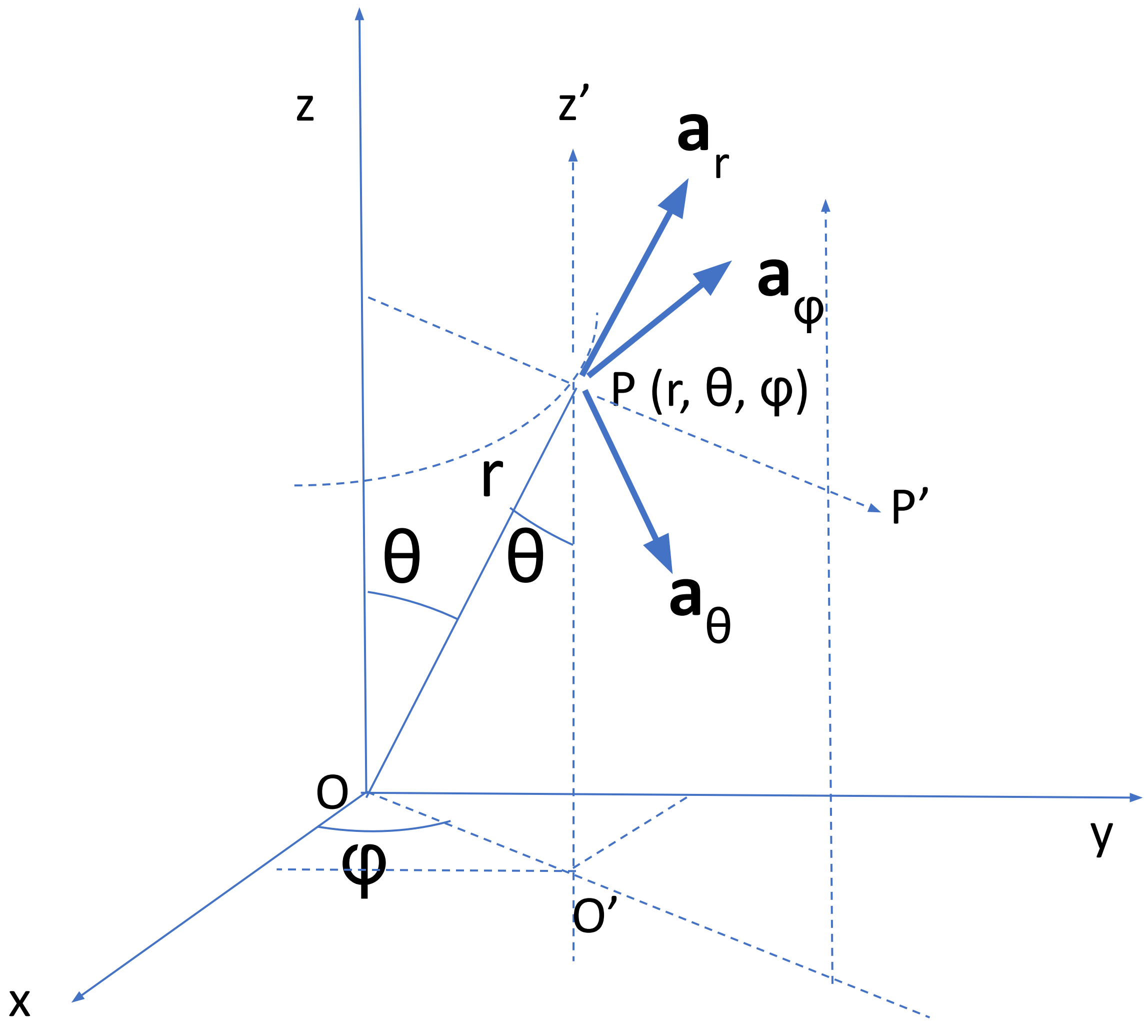


In spherical coordinates, the point P in space is defined by the coordinates, r, θ, and ϕ.

The distance OP from origin is r. The angle made by the line OP with z-axis is θ. The angle made by the projection of OP on xy-plane with x-axis is ϕ.

Note that the unit vectors, $\boldsymbol{a}_r$ $\boldsymbol{a}_\theta$ and $\mathbf{a}_\phi$ are mutually perpendicular.

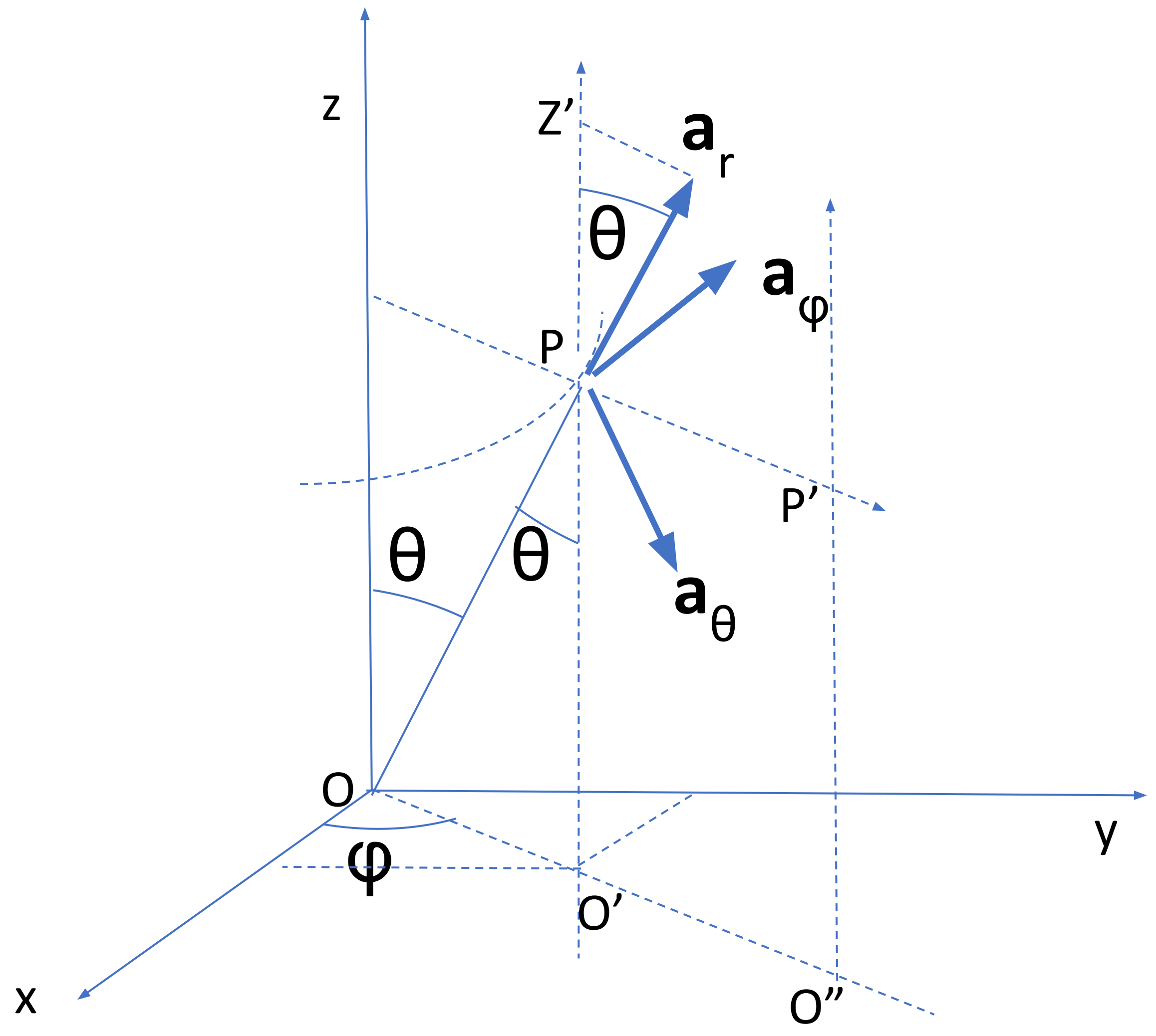
Spherical coordinate system



The direction of $\boldsymbol{a}_r$ is the direction of position vector $\boldsymbol{r}$. The direction of $\boldsymbol{a}_\theta$ is that of perpendicular to OP in the increasing side of θ. The direction of $\boldsymbol{a}_\varphi$ is that of normal to the plane formed by z-axis and OP. Notice that both $\boldsymbol{a}_r$ and $\boldsymbol{a}_\theta$ are in this plane (and they are mutually perpendicular).

In order to transform a vector given in spherical coordinates to vector in cartesian coordinates, one should know the DOT products between unit vectors in one system to that in the other system.

Spherical coordinate system



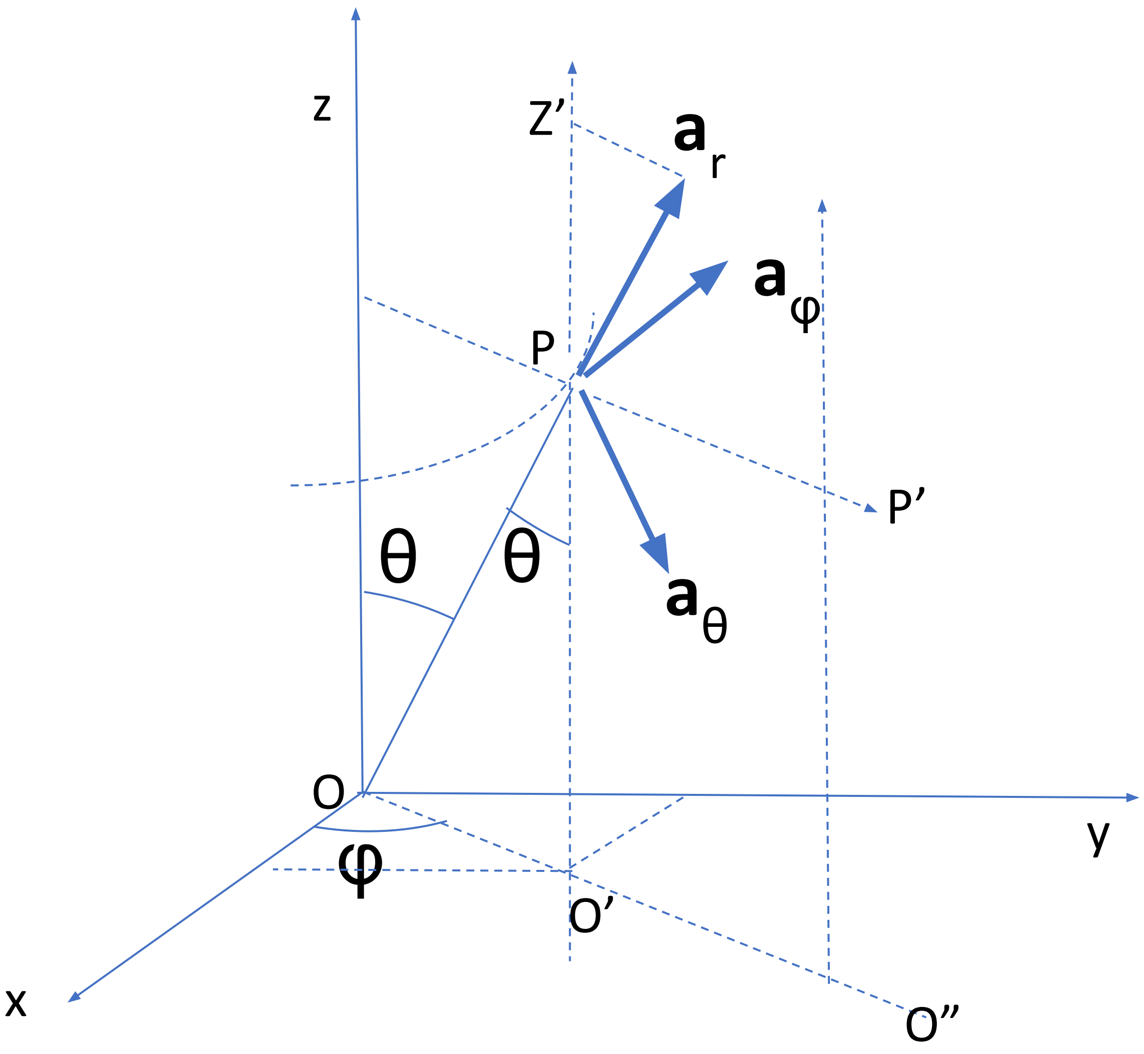
DOT product, $\boldsymbol{a}_r \cdot \boldsymbol{a}_x$: Consider the plane formed by z-axis and the line P'O". Unit vectors $\boldsymbol{a}_r$, $\boldsymbol{a}_\theta$ are in this plane, while $\boldsymbol{a}_\varphi$ is perpendicular to the plane.

Notice that angle between $\boldsymbol{a}_r$ and PZ' is θ, so angle between $\boldsymbol{a}_r$ and PP' is 90 – θ (since PZ' and PP' are perpendicular). Hence projection of $\boldsymbol{a}_r$ on PP' is cos(90 – θ), which is sin(θ).

Since PP' is parallel to OO", projection of $\boldsymbol{a}_r$ on OO" is also sin(θ). As OO" makes an angle ϕ with x-axis, further projection on x-axis will be sin(θ) cos(ϕ), so,

$$\boldsymbol{a}_r \cdot \boldsymbol{a}_x = \sin(\theta)\cos(\phi)$$

Spherical coordinate system



DOT products, $\boldsymbol{a}_r \cdot \boldsymbol{a}_y$; $\boldsymbol{a}_r \cdot \boldsymbol{a}_z$

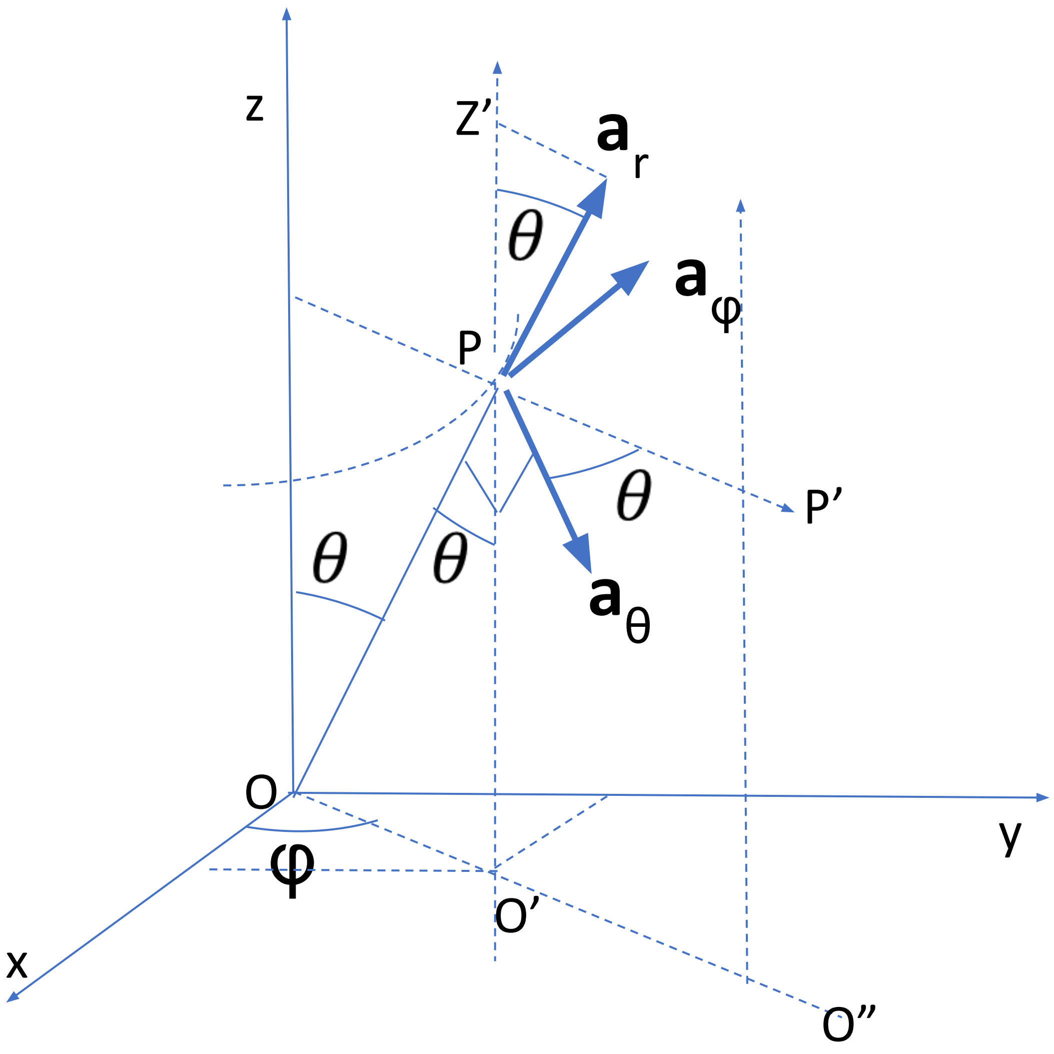
We know from the previous slide that projection of $\boldsymbol{a}_r$ on OO" is sin(θ). As OO" makes an angle 90 – φ with y-axis, further projection on y-axis will be sin(θ) cos(90 – φ). Thus,

$$\boldsymbol{a}_r \cdot \boldsymbol{a}_y = \sin(\theta)\sin(\varphi)$$

Since angle between $\boldsymbol{a}_r$ and $\boldsymbol{a}_z$ is θ, we get,

$$\boldsymbol{a}_r \cdot \boldsymbol{a}_z = \cos(\theta)$$

Spherical coordinate system



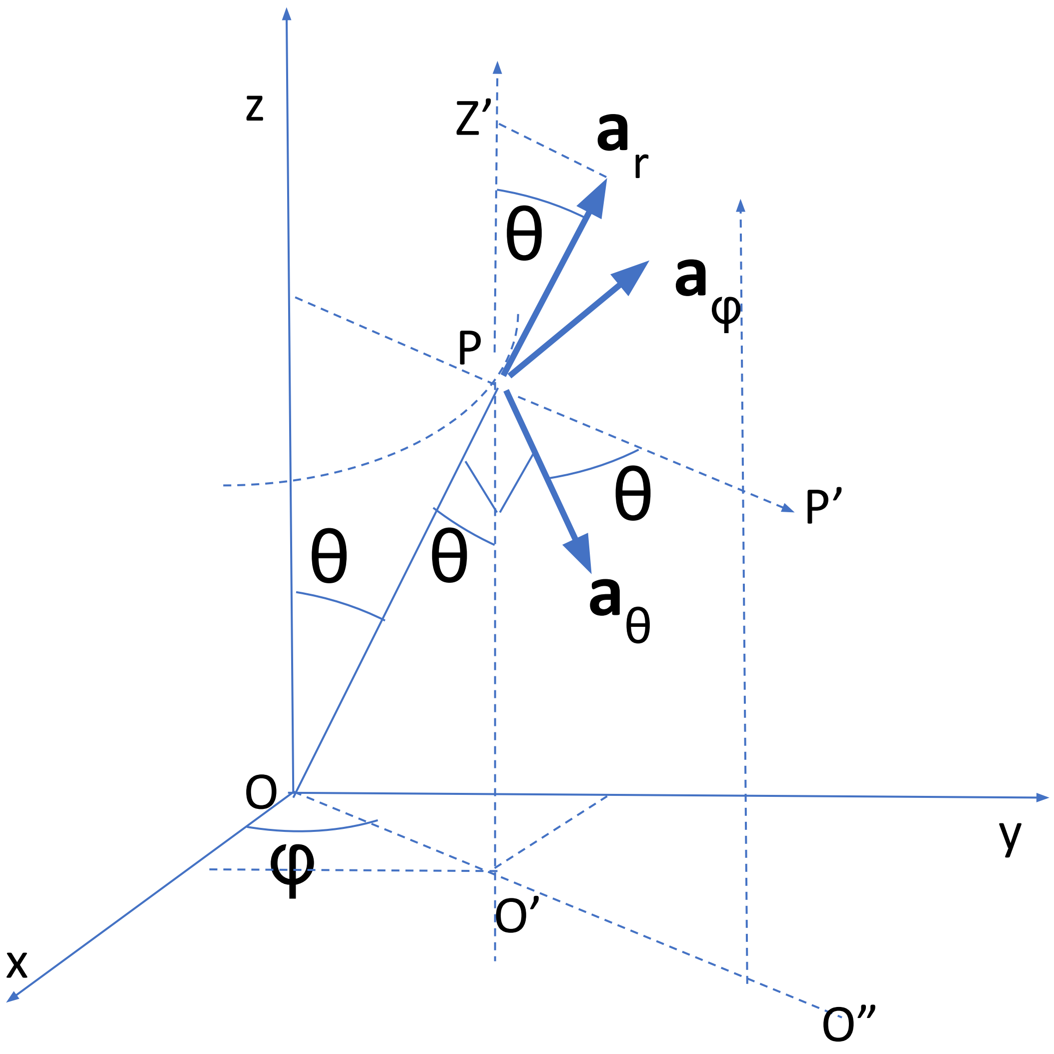
DOT product, $\boldsymbol{a}_\theta \cdot \boldsymbol{a}_x$

The angle between $\boldsymbol{a}_r$ and $\boldsymbol{a}_\theta$ (so OP and $\boldsymbol{a}_\theta$) is 90˚; so the angle between $\boldsymbol{a}_\theta$ and PO' is 90 – θ. Since PO' and PP' are perpendicular, the angle between $\mathbf{a}_\theta$ and PP' is θ. Hence projection of $\mathbf{a}_\theta$ on PP' is cos(θ).

Since PP' is parallel to OO", projection of $\boldsymbol{a}_\theta$ on OO" is also cos(θ). As OO" makes an angle ϕ with x-axis, further projection on x-axis will be cos(θ) cos(ϕ), Thus,

$$\boldsymbol{a}_\theta \cdot \boldsymbol{a}_x = \cos(\theta)\cos(\phi)$$

Spherical coordinate system



DOT products, $\boldsymbol{a}_\theta \cdot \boldsymbol{a}_y$ *and* $\boldsymbol{a}_\theta \cdot \boldsymbol{a}_z$

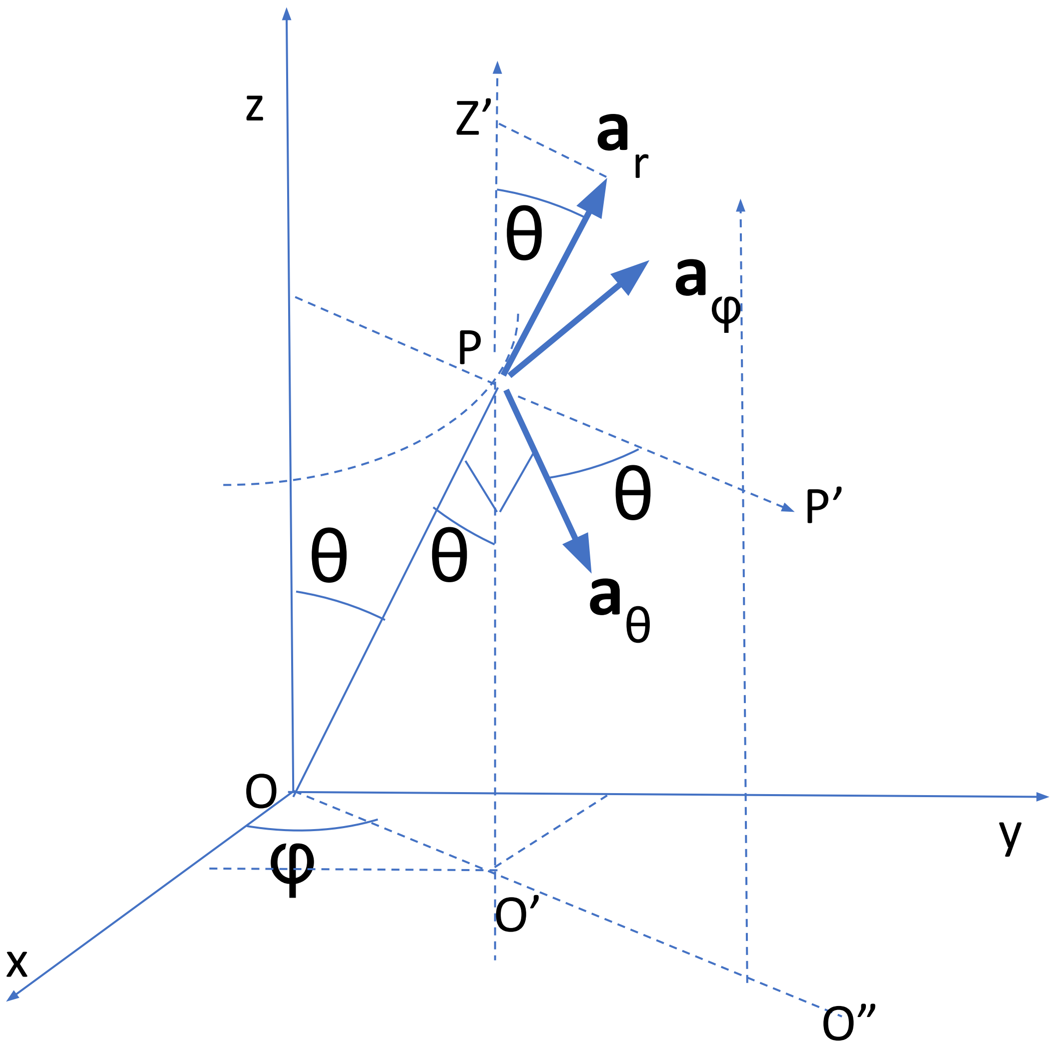
From the earlier discussion, we know that projection of $\boldsymbol{a}_\theta$ on OO" is cos(θ). As OO" makes an angle 90 – ϕ with y-axis, further projection on y-axis will be cos(θ) cos(90 – ϕ), Thus,

$$\boldsymbol{a}_\theta \cdot \boldsymbol{a}_y = \cos(\theta)\sin(\phi)$$

The angle between $\mathbf{a}_\theta$ and $\mathbf{a}_z$ is 90 + θ; hence

$$\boldsymbol{a}_\theta \cdot \boldsymbol{a}_z = \cos(90 + \theta) = -\sin(\theta)$$

Spherical coordinate system



DOT products, $\boldsymbol{a}_\phi \cdot \boldsymbol{a}_x$; $\boldsymbol{a}_\phi \cdot \boldsymbol{a}_y$; $\boldsymbol{a}_\phi \cdot \boldsymbol{a}_z$

We know that $\boldsymbol{a}_\phi$ is perpendicular to the plane formed by z-axis the line OP. This plane makes angle ϕ with x-axis. So the angle between $\boldsymbol{a}_\phi$ and $\boldsymbol{a}_x$ is 90 + ϕ. Therefore,

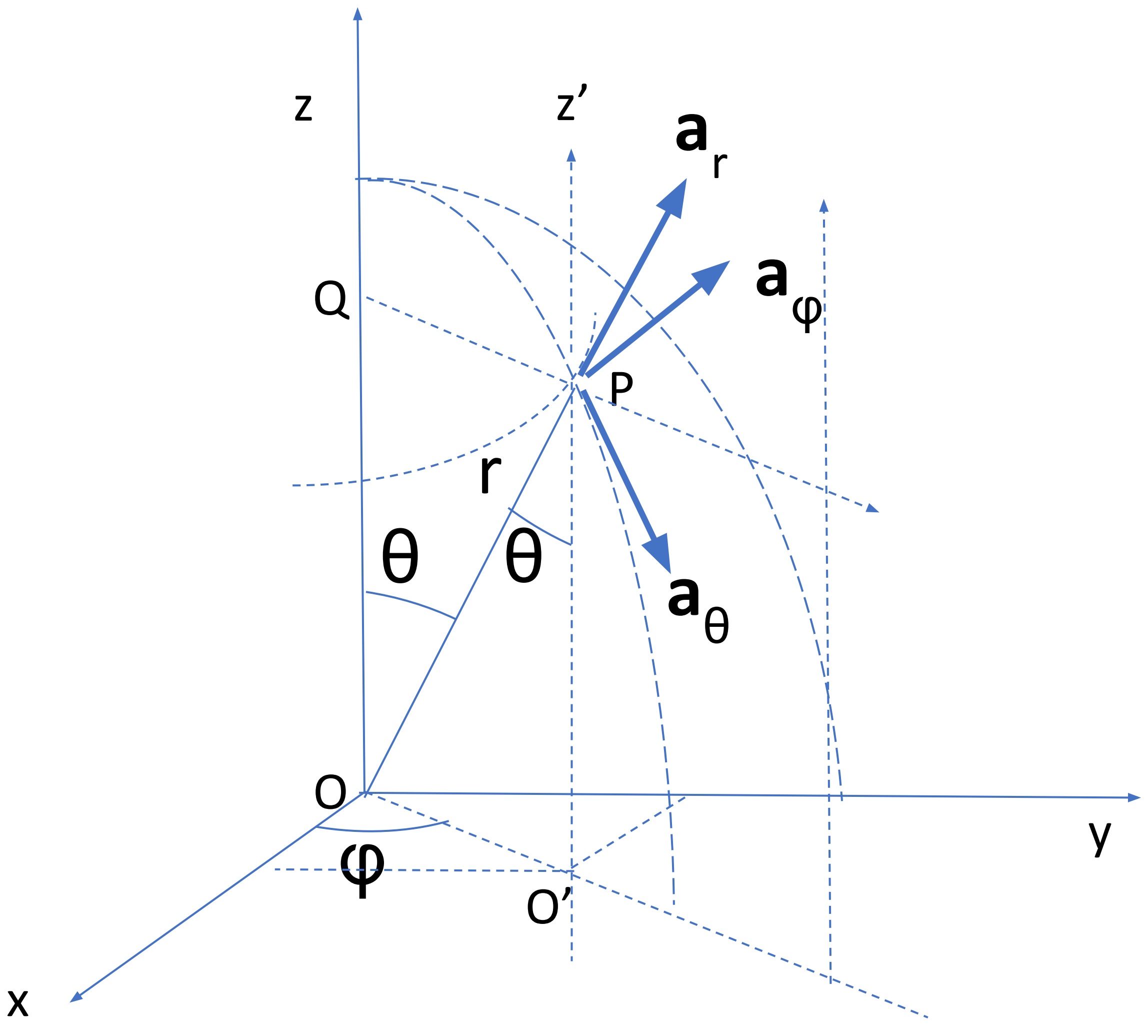
$$\boldsymbol{a}_\phi \cdot \boldsymbol{a}_x = \cos(90+\phi) = -\sin(\phi)$$

Since the angle between $\boldsymbol{a}_\phi$ and $\boldsymbol{a}_x$ is 90 + ϕ, the angle between $\boldsymbol{a}_\phi$ and $\boldsymbol{a}_y$ is ϕ. Therefore,

$$\boldsymbol{a}_\phi \cdot \boldsymbol{a}_y = \cos(\phi)$$

Since $\boldsymbol{a}_\phi$ is perpendicular to $\boldsymbol{a}_z$ the DOT product, $\boldsymbol{a}_\phi \cdot \mathbf{a}_z = 0$.

Spherical coordinate system



Scalar transformation of coordinates:

Note that OO' = QP = r sin(θ), and x = OO' cos(ϕ), hence

$$x = r \sin(\theta) \cos(\phi)$$

Similarly, y = OO' sin(ϕ), hence

$$y = r \sin(\theta) \sin(\phi),$$

Note that z = OQ = r cos(θ). Using these relations, the reverse transformations are,

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \cos^{-1} \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\varphi = \tan^{-1} \frac{y}{x}$$

Example: Express the vector, $\boldsymbol{G} = \left(\frac{xz}{y}\right)\boldsymbol{a}_x$ in components of spherical coordinates.

Solution: First we express $\boldsymbol{G} = G_r\boldsymbol{a}_r + G_\theta\boldsymbol{a}_\theta + G_\varphi\boldsymbol{a}_\varphi$, where

$G_r = \boldsymbol{G} \cdot \boldsymbol{a}_r$, $G_\theta = \boldsymbol{G} \cdot \boldsymbol{a}_\theta$, $G_\emptyset = \boldsymbol{G} \cdot \boldsymbol{a}_\emptyset$

$$G_r = \left(\frac{xz}{y}\right)\boldsymbol{a}_x \cdot \boldsymbol{a}_r = \left(\frac{xz}{y}\right)\sin\theta\ \cos\varphi$$

Using coordinate transformations, $x = r\ \sin\theta\cos\emptyset$, $y = r\ \sin\theta\sin\emptyset$, $z = r\cos\theta$,

$$G_r = r\ \sin\theta\ \cos\theta\ \frac{cos^2\emptyset}{\sin\emptyset}$$

$$G_\theta = \left(\frac{xz}{y}\right)\boldsymbol{a}_x \cdot \boldsymbol{a}_\theta = \left(\frac{xz}{y}\right)\cos\theta\ \cos\emptyset = r\ cos^2\theta\frac{cos^2\emptyset}{\sin\emptyset}$$

$$G_\emptyset = \left(\frac{xz}{y}\right)\boldsymbol{a}_x \cdot \boldsymbol{a}_\emptyset = \left(\frac{xz}{y}\right)(-\sin\emptyset) = -r\ \cos\theta\ \cos\emptyset$$

Introducing the concept of the 'Del' operation

Consider a scalar function, $f(x, y, z)$, in space. Let an incremental change in f be df in an incremental length, dl, where the vector, $\boldsymbol{dl} = \boldsymbol{a}_x\, dx + \boldsymbol{a}_y\, dy + \boldsymbol{a}_z\, dz$; note that df is defined as,

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz$$

$$df = \left(\boldsymbol{a}_x \frac{\partial f}{\partial x} + \boldsymbol{a}_y \frac{\partial f}{\partial y} + \boldsymbol{a}_z \frac{\partial f}{\partial z}\right) \cdot (\boldsymbol{a}_x\, dx + \boldsymbol{a}_y\, dy + \boldsymbol{a}_z\, dz)$$

$$= \left(\boldsymbol{a}_x \frac{\partial}{\partial x} + \boldsymbol{a}_y \frac{\partial}{\partial y} + \boldsymbol{a}_z \frac{\partial}{\partial z}\right) f \cdot (\boldsymbol{a}_x\, dx + \boldsymbol{a}_y\, dy + \boldsymbol{a}_z\, dz)$$

Let us denote

$$\boldsymbol{\nabla} = \boldsymbol{a}_x \frac{\partial}{\partial x} + \boldsymbol{a}_y \frac{\partial}{\partial y} + \boldsymbol{a}_z \frac{\partial}{\partial z}$$

which is known as Del operator.

Hence *df* becomes,

$$df = \boldsymbol{\nabla} f \cdot \boldsymbol{dl} = |\boldsymbol{\nabla} f||\boldsymbol{dl}|\cos(\theta)$$

Notice that when ***dl*** aligns with $\boldsymbol{\nabla} f$ (means θ is zero) the change *df* is maximum. Hence $\boldsymbol{\nabla} f$ indicates the direction of maximum change in the scalar function, *f(x, y, z)*.

Gradient of scalar function, $\boldsymbol{\nabla} f$ *:*

$\boldsymbol{\nabla} f$ is known as gradient of function *f*, or Grad (*f*), and its direction indicates the direction of maximum change in *f*. The direction normal to $\boldsymbol{\nabla} f$ is the direction of zero change in *f*, or the contour of constant *f*.
Thus $\boldsymbol{\nabla} f = 0$ yields the surface of constant *f*.

Divergence of vector function:

The divergence of vector at a point, $\boldsymbol{\nabla} \cdot \boldsymbol{D}$, is the net outflow of that vector from an infinitesimal volume about that point.

Divergence Theorem:

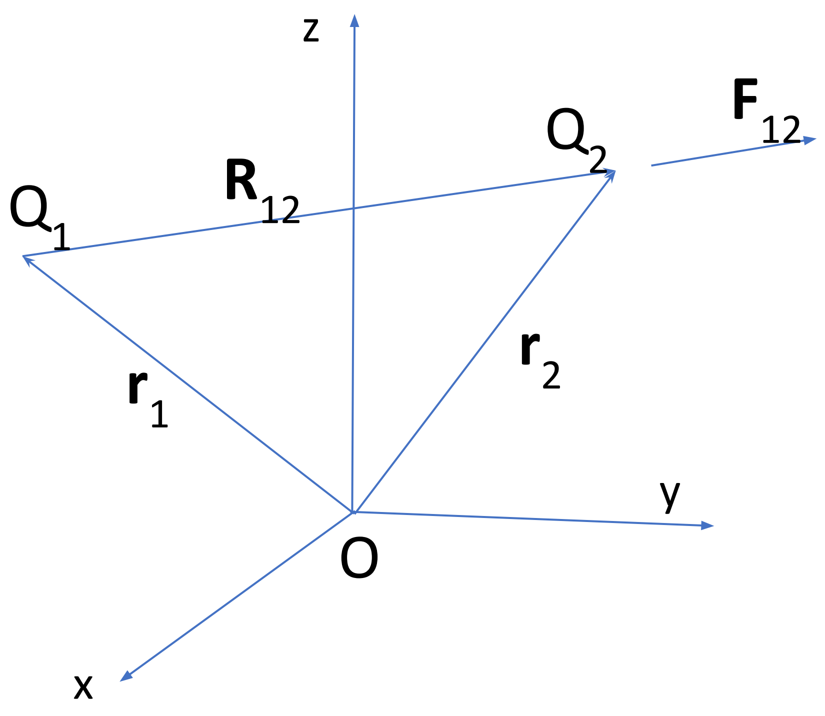
$$\int_{Vol} \boldsymbol{\nabla} \cdot \boldsymbol{D}\, dv = \oint_{S} \boldsymbol{D} \cdot \boldsymbol{ds}$$

Curl of a vector:

The curl of a vector, ***A***, is given by $\boldsymbol{\nabla} \times \boldsymbol{A}$ and in cartesian coordinates it is,

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Electrostatics



The electric force exerted by charge Q_1 on charge Q_2 is:

$$\boldsymbol{F}_{12} = \frac{Q_1 Q_2}{4\pi\epsilon\, {R_{12}}^2}\boldsymbol{a}_{12} = \frac{Q_1 Q_2}{4\pi\epsilon\, {R_{12}}^3}\boldsymbol{R}_{12},$$

See the Figure on the left:

$\mathbf{r}_1$: position vector of Q_1 point

$\mathbf{r}_2$: position vector of Q_2 point

$\mathbf{R}_{12}$: distance vector from Q_1 to Q_2

ϵ : permittivity of the medium. For free space,

$$\epsilon = \epsilon_0 = \frac{10^{-9}}{36\pi} \quad F/m$$

$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1$; $\mathbf{a}_{12} = \mathbf{R}_{12} / R_{12}$

where $\mathbf{a}_{12}$ is unit vector in the direction of $\mathbf{R}_{12}$ and R_{12} is the magnitude of $\mathbf{R}_{12}$.

Electrostatics

The electric force exerted by the charges, Q_1 , Q_2 , Q_3 , . . . on the charge Q_A , is given by,

$$\boldsymbol{F}_A = \boldsymbol{F}_{1A} + \boldsymbol{F}_{2A} + \boldsymbol{F}_{3A} + \cdots = \frac{Q_1 Q_A}{4\pi\epsilon {R_{1A}}^2}\boldsymbol{a}_{1A} + \frac{Q_2 Q_A}{4\pi\epsilon {R_{2A}}^2}\boldsymbol{a}_{2A} + \frac{Q_3 Q_A}{4\pi\epsilon {R_{3A}}^2}\boldsymbol{a}_{3A} + \ldots$$

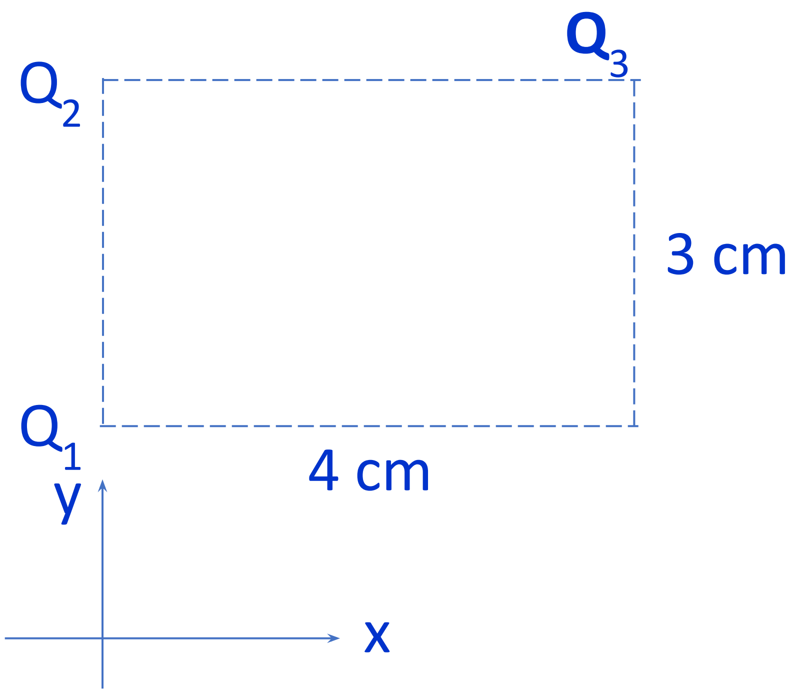
where $$\boldsymbol{F}_{1A} = \frac{Q_1 Q_A}{4\pi\epsilon {R_{1A}}^2}\boldsymbol{a}_{1A} \ ,$$

$$\boldsymbol{R}_{1A} = \mathbf{r}_A - \mathbf{r}_1 \ ,$$

$$\mathbf{a}_{1A} = \boldsymbol{R}_{1A} / R_{1A} \text{ , and so on.}$$

where $\mathbf{a}_{1A}$ is unit vector in the direction of $\boldsymbol{R}_{1A}$ and R_{1A} is the magnitude of $\boldsymbol{R}_{1A}$.

Electrostatics



Example: Three point-charges are located at the corners of a rectangle in free space as shown in the figure. The charges are: Q_1 = 3 μC, Q_2 = – 2 μC, and Q_3 = 5 μC . Find the net force on Q_3.

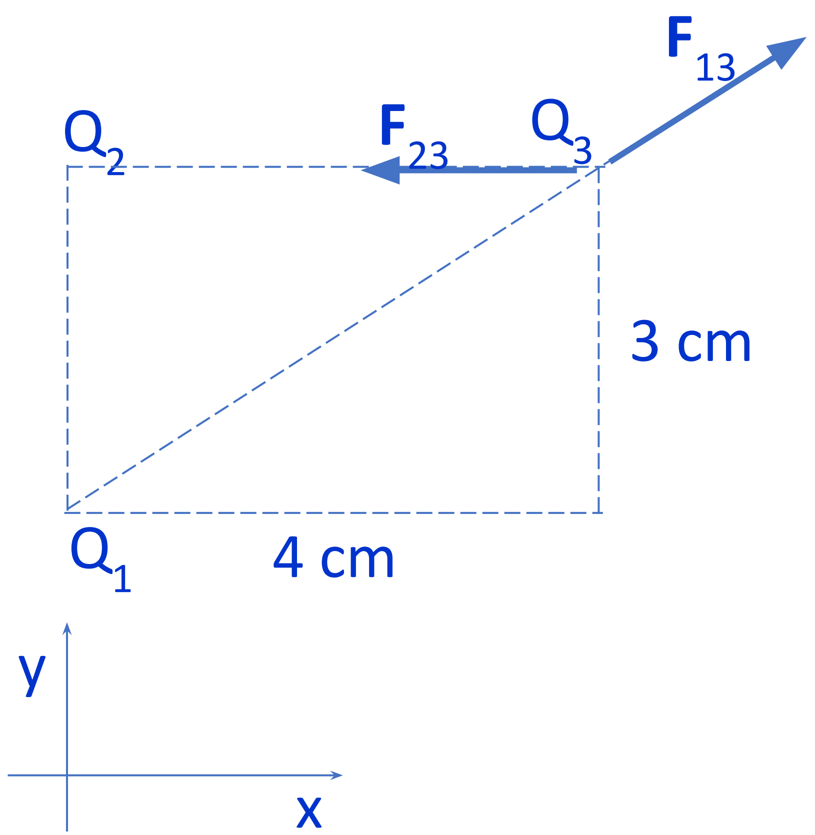
Solution: Let Q1 be at origin. So, $\mathbf{r}_1 = 0$, $\mathbf{r}_2 = 0.03\,\mathbf{a}_y$ and $\mathbf{r}_3 = 0.04\,\mathbf{a}_x + 0.03\,\mathbf{a}_y$. Force exerted by Q_1 on Q_3 is

$$\boldsymbol{F}_{13} = \frac{Q_1 Q_3}{4\pi\epsilon 0\, R_{13}{}^3}\boldsymbol{R}_{13},$$ where

$\mathbf{R}_{13} = \mathbf{r}_3 - \mathbf{r}_1 = 0.04\,\mathbf{a}_x + 0.03\,\mathbf{a}_y$, and $R_{13} = 0.05$ m. Since for free space, $\epsilon_0 = 10^{-9}/36\pi$, we obtain,

$\mathbf{F}_{13} = 43.2\,\mathbf{a}_x + 32.4\,\mathbf{a}_y$ Newtons.

Electrostatics



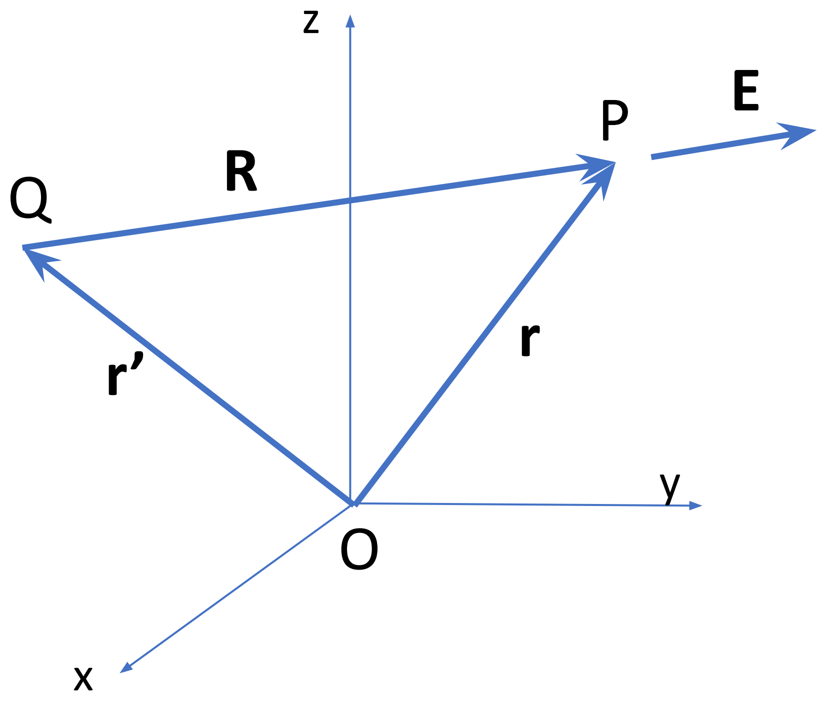
Similarly, $\mathbf{R}_{23} = \mathbf{r}_3 - \mathbf{r}_2 = 0.04\,\mathbf{a}_x$, and $R_{23} = 0.04$ m.
Force exerted by Q_2 on Q_3 is

$$\boldsymbol{F}_{23} = \frac{Q_2 Q_3}{4\pi\epsilon_0 R_{23}{}^3}\boldsymbol{R}_{23},$$

Thus, $\mathbf{F}_{23} = -56.25\,\mathbf{a}_x$ Newtons. Hence the net force on Q_3 is:

$$\mathbf{F}_3 = \mathbf{F}_{13} + \mathbf{F}_{23} = -13.05\,\mathbf{a}_x + 32.4\,\mathbf{a}_y \text{ Newtons.}$$

Electrostatics



Electric field: The electric field intensity is the force exerted by the charge Q on a unit positive charge.

$$\boldsymbol{E} = \frac{Q}{4\pi \in |\boldsymbol{r}-\boldsymbol{r}'|^2}\,\boldsymbol{a}_R = \frac{Q(\boldsymbol{r}-\boldsymbol{r}')}{4\pi \in |\boldsymbol{r}-\boldsymbol{r}'|^3}$$

r : position vector of the field point, P

r' : position vector of the point, where charge Q is placed

R : distance vector from the charge Q to the field point, and ***R*** = ***r*** − ***r'***

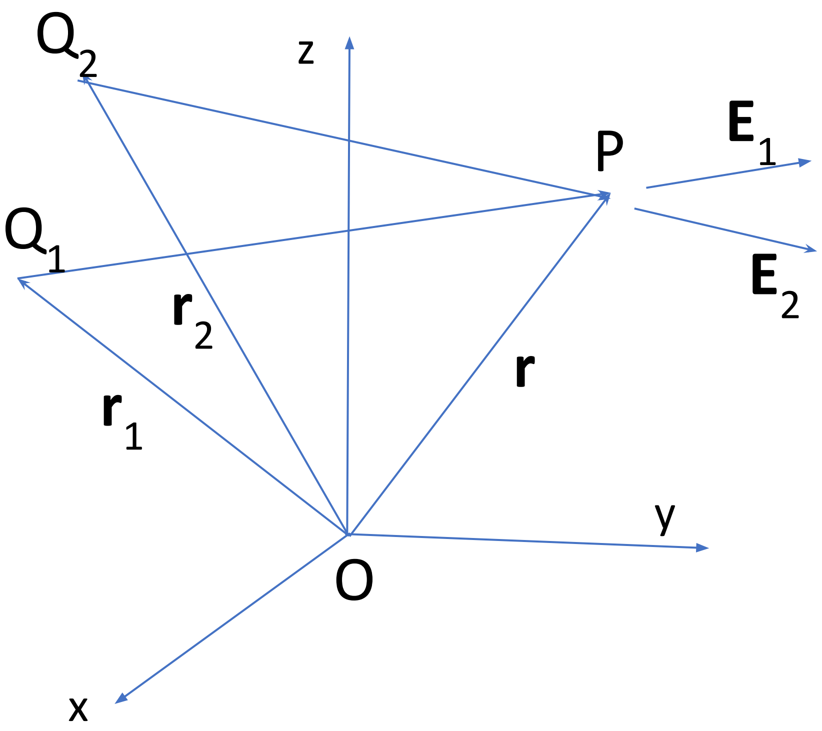
$\boldsymbol{a}_R$: Unit vector in the direction of ***R***

∈ : permittivity of the medium

If the charge is placed at origin, ***r'*** = 0, and ***R*** = ***r***, then ***E*** is:

$$\boldsymbol{E} = \frac{Q}{4\pi \in |\boldsymbol{r}|^2}\,\boldsymbol{a}_R = \frac{Q\boldsymbol{r}}{4\pi \in |\boldsymbol{r}|^3}$$

Electrostatics



If there are more point charges, the net electric field is the vector sum of electric fields due to individual charges.

$$\boldsymbol{E} = \frac{1}{4\pi \in} \sum_{k=1}^{N} \frac{Q_k(\boldsymbol{r} - \boldsymbol{r}_k)}{|\boldsymbol{r} - \boldsymbol{r}_k|^3}$$

r : position vector of the field point

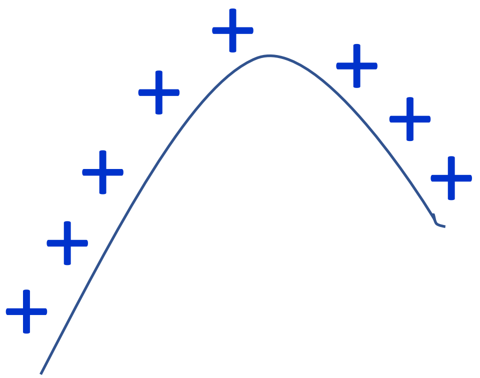
$\mathbf{r}_k$: position vector of the point, where charge Q_k is placed

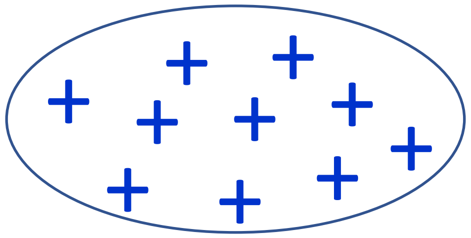
$\in$: permittivity of the medium

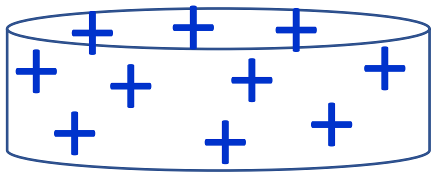
$\mathbf{E}_1$: Field due to charge Q_1

$\mathbf{E}_2$: Field due to charge Q_2

Electrostatics







If there is a charge density of ρ_L C / m over a length of a wire, then the charge in an infinitesimal length *dl* is: $dQ = \rho_L\, dl$, and total charge is:

$$Q = \int \rho_L \, dl$$

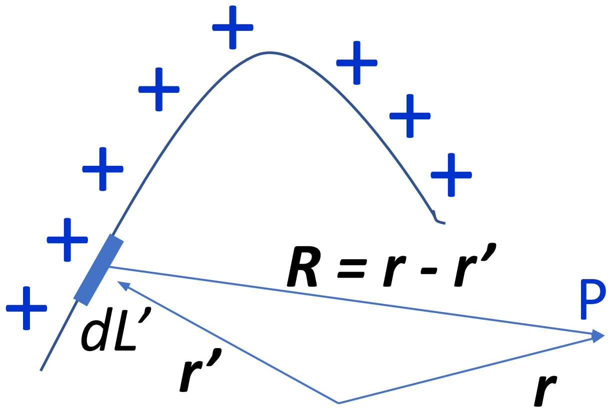
If a charge density of ρ_s C / m² is defined over a surface, then the charge in an infinitesimal surface area *ds* is: $dQ = \rho_s\, ds$, and total charge is:

$$Q = \iint \rho_s \, ds$$

If a charge density of ρ_v C / m³ is defined in a volume, then the charge in an infinitesimal volume *dv* is: $dQ = \rho_v\, dv$, and total charge is:

$$Q = \iiint \rho_v \, dv$$

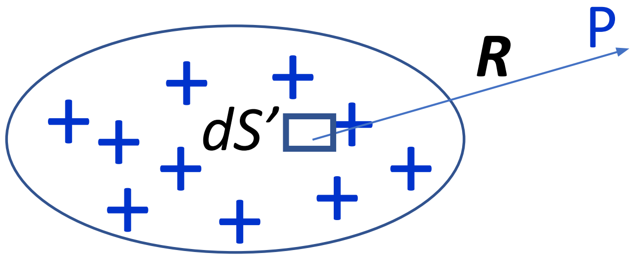
Electrostatics



For a line charge, the electric field is:

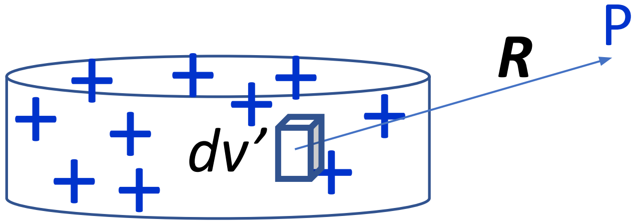
$$\boldsymbol{E} = \frac{1}{4\pi\epsilon}\int \frac{\rho_L \boldsymbol{R}\, dL'}{R^3}$$

where **R** = **r** – **r'** is the distance vector from the elemental charge, $\rho_L\, dL'$, to the field point, and R is magnitude of **R**.



In case of surface charge, the electric field is:

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon}\int \frac{\rho_S \boldsymbol{R}\, dS'}{R^3}$$

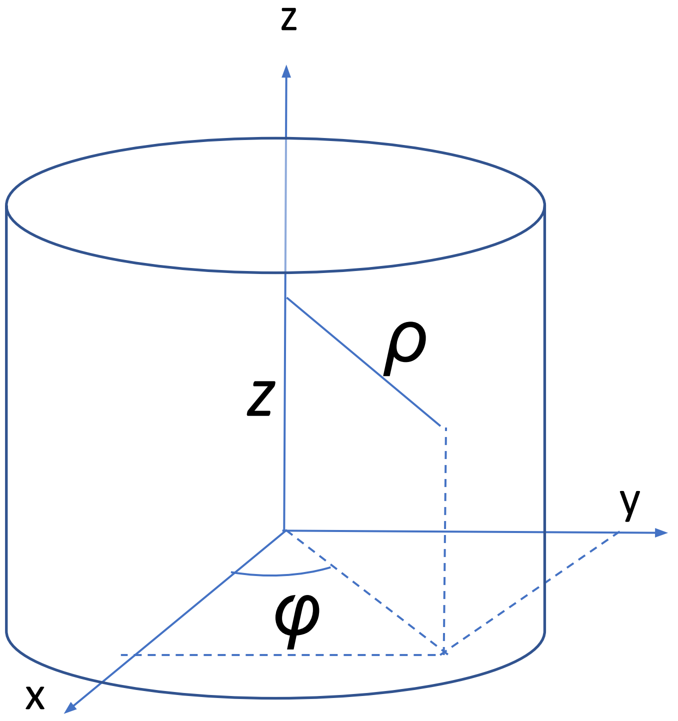


For a volume charge, the electric field is:

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon}\int \frac{\rho_v \boldsymbol{R}\, dv'}{R^3}$$

Electrostatics

Example: Find the total charge in the cylinder of height 30 cm and radius 10 cm, if the charge density inside the cylinder is given by: $\rho_v = 100e^{-z}(x^2 + y^2)^{-0.25}$ C/m³



Solution: The problem can be solved easily in cylindrical coordinates. Hence the charge density is expressed as:

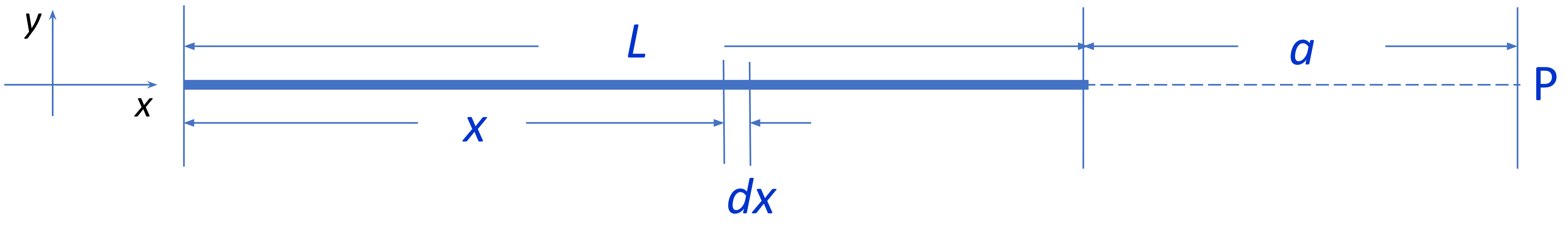
$$\rho_v = 100e^{-z}(x^2 + y^2)^{-0.25} = 100e^{-z}\,\rho^{-0.5}$$

since $\rho^2 = x^2 + y^2$, total charge enclosed in the cylinder is: $Q = \iiint \rho_v\, dv$, where $dv = \rho\, d\rho d\varphi dz$, the infinitesimal volume in cylindrical coordinates. So,

$Q = \iiint \rho_v\, \rho\, d\rho d\varphi dz = 100 \iiint e^{-z}\, \rho^{+0.5}\, d\rho d\varphi dz$,

The limits of integration are: $\rho = 0$ to 0.1 m, $\varphi = 0$ to 2π rad, and $z = 0$ to 0.3 m. Integrating, we get $Q = 3.43$ C.

Example: A charge of Q is uniformly distributed over a thin wire of length L. Determine the field at the point P shown in the figure.



Solution: Let ρ_L C / m be the uniform charge density on the wire. Hence charge in the infinitesimal length dx will be $(\rho_L \; dx)$ C. Infinitesimal field produced by this charge is,

$$\boldsymbol{dE} = \frac{\rho_L \, dx \, \boldsymbol{R}}{4\pi\epsilon \, R^3}$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}' = (L + a)\, \mathbf{a}_x - x\, \mathbf{a}_x = (L + a - x)\, \mathbf{a}_x$, and $R = (L + a - x)$. Hence,

$$\boldsymbol{dE} = \frac{\rho_L \, dx}{4\pi\epsilon \, (L + a - x)^2} \, \mathbf{a}_x$$

Electrostatics

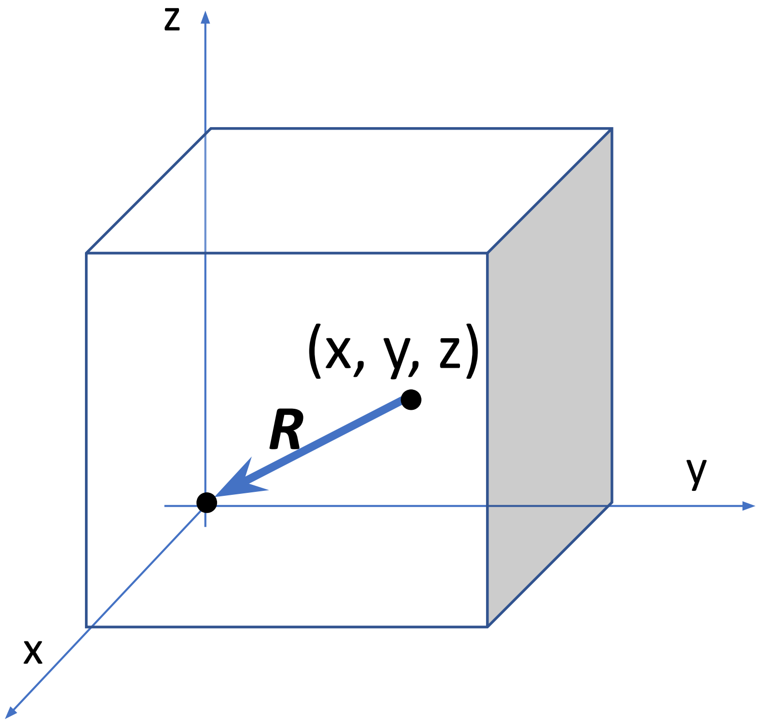
The net electric field due to total charge on the wire can be obtained by integrating elemental field as below.

$$\boldsymbol{E} = \ \mathbf{a}_x \ \frac{\rho_L}{4\pi\epsilon} \int_0^L \frac{dx}{(L+a-x)^2} = \mathbf{a}_x \ \frac{\rho_L}{4\pi\epsilon} \left[\frac{1}{L+a-x}\right]_0^L$$

$$\boldsymbol{E} = \mathbf{a}_x \ \frac{\rho_L}{4\pi\epsilon} \left(\frac{1}{a} - \frac{1}{L+a}\right) = \mathbf{a}_x \ \frac{\rho_L \, L}{4\pi\epsilon \, a(L+a)}$$

Electrostatics

Example: A continuous volume charge, $\rho_v = (x^2 + y^2 + z^2)^{5/2}$, is distributed in the region, $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, and is zero elsewhere. Find the electric field at the origin (see figure).



Solution: Let an elemental volume dv be at point (x, y, z) within the cube as shown. Notice that field point is at origin.

So, $\boldsymbol{r} = 0$, and $\boldsymbol{r}' = x\,\mathbf{a}_x + y\,\mathbf{a}_y + z\,\mathbf{a}_z$, hence

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}' = -(x\,\mathbf{a}_x + y\,\mathbf{a}_y + z\,\mathbf{a}_z),\text{ and}$$

$$R = (x^2 + y^2 + z^2)^{1/2}$$

For a volume charge, the electric field is:

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon}\int \frac{\rho_v \boldsymbol{R}\,dv}{R^3}$$

Electrostatics

$$\boldsymbol{E} = \frac{-1}{4\pi\epsilon} \int_0^1\int_0^1\int_0^1 (x^2 + y^2 + z^2)\ (x\boldsymbol{a}_x + y\boldsymbol{a}_y + z\boldsymbol{a}_z)\ dxdydz$$

Let us determine x component of electric field, E_x .

$$E_x = \frac{-1}{4\pi\epsilon} \int_0^1\int_0^1\int_0^1 [(x^3 + xy^2 + xz^2)\ dx]\ dydz$$

Integrating with respect to *x* first and applying limits,

$$E_x = \frac{-1}{4\pi\epsilon} \int_0^1\int_0^1 \left[\frac{1}{4} + \frac{y^2}{2} + \frac{z^2}{2}\right] dy\, dz$$

Integrating with respect to *y* and then *z* (and applying limits),

$$E_x = \frac{-1}{4\pi\epsilon} \int_0^1 \left(\frac{5}{12} + \frac{z^2}{2}\right) dz = \frac{-1}{4\pi\epsilon}\left(\frac{7}{12}\right) = -5.25 \times 10^9\ V/m$$

Due to symmetry, E_y and E_z components also have same magnitude as E_x.

Electrostatics

Gauss Law: Gauss law states that the total flux, ψ, passing through any closed surface is equal to the total charge, Q, enclosed by the surface.

$$\psi = Q = \int_{Vol} \rho_v \, dv$$

If ***D*** is the flux density (or displacement density) at a point on a surface, then the flux passing through an elemental surface ds is,

$$d\psi = \boldsymbol{D} \cdot \boldsymbol{ds}$$

Total flux passing through closed surface is then expressed as,

$$\psi = \oint_S \boldsymbol{D} \cdot \boldsymbol{ds}$$

$$\oint_S \boldsymbol{D} \cdot \boldsymbol{ds} = Q = \int \rho_v \, dv$$

Electrostatics

Divergence Theorem: From divergence theorem, we know that

$$\oint_{\boldsymbol{S}} \boldsymbol{D} \cdot \boldsymbol{ds} = \int_{Vol} \boldsymbol{\nabla} \cdot \boldsymbol{D}\, dv$$

Therefore,

$$\int_{Vol} \boldsymbol{\nabla} \cdot \boldsymbol{D}\, dv = \int_{Vol} \rho_v\, dv = Q$$

Hence, $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_v$

Divergence of displacement density at a point is equal to the volume charge density at that point.

Electrostatics

Work: The force experienced by a charge Q in an electric field $\boldsymbol{E}$ is $Q\boldsymbol{E}$, and the work done against this force in moving the charge a small distance $\boldsymbol{dl}$ is: $dW = -Q\boldsymbol{E} \cdot \boldsymbol{dl}$

Hence total work done in moving the charge from point B to point A against the field is:

$$W = -Q \int_B^A \boldsymbol{E} \cdot \boldsymbol{dl}$$

Potential difference: The work done against the electric field in moving a unit charge from point B to point A is termed as potential difference between A and B.

$$V_{AB} = W/Q = -\int_B^A \boldsymbol{E} \cdot \boldsymbol{dl}$$

Traversing through any path, if final point A coincides with initial point B, then there is no potential difference. Hence,

$$\oint \boldsymbol{E} \cdot \boldsymbol{dl} = 0$$

Electrostatics

Potential: The work done against the electric field in moving a unit charge from infinity to point *A* is termed as potential at point *A*.

$$V_A = -\int_{\infty}^{A} \boldsymbol{E} \cdot \boldsymbol{dl}, \qquad \text{similarly} \qquad V_B = -\int_{\infty}^{B} \boldsymbol{E} \cdot \boldsymbol{dl}$$

The potential difference between *A* and *B* is, by definition,

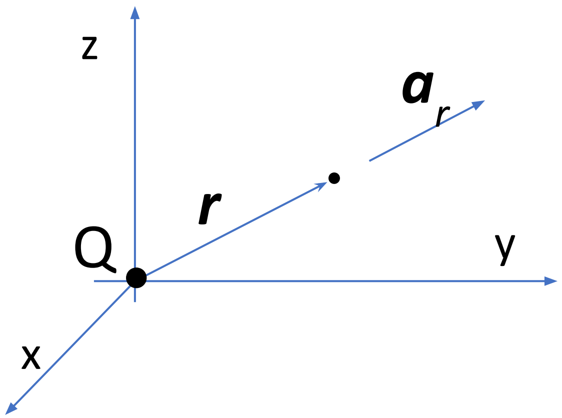
$$V_{AB} = -\int_{B}^{A} \boldsymbol{E} \cdot \boldsymbol{dl}$$

However, using the definition of potentials, it can be expressed as,

$$V_{AB} = V_A - V_B = -\int_{\infty}^{A} \boldsymbol{E} \cdot \boldsymbol{dl} - \left(-\int_{\infty}^{B} \boldsymbol{E} \cdot \boldsymbol{dl}\right) = \int_{\infty}^{B} \boldsymbol{E} \cdot \boldsymbol{dl} - \int_{\infty}^{A} \boldsymbol{E} \cdot \boldsymbol{dl}$$

Hence,

$$V_{AB} = V_A - V_B = \int_{A}^{B} \boldsymbol{E} \cdot \boldsymbol{dl} = -\int_{B}^{A} \boldsymbol{E} \cdot \boldsymbol{dl}$$



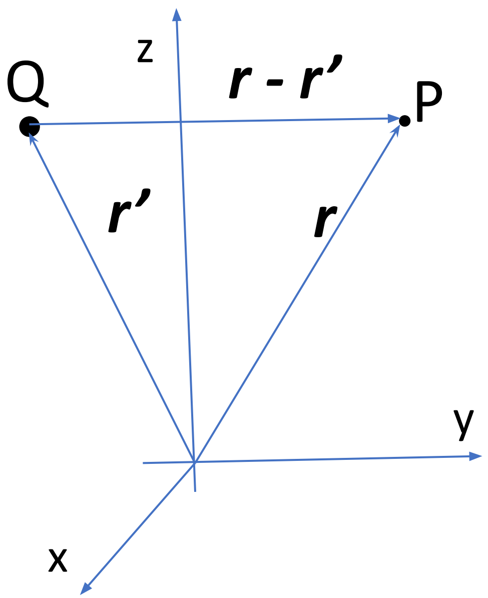
Potential due to a point charge at origin: Consider a point charge Q placed at origin.
The electric field at a distance r from the origin is

$$\boldsymbol{E} = \frac{Q}{4\pi\epsilon r^2}\boldsymbol{a}_r$$

The incremental distance in radial direction is, $\boldsymbol{dl} = dr\,\boldsymbol{a}_r$, since there is no change in θ or ϕ, as the unit charge is moved from ∞ to point A in radial direction; hence the potential is,

$$V = -\int_{\infty}^{A} \boldsymbol{E}\cdot\boldsymbol{dl} = \frac{-Q}{4\pi\epsilon}\int_{\infty}^{r}\frac{1}{r^2}\,dr = \frac{Q}{4\pi\epsilon r}$$

Electrostatics



*Potential due to a point charge in space at **r'***:

The potential at point P is,

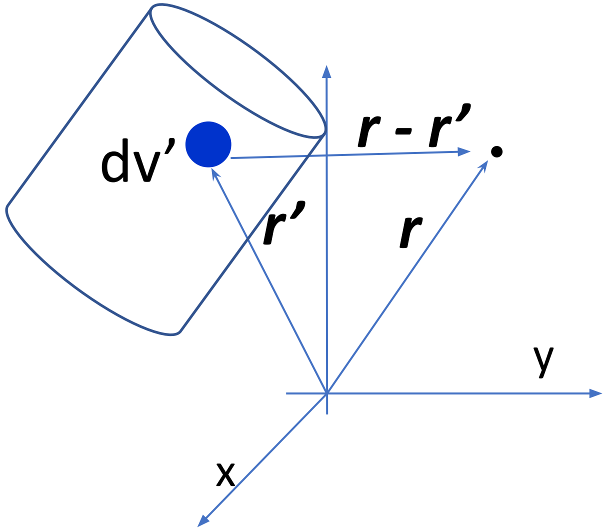
$$V = \frac{Q}{4\pi\epsilon|\boldsymbol{r} - \boldsymbol{r}'|}$$

Potential due to several point charges in space:

The potential at point P is,

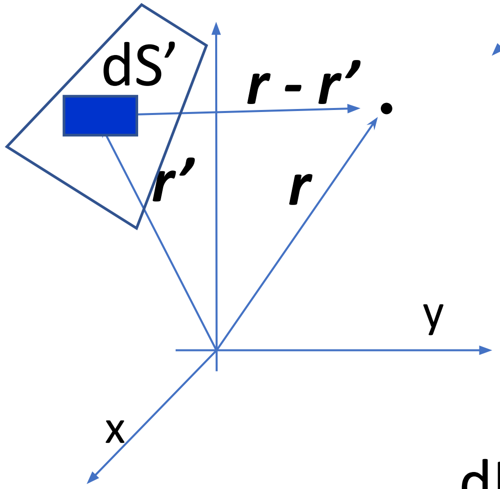
$$V = \frac{1}{4\pi\epsilon}\sum_{k=1}^{N}\frac{Q_k}{|\boldsymbol{r} - \boldsymbol{r}_k|}$$

Electrostatics



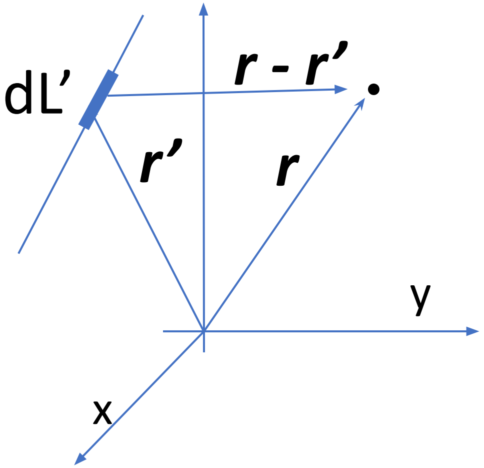
Potential due to continuous volume charge distribution:

$$V = \frac{1}{4\pi\epsilon}\int_{Vol}\frac{\rho_v\, dv'}{|\boldsymbol{r}-\boldsymbol{r}'|}$$



Potential due to continuous surface charge distribution:

$$V = \frac{1}{4\pi\epsilon}\int_{S}\frac{\rho_s\, dS'}{|\boldsymbol{r}-\boldsymbol{r}'|}$$



Potential due to continuous line charge distribution:

$$V = \frac{1}{4\pi\epsilon}\int_{L}\frac{\rho_L\, dL'}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

Electrostatics

Electric field in terms of potential : The incremental work in moving a unit charge for an incremental distance is called incremental potential.

$$dV = -\boldsymbol{E} \cdot \boldsymbol{dl}$$

However, *dV* can be expressed as,

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

$$= \left(\boldsymbol{a}_x \frac{\partial}{\partial x} + \boldsymbol{a}_y \frac{\partial}{\partial y} + \boldsymbol{a}_z \frac{\partial}{\partial z} \right) V \cdot (\boldsymbol{a}_x\, dx + \boldsymbol{a}_y\, dy + \boldsymbol{a}_z\, dz)$$

Thus,

$$dV = \boldsymbol{\nabla} V \cdot \boldsymbol{dl} = -\boldsymbol{E} \cdot \boldsymbol{dl}$$

which yields, $\boldsymbol{E} = -\boldsymbol{\nabla} V$, performing curl operation, $\boldsymbol{\nabla} \times \boldsymbol{E} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} V = 0$

Since Curl of a Gradient is zero (a vector identity)

Electrostatics

Maxwell's equations for electrostatic case:

$$\nabla \times \boldsymbol{E} = 0, \qquad \nabla \cdot \boldsymbol{D} = \rho_v$$

Energy stored in the electric field: The energy stored in the electric field is:

$$W_E = \frac{1}{2} \int_{Vol} \rho_v \, V \, dv = \frac{1}{2} \int_{Vol} \epsilon_0 \, E^2 \, dv$$

From the above expression, we note the quantity inside the volume integral has the dimension of density, so we define an energy density as:

$$w_E = \frac{1}{2} \epsilon_0 \, E^2$$

Electrostatics

Example: A potential field, $V = 2x^2y - 5z$, exists in free space. Determine potential, electric field, direction of electric field, displacement density, charge density, and energy density, at point *P*(– 4, 3, 6).

Solution: The potential at P is $V_P = 2(-4)^2(3) - 5(6) = 66$ V. The electric field in space is,

$$\boldsymbol{E} = -\boldsymbol{\nabla} V = -\left(4xy\,\boldsymbol{a}_x + 2x^2\boldsymbol{a}_y - 5\boldsymbol{a}_z\right) = -4xy\,\boldsymbol{a}_x - 2x^2\boldsymbol{a}_y + 5\boldsymbol{a}_z$$

The field at *P* is

$$\boldsymbol{E}_P = 48\,\boldsymbol{a}_x - 32\,\boldsymbol{a}_y + 5\boldsymbol{a}_z \ \text{V/m},$$

The magnitude of $\boldsymbol{E}_P$ is $|\boldsymbol{E}_P| = \sqrt{48^2 + (-32)^2 + 5^2} = 57.9$ V/m

The direction of $\boldsymbol{E}_P$ is given by the unit vector,

$$\boldsymbol{a}_{E,P} = \frac{48\,\boldsymbol{a}_x - 32\,\boldsymbol{a}_y + 5\boldsymbol{a}_z}{57.9} = 0.829\,\boldsymbol{a}_x - 0.553\,\boldsymbol{a}_y + 0.086\,\boldsymbol{a}_z$$

The displacement density is,

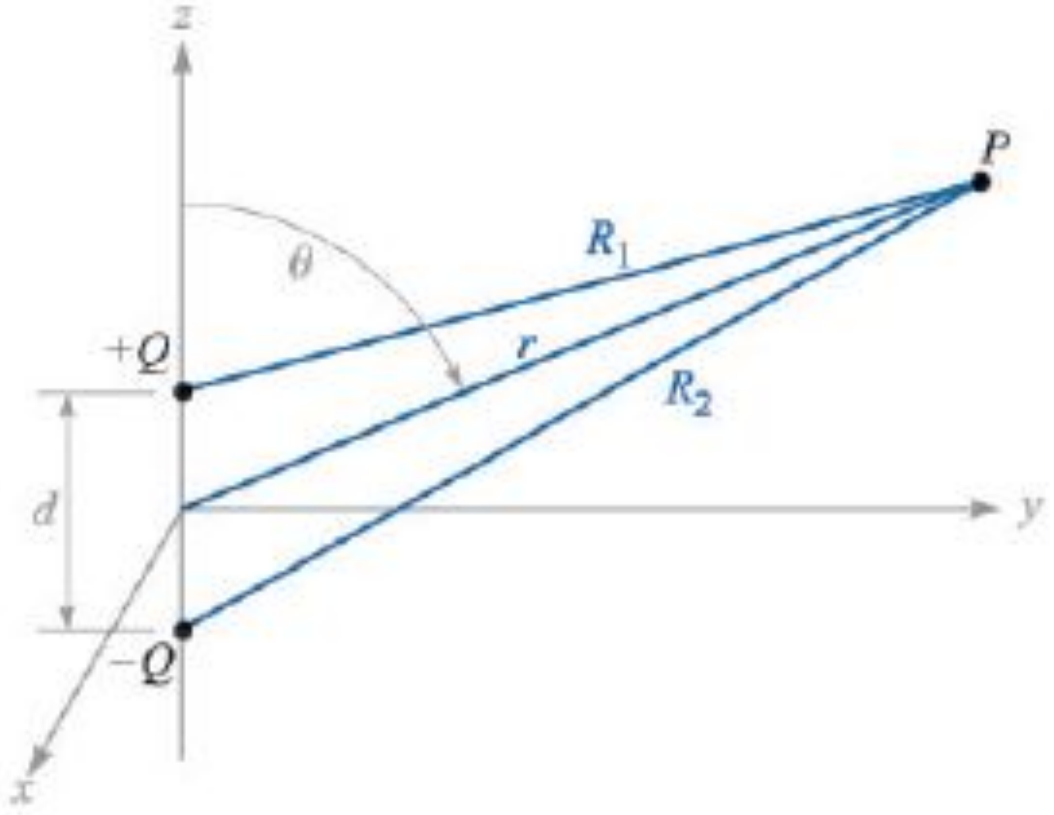
$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} = \frac{10^{-9}}{36\pi}\left(-4xy\,\boldsymbol{a}_x - 2x^2\boldsymbol{a}_y + 5\boldsymbol{a}_z\right)$$

$$\boldsymbol{D} = (-35.4xy\,\boldsymbol{a}_x - 17.71x^2\boldsymbol{a}_y + 44.3\boldsymbol{a}_z) \times 10^{-12} \quad \text{C/m}^3$$

The charge density is, $\rho_v = \boldsymbol{\nabla} \cdot \boldsymbol{D} = -35.4 \times 10^{-12}\, y \quad \text{C/m}^3$

The energy density is, $w_E = \frac{1}{2}\epsilon_0 E^2 = \frac{10^{-9}}{72\pi}(16x^2y^2 + 4x^4 + 25) \quad \text{J/m}^3$

Potential and field due to dipole

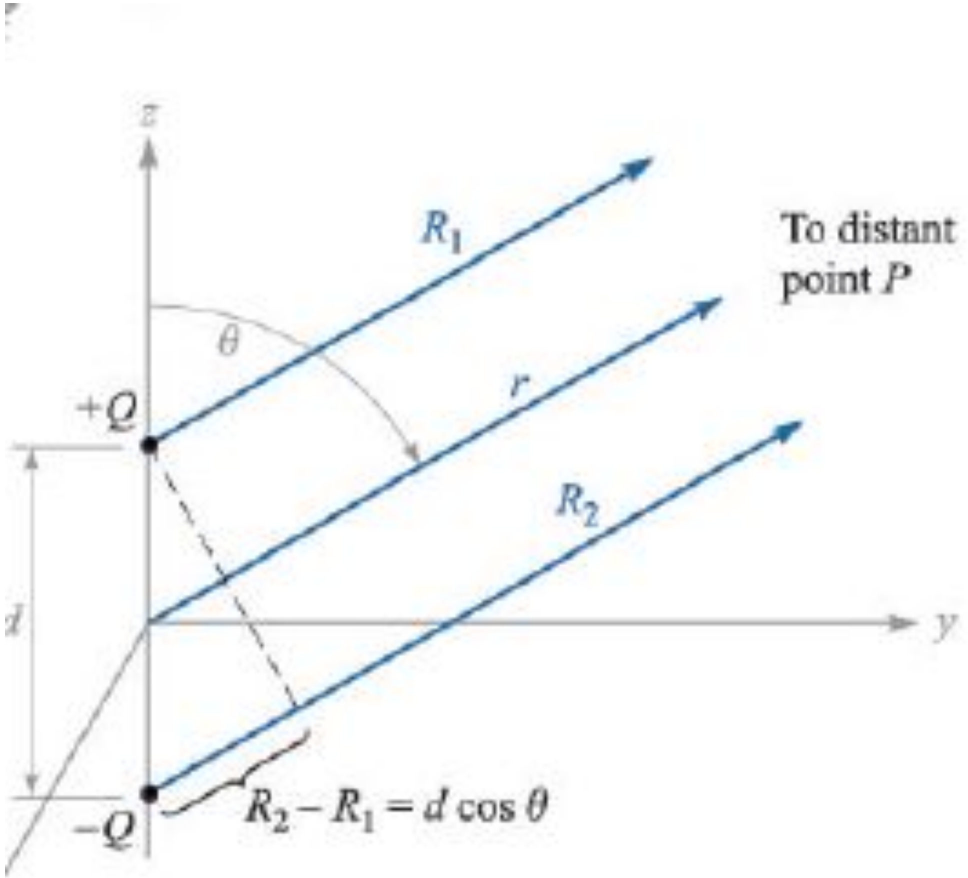


Dipole is a pair of charges of equal magnitude, but with opposite polarity, separated by a small distance.

In the figure shown, two charges, +Q and –Q, are separated by a small distance, d, compared to the distance, r, of the field point. The potential, V, at point P is given by scalar sum of potentials at P due to charges, +Q and –Q,

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\epsilon_0}\,\frac{R_2 - R_1}{R_1 R_2}$$

Potential and field due to dipole



When $r \gg d$,

$R_2 \approx R_1 + d\cos\theta$, and $R_1 R_2 = r^2$,

hence

$$R_2 - R_1 \doteq d\cos\theta$$

$$V = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2}$$

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi\right)$$

$$\mathbf{E} = \frac{Qd}{4\pi\epsilon_0 r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

Potential and field due to dipole

Let us define a vector length ***d*** which starts at - Q and ends at + Q, and then define a dipole moment ***p*** = *Q**d*** ; the potential is then

$$\text{Since } \mathbf{d} \cdot \mathbf{a}_r = d \cos\theta, \qquad V = \frac{\mathbf{p} \cdot \mathbf{a}_r}{4\pi\epsilon_0 r^2}$$

In general, potential due to dipole placed any where is given by,

$$V = \frac{1}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2} \mathbf{p} \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

where ***r'*** is the position vector of dipole centre, and ***r*** is the position vector of the field point.

Potential and field due to dipole

Example: Given a dipole moment $\boldsymbol{p} = 3\boldsymbol{a}_x - 2\boldsymbol{a}_y + \boldsymbol{a}_z$ nC m in free space, find potential at (2, 3, 4).

Solution: The position vector of field point is $\boldsymbol{r} = 2\boldsymbol{a}_x + 3\boldsymbol{a}_y + 4\boldsymbol{a}_z$ and its magnitude is $\mathrm{r} = \sqrt{4 + 9 + 16} = 5.385$, and $r^2 = 29$.

Unit vector of $\boldsymbol{r}$ is

$$\boldsymbol{a}_r = \frac{1}{5.385}(2\boldsymbol{a}_x + 3\boldsymbol{a}_y + 4\boldsymbol{a}_z)$$

The potential at (2, 3, 4) is $V = \dfrac{\mathbf{p} \cdot \mathbf{a}_r}{4\pi\epsilon_0 r^2}$

where $\boldsymbol{p} \cdot \boldsymbol{a}_r = \dfrac{10^{-9}}{5.385}(3\boldsymbol{a}_x - 2\boldsymbol{a}_y + \boldsymbol{a}_z) \cdot (2\boldsymbol{a}_x + 3\boldsymbol{a}_y + 4\boldsymbol{a}_z) = 0.7428 \times 10^{-9}$

and $4\pi\epsilon_0 r^2 = 4\pi\left(\dfrac{10^{-9}}{36\pi}\right)29 = 3.222 \times 10^{-9}$, hence $V = 0.2305$ volts.

Continuity Equation

If a current density, **J**, is passing through an infinitesimal surface area, **dS**, then the infinitesimal current is, **J · dS**, and the total current passing out of a closed surface is

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S}$$

This current is result of *rate of decrease* of charge, Q_i, inside the closed surface.

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{dQ_i}{dt}$$

By divergence theorem, we know that

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_{\mathrm{vol}} (\nabla \cdot \mathbf{J})\, dv$$

Continuity Equation

If the volume inside the closed surface has a volume charge density ρ_v then the charge *Qi* can be replaced by the volume integral of charge density.

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = -\frac{d}{dt}\int_{\text{vol}} \rho_v \, dv$$

Interchanging the order of differentiation and integration yields,

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = \int_{\text{vol}} -\frac{\partial \rho_v}{\partial t}\, dv$$

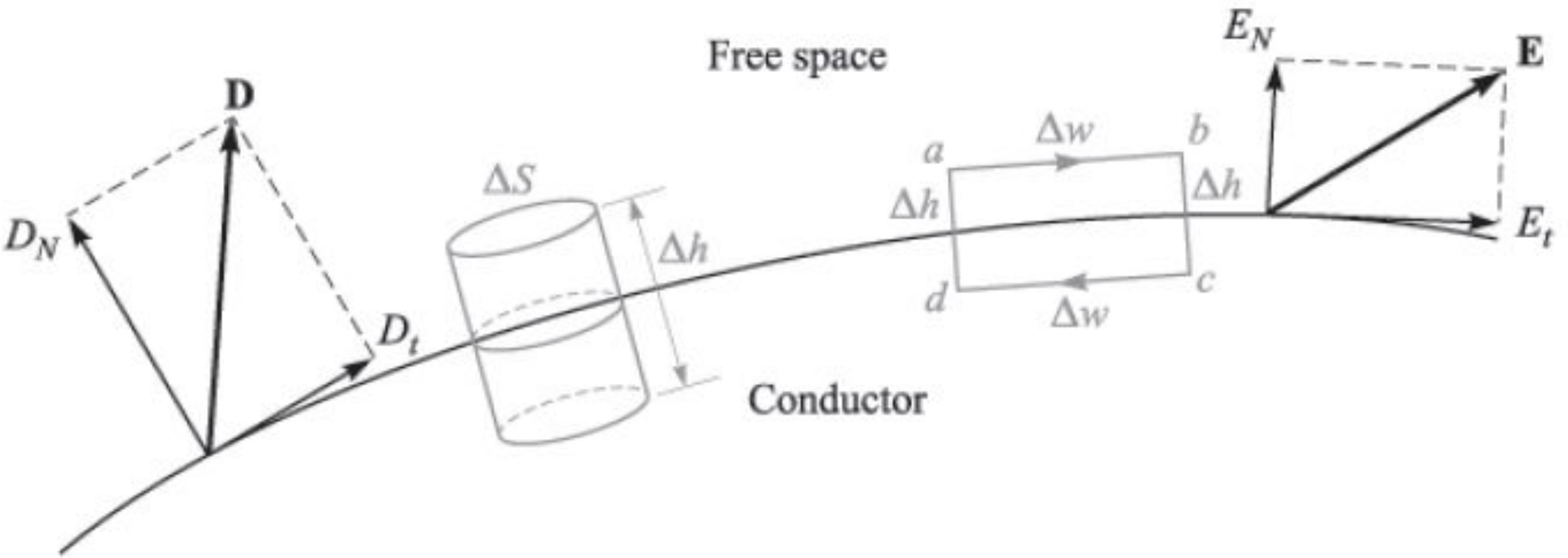
Thus, divergence of current density is equal to the *rate of decrease* of charge density.

$$(\nabla \cdot \mathbf{J}) = -\frac{\partial \rho_v}{\partial t}$$

Boundary conditions

Metal (conductor) to free space boundary: Consider a closed path, *abcda;* we know that line integral of electric field around a closed path is zero. Since electric field inside the conductor is zero, we need to consider the fields (both tangential and normal) in the free space only. Hence,

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0 \quad \Longrightarrow \quad E_t \Delta w - E_{N,\text{at}\,b} \tfrac{1}{2} \Delta h + E_{N,\text{at}\,a} \tfrac{1}{2} \Delta h = 0$$



Boundary conditions

We let Δh to approach zero, while keeping Δw small but finite; this results in

$$E_t \Delta w = 0 \quad \Longrightarrow \quad E_t = 0$$

Now consider the small volume in the figure. If Q is the charge inside this volume, then by Gauss law,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

Integrating over the three surfaces yields,

$$\int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{sides}} = Q$$

Boundary conditions

Since fields inside the conductor are zero, the integration over the bottom surface is zero. As we let Δh to approach zero, the integration over the side surface is also zero. If there exists a surface charge with density ρ_s then,

$$D_N \Delta S = Q = \rho_S \Delta S \quad \Longrightarrow \quad D_N = \rho_S$$

Therefore, the boundary conditions for conductor-to-free space boundary are, the tangential components of electric field and displacement density are zero; and the normal component of displacement density is equal to the surface charge density.

$$D_t = E_t = 0 \qquad \text{and} \qquad D_N = \epsilon_0 E_N = \rho_S$$

Boundary conditions

Example: Given the potential, $V = 100(x^2 - y^2)$, and a point P (2, - 1, 3), which is on conductor-free space boundary, determine potential, electric field, displacement density, and surface charge density at P.

Solution: The potential at P is, $V_P = 100(2^2 - (-1)^2) = 300 \quad V$, and since conductor surface is an equipotential surface, the potential every where on conductor-free space boundary is 300 V. Hence equation for this boundary is

$$300 = 100(x^2 - y^2) \quad \Rightarrow \quad x^2 - y^2 = 3$$

The electric field is, $\boldsymbol{E} = -\boldsymbol{\nabla} V$, hence

$$\mathbf{E} = -100\nabla(x^2 - y^2) = -200x\mathbf{a}_x + 200y\mathbf{a}_y$$

$$\mathbf{E}_p = -400\mathbf{a}_x - 200\mathbf{a}_y \quad \text{V/m}$$

Boundary conditions

The displacement density, ***D =*** ϵ_0 ***E,***

$$\mathbf{D}_P = 8.854 \times 10^{-12}\mathbf{E}_P = -3.54\mathbf{a}_x - 1.771\mathbf{a}_y \quad \mathrm{nC/m^2}$$

Since there can not be tangential component of D at point P, the normal component of D is given by,

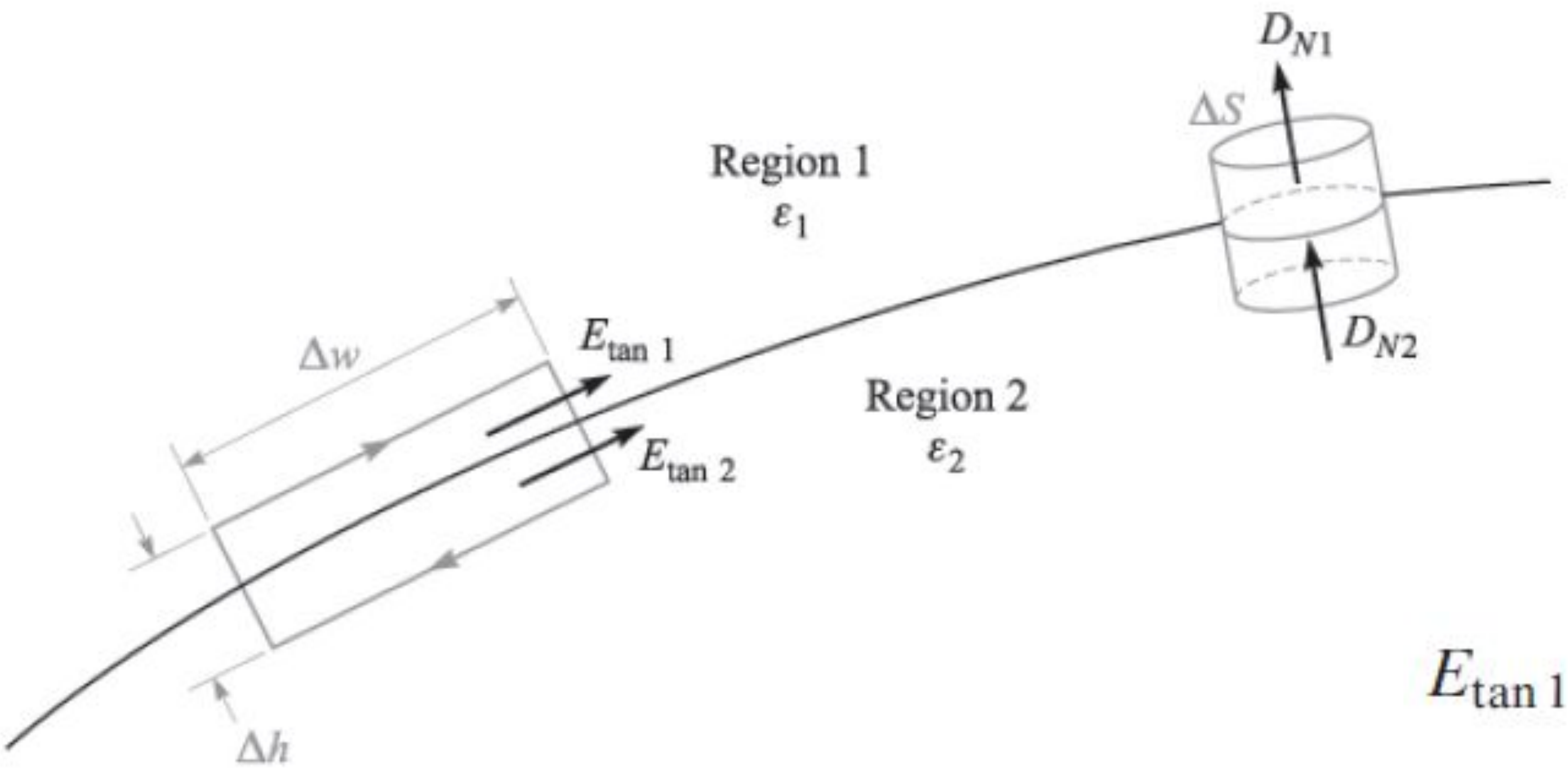
$$D_N = |\mathbf{D}_P| = 3.96 \quad \mathrm{nC/m^2}$$

The surface charge density at P is obtained as,

$$\rho_{S,P} = D_N = 3.96 \quad \mathrm{nC/m^2}$$

Boundary conditions

Dielectric to dielectric boundary:



Using

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

around the closed path shown, note that as Δh goes to zero we obtain,

$$E_{\tan 1}\,\Delta w - E_{\tan 2}\,\Delta w = 0$$

$$E_{\tan 1} = E_{\tan 2}$$

Boundary conditions

The tangential displacement densities across the boundary are related as,

$$\frac{D_{\tan 1}}{\epsilon_1} = E_{\tan 1} = E_{\tan 2} = \frac{D_{\tan 2}}{\epsilon_2}$$

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{\epsilon_1}{\epsilon_2}$$

Consider the small volume shown in the figure. Applying Gauss law, the flux coming out of the closed surface is equal to the charge inside that surface. Hence,

$$D_{N1}\Delta S - D_{N2}\Delta S = \Delta Q = \rho_S \Delta S$$

Boundary conditions

However, at the boundary of two perfect dielectrics surface charges do not exist, therefore ρ_s is zero. Therefore,

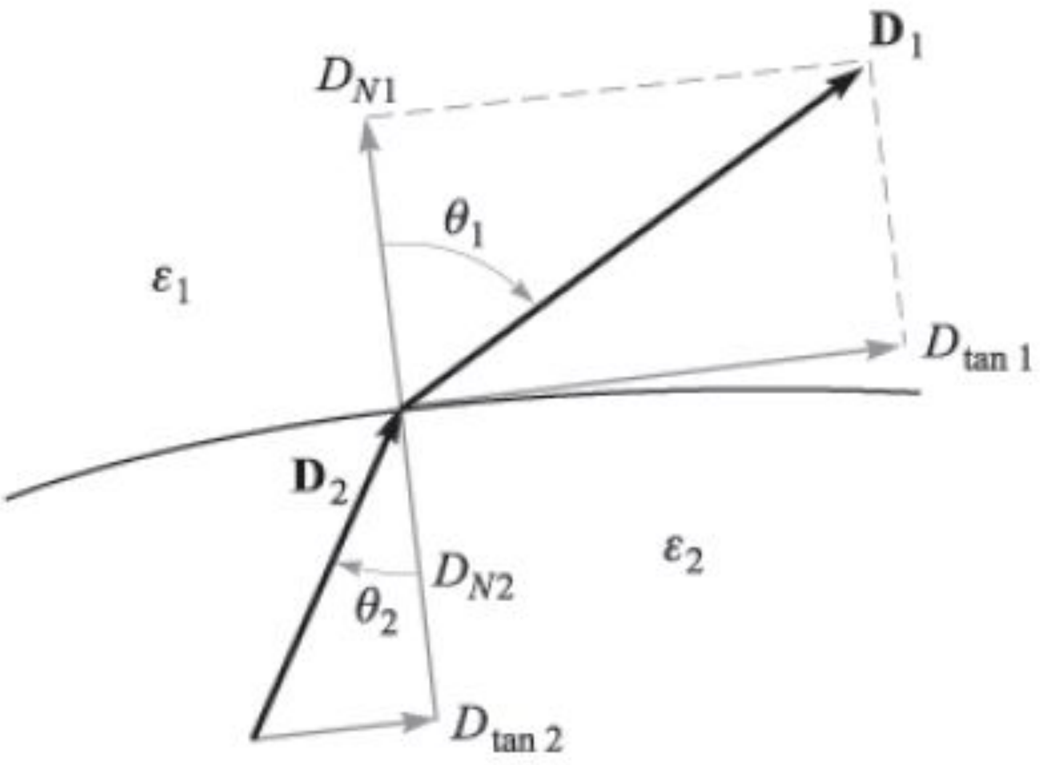
$$D_{N1} = D_{N2} \quad \Longrightarrow \quad \epsilon_1 E_{N1} = \epsilon_2 E_{N2}$$

Conclusion:

At the boundary of two perfect dielectrics, the tangential component of electric field is continuous, while the tangential component of displacement density is discontinuous; the normal component of displacement density is continuous, while the normal component of electric field is discontinuous.

Boundary conditions

Dielectric to dielectric boundary: Expressions for D_2 and E_2 in terms of D_1 and E_1



$$D_{N1} = D_1 \cos\theta_1 = D_2 \cos\theta_2 = D_{N2}$$

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

$$\epsilon_2 D_1 \sin\theta_1 = \epsilon_1 D_2 \sin\theta_2$$

Boundary conditions

Dielectric to dielectric boundary: Expressions for D_2 and E_2 in terms of D_1 and E_1

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

$$D_2 = D_1\sqrt{\cos^2\theta_1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \sin^2\theta_1}$$

$$E_2 = E_1\sqrt{\sin^2\theta_1 + \left(\frac{\epsilon_1}{\epsilon_2}\right)^2 \cos^2\theta_1}$$

Thank you