## Electrostatic Boundary Value Problems



### Poisson's AND LAPLACE'S EQUATIONS

Poisson's and Laplace's equations are easily derived from Gauss's law for a linear, isotropic material medium.

$$\nabla \cdot D = \nabla \cdot \varepsilon E = \rho_v$$

$$E = -\nabla V$$

$$\nabla \cdot (-\varepsilon \nabla V) = \rho_v$$

For an inhomogeneous medium

$$\nabla \cdot (-\varepsilon \nabla V) = \rho_v$$

For an a homogeneous medium

$$\nabla \cdot (\nabla V) = -\frac{\rho_v}{\varepsilon}$$

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}}$$ This is known as Poisson's Equation

When $\rho_v = 0$, for a charge free region

$$\boxed{\nabla^2 V = 0}$$ This is known as Laplace's equation

$\nabla^2$ is a laplacian operator

$\nabla^2 V = 0$ can be expressed in three different coordinate system, they are

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$ [In cartesian coordinate system]

(2)

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$ [In cylindrical coordinate system]

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

[In spherical coordinate system]

Laplace's equation is of primary importance in solving electrostatic problems

## General Procedure For solving Poisson's OR Laplace's Equation

The following general procedure may be taken in solving a given boundary-value problem involving Poisson's equation or Laplace's equation.

(1) Solve Laplace's equation $\nabla^2 V = 0$ if $\rho_v = 0$, or poisson's equation $\nabla^2 V = -\frac{\rho_v}{\varepsilon}$ if $\rho_v \neq 0$ using either (a) direct integration when V is a function of one variable or (b) seperation of variables if V is a function of more than one variable. The solution at this point is not unique but is expressed in terms of unknown integration constants to be determined.

2. Apply the boundary conditions to determine a unique solution for V. Imposing the given boundary conditions makes the solution unique.

3. Having obtained V, find $\vec{E}$ using $\vec{E} = -\nabla V$, $\vec{D}$ from $\vec{D} = \varepsilon \vec{E}$ and $\vec{J} = \sigma \vec{E}$

4. If required, find the charge Q induced on a conductor using $Q = \int_s \rho_s \, ds$

where $\rho_s = D_n$ = Normal component of $\vec{D}$

Then, Capacitance of two Conductors $= C = \frac{Q}{V}$

Resistance of an object can be found by using $R = \frac{V}{I}$ where $I = \int_s \vec{J} \cdot \vec{ds}$

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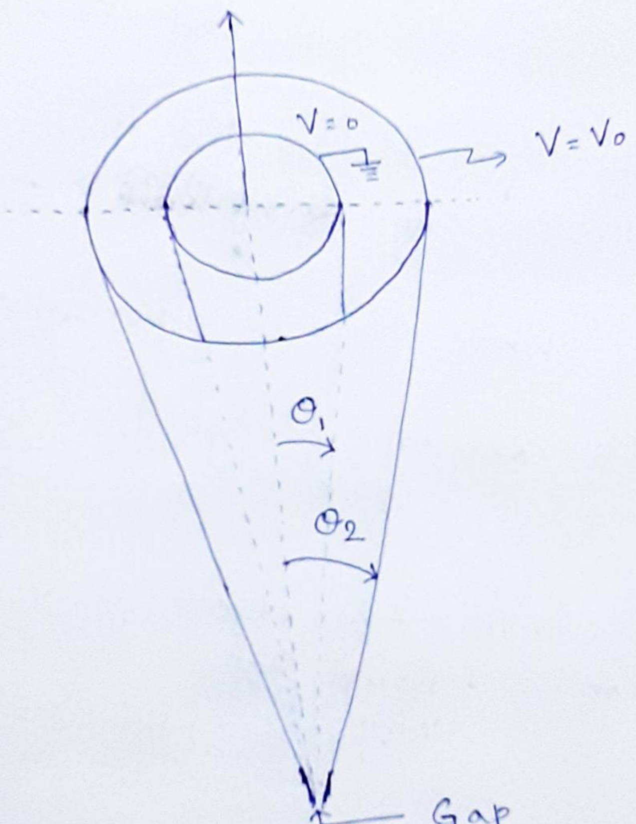
Note:

1. $\nabla^2 V = 0$
2. $V$
3. $E = -\nabla V$, $D = \varepsilon E$, $J = \sigma E$
4. $Q = \int_s D \cdot ds$ $\quad I = \int_s J \cdot ds$
5. $C = \frac{Q}{V}$, $R = \frac{V}{I}$

1) Two conducting cones ($\theta = \pi/10$ and $\theta = \pi/6$) of infinite extent are seperated by an infinitesimal gap at $r=0$, If $V(\theta = \frac{\pi}{10}) = 0$ and $V(\theta = \frac{\pi}{6}) = 50V$, find V and $\vec{E}$ between the cones. (4)

Solution:




Since "V" depends only on "θ", Laplace's equation in spherical coordinates should be considered

$$\nabla^2 V = 0$$

$$\cancel{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)}^{0} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \cancel{\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}}^{0} = 0$$

$$\therefore \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0$$

$$\therefore \quad \frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right) = 0$$

Since it is a variable of '$\theta$' only

$$\frac{d}{d\theta}\left[\sin\theta \frac{dV}{d\theta}\right] = 0$$

Integrating once

$$\sin\theta \frac{dV}{d\theta} = A$$

$$\frac{dV}{d\theta} = \frac{A}{\sin\theta}$$

$$dV = \frac{A \quad d\theta}{2\cos\theta/2 \, \sin\theta/2}$$

$$\int dV = \int \frac{A \, d\theta}{2\cos\theta/2 \, \sin\theta/2}$$

Divide Numerator and denominator by $\cos\theta/2$

$$\int dV = A \int \frac{\frac{1}{\cos\theta/2} \, d\theta}{2\cos\theta/2 \times \frac{\sin\theta/2}{\cos\theta/2}}$$

$$V = A \int \frac{1/2 \sec^2\theta/2 \, d\theta}{\tan\theta/2}$$

put $x = \tan\theta/2 \Rightarrow dx = \sec^2\theta/2 \times \frac{1}{2} d\theta$

$$V = A \int \frac{dx}{x} = A \ln(x) + B$$

$$V = A \ln\left(\tan\theta/2\right) + B$$

Now we have to apply the boundary conditions

Consider $V = A \ln(\tan \theta/2) + B$ (6)

~~At~~ Let $\theta_1 = \frac{\pi}{10}$ $\theta_2 = \frac{\pi}{6}$

$$V\Big|_{\theta=\theta_1} = 0$$

$$V\Big|_{\theta=\theta_1} = A \ln(\tan \theta_1/2) + B$$

$$\therefore 0 = A \ln(\tan \theta_1/2) + B$$

$$B = -A \ln(\tan \theta_1/2)$$

$$V = A \ln(\tan \theta/2) - A \ln(\tan \theta_1/2)$$

$$V = A \ln\left[\frac{\tan \theta/2}{\tan \theta_1/2}\right]$$

At ~~V~~ $\theta = \theta_2 \Rightarrow V = V_0$

$$V = V_0$$

$$V = A \ln\left[\frac{\tan \theta_2/2}{\tan \theta_1/2}\right]$$

$$\therefore A \ln\left[\frac{\tan \theta_2/2}{\tan \theta_1/2}\right] = V_0$$

$$A = \frac{V_0}{\ln\left[\frac{\tan \theta_2/2}{\tan \theta_1/2}\right]}$$

$$\therefore V = A \ln\left[\frac{\tan \theta/2}{\tan \theta_1/2}\right]$$

$$V = \frac{V_0}{\ln\left[\frac{\tan \theta_2/2}{\tan \theta_1/2}\right]} \times \ln\left[\frac{\tan \theta/2}{\tan \theta_1/2}\right]$$

$$\vec{E} = -\nabla V = -\frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta$$

$$\vec{E} = -\frac{1}{r}\frac{d}{d\theta}\left[\frac{V_0}{\ln\left(\frac{\tan\theta_2/2}{\tan\theta_1/2}\right)} \times \ln\left(\frac{\tan\theta/2}{\tan\theta_1/2}\right)\right]\hat{a}_\theta$$

$$\vec{E} = -\frac{1}{r}\frac{V_0}{\ln\left[\frac{\tan\theta_2/2}{\tan\theta_1/2}\right]} \times \frac{1}{\tan\theta/2} \times \sec^2\theta/2 \times \frac{1}{2}\hat{a}_\theta$$

$$\vec{E} = -\frac{V_0}{r\ln\left[\frac{\tan\theta_2/2}{\tan\theta_1/2}\right]} \times \frac{1}{\left(2\frac{\sin\theta/2}{\cancel{\cos\theta/2}} \times \cos\theta/2 \times \cancel{\cos\theta/2}\right)}\hat{a}_\theta$$

$$\vec{E} = -\frac{V_0}{r\sin\theta\ln\left[\frac{\tan\theta_2/2}{\tan\theta_1/2}\right]}\hat{a}_\theta$$

At $\theta_1 = \frac{\pi}{10}$, $\theta_2 = \frac{\pi}{6}$ and $V_0 = 50$

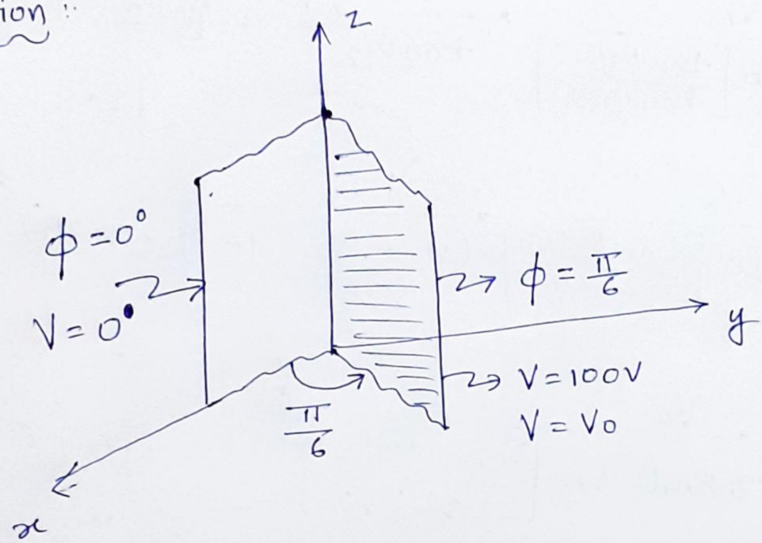
$$V = \frac{V_0}{\ln\left[\frac{\tan\theta_2/2}{\tan\theta_1/2}\right]}\ln\left[\frac{\tan\theta/2}{\tan\theta_1/2}\right]$$

$$V = 95.1\ln\left[\frac{\tan\theta/2}{0.1584}\right] V$$

$$\vec{E} = -\frac{95.1}{r\sin\theta}\hat{a}_\theta \; V/m$$

2) Semi-infinite conducting planes at $\phi = 0$ and $\phi = \pi/6$ are seperated by an infinitesimal (very very small) insulating gap as shown in figure. If $V(\phi=0) = 0$ and $V(\phi = \pi/6) = 100$. Calculate "V" and "E" in the region between the planes. (8)

Solution :




It is in cylindrical coordinate System

$$\nabla^2 V = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Since "V" is variable along $\phi$

$$\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

$$\frac{\partial^2 V}{\partial \phi^2} = 0$$

Integrating $\int \frac{\partial^2 V}{\partial \phi^2} = \int 0$

$$\frac{\partial V}{\partial \phi} = A$$

Integrating $V = A\phi + B$

$$V = A\phi + B$$

At $\phi = 0$ , $V = 0$

$$0 = A(0) + B$$

$$\boxed{B = 0}$$

At $\phi = \pi/6$ , $V = V_0 = 100$

$$100 = V_0 = \cancel{A(\phi/6)}\ A\left(\frac{\pi}{6}\right) + B$$

$$100 = A\left[\frac{\pi}{6}\right] + \cancel{B}^{\,0}$$

$$\frac{100}{\pi/6} = A \qquad \boxed{A = \frac{100}{\pi/6}}$$

$$V = \frac{100}{\pi/6}\phi + 0$$

$$\bar{E} = -\nabla V = -\frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi$$

$$\bar{E} = -\frac{1}{\rho}\frac{\partial}{\partial \phi}\left(\frac{100}{\pi/6}\phi\right)\hat{a}_\phi$$

$$\bar{E} = -\frac{1}{\rho}\frac{600}{\pi}\hat{a}_\phi$$

$$\bar{E} = -\frac{600}{\pi\rho}\hat{a}_\phi$$

# RESISTANCE AND CAPACITANCE



## RESISTANCE

The procedure for finding resistance is as follows

1. Choose a suitable coordinate system
2. Assume $V_0$ as the potential difference between conductor terminals
3. Solve Laplace's equation $\nabla^2 V = 0$ to obtain $V$. Then determine $\vec{E}$ from $\vec{E} = -\nabla V$ and find $I$ from $I = \int_s \sigma \vec{E} \cdot d\vec{s}$ where $\sigma \vec{E} = \vec{J}$
4. Finally, obtain $R$ as $V_0/I$

## CAPACITANCE

The procedure for finding capacitance is as follows

1. Choose a suitable coordinate system
2. Assume $V_0$ as the potential difference between conductor terminals.
3. Solve Laplace's Equation $\nabla^2 V = 0$ to obtain $V$, then determine $\vec{E}$ from $\vec{E} = -\nabla V$ and find $I$ from $I = \int_s \vec{J} \cdot d\vec{s} = \int_s \sigma \vec{E} \cdot ds$

## CAPACITANCE

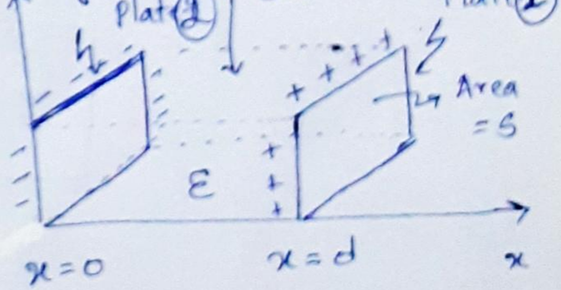
Following steps are involved in finding the capacitance

1) Choose a suitable coordinate system.
2) Let the two conducting plate carry charges $+Q$ and $-Q$.
3) Determine $\vec{E}$ by using Coulomb's or Gauss's law and find $V$ from $V = -\int_L \vec{E} \cdot \vec{dl}$. The negative sign may be ignored in this case because we are interested in the absolute value of $V$.
4) Finally, obtain $C$ from $C = Q/V$

## A] PARALLEL PLATE CAPACITOR

The above explained procedure for calculation of capacitor may be applied as follows

*) It is in Cartesian coordinate system

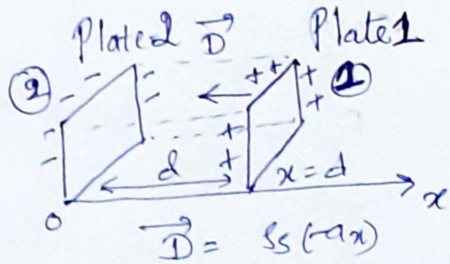



*) Let Plate-1 and Plate-2 carry charges $+Q$ and $-Q$ respectively. These charges are distributed uniformly so that

$$\rho_s = \frac{Q}{S}$$

where $Q$ = Total charge
$S$ = Surface Area
$\rho_s$ = charge density

$$Q = \rho_s S$$




*) $\vec{D} = \rho_s(-\hat{a}_x)$

$$\varepsilon \vec{E} = -\rho_s \hat{a}_x$$

$$\vec{E} = -\frac{\rho_s}{\varepsilon} a_x$$

$$\vec{E} = -\frac{Q}{\varepsilon S}\hat{a}_x \quad \text{where} \quad \rho_s = \frac{Q}{S}$$

$$V = -\int_2^1 \vec{E}\cdot\vec{dl} = -\int_2^1 -\frac{Q}{\varepsilon S}\hat{a}_x \cdot dx\,\hat{a}_x$$

$$V = +\int_0^d \frac{Q}{\varepsilon S}dx = \frac{Qd}{\varepsilon S}$$

where S = Surface area of the plate

$$V = \frac{Qd}{\varepsilon S}$$

*) Consider $V = \dfrac{Qd}{\varepsilon S}$

But $C = \dfrac{Q}{V} = \dfrac{\cancel{Q}}{\dfrac{\cancel{Q}d}{\varepsilon S}}$

$$\boxed{C = \frac{\varepsilon S}{d}}$$

where $\varepsilon = \varepsilon_0 \varepsilon_r$

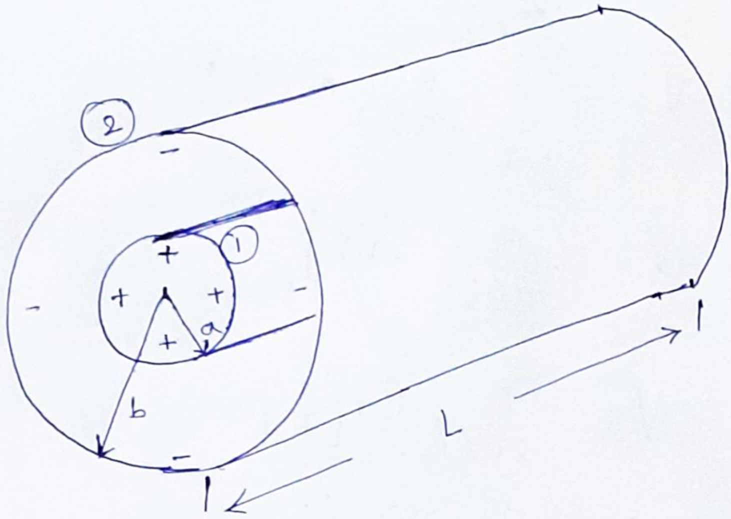
If $\varepsilon = \varepsilon_0 \varepsilon_r$

$$\boxed{C = \frac{\varepsilon_0 \varepsilon_r S}{d}}$$

Energy ~~of~~ stored in Capacitor $= \frac{1}{2}CV^2$

$$= \frac{1}{2}QV = \frac{Q^2}{2C}$$

## B] COAXIAL CAPACITOR

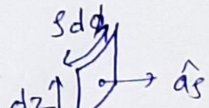



Consider a coaxial capacitor which is nothing but a coaxial cable or coaxial cylindrical capacitor. Let the Length of the coaxial capacitor be "L". The inner radius of the conductor is 'a' and outer radius of the conductor is 'b'.

*) It is in cylindrical coordinate system

*) $$Q = \oint_S \vec{D} \cdot \vec{ds} = \int_S \varepsilon \vec{E} \cdot \vec{ds}$$

$$Q = \int_S \varepsilon E_\rho \hat{a}_\rho \cdot \rho \, d\phi \, dz \, \hat{a}_\rho$$




$$Q = \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \varepsilon E_\rho \rho \, d\phi \, dz$$

*) $$Q = \varepsilon E_\rho \rho 2\pi L = \varepsilon E_\rho 2\pi \rho L$$

$$E_\rho = \frac{Q}{2\pi\varepsilon\rho L} \Rightarrow \boxed{\vec{E} = \frac{Q}{2\pi\varepsilon\rho L} \hat{a}_\rho}$$

$$V = -\int_{2}^{1} \vec{E} \cdot \vec{dl}$$

$$V = -\int_{\rho=b}^{\rho=a} \frac{Q}{2\pi\varepsilon\rho L} \hat{a}_\rho \cdot d\rho \hat{a}_\rho$$

$$V = \frac{-Q}{2\pi\varepsilon L} \ln\rho \Big|_{\rho=b}^{\rho=a}$$

$$V = \frac{-Q}{2\pi\varepsilon L} \ln\left[\frac{a}{b}\right]$$

$$V = \frac{Q}{2\pi\varepsilon L} \ln\left(\frac{b}{a}\right)$$

*)

$$C = \frac{Q}{V}$$

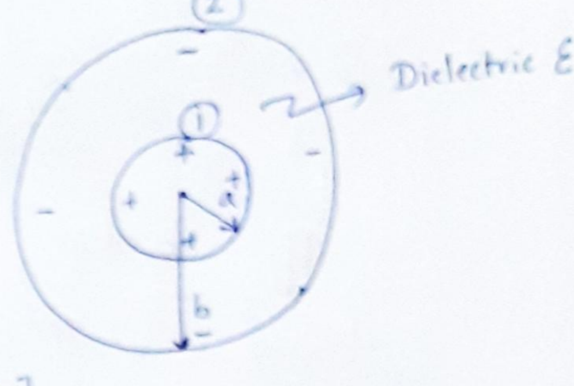
$$C = \frac{\cancel{Q}}{\frac{\cancel{Q}}{2\pi\varepsilon L} \ln(b/a)}$$

$$C = \frac{2\pi\varepsilon L}{\ln(b/a)}$$

Capacitance $C = \dfrac{2\pi\varepsilon L}{\ln(b/a)}$

## c. SPHERICAL CAPACITOR

A spherical capacitor is the case of two concentric spherical conductors. Consider the inner sphere of "a" and the outer sphere of radius "b" [where (b>a)].




These two spheres are seperated by a dielectric medium with permittivity "ε" as shown in figure. We assume charges +Q and −Q on the inner and outer spheres, respectively.

*) It is in spherical coordinate system

*) $$Q = \oint_s \vec{D}\cdot\vec{ds} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \varepsilon E_r \hat{a}_r \cdot r^2 \sin\theta \, d\phi \, d\theta \, \hat{a}_r$$ [From Gauss's Law]

$$Q = \varepsilon E_r 4\pi r^2$$

$$E_r = \frac{Q}{4\pi\varepsilon r^2}$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon r^2}\hat{a}_r$$

*) The potential difference between the conductors is

$$V = -\int_2^1 \vec{E}\cdot\vec{dl} = -\int_b^a \frac{Q}{4\pi\varepsilon r^2}\hat{a}_r \cdot dr\hat{a}_r$$

$$V = \frac{Q}{4\pi\varepsilon r}\Big|_b^a = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$$

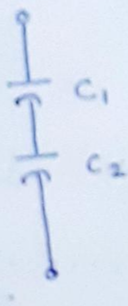
Thus the Capacitance of the spherical Capacitor is

$$C = \frac{Q}{V} = \frac{\cancel{Q}}{\frac{\cancel{Q}}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]} = \frac{4\pi\varepsilon}{\left[\frac{1}{a} - \frac{1}{b}\right]}$$

If $b \to \infty$

then $C = \dfrac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{\infty}} = 4\pi\varepsilon a$

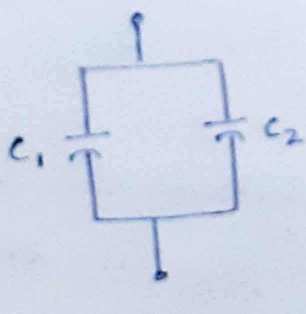
## Capacitors in Series




$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$C = \frac{C_1 C_2}{C_1 + C_2}$$

## Capacitors in Parallel




$$C = C_1 + C_2$$

## Expressions for finding the Resistance and the capacitance

$$R = \frac{V}{I} = \frac{\int_L \vec{E}\cdot\vec{dl}}{\int_S \vec{J}\cdot\vec{ds}} = \frac{\int_L \vec{E}\cdot\vec{dl}}{\int_S \sigma\vec{E}\cdot\vec{ds}}$$

$$C = \frac{Q}{V} = \frac{\oint_S \vec{D}\cdot\vec{ds}}{\int_L \vec{E}\cdot\vec{dl}} = \frac{\varepsilon\oint_S \vec{E}\cdot\vec{ds}}{\int_L \vec{E}\cdot\vec{dl}}$$

$$RC = \frac{\int_L \cancel{\vec{E}\cdot\vec{dl}}}{\int_S \sigma\cancel{\vec{E}\cdot\vec{ds}}} \times \frac{\varepsilon\oint_S \cancel{\vec{E}\cdot ds}}{\int_L \cancel{\vec{E}\cdot\vec{dl}}}$$

$$\boxed{RC = \frac{\varepsilon}{\sigma}} \rightarrow (1)$$

$RC = \frac{\varepsilon}{\sigma}$ = Relaxation time "$T_r$" of the medium seperating the conductors.

Equation (1) is valid when the medium is homogeneous. Relaxation time can be calculated as follows

(i) Parallel-plate capacitor

$$C = \frac{\varepsilon S}{d} \qquad R = \frac{d}{\sigma S}$$

$$RC = \frac{\cancel{d}}{\sigma \cancel{S}} \times \frac{\varepsilon \cancel{S}}{\cancel{d}} = \frac{\varepsilon}{\sigma}$$

$$R = \frac{\int_L E\cdot dl}{\int_S \sigma\vec{E}\cdot ds} =$$

$$= \frac{E/x\,\cancel{(dx)} \rightarrow d}{\sigma\, E/x\, \cancel{(dx\,dy)} \rightarrow S}$$

$$R = \frac{d}{\sigma S}.$$

(ii) Cylindrical Capacitor

$$C = \frac{2\pi \varepsilon L}{\ln(b/a)} \qquad R = \frac{\ln(b/a)}{2\pi \sigma L}$$

$$RC = \frac{\varepsilon}{\sigma}$$

(iii) Spherical Capacitor

$$C = \frac{4\pi \varepsilon}{\left[\frac{1}{a} - \frac{1}{b}\right]} \qquad R = \frac{\left(\frac{1}{a} - \frac{1}{b}\right)}{4\pi \sigma}$$

$$RC = \frac{\varepsilon}{\sigma}$$

(iv) For an isolated spherical conductor

$$C = 4\pi \varepsilon a \qquad R = \frac{1}{4\pi \sigma a}$$

$a$ = Radius of the spherical conductor

$$RC = \frac{\varepsilon}{\sigma}$$

*) It should be noted that resistance calculated for parallel plate capacitor, cylindrical capacitor, spherical capacitor and an isolated spherical conductor is not the resistance of the capacitor conductor but it is the resistance between the plates. Therefore '$\sigma$' in those equations is the conductivity of the dielectric medium seperating the conductors

1. A metal bar of conductivity $\sigma$ is bent to form a flat $90^{0}$ sector of inner radius "a", outer radius "b", and thickness "t" as shown in figure. show that

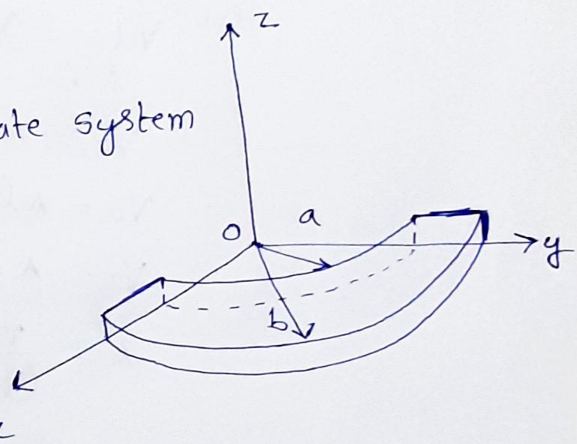
(a) the resistance of the bar between the vertical curved surfaces at $\rho = a$ and $\rho = b$ is

$$R = \frac{2 \ln b/a}{\sigma \pi t}$$

(b) the resistance between the two horizontal surfaces at $z = 0$ and $z = t$ is

$$R' = \frac{4t}{\sigma \pi (b^2 - a^2)}$$

Solution:




It is in cylindrical coordinate system

$$\nabla^2 V = 0$$

$$\nabla^2 V = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0$$

$$\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0$$

Integrating on both sides

$$\int \frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = \int 0$$

$$\rho \frac{dV}{d\rho} = A$$

Integrating once again

$$\int \frac{\partial V}{\partial \rho} = \int \frac{A}{\rho}$$

$$V = A\ln(\rho) + B$$

where "A" and "B" are constants

$$V = A\ln(\rho) + B$$

$$V\Big|_{\rho=a} = A\ln(a) + B$$

$$V\Big|_{\rho=a} = 0$$

$$A\ln(a) + B = 0 \Rightarrow B = -A\ln(a)$$

$$V\Big|_{\rho=b} = A\ln(b) + B$$

$$V\Big|_{\rho=b} = V_0$$

$$V_0 = A\ln(b) + B$$

$$V_0 = A\ln(b) - A\ln(a)$$

$$V_0 = A\ln(b/a)$$

$$A = \frac{V_0}{\ln(b/a)}$$

$$V = A\ln(\rho) + B$$

$$V = \frac{V_0}{\ln(b/a)}\ln(\rho) + \left(-\frac{V_0}{\ln(b/a)}\ln(a)\right)$$

$$V = \frac{V_0}{\ln(b/a)}\ln\left(\frac{\rho}{a}\right)$$

$$E = -\nabla V = -\frac{\partial V}{\partial \rho}a_\rho = -\frac{V_0}{\ln(b/a)} \times \frac{1}{\rho}\hat{a}_\rho$$

$\vec{J} = \sigma \vec{E}$ $\quad d\vec{s} = -\rho \, d\phi \, dz \, \hat{a}_\rho$

$$I = \int_S \vec{J} \cdot d\vec{s} = \int_{\phi=0}^{\pi/2} \int_{z=0}^{t} -\frac{V_0 \sigma}{\ln(b/a)} \times \frac{1}{\rho} \hat{a}_\rho \cdot \left[-\rho \, d\phi \, dz \, \hat{a}_\rho\right]$$

$$= \frac{\pi}{2} t \frac{V_0 \sigma}{\ln(b/a)}$$

$$R = \frac{V_0}{I} = \frac{\cancel{V_0}}{\frac{\pi}{2} t \frac{\cancel{V_0} \sigma}{\ln(b/a)}} = \frac{2\ln(b/a)}{\sigma \pi t}$$

(b) Let ~~the~~ $V_0$ be the potential difference between the two horizontal surfaces so that $V(z=0) = 0$ and $V(z=t) = V$, $V = V(z)$

Laplace's equation $\frac{\partial^2 V}{\partial z^2} = 0$

Integrating

$$\frac{\partial V}{\partial z} = A$$

Integrating

$$V = Az + B$$

Applying boundary conditions

$$V\big|_{z=0} = 0$$

$$V\big|_{z=0} = A(0) + B \Rightarrow B = 0$$

$$V\big|_{z=t} = V_0$$

$$V\big|_{z=t} = At \Rightarrow V_0 = At \quad A = \frac{V_0}{t}$$

$$V = \frac{V_0}{t} z$$

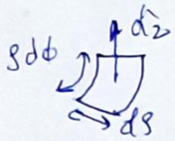
$$\vec{E} = -\nabla V$$

$$\vec{E} = -\frac{dV}{dz}\hat{a}_z = -\frac{V_0}{t}\hat{a}_z$$

$$\vec{J} = \sigma \vec{E}$$

$$\vec{J} = \sigma\left(-\frac{V_0}{t}\hat{a}_z\right) = -\frac{\sigma V_0}{t}\hat{a}_z$$

$$\vec{ds} = -\rho\, d\phi\, d\rho\, \hat{a}_z$$




$$I = \int_s \vec{J} \cdot \vec{ds}$$

$$I = \int_{\rho=a}^{\rho=b} \int_{\phi=0}^{\pi/2} \left(-\frac{V_0 \sigma}{t}\hat{a}_z\right) \cdot \left(-\rho\, d\phi\, d\rho\, \hat{a}_z\right)$$

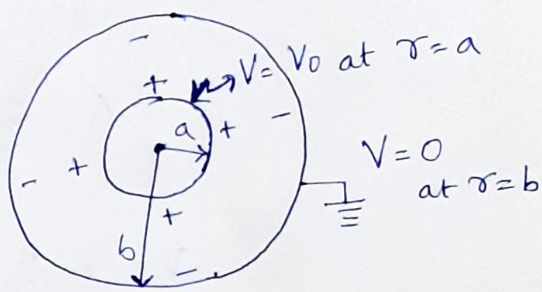
$$I = \frac{V_0 \sigma}{t} \frac{\pi}{2} \times \frac{\rho^2}{2}\Big|_a^b$$

$$I = \frac{V_0 \sigma \pi}{4t}\left[b^2 - a^2\right]$$

$$R' = \frac{V_0}{I} = \frac{\cancel{V_0}}{\frac{\cancel{V_0}\sigma\pi\left[b^2 - a^2\right]}{4t}} = \frac{4t}{\sigma\pi\left(b^2 - a^2\right)}$$

3) Conducting spherical shells with radii $a = 10\,cm$ and $b = 30\,cm$ are maintained at a potential difference of 100V such that ~~(V(r=a)~~ $V(r=b) = 0$ and $V(r=a) = 100\,V$. Determine $V$ and $\vec{E}$ in the region between the shells. If $\varepsilon_r = 2.5$ in the region, determine the total charge induced on the shells and the capacitance of the capacitor.

Solution:




$$\nabla^2 V = 0$$

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

Integrating

$$\int \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = \int 0$$

$$r^2 \frac{\partial V}{\partial r} = A$$

$$\frac{\partial V}{\partial r} = \frac{A}{r^2}$$

Integrating again

$$V = -\frac{A}{r} + B$$

using boundary conditions



At $r = a$ $\quad V = V_0$

At $r = b$ $\quad V = 0$

$$V = -\frac{A}{r} + B$$

At $r = b$ $\quad V = 0 \Rightarrow 0 = -\frac{A}{b} + B$

$$B = \frac{A}{b}$$

At $r = a$ $\quad V = V_0 \Rightarrow V_0 = -\frac{A}{a} + \frac{A}{b}$

$$V_0 = A\left[\frac{1}{b} - \frac{1}{a}\right]$$

$$A = \frac{V_0}{\left(\frac{1}{b} - \frac{1}{a}\right)}$$

$$V = -\frac{A}{r} + B$$

$$V = -\frac{V_0}{\left(\frac{1}{b} - \frac{1}{a}\right)} \times \frac{1}{r} + \frac{1}{b} \times \frac{V_0}{\left(\frac{1}{b} - \frac{1}{a}\right)}$$

$$V = \frac{V_0}{\left(\frac{1}{b} - \frac{1}{a}\right)} \times \left[-\frac{1}{r} + \frac{1}{b}\right]$$

$$V = \frac{V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)} \times \left[\frac{1}{r} - \frac{1}{b}\right]$$

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{a}_r = -\frac{A}{r^2}\hat{a}_r$$

$$\vec{E} = \frac{-V_0}{r^2\left[\frac{1}{b} - \frac{1}{a}\right]}\hat{a}_r = \frac{V_0}{r^2\left[\frac{1}{a} - \frac{1}{b}\right]}\hat{a}_r$$

$$Q = \int_S \vec{D} \cdot \vec{ds} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \varepsilon E_r \hat{a}_r \cdot r^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r$$

$$Q = \frac{4\pi \varepsilon_0 \varepsilon_r V_0}{\cancel{r^2}\left[\frac{1}{a} - \frac{1}{b}\right]} \times \cancel{r^2}$$

$$C = \frac{Q}{V_0} = \frac{4\pi \varepsilon_0 \varepsilon_r \cancel{V_0}}{\cancel{V_0}\left[\frac{1}{a} - \frac{1}{b}\right]} = \frac{4\pi \varepsilon_0 \varepsilon_r}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

Put $a = 10 \text{cm}$ $\quad b = 30 \text{cm}$ $\quad V_0 = 100 \text{V}$ $\quad \varepsilon_0 = \frac{10^{-9}}{36\pi}$

$a = 0.1 \text{m}$ $\quad b = 0.3 \text{m}$

$\varepsilon_r = 2.5$

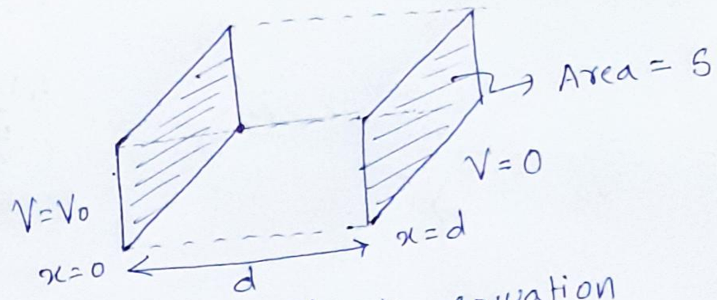
$$\vec{E} = \frac{100}{r^2\left[\frac{1}{0.1} - \frac{1}{0.3}\right]} \hat{a}_r = \frac{15}{r^2} \hat{a}_r \text{ V/m}$$

$$Q = \pm \frac{4\pi \varepsilon_0 \varepsilon_r V_0}{\left[\frac{1}{a} - \frac{1}{b}\right]} = \pm 4.167 \times 10^{-9} \text{ C}$$

$$C = \frac{|Q|}{V_0} = \frac{4.167 \times 10^{-9}}{100} = 41.67 \times 10^{-12} \text{ F}$$

4) Find the capacitance between parallel plates using Laplace's equation. The distance between the plates is "d". $V = V_o$ at $x = 0$ and $V = 0$ at $x = d$.

Solution:




Sol Consider Laplace's equation

$$\nabla^2 V = 0$$

$$\frac{d^2V}{dx^2} + \cancel{\frac{d^2V}{dy^2}}^{0} + \cancel{\frac{d^2V}{dz^2}}^{0} = 0$$

$$\frac{d^2V}{dx^2} = 0$$

Integrating on both sides

$$\int \frac{d^2V}{dx^2} = \int 0$$

Integrating again $\displaystyle\int \frac{dV}{dx} = \int A$

$$V = Ax + B$$

At $x = 0$ $\quad V = V_o$

$\therefore V_o = A(0) + B \Rightarrow B = V_o$

At $x = d$ $\quad V = 0$

$$0 = Ad + \cancel{B}^{V_o}$$

$$Ad + V_0 = 0$$

$$A = -\frac{V_0}{d}$$

$$V = \left(-\frac{V_0}{d}\right)x + V_0$$

$$\vec{E} = -\nabla V = -\frac{dV}{dx}\hat{a}_x$$

$$\vec{E} = -\left[\frac{d}{dx}\left[\left(-\frac{V_0}{d}x\right) + V_0\right]\right]\hat{a}_x$$

$$\vec{E} = \frac{V_0}{d}\hat{a}_x$$

$$\vec{D} = \varepsilon\vec{E} = \varepsilon\vec{E}$$

$$Q = \int_S \vec{D}\cdot d\vec{S} = \int_S \varepsilon\vec{E}\cdot d\vec{S} =$$

[where $S$ = Area of the plate]

$$Q = \varepsilon E_x S$$

$$Q = \varepsilon\frac{V_0}{d}S$$

$$C = \frac{Q}{V_0} = \frac{Q}{V_0} = \frac{\varepsilon\frac{V_0}{d}S}{V_0}$$

$$\boxed{C = \frac{\varepsilon S}{d} = \frac{\varepsilon_0\varepsilon_r S}{d}}$$

Where $S$ = Area of the plate
$d$ = distance between the plates

5) Determine the capacitance of each capacitors shown in figure. Take $\varepsilon_{r1} = 4$, $\varepsilon_{r2} = 6$, $d = 5mm$ and $S = 30cm^2$

(a)

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$$

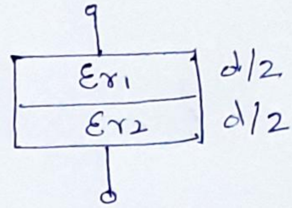



Figure 1

$$C_1 = \frac{\varepsilon_0 \varepsilon_{r1} S}{\frac{d}{2}} = \frac{\frac{10^{-9}}{36\pi} \times 4 \times 30 \times 10^{-4}}{\frac{5 \times 10^{-3}}{2}}$$

$$C_1 = 42.44pF$$

$$C_2 = \frac{\varepsilon_0 \varepsilon_{r2} S}{d/2} = \frac{\frac{10^{-9}}{36\pi} \times 6 \times 30 \times 10^{-4}}{5 \times 10^{-3}/2} = 63.66pF$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = 25.46pF$$

(b)

$$C_{eq} = C_1 + C_2$$

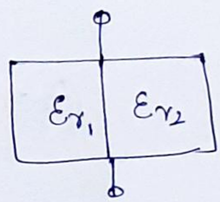



Figure 2

$$C_1 = \frac{\varepsilon_0 \varepsilon_{r1} S/2}{d} \quad \left[\text{Area} = S/2\right]$$

$$C_1 = \frac{(10^{-9}/36\pi) \times 4 \times (30 \times 10^{-4}/2)}{5 \times 10^{-3}}$$

$$= 10.61 pF$$

$$C_2 = \frac{\varepsilon_0 \varepsilon_{r2} S/2}{d} = \frac{(10^{-9}/36\pi) \times 6 \times (30 \times 10^{-4}/2)}{5 \times 10^{-3}}$$

$$= 15.91pF$$

$$C = C_1 + C_2 = 26.52pF$$