CONTROL SYSTEMS

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THE PERFORMANCE OF FEEDBACK CONTROL SYSTEMS

Time Domain Analysis

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Mathematical model of physical systems : → Analogous systems

Electrical / Mechanical / Electromechanical

Idealizing assumptions (Linear, Lumped, Time invariant)

ODE with constant co-efficients

Laplace Transform

Algebraic relations

State space representation

BD ⟷ SFG

$\boxed{\text{Transfer functions}}$ → mathematical mode

To analyse the system

Time domain

Frequency domain

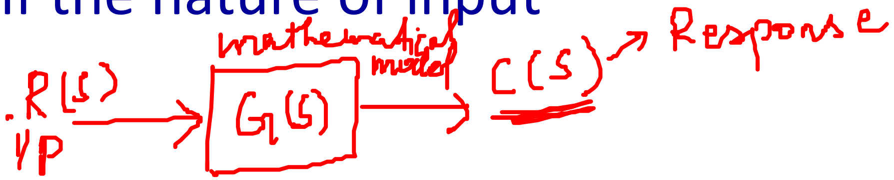
$\boxed{\text{design}}$

TIME DOMAIN ANALYSIS

Introduction

- In time-domain analysis the response of a dynamic system to an input is expressed as a function of time.

- It is possible to compute the time response of a system if the nature of input and the mathematical model of the system are known.



- Usually, the input signals to control systems are not known fully ahead of time.

- It is therefore difficult to express the actual input signals mathematically by simple equations.

Standard Test Signals

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- **The characteristics of actual input signals are -**
- **sudden shock**
- **a sudden change**
- **a constant velocity**
- **and constant acceleration.**

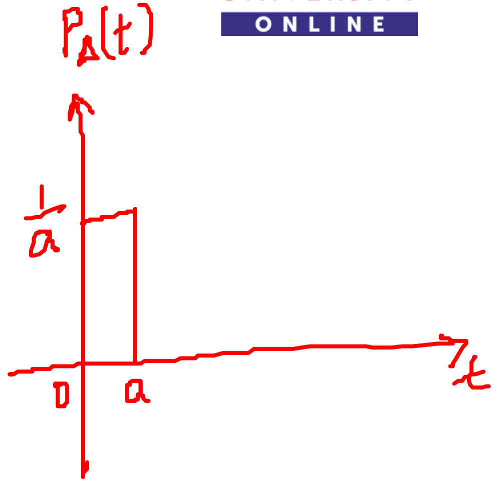
- **The dynamic behavior of a system is therefore judged and compared under application of standard test signals – an impulse, a step, a constant velocity, and constant acceleration.**

- **The other standard signal of great importance is a sinusoidal signal.**

Impulse signal

The impulse signal imitate the sudden shock characteristic of actual input signal.

Continuous-time Dirac Delta function (distribution), $\delta(t) = 0,\ t \neq 0$ and $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$



$$P_\Delta(t) = \frac{1}{a}\left(1(t) - 1(t-a)\right)$$

Discrete-time

$$\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0 \end{cases} \rightarrow \text{unit sample}$$

$$LT\{P_\Delta(t)\} = \frac{1}{a}\int_0^{\infty} \left(1(t) - 1(t-a)\right) e^{-st}\,dt$$

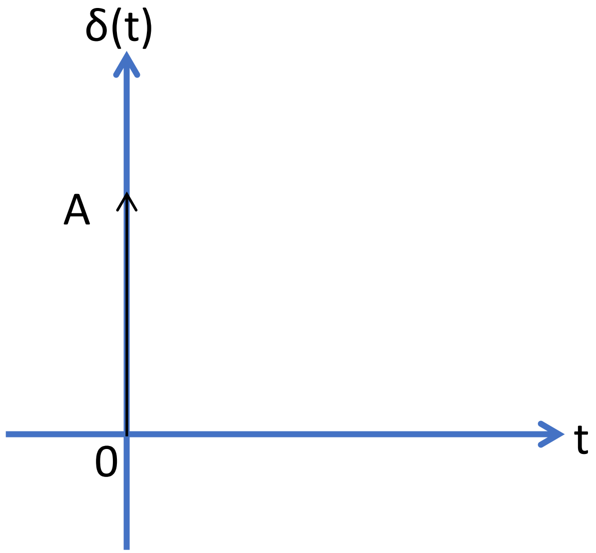
unit step

$$= \lim_{a \to 0} \frac{1}{a}\left[\frac{1}{s} - \frac{e^{-as}}{s}\right] = \lim_{a \to 0} \frac{1}{as}\left[1 - e^{-as}\right] = \lim_{a \to 0} \frac{1}{as}\left[1 - \left(1 - as + \frac{1}{2}a^2s^2 - \cdots\right)\right]$$

$$= \lim_{a \to 0}\left(1 - \frac{as}{2} + \cdots\right) = 1$$

- **Impulse signal**
  - **The impulse signal imitate the sudden shock characteristic of actual input signal.**

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$



  - **If A=1, the impulse signal is called unit impulse signal.**

TIME DOMAIN ANALYSIS

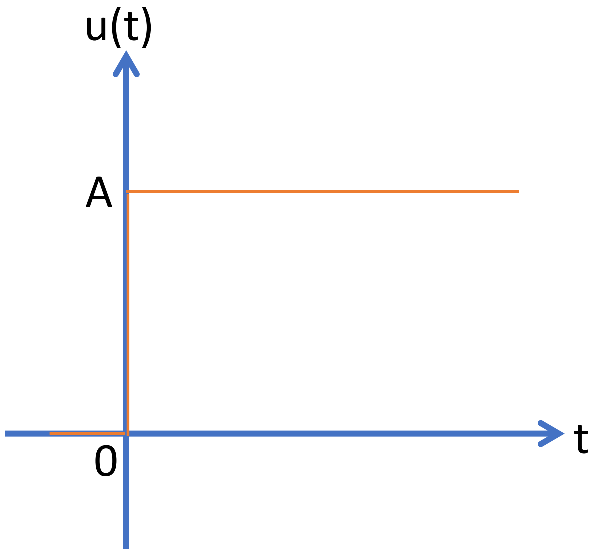
Standard Test Signals

$\dot{x} = Ax + Bu \rightarrow$ control i/p

$y = Cx + Du$

- **Step signal**
  - **The step signal imitate the sudden change characteristic of actual input signal.**

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

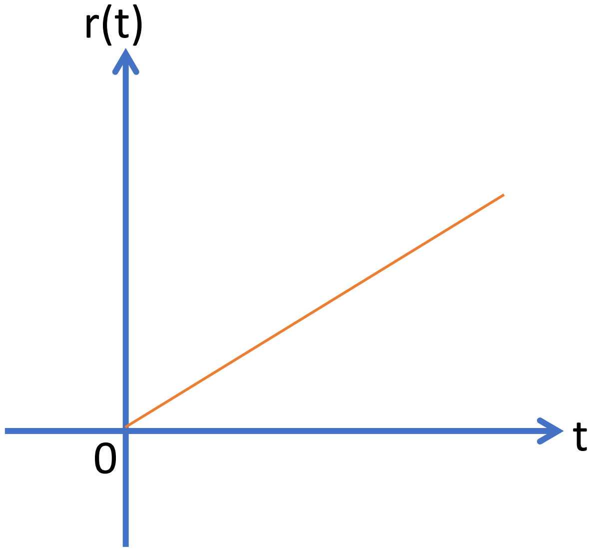


  - **If A=1, the step signal is called unit step signal**

$$L\{u(t)\} = \frac{A}{s} \quad \text{or} \quad \frac{1}{s} \; (\because A=1)$$

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \Rightarrow \text{unit ramp}$$

- **Ramp signal**
  - The ramp signal imitate the velocity characteristic of actual input signal.

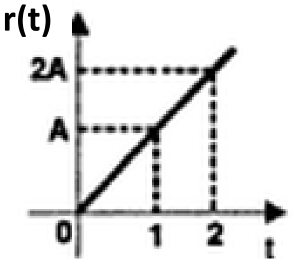


$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

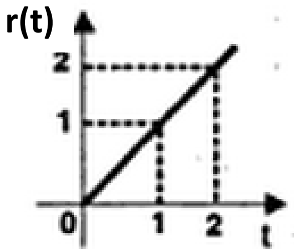
  - If *A=1*, the ramp signal is called unit ramp signal

$$R(s) = \frac{A}{s^2}, \quad L\{t^n\} = \frac{n!}{s^{n+1}}$$

ramp signal with slope A



unit ramp signal



unit Parabolic $p(t) = \begin{cases} t^2/2, & t \geq 0 \\ 0, & t < 0 \end{cases}$

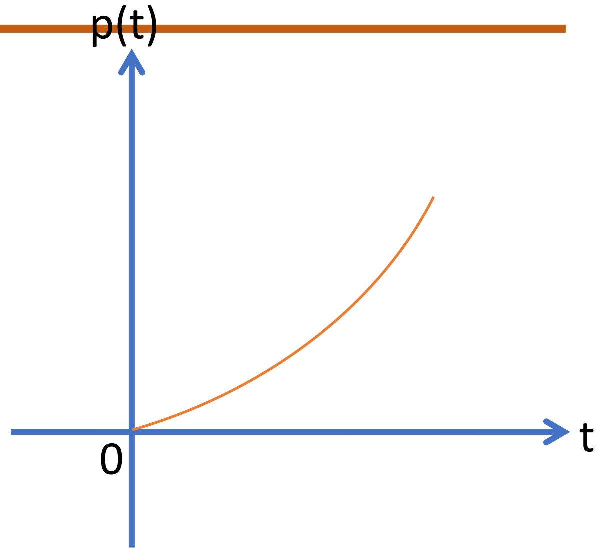
$R(s) = \frac{2!}{2s^3} = \frac{1}{s^3}$

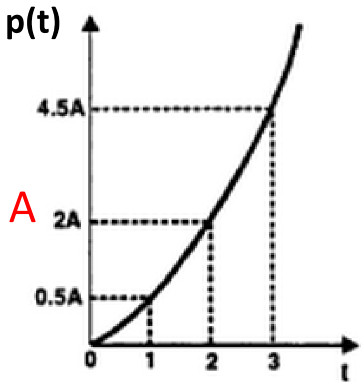
- **Parabolic signal**
  - The parabolic signal imitate the constant acceleration characteristic of actual input signal.

$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

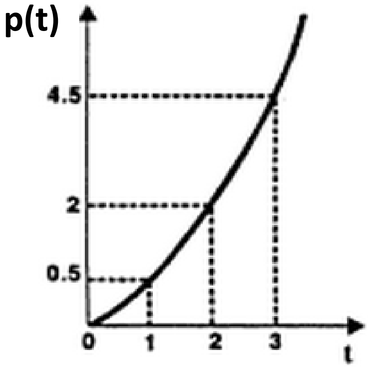
  - If *A=1*, the parabolic signal is called unit parabolic signal.

$t^n \underset{LT}{\rightleftharpoons} \frac{n!}{s^{n+1}}$





parabolic signal with slope A



Unit parabolic signal

Relation between Standard Test Signals

- **Impulse**

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

- **Step**

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- **Ramp**

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- **Parabolic**

$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$\int$ $\int$ $\int$

$\frac{d}{dt}$ $\frac{d}{dt}$ $\frac{d}{dt}$

Laplace Transform of Standard Test Signals

- **Impulse**

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$L\{\delta(t)\} = \delta(s) = A$$

- **Step**

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{u(t)\} = U(s) = \frac{A}{S}$$

- **Ramp**

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{r(t)\} = R(s) = \frac{A}{s^2}$$

- **Parabolic**

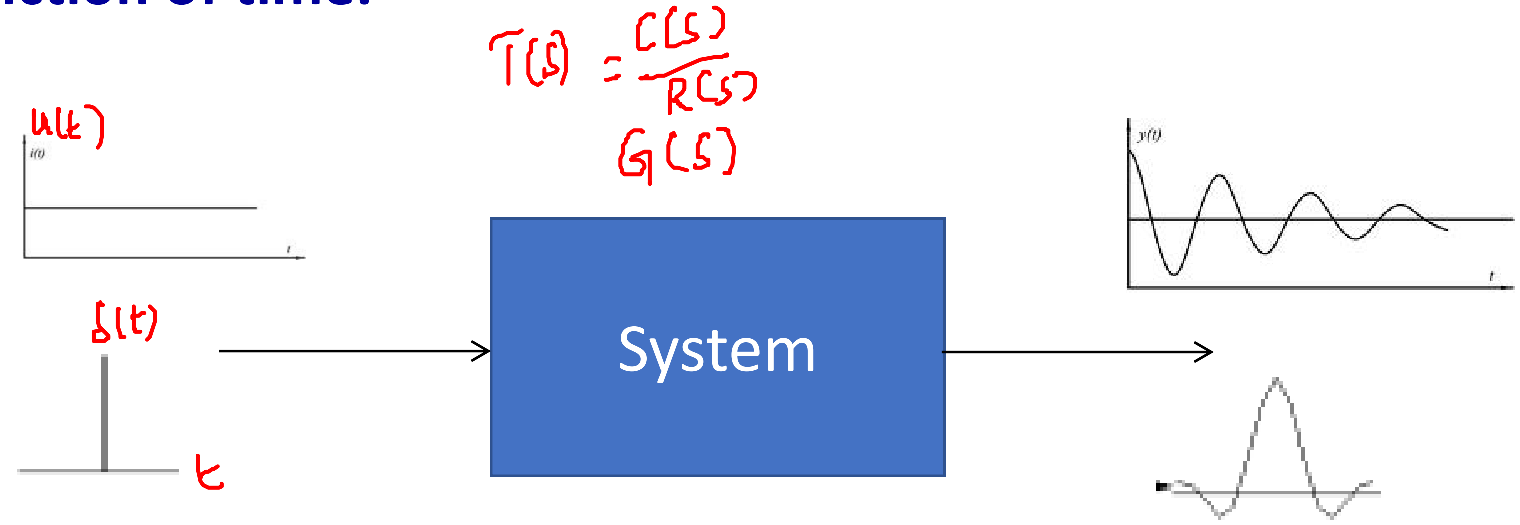
$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{p(t)\} = P(s) = \frac{A}{S^3}$$

TIME DOMAIN ANALYSIS

Time Response of Control Systems

- **Time response of a dynamic system to an input expressed as a function of time.**

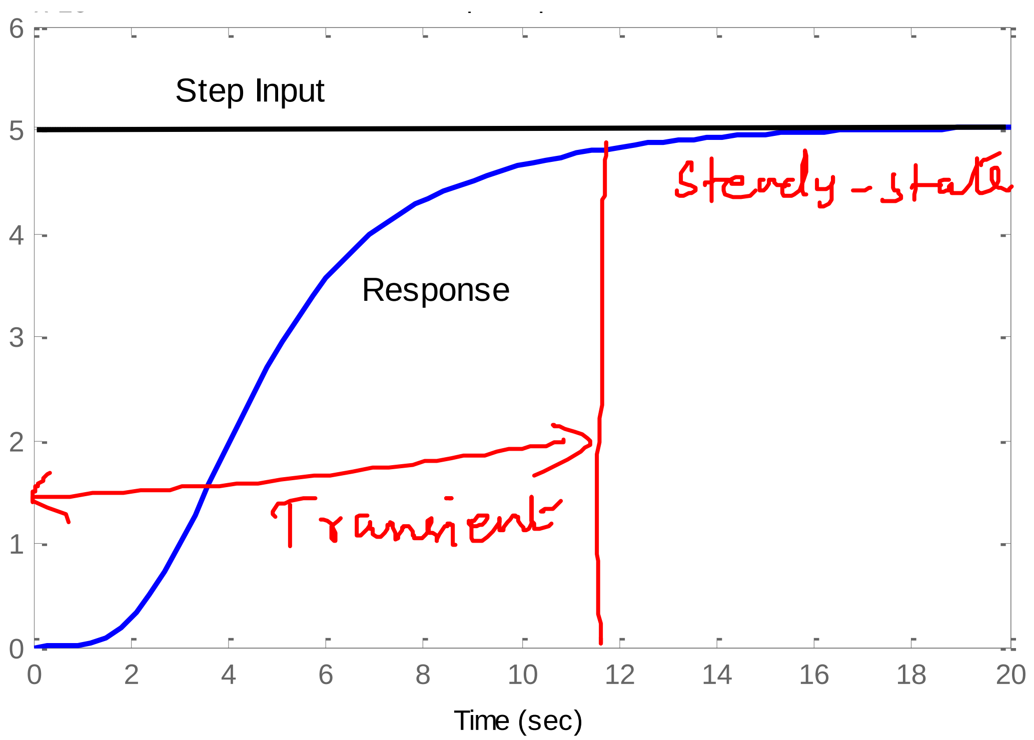


- **The time response of any system has two components**
  - **Transient response**
  - **Steady-state response.**

- **When the response of the system is changed from equilibrium it takes some time to settle down. This is called transient response.**

- **The response of the system after the transient response is called steady state response.**

Transient	steady-state
natural	forced response
Complementary	Particular solution
zero-input	zero state



TIME RESPONSE OF CONTROL SYSTEM

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TIME DOMAIN ANALYSIS

Time Response of Control Systems

- Transient response depend upon the system poles only and not on the type of input.

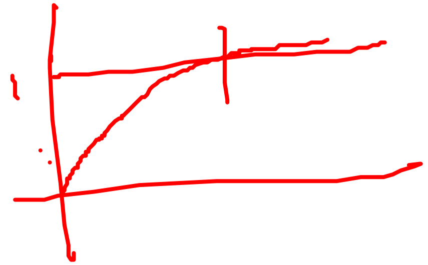
$$\frac{C(s)}{R(s)} = G(s) = \frac{1}{s+a} \quad , \quad R(s) = 1$$

- It is therefore sufficient to analyze the transient response using a step input.

$$C(s) = \frac{1}{s+a} \Rightarrow s = -a \Rightarrow \text{pole}$$

$$c(t) = e^{-at}$$

- The steady-state response depends on system dynamics and the input quantity.



- It is then examined using different test signals by final value theorem.

$$\lim_{t \to \infty} c(t) = \lim_{s \to 0} s\, C(s) = \lim_{s \to 0} s \cdot G(s) R(s)$$

- **The first order system has only one pole.**

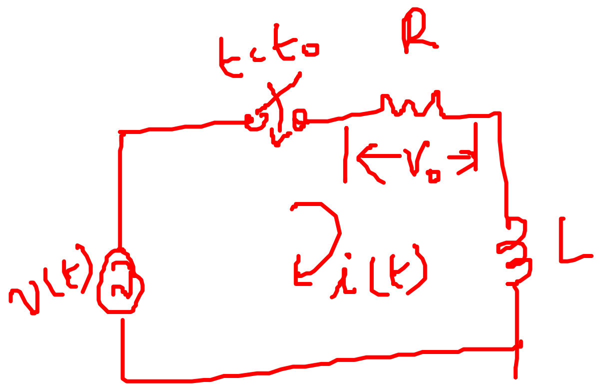
RL / RC

$\tau = RC$

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

τ

$K = 1$



$V_o(s) = I(s) \cdot R$

$\frac{V_o(s)}{V(s)} = \frac{R}{R+Ls}$

$= \frac{1}{1+\frac{L}{R}s} = \frac{1}{1+\tau s}$ ⇒ Time-constant form

time constant, $\tau = \frac{L}{R}$

$\frac{1/\tau}{s+1/\tau}$

$Ri(t) + L\frac{di}{dt} = v(t)$

$(R+Ls)I(s) = V(s)$

$\frac{I(s)}{V(s)} = \frac{1}{R+Ls}$

$= \frac{1/L}{s+R/L}$ ⇒ Pole-zero form

Sly: $a = \frac{1}{\tau}$, $\frac{V_o(s)}{V(s)} = \frac{a}{s+a}$

- The first order system has only one pole.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

- Where *K* is the D.C gain and *T* is the time constant of the system.

- Time constant is a measure of how quickly a 1st order system responds to a unit step input.

- The first order system given below.

$$G(s) = \frac{10}{3s+1}$$

- D.C gain is 10 and time constant is 3 seconds.

- For the following system

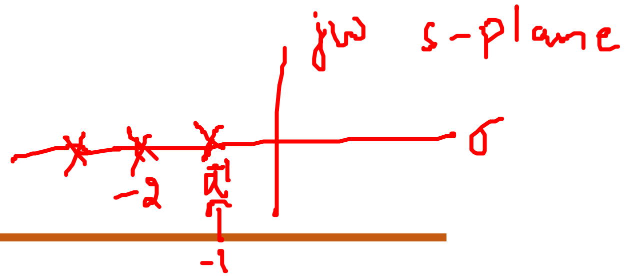
$$G(s) = \frac{3}{s+5} = \frac{3/5}{1/5s+1}$$

$$= \frac{\overset{K}{3/5}}{\underset{\tau}{\frac{1}{5}}s + 1}$$

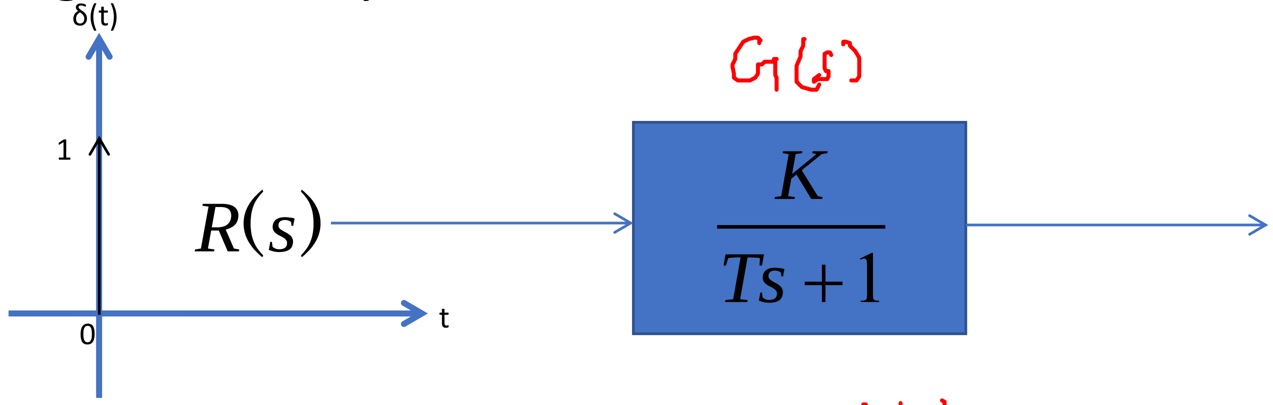
- D.C Gain of the system is 3/5 and time constant is 1/5 seconds.

TIME RESPONSE OF CONTROL SYSTEM

Impulse Response of First Order System



- Consider the following 1st order system



$= \frac{K}{Ts+1} \cdot R(s)$

$$\frac{V_0(s)}{V(s)} = \frac{a}{s+a}$$

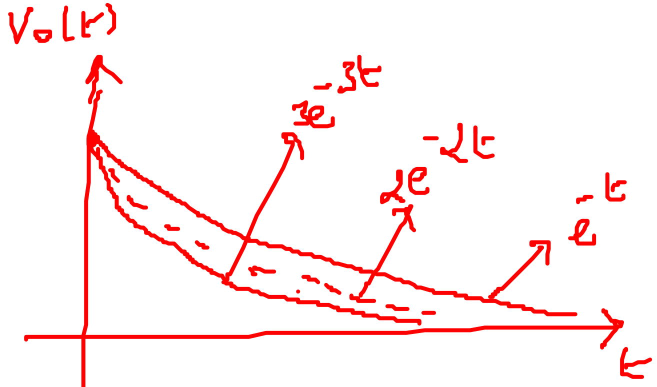
$V(s) = 1$

$$V_0(s) = \frac{a}{s+a}$$

$$R(s) = \delta(s) = 1$$

$$V_0(t) = a e^{-at}, \quad \because a = \frac{1}{\tau}$$

$$= \frac{1}{\tau} e^{-t/\tau}$$

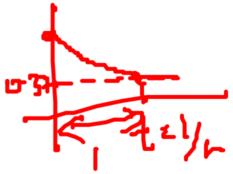


As τ ↓ses ⇒ pole is moving far from jω-axis ⇓ Response dies down faster

If $a = 1$, $\frac{1}{\tau} = 1 \Rightarrow \tau = 1 \Rightarrow e^{-t}$

$a = 2$, $1/\tau = 2 \Rightarrow \tau = 1/2 \Rightarrow 2e^{-2t}$

$a = 3$, $1/\tau = 3 \Rightarrow \tau = 1/3 \Rightarrow 3e^{-3t}$



τ can be described as the time for e^{-at} to decay to 37% of its initial value

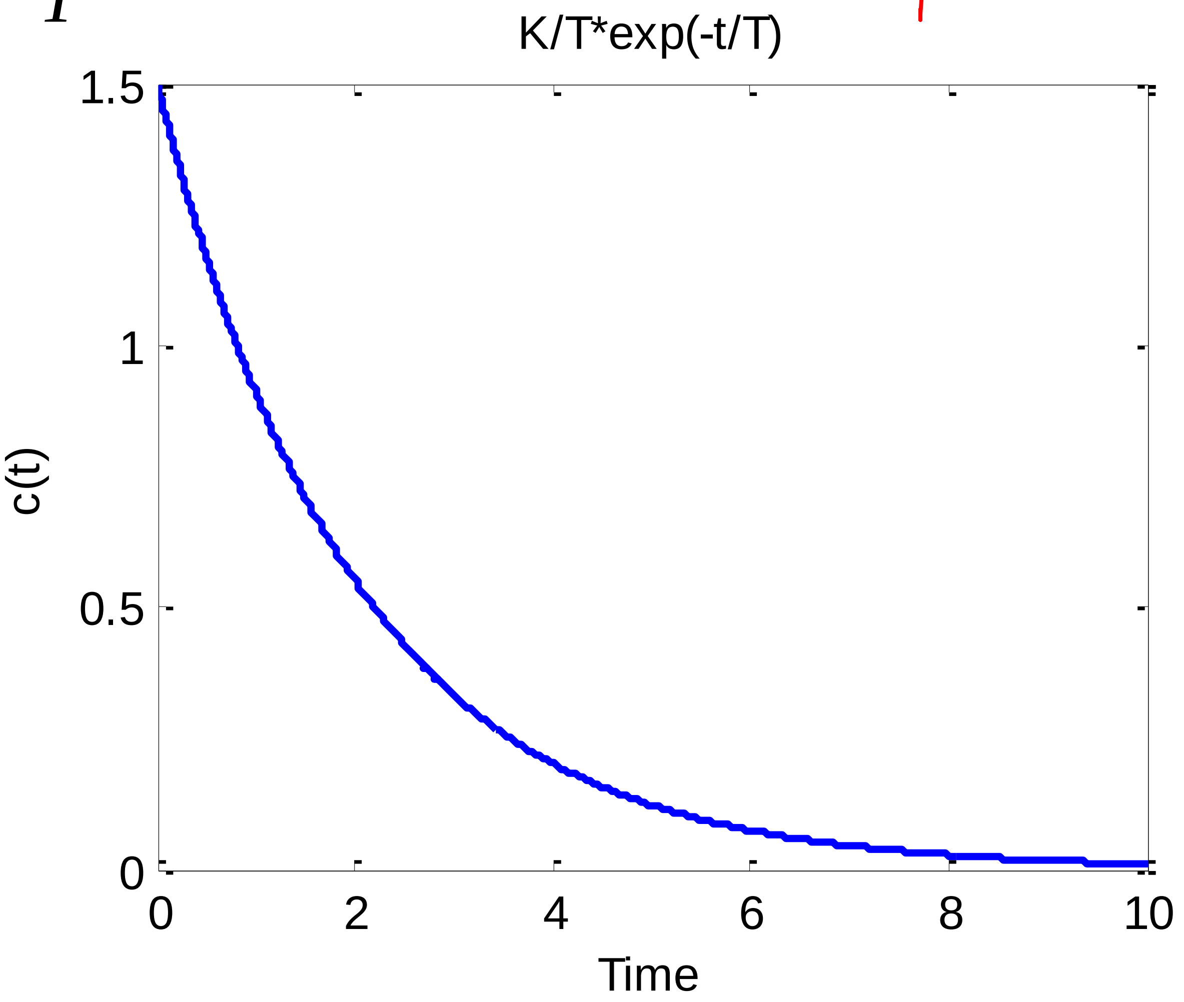
i.e if $t = \frac{1}{a}$ in $e^{-at} = e^{-1} = 0.37$

$\frac{1}{\tau} = \frac{1}{sec}$ or freq $\Rightarrow$ We can call the parameter 'a' the exponential frequency

Impulse Response of First Order System

- If K=3 and T=2s then $c(t) = \frac{K}{T} e^{-t/T}$ $= \frac{3}{2} e^{-t/2}$, $t = 0:0.1:10$

plot(t, c)



Step Response of 1st Order System

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Step Response of First Order System

$\frac{K}{T} = A(s+1/T) + Bs$

$s=0, \quad A = \frac{K/T}{1/T} = K$

$s=-1/T, \quad B = \frac{K/T}{-1/T} = -K$

- Consider the following 1st order system

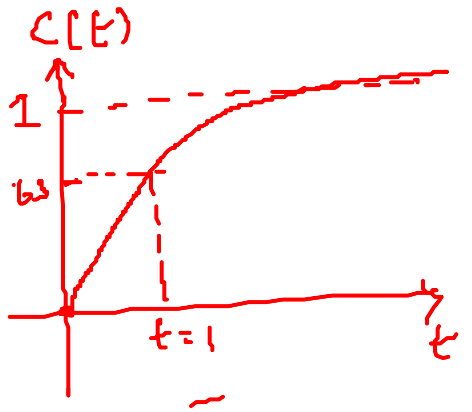
$G(s)$

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s) = G(s)\,R(s)$$

$R(s) = \frac{1}{s}$

$= \frac{K}{Ts+1}\cdot\frac{1}{s}$

$= \frac{K/T}{s(s+\frac{1}{T})}$



$$R(s) = U(s) = \frac{1}{s}$$

$$C(s) = \frac{K}{s(Ts+1)}$$

$C(s) = \frac{A}{s} + \frac{B}{s+\frac{1}{T}}$

- In order to find out the inverse Laplace of the above equation, we need to break it into partial fraction expansion

$= \frac{K}{s} - \frac{K}{s+1/T}$

$$C(s) = \frac{K}{s} - \frac{KT}{Ts+1}$$

$c(t) = K - K\cdot e^{-t/T}$

$K=1, T=1$

$c(t) = 1 - e^{-t/T}$

Step Response of First Order System

$$C(s) = K\left(\frac{1}{s} - \frac{T}{Ts+1}\right)$$

- Taking Inverse Laplace of above equation

$$c(t) = K\left(u(t) - e^{-t/T}\right)$$

- Where u(t)=1

$$c(t) = K\left(1 - e^{-t/T}\right)$$

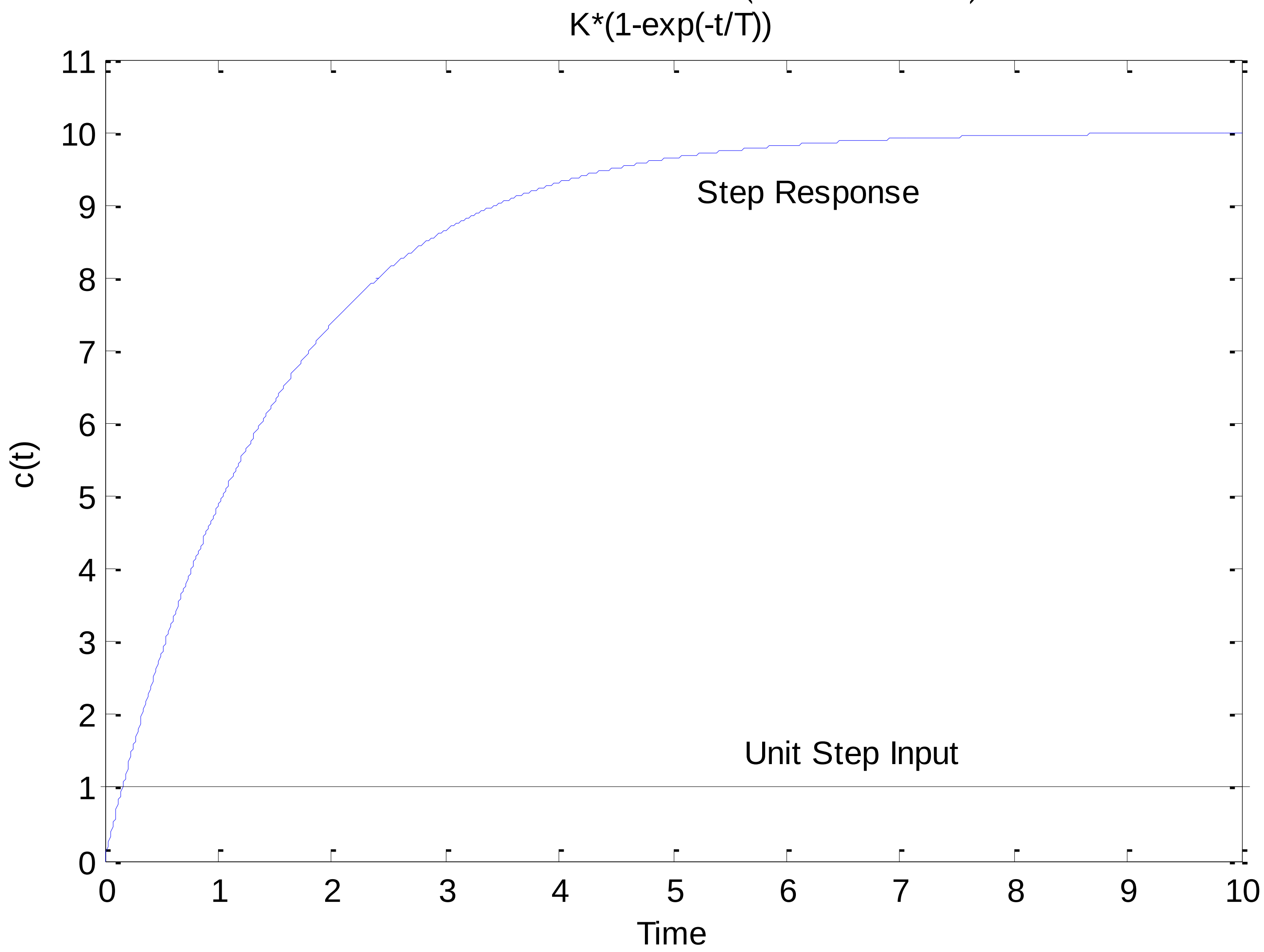
- When t=T (time constant)

$$c(t) = K\left(1 - e^{-1}\right) = 0.632K$$

time constant is the time it takes for the step response to rise to 63% of its final value

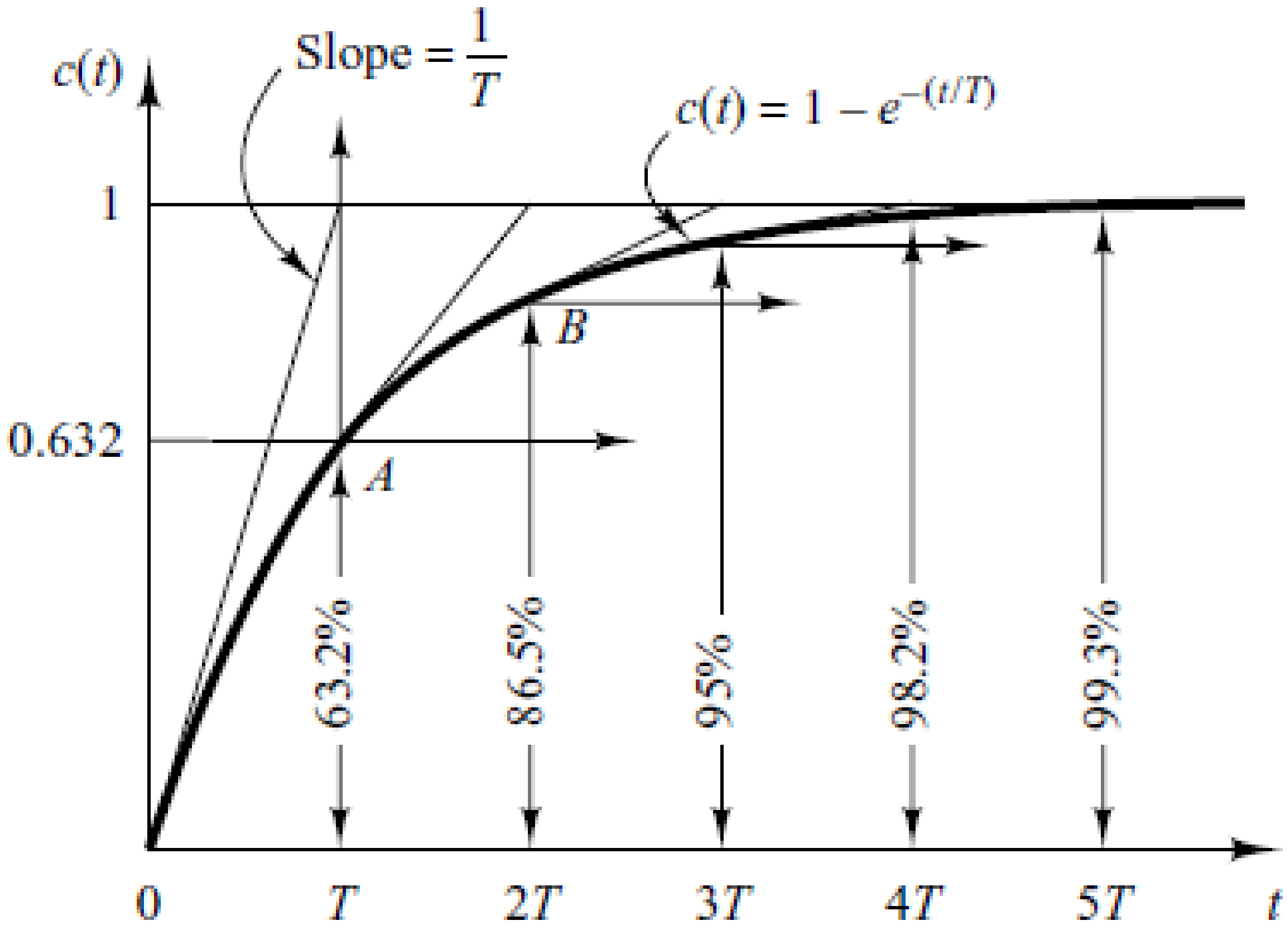
Step Response of First Order System

- If K=10 and T=1.5s then $$c(t) = K\left(1 - e^{-t/T}\right)$$



Step Response of First Order System

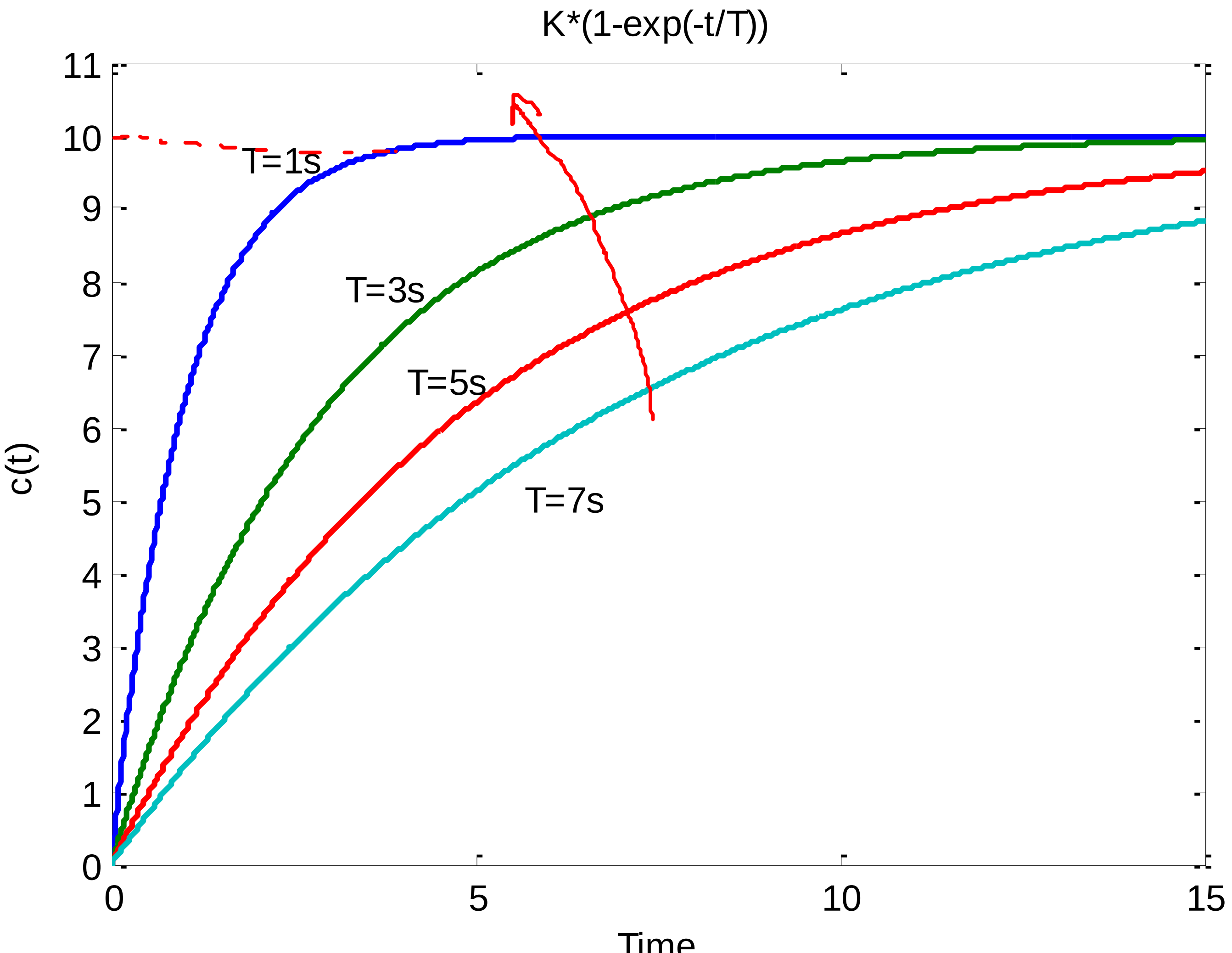
System takes five time constants to reach its final value.



Step Response of First Order System

- If K=10 and T=1, 3, 5, 7

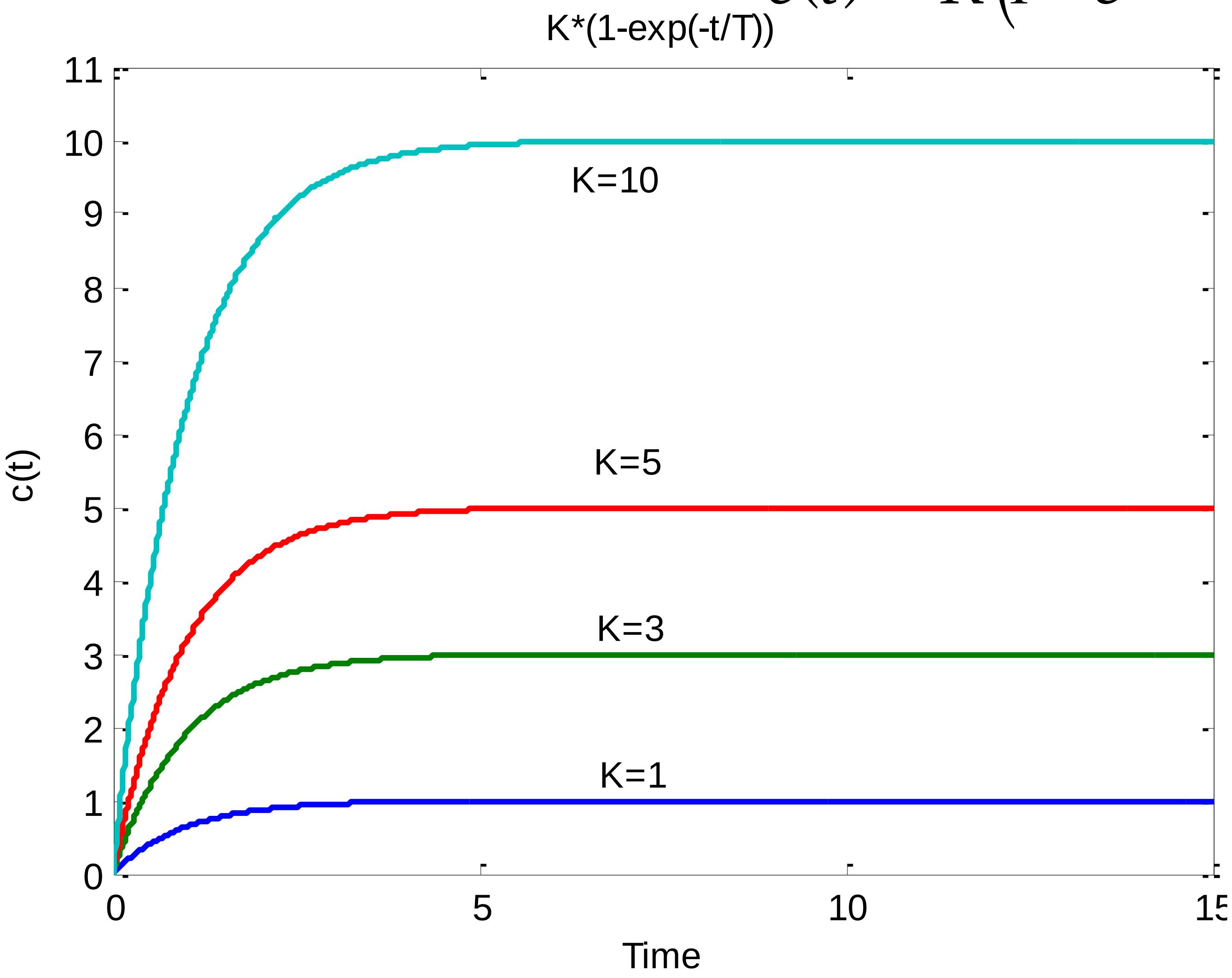
$$c(t) = K\left(1 - e^{-t/T}\right)$$



Step Response of First Order System

- If K=1, 3, 5, 10 and T=1

$$c(t) = K\left(1 - e^{-t/T}\right)$$



Relation Between Step and impulse response

- The step response of the first order system is

$$c(t) = K\left(1 - e^{-t/T}\right) = K - Ke^{-t/T}$$

$\delta(t) = \frac{d\,u(t)}{dt}$

$u(t) = \int_{0^-}^{\infty} \delta(t)\,dt$

- Differentiating c(t) with respect to t yields

$c(t) =$

$$\frac{dc(t)}{dt} = \frac{d}{dt}\left(K - Ke^{-t/T}\right)$$

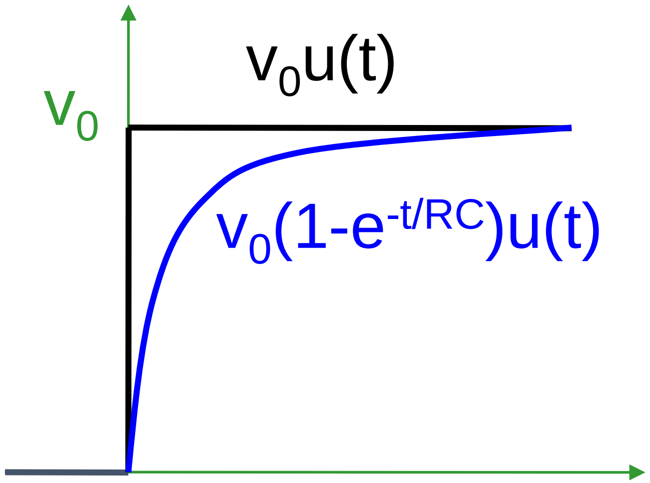
$$\frac{dc(t)}{dt} = \frac{K}{T}e^{-t/T}$$

$\rightarrow$ impulse response

Analysis of Simple RC Circuit

$$RC\frac{dv(t)}{dt}+v(t)=v_0u(t)$$

$$v(t)=Ke^{-t/RC}+v_0u(t)$$



match initial state:

$$v(0)=0 \quad \Rightarrow K+v_0u(t)=0 \Rightarrow K+v_0=0$$

output response for step-input:

$$v(t)=v_0(1-e^{-t/RC})u(t)$$

Analysis of Simple RC Circuit

- $v(t) = v_0(1 - e^{-t/RC})$ -- waveform under step input $v_0u(t)$
- $v(t)=0.5v_0 \Rightarrow t = 0.69RC$
  - i.e., delay = 0.69RC (50% delay)

$v(t)=0.1v_0 \Rightarrow t = 0.1RC$

$v(t)=0.9v_0 \Rightarrow t = 2.3RC$

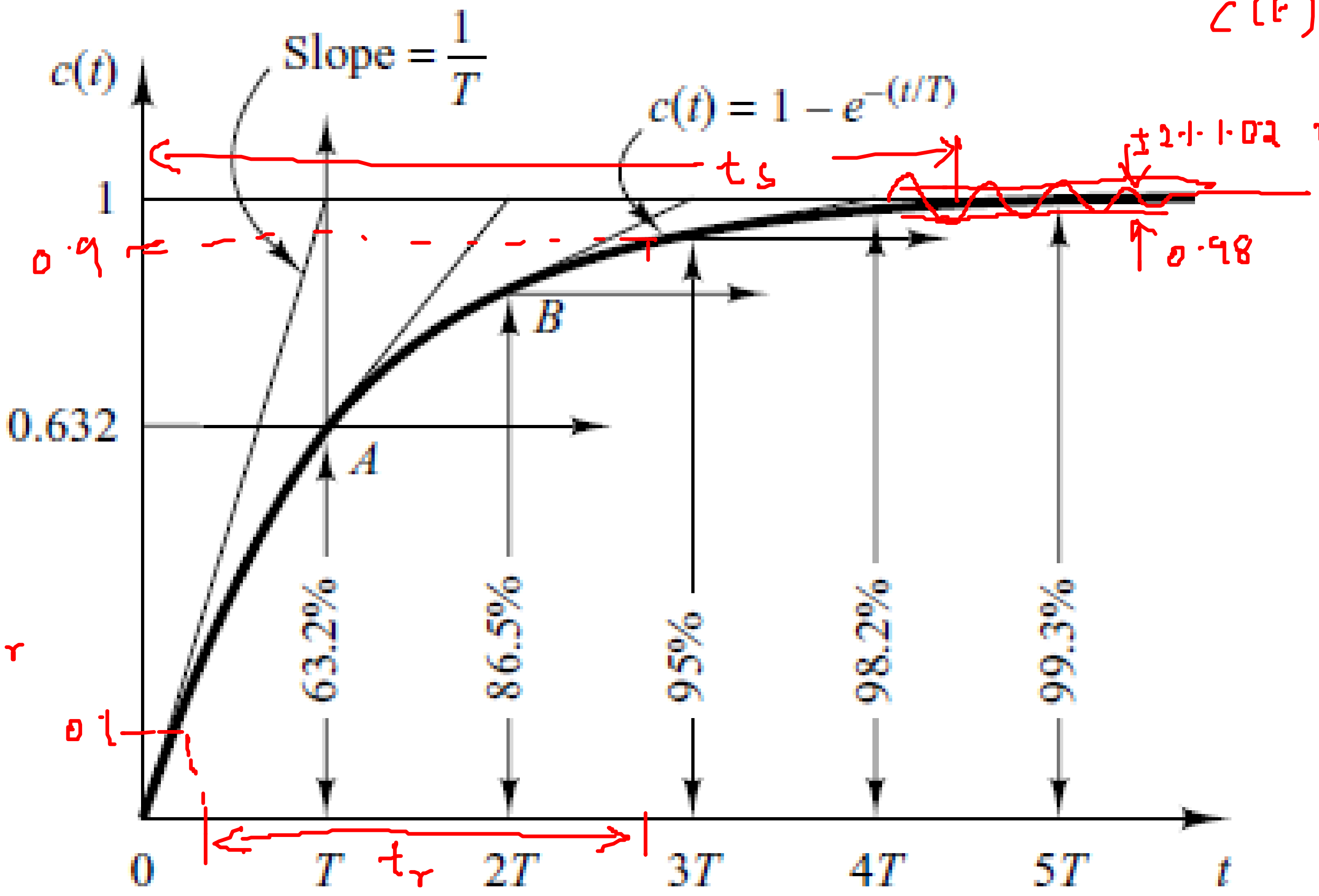
i.e., rise time = 2.2RC (if defined as time from 10% to 90% of Vdd)

Step Response of First Order System

System takes five time constants to reach its final value.

Performance specifications

1. Rise time t_r
2. Settling time t_s
3. Steady-state error e_{ss}



$t = T$

$c(t) = 1 - e^{-1} = 0.63$

error $e(t) = V_i(t) - c(t)$

$= 1 - (1 - e^{-t/T})$

$= e^{-t/T}$

$e_{ss} = \lim_{t \to \infty} e(t)$

$= \lim_{t \to \infty} e^{-t/T}$

$= 0$

Rise Time: Time for the waveform to go from 0.1 to 0.9 of its final value

$c(t) = 1 - e^{-t/T}$, if $a = 1/T$, $c(t) = 1 - e^{-at}$

at $c = 0.9$, $t = t_{r_1}$, $c(t_{r_1}) = 1 - e^{-at_{r_1}}$ $\Rightarrow$ $0.9 = 1 - e^{-at_{r_1}}$

$e^{-at_{r_1}} = 0.1$

$-at_{r_1} = \ln(0.1)$

$t_{r_1} = \frac{2.30}{a}$

let $c(t) = 0.1 \Rightarrow$ $t = t_{r_2}$, $0.1 = 1 - e^{-at_{r_2}}$ $\Rightarrow$ $e^{-at_{r_2}} = 0.9$

$-at_{r_2} = \ln(0.9)$

$t_{r_2} = \frac{0.11}{a}$

$t_r = t_{r_1} - t_{r_2} = \frac{2.30}{a} - \frac{0.11}{a} = \frac{2.19}{a} \Rightarrow \boxed{t_r = \frac{2.19}{a}}$, $\because a = \frac{1}{T}$

TIME RESPONSE OF CONTROL SYSTEMS

Step response of first order system – Settling Time

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Settling time, t_s : Time required for the system to settle within certain % of the input amplitude.

Find t_s s.t response remains within 2% tolerance i.e b/w 0.98 & 1.02

$$c(t) = 1 - e^{-at}$$

$$c(t_s) = 1 - e^{-at_s}$$

$$0.98 = 1 - e^{-at_s}$$

$$e^{-at_s} = 0.02$$

$$-at_s = \ln(0.02)$$

$$= -4$$

$$t_s = 4/a = \frac{4}{1/T} = 4T$$

5% tolerance

0.95

$$0.95 = 1 - e^{-at_s}$$

$$e^{-at_s} = 0.05$$

$$-a t_s = \ln(0.05)$$

$$t_s \approx \frac{3}{a} = 3\tau$$

Analysis of Simple RC Circuit - Example

- Impulse response of a 1st order system is given below.

$$c(t) = 3e^{-0.5t}$$

- Find out
  - Time constant T
  - DC Gain K
  - Transfer Function
  - Step Response

$$\boxed{\frac{C(s)}{R(s)} = \frac{3}{s+0.5}}$$

$$C(s) = \frac{3}{s+0.5} \cdot \cancel{R(s)} \quad 1$$

$$C(s) = \frac{3}{s+0.5} = \frac{3/0.5}{\frac{1}{0.5}s+1} = \frac{6}{2s+1}$$

K

τ or T

Analysis of Simple RC Circuit – Example 1

- **The Laplace Transform of Impulse response of a system is actually the transfer function of the system.**
- **Therefore taking Laplace Transform of the impulse response given by following equation.**

$$c(t) = 3e^{-0.5t}$$

$$C(s) = \frac{3}{S+0.5} \times 1 = \frac{3}{S+0.5} \times \delta(s)$$

$$\frac{C(s)}{\delta(s)} = \frac{C(s)}{R(s)} = \frac{3}{S+0.5}$$

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

Analysis of Simple RC Circuit – Example 1

- **Impulse response of a 1st order system is given below.**

$$c(t) = 3e^{-0.5t}$$

- **Time constant T=2**
- **DC Gain K=6**
- **Transfer Function**

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

$\Rightarrow \quad C(s) = \frac{6}{2s+1} \cancel{R(s)} \xrightarrow{} 1/s \quad = \frac{3}{s(s+1/2)}$

$= \frac{3}{s(s+1/2)} = \frac{A}{s} + \frac{B}{s+1/2}$

$C(s) = \frac{6}{s} - \frac{6}{s+1/2} \Rightarrow c(t) = 6 - 6 \cdot e^{-\frac{1}{2}t}$

Analysis of Simple RC Circuit – Example 1

- **For step response integrate impulse response**

$$c(t) = 3e^{-0.5t}$$

$$\int c(t)dt = 3\int e^{-0.5t}dt$$

$$c_s(t) = -6e^{-0.5t} + C$$

- **We can find out C if initial condition is known e.g. $c_s(0)=0$**

$$0 = -6e^{-0.5\times 0} + C$$
$$C = 6$$

$$c_s(t) = 6 - 6e^{-0.5t}$$

Analysis of Simple RC Circuit – Example 1

- If initial conditions are not known then partial fraction expansion is a better choice

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

since $R(s)$ is a step input, $R(s) = \frac{1}{s}$

$$C(s) = \frac{6}{s(2S+1)}$$

$$\frac{6}{s(2S+1)} = \frac{A}{s} + \frac{B}{2s+1}$$

$$\frac{6}{s(2S+1)} = \frac{6}{s} - \frac{6}{s+0.5}$$

$$c(t) = 6 - 6e^{-0.5t}$$

Ramp Response of 1st Order System

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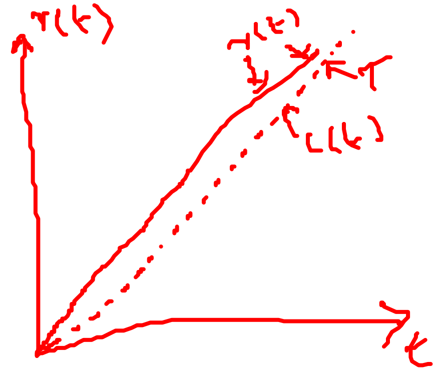
TIME RESPONSE OF CONTROL SYSTEMS

Ramp Response of First Order System

$r(t) = t, \quad t \geq 0$



- Consider the following 1st order system

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

error, $e(t)$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} (r(t) - c(t))$$
$$= \lim_{t\to\infty} (\cancel{t} - \cancel{t} + T - Te^{-t/T})$$
$$= T$$

$$R(s) = \frac{1}{s^2}$$

$$C(s) = \frac{K}{s^2(Ts+1)} \qquad \frac{K/T}{s^2(s+\frac{1}{T})} \Rightarrow \frac{A}{s^2} + \frac{B}{s} + \frac{C}{Ts+1}$$

- The ramp response is given as

$$A = s^2 C(s)\Big|_{s=0} = K \qquad C = \left(s+\frac{1}{T}\right)C(s)\Big|_{s=-\frac{1}{T}} = KT$$

$$c(t) = K\left(t - T + Te^{-t/T}\right) \qquad B = \frac{d}{ds}\left[s^2 C(s)\right]\Big|_{s=0} = -KT$$

Parabolic Response of First Order System

- Consider the following 1st order system

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = \frac{1}{s^3} \quad \text{Therefore,} \quad C(s) = \frac{K}{s^3(Ts+1)}$$

Practical Determination of Transfer Function of 1st Order Systems

- **If we can identify *T* and *K* empirically we can obtain the transfer function of the system.**

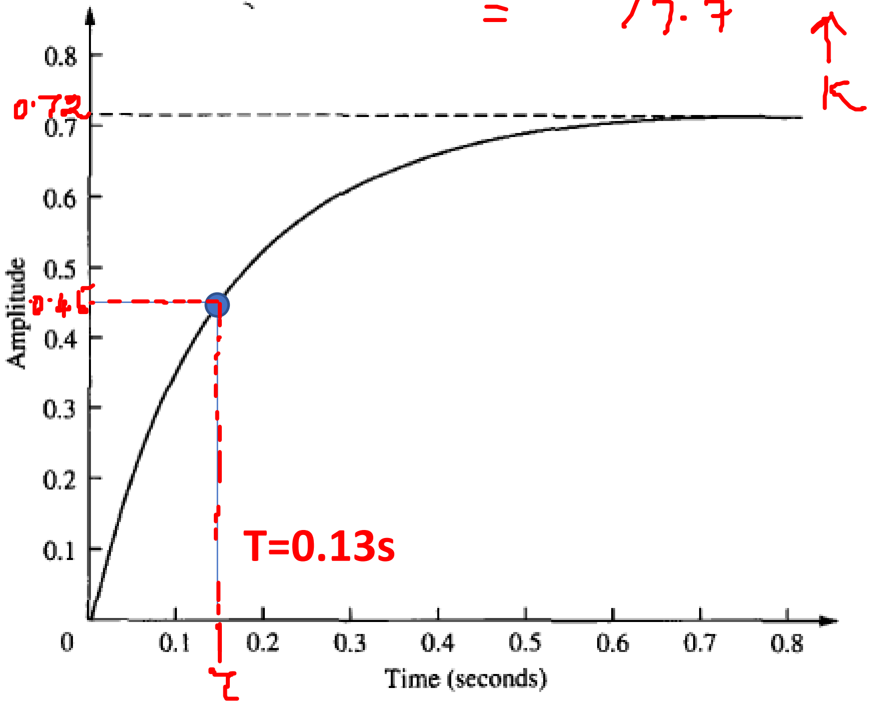
$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

$\lim_{s \to 0} s.C(s) = \lim_{s \to 0} \cancel{s}. \frac{5.5}{s+7.7} . \cancel{R(s)} \; 1/s$

$= 5.5/7.7 = 0.72$

K

Practical Determination of Transfer Function of 1st Order Systems



- For example, assume the unit step response given in figure.
- From the response, we can measure the time constant, that is, the time for the amplitude to reach 63% of its final value.
- Since the final value is about 0.72 the time constant is evaluated where the curve reaches 0.63 x 0.72 = 0.45, or about 0.13 second.

- Thus transfer function is obtained as:

$$\frac{C(s)}{R(s)} = \frac{0.72}{0.13s+1} = \frac{5.5}{s+7.7}$$

First Order System with Zero

$$\frac{C(s)}{R(s)} = \frac{K(1+\alpha s)}{Ts+1}$$

$1+\alpha s = 0 \Rightarrow s = -1/\alpha$ - zero

$1+Ts = 0 \Rightarrow s = -1/T$ - pole

- Zero of the system lie at *-1/α* and pole at *-1/T*.

- Step response of the system would be:

$C(s) = \frac{K(1+\alpha s)}{Ts+1} \cdot R(s)$, $R(s) = 1/s$

$$C(s) = \frac{K(1+\alpha s)}{s(Ts+1)} = \frac{\frac{K}{T}(1+\alpha s)}{s(s+1/T)}$$

$$C(s) = \frac{K}{s} + \frac{K(\alpha - T)}{(Ts+1)}$$

$\Rightarrow c(t) = K + K(\alpha - T) \cdot e^{-t/T}$

SECOND ORDER SYSTEMS

Performance of Second Order Systems

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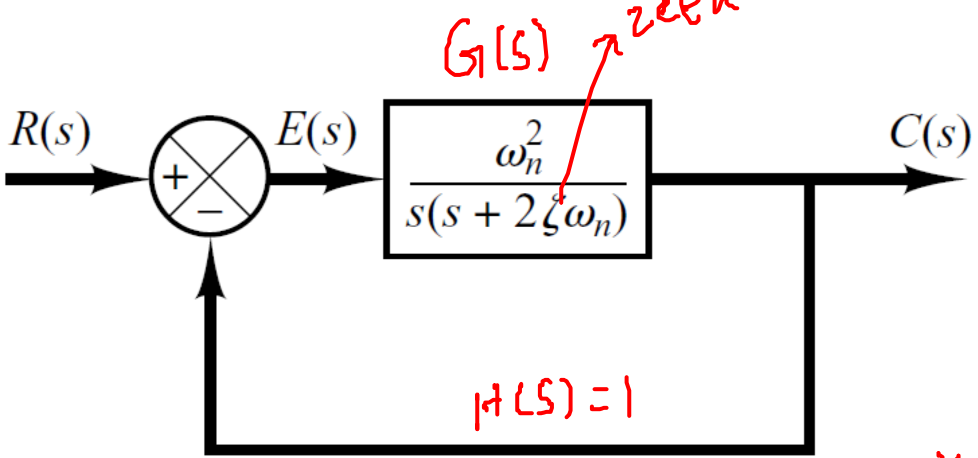
SECOND ORDER SYSTEMS

Introduction

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

A general second-order system is characterized by the following transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$



series | parallel

ex, RLC

M − S − D

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2+Bs+K} = \frac{1/M}{s^2+\frac{B}{M}s+\frac{K}{M}}$$

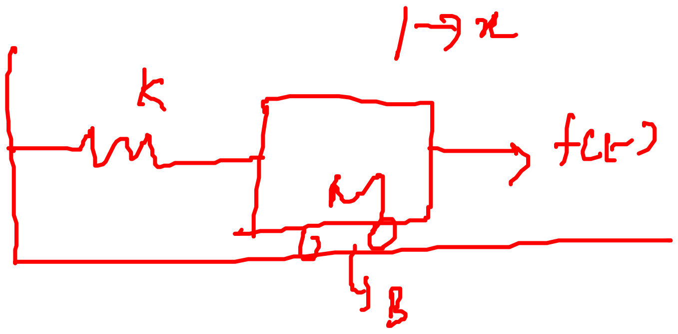
$\frac{K}{M} = \omega_n^2$

ω_n -> un-damped natural frequency of the second order system, which is the frequency of oscillation of the system without damping.

ζ-> damping ratio of the second order system, which is a measure of the degree of resistance to change in the system output.

SECOND ORDER SYSTEMS

Example – Mass Spring Damper/RLC series/RLC parallel



$$f(t) = M\ddot{x} + B\dot{x} + kx$$

$$F(s) = (Ms^2 + Bs + k)X(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + k} \Rightarrow a_2 s^2 + a_1 s + a_0$$

↳ actual or effective damping

$$s_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}$$

If $a_1^2 = 4a_0 a_2 \Rightarrow s_{1,2}$ = repeated real roots

$a_1 = \sqrt{4a_0 a_2} = 2\sqrt{a_0 a_2} \Rightarrow$ critical damping

If $a_1 = 0$, $s_{1,2} = \pm j\frac{\sqrt{4a_0 a_2}}{2a_2} = \pm j\sqrt{\frac{a_0}{a_2}}$

$a_2 s^2 + a_0 \Rightarrow s^2 = -\frac{a_0}{a_2} \Rightarrow s = \pm j\sqrt{\frac{a_0}{a_2}} \rightarrow \omega = \omega_n \Rightarrow$ natural frequency

Example – Mass Spring Damper/RLC series/RLC parallel

Damping ratio (ζ) zeta $= \dfrac{\text{Actual damping}}{\text{critical damping}} = \dfrac{a_1}{2\sqrt{a_0 a_2}} = \dfrac{a_1/a_2}{2\underbrace{\sqrt{a_0/a_2}}_{\omega_n}}$

$$\frac{1}{a_2 s^2 + a_1 s + a_0} = \frac{1/a_2}{s^2 + \frac{a_1}{a_2} s + \frac{a_0}{a_2}} = \frac{1/a_2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\zeta = \frac{a_1/a_2}{2\omega_n} \Rightarrow 2\zeta\omega_n = \frac{a_1}{a_2}$$

$$\omega_n = \sqrt{\frac{a_0}{a_2}}$$

Example

- **Determine the un-damped natural frequency and damping ratio of the following second order system.**

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4} \approx \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\omega_n^2 = 4$

$\omega_n = 2$

$\cancel{2}\zeta\omega_n = \cancel{2}$

$\zeta = 1/2 = 0.5$

- **Compare the numerator and denominator of the given transfer function with the general 2nd order transfer function.**

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \underbrace{2\zeta\omega_n}_{2} s + \omega_n^2}$$

$$\omega_n^2 = 4 \;\Rightarrow \omega_n = 2$$

$$\cancel{s^2} + 2\zeta\omega_n s + \cancel{\omega_n^2} = \cancel{s^2} + 2s + \cancel{4}$$

$$\Rightarrow 2\zeta\omega_n s = 2s$$

$$\Rightarrow \zeta\omega_n = 1$$

$$\Rightarrow \zeta = 0.5$$

SECOND ORDER SYSTEMS

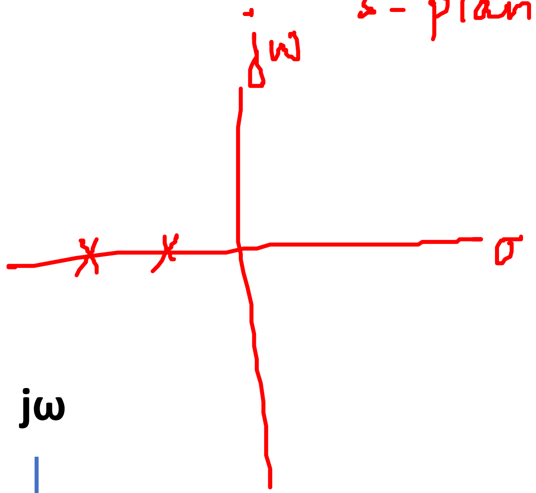
Introduction

$$s = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$s_{1,2} = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

$s = \sigma + j\omega$

s-plane



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{\underset{a}{s^2} + \underbrace{2\zeta\omega_n}_{b} s + \underbrace{\omega_n^2}_{c}}$$

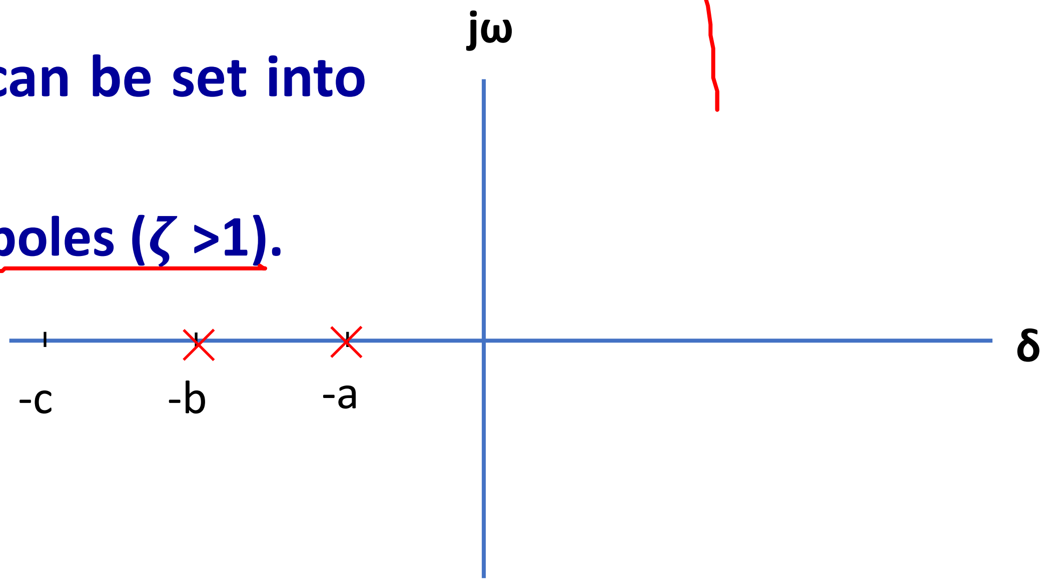
- **Two poles of the system are**

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- **According the value of ζ, a second-order system can be set into one of the four categories:**

1. Overdamped **- when the system has two real distinct poles (ζ >1).**



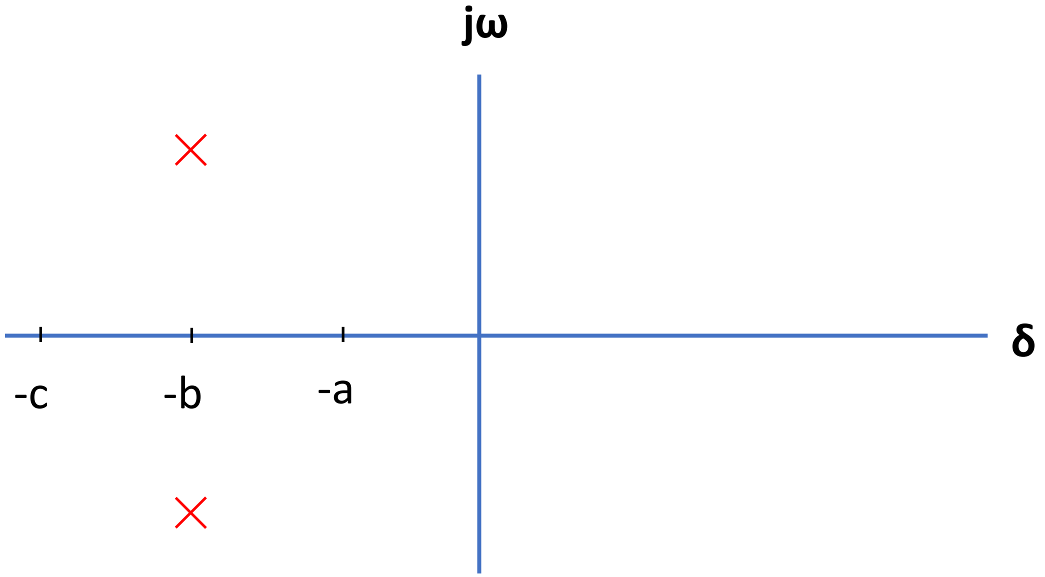
- **Two poles of the system are**

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

2. *Underdamped* –

when the system has two complex conjugate poles ($0 < \zeta < 1$)



- **Two poles of the system are**

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

$\pm j\omega_n$

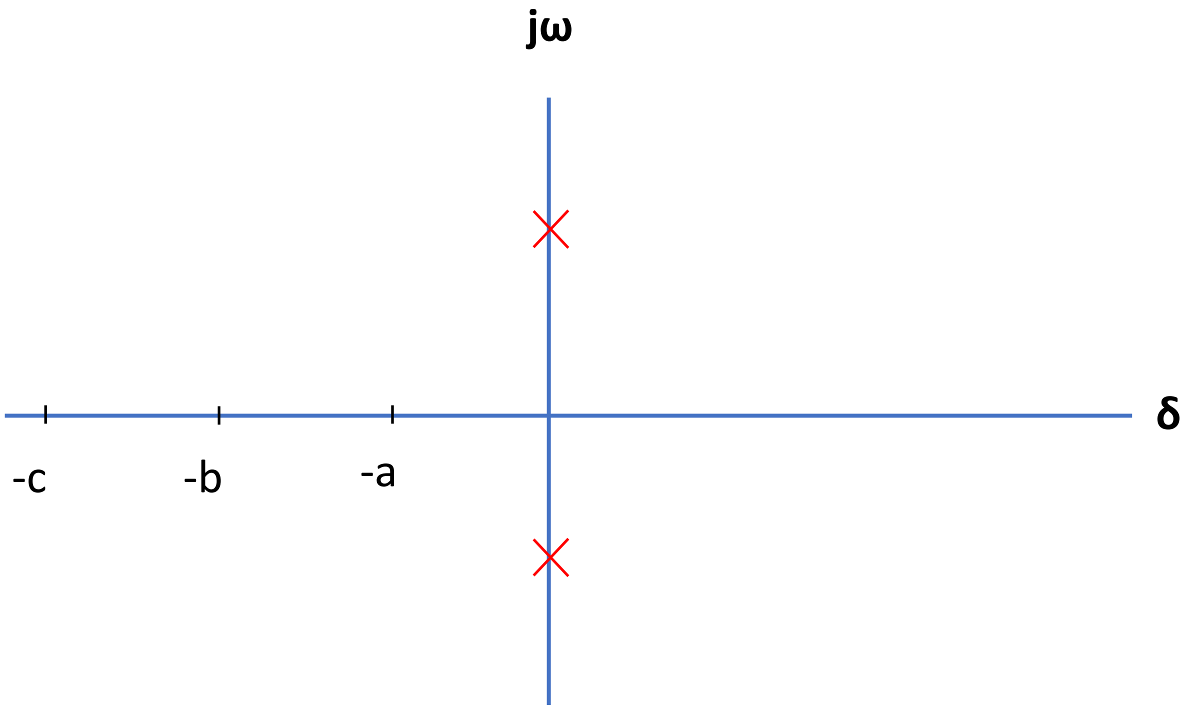
3. *Undamped* - when the system has two imaginary poles (ζ = 0).

$s^2 + 2\cancel{\zeta}^{\,0}\omega_n s + \omega_n^2$

$s^2 + \omega_n^2$

$s = \sqrt{-\omega_n^2}$

$= \pm j\omega_n$

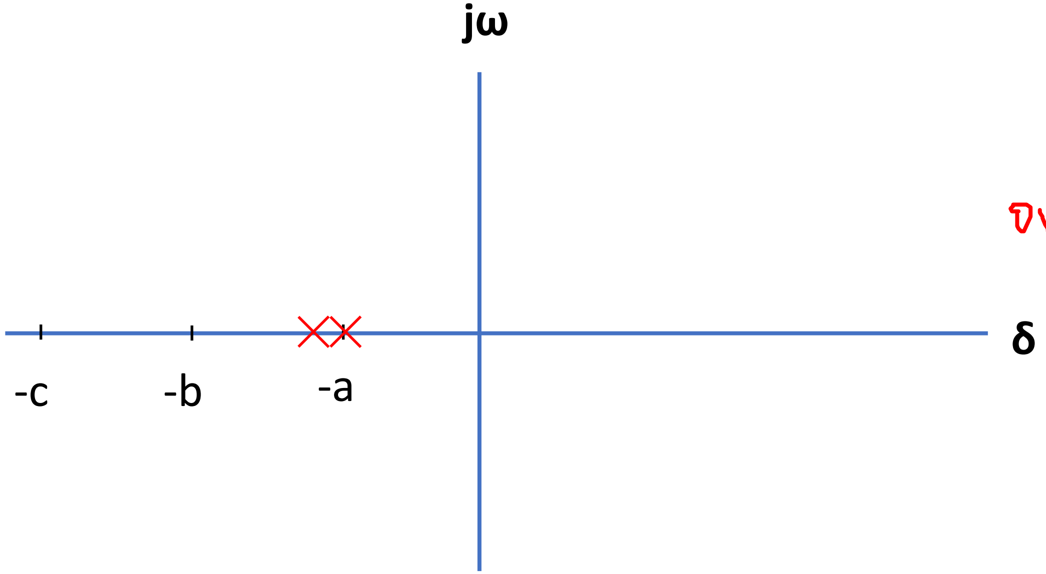


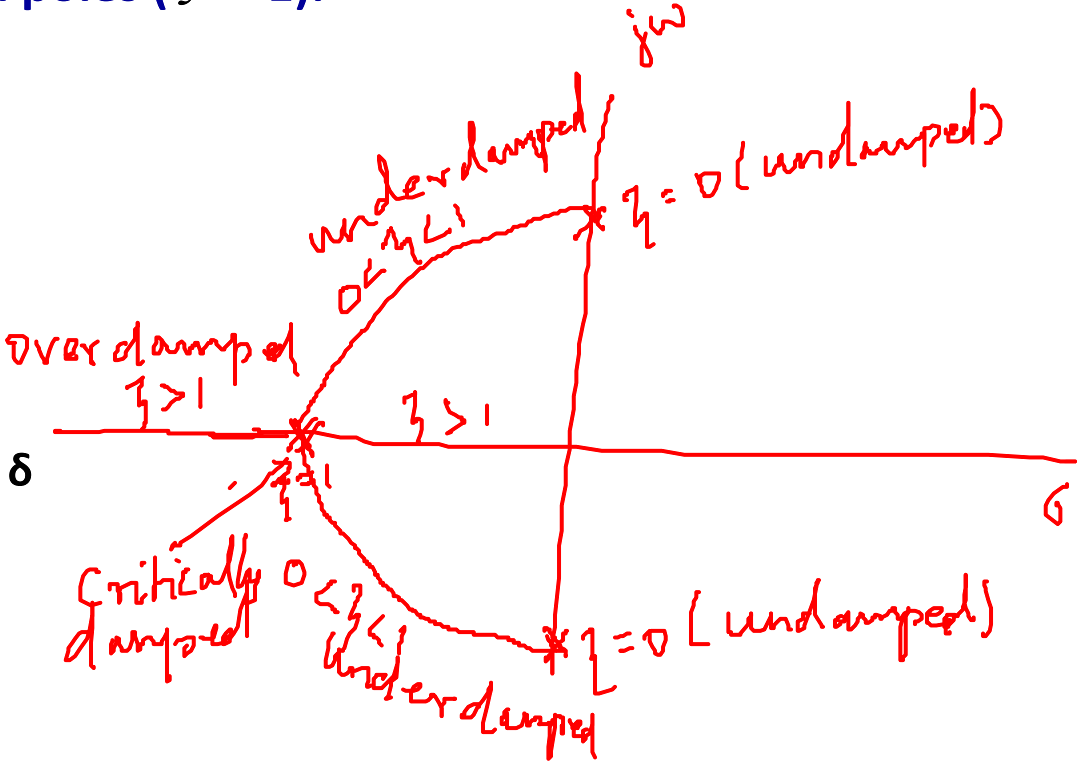
SECOND ORDER SYSTEMS
Introduction

- **Two poles of the system are**

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$
$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

4. *Critically damped* - when the system has two real but equal poles (ζ = 1).





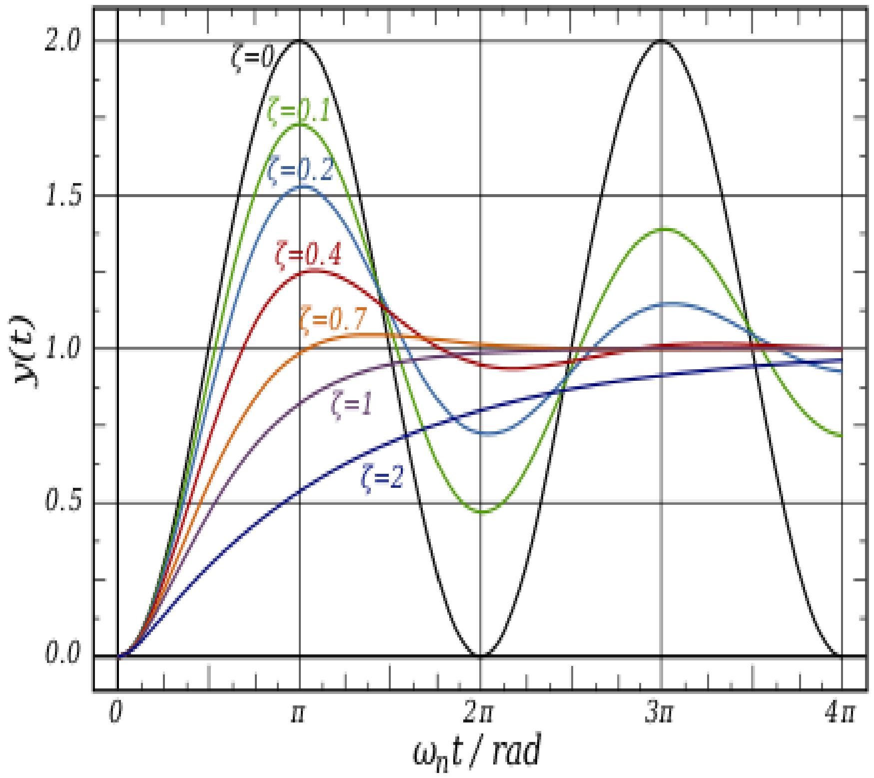
SECOND ORDER SYSTEMS

Introduction

How will the damping ratio affect the step response of the second order system?

$0 < \zeta < 1 \rightarrow$ underdamped

- **The degree of damping will indicate the nature of transients.**
- **For the ratio equal to Zero, the system will have no damping at all and continue to oscillate indefinitely.**
- **The ratio when increased from 0 to 1 (0 to 100%), will reduce the oscillations, with exactly no oscillations and best response at damping ratio equal to 1.**
- **On further increasing the damping ratio, the degree of damping has been overdone, this will cause sluggish performance/longer transients in the system.**



SECOND ORDER SYSTEMS

Impulse Response of underdamped System

$\sin(at) = \frac{a}{s^2+a^2}$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad R(s) = \delta(s) = 1$$

$\omega_d \triangleq \omega_n\sqrt{1-\zeta^2}$

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \cancel{R(s)}^{\,1}$$

$$= \frac{\omega_n^2}{\underbrace{s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2}_{} - \zeta^2\omega_n^2 + \omega_n^2} = \frac{\omega_n^2 \; \omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} \cdot \frac{1}{\omega_n\sqrt{1-\zeta^2}}$$

$(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)$

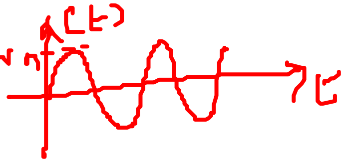
$$c(t) = \frac{\omega_n^2}{\omega_n\sqrt{1-\zeta^2}} \cdot e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\, t\right), \quad t \geq 0$$

$$c(t) = \frac{\omega_n^2}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$$

SECOND ORDER SYSTEMS

Impulse Response of underdamped System

$\zeta = 0, \quad C(s) = \dfrac{\omega_n^2}{s^2+\omega_n^2} \Rightarrow c(t) = \omega_n \sin \omega_n t$



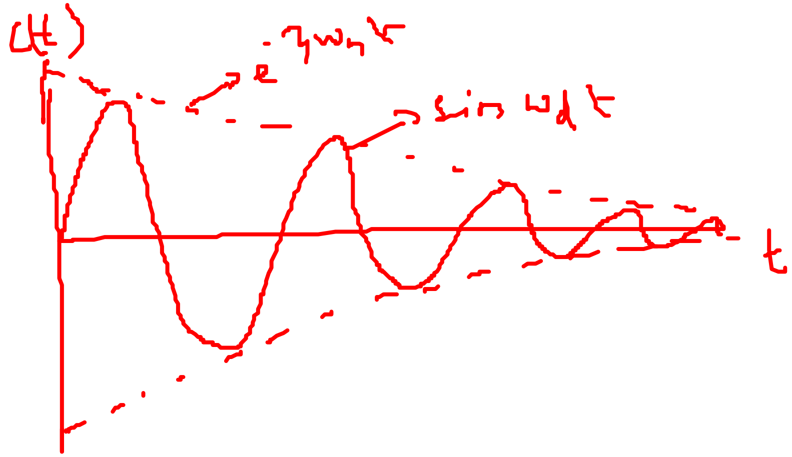
Impulse Response, $R(s) = \delta(s) = 1$

$$C(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

+ & − by $\zeta^2\omega_n^2$ in denom

$$= \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2+\zeta^2\omega_n^2-\zeta^2\omega_n^2}$$

$$= \frac{\omega_n^2}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)} \times \frac{\sqrt{\omega_n^2(1-\zeta^2)}}{\sqrt{\omega_n^2(1-\zeta^2)}}$$



$$L^{-1}[C(s)] = \frac{\omega_n^2 e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}} \sin\left(\sqrt{\omega_n^2(1-\zeta^2)}\, t\right)$$

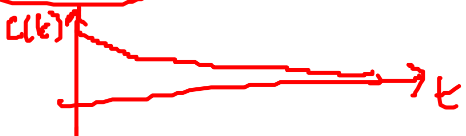
$$\boxed{c(t) = \frac{\omega_n^2 e^{-\zeta\omega_n t}}{\omega_d} \sin(\omega_d t)}, \quad \omega_d = \sqrt{\omega_n^2(1-\zeta^2)}$$

$\zeta = 0, \quad c(t) = \omega_n \sin \omega_n t$

$\zeta = 1, \quad c(t) = \omega_n^2 e^{-\omega_n t}\, t$

$C(s) = \dfrac{\omega_n^2}{(s+\omega_n)^2}$

$\zeta > 1, \quad \dfrac{\omega_n^2}{(s+s_1)(s+s_2)} = \dfrac{A}{s+s_1} + \dfrac{B}{s+s_2} = e^{-s_1 t} + e^{-s_2 t}$



$a = \dfrac{1}{\tau} \Rightarrow$ exponential frequency

Step Response of underdamped System

$0 < \zeta < 1$

$\cos(at) = \frac{s}{s^2 + a^2}$

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$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \xrightarrow{\text{Step Response}} C(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

- **The partial fraction expansion of above equation is given as**

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2\left(1 - \zeta^2\right)$$

$$(s + \cancel{2}\zeta\omega_n)^2$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{\left(s + \zeta\omega_n\right)^2 + \omega_n^2\left(1 - \zeta^2\right)}$$

Step Response of underdamped System

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

- **Above equation can be written as**

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

- **Where** $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ **, is the frequency of transient oscillations and is called damped natural frequency.**

- **The inverse Laplace transform of above equation can be obtained easily if C(s) is written in the following form:**

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

SECOND ORDER SYSTEMS

Step Response of underdamped System

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{\left(s+\zeta\omega_n\right)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\left(s+\zeta\omega_n\right)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{\left(s+\zeta\omega_n\right)^2 + \omega_d^2} - \frac{\frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{\left(s+\zeta\omega_n\right)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{\left(s+\zeta\omega_n\right)^2 + \omega_d^2} - \frac{\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{\left(s+\zeta\omega_n\right)^2 + \omega_d^2}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t$$

SECOND ORDER SYSTEMS

Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

0 < 3 < 1

- When $\zeta = 0$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$
$$= \omega_n$$

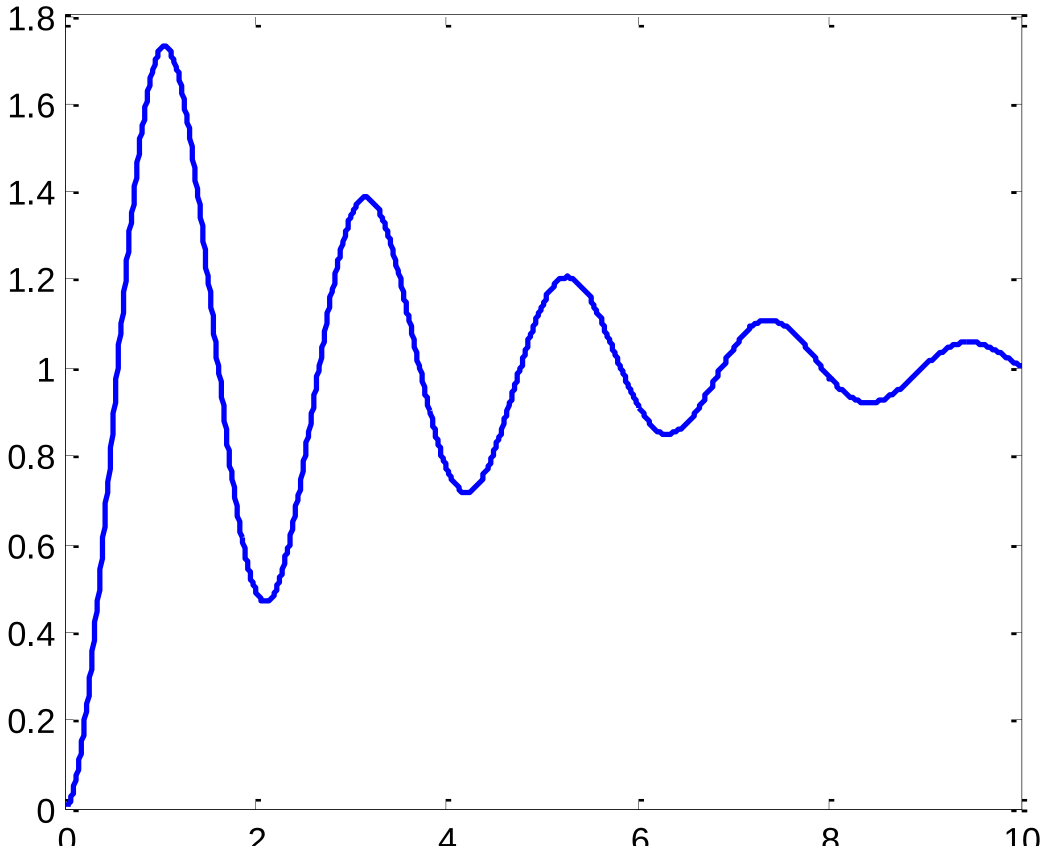
$$c(t) = 1 - \cos\omega_n t$$

SECOND ORDER SYSTEM

Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

if $\zeta = 0.1$ and $\omega_n = 3$

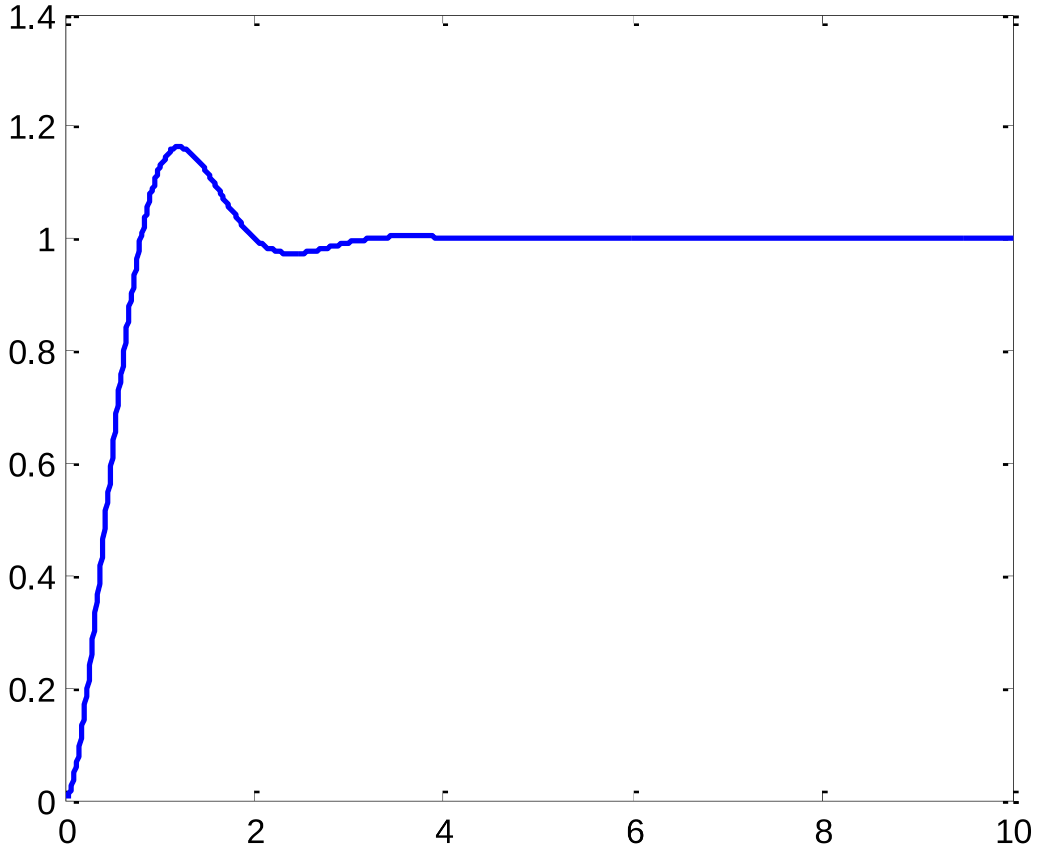


SECOND ORDER SYSTEM

Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

if $\zeta = 0.5$ and $\omega_n = 3$

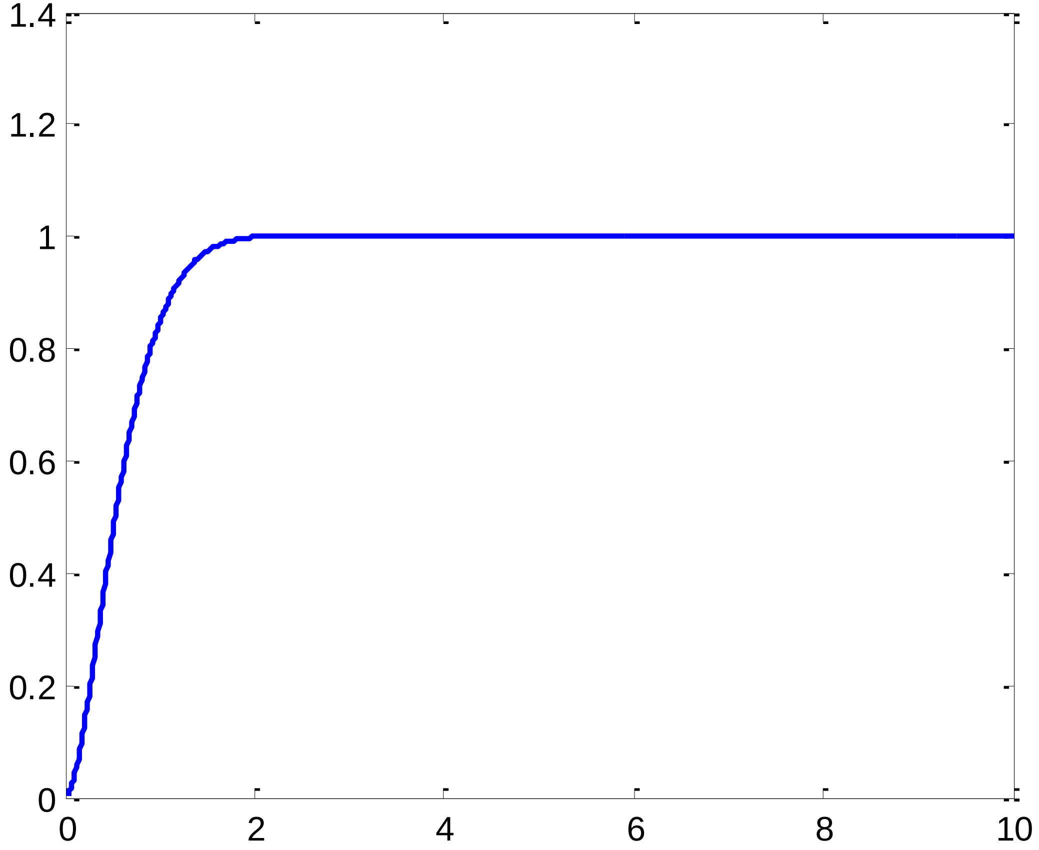


SECOND ORDER SYSTEM

Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

if $\zeta = 0.9$ and $\omega_n = 3$



SECOND ORDER SYSTEM

Step Response of critically damped System

when $\zeta = 1$, $R(s) = 1/s$

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \xcancel{R(s)}^{\,1/s} = \frac{\omega_n^2}{s(s^2 + 2\omega_n s + \omega_n^2)} = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

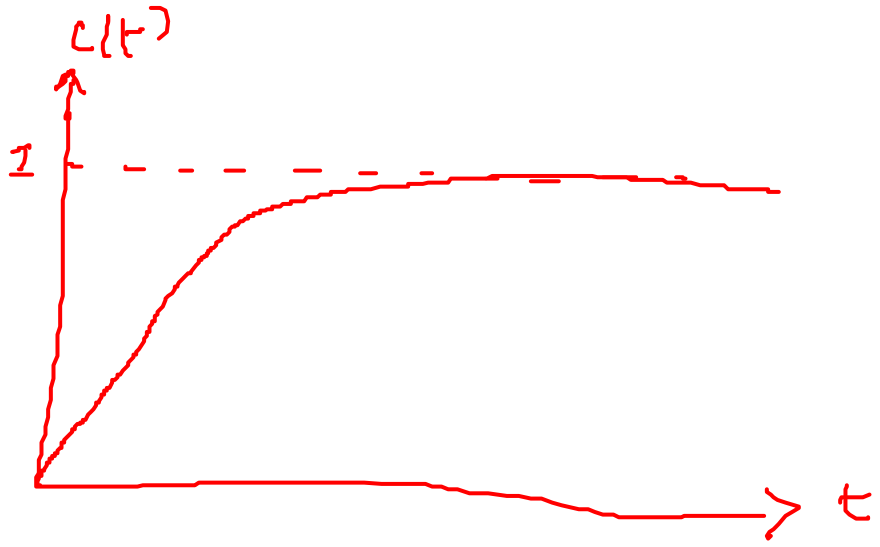
$$\frac{\omega_n^2}{s(s+\omega_n^2)} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

$$\omega_n^2 = A(s+\omega_n)^2 + Bs(s+\omega_n) + C(s)$$

$A = 1$, $B = -1$, $C = -\omega_n$

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

$$c(t) = 1 - e^{-\omega_n t} - \omega_n t\, e^{-\omega_n t}$$, assume $\omega_n = 1$



SECOND ORDER SYSTEM

S – Plane (Underdamped System)

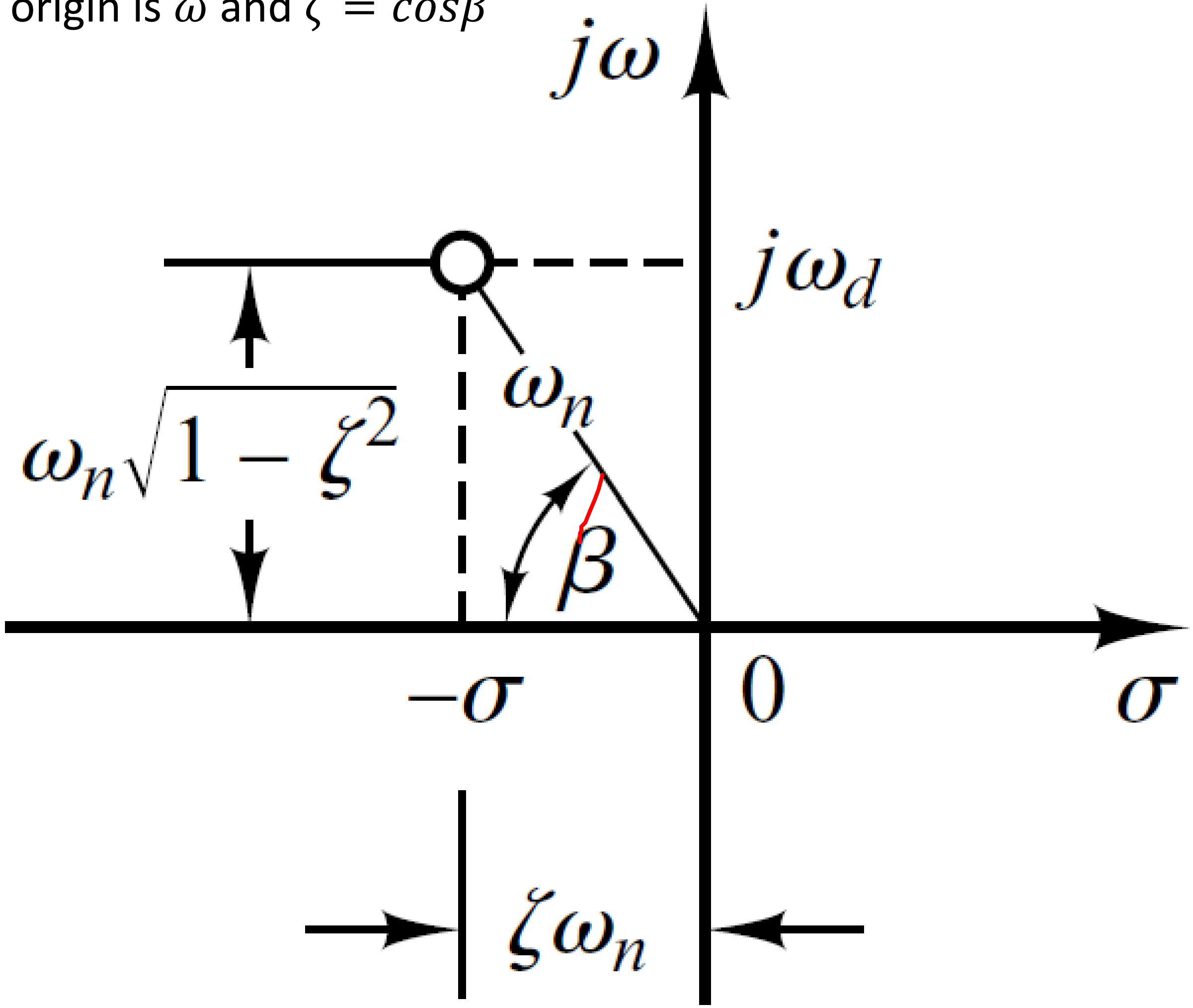
Since $\omega^2\zeta^2 - \omega^2(\zeta^2 - 1) = \omega^2$, the distance from the pole to the origin is ω and $\zeta = cos\beta$

$$\underbrace{-\omega_n\zeta}_{\sigma} + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

$$\sin\beta = \frac{\omega_n\sqrt{1-\zeta^2}}{\omega_n} = \frac{\sqrt{1-\zeta^2}}{}$$

$$\cos\beta = \frac{\zeta\omega_n}{\omega_n} = \frac{\zeta}{}$$



SECOND ORDER SYSTEM

S – Plane (Underdamped System)

$\zeta = 0 \Rightarrow c(t) = \omega_n \cos \omega_n t$

$0 < \zeta < 1 \Rightarrow$

$\zeta = 1 \Rightarrow c(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$

PES UNIVERSITY ONLINE

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\underbrace{\sqrt{1-\zeta^2}}_{\sin\beta}\cos \omega_d t + \underbrace{\zeta}_{\cos\beta} \sin \omega_d t\right]$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t\right]$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin\beta \cos \omega_d t + \cos\beta \sin \omega_d t\right)$$

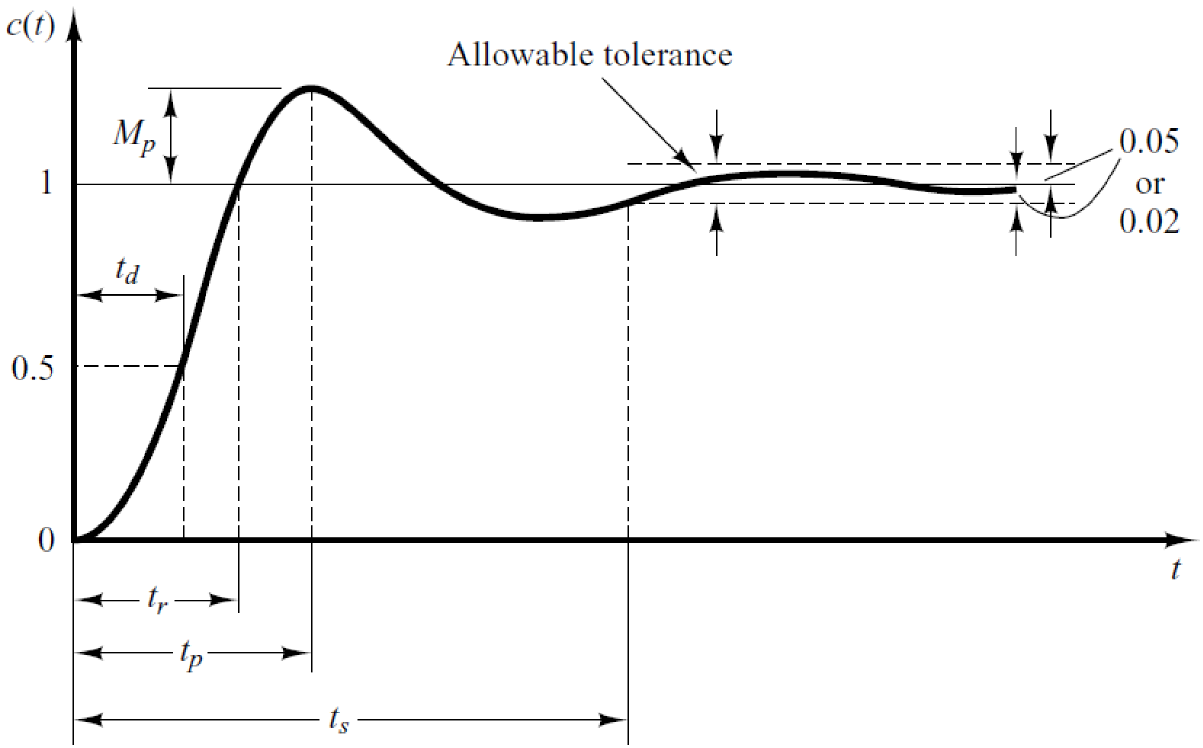
$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\beta + \omega_d t)$$

- Rise time: set c(t)=1, we have $t_r = \frac{\pi-\beta}{\omega_d}$
- Peak time: set $\frac{dc(t)}{dt} = 0$, we have $t_p = \frac{\pi}{\omega_d}$
- Maximum overshoot: $M_p = c(t_p) - 1 = e^{-(\zeta\omega_n/\omega_d)\pi}$ (for unity output)
- Settling time: the time for the outputs always within 2% of the final value is approximately $\frac{4}{\zeta\omega_{\cancel{d}\,n}}$

SECOND ORDER SYSTEMS
Underdamped System

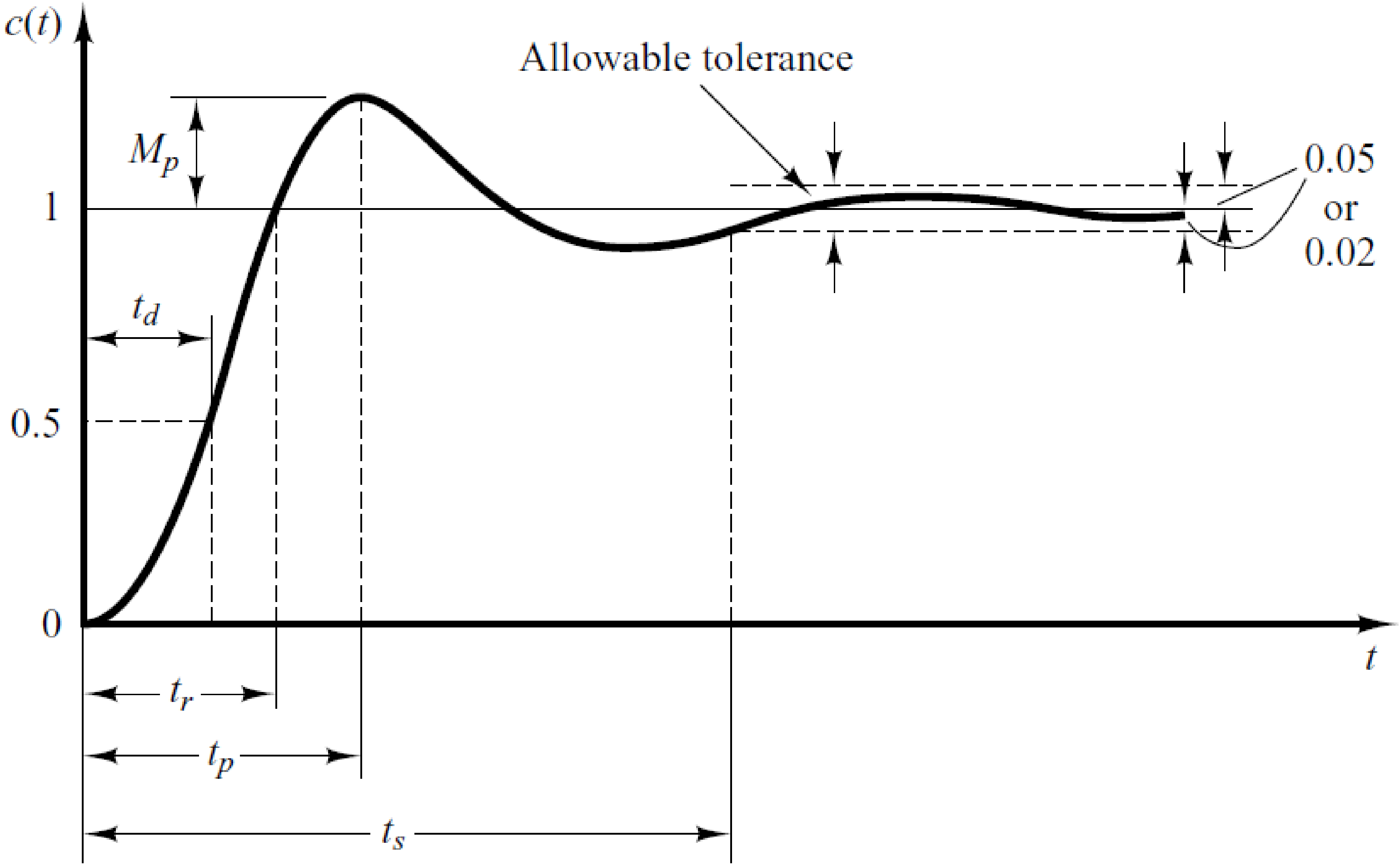
For $0 < \zeta < 1$ and $\omega_n > 0$, the 2nd order system's response due to a unit step input is as follows.

Important timing characteristics: delay time, rise time, peak time, maximum overshoot, and settling time.



Delay Time

- **The delay (t_d) time is the time required for the response to reach half the final value the very first time.**



SECOND ORDER SYSTEMS

Rise Time

$\phi = \beta = \cos^{-1}\zeta \;|\; \beta = \sin^{-1}(\sqrt{1-\zeta^2})$

- **The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value.**
- **For underdamped second order systems, the 0% to 100% rise time is normally used.**
- **For overdamped systems, the 10% to 90% rise time is commonly used.**

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\omega_d}\,\omega_n \sin(\omega_d t + \phi) \qquad \phi \rightarrow \beta$$

$$t = t_r \Rightarrow c(t_r) = 1$$

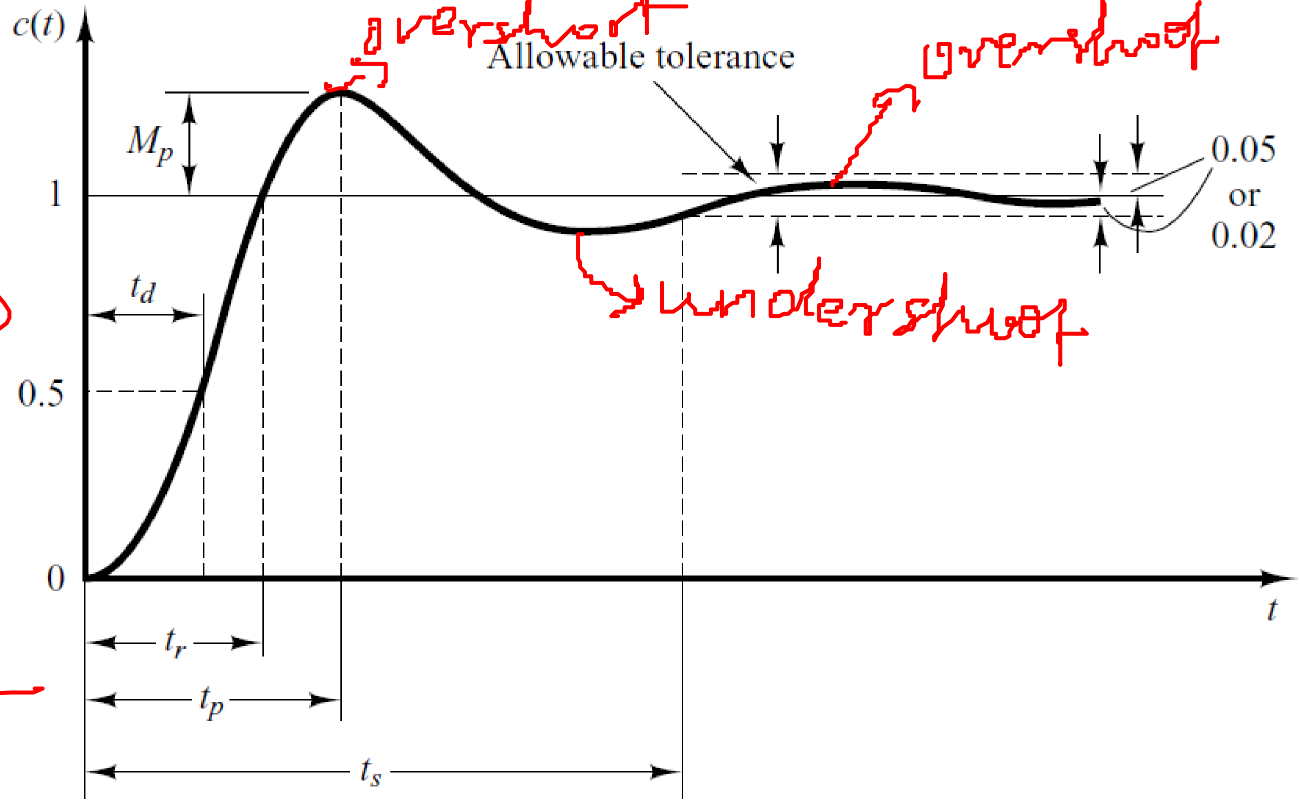
$$c(t_r) = 1 = 1 - \frac{e^{-\zeta\omega_n t_r}}{\omega_d}\,\omega_n \sin(\omega_d t_r + \phi)$$

$$\frac{e^{-\zeta\omega_n t_r}}{\omega_d}\cdot\omega_n \sin(\omega_d t_r + \phi) = 0 \quad (\pi, 2\pi,$$

$$\Rightarrow \sin(\omega_d t_r + \phi) = 0$$

$$\omega_d t_r + \phi = n\pi \Rightarrow t_r = \frac{n\pi - \phi}{\omega_d}$$

$$t_r = (\pi - \phi)/\omega_d \quad \because n=1$$



SECOND ORDER SYSTEMS

Peak Time

- **The peak time (tp) is the time required for the response to reach the first peak of the overshoot.**

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$\left.\frac{dc}{dt}\right|_{t=t_p} = \zeta\omega_n e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) + e^{-\zeta\omega_n t}\left(\omega_d\sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right)$$

$$= \zeta\omega_n e^{-\zeta\omega_n t}\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t + e^{-\zeta\omega_n t}\omega_d\sin\omega_d t$$

$$= e^{-\zeta\omega_n t}\sin\omega_d t\left(\underbrace{\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}} + \omega_n\sqrt{1-\zeta^2}}_{\omega_n}\right)$$

$$= e^{-\zeta\omega_n t}\frac{\omega_n}{\sqrt{1-\zeta^2}}\cdot\left.\sin\omega_d t\right|_{t=t_p} = 0 \Rightarrow \sin\omega_d t_p = 0$$

$$\omega_d t_p = n\pi$$

$$\boxed{\therefore\ t_p = \frac{n\pi}{\omega_d}}$$

SECOND ORDER SYSTEMS

Peak Time

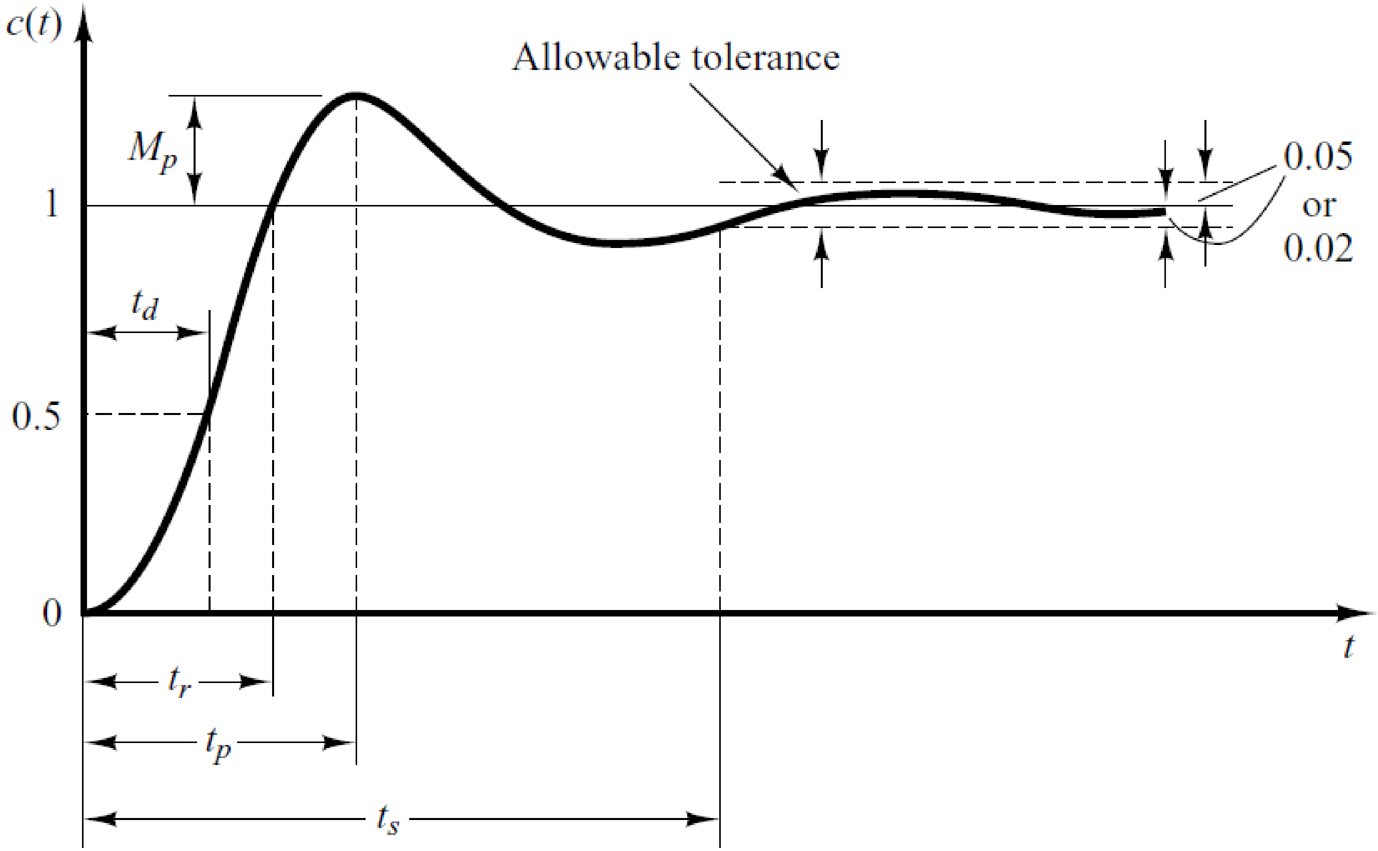
- **The peak time (tp) is the time required for the response to reach the first peak of the overshoot.**

$$\frac{dc(t)}{dt} = 0$$

$\therefore \sin \omega_d t_p = 0$

$\omega_d t_p = n\pi \quad , \quad n = 1$

$$t_p = \frac{\pi}{\omega_d}$$



SECOND ORDER SYSTEMS

Maximum Overshoot

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi) \Big|_{t=t_p}$$

The maximum overshoot is the maximum peak value of the response curve measured from unity.

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

$$M_p = c(t_p) - 1 \quad , \; t_p = \pi/\omega_d$$

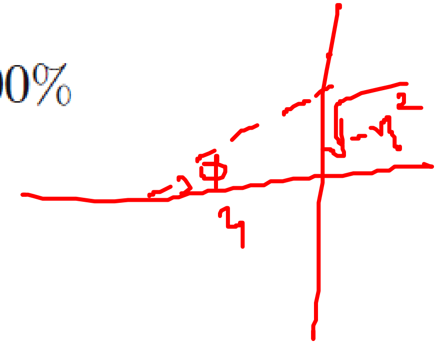
$$= -\frac{e^{-\zeta\omega_n \pi/\omega_d}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \phi)$$

$$= -\frac{e^{-\zeta\omega_n \pi/\omega_n\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}} \sin(\pi + \phi)$$

$$= -\frac{e^{-\zeta\pi/\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}} (-\sin\phi)$$

$$\sqrt{1-\zeta^2} \leftarrow \sin\phi$$

$$= e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$



SECOND ORDER SYSTEMS

Settling Time

$$c(t)\Big|_{t=t_s} = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi) = 1.02$$

$$-\frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_s + \phi) = 0.02$$

- **The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value**

(usually 2% or 5%).

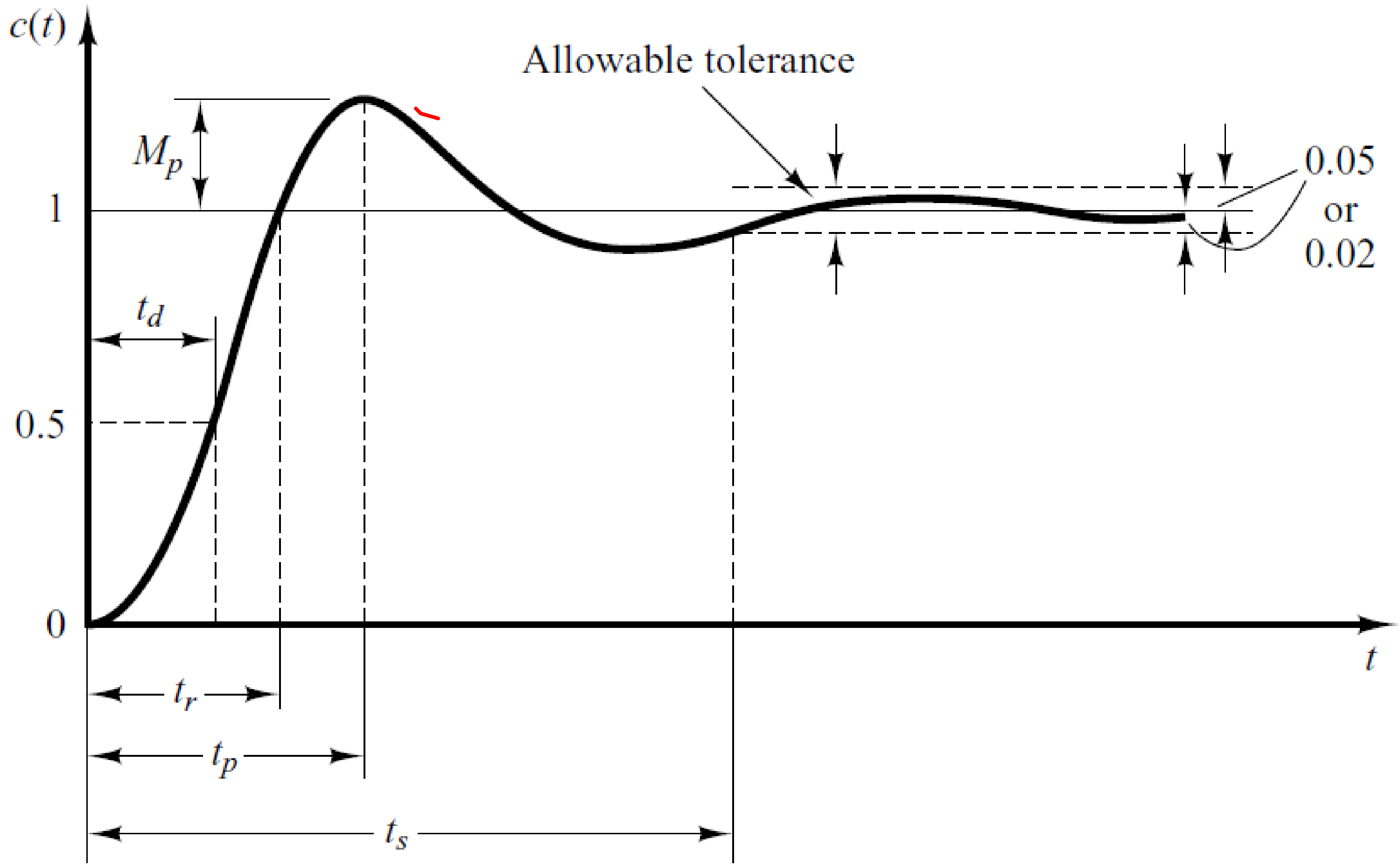
$$\frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} < 0.02$$

$$\zeta\omega_n t_s = -\ln(0.02)$$

$$t_s = \frac{4}{\zeta\omega_n} = 4\tau \ (2\%)$$

$$t_s = \frac{3}{\zeta\omega_n} = 3\tau \ (5\%)$$

$$\tau = 1/\zeta\omega_n$$



SECOND ORDER SYSTEMS

Examples

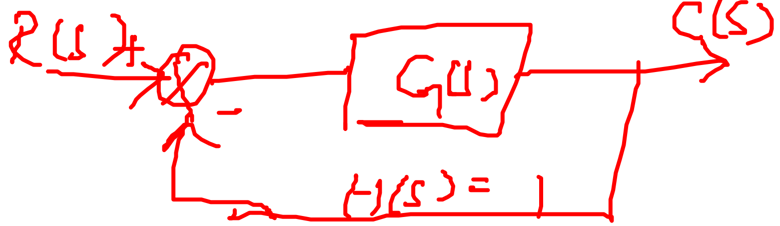
$t_s = \frac{4}{\zeta \omega_n} = 8\ sec\ [2\%]$ or $6\ sec\ [5\%]$

- **For a unity negative feedback system having open loop transfer function**

$G(s) = \frac{1}{s(s+1)}$

Find.

- **Damping ratio** , ζ
- **Damped frequency and natural frequency** ω_d ω_n
- **Peak time** , t_p
- **Peak overshoot for step input** , M_p
- t_r , t_s



$$\frac{C(s)}{R(s)} = \frac{\frac{1}{s(s+1)}}{1 + \frac{1}{s(s+1)}} = \frac{1}{s^2+s+1} \approx \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

sol: $\omega_n^2 = 1 \Rightarrow \omega_n = 1\ rad/sec$

$2\zeta\omega_n = 1 \Rightarrow \zeta = 1/2$

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 0.866$

$t_p = \frac{\pi}{\omega_d} = 3.627\ sec$

$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} = 0.163$

$t_r = \frac{\pi - \phi}{\omega_d} = 2.41\ sec$, $\zeta = \cos\phi$

$\phi = \cos^{-1}\zeta = \pi/3$

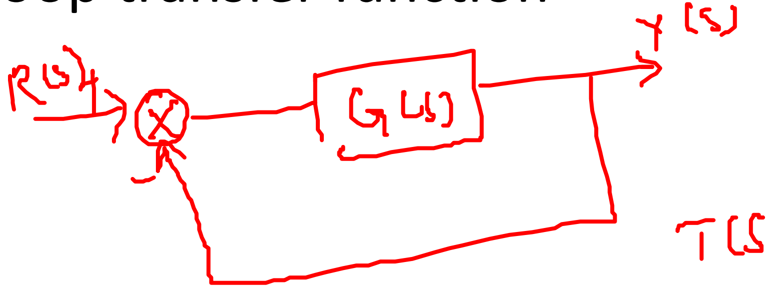
PERFORMANCE OF SECOND ORDER SYSTEMS

Example

For a unity feedback systems having open loop transfer function

$$G(s) = \frac{1}{s(s+1)}$$

Find $\zeta, \omega_d, \omega_n, t_p, M_p$ for step input.



$$T(s) = \frac{Y(s)}{R(s)}$$

$$= \frac{G(s)}{1+G(s)}$$

$$= \frac{1/s(s+1)}{1+\frac{1}{s(s+1)}}$$

$$= \frac{1}{s(s+1)+1}$$

$$s^2+s+1 = s^2+2\zeta\omega_n s+\omega_n^2$$

$$\Rightarrow \omega_n^2 = 1 \Rightarrow \omega_n = 1 \text{ rad/sec}$$

$$\Rightarrow 2\zeta\omega_n = 1 \Rightarrow \zeta = \frac{1}{2\omega_n} = 1/2$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{1-1/4} = \frac{\sqrt{3}}{2} \text{ rad/sec}$$

$$t_p = \pi/\omega_d = \frac{\pi}{\sqrt{3}/2} = \frac{2\pi}{\sqrt{3}} \text{ sec} = 3.627 \text{ sec}$$

$$M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$= e^{-\frac{\pi/2}{\sqrt{3}/2}} = e^{-\pi/\sqrt{3}}$$

$$= 0.163$$

$$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = T(s) = \frac{1}{s^2+s+1}$$

PERFORMANCE OF SECOND ORDER SYSTEMS

Example

The closed loop poles of a system are given as $-2 \pm j3$. Find $\zeta, \omega_d, \omega_n, t_p, M_p$ for step input.

Sol:- $(s+2-j3)(s+2+j3) = 0$

$(s+2)^2 + 9 = 0 \Rightarrow s^2 + 4s + 13 = 0$

$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

$\omega_n^2 = 13 \Rightarrow \omega_n = \sqrt{13} = 3.605$ rad/sec

$2\zeta\omega_n = 4 \Rightarrow \zeta = \frac{4}{2 \times 3.605} = 0.55$

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 3$ rad/sec

$t_p = \frac{\pi}{\omega_d} = 1.04$ sec

$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$

$= 12.3\ \%$

$t_s = \frac{4}{\zeta\omega_n}$ (2%) $= 2$ sec

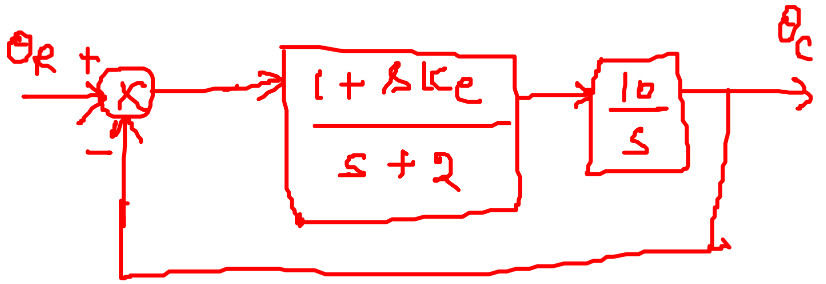
$t_r = \frac{\pi - \phi}{\omega_d} = 0.71$

$\phi = \cos^{-1}\zeta = 0.98$

PERFORMANCE OF SECOND ORDER SYSTEMS

Example

Determine the values of t_s, M_p & e_{ss} (unit ramp input) with and without error rate control K_e. Given $\zeta = 0.6$



$$\frac{\theta_c(s)}{\theta_R(s)} = \frac{\frac{10(1+sK_e)}{s(s+2)}}{1 + \frac{10(1+sK_e)}{s(s+2)}}$$

$$= \frac{10 + 10sK_e}{s^2 + 2s + 10sK_e + 10} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\Rightarrow \omega_n^2 = 10 \Rightarrow \omega_n = \sqrt{10}$

$2\zeta\omega_n = 2 + 10K_e$

$$\Rightarrow K_e = \frac{2\zeta\omega_n - 2}{10} = \frac{2 \times 0.6 \times 3.16 - 2}{10} = 0.18$$

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.6 \times \sqrt{10}} = 2.11 \text{ sec}$$

PERFORMANCE OF SECOND ORDER SYSTEMS

Example

$$\% M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100 = 9.47\%$$

$$e_{ss} = \frac{1}{K_v}, \quad K_v = \lim_{s \to 0} s\,G(s) = \lim_{s \to 0} \cancel{s} \cdot \frac{10(1+sK_e)}{\cancel{s}(s+2)} = \frac{10}{2} = 5$$

$$= \frac{1}{5}$$

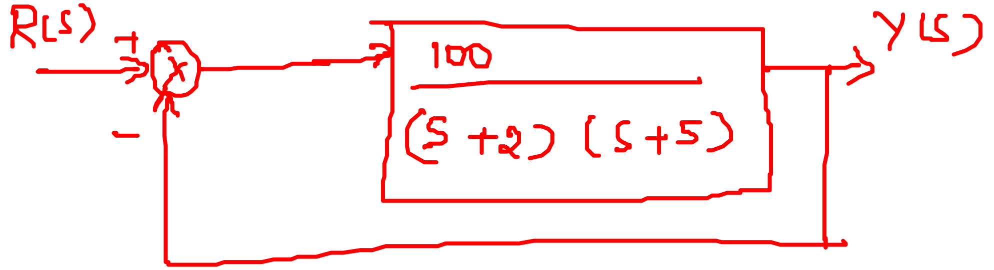
without $K_e \Rightarrow \underline{K_e = 0}$, $\quad G(s) = \frac{10}{s(s+2)}$, $\quad \frac{\theta_C}{\theta_R} = \frac{10}{s^2 + 2s + 10}$

$$\omega_n = \sqrt{10}, \quad 2\omega_n\zeta = 2 \Rightarrow \zeta = \frac{1}{\sqrt{10}} = 0.32$$

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.32 \times \sqrt{10}} = \underline{4 \text{ secs}}, \quad \% M_p = e^{-\pi \times 0.32/\sqrt{1-0.32^2}} \times 100 = \underline{35\%}$$

PERFORMANCE OF SECOND ORDER SYSTEMS

Example



Determine e_{ss} (step i/p), % overshoot, ζ & ω_n.

Sol: $e_{ss} = \dfrac{A}{1+K_p}$, $K_p = \lim\limits_{s \to 0} G(s) = 10$

$= \dfrac{A}{1+10} = A/11$

$\dfrac{Y(s)}{R(s)} = \dfrac{100}{s^2+7s+110}$

final value

$\cancel{Y(s)} = \lim\limits_{s \to 0} s \cdot Y(s)$

$= \lim\limits_{s \to 0} s \cdot \underset{A/s}{\cancel{R(s)}} \cdot \dfrac{100}{s^2+7s+110}$

$= \dfrac{100A}{110} = 0.909A$

$2\zeta\omega_n = 7$

$\zeta = \dfrac{7}{2\sqrt{110}}$

$= 0.334$

% P.O $= \left(100 \times e^{-\pi\zeta/\sqrt{1-\zeta^2}}\right) \times 0.909$

$= 29.86$ %

STEADY – STATE ERROR

Example

$T(s) = \frac{400}{s^2 + 12s + 400}$

find ζ, ω_n, ω_d

Determine the impulse & step response

Impulse response

$$y(t) = \frac{\omega_n^2}{\omega_d} e^{-\zeta \omega_n t} \sin(\omega_d t)$$

$\cos\phi = \zeta$

$\sin\phi = \sqrt{1-\zeta^2}$

Step response

$$y(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin(\phi + \omega_d t)$$

$\omega_d = \omega_n \sqrt{1-\zeta^2}$

The Performance of Feedback Control Systems:

Example

$$T(s) = \frac{9}{s^2+2s+9}$$

Find the step response

$\zeta = 0.33$, $\omega_n = 3$, $\omega_d = 2.83$ rad/sec

$\phi = 1.23$

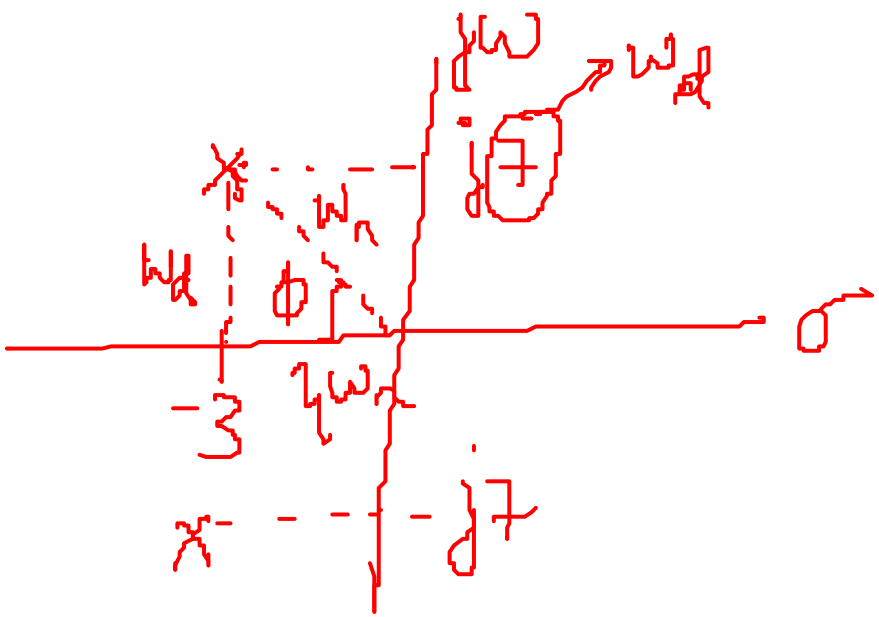
$$c(t) = 1 - \frac{e^{-t}}{0.94}\sin(1.23 + 2.83t)$$

$$T(s) = \frac{9}{s^2+9} \Rightarrow C(s) = \frac{9}{s(s^2+9)} =$$

The Performance of Feedback Control Systems:

Example

$$-\zeta\omega_n \pm \underbrace{j\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$$



~~for~~ find ζ, ω_n, t_p

$\% M_p, t_s$

$\zeta = \cos\phi$, $\phi = \tan^{-1}\frac{7}{3} = 1.16$

$= 0.394$

$\omega_n = \sqrt{7^2+3^2} = 7.616$ rad/sec

$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449$ sec,

$\% M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$

$= 26\%$

$t_s = 4/3 = 1.33$ sec.

The Performance of Feedback Control Systems:

Effects of a Third Pole and a Zero on the Second Order System Response

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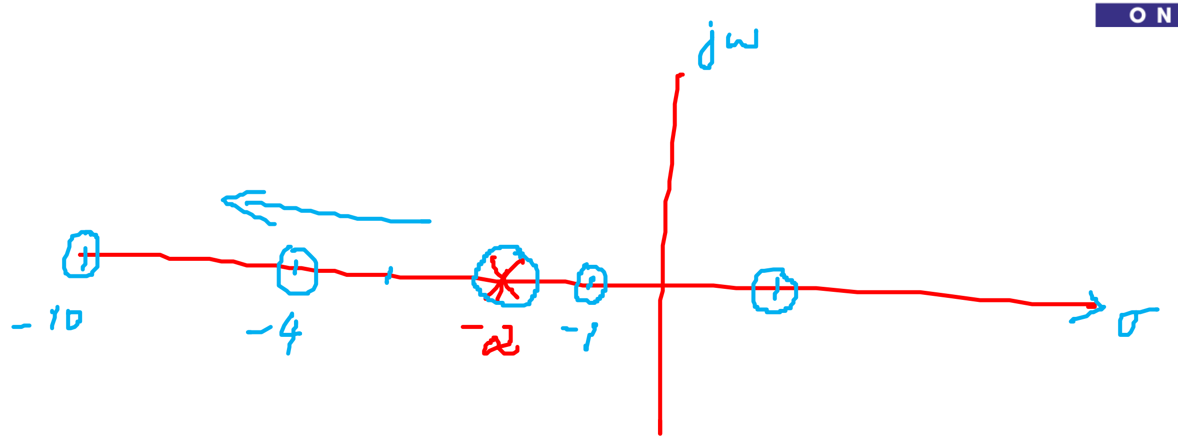
PERFORMANCE OF SECOND ORDER SYSTEMS

Effects of adding Zero – First Order System

- First Order System, $T(s) = \frac{p}{z}\frac{(s+z)}{(s+p)}$
- Step response of FOS, R(s) = 1/s
- Y(s)= $T(s)R(s) = \frac{p}{z}\frac{(s+z)}{s(s+p)}$
- Then $Y(s) = \frac{1}{s} + \frac{(\frac{p}{z}-1)}{(s+p)}$
- If p=z, Y(s) = 1/s
- p = 2;z1 = 2; z2 = 4; z3 = 10; z4 = 1;z5 = -1; z6=-4;

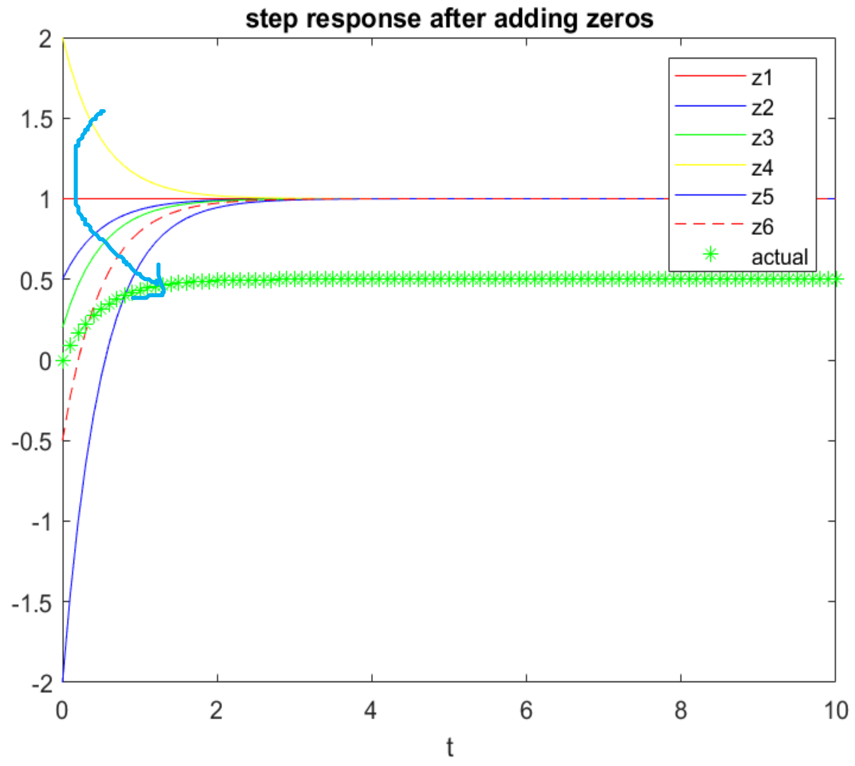


PERFORMANCE OF SECOND ORDER SYSTEMS

Effects of adding Zero – First Order System

$z_1 = 2$, $z_2 = 4$, $z_3 = 10$
$z_4 = 1$

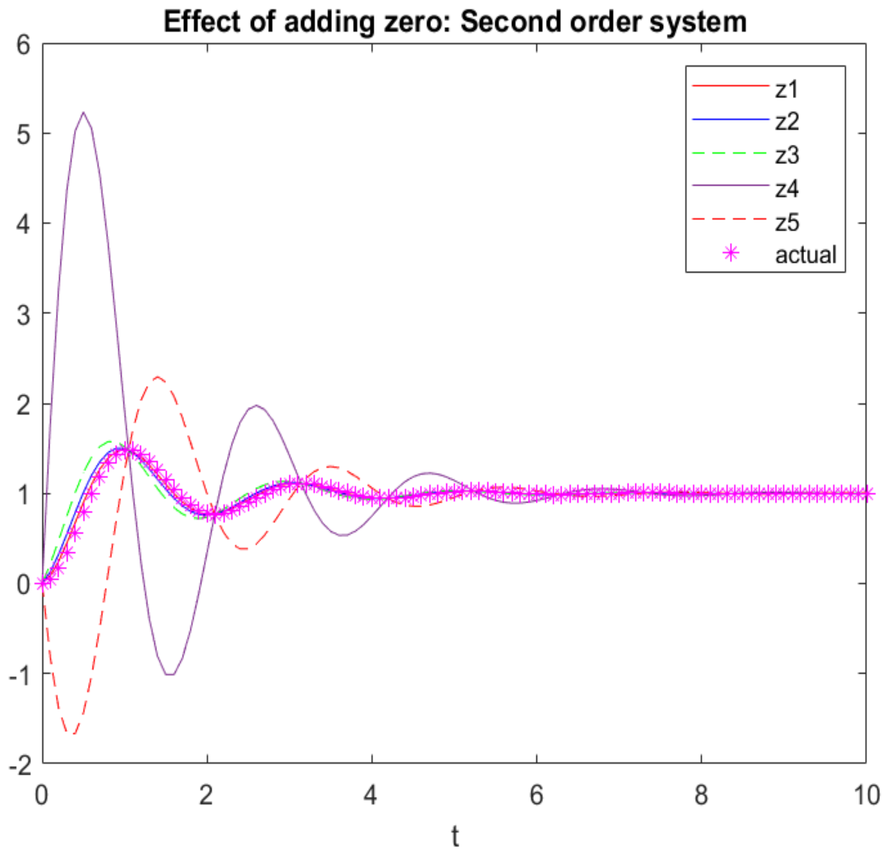
- $Y(s) = \frac{1}{s} + \frac{(\frac{p}{z} - 1)}{(s+p)}$, If p=z, Y(s) = 1/s
- p = 2;z1 = 2; z2 = 4; z3 = 10; z4 = 1;z5 = -1; z6=-4;
- Actual response => without adding zero
- As zero moves far from the origin to wards -∞ on s-plane => the effect of zero is negligible, as zero closer to origin=> **rise time decreases as well as as settling time=> overshoot increases**
- When zero lies on RHS of s-plane, the response starts from –ve value, which is referred as **undershoot**.



PERFORMANCE OF SECOND ORDER SYSTEMS

Effects of adding Zero – Second Order System

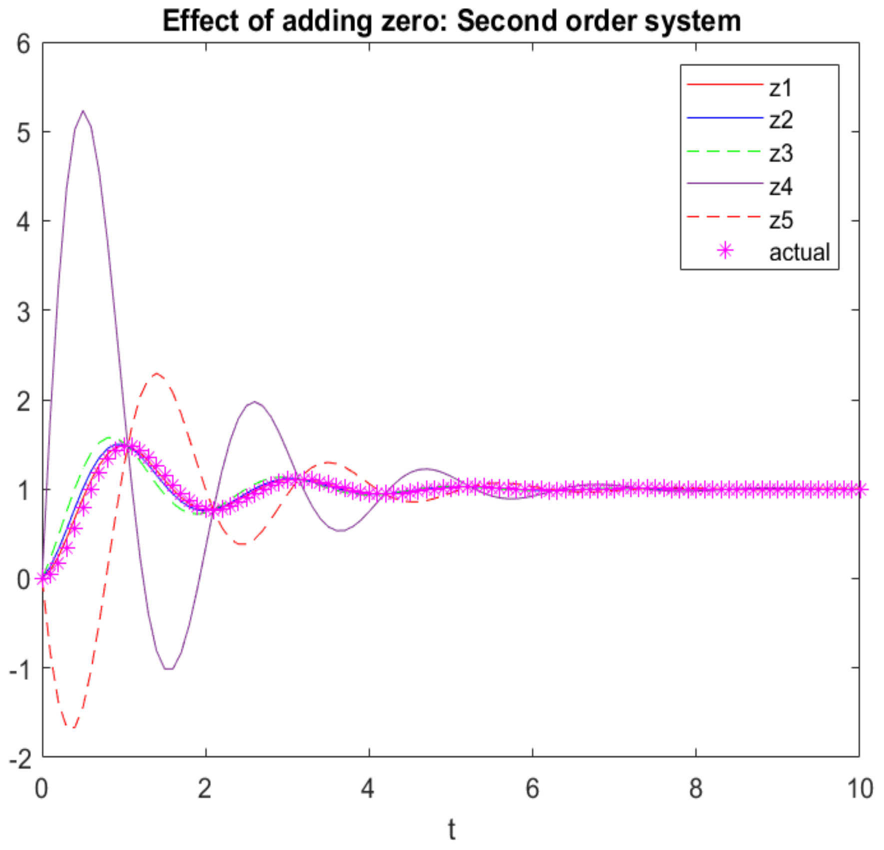
- Overdamped system, $G(s) = K\frac{(\frac{s}{z}+1)}{(\frac{s}{p_1}+1)(\frac{s}{p_2}+1)}$
- Step response of SOS, R(s) = 1/s
- Then, $Y(s) = \frac{K_1}{s} + \frac{K_2}{(s+p_1)} + \frac{K_3}{(s+p_2)}$
- z1 = 20; z2 = 10; z3 = 5; z4 = 0.5; z5 = -1;
- Actual response => without adding zero
- For underdamped, poles will be complex.



PERFORMANCE OF SECOND ORDER SYSTEMS

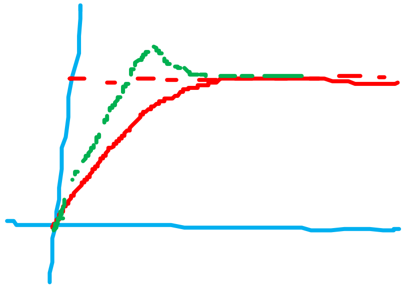
Effects of adding Zero – Second Order System

- Zero do not add dynamics, only changes the residues
- Do not affect the stability
- System becomes faster
- Greater overshoot, but in case of non-minimum phase we get undershoot.
- Closer the zero is to dominant poles, greater its effect on the transient response.
- As the zero moves away from the dominant pole, the response approaches that of two pole system.



PERFORMANCE OF SECOND ORDER SYSTEMS

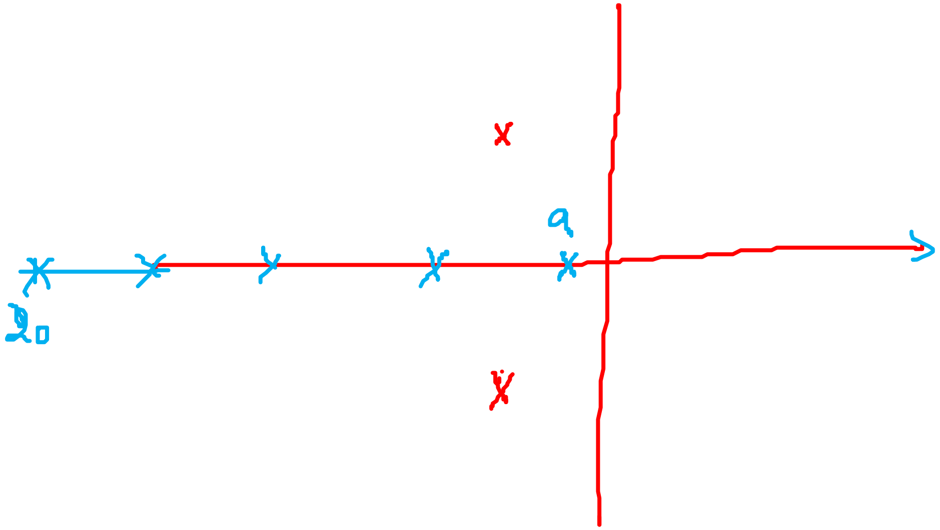
Effects of adding pole – Second Order System



- $G(s) = K\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

- Step response, R(s) = 1/s and adding a pole

- $Y(s) = K\dfrac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)(s+a)}$



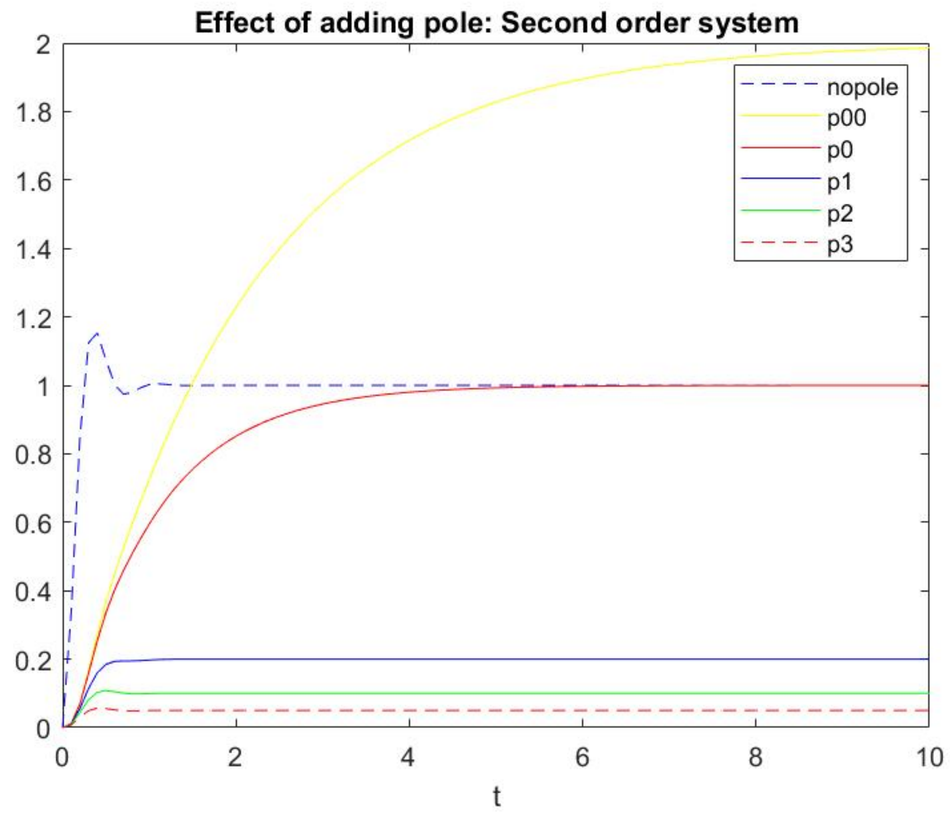
- Then, Y$(s) = \dfrac{K_1}{s} + \dfrac{K_2}{(s+p_1)} + \dfrac{K_3}{(s+p_2)} + \dfrac{K_4}{(s+a)}$

- p00 = 0.5; p0 = 1; p1 = 5; p2 = 10; p3 = 20;

PERFORMANCE OF SECOND ORDER SYSTEMS

Effects of adding pole – Second Order System

- p00 = 0.5; p0 = 1; p1 = 5; p2 = 10; p3 = 20;
- System response is slower.
- As pole moves far from the origin towards -∞ on s-plane => the third pole will not have any effect on the response
- **As the pole move closer to the origin, third pole will become dominant and effect of the same is more and effect of complex pole diminishes => Overshoot decreases**



The Performance of Feedback Control Systems:

Steady-State Error, e_{ss} error $e(t)$,

$e(t) = r(t) - y(t)$ or $E(s) = R(s) - Y(s)$

$e_{ss} = \lim_{t \to \infty} e(t)$ or $\lim_{s \to 0} sE(s)$

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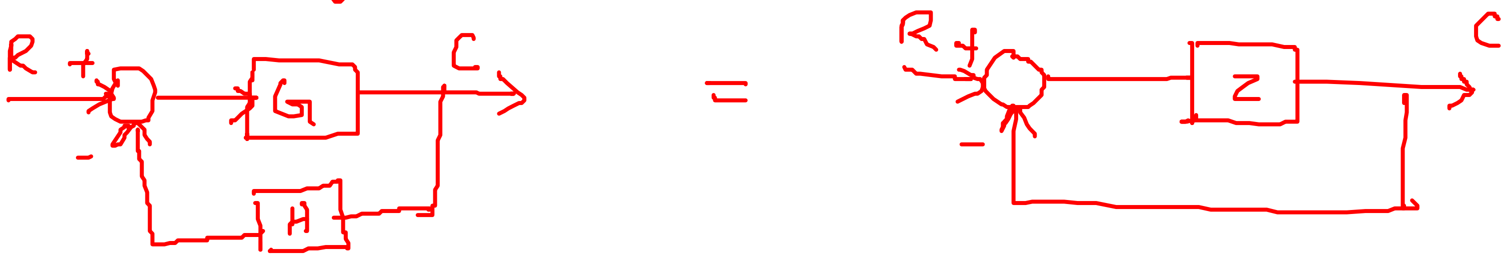
PERFORMANCE OF SECOND ORDER SYSTEMS

The Steady – State Error of Non unity Feedback Systems

Non-unity feedback system:



$$\frac{C}{R} = \frac{G}{1+GH} \quad = \quad \frac{C}{R} = \frac{Z}{1+Z}$$

$$\frac{G}{1+GH} = \frac{Z}{1+Z}$$

$$\frac{G(1+Z)}{1+GH} = Z \Rightarrow \frac{G}{1+GH} = Z - \frac{ZG}{1+GH}$$

$$= Z\left(1 - \frac{G}{1+GH}\right)$$

$$Z = \frac{\frac{G}{1+GH}}{\frac{1+GH-G}{1+GH}}$$

$$\boxed{Z = \frac{G}{1+GH-G}}$$

Type of Z & order of Z

Type ⇒ no. of poles at the origin

order ⇒ highest degree of denominator Poly

The Steady – State Error of Non unity Feedback Systems

error constants for non-unity feedback systems are

$$K_p = \lim_{s \to 0} Z(s)$$

$$K_v = \lim_{s \to 0} s\, Z(s)$$

$$K_a = \lim_{s \to 0} s^2\, Z(s)$$

when $H(s) = 1$, $Z(s) = G(s)$

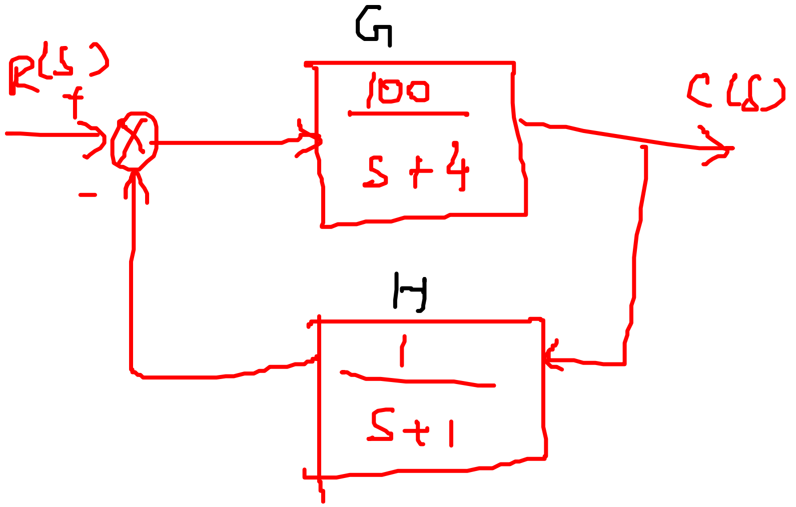
$k_p \to$ proportional error constant

$k_v \to$ velocity error constant

$k_a \to$ acceleration error constants

STEADY – STATE ERROR

Example



Determine static error constants & e_{ss} for unit step & ramp i/p

$$Z = \frac{G}{1 + GH - G} = \frac{\frac{100}{s+4}}{1 + \frac{100}{s+4} \cdot \frac{1}{s+1} - \frac{100}{s+4}}$$

$$= \frac{100/s+4}{\frac{(s+4)(s+1) + 100 - 100(s+1)}{(s+1)(s+4)}} = \frac{100(s+1)}{(s+1)(s+4) - 100(s+1) + 100}$$

$$K_p = \lim_{s \to 0} Z(s) = \frac{100}{4 - 100 + 100} = \frac{100}{4} = 25$$

$$K_v = \lim_{s \to 0} s \cdot Z(s) = 0$$

$$K_a = \lim_{s \to 0} s^2 Z(s) = 0$$

$$e_{ss} = \frac{1}{1 + K_p} = \frac{1}{1+25} = 1/26$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty$$

type = 0, order = 2

THANK YOU

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