Unit-IV : IIR filters

* IIR filters have infinite length impulse response
* IIR filters have poles as well as zeros.
* IIR filters have more effective filtering characteristics compared to FIR filters.
* IIR filters normally have non-linear phase response.
* The analog filter design theory is well established. Therefore, IIR digital filters are designed from analog filters.

The analog filter is described by the system function

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k} \quad \text{---} (1)$$

where β_k and α_k are the filter co-efficients

The system function can also be obtained from the impulse response $h(t)$

$$H_a(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt \quad \text{---} (2)$$

Thus, $h(t)$ is related to $H_a(s)$ by the Laplace transform.

Alternatively, the analog filter can be described by a linear constant co-efficient differential equation

$$\text{i.e., } \sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k} \quad \text{---} (3)$$

where, $x(t)$ is the i/p & $y(t)$ is the o/p.

Each of these three equivalent characterizations of an analog filter leads to alternative methods for converting analog filter into digital filter.

Design procedure for an IIR filter

① Selecting transformation method.

② Mapping the specification of digital filter to an equivalent analog filter.

③ Design an analog filter

④ transform the analog filter to a digital filter.

The transformation techniques should have the following desirable properties.

1. The $j\Omega$-axis of s-plane should map onto the unit circle in the Z-plane.
Thus, there will be a direct relationship between the two frequency variables in the two domains. [Preserve frequency response]

2. The left half of s-plane should be mapped inside the unit circle in Z-plane.
Bcoz of this, a stable analog filter is converted to a stable digital filter.
[Stability maintained].

Transformation Methods.

① Backward Difference Method

② Impulse Invariance Method

③ Matched Z-transformation

④ Bilinear Transformation.

* A physically realizable and stable IIR filters cannot have linear phase.
* If an application requires linear-phase then FIR filter is the solution.

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fig ⓑ $y(n)$ -
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$\boxed{S =}$
$\therefore H(z)$
Since,
$S = \sigma + j$

1] Backward difference Method

Simplest way of converting an analog filter into a digital filter is to approximate the differential equation of the analog filter by an equivalent difference equation.

for the derivative $\left.\frac{dy(t)}{dt}\right|_{t=nT}$ we substitute the backward difference equation.

i.e., $\left.\frac{d\,y(t)}{dt}\right|_{t=nT} = \frac{y(nT)-y(nT-T)}{T} = \frac{y(n)-y(n-1)}{T}$ ——— ①

where T is the sampling interval, & $y(n)=y(nT)$

we know that Laplace transform of

$\frac{d\,y(t)}{dt} = s\,Y(s)$ ——— ②

Apply z-transform to equⁿ ① we get

$\left.\frac{d\,y(t)}{dt}\right|_{t=nT} = \frac{y(n)-y(n-1)}{T}$ is $\frac{(1-z^{-1})Y(z)}{T}$ ——— ③

fig ⓐ $y(t) \longrightarrow \boxed{H(s)=s} \longrightarrow \frac{dy(t)}{dt}$ [equⁿ ②]

fig ⓑ $y(n) \longrightarrow \boxed{H(z)=\frac{1-z^{-1}}{T}} \longrightarrow \frac{y(n)-y(n-1)}{T}$ [equⁿ ③]

Comparing fig ⓐ and fig ⓑ we get

$$\boxed{s = \frac{1-z^{-1}}{T}} \text{ ——— ④}$$

$\therefore\ H(z) = \left.H_a(s)\right|_{s \to \frac{1-z^{-1}}{T}}$

Since, $s = \frac{1-z^{-1}}{T}$

$z = \frac{1}{1-sT}$ ——— ⑤

$s = \sigma + j\Omega$ ——— ⑥

Using ⑥ in ⑦ we get

$$z = \frac{1}{1-(\sigma+j\Omega)T} = \frac{1}{1-\sigma T-j\Omega T} \quad \text{—— (8)}$$

$$|z| = \frac{1}{\sqrt{(1-\sigma T)^2+(\Omega T)^2}} \quad \text{—— (9)}$$

(i) when, $\sigma<0$.

$|z|<1$.

If $H(s)$ is stable all its poles will have -ve real parts, so the corresponding poles of $H(z)$ will have magnitudes less than '1'.

(ii) when, $\sigma=0$.

from ⑧ $\quad z = \frac{1}{1-j\Omega T} = \frac{0.5+0.5+0.5j\Omega T-0.5j\Omega T}{1-j\Omega T}$

$$z = \frac{0.5(1-j\Omega T)+0.5(1+j\Omega T)}{(1-j\Omega T)}$$

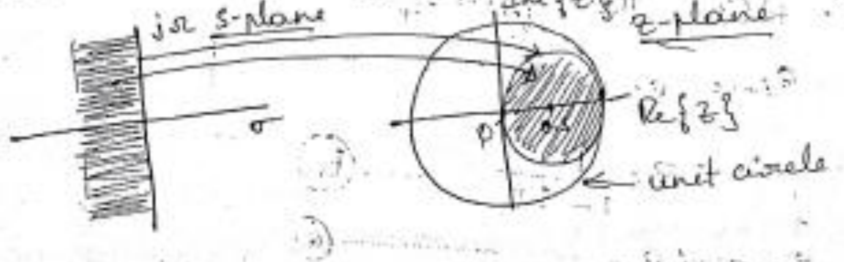
$$z = 0.5+0.5\left[\frac{1+j\Omega T}{1-j\Omega T}\right]$$

$$z-0.5 = 0.5\left[\frac{1+j\Omega T}{1-j\Omega T}\right]$$

$$|z-0.5| = 0.5$$

~~Thus, the image of $j\Omega$ axis of s-plane in the z-plane is a z-plane~~

Thus; $j\Omega$-axis of the s-plane is mapped to a circle of radius 0.5, centered at $z=0.5$ in z-plane.



(iii) $\sigma > 0$, $|z| > 0.5$ (58)

The right half of s-plane maps outside the circle of radius 0.5

(*) Thus, this method maps a stable analog filter to a stable digital filter. But, the $j\Omega$-axis does not map onto a unit circle on z-plane.

(*) It restricts the digital filter pole locations that are confined to relatively small frequencies

(*) Therefore, this mapping is restricted to the design of lowpass filters and bandpass filters having relatively small resonant frequencies.

Example:

① Using backward difference method convert the given impulse function $H_a(s)$ to $H(z)$:

$$H_a(s) = \frac{1}{(s+0.1)^2+3^2} \qquad H(z) = \frac{1}{\left[\left(\frac{1-z^{-1}}{T}\right)+0.1\right]^2+3^2}$$

$$= \frac{T^2}{z^{-2} - 2(1+0.1T)z^{-1} + (1+0.2T+9.01T^2)}$$

$$= \frac{\frac{T^2}{(1+0.2T+9.01T^2)}}{1 - \frac{2(1+0.1T)}{(1+0.2T+9.01T^2)}z^{-1} + \frac{z^{-2}}{(1+0.2T+9.01T^2)}}$$

If $T = 1$ sec

$$H(z) = \frac{0.0979}{1 - 0.2155z^{-1} + 0.09792z^{-2}}$$

② $H_a(s) = \frac{1}{s^2+16}$

$$\therefore H(z) = \frac{1}{\left[\frac{1-z^{-1}}{T}\right]^2+16} = \frac{T^2}{1 - 2z^{-1} + z^{-2} + 16T^2}$$

At $T = 1$ sec $\quad H(z) = \frac{1}{z^{-2} - 2z^{-1} + 17}$

2] Impulse Invariance Transformation

Let $h_a(t)$ be the impulse response of an analog filter.

The unit sample response of the corresponding digital filter is obtained by sampling $h_a(t)$ uniformly i.e., at $t = nT$

$$h_a(t)\big|_{t=nT} = h_a(nT) = h(n), \quad n = 0, 1, 2$$

Let $H_a(s)$ be the transfer function of an analog filter. If we assume that the poles are distinct, then its partial fraction expansion is

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s - s_k} \quad \text{--- (1)}$$

Where s_k are the poles of the analog filter and C_k are the coefficients of partial fraction expansion.

$h_a(t)$ can be obtained from $H_a(s)$ taking inverse Laplace transform

$$h_a(t) = \sum_{k=1}^{N} C_k e^{s_k t} \quad \text{--- (2)}, \quad t \geq 0$$

The unit sample response of the digital filter is obtained by sampling $h_a(t)$ uniformly, every T seconds $(t = nT)$

$$\therefore h(n) = h_a(t)\big|_{t=nT} = h_a(nT) = \sum_{k=1}^{N} C_k e^{s_k nT} \quad \text{--- (3)}$$

Now the system function of the IIR filter can be obtained by taking Z-transform of $h(n)$

$H(z)$.

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(59)

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \quad \text{—— (4)}$$

using (3) in (4) we get

$$H(z) = \sum_{n=0}^{\infty} \left[\sum_{k=1}^{N} C_k e^{S_k nT} \right] z^{-n}$$

Interchanging the summation we get

$$H(z) = \sum_{k=1}^{N} C_k \sum_{n=0}^{\infty} \left(e^{S_k T} z^{-1}\right)^{n}$$

$$\therefore \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$H(z) = \sum_{k=1}^{N} C_k \left[\frac{1}{1-e^{S_k T} z^{-1}} \right] \quad \text{—— (5)}$$

Comparing (5) and (1) we find that.

$$\frac{1}{S-S_K} \xrightarrow{(IIT)} \frac{1}{1-e^{S_K T} z^{-1}} \quad \text{—— (6)}$$

Equation (6) shows that the analog pole at $S = S_K$ is mapped to a digital pole at

$$Z_k = e^{S_k T} \quad \text{for } k=1, \dots N, \quad \text{—— (7)}$$

$Z_k = e^{S_k T}$ the poles of an analog filter are related to the corresponding poles of the digital filter by the following relation

$$z = e^{ST} \quad \text{—— (8)}$$

Let $S = \sigma + j\Omega$ & $z = re^{j\omega}$ [in polar form]

$$\Rightarrow re^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$re^{j\omega} = e^{\sigma T} e^{j\Omega T}$$

$$\therefore \boxed{r = e^{\sigma T}} \quad (9) \quad \& \quad \boxed{\omega = \Omega T} \quad \text{—— (10)}$$

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from equ^n (9) we have

(i) If $\sigma < 0$; $r < 1$,

$\therefore$ The left side of s-plane is mapped inside the unit circle.

(ii) If $\sigma > 0$; $r > 1$

$\therefore$ the right side of the s-plane is mapped outside the unit circle.

(iii) If $\sigma = 0$; $r = 1$

The $j\Omega$ axis is mapped onto the unit circle.

Since, the left side of the s-plane is mapped inside the unit circle, a stable analog system is converted into a stable digital system.

when $\sigma = 0$; $j\Omega$ axis will be mapped onto a circle of radius $[r = 1]$ in z-plane.

However, the mapping of $j\Omega$ axis is many to one.

The mapping, $\omega = \Omega T$ implies that

$(-\pi \le \omega \le +\pi) \Rightarrow$ range of ω corresponds to

$\left(\frac{-\pi}{T} \le \Omega \le +\frac{\pi}{T}\right)$ it maps into the interval $-\pi \le \omega \le +\pi$.

In general, the interval $\frac{(2k-1)\pi}{T} \le \Omega \le \frac{(2k+1)\pi}{T}$ on is also maps into the interval $-\pi \le \omega \le +\pi$,

where k is an integer.

Thus, the mapping of analog frequency Ω, to digital frequency, ω, is many-to-one.

$\Rightarrow$ we know that $\pi \le \omega \le 3\pi$ is same as $-\pi \le \omega \le \pi$ on the unit circle.

$\therefore \pi \le \omega \le 3\pi$ corresponds to $\frac{\pi}{T} \le \Omega \le \frac{3\pi}{T}$.

Thus, $\frac{\pi}{T} \le \Omega \le \frac{3\pi}{T}$ maps ~~to~~ on $-\pi \le \omega \le +\pi$.

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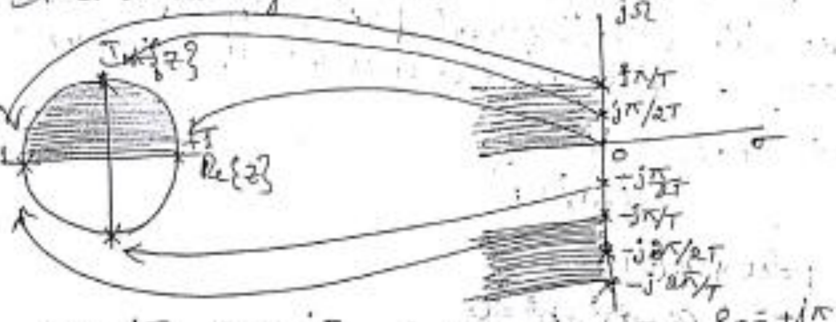
This effect ~~takes place~~ is because of sampling. (60)
It is basically aliasing in frequency domain.



$s_1 = -j\frac{\pi}{T}$, $s_2 = -j\frac{\pi}{2T}$, $s_3 = 0$, $s_4 = +j\frac{\pi}{2T}$, $s_5 = +j\frac{\pi}{T}$

$z_1 = e^{s_1 T} = e^{-j\frac{\pi}{T}T} = -1$, $z_2 = e^{s_2 T} = e^{-j\frac{\pi}{2T}T} = -j$

$z_3 = e^0 = 1$, $z_4 = e^{s_4 T} = e^{j\frac{\pi}{2T}T} = +j$, $z_5 = e^{s_5 T} = e^{j\frac{\pi}{T}T} = -1$

fig: Mapping of $j\Omega$-axis onto unit circle

Note :- ① As the impulse response is sampled, we know that the spectrum of the sampled signal consists of replicas of the original spectrum at multiples of $\Omega_s = \frac{2\pi}{T}$.
② There is no aliasing if $\Omega_s > 2\Omega_h$, where Ω_h is the highest frequency component of the analog filter frequency spectrum. However, no practical analog filter is bandlimited. ∴, this method will always cause aliasing, and the frequency response of the designed filter will be distorted.
③ This method is suitable for designing LPFs and BPFs, where the stopbands overlap and the effect of aliasing is small.
④ However, it is unsuitable for the designing of HPFs and BSFs, where the passbands overlap, thereby magnifying the effect of aliasing.

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Examples

(1) Convert the given analog filter system functions to digital filter system functions using impulse invariance transformation.

(a) $H(s) = \frac{b}{(s+a)^2 + b^2}$

Poles of $H(s)$ are

$$(s+a)^2 + b^2 = 0$$

$$s = -a \pm jb$$

$$s_1 = -a+jb, \quad s_1 = -a-jb$$

In factored form

$$H_a(s) = \frac{b}{(s+a-jb)(s+a+jb)}$$

$$= \frac{C_1}{s+a-jb} + \frac{C_2}{s+a+jb}$$

$$C_1 = H(s) \times (s+a-jb)\Big|_{s=-a+jb} = \frac{1}{j2}$$

$$C_2 = C_1^* = -\frac{1}{j2}$$

$$H(z) = \frac{\frac{1}{j2}}{1 - e^{-(a-jb)T} z^{-1}} + \frac{-\frac{1}{j2}}{1 - e^{-(a+jb)T} z^{-1}}$$

$$\boxed{H(z) = \frac{e^{-aT} \sin bT \, z^{-1}}{1 - 2e^{-aT} \cos bT \, z^{-1} + e^{-2aT} z^{-2}}} \quad (1)$$

(b) $H(s) = \frac{s+a}{(s+a)^2 + b^2}$

$$H(s) = \frac{(s+a)}{(s+a-jb)(s+a+jb)}$$

$$= \frac{C_1}{(s+a-jb)} + \frac{C_2}{(s+a+jb)}$$

$$C_1 = \tfrac{1}{2}, \quad C_2 = C_1^* = \tfrac{1}{2}$$

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$$\therefore H(z) = \frac{1/2}{1 - e^{-(a+jb)T}} + \frac{1/2}{1 - e^{-(a-jb)T}}$$

$$\boxed{H(z) = \frac{1 - e^{-aT}\cos bT z^{-1}}{1 - 2\cos bT\, e^{-aT} z^{-1} + e^{-2aT} z^{-2}}}$$

(c) $H(s) = \frac{s}{(s+a)^2 + b^2}$

$$= \frac{s + a - a}{(s+a)^2 + b^2} = \frac{s+a}{(s+a)^2+b^2} - \frac{a}{(s+a)^2+b^2}$$

$$= \frac{s+a}{(s+a)^2+b^2} - \frac{a}{b}\frac{b}{(s+a)^2+b^2}$$

$$= \frac{1 - e^{-aT}\cos bT z^{-1}}{1 - 2e^{-aT}z^{-1}\cos bT + e^{-2aT}z^{-2}} - \frac{a}{b}\frac{e^{-aT}\sin bT z^{-1}}{1 - 2e^{-aT}z^{-1}\cos bT + e^{-2aT}z^{-2}}$$

$$\boxed{H(z) = \frac{1 - \left[\cos(bT) + \frac{a}{b}\sin(bT)\right] e^{-aT} z^{-1}}{1 - 2e^{-aT}\cos(bT) z^{-1} + e^{-2aT} z^{-2}}}$$

(d) $H(s) = \frac{s + 0.1}{(s+0.1)^2 + 3^2}$ $a = 0.1$, $b = 3$

$$\boxed{H(z) = \frac{1 - e^{-0.1T}\cos(3T) z^{-1}}{1 - 2\cos(3T) e^{-0.1T} z^{-1} + e^{-2\times 0.1T} z^{-2}}}$$

(e) $H(s) = \frac{1}{(s+1)(s+2)}$ Assume $T = 0.2$ sec

$$H(s) = \frac{c_1}{(s+1)} + \frac{c_2}{(s+2)}$$

$c_1 = 1$ & $c_2 = -1$

$$H(z) = \frac{1}{1 - e^{-(1)(0.2)}z^{-1}} - \frac{1}{1 - e^{-(2)(0.2)}z^{-1}} = \frac{0.148z}{z^2 - 1.48z + 0.548}$$

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(4) $H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)}$

$$= \frac{A}{(s+0.5)} + \frac{B}{(s+0.25s-j1.392)} + \frac{C}{(s+0.25s+j1.392)}$$

$$= \frac{0.5}{(s+0.5)} + \frac{-0.25-j0.045}{(s+0.25s-j1.392)} + \frac{-0.25+j0.045}{(s+0.25s+j1.392)}$$

$$= \cancel{\frac{0.5}{(s+0.5)}} + \cancel{\frac{-0.5s}{s^2+0.5s+2}}$$

$$= \frac{0.5}{(s+0.5)} + \frac{-0.5s}{(s+0.25)^2+(1.392)^2}$$

$$= \frac{0.5}{(s+0.5)} + \frac{-0.5(s+a-a)}{(s+0.25)^2+(1.392)^2}$$

$$= \frac{0.5}{(s+0.5)} - 0.5\left[\left(\frac{s+0.25}{(s+0.25)^2+1.392^2}\right) - \left(\frac{0.25}{(s+0.25)^2+1.392^2}\right)\right]$$

$$H(z) = \frac{0.5}{1-e^{-0.5T}z^{-1}} - 0.5\left[\left(\frac{1-e^{-0.25T}\cos(1.392T)z^{-1}}{1-2e^{-0.25T}\cos(1.392T)z^{-1}+e^{-0.5T}z^{-2}}\right)\right.$$

$$\left. - \left(0.18\ \frac{e^{-0.25T}\sin(1.392T)z^{-1}}{1-2e^{-0.25T}\cos(1.392T)z^{-1}+e^{-0.5T}z^{-2}}\right)\right]$$

2. Design a low pass Butterworth filter using impulse invariant method for satisfying the following constraints:

Passband ω_p : 0.162 rad , Passband ripple: 3 dB

Stopband ω_s : 1.63 rad, stopband attenuation: 30 dB

Sampling frequency: 8 kHz.

Solution: $\boxed{\omega = \Omega T}$

$$\Omega_p = \frac{\omega_p}{T_s} = 8000 \times 0.162 = 1296 \text{ rad/sec}$$

$$\Omega_s = \frac{\omega_s}{T_s} = 8000 \times 1.63 = 13{,}040 \text{ rad/sec}$$

(62)

$$\Omega_r = \frac{\Omega_s}{\Omega_p} = \frac{13{,}040}{1296} = 10.06$$

$$N = \left\lceil \frac{\log\left[(10^{0.3}-1)/(10^{3}-1)\right]}{2\log\left(\frac{1}{10.06}\right)} \right\rceil = 2$$

$$\Omega_c = \frac{1}{(10^{0.3}-1)^{1/2 \times 2}} = 1.00$$

$$\Omega_{cp} = 1 \times 1296 = 1296 \text{ rad/sec}$$

$$H_2(s) = \frac{1}{s^2+\sqrt{2}s+1} = \frac{1}{(s-(-0.707+j0.707))(s-(-0.707-j0.707))}$$

$$H_a(s) = H_2(s)\Big|_{s \to \frac{s}{1296}} = \frac{1}{\left(\frac{s}{1296}-(-0.707+j0.707)\right) \times \left(\frac{s}{1296}-(-0.707-j0.707)\right)}$$

$$= \frac{(1296)^2}{(s-(-916.27+j916.27))(s-(-916.27-j916.27))}$$

$$= \frac{1.679\times10^6}{(s+916.27)^2+(916.27)^2}$$

$$= \frac{1.679\times10^6}{916.27} \times \frac{916.27}{(s+916.27)^2+(916.27)^2}$$

$$H(z) = 1.833\times10^3\left[\frac{e^{-916.27T}\sin(916.27T)z^{-1}}{1-2e^{-916.27T}\cos(916.27T)z^{-1}+e^{-2\times916.27T}z^{-2}}\right]$$

3] Design a Chebyshev-Type I ~~Butterworth~~ filter using impulse invariant method for the specifications given above [Question #2];

$\omega_p = 0.2\pi$ rad, Passband ripple: 3 dB

$\omega_s = 1.63$ rad, Stop band attenuation: 30 dB

$F_s = 8$ kHz.

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$\Omega_p = \frac{\omega_p}{T_s} = 1{,}296$ rad/sec $\quad \Omega_s = \frac{\omega_s}{T_s} = 13{,}040$ rad/sec

$\Omega_r = \frac{\Omega_s}{\Omega_p} = 10.06$

$\epsilon = \sqrt{10^{0.3}-1} = 0.997$, $A = 10^{30/20} = 10^{1.5}$

$g = \left[\frac{(10^{1.5})^2 - 1}{0.997^2}\right]^{1/2}$

$\therefore N = \left\lceil \frac{\log_{10}\left[g+\sqrt{g^2-1}\right]}{\log_{10}\left(\Omega_r + \sqrt{\Omega_r^2 - 1}\right)} \right\rceil = 2$

$S_k = \sigma_k + j\Omega_k \quad k = 1, 2$

$S_1 = -0.322 + j\,0.777$

$S_2 = -0.322 - j\,0.777$

$H_2(s) = \frac{k}{(s+0.322-j0.777)(s+0.322+j0.777)}$

$b_0 = 0.7067$, $K = \frac{b_0}{\sqrt{1+\epsilon^2}} = 0.500$ Since, N is even

$\therefore H_2(s) = \frac{0.5}{(s+0.322)^2 + (0.777)^2}$

$H_{LP}(s) = H_2(s)\Big|_{s \to \frac{s}{1296}}$

$= \frac{0.5}{\left(\frac{s}{1296} + 0.322\right)^2 + (0.777)^2}$

$= \frac{8.39\times10^2 \times 1\times10^3}{(s+417.31)^2 + (1\times10^3)^2}$

$H(z) = 8.39\times10^2\left[\frac{e^{-417.31T}z^{-1}\sin(1\times10^3 T)}{1 - 2e^{-417.31T}\cos(10^3 T)z^{-1} + e^{-2(417.31)T}z^{-2}}\right]$

$T = \frac{1}{Fs}$

$= \frac{1}{8000}$

4) Design a low pass Butterworth filter using impulse invariant method for satisfying the following constraints: (63)

Passband: 0–400 Hz, Passband ripple: 2 dB

Stopband: (2.1–4) KHz; stopband attenuation: 20 dB

Sampling frequency: 10 KHz

Solution:

$$\omega_p = \frac{2\pi(400)}{10\times10^3} = 0.25 \text{ rad}, \quad \omega_s = \frac{2\pi(2100)}{10000} = 1.319 \text{ rad}$$

$$\Omega_p = \frac{\omega_p}{T_s} = 10{,}000 \times 0.25 = 2500 \text{ rad/sec}$$

$$\Omega_s = \frac{\omega_s}{T_s} = 10{,}000 \times 1.319 = 13{,}190 \text{ rad/sec}$$

$$\Omega_r = \frac{\Omega_s}{\Omega_p} = \frac{13{,}190}{2500} = 5.276$$

$$N = 2, \quad \Omega_c = \frac{1}{(10^{0.2}-1)^{1/4}} = 1.14$$

$$\therefore \Omega_{cp} = 2500 \times 1.14$$

$$\Omega_{cp} = 2{,}858.71$$

$$H_2(s) = \frac{1}{s^2+\sqrt{2}s+1} = \frac{1}{(s+0.707+j0.707)(s+0.707-j0.707)}$$

$$= \frac{1}{(s+0.707)^2+(0.707)^2}$$

$$H_{LP}(s) = H_2(s) = \frac{(2858.71)^2}{(s+2021.1)^2+(2021.1)^2}$$

$$= 4043.65\left[\right.$$

$$H(z) = 4043.65\left[\frac{e^{-2021.1T} z^{-1}\sin((2021.1)T)}{1-2e^{-2021.1T}z^{-1}\cos(2021.1T)+e^{-2\times2021.1T}z^{-2}}\right]$$

$$T = \frac{1}{10{,}000}$$

5) For the specification given above, design a low pass Chebyshev Type-I filter using IIT method.

$f_p = 400$ Hz $\quad A_p = 2$ dB $\quad F_s = 10$ KHz

$f_s = 2.1$ KHz $\quad A_s = 20$ dB

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Margin: 13,190 rad/sec; 1.5; 2; (2.1); j0.777); is given

$\omega_p = 0.25$ rad | $\Omega_p = \frac{\omega_p}{T_s} = 2500$ rad/sec

$\omega_s = 1.319$ rad | $\Omega_s = \frac{\omega_s}{T_s} = 13,190$ rad/sec

$\Omega_r = \frac{\Omega_s}{\Omega_p} = 5.276$ rad/sec

$\epsilon = \sqrt{10^{0.2}-1} = 0.764$, $A = 10^{20/20} = 10$, $g = \left[\frac{10^2-1}{0.76^2}\right]^{1/2}$

$N = 2$

$s_k = \sigma_k + j\Omega_k \quad k = 1, 2$

$s_1 = -0.4022 + j0.813$

$s_2 = -0.4022 - j0.813$

$$H_2(s) = \frac{K}{(s+0.4022-j0.813)\cdot(s+0.4022+j0.813)}$$

$$K = \frac{b_0}{\sqrt{1+\epsilon^2}} = 0.653$$

$$H_2(s) = \frac{0.653}{(s+0.4022)^2+(0.813)^2}$$

$$H_{LP}(s) = H_2(s) \Big|_{s \to \frac{s}{2500}}$$

$$= \frac{0.653}{\left(\left(\frac{s}{2500}\right)+0.4022\right)^2+(0.813)^2}$$

$$= \frac{2.7\times10^3 \times 2032.5}{(s+1005)^2+(2032.5)^2}$$

$$H(z) = 2.7\times10^3\left[\frac{e^{-1005T}z^{-1}\sin((2032.5)T)}{1-2e^{-1005T}z^{-1}\cos((2032.5)T)+e^{-2(1005)T}z^{-2}}\right]$$

Matched-z-transformation

Analog filter system function $H_a(s)$ is defined as

$$H_a(s) = \frac{\prod_{k=1}^{M}(s-z_k)}{\prod_{k=1}^{N}(s-p_k)} \xrightarrow{\text{is transformed to}} H(z) = \frac{\prod_{k=1}^{M}(1-e^{z_k T}z^{-1})}{\prod_{k=1}^{N}(1-e^{p_k T}z^{-1})}$$

where 'T' is the sampling interval.

* Poles in the left half of s-plane are mapped inside the unit circle in the z-plane. ∴ the stability is preserved.
* Unlike Bilinear transformation, impulse invariance and backward difference method, an all-pole analog filter will be transformed as an all-pole digital filter.
* ∴ there are no zeros to help shape the frequency response. Even if $H(s)$ does have zeros, if their imaginary parts are greater than $\frac{\omega_s}{2}$, the resulting zeros in $H(z)$ will produce serious aliasing errors.

Examples

① $H(s) = \frac{(s+2)}{(s+1)(s+3)}$

$T = 0.1$ sec

$$H(z) = \frac{(1-e^{-2T}z^{-1})}{(1-e^{-T}z^{-1})(1-e^{-3T}z^{-1})}$$

$$H(z) = \frac{(1-0.819z^{-1})}{(1-0.905z^{-1})(1-0.741z^{-1})}$$

② $H(s) = \frac{4s(s+1)}{(s+2)(s+3)}$

$F_s = 4\,Hz$ ∴ $T = \frac{1}{4}$

$$H(z) = \frac{4(1-z^{-1})(1-e^{-T}z^{-1})}{(1-e^{-2T}z^{-1})(1-e^{-3T}z^{-1})}$$

Observations:

① Stable poles of $H(s)$, at $s=-1$ & $s=-3$, become stable poles of $H(z)$, at $z = e^{-T} = 0.905$ & $z = e^{-3T} = 0.741$

② $H(j\Omega)|_{\Omega=0} = H(s)|_{s=0} = \frac{2}{1\times 3} = 0.66$

but $H(\omega)|_{\omega=0} = H(z)|_{z=1} = \frac{1-0.819}{(1-0.905)(1-0.741)} = 7.3562$

∴ the analog freq plot must be scaled by $1/0.66 = 1.515$ & digital plot by $1/7.3562 = 0.1359$ to make them equal at zero frequency.

Bilinear Transformation

It is used for the transformation of an analog filter to a digital filter.

Consider the derivative,

$$\frac{d\,y(t)}{dt} = x(t) \quad \text{——} ①$$

taking the Laplace transform on both sides

$$s\,Y(s) = X(s) \quad \text{——} ②$$

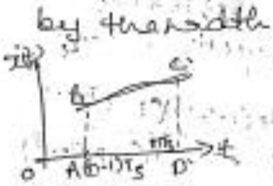
Integrating both sides of equation ① to find $y(t)$ within the limits $(n-1)T$ and nT, where T is the sampling interval.

$$\int_{(n-1)T}^{nT} \frac{d\,y(t)}{dt} = \int_{(n-1)T}^{nT} x(t)\,dt$$

$$y(nT) - y(n-1)T = \int_{(n-1)T}^{nT} x(t)\,dt \quad \text{——} ③$$

$\int_{(n-1)T}^{nT} x(t)dt$, is approximated by the trapezoidal rule

If T is small, the area (integral) can be approximated by the mean height of $x(t)$ between the two limits and then multiplying by the width.



i.e., $$\int_{(n-1)T}^{nT} x(t)dt = \left[\frac{x(nT)+x(n-1)T}{2}\right]T \quad \text{——} ④$$

using ④ in ③ we get

$$y(nT) - y(n-1)T = \left[\frac{x(nT)+x(n-1)T}{2}\right]T$$

Since, $x(n) \equiv x(nT)$ & $y(n) \equiv y(nT)$

$$y(n) - y(n-1) = \left[\frac{x(n)+x(n-1)}{2}\right] \quad \text{——} ⑤$$

Taking Z-transform of ⑤ we get

$$Y(z) - z^{-1}Y(z) = \left[\frac{X(z)+z^{-1}X(z)}{2}\right]T$$

(65)

$$(1-z^{-1})Y(z) = \left[\frac{(1+z^{-1})}{2}X(z)\right]T$$

$$X(z) = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]Y(z) \quad \text{——— (6)}$$

Comparing (2) and (6) we get

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] \quad \text{——— (7)}$$

$$s = \frac{2}{T}\left[\frac{z-1}{z+1}\right] \quad \text{——— (8)}$$

Sub $s = \sigma + j\Omega$ & $z = re^{j\omega}$ in (8)

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right] = \frac{2}{T}\left[\frac{(r\cos\omega-1)+jr\sin\omega}{(r\cos\omega+1)+jr\sin\omega}\right]$$

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{(r\cos\omega-1)+jr\sin\omega}{(r\cos\omega+1)+jr\sin\omega}\right] \times \left[\frac{(r\cos\omega+1)-jr\sin\omega}{(r\cos\omega+1)-jr\sin\omega}\right]$$

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{r^2-1}{r^2+1+2r\cos\omega} + j\,\frac{2r\sin\omega}{r^2+1+2r\cos\omega}\right]$$

$$\sigma = \frac{2}{T}\left[\frac{r^2-1}{r^2+1+2r\cos\omega}\right] \quad \text{——— (9)} \qquad \& \qquad \Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{r^2+1+2r\cos\omega}\right] \quad \text{——— (10)}$$

from (9) & (10)

(i) If $r<1$, $\sigma<0$. This implies that the left hand side of the s-plane is mapped inside the unit circle.

(ii) If $r>1$, $\sigma>0$.
This means, the right hand side of s-plane is mapped outside the unit circle.

(iii) If $r=1$, $\sigma=0$.
∴ the imaginary $j\Omega$ axis is mapped to the circle of unit radius centered at zero in z-plane.
∴ the stability is preserved.

when $r=1$, $\sigma=0$ and $\Omega = \frac{2}{T}\left[\frac{2\sin\omega}{1^2+1+2\cos\omega}\right]$.

$$\Omega = \frac{2}{T}\left(\frac{\sin\omega}{1+\cos\omega}\right)$$

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right) \quad \text{——— (11)}$$

$$\therefore \quad \omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right) \quad \text{——— (12)}$$

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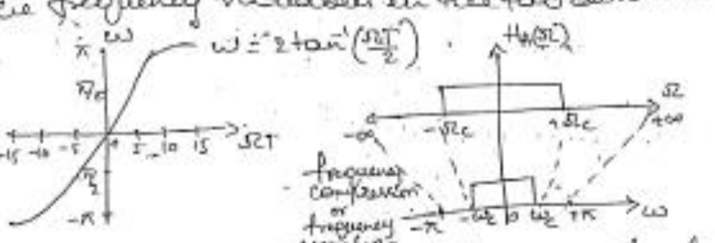
Equation (12) describes the relationship between the frequency variables in the two domains.



* The entire range of Ω is mapped only once into the range $-\pi \le \omega \le \pi$. There is point to point mapping, hence, no aliasing.
* The mapping is highly non-linear and a frequency compression is observed due to the non-linearity of the arc-tangent function.

Examples:

(1) Using Bilinear transformation design a second order lowpass Butterworth filter with cut-off frequency of 1 kHz and sampling frequency of 10^4 samples/sec.

Solution: $N = 2$, $F_c = 1000$ Hz, $F_s = 10{,}000$ Hz

$$\omega_c = \frac{2\pi F_c}{F_s} = \frac{2\pi \times 1000}{10{,}000} = 0.2\pi \text{ rad/sample}$$

$$\Omega_c = \frac{2}{(1/10000)} \tan\left(\frac{0.2\pi}{2}\right)$$

$$\Omega_c = 6498.4 \text{ rad/sec}$$

$$H_2(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

$$H_{LP}(s) = H_2(s)\Big|_{s \to s/6498.4}$$

$$= \frac{1}{\left(\frac{s}{6498.4}\right)^2 + \sqrt{2}\left(\frac{s}{6496.4}\right) + 1}$$

$$= \frac{(6498.4)^2}{s^2 + 9190.125s + 42.23\times10^6}$$

$$H(z) = H_{LP}(s)\Big|_{s \to \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$H(z) = \frac{(6498.4)^2}{\left(2\times10^4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^2 + 9190.125\left(2\times10^4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) + 42.23\times10^6}$$

$$H(z) = \frac{0.0676\,(1+2z^{-1}+z^{-2})}{1-1.143z^{-1}+0.4128z^{-2}}$$

(2) Design a digital lowpass filter using Chebyshev filter design procedure that meets the following specifications: $\boxed{\text{Assume } T = 2\,secs}$

$A_p = 1\,dB$, $\omega_p = 0.2\pi$ radians/sample

$A_s = 15\,dB$, $\omega_s = 0.3\pi$ radians/sample

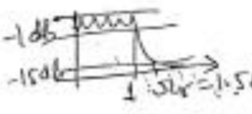
Use bilinear transformation.

Solution:

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = 0.325$$

$$\Omega_s = 0.509$$

$$\Omega_r = \frac{\Omega_s}{\Omega_p} = \frac{0.509}{0.325} = 1.56\ \text{rad/sec}$$



$$\epsilon = \sqrt{10^{0.1}-1} = 0.508$$

$$A = 10^{15/20}, \quad g = \left[\frac{A^2-1}{\epsilon^2}\right]^{1/2}$$

order estimated

$N = 3$

$$H_3(s) = \frac{k}{(s-s_1)(s-s_2)(s-s_3)}$$

$s_1 = -0.247 + j0.965$ | $s_3 = -0.247 - j0.965$

$s_2 = -0.494$

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$$H_3(s) = \frac{K}{s^3+0.988s^2+1.238s+0.491}$$

$K = b_0$ since 'N' is odd

$$H_3(s) = \frac{0.491}{s^3+0.988s^2+1.238s+0.491}$$

$$H_{LP}(s) = H_3(s)\Big|_{s \to \frac{s}{0.325}}$$

$$H_{LP}(s) = \frac{0.491}{\left(\frac{s}{0.325}\right)^3+0.988\left(\frac{s}{0.325}\right)^2+1.238\left(\frac{s}{0.325}\right)+0.491}$$

$$= \frac{0.0168}{s^3+0.321s^2+0.13s+0.0168}$$

$$H(z) = H_{LP}(s)\Big|_{s \to \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \quad T = 2\,sec.$$

$$H(z) = \frac{0.0168}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^3+0.321\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2+0.138\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.0168}$$

$$H(z) = \frac{0.0114\,(1+z^{-1})^3}{1-2.139z^{-1}+1.771z^{-2}-0.539z^{-3}}$$

(3) Design a ~~second~~ first order digital highpass butterworth filter for cut-off frequency = 30Hz and Sampling frequency = 150 Hz. Use bilinear transformation.

$F_c = 30\,Hz$ $\quad \omega_c = \frac{2\pi(30)}{150} = 0.4\pi$ rad/sample (67)

$F_S = 150\,Hz$

$T = \frac{1}{150}$

$\boxed{N=1}$

$\Omega_c = \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right) = 217.963$ rad/sec

$$H_1(S) = \frac{1}{(S+1)}$$

$$H_{HP}(S) = H(S)\Big|_{S \to \frac{217.963}{S}} = \frac{1}{\left(\frac{217.963}{S}+1\right)}$$

$$H_{HP}(S) = \frac{S}{S+217.963}$$

$$H(z) = H_{HP}(S)\Big|_{S \to \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{300\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{300\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+217.963}$$

$$\boxed{H(z) = \frac{0.5792(1-z^{-1})}{1-0.1583z^{-1}}}$$

(4) Design a bandstop Butterworth and Chebyshev type-I filter to meet the following specifications.

(a) Passband edge frequencies 100 & 600 Hz

(b) 20 dB attenuation at 200 & 400 Hz (stopband)

(c) Passband ripple for Chebyshev filter is 1.1 dB

(d) Passband attenuation for Butterworth filter is 3 dB

(e) The sampling frequency is 2 KHz.

Solution: $f_l = 100\,Hz$, $f_1 = 200\,Hz$, $f_2 = 400\,Hz$,

$f_u = 600\,Hz$,

$\Omega_l = 2$

Realization of filters (68)

① Direct form I ③ Cascaded ⑤ Lattice
② Direct form II ④ Parallel.

[I] Direct form I realization

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

$$H(z) = H_1(z) \cdot H_2(z)$$

$$H_1(z) = \frac{Y_1(z)}{X_1(z)} = \sum_{k=0}^{M} b_k z^{-k} \Rightarrow \text{All zero system}$$

$$H_2(z) = \frac{Y_2(z)}{X_2(z)} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}} \Rightarrow \text{All pole system}$$

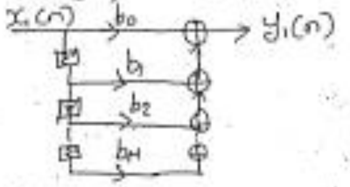
$$\frac{Y_1(z)}{X_1(z)} = \sum_{k=0}^{M} b_k z^{-k}.$$

$$Y_1(z) = \left[\sum_{k=0}^{M} b_k z^{-k}\right] X_1(z)$$

$$Y_1(z) = \left[b_0 + b_1 z^{-1} + b_2 z^{-2} - \dots + b_M z^{-M}\right] X_1(z)$$

Applying inverse Z-transform

$$y_1(n) = b_0 x_1(n) + b_1 x_1(n-1) + b_2 x_1(n-2) + \dots + b_M x_1(n-M)$$



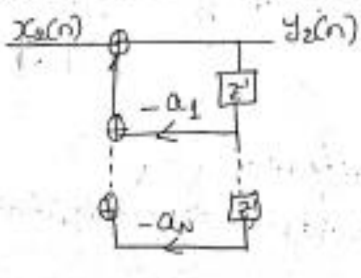
$$H_2(z) = \frac{Y_2(z)}{X_2(z)} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

$$Y_2(z)\left[1+\sum_{k=1}^{N} a_k z^{-k}\right] = X_2(z)$$

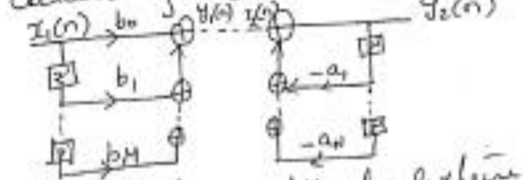
$$Y_2(z) = X_2(z) - Y_2(z)\left[\sum_{k=1}^{N} a_k z^{-k}\right]$$

Applying inverse Z-transform

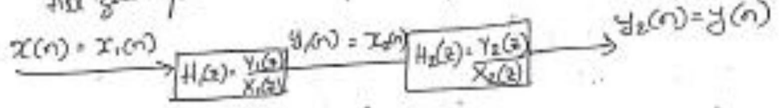
$$y_2(n) = x_2(n) - a_1 y_2(n-1) - a_2 y_2(n-2) \dots - a_N y_2(n-N)$$



Cascading $H_1(z)$ & $H_2(z)$



All zero system   All pole system



Direct form II Realization

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$H(z) = \frac{Y(z)}{V(z)} \cdot \frac{V(z)}{X(z)} = \frac{V(z)}{X(z)} \cdot \frac{Y(z)}{V(z)}$$

$$H(z) = H_1(z) \cdot H_2(z)$$

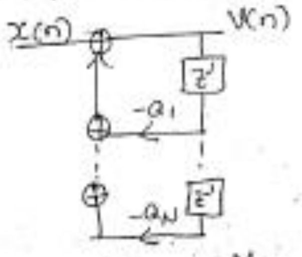
Let $H_1(z) = \dfrac{V(z)}{X(z)}$ & $H_2(z) = \dfrac{Y(z)}{V(z)}$

$$H_1(z) = \frac{V(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}} \Rightarrow \text{All pole system}$$

$$V(z) = X(z) - V(z)\left[\sum_{k=1}^{N} a_k z^{-k}\right]$$

Applying inverse z-transform

$$v(n) = x(n) - a_1 v(n-1) - a_2 v(n-2) \cdots - a_N v(n-N)$$



$$H_2(z) = \frac{Y(z)}{V(z)} = \sum_{k=0}^{M} b_k z^{-k} \Rightarrow \text{All zero system}$$

$$Y(z) = \left[\sum_{k=0}^{M} b_k z^{-k}\right] V(z)$$

Applying inverse z-transform

$$y(n) = b_0 v(n) + b_1 v(n-1) + b_2 v(n-2) \cdots + b_M v(n-M)$$

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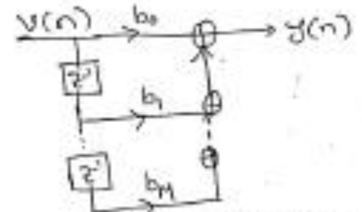
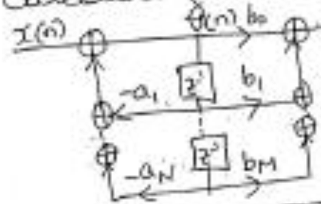
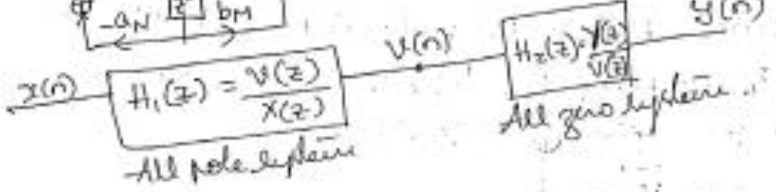


Cascading $H_1(z)$ & $H_2(z)$ we get





Examples

1. $$H(z) = \frac{7z^2 - 5.25z + 1.375}{z^2 - 0.75z + 0.125}$$

(or) $$H(z) = \frac{7 - 5.25z^{-1} + 1.375z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}}$$

Direct form I

$$H(z) = H_1(z) \cdot H_2(z)$$

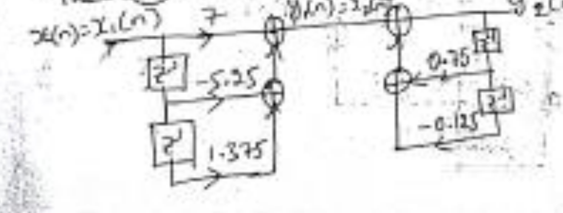
$$H_1(z) = \frac{Y_1(z)}{X_1(z)} = 7 - 5.25z^{-1} + 1.375z^{-2}$$

$$\therefore y_1(n) = 7x_1(n) - 5.25x_1(n-1) + 1.375x_1(n-2) \quad \text{——(1)}$$

$$H_2(z) = \frac{Y_2(z)}{X_2(z)} = \frac{1}{1 - 0.75z^{-1} + 0.125z^{-2}}$$

$$y_2(n) = x_2(n) + 0.75y_2(n-1) - 0.125y_2(n-2) \quad \text{——(2)}$$

~~from (2)~~ cascading $H_1(z)$ & $H_2(z)$ we get



Direct form II

$$H(z) = H_1(z)\, H_2(z)$$

$$H_1(z) = \frac{V(z)}{X(z)} = \frac{1}{1 - 0.75z^{-1} + 0.125z^{-2}}$$

$$V(z)\left[1 - 0.75z^{-1} + 0.125z^{-2}\right] = X(z)$$

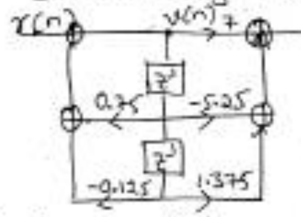
$$\therefore v(n) = x(n) + 0.75\, v(n-1) - 0.125\, v(n-2) \quad \text{—— ①}$$

$$H_2(z) = \frac{Y(z)}{V(z)} = 7 - 5.25z^{-1} + 1.375z^{-2}$$

$$Y(z) = \left[7 - 5.25z^{-1} + 1.375z^{-2}\right] V(z)$$

$$y(n) = 7v(n) - 5.25\, v(n-1) + 1.375\, v(n-2) \quad \text{—— ②}$$

Cascading $H_1(z)$ & $H_2(z)$ we get



③ for the difference equation

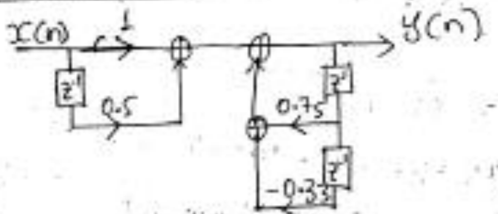
$$y(n) = 0.75\, y(n-1) - 0.33\, y(n-2) + x(n) + 0.5\, x(n-1)$$

draw direct form I & II structures

$$y(n) - 0.75\, y(n-1) + 0.33\, y(n-2) = x(n) + 0.5\, x(n-1)$$

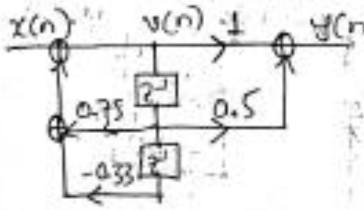
$$\boxed{Y(z)\left[1 - 0.75z^{-1} + 0.33z^{-2}\right] = X(z)\left[1 + 0.5z^{-1}\right]}$$

Direct form I



$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.5z^{-1}}{1 - 0.75z^{-1} + 0.33z^{-2}}$$

Direct form II



Cascaded form realization

(70)

$H(z) = H_1(z) \cdot H_2(z)$

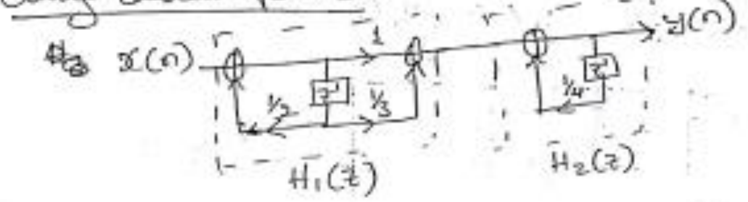
first / second order system function

Example(s)

1) $H(z) = \frac{Y(z)}{X(z)} = \frac{(1+\frac{1}{3}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = H_1(z) \cdot H_2(z)$

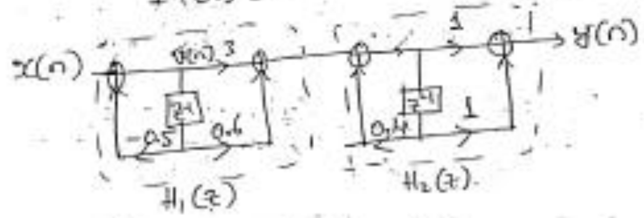
$H_1(z) = \frac{(1+\frac{1}{3}z^{-1})}{(1-\frac{1}{2}z^{-1})}$ $\quad H_2(z) = \frac{1}{(1-\frac{1}{4}z^{-1})}$

Using direct-form II



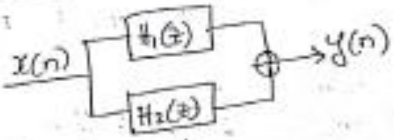
2) $H(z) = \frac{3+3.6z^{-1}+0.6z^{-2}}{1+0.1z^{-1}-0.2z^{-2}} = \frac{(3+0.6z^{-1})(1+z^{-1})}{(1+0.5z^{-1})(1-0.4z^{-1})}$

$H_1(z) = \frac{3+0.6z^{-1}}{1+0.5z^{-1}}$ $\quad H_2(z) = \frac{1+z^{-1}}{1-0.4z^{-1}}$



Parallel form realization

$H(z) = H_1(z) + H_2(z)$



Examples

PTO

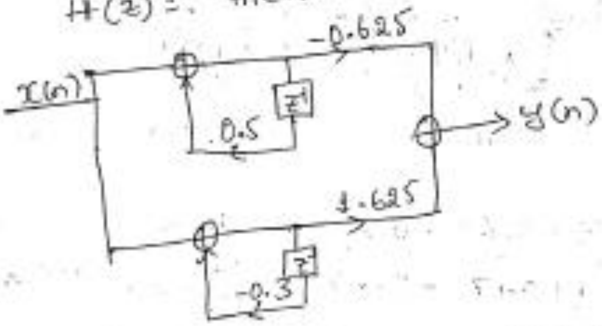
1] $$H(z) = \frac{1 - z^{-1}}{1 - 0.2z^{-1} - 0.15z^{-2}}$$

$$\frac{H(z)}{z} = \frac{(z-1)}{(z^2 - 0.2z - 0.15)} = \frac{(z-1)}{(z-0.5)(z+0.3)}$$

$$\frac{H(z)}{z} = \frac{-0.625}{(z-0.5)} + \frac{1.625}{(z+0.3)}$$

$$H(z) = \frac{-0.625}{(1-0.5z^{-1})} + \frac{1.625}{(1+0.3z^{-1})}$$

$$H(z) = H_1(z) + H_2(z)$$



2] $$H(z) = \frac{1 - 2/3\, z^{-1}}{1 - 7/8\, z^{-1} + 3/32\, z^{-2}} \cdot \frac{1 + \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

$$\frac{H(z)}{z} = \frac{(z - 2/3)}{(z^2 - 7/8\, z + 3/32)} \cdot \frac{(z^2 + 7/4\, z - 1/2)}{(z^2 - z + 1/2)}$$

$$\frac{H(z)}{z} = \frac{(z - 2/3)}{(z - 3/4)(z - \frac{1}{8})} \cdot \frac{(z^2 + 7/4\, z - 1/2)}{(z - 1/2 - j/2)(z - \frac{1}{2} + j/2)}$$

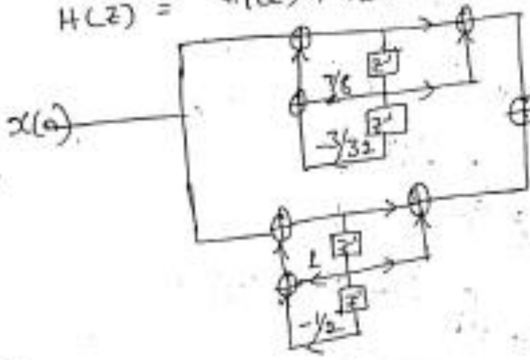
$$\frac{H(z)}{z} = \frac{A_1}{(z - 3/4)} + \frac{A_2}{(z - 1/8)} + \frac{A_3}{(z - \frac{1}{2} - j/2)} + \frac{A_4}{(z - 1/2 - j/2)}$$

(71)

$\frac{H(z)}{z} =$

$H(z) = \frac{}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}} + \frac{}{1-z^{-1}+\frac{1}{2}z^{-2}}$

$H(z) = H_1(z) + H_2(z)$



3] $H(z) = \frac{6z^2+7z+1}{z^2 - 0.75z + 0.125}$

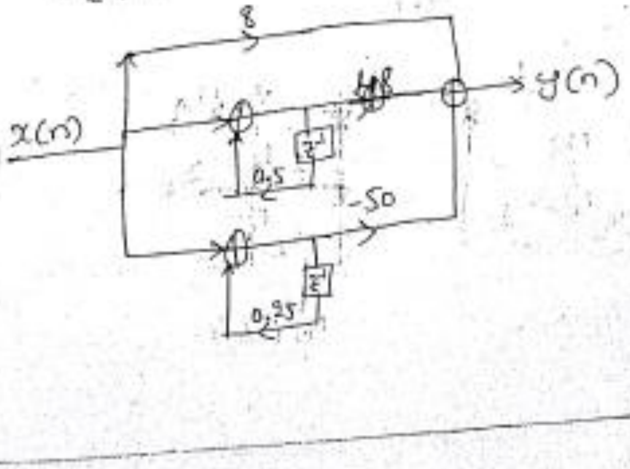
$\frac{H(z)}{z} = \frac{6z^2+7z+1}{z(z^2-0.75z+0.125)}$

$= \frac{A}{z} + \frac{B}{(z-0.5)} + \frac{C}{(z-0.25)}$

$= \frac{8}{z} + \frac{48}{z-0.5} + \frac{-50}{z-0.25}$

$H(z) = 8 + \frac{48}{1-0.5z^{-1}} - \frac{50}{1-0.25z^{-1}}$

$H(z) = H_1(z) + H_2(z) + H_3(z)$



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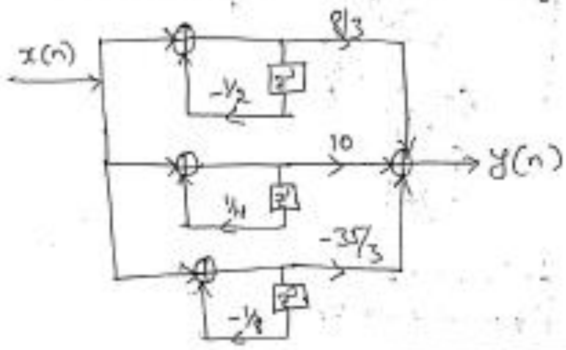
Lattice Realization

4] $H(z) = \dfrac{(1+z^{-1})(1+2z^{-1})}{(1+0.5z^{-1})(1-\frac{1}{4}z^{-1})(1+\frac{1}{8}z^{-1})}$

$$\frac{H(z)}{z} = \frac{A}{z+0.5} + \frac{B}{z-\frac{1}{4}} + \frac{C}{z+\frac{1}{8}}$$

$$H(z) = \frac{8/3}{1+0.5z^{-1}} + \frac{10}{1-\frac{1}{4}z^{-1}} + \frac{-35/3}{1+\frac{1}{8}z^{-1}}$$



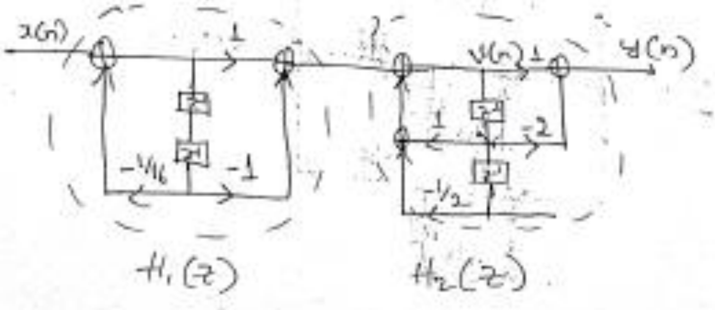
3] Cascaded form realization

$$H(z) = \frac{(z-1)(z-2)(z+1)z}{(z-0.5-j0.5)(z-0.5+j0.5)(z-j\frac{1}{4})(z+j\frac{1}{4})}$$

$$= \frac{(z^2-1)(z^2-2z)}{(z^2+\frac{1}{16})(z^2-z+0.5)}$$

$$= \frac{(1-z^{-2})(1-2z^{-1})}{(1+\frac{1}{16}z^{-2})(1-z^{-1}+0.5z^{-2})}$$

$H_1(z)$ $H_2(z)$

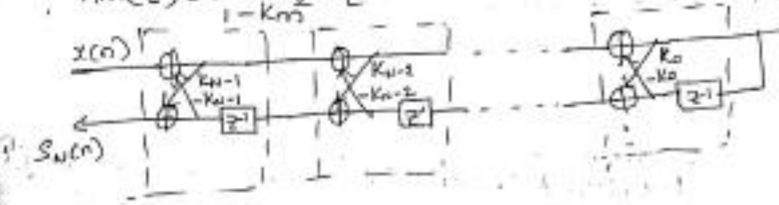


Lattice Realization

$K_m = \dfrac{a_{m+1}[m+1]}{a_{m+1}[0]}$ $\quad H(z) = \dfrac{1}{A(z)}$

Reflection Coefficients

$$A_m(z) = \frac{1}{1-K_m^2}\left[A_{m+1}(z) - K_m \tilde{A}_{m+1}(z)\right]$$



$\boxed{1}$ $A(z) = \dfrac{1}{1-1.8z^{-1}+1.62z^{-2}-0.729z^{-3}}$

$H(z) = 1/A_3(z)$

$A_3(z) = 1-1.8z^{-1}+1.62z^{-2}-0.729z^{-3}$

$\tilde{A}_3(z) = -0.729+1.62z^{-1}-1.8z^{-2}+z^{-3}$

$K_2 = \dfrac{-0.729}{1}$

$$A_2(z) = \frac{1}{1-(-0.729)^2}\left[1-1.8z^{-1}+1.62z^{-2}-0.729z^{-3}-(-0.729)(-0.729+1.62z^{-1}-1.8z^{-2}+z^{-3})\right]$$

$$A_2(z) = \frac{1}{1-0.5314}\left[1-1.8z^{-1}+1.62z^{-2}-0.5314+1.1809z^{-1}-1.36z^{-2}\right]$$

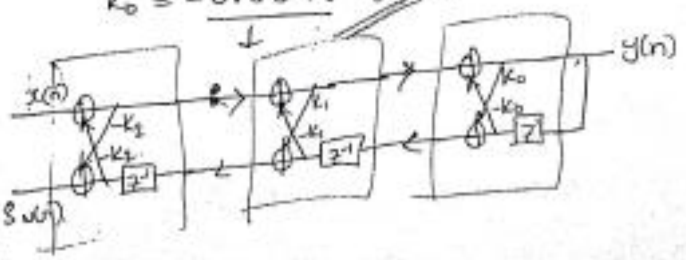
$A_2(z) = 1-1.32z^{-1}+0.55z^{-2}$

$\tilde{A}_2(z) = 0.55-1.32z^{-1}+z^{-2}$

$K_1 = \dfrac{0.55}{1}$ $\quad (0.55)$

$A_1(z) = 1-0.8516z^{-1}$

$K_0 = \dfrac{-0.8516}{1}$ $\quad (0.79)$



2) $H(z) = \frac{1}{(1-0.98z^{-1})^3} = \frac{1}{1-2.9400z^{-1}+2.8812z^{-2}-0.941192z^{-3}}$

$k_2 = -0.941192, \quad k_1 = 0.999456, \quad k_0 = -0.999312$

$A_2(z) = 1 - 1.99932z^{-1} + 0.999456z^{-2}$

$A_1(z) = 1 - 0.999932z^{-1}$

3) $H(z) = \frac{1}{1-0.091z^{-1}-1.4299z^{-2}+0.2789z^{-3}+0.4048z^{-4}-0.1823z^{-5}+0.0316z^{-6}}$

$k_5 = 0.0316, \quad k_4 = -0.1796, \quad k_3 = 0.4496, \quad k_2 = 0.0168$

$k_1 = -0.9949, \quad k_0 = 0.8984$

REALIZATION OF FIR FILTERS

[I] Direct Form-I and Direct Form-II Structures

(1) The general Differential equation is given by:-

$$\boxed{\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)} \quad \text{---} [1]$$

Assuming, $\boxed{a_0 = 1}$ we can write the expression for $y(n)$ as

$$\boxed{y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)} \quad \text{---} [2]$$

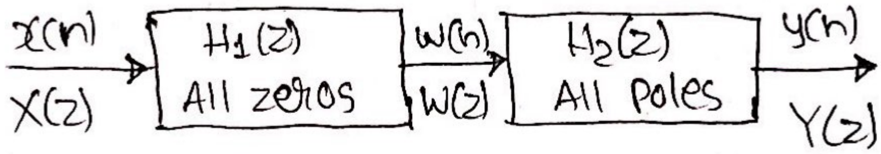
Taking Z.T we get

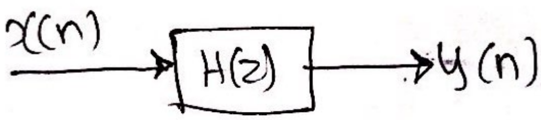
$$\therefore Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$\therefore Y(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$\therefore \boxed{\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}} \quad \text{---} [3]$$

1] Direct-Form-I Structure:- $\boxed{H(z) = H_1(z) H_2(z)}$





We define, $\boxed{H_1(z) = \sum_{k=0}^{M} b_k z^{-k}}$ —[4]

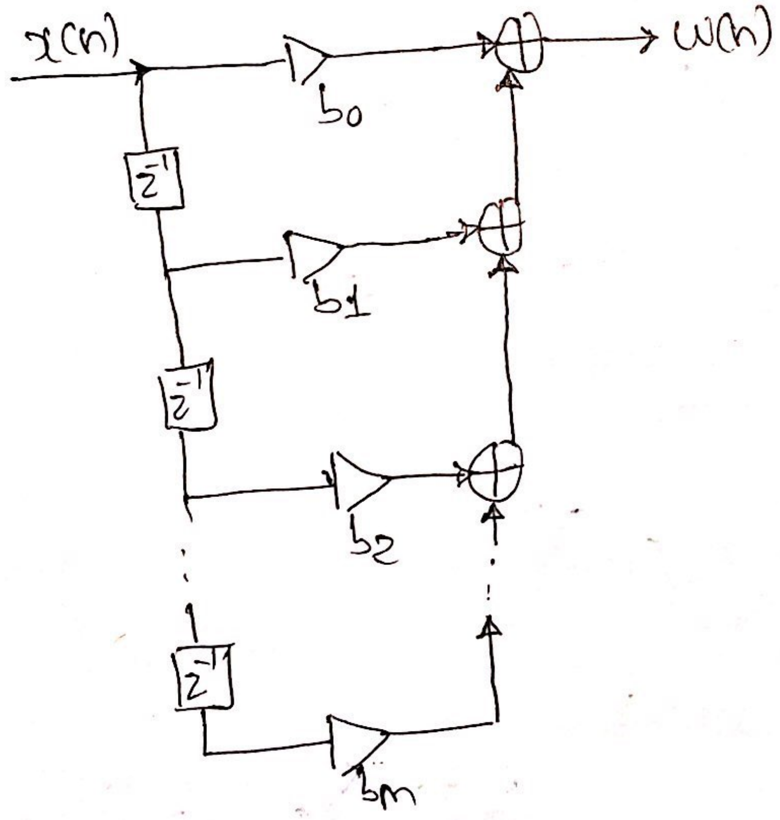
$\boxed{W(z) = H_1(z)X(z)}$ —[5]-a

$\boxed{W(z) = X(z)\sum_{k=0}^{M} b_k z^{-k}}$ —[5]-b

Taking inverse Z.T. of eq-[5]-b

$\therefore$ $\boxed{w(n) = \sum_{k=0}^{M} b_k x(n-k)}$ —[6] → <u>$x(n)$ is input $w(n)$ is output</u>

Implementing eq-[6] as a block diagram containing adders, multipliers and delay elements:-



Now consider, 2nd system:- $H_2(z)$ defined as

$$\boxed{H_2(z) = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}}$$ —[7]

$$\boxed{\frac{Y(z)}{W(z)} = H_2(z) = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}} \quad \text{—[8]-a}$$

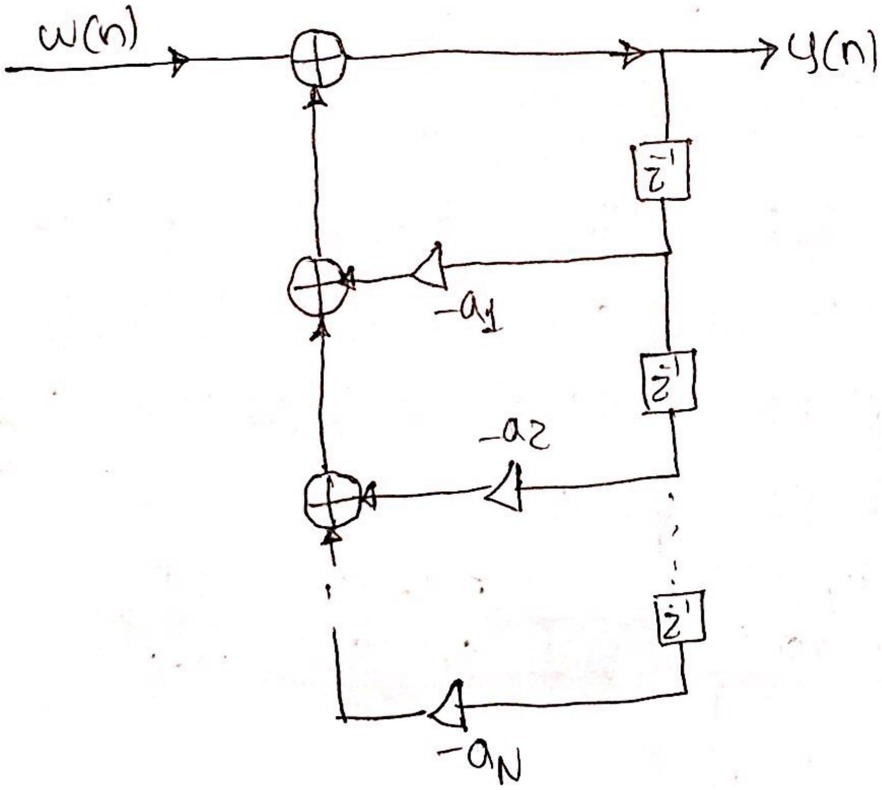
$$\therefore \boxed{Y(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = W(z)} \quad \text{—[8]-b}$$

Taking inverse Z.T. of eq-[8]-b

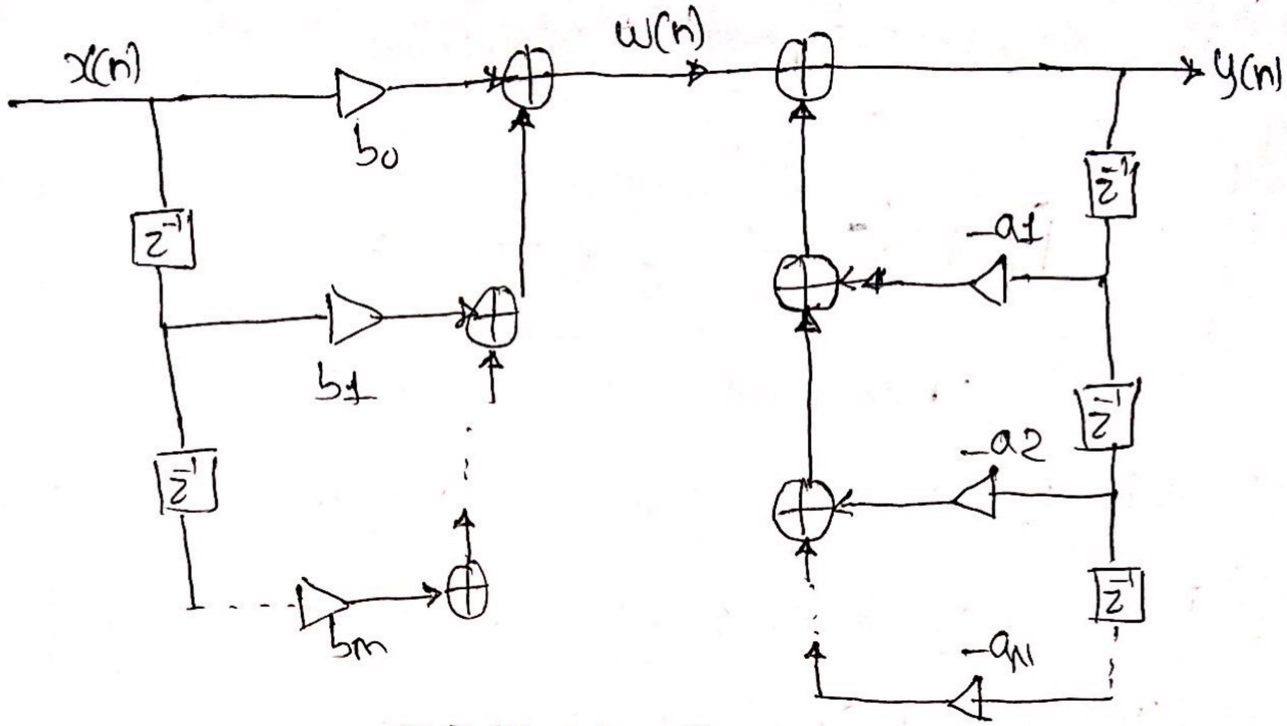
$$\therefore y(n) + a_1 y(n-1) + a_2 y(n-2) + \cdots + a_N y(n-N) = w(n)$$

$$\therefore \boxed{y(n) = -a_1 y(n-1) - a_2 y(n-2) \cdots + a_N y(n-N) + w(n)} \quad \text{—[9]}$$

Implementing eq-[9] as a block diagram



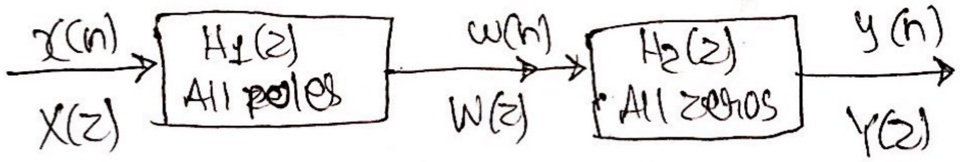
Combining the both block diagrams implemented by eq-[6] and eq-[9] we get the Direct Form-I [DF-I] structure for IIR system representing eq-[2].



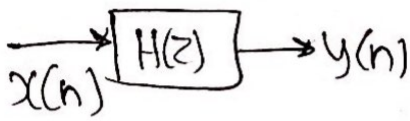
DF-I structure

2] Direct-Form-II structure:-

For the IIR system if $H_1(z)$ and $H_2(z)$ in the block -diagram of DF-I structure are interchanged, the output y(n) does not change for the input x(n). Now DF-II is given by the block diagram



Now
$$\boxed{H_1(z) = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}} \text{ —[10]-a}$$



$$\boxed{H_2(z) = \sum_{k=0}^{m} b_k z^{-k}} \text{ —[10]-b}$$

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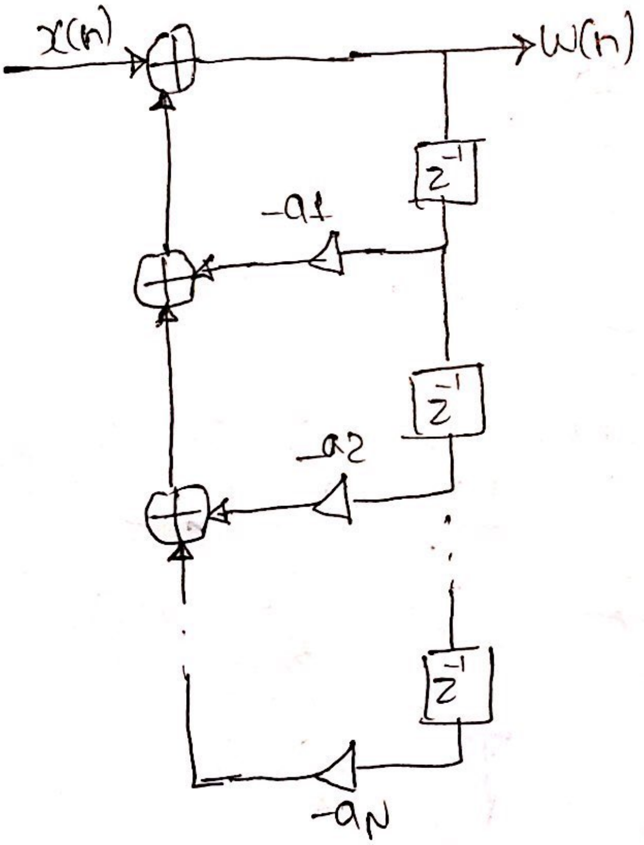
$$\boxed{\frac{W(z)}{X(z)} = H_1(z) = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}} \quad \text{—[11]-a}$$

$$\boxed{W(z)\left[1+\sum_{k=1}^{N} a_k z^{-k}\right] = X(z)} \quad \text{—[11]-b}$$

Taking inverse Z.T. of eq-[11]-b

$$w(n) + a_1 w(n-1) + a_2 w(n-2) + \cdots + a_N w(n-N) = x(n)$$

$$\boxed{w(n) = -a_1 w(n-1) - a_2 w(n-2) \cdots + a_N w(n-N) + x(n)} \quad \text{—[12]}$$

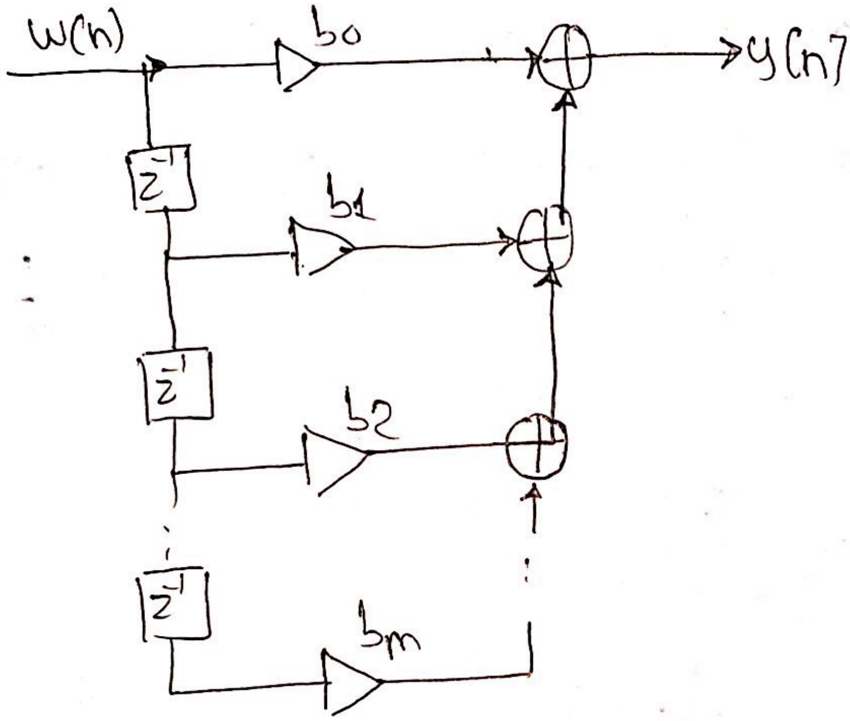


Now consider, 2nd system $\boxed{H_2(z) = \frac{Y(z)}{W(z)}}$ —[13]-a

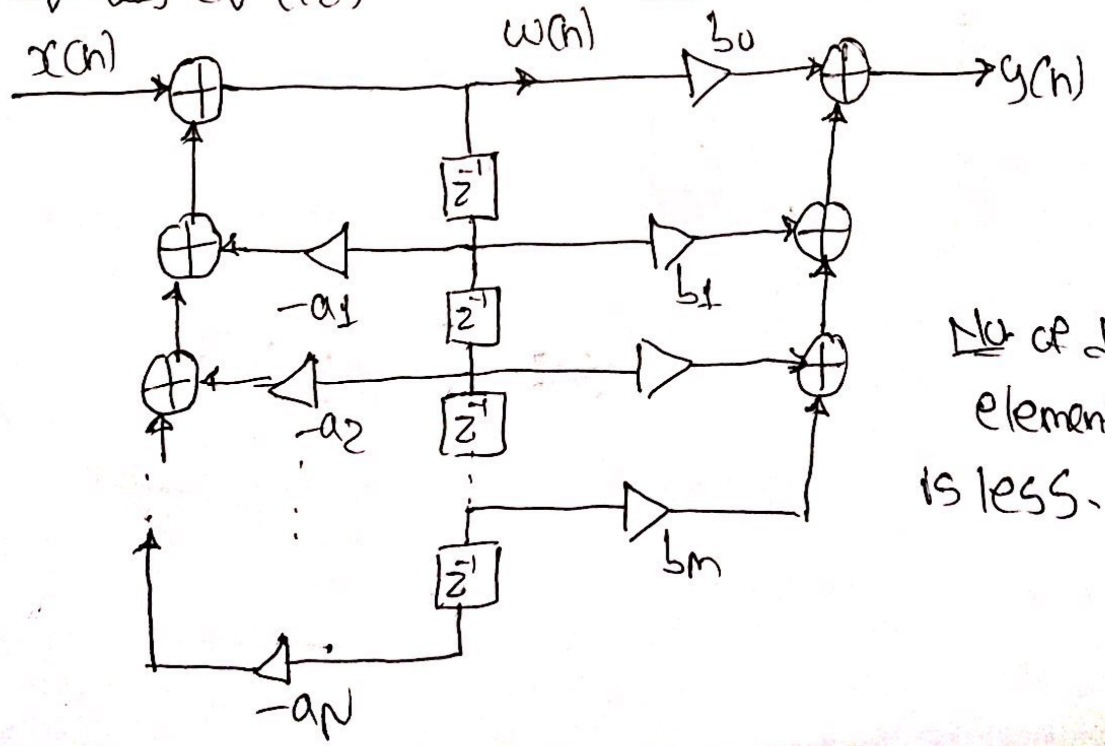
$$\therefore \boxed{Y(z) = W(z) \sum_{k=0}^{m} b_k z^{-k}} \quad \text{— [13]-b}$$

$$\therefore \boxed{Y(z) = b_0 w(n) + b_1 w(n-1) + \cdots + b_m w(n-m)} \quad \text{— [13]-c}$$

Implementing eq-[13]-c as a block diagram:-



Combining both the block diagrams for eq-[13]-c and ~~eq-[6]~~ eq-[12] we get DF-II structure.

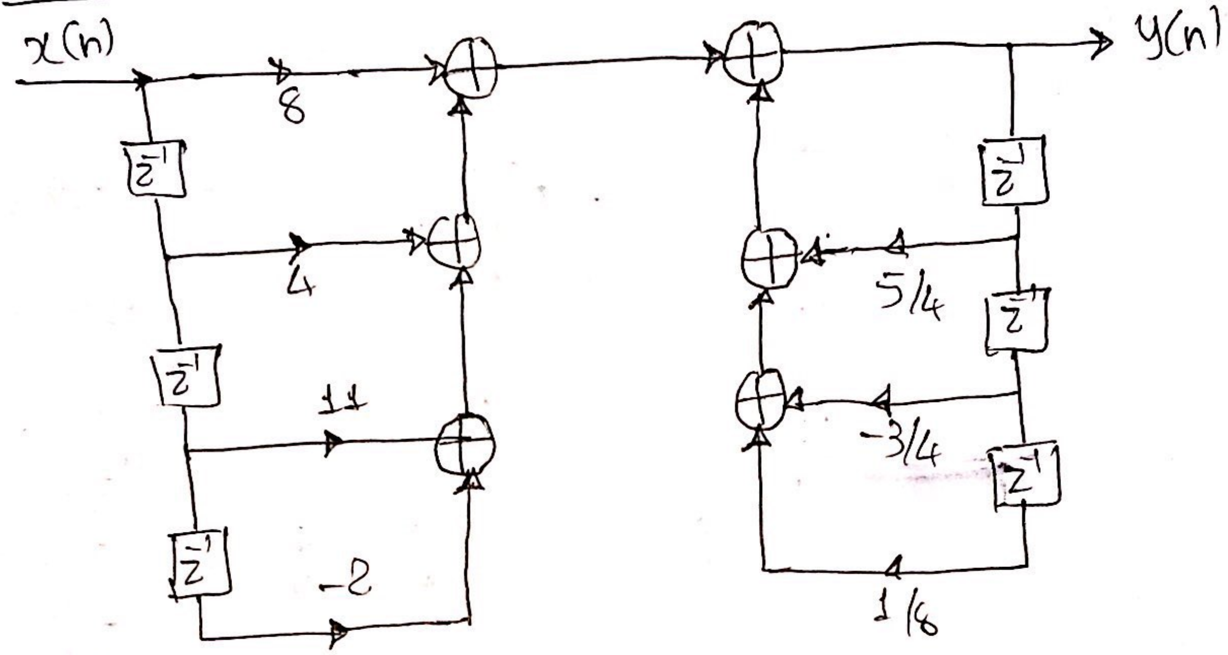


No. of delay elements required is less.

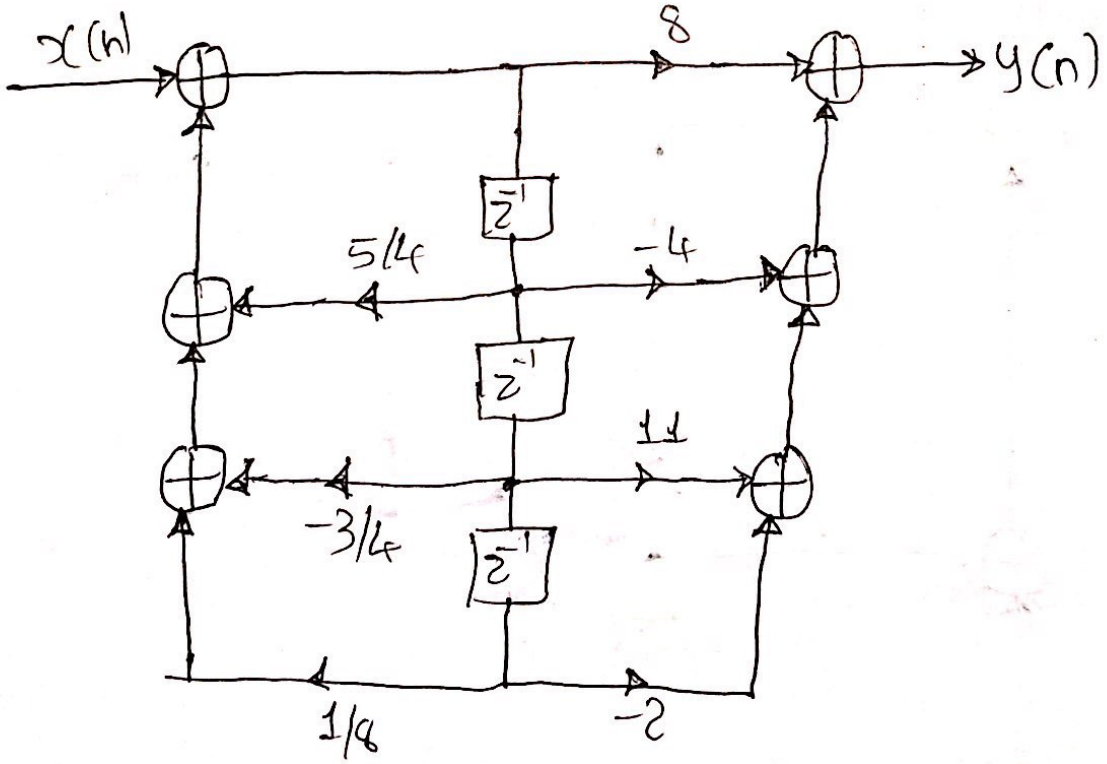
Problems on DF-I and DF-II :-

① $H(z) = \dfrac{8-4z^{-1}+11z^{-2}-2z^{-3}}{1-5/4\, z^{-1}+\frac{3}{4}z^{-2}-\frac{1}{8}z^{-3}}$

a] DF-I :-



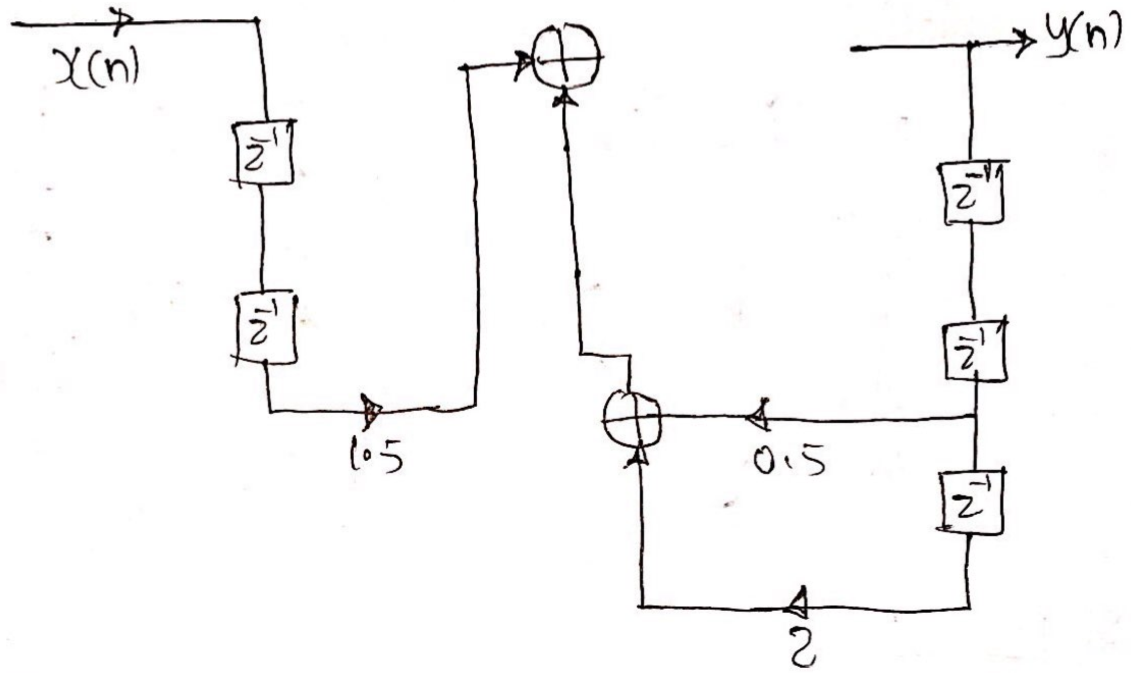
b] DF-II :-



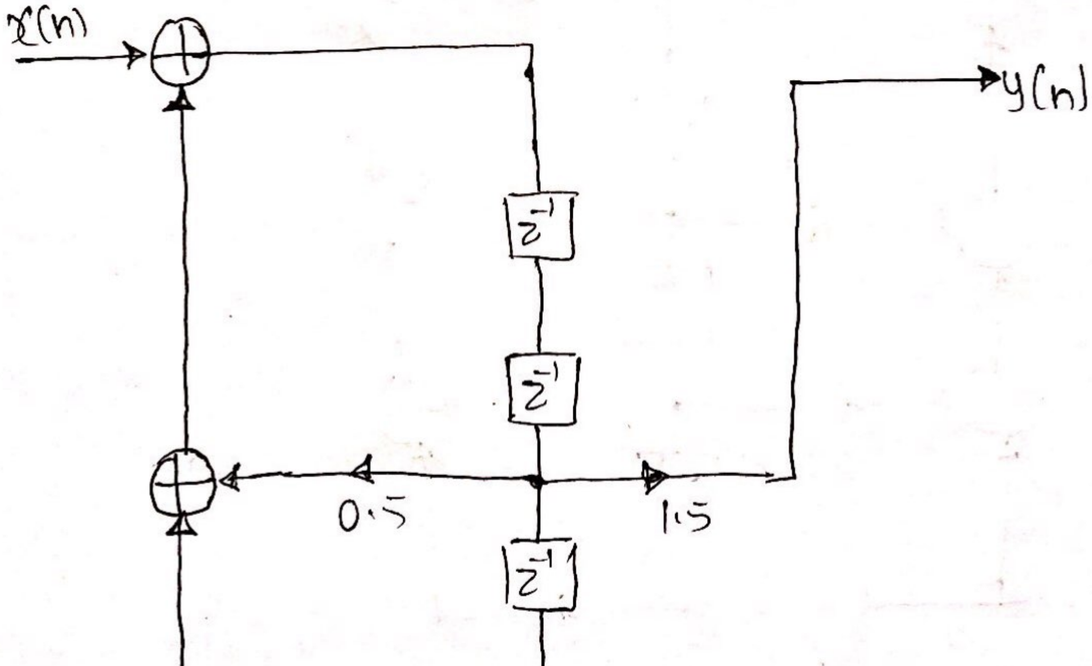
② $2y(n) - y(n-2) - 4x(n-3) = 3x(n-2)$

$$\boxed{y(n) = \frac{1}{2}y(n-2) + 2x(n-3) + 1.5x(n-2)}$$

a] DF-I:-



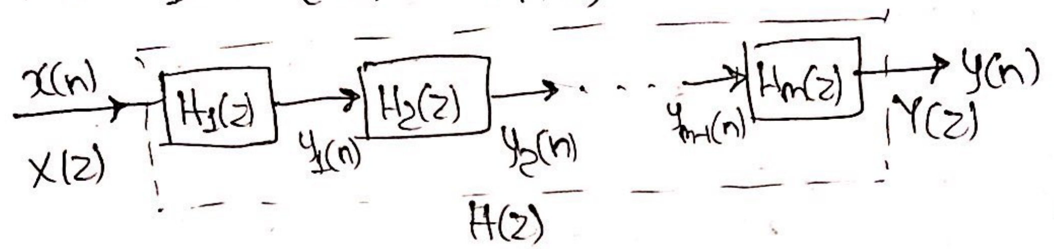
b] DF-II:-



[II]

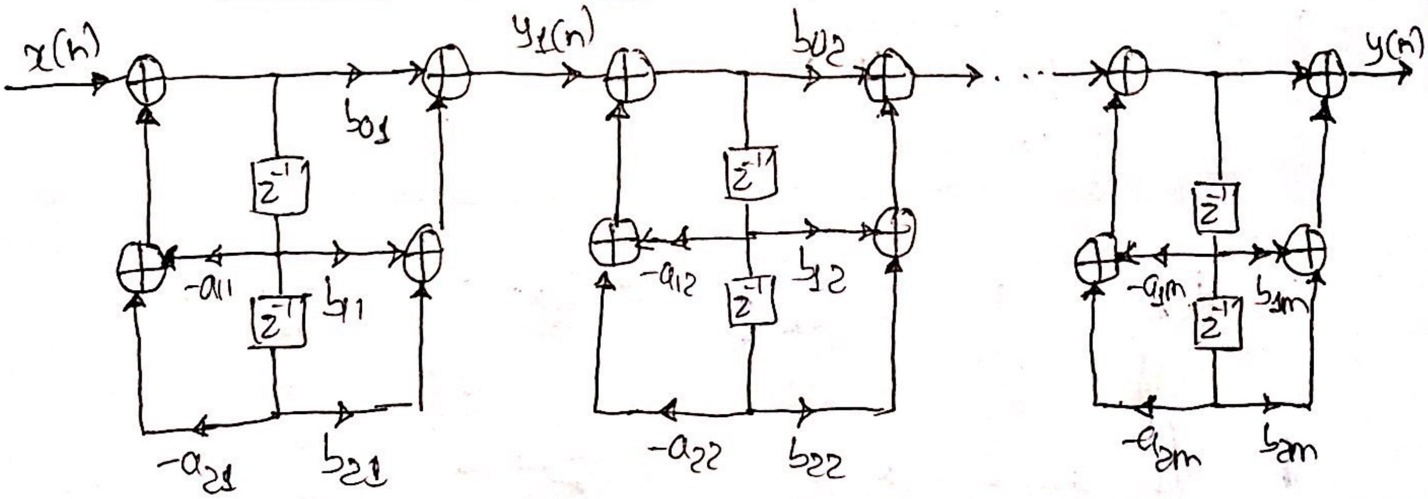
Cascade Realization

1. In this type of realization, $H(z)$ is broken into the product of TFs $H_1(z), H_2(z), \ldots H_m(z)$.



It could be broken down into many different forms. The most common form of cascade realization to have blocks in the form above, and each should be bi-quadratic section. Let, $H_k(z)$ be of the form

$$\boxed{H_k(z) = \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 + a_{1k} z^{-1} + a_{2k} z^{-2}}} \quad \text{—[1]} ; \; k = 1, 2, \ldots m$$



Special cases:- i] If, $H_k(z) = C$; if $b_{1k} = b_{2k} = a_{1k} = a_{2k} = 0$

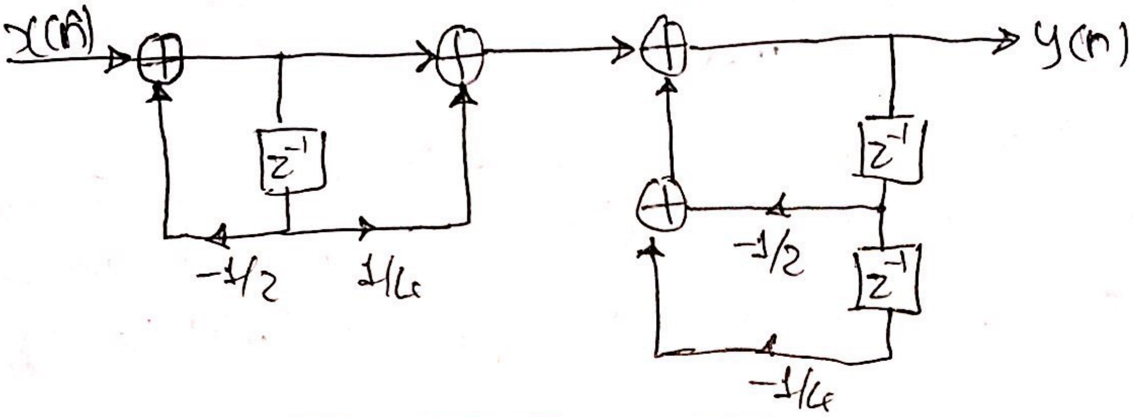
ii] If, $H_k(z) = \dfrac{b_{0k}}{1 + a_{1k} z^{-1}}$; $b_{1k} = b_{2k} = a_{2k} = 0$

iii] If $H_k(z) = b_{1k} z^{-1}$; if $b_{0k} = b_{2k} = a_{1k} = a_{2k} = 0$

iv] If $H_k(z) = \dfrac{b_{0k} + b_{1k} z^{-1}}{1 + a_{1k} z^{-1}}$; $b_{2k} = a_{2k} = 0$

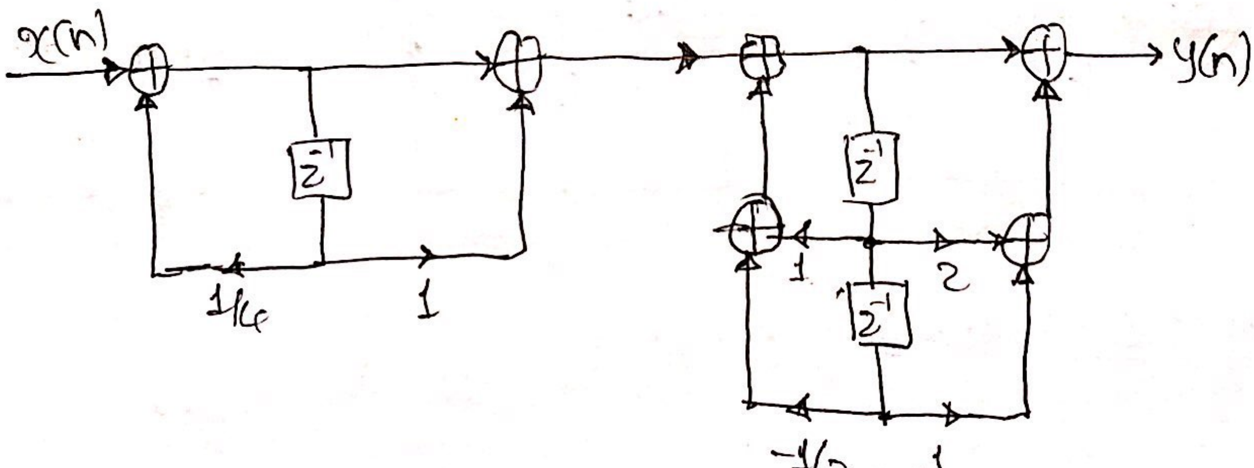
① $$H(z) = \frac{1 + 1/4 z^{-1}}{(1 + 1/2 z^{-1})(1 + 1/2 z^{-1} + 1/4 z^{-2})}$$

$$\boxed{H_1(z) = \frac{1 + 1/4 z^{-1}}{1 + 1/2 z^{-1}}} \; ; \; \boxed{H_2(z) = \frac{1}{1 + 1/2 z^{-1} + 1/4 z^{-2}}}$$



② $$H(z) = \frac{(1 + z^{-1})^3}{(1 - 1/4 z^{-1})(1 - z^{-1} + 1/2 z^{-2})}$$

$$\boxed{H_1(z) = \frac{1 + z^{-1}}{1 - 1/4 z^{-1}}} \quad \boxed{H_2(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - z^{-1} + 1/2 z^{-2}}}$$



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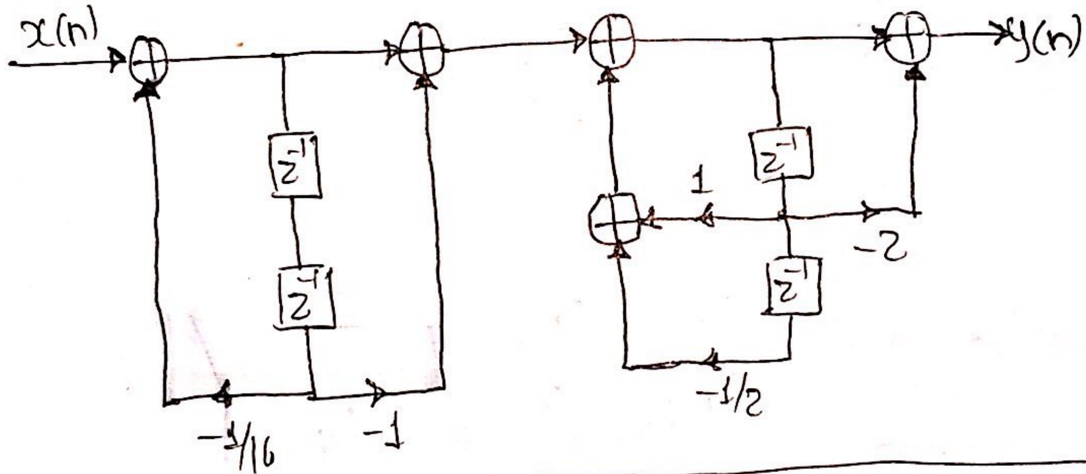
(3) $$H(z) = \frac{(z-1)(z-2)(z+1)z}{(z-1/2-j/2)(z-1/2+j/2)(z-j/4)(z+j/4)}$$

$$H(z) = \frac{(z^2-1)(z^2-2z)}{[(z-1/2)^2+1/4]+[z^2+1/16]}$$

$$H(z) = \frac{z^4(1-z^{-2})(1-2z^{-1})}{[z^2-z+1/2][z^2+1/16]} = \boxed{\frac{z^4[1-z^{-2}][1-2z^{-1}]}{z^4[1-z^{-1}+1/2z^{-2}][1+1/16z^{-2}]}}$$

$$\boxed{H_1(z) = \frac{1-z^{-2}}{1+1/16z^{-2}}} \quad \boxed{H_2(z) = \frac{1-2z^{-1}}{1-z^{-1}+1/2z^{-2}}}$$



[III] Parallel Realization

(1) H(z) could be written as the sum of $H_1(z), H_2(z), \dots H_m(z)$.

$$\boxed{H(z) = H_1(z) + H_2(z) + \dots + H_m(z)} \text{ — [1]}$$

$$H(z) = \frac{Y(z)}{X(z)} \quad \therefore \boxed{Y(z) = H_1(z)X(z) + H_2(z)X(z) + \dots + X(z)H_m(z)}$$

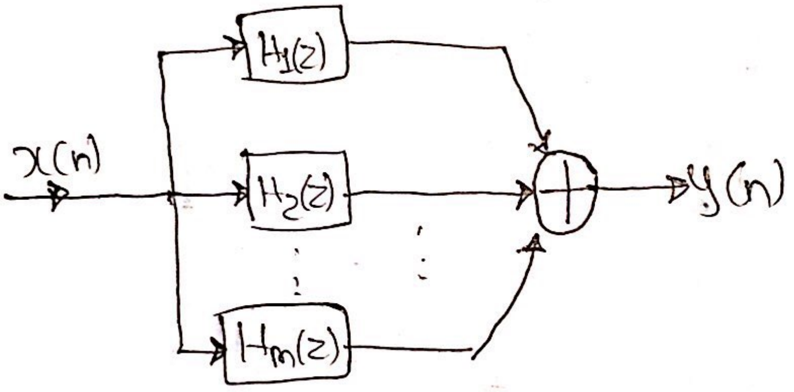
└ [2]

Eq. [2] can be realized by passing the input sequence $x[n]$ through 'm' discrete time filters in parallel and summing the outputs to obtain $y(n)$, $H_k(z)$ can be considered to be bi-quadratic sections.

One parallel-form can be written as:-

$$\boxed{H_k(z) = \frac{b_{0k} + b_{1k} z^{-1}}{1 + a_{1k} z^{-1} + a_{2k} z^{-2}}} \quad ; 1 \le k \le m$$

i] If $H_k(z) = C$; ii] $H_k(z) = \dfrac{b_{0k}}{1 + a_{1k} z^{-1}}$; iii] $H_k(z) = b_{1k} z^{-1}$.



Problems:- ① $H(z) = \dfrac{(1+z^{-1})(1+2z^{-1})}{(1+1/2 z^{-1})(1-1/4 z^{-1})(1/8 z^{-1}+1)}$

Put, $\boxed{z^{-1} = v}$ $\rightarrow$ Resolve into partial fractions for parallel realization of $H(z)$

$$\therefore H(z) = \frac{(1+v)(1+2v)}{(1+1/2 v)(1-1/4 v)(1+1/8 v)}$$

$$\therefore H(z) = \frac{A}{1+1/2 v} + \frac{B}{1-1/4 v} + \frac{C}{1+1/8 v} = (1+v)(1+2v)$$

$$A(1-1/4 v)(1+1/8 v) + B(1+1/2 v)(1+1/8 v) + C(1+1/2 v)(1-1/4 v) = (1+v)(1+2v)$$

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Finding A, B, C :- $v=4$; $A(0)+B(3)(3/2)=(5)(9)+C(0)$

$\therefore \boxed{B=10}$

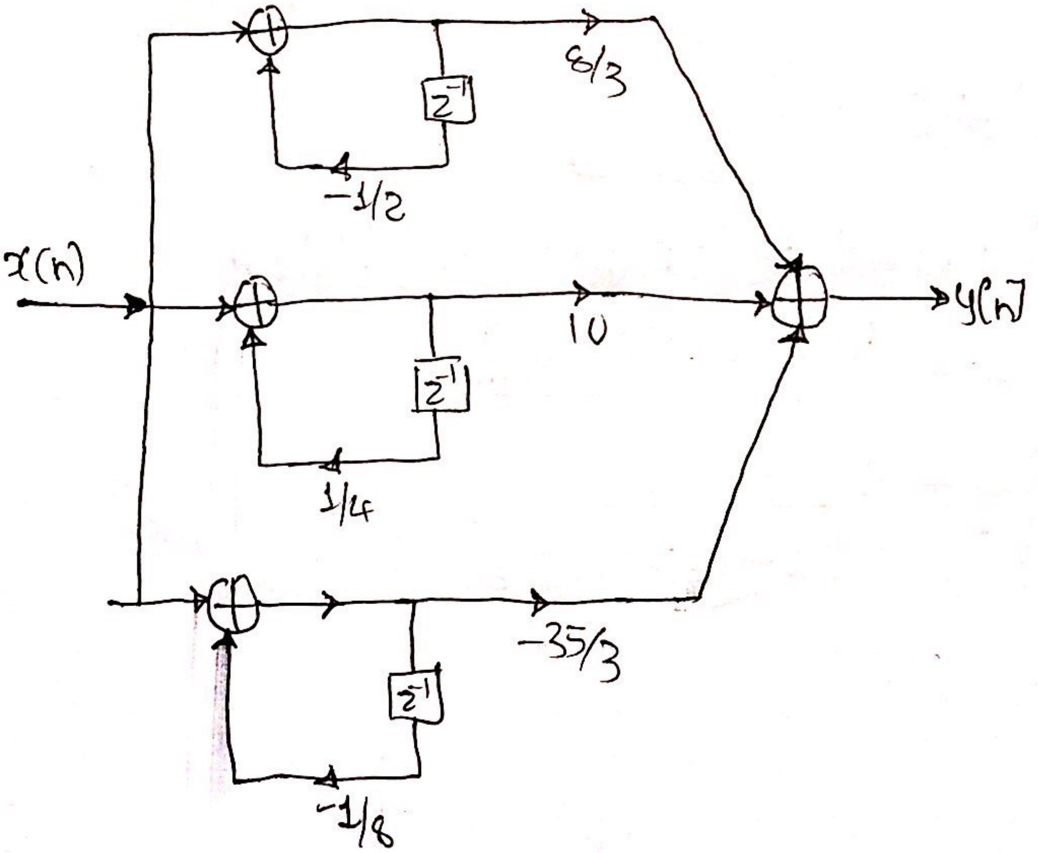
$v=-8$; $A(0)+B(0)+C(1-4)(3)=(-7)(-15)$

$\boxed{C=\frac{-35}{3}}$

$v=-2$; $B(0)+C(0)+A(3/2)(3/4)=(-1)(-3)$

$\therefore \boxed{A=8/3}$

$$H(z)=\frac{8/3}{1+\frac{1}{2}z^{-1}}+\frac{10}{1-1/4z^{-1}}+\frac{-35/3}{1+1/8z^{-1}}$$



(2) $H(z) = \dfrac{1 + 1/4\, z^{-1}}{(1 + 1/2\, z^{-1})(1 + \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2})}$

Put, $\boxed{z^{-1} = V}$

$\therefore H(z) = \dfrac{1 + 1/4\, V}{(1 + 1/2\, V)(1 + 1/2\, V + 1/4\, V^2)}$

$\therefore H(z) = \dfrac{A}{1 + \frac{1}{2} V} + \dfrac{BV + C}{1 + \frac{1}{2} V + \frac{1}{4} V^2} = 1 + 1/4\, V$

$\therefore A(1 + 1/2\, V + 1/4\, V^2) + (BV + C)(1/2\, V + 1) = 1 + \frac{1}{4} V$

Put,

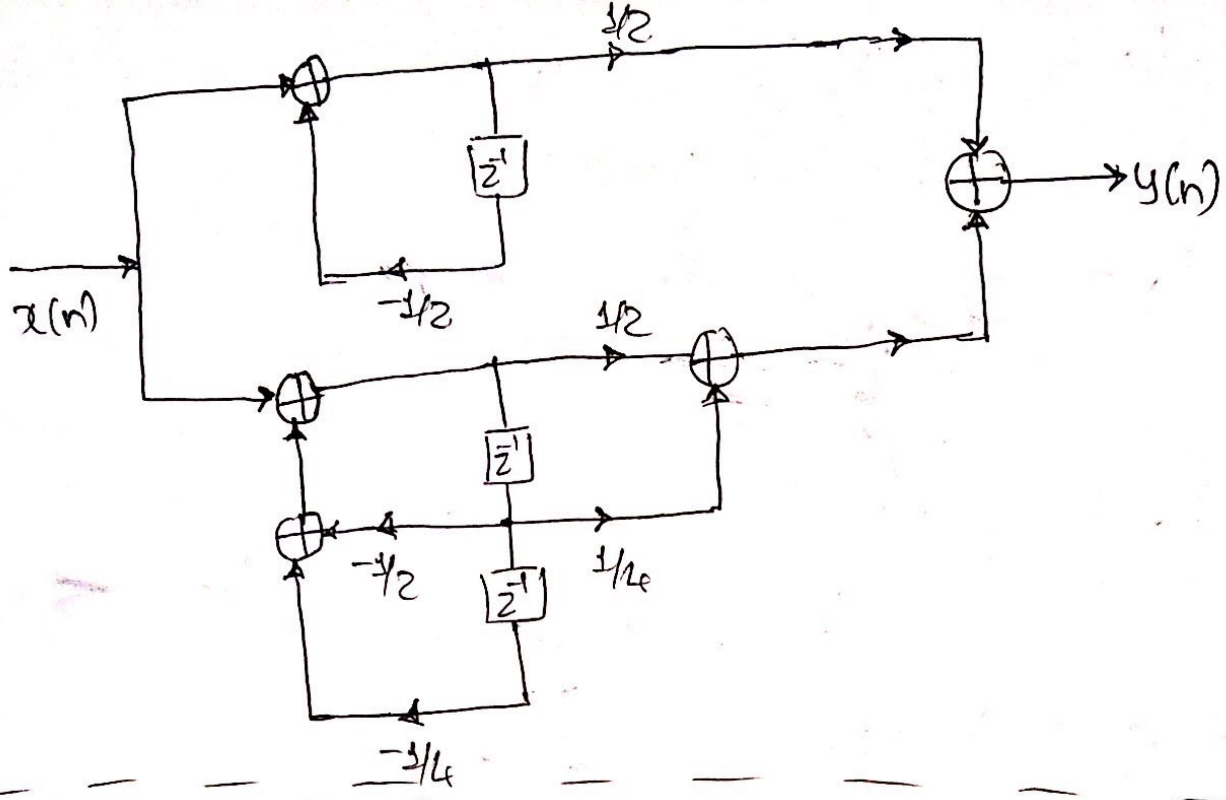
$V = -2;\quad B(0) + A(\frac{1}{4} \times 4) = 1 - \frac{1}{2} = \frac{1}{2} \quad \boxed{A = 1/2}$

$V = 0;\quad (1/2) + C = 1 \quad \boxed{C = 1/2}$

$V = 2;\quad (1/2)(1 + 1 + 1) + (2B + 1/2)(2) = 1 + \frac{1}{2} = \frac{3}{2}$

$4B + 1 = 0 \quad \boxed{B = -1/4}$

$H(z) = \dfrac{1/2}{1 + \frac{1}{2} z^{-1}} + \dfrac{1/2 - 1/4\, z^{-1}}{1 + \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2}}$



(3) $$H(z) = \frac{8z^3 - 4z^2 + 11z - 2}{(z - 1/4)(z^2 - z + 1/2)}$$

$$H(z) = \frac{8z^3 - 4z^2 + 11z - 2}{z^3 - 5/4 z^2 + 1/4 z + 1/8} =$$

$$H(z) = \frac{z^3[8 - 4z^{-1} + 11z^{-2} - 2z^{-3}]}{z^3[1 - 5/4 z^{-1} + 1/4 z^{-2} + 3/8 z^{-3}]}$$ Put, $\boxed{z^{-1} = v}$

$$\therefore H(z) = \frac{8 - 4v + 11v^2 - 2v^3}{1 - 5/4 v + 3/4 v^2 - 1/8 v^3}$$ Dividing Nr./Dr. we get

$$H(z) = 16 + \frac{-v^2 + 16v - 8}{(1 - 1/4 v)(1 - v + v^2/2)}$$

$$\cancel{H(z) = 16 + \frac{B\nu + C}{(1-1/4\nu)(1-\nu+\nu^2/2)}}$$

$$H(z) = 16 + \frac{A}{(1-1/4\nu)} + \frac{(B\nu+C)}{(1-\nu+\nu^2/2)}$$

$$H(z) = 16 + \frac{A(1-\nu+\nu^2/2) + (B\nu+C)(1-1/4\nu)}{(1-1/4\nu)(1-\nu+\nu^2/2)}$$

For the partial fractions:-

$$A(1-\nu+\nu^2/2) + (B\nu+C)(1-1/4\nu) = -\nu^2+16\nu-8$$

$$\nu=4; \; A(-3+8) = -16+64-8 = 40$$

$$\boxed{A=8}$$

$$8 - 8\nu + 4\nu^2 + B\nu - \frac{B}{4}\nu^2 + C - \frac{C\nu}{4} = -\nu^2+16\nu-8$$

$$\therefore \left(4-\frac{B}{4}\right)\nu^2 + \left(B-8-\frac{C}{4}\right)\nu + (8+C) = -\nu^2+16\nu-8$$

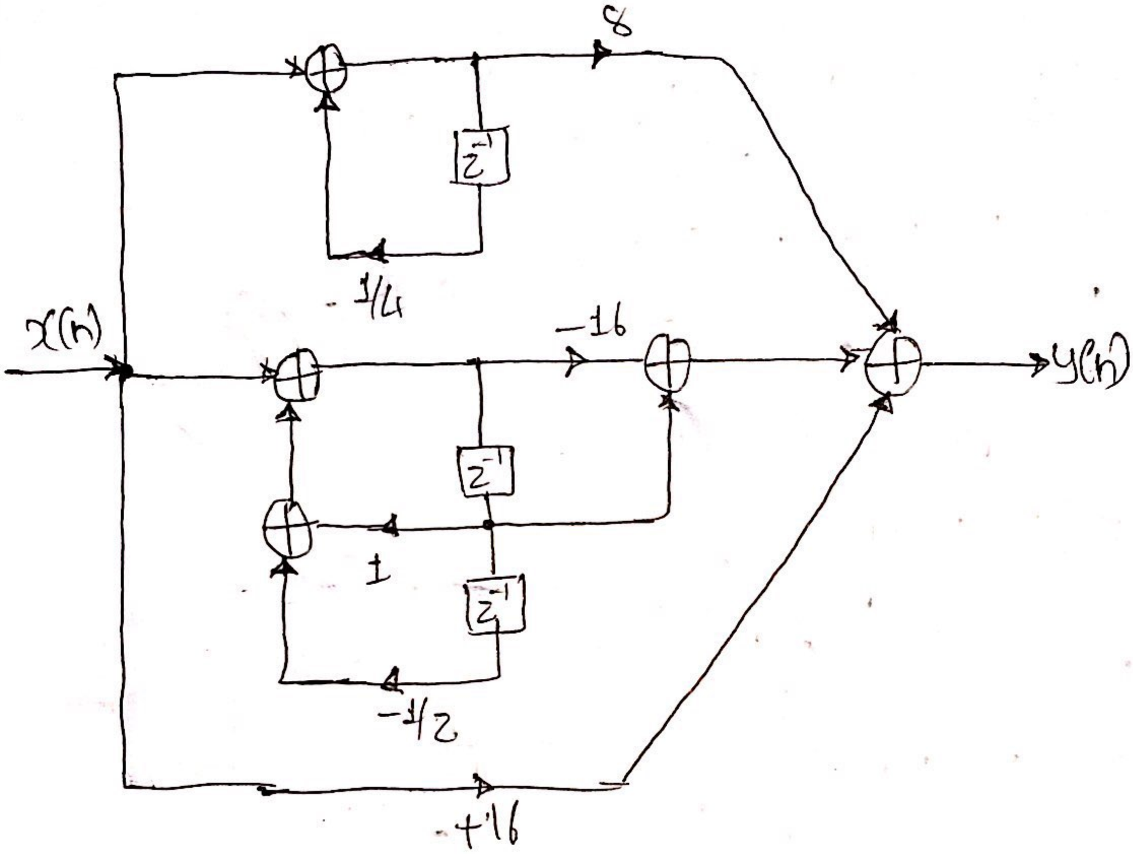
$$8+C=-8 \qquad \boxed{C=-16}$$

$$4-\frac{B}{4}=-1 \qquad \boxed{B=20}$$

$$\therefore H(z) = 16 + \frac{8}{(1-1/4\nu)} + \frac{20\nu - 16}{(1-\nu+\nu^2/2)}$$

as $\boxed{z^{-1} = v}$

$$\therefore \boxed{H(z) = 16 + \frac{8}{1 - 1/4\, z^{-1}} + \frac{20z^{-1} - 16}{1 - z^{-1} + \frac{1}{2} z^{-2}}}$$



[IV] IIR Lattice structures [All-pole Realizations]

① Concept:- a] The Lattice realization of IIR filter is obtained by first realizing the lattice structure for an all-pole filter and then inserting the appropriate multipliers and adders to realize the zeros. Here, we realize an all-pole filter.

b] The all-pole filter is assumed to be

$$\boxed{H(z) = \frac{1}{A(z)} = \frac{1}{1 + \sum_{k=1}^{N} a(i) z^{-i}}} \quad \text{— [1]}$$

In the FIR filters we had derived the lattice structure for the polynomial $A(z)$. The TF $H(z)$ of an all-pole filter has the same structure except that the input and output are interchanged, which can be interpreted as follows:-

For FIR filter:- $H(z) = \frac{Y(z)}{X(z)} = 1 + \sum_{i=1}^{N-1} a(i) z^{-i}$

$$\therefore \boxed{Y(z) = X(z) + \sum_{i=1}^{N-1} a(i) z^{-i} X(z)} \quad \text{—[2]-a}$$

But for an all-pole IIR filter:-

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + \sum_{i=1}^{N-1} a(i) z^{-i}}} \quad \text{—[1]}$$

→ see previous page

$$\therefore \boxed{X(z) = Y(z) + \sum_{i=1}^{N-1} a(i) z^{-i} Y(z)} \quad \text{—[2]-b}$$

Note that eq-[2]-b $X(z)$ and $Y(z)$ positions are interchanged, compared with eq-[2]-a.

c] Thus, $f_N(n)$ is the input and $f_0(n)$ is the output. Hence, given the input $f_N(n) = x(n)$, we must find $f_{N-1}(n), f_{N-2}(n), \ldots f_0(n)$ successively.

The lattice equations for mth stage as given for FIR filter is

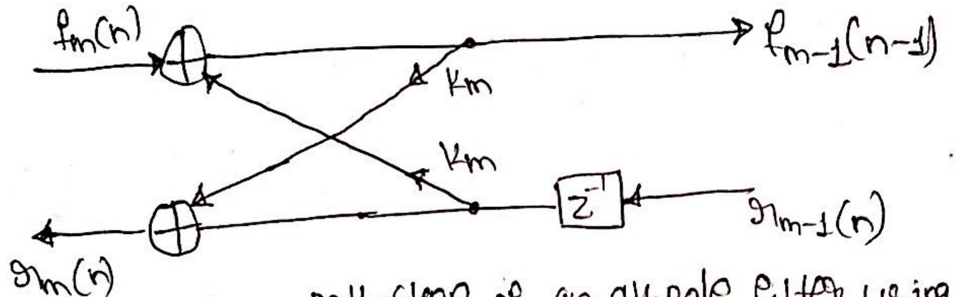
$$\boxed{f_m(n) = f_{m-1}(n) + k_m g_{m-1}(n-1)} \quad \text{—[3]-a}$$

$$\boxed{g_m(n) = k_m f_{m-1}(n) + g_{m-1}(n-1)} \quad \text{—[3]-b}$$

For all-pole filters eq-[3]a is modified as

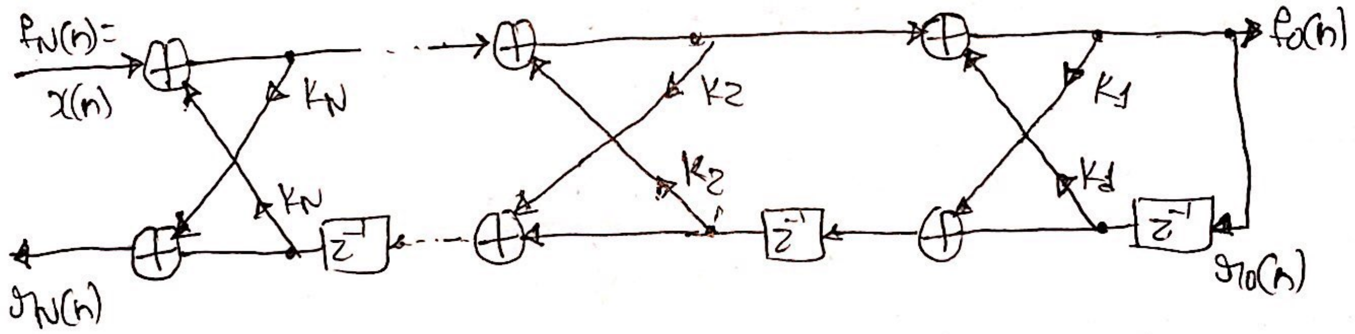
$$\boxed{f_{m-1}(n) = f_m(n) - K_m\, g_{m-1}(n-1)} \quad \text{—[4] and} \quad \boxed{g_m(n) = K_m\, f_{m-1}(n) + g_{m-1}(n-1)} \quad \text{—[5]}$$

Lattice structure developed using eq-[4], [5] is



mth-stage of an all-pole filter using lattice structure

Now the lattice structure for Nth order all-pole filter realized as the cascade first-order stages is ashown in the figure below.



d] The formulae for Lattice coefficients for $A(z)$ are as follows:— For $m = N, N-1, \ldots 1$ do

$$\boxed{K_m = a_m(m)} \quad \boxed{a_{m-1}(i) = \frac{a_m(i) - a_m(m)\, a_m(m-i)}{1 - K_m^2}} \; ; \; 1 \le i \le m-1$$

Problems:- ① $H(z) = \frac{1}{1-0.2971z^{-1}+0.3564z^{-2}-0.0276z^{-3}} = \frac{1}{A(z)}$

The lattice coefficients for $A(z)$ are given by:-

$$\boxed{A(z) = 1-0.2971z^{-1}+0.3564z^{-2}-0.0276z^{-3}}$$

$$\boxed{a_3(0)=1} \boxed{a_3(1)=-0.2971} \boxed{a_3(2)=0.3564} \boxed{a_3(3)=-0.0276}$$

For $m=N, N-1, \ldots 1$ do $\boxed{K_m = a_m(m)}$

For $m=3, 2, 1$,. $\quad a_{m-1}(i) = \frac{a_m(i) - a_m(m)\, a_m(m-i)}{1-k_m^2}$

For $\underline{m=3}$; $\boxed{a_3(3) = k_3 = -0.0276}$; $1 \leq i \leq 2$

$\underline{i=1}$; $\quad a_2(1) = \frac{a_3(1) - a_3(3)\, a_3(3-1)}{1-k_3^2}$

$\therefore a_2(1) = \frac{-0.2971-(-0.0276)(0.3564)}{1-(-0.0276)^2} = \frac{-0.2873}{0.9992}$

$$\boxed{a_2(1) = -0.2875}$$

$\underline{i=2}$; $a_2(2) = \frac{a_3(2) - a_3(3)\, a_3(3-2)}{1-(-0.0276)^2} = \frac{a_3(2) - a_3(3) a_3(1)}{0.9992}$

$\underline{m=3}$;

$a_2(2) = \frac{0.3564-(-0.0276)(-0.2971)}{0.9992}$

$\underline{\underline{a_2(2)}} = \frac{0.3482}{0.9992} = \boxed{0.3485}$

For $\underline{m=2}$; $\boxed{K_2 = a_2(2) = 0.3485}$

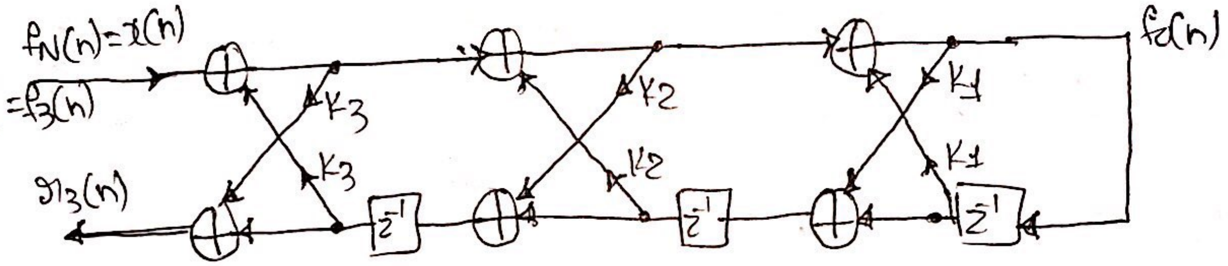
$$\boxed{a_1(i) = \frac{a_2(i) - a_2(2)\, a_2(2-i)}{1-K_2^2}} \;,\; 1 \le i \le (m-1)$$

$$1 \le i \le 1$$

$$\therefore\ a_1(1) = \frac{a_2(1) - a_2(2) a_2(1)}{1-(0.3485)^2} = \frac{-0.2875 - (0.3485)(-0.2875)}{0.8785}$$

$$\therefore\ \underline{a_1(1)} = K_1 = \frac{-0.18\cancel{73}}{0.8785} = \boxed{-0.2136}$$

$\underline{m=1}$; $\boxed{K_1 = a_1(1) = -0.2136}$



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② Find the lattice-structure for the all-pole filter

$$\cancel{H(z)} \; y(n) + \frac{3}{4} y(n-1) + \frac{1}{4} y(n-2) = x(n)$$

$\underline{\text{Solution:-}}$ Taking Z.T.:-

$$Y(z) + \frac{3}{4} z^{-1} Y(z) + \frac{1}{4} z^{-2} Y(z) = X(z)$$

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + \frac{3}{4} z^{-1} + \frac{1}{4} z^{-2}} = \frac{1}{A(z)}} \quad (1)$$

where, $A(z) = 1 + \frac{3}{4} z^{-1} + \frac{1}{4} z^{-2}$

$\boxed{a_2(0) = 1}$ $\boxed{a_2(1) = 3/4}$ $\boxed{a_2(2) = 1/4}$

For $m = N, N-1, \cdots 1 \rightarrow \boxed{m=2,1}$ $\boxed{K_m = a_m(m)}$

$$\boxed{a_{m-1}(i) = \frac{a_m(i) - a_m(m)\, a_m(m-i)}{1-K_m^2}}$$

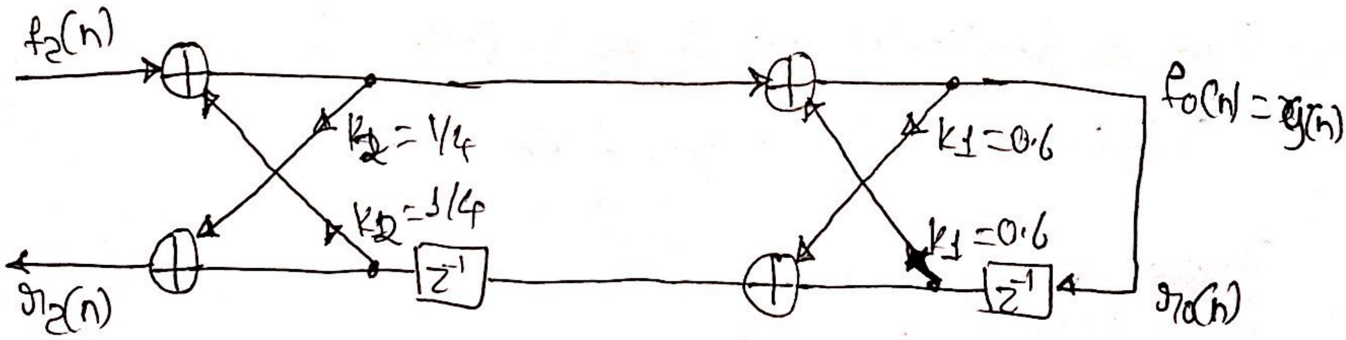
① For $m=N=2$; $\boxed{K_2 = a_2(2) = 1/4}$; $1 \le i \le m-1$; $1 \le i \le 1$

$m=2$; $i=1$; $a_1(1) = \dfrac{a_2(1) - a_2(2)\, a_2(1)}{1-K_2^2}$

$$a_1(1) = \frac{3/4 - (1/4)(3/4)}{1 - 1/16} = \frac{3/4 - 3/16}{15/16} = \frac{(12-3)/16}{15/16} = \frac{9}{15} = \frac{3}{5}$$

$\boxed{a_1(1) = K_1 = 0.6}$

for $m=1$; $\boxed{K_1 = a_1(1) = 0.6}$



Approximation of Derivatives method – PROBLEMS

① Show that, the backward difference substitution $s = \frac{1-z^{-1}}{T}$ maps the left-half of s-plane into points inside the circle of radius 0.5, right-half of s-plane into points outside the circle of 0.5, and $j\Omega$-axis maps to the points into the circle of radius 0.5 and center $z = 1/2$.

Solution:– Given:– $\boxed{s = \frac{1-z^{-1}}{T}}$ — [1]

Re-arranging eq–[1] we get

$$s = \frac{1 - 1/z}{T} = \frac{z-1}{zT}$$

$\therefore szT = z-1$

$\therefore z(1-sT) = 1$; $\boxed{z = \frac{1}{1-sT}}$ — [2]

Put, $\boxed{s = \sigma + j\Omega}$ in eq–[2]

$$\boxed{z = \frac{1}{(1-\sigma T) - j\Omega T}} \text{ — [3]}$$

Rationalizing eq–[3]:–

$$\therefore z = \frac{(1-\sigma T) + j\Omega T}{[(1-\sigma T) + j\Omega T][(1-\sigma T) - j\Omega T]}$$

$$\therefore z = \frac{(1-\sigma T) + j\Omega T}{(1-\sigma T)^2 + \Omega^2 T^2}$$

$$\boxed{z = \frac{(1-\sigma T)}{(1-\sigma T)^2+\Omega^2T^2} + j\,\frac{\Omega T}{(1-\sigma T)^2+\Omega^2T^2}} \text{ — [4]}$$

$$\boxed{z = x + jy} \text{ — [5]}$$

$$\boxed{|z|^2 = x^2+y^2 = \frac{(1-\sigma T)^2+\Omega^2T^2}{[(1-\sigma T)^2+\Omega^2T^2]^2}} \text{ — [6]}$$

$$\boxed{x^2+y^2 = \frac{1}{[(1-\sigma T)^2+\Omega^2T^2]}} \text{ — [7]-a}$$

From eq-[4] and [5] $\boxed{x = \frac{1-\sigma T}{(1-\sigma T)^2+\Omega^2T^2}}$ — [7]-b

$$\therefore \boxed{x^2+y^2 = \frac{x}{(1-\sigma T)}} \text{ — [8]}$$

$$x^2 - \frac{x}{(1-\sigma T)} + y^2 = 0$$

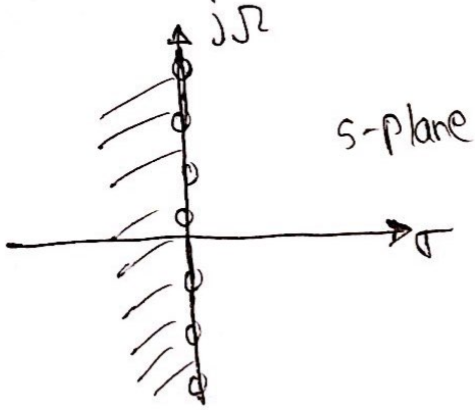
$$\boxed{\left[x - \frac{1}{2(1-\sigma T)}\right]^2 + y^2 = \left[\frac{1}{2(1-\sigma T)}\right]^2} \text{ — [9]}$$

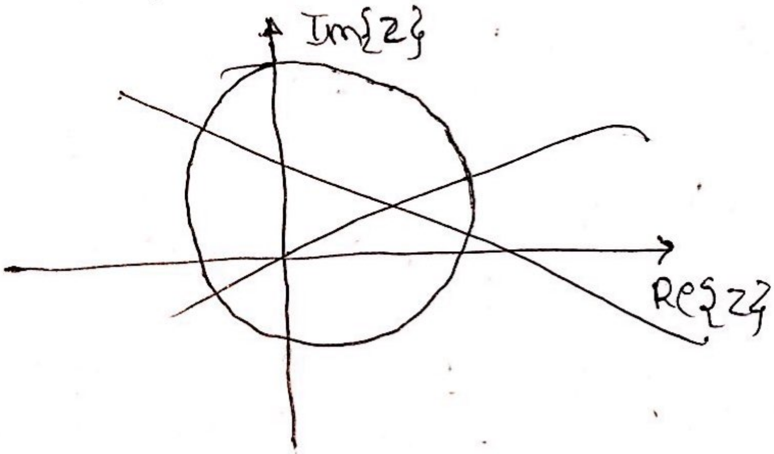
This represents a circle of radius $\left[\frac{1}{2(1-\sigma T)}\right]$ and center $z = \left[\frac{1}{2(1-\sigma T)}, 0\right]$.

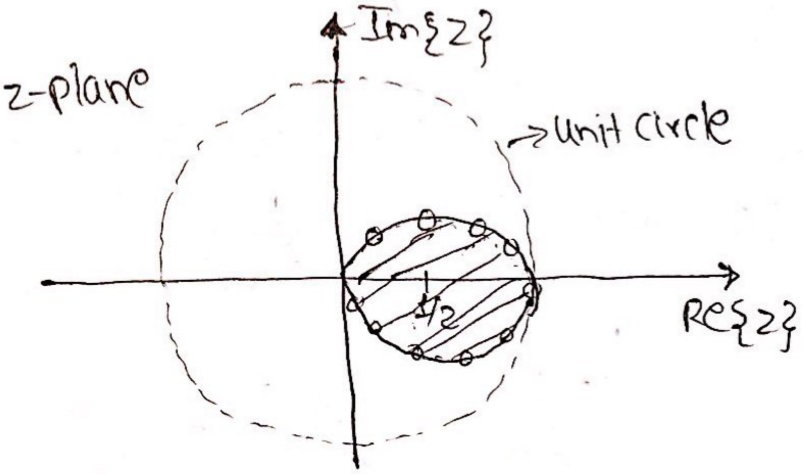
case-i] For $\sigma = 0$; $s = j\Omega$ substituting $\boxed{\sigma = 0}$ in eq-[9] we get:-

$$\boxed{\left[x - \frac{1}{2}\right]^2 + y^2 = \left[\frac{1}{2}\right]^2} \quad [10]$$

~~This~~ Thus, the $j\Omega$-axis in s-plane gets mapped to a circle of radius $(1/2)$ and center $z = (1/2, 0)$.







case-ii] For $\sigma < 0$; in eq-[9] i.e., Left-Half of s-plane maps to points inside the circle of radius 0.5 and center $(1/2, 0)$.

case-iii] For $\sigma > 0$; in eq-[9] i.e, Right-Half of s-plane maps to points outside the circle of radius 0.5 and center $(1/2, 0)$.

② Use backward difference for derivative to convert analog LPF with system function $H(S) = \frac{1}{S+2}$.

Solution:- Given:- $\boxed{H(S) = \frac{1}{S+2}}$ — [1]

Substituting the backward difference formula $S = \frac{1-z^{-1}}{T}$

$$\therefore H(z) = H(S)\Big|_{S=\frac{1-z^{-1}}{T}} = \frac{1}{\frac{1-z^{-1}}{T}+2}$$

$$\therefore \boxed{H(z) = \frac{T}{1-z^{-1}+2T}}$$ if $T = 1sec$.

$$\boxed{H(z) = \frac{1}{3-z^{-1}}}$$ — [2]

③ Use backward difference of derivative to convert the analog filter with TF $H(S) = \frac{1}{S^2+16}$

$$H(z) = H(S)\Big|_{S=\frac{1-z^{-1}}{T}} = \frac{1}{\left[\frac{1-z^{-1}}{T}\right]^2+16}$$

$$H(z) = \frac{T^2}{1-2z^{-1}+z^{-2}+16T^2}$$ if $T = 1sec$.

$$\boxed{H(z) = \frac{1}{z^{-2}-2z^{-1}+17}}$$

Note:- Any analog filter $H(S)$ Butterworth or Chebyshev type-1, can be converted into $H(z)$ by substituting $\boxed{S^i = \left[\frac{1-z^{-1}}{T}\right]^i}$

PROBLEMS ON IMPULSE INVARIANCE METHOD

① Given $H(S) = \frac{1}{S+1}$ Find $H(z)$ using Impulse Invariance method [IIM].

$\cancel{H(z) = H(S)}$ $\boxed{H(z) = \frac{1}{1-e^{-T}z^{-1}}}$ Taking $\boxed{T=1sec.}$

$\therefore \boxed{H(z) = \frac{1}{1-0.3678z^{-1}}}$

---

② $H(S) = \frac{2}{(S+1)(S+2)}$ find $H(z)$ using IIM assume, $T=1sec.$

$H(S) = \frac{A}{S+1} + \frac{B}{S+2} = 2$; $A(S+2) + B(S+1) = 2$

$S=-2$; $B(-1) = 2$ $\boxed{B=-2}$

$S=-1$; $A(1) = 2$ $\boxed{A=2}$

$\therefore \boxed{H(S) = \frac{2}{S+1} - \frac{2}{S+2}}$ $H(z) = \frac{2}{1-e^{-T}z^{-1}} - \frac{2}{1-e^{-2T}z^{-1}}$

$\therefore H(z) = \frac{2}{1-e^{-1}z^{-1}} - \frac{2}{1-e^{-2}z^{-1}} = \frac{2}{1-0.3678z^{-1}} - \frac{2}{1-0.1353z^{-1}}$

$\therefore \boxed{H(z) = \frac{0.465z^{-1}}{1-0.503z^{-1}+0.04976z^{-2}}}$

---

③ Given $H(S) = \frac{10}{S^2+7S+10}$ Using IIM find $H(z)$; take $T=0.25sec.$

$H(S) = \frac{10}{S(S+5)+2(S+5)} = \frac{10}{(S+2)(S+5)}$

$$H(s) = \frac{A}{s+2} + \frac{B}{s+5} = 10 \quad ; \quad H(s) = \frac{3.33}{s+2} - \frac{3.33}{s+5}$$

$$A(s+5) + B(s+2) = 10$$

$s=-5$; $B(-3) = 10$ $\boxed{B = -3.33}$

$s=-2$; $A(3) = 10$ $\boxed{A = 3.33}$

$$H(z) = \frac{3.33}{1-e^{-2T}z^{-1}} - \frac{3.33}{1-e^{-5T}z^{-1}}$$

$$H(z) = \frac{3.33}{1-e^{-0.4}z^{-1}} - \frac{3.33}{1-e^{-1}z^{-1}} = \frac{3.33}{1-0.6703z^{-1}} - \frac{3.33}{1-0.3678z^{-1}}$$

$$H(z) = \frac{3.33(1-0.3678z^{-1}) - 3.33(1-0.6703z^{-1})}{(1-0.6703z^{-1})(1-0.3678z^{-1})}$$

$$\therefore H(z) = \frac{3.33 - 1.2247z^{-1} - 3.33 + 2.232z^{-1}}{1 - 1.0381z^{-1} + 0.2465z^{-2}}$$

$$\therefore \boxed{H(z) = \frac{1.0073z^{-1}}{1 - 1.0381z^{-1} + 0.2465z^{-2}}}$$

---

(4) Apply IIM to $H(s) = \frac{(s+a)}{(s+a)^2+b^2}$

$$\boxed{h(t) = e^{-at}\cos(bt)\,u(t)} \quad [1]$$

$$\boxed{h(nT) = e^{-anT}\cos(bnT)\,u(nT)} \quad [2]$$

$$H(z) = \sum_{n=0}^{\infty} h(nT)\,z^{-n} = \sum_{n=0}^{\infty} \left[e^{-aTn}\left(\frac{e^{jbnT}+e^{-jbnT}}{2}\right)z^{-n}\right]$$

$$\therefore H(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left[e^{-(a-jb)T}z^{-1}\right]^n + \frac{1}{2}\sum_{n=0}^{\infty}\left[e^{-(a+jb)T}z^{-1}\right]^n$$

$$H(z) = \frac{1}{2}\left[\frac{1}{1-e^{-(a-jb)T}z^{-1}} + \frac{1}{1-e^{-(a+jb)T}z^{-1}}\right]$$

$$\therefore H(z) = \frac{1}{2}\left[\frac{1-e^{-aT}e^{-jbT}z^{-1} + 1-e^{-aT}e^{jbT}z^{-1}}{1-e^{-aT}e^{-jbT}z^{-1}-e^{-aT}e^{jbT}z^{-1}+e^{-2aT}z^{-2}}\right]$$

$$\therefore H(z) = \frac{1}{2}\left[\frac{2-(e^{-aT}\cos(bT)z^{-1})2}{1-e^{-aT}2\cos(bT)z^{-1}+e^{-2aT}z^{-2}}\right]$$

$$\therefore \boxed{H(z) = \frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}}$$

---

(5) $H(s) = \dfrac{b}{(s+a)^2+b^2}$ Apply IIM.

$\boxed{h(t) = e^{-at}\sin(bt)\,u(t)}$ $\boxed{h(nT) = e^{-anT}\sin(bnT)\,u(nT)}$

$$\therefore H(z) = \sum_{n=0}^{\infty} e^{-anT}\sin(bnT)\,z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-anT}\left[\frac{e^{jbnT}-e^{-jbnT}}{2j}\right]z^{-n}$$

$$\therefore H(z) = \frac{1}{2j}\sum_{n=0}^{\infty}\left[e^{-(a-jb)T}z^{-1}\right]^n - \frac{1}{2j}\sum_{n=0}^{\infty}\left[e^{-(a+jb)T}z^{-1}\right]^n$$

$$\therefore H(z) = \frac{1}{2j}\left[\frac{1}{1-e^{-(a-jb)T}z^{-1}} - \frac{1}{1-e^{-(a+jb)T}z^{-1}}\right]$$

$$H(z)=\frac{1}{2j}\left[\frac{1-e^{-aT}e^{-jbT}z^{-1}-1+e^{-aT}e^{jbT}z^{-1}}{(1-e^{-aT}e^{jbT}z^{-1})(1-e^{-aT}e^{-jbT}z^{-1})}\right]$$

$$\therefore H(z)=\frac{1}{2j}\left[\frac{e^{-aT}z^{-1}[e^{jbT}-e^{-jbT}]}{1-e^{-aT}e^{-jbT}z^{-1}-e^{-aT}e^{jbT}z^{-1}+e^{-2aT}z^{-2}}\right]$$

$$\therefore \boxed{H(z)=\frac{e^{-aT}z^{-1}\sin(bT)}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-a2T}z^{-2}}}$$

---

(6) $H(s)=\dfrac{s+2}{(s+1)(s+3)}=\dfrac{A}{s+1}+\dfrac{B}{s+3}=s+2$

$A(s+3)+B(s+1)=s+2$

$s=-3;\ B(-2)=-1\ \boxed{B=1/2}\quad s=-1;\ \boxed{A=1/2}$

$\therefore H(s)=\dfrac{1/2}{s+1}+\dfrac{1/2}{s+3}\qquad \boxed{h(t)=\frac{1}{2}[e^{-t}+e^{-3t}]u(t)}$

$$\boxed{h(nT)=\frac{1}{2}[e^{-nT}+e^{-3nT}]u(nT)}$$

$$H(z)=\sum_{n=0}^{\infty}h(nT)z^{-n}=\sum_{n=0}^{\infty}\frac{1}{2}[e^{-nT}z^{-n}+e^{-3nT}z^{-n}]$$

$$H(z)=\frac{1}{2}\sum_{n=0}^{\infty}[e^{-T}z^{-1}]^{n}+\frac{1}{2}\sum_{n=0}^{\infty}[e^{-3T}z^{-1}]^{n}$$

$$\therefore H(z)=\frac{1}{2}\,\frac{1}{1-e^{-T}z^{-1}}+\frac{1}{2}\,\frac{1}{1-e^{-3T}z^{-1}}$$

$$H(z)=\frac{1}{2}\left[\frac{1-e^{-3T}z^{-1}+1-e^{-T}z^{-1}}{(1-e^{-T}z^{-1})(1-e^{-3T}z^{-1})}\right]=\frac{1}{2}\left[\frac{2-z^{-1}(e^{-3T}+e^{-T})}{1-e^{-3T}z^{-1}-e^{-T}z^{-1}+e^{-4T}z^{-2}}\right]$$

$$=\frac{1}{2}\left[\frac{2-e^{-2T}z^{-1}(e^{-T}+e^{T})}{1-z^{-1}e^{-2T}(e^{-T}+e^{T})+e^{-4T}z^{-2}}\right]=\boxed{\frac{1-e^{-2T}z^{-1}\cosh T}{1-z^{-1}e^{-2T}2\cosh T+e^{-4T}z^{-2}}}$$

(4)

(7) Given $H(S) = \frac{S+0.2}{(S+0.2)^2+9}$ assume $T=1S$ find $H(z)$ using IIM.

<u>using the result of problem-[4]:-</u>

$\boxed{a=0.2}$ $\boxed{b=3}$

$$\therefore H(z) = \frac{1-e^{-aT}\cos bT\, z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$\therefore H(z) = \frac{1-e^{-0.2}\cos(3)z^{-1}}{1-2e^{-0.2}\cos(3)z^{-1}+e^{-0.4}z^{-2}}$$

$$\therefore H(z) = \frac{1-(0.8187)(-0.99)z^{-1}}{1+1.621z^{-1}+0.6703z^{-2}} = \boxed{\frac{1+0.8105z^{-1}}{1+1.621z^{-1}+0.6703z^{-2}}}$$

(8) Design a 3rd order Butterworth digital filter using IIM technique, assuming sampling period $T=1sec$.

<u>Solution:-</u> For $\boxed{N=3}$ the TF of a normalized Butterworth filter is given by:-

$$\boxed{H(S) = \frac{1}{(S+1)(S^2+S+1)}} \quad -[1]$$

$$H(S) = \frac{A}{S+1} + \frac{BS+C}{S^2+S+1} = 1$$

$\cancel{H(S)=}$ $A(S^2+S+1)+(BS+C)(S+1)=1$

$S=-1;$ $\boxed{A=1}$

|||ly $\boxed{C=0}$ $\boxed{B=-1}$

$$H(s)=\frac{1}{s+1}-\frac{s}{s^2+s+1}=\frac{1}{s+1}-\left[\frac{s+0.5-0.5}{s^2+s+1}\right]$$

$$\therefore H(s)=\frac{1}{s+1}-\frac{(s+0.5)}{(s+0.5)^2+(0.866)^2}+\left[\frac{0.5\times0.866}{(s+0.5)^2+(0.866)^2}\right]\frac{1}{0.866}$$

$$\therefore H(s)=\frac{1}{s+1}-\frac{(s+0.5)}{(s+0.5)^2+(0.866)^2}+0.577\left[\frac{0.866}{(s+0.5)^2+(0.866)^2}\right]$$

$$\therefore H(z)=\frac{1}{1-e^{-T}z^{-1}}-\left[\frac{1-e^{-0.5T}\cos(0.866T)z^{-1}}{1-2e^{-0.5T}\cos(0.866T)z^{-1}+e^{-0.5T}z^{-2}}\right]$$

$$+\;0.577\left[\frac{e^{-0.5T}z^{-1}\sin(0.866T)}{1-2e^{-0.5T}\cos(0.866T)z^{-1}+e^{-0.5T}z^{-2}}\right]$$

For T=1sec. and simplifying

$$\therefore H(z)=\frac{1}{1-e^{-1}z^{-1}}-\left[\frac{1-(0.6065)(0.6478)z^{-1}}{1-0.7858z^{-1}+0.3678z^{-2}}\right]$$

$$+\;0.577\left[\frac{0.7617\times0.6065z^{-1}}{1-0.7858z^{-1}+0.3678z^{-2}}\right]$$

$$\therefore \boxed{H(z)=\frac{1}{1-0.3186z^{-1}}+\frac{(-1+0.6593z^{-1})}{(1-0.7858z^{-1}+0.3678z^{-2})}}$$

9) Design a Butterworth LPF using IIM technique for the specifications:-

$$0.8 \leq |H(e^{j\omega})| \leq 1 \; ; \; 0 \leq \omega \leq 0.2\pi$$

$$|H(e^{j\omega})| \leq 0.2 \; ; \; 0.6\pi \leq \omega \leq \pi .$$

$\boxed{\omega_p = 0.2\pi \text{ rad}}$ $\boxed{\omega_s = 0.6\pi \text{ rad}}$ $K_2 = 20\log_{10}\delta_s$

$\boxed{K_2 = -14 dB}$

$K_1 = 20\log_{10}(1-\delta_p)$

$\boxed{K_1 = 20\log_{10}(0.8) = -1.9 dB = -2 dB}$

For Normalized LPF:- $\boxed{\omega_p = 1 \text{ rad}}$, $\boxed{\omega_{r} = \frac{\omega_s}{\omega_p} = 3 \text{ rad}}$

$$\therefore N \geq \frac{\log\left[\left(10^{-K_1/10}-1\right)/\left(10^{-K_2/10}-1\right)\right]}{2\log_{10}(\omega_r/\omega_p)}$$

$$\therefore N \geq \frac{\log_{10}\left[\left(10^{0.2}-1\right)/\left(10^{1.4}-1\right)\right]}{2\log_{10}(3/1)} = \frac{\log_{10}\left[(0.5849/24.118)\right]}{\cancel{0.9542}\ 0.9542}$$

$$\cancel{\therefore N \geq \frac{\log_{10}[0.0242]}{0.9542} =}$$

$$N \geq \frac{-1.6152}{-0.9542} = 1.692 \quad \boxed{N \simeq 2}$$

Cut-off freq:- (Norm LPF) $\Omega_{c} = \frac{1}{\left[10^{-K_1/10}-1\right]^{1/2N}} = \frac{1}{0.8745} = \boxed{1.143 \text{ rad}}$

From Butterworth tables:-

$$\boxed{H_N(s) = H_2(s) = \frac{1}{s^2 + \sqrt{2}s + 1}}$$

$$\therefore H_{LP}(s) = H_2(s)\Big|_{s \to \frac{s}{(\omega p)\Omega_{cn}}} = \frac{s}{(0.2\pi)(1.143)} = \frac{s}{0.23\pi}$$

$$\therefore \boxed{H_{LP}(s) = \frac{0.5266}{s^2 + 1.03s + 0.5266}}$$

$$\therefore \underline{\underline{H_{LP}(s)}} = \frac{(0.5266)(0.5112)}{\left[(s+0.515)^2 + (0.5112)^2\right] 0.5112}$$

$$\therefore H_{LP}(s) = 1.03\left[\frac{0.5112}{(s+0.515)^2 + (0.5112)^2}\right] \quad \boxed{T = 1s}$$

$$\therefore H_{LP}(z) = 1.03\left[\frac{e^{-0.515} z^{-1} \sin(0.5112)}{1 - 2e^{-(0.515)}\cos(0.5112) z^{-1} + e^{-2(0.515)} z^{-2}}\right]$$

$$\therefore H_{LP}(z) = 1.03\left[\frac{(0.4892)(0.5975) z^{-1}}{1 - 1.0422 z^{-1} + 0.36 z^{-2}}\right]$$

$$\therefore \boxed{H_{LP}(z) = \frac{0.30106 z^{-1}}{1 - 1.0422 z^{-1} + 0.36 z^{-2}}}$$

PROBLEMS ON MATCHED Z-TRANSFORM

① Given $H(S) = \frac{S+2}{(S+1)(S+3)}$ Find $H(z)$ using matched Z-Transform (MZT).

The system uses a sampling frequency of $F_S = 10\,Hz\,(T = 0.1\,sec)$

W.K.T.:- $S + S_i \Rightarrow 1 - e^{-S_i T} z^{-1}$

$\Rightarrow \quad S+1 \rightarrow 1 - e^{-T} z^{-1} = \boxed{1 - 0.905 z^{-1}}$

$\Rightarrow \quad S+2 \rightarrow 1 - e^{-2T} z^{-1} = \boxed{1 - 0.819 z^{-1}}$

$S+3 \rightarrow 1 - e^{-3T} z^{-1} = \boxed{1 - 0.741 z^{-1}}$

$$\therefore \boxed{H(z) = \frac{1 - 0.819 z^{-1}}{(1 - 0.905 z^{-1})(1 - 0.741 z^{-1})}}$$

Observations:- a] The stable poles of $H(S)$ at $s = -1$ and $s = -3$ becomes stable poles of $H(z)$ at $z = e^{-T} = 0.905$ and $z = e^{-3T} = 0.741$.

b] It is common practice to have $|H(j\Omega)|\,|_{\Omega=0}$ and $|H(\omega)|_{\omega=0}$ to have unit values. In this case,

$$|H(j\Omega)|\,|_{\Omega=0} = H(S)|_{S=0} = \frac{2}{1 \times 3} = \boxed{0.667}$$

and $$|H(\omega)|_{\omega=0} = H(z)|_{z=1} = \frac{(1 - 0.819)}{(1 - 0.915)(1 - 0.741)} = \boxed{7.3562}$$

Hence, the analog frequency plot should be scaled by $\frac{1}{0.667} = 1.515$ and digital frequency plot should be scaled by $\frac{1}{7.3562} = 0.1359$ to make them equal at zero frequency.

(2) $H(s) = \dfrac{4s(s+1)}{(s+2)(s+3)}$ Find $H(z)$ using MZT

with $\boxed{F_s = 4Hz}$

$\therefore \boxed{T = 0.25 sec.}$

$$\therefore H(z) = \frac{4(1-z^{-1})(1-e^{-T}z^{-1})}{(1-e^{-2T}z^{-1})(1-e^{-3T}z^{-1})}$$

$$\therefore \boxed{H(z) = \frac{4(1-z^{-1})(1-0.7788z^{-1})}{(1-0.605z^{-1})(1-0.4723z^{-1})}}$$

PROBLEMS ON Bi-Linear Transformation

(1) Design a Low-pass Chebyshev Type-1 filter for the following specifications:- Passband edge frequency $= f_p = 1KHz$

Stopband edge frequency $= f_s = 3KHz$

Sampling frequency $= F_s = 10KHz$

Passband ripple $= k_1 = 1dB$

Stopband ripple $= k_2 = -40dB$

Solution:- The normalized band edge frequencies (digital) are given by:-

$$\omega_p = \frac{2\pi f_p}{F_s} = \frac{2\pi(1K)}{(10K)} = \boxed{0.2\pi \, rad} \qquad \omega_s = \frac{2\pi f_s}{(F_s)} = \frac{2\pi(3K)}{(10K)} = \boxed{0.6\pi} \, rad$$

Pre-warped frequencies:- For simplicity, $\boxed{T = 2sec}$

$$\Omega_p' = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = \boxed{0.32492 \, rad/sec}$$

$$\Omega_s' = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = \boxed{1.3764 \, rad/sec}$$

For normalized prototype LPF:- $\boxed{\Omega_p = 1 \, rad/sec}$

$$\Omega_r = \frac{\Omega_s'}{\Omega_p'} = \frac{1.3764}{0.32492} = \boxed{4.236 \, rad/sec}$$

$$\text{Filter order} = N \geq \frac{\cosh^{-1}\left[(10^{-k_2/10}-1)/(10^{-k_1/10}-1)\right]^{1/2}}{\cosh^{-1}\left[\Omega_r/\Omega_p\right]}$$

$$\therefore N \geq \frac{\cosh^{-1}\left[(10^{4}-1)/(10^{0.1}-1)\right]^{1/2}}{\cosh^{-1}(4.236)} = \frac{5.973}{2.122} = \boxed{2.81}$$

$\boxed{N \simeq 3}$ rd order

From the normalized Chebyshev Type-1 tables for $\boxed{N=3}$ odd; $K_1 = 1dB$ ripple

$$\therefore \boxed{H_N(s) = \frac{K_0}{s^3+0.988s^2+1.238s+0.49131}}$$ here, $\boxed{K_0 = b_0 = 0.49131}$

$$\therefore \boxed{H_N(s) = \frac{0.49131}{s^3+0.988s^2+1.238s+0.49131}} \quad [1]$$

The TF corresponding to $\Omega_p' = 0.32492$ rad/sec, is got by

$$\therefore H_{LP}(s) = H_N(s)\Big|_{s \to \frac{s}{\Omega_p'} = \frac{s}{0.32492}}$$

$$\therefore \boxed{H_{LP}(s) = \frac{0.01689}{s^3+0.32099s^2+0.13068s+0.016849}} \quad [2]$$

The digital TF $H_{LP}(z)$ is given by:- BLT substitution

$$s \to \frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})} = \frac{2}{T}\frac{(z-1)}{(z+1)} = \frac{(z-1)}{(z+1)}$$ as $\boxed{T = 2sec.}$ for simplicity.

$$\therefore \boxed{H_{LP}(z) = H_{LP}(s)\Big|_{s=\frac{z-1}{z+1}} = \frac{0.011474z^3+0.034421z^2+0.034421z+0.0011474}{z^3-2.178z^3+1.7698z^2+0.53976}}$$

---

② Design a Chebyshev Type-1 HPF (digital using BLT) for the following specifications:-

Passband edge freq. $= f_p = 700Hz$

Stopband edge freq. $= f_s = 500Hz$

Sampling freq. $= F_s = 2kHz$

Passband ripple $= K_1 = -1dB$

Stopband attn. $= K_2 = -40dB$

<u>Solution</u>:- Normalized angular band-edge frequencies are:-

$\underline{\omega_p} = \frac{2\pi f_p}{F_s} = \frac{2\pi(700)}{2000} = \boxed{0.7\pi \text{ rad}}$ $\quad$ $\underline{\omega_s} = \frac{2\pi(500)}{2000} = \boxed{0.5\pi \text{ rad}}$

Pre-warped frequencies:- $\Omega_p' = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = \tan\left(\frac{0.7\pi}{2}\right)$

$\boxed{T = 2\text{ sec.}}$ assumed $\quad \boxed{\Omega_p' = 1.9626 \text{ rad/sec}}$

$\underline{\Omega_s'} = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = \boxed{1 \text{ rad/sec.}}$

For the prototype analog LPF:- $\boxed{\Omega_p = 1 \text{ rad/sec.}}$

$$\boxed{\Omega_r = \frac{\Omega_p'}{\Omega_s'} = 1.9626 \text{ rad/sec.}}$$

Filter order:- $N \geq \frac{\cosh^{-1}\left[(10^{-k_2/10}-1)/(10^{-k_1/10}-1)\right]^{1/2}}{\cosh^{-1}\left[\Omega_r/\Omega_p\right]}$

$$\therefore N \geq \frac{\cosh^{-1}\left[(10^{4}-1)/(10^{0.1}-1)\right]^{1/2}}{\cosh^{-1}[1.9626]} = \frac{\cosh^{-1}(\ ) \ 5.973}{1.295} = 4.612$$

$\boxed{N \cong 5}$ odd $\quad \boxed{\epsilon = [10^{-k_1/10}-1]^{1/2} = 0.5088}$

$\boxed{\sinh^{-1}\left|\frac{1}{\epsilon}\right| = 1.428} = \underline{\theta}$ $\quad \boxed{a = \sinh\theta = 1.9652}\boxed{b = \cosh\theta = 2.205}$

Poles:- $\boxed{s_k = -a \cancel{\cos} \sin\left[\frac{(2k-1)\pi}{2N}\right] + jb\cos\left[\frac{(2k-1)\pi}{2N}\right]}$

$\cancel{a = \sinh\theta} = -1.9652\sin\left[\frac{(2k-1)\pi}{2N}\right] + j2.205\cos\left[\frac{(2k-1)\pi}{2N}\right]$

For $k = 1, 2, 3, \ldots 5$

$\therefore \boxed{s_1, s_5 = -0.08946 \pm j0.99014}\boxed{s_{2,4} = -0.23421 \pm j0.61194}$

$\boxed{s_3 = -0.2895}$

$$H_5(S) = \frac{K_0}{(S+0.2895)(S+0.08946+j0.99014)(S+0.08946-j0.99014)(S+0.23421+j0.61194)(S+0.23421-j0.61194)}$$

$$\therefore H_5(S) = \frac{K_0}{S^5+0.93682S^4+1.6888S^3+0.9744S^2+0.58053S+0.12283}$$

$\boxed{K_0 = 0.12283}$ as 'N' odd.

$$\therefore \boxed{H_5(S) = \frac{0.12283}{S^5+0.93682S^4+1.6888S^3+0.9744S^2+0.58053S+0.12283}}$$

Analog ~~Digital~~ TF $H_{HP}(S)$ is given by:- $S \rightarrow \frac{\Omega_P'}{S} = \frac{1.9626}{S}$

$$\therefore \boxed{H_{HP}(S) = \frac{0.12283\,S^5}{S^5+9.2762S^4+30.557S^3+103.94S^2+113.16S+237.67}}$$

Digital TF $H_{HP}(z)$ is given by:-

$$H_{HP}(z) = H_{HP}(S)\Big|_{S \rightarrow \frac{(z-1)}{(z+1)}}$$

$$\therefore \boxed{H_{HP}(z) = \frac{0.00202z^5-0.01012z^4+0.02022z^3-0.02022z^2+0.01012-0.0202}{z^5+3.1624z^4+4.7607z^3+4.0528z^2+1.9344z+0.41529}}$$

(3) Using BLT, design a digital Chebyshev Type-1 filter with following specifications:-

Lower passband edge freq. = $f_l = 200Hz$

Upper passband edge freq. = $f_u = 400Hz$

Lower stopband edge freq. = $f_1 = 100Hz$

Upper stopband edge freq. = $f_2 = 500Hz$

Passband ripple = $K_1 = -1dB$

Stopband attn. = $K_2 = -10dB$

Sampling freq. = $F_s = 2KHz$

Solution:- $\omega_l = \frac{2\pi(200)}{2000} = \boxed{0.2\pi\ rad}$ $\quad \omega_u = \frac{2\pi f_u}{F_s} = \frac{2\pi(400)}{2000} = \boxed{0.4\pi\ rad}$

$\omega_1 = \frac{2\pi f_1}{2000} = \frac{2\pi(100)}{2000} = \boxed{0.1\pi\ rad}$ $\quad \omega_2 = \frac{2\pi(f_2)}{F_s} = \frac{2\pi(500)}{2000} = \boxed{0.5\pi\ rad}$

Pre-warped frequencies:- $\Omega_l' = \frac{2}{T}\tan\left(\frac{\omega_l}{2}\right) = \boxed{0.325\ rad/sec}$

$\boxed{T = 2sec.}$ assumed

$\Omega_u' = \frac{2}{T}\tan\left(\frac{\omega_u}{2}\right) = \boxed{0.7265\ rad/sec.}$

$\Omega_1' = \frac{2}{T}\tan\left(\frac{\omega_1}{2}\right) = \boxed{0.1584\ rad/sec.}$

$\Omega_2' = \frac{2}{T}\tan\left(\frac{\omega_2}{2}\right) = \boxed{1\ rad/sec.}$

For normalized LPF:- $\boxed{\Omega_p = 1\ rad/sec.}$

$\Omega_r = \min\{|A|, |B|\}$

$$A = \frac{-\Omega_1'^2 + \Omega_l'\Omega_u'}{\Omega_1'(\Omega_u' - \Omega_l')} = \frac{-0.0251 + 0.2361}{0.1584 \times 0.4012} = \frac{0.211}{0.0635} = \boxed{3.3202\ rad/sec.}$$

$$B = \frac{+\Omega_2'^2 - \Omega_l'\Omega_u'}{\Omega_2'(\Omega_u' - \Omega_l')} = \frac{1 - 0.2361}{1 \times 0.4012} = \boxed{1.9025\ rad/sec}$$

Filter order:- $N \geq \dfrac{\cosh^{-1}\left[(10^{-K_2/10}-1)/(10^{-K_1/10}-1)\right]^{1/2}}{\cosh^{-1}\left[\Omega_{s1}/\Omega_p\right]}$

$N \geq \dfrac{\cosh^{-1}\left[\overset{5.895}{\cancel{34.759}}\right]}{\cosh^{-1}\left[1.90258\right]} = \cancel{\dfrac{6.241}{1.2587} = 3.36}\ \cancel{\boxed{N \simeq 4}}$

$N \geq \dfrac{2.46}{1.2587} = 1.95 \qquad \boxed{N \simeq 2}$

From the tables $K_1 = -1dB$, $\boxed{N=2}$ (even) $\boxed{\epsilon = 0.5088}$

$\therefore \boxed{H_2(s) = \dfrac{K_0}{s^2 + 1.0977s + 1.1025}} \qquad K_0 = \dfrac{b_0}{[1+\epsilon^2]^{1/2}} = \dfrac{1.1025}{[1+0.2588]^{1/2}}$

$\therefore \boxed{K_0 = 0.9826}$

$\therefore \boxed{H_2(s) = \dfrac{0.9826}{s^2 + 1.0977s + 1.1025}}$

$\therefore H_{BP}(s) = H_2(s)\Big|_{s \to \frac{s^2 + \Omega_l' \Omega_u'}{s(\Omega_u' - \Omega_l')}} = \frac{s^2 + 0.2361}{s(0.4017)}$

$\therefore \boxed{H_{BP}(s) = \dfrac{0.1584 s^2}{s^4 + 0.4407s^3 + 0.6497s^2 + 0.104s + 0.0557}}$

Digital TF $H_{BP}(z)$ is obtained by:-

$H_{BP}(z) = H_{BP}(s)\Big|_{s \to \frac{(z-1)}{(z+1)}}$

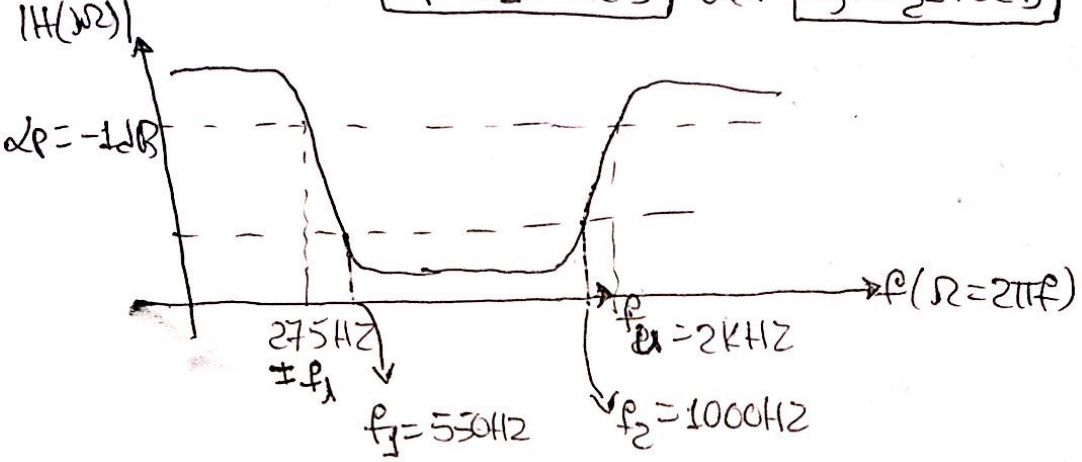
$\therefore \boxed{H_{BP}(z) = \dfrac{0.0704z^4 - 0.1408z^2 + 0.0704}{z^4 - 1.9779z^3 + 2.2375z^2 - 1.3793z + 0.5158}}$

(0.5158)

(4) Design a Chebyshev type-1 Band-Reject filter with following specifications:- Passband:- DC to 275Hz and 2KHz to ∞.

Stopband:- 550Hz to 1000Hz

$\boxed{\alpha_p = k_1 = -1dB}$ and $\boxed{\alpha_s = k_2 = 15dB}$ $\boxed{F_s = 8KHz}$



$$\underline{\omega_l} = \frac{2\pi f_l}{F_s} = \frac{2\pi(275)}{8000} = \boxed{0.06875\pi \text{ rad}}$$

$$\omega_u = \frac{2\pi(f_u)}{F_s} = \frac{2\pi(2KHz)}{8K}$$

$$\boxed{\omega_u = 0.5\pi \text{ rad}}$$

$$\omega_1 = \frac{2\pi f_1}{F_s} = \frac{2\pi(550)}{8000} = \boxed{0.1375\pi \text{ rad}}$$

$$\underline{\omega_2} = \frac{2\pi f_2}{F_s} = \frac{2\pi(1000)}{8000} = \boxed{0.25\pi \text{ rad}}$$

Pre-warped frequencies:-

$$\underline{\Omega_l'} = \frac{2}{T}\tan\left(\frac{\omega_l}{2}\right) = \tan\left(\frac{0.06875\pi}{2}\right) = \boxed{0.1084 \text{ rad/sec}}$$

$\boxed{T = 2sec.}$ $$\underline{\Omega_u'} = \frac{2}{T}\tan\left(\frac{\omega_u}{2}\right) = \tan\left(\frac{0.5\pi}{2}\right) = \boxed{1 \text{ rad/sec.}}$$

$$\underline{\Omega_1'} = \frac{2}{T}\tan\left(\frac{\omega_1}{2}\right) = \tan\left(\frac{0.1375\pi}{2}\right) = \boxed{0.2194 \text{ rad/sec.}}$$

$$\underline{\Omega_2'} = \frac{2}{T}\tan\left(\frac{\omega_2}{2}\right) = \tan\left(\frac{0.25\pi}{2}\right) = \boxed{0.4142 \text{ rad/sec.}}$$

For prototype normalized LPF:- $\boxed{\Omega_p = 1 \text{ rad/sec.}}$

$$\boxed{\Omega_s = \min\{|A|, |B|\}}$$

$$\underline{\underline{A}} = \frac{\Omega_1(\Omega_u - \Omega_l')}{-\Omega_1'^2 + \Omega_l'\Omega_u'} = \frac{0.2194(0.8916)}{-0.0481 + 0.1084} = \boxed{3.244 \text{ rad/sec}}$$

$$\underline{\underline{B}} = \frac{\Omega_2'(\Omega_u' - \Omega_l')}{-\Omega_2'^2 + \Omega_l'\Omega_u'} = \frac{0.4142 \times 0.8916}{-0.1715 + 0.1084} = \frac{0.3693}{-0.0631} = -5.85 \text{ rad/sec}$$

$$\boxed{|B| = 5.85 \text{ rad/sec}}$$

$$\boxed{\Omega_r = 3.246 \text{ rad/sec}}$$

$$N \geq \frac{\cosh^{-1}\left[\left(10^{-k_2/10} - 1\right) / \left(10^{-k_1/10} - 1\right)\right]^{1/2}}{\cosh^{-1}(\Omega_r/1)}$$

$$\therefore N \geq \frac{\cosh^{-1}[30.622/0.2589]^{1/2}}{\cosh^{-1}[3.246]} = \cancel{\frac{5.466}{1.845}} - \cancel{2.96}\ \frac{3.077}{1.845} = 1.668$$

$$\boxed{N \simeq 2} \qquad H_N(s) = H_2(s) = \frac{K_0}{s^2 + 1.0975 + 1.102}$$

$$\boxed{K_0 = \frac{b_0}{[1+\epsilon^2]^{1/2}}} \quad \boxed{\epsilon = [10^{-k_1/10} - 1]^{1/2} = 0.5088} \quad \boxed{K_0 = \frac{1.102}{1.1219} = 0.982}$$

$$\therefore \boxed{H_2(s) = \frac{0.982}{s^2 + 1.0975 + 1.102}}$$

$$H_{BS}(s) = H_2(s)\Big|_{s \to \frac{s(\Omega_u' - \Omega_l')}{s^2 + \Omega_l'\Omega_u'}} = \frac{s(0.8916)}{s^2 + 0.1084}$$

$$\therefore \boxed{H_{BS}(s) = \frac{0.8915(s^4 + 0.2168s^2 + 0.0117)}{s^4 + 0.8878s^3 + 0.9328s^2 + 0.09618s + 0.0117}}$$

The digital TF $H_{BS}(z)$ is given by:-

$$H_{BS}(z) = H_{BS}(s)\Big|_{s = \frac{z-1}{z+1}}$$

Scanned with CamScanner

$$\boxed{H_{BS}(z) = \frac{0.3732\left[z^4 - 3.2176z^3 + 4.588z^2 - 3.2176z + 1\right]}{z^4 - 1.8869z^3 + 1.429z^2 - 0.8077z + 0.3292}}$$

Butterworth Filter Problems

(5) Design a Butterworth IIR digital HPF for the following specifications:-

Passband edge frequency = $f_p = 40\,Hz$

Stopband edge frequency = $f_s = 25\,Hz$

Sampling freq. = $F_s = 100\,Hz$

Passband ripple = $K_1 = -1\,dB$

Stopband ripple = $K_2 = -20\,dB$

Solution:- The angular band-edge frequencies are:-

$$\omega_p = \frac{2\pi(f_p)}{F_s} = \frac{2\pi(40)}{100} = \boxed{0.8\pi \text{ rad}} \qquad \omega_s = \frac{2\pi(f_s)}{F_s} = \frac{2\pi(25)}{100}$$

$$\boxed{\omega_s = 0.5\pi \text{ rad}}$$

Pre-warped frequencies:- $\boxed{T = 2\,sec.}$

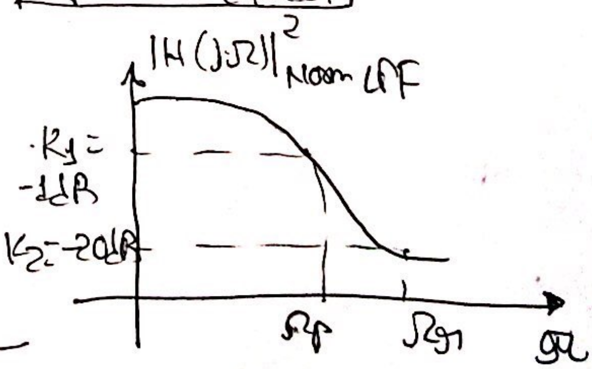
$$\Omega_p' = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = \tan\left(\frac{0.8\pi}{2}\right)$$

$$\boxed{\Omega_p' = 3.077 \text{ rad/sec}}$$

$$\Omega_s' = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = \boxed{1 \text{ rad/sec.}}$$

For prototype Normalized LPF:- $\boxed{\Omega_p = 1 \text{ rad/sec.}}$

$$\boxed{\Omega_r = \frac{\Omega_p'}{\Omega_s'} = 3.077 \text{ rad/sec}}$$



Filter order:-

$$N \geq \frac{\log_{10}\left[\left(10^{-K_1/10} - 1\right)/\left(10^{-K_2/10} - 1\right)\right]}{2\log_{10}\left(\cancel{\Omega_p'/\Omega_s'}\right)\left(\Omega_p/\Omega_r\right)}$$

$$N \geq \frac{\log_{10}\left[0.2589/99\right]}{2\log_{10}\left[1/3.077\right]} = \frac{-2.5825}{-0.976} = 2.645$$

$$\boxed{N \cong 3}$$

$$\boxed{H_3(s) = H_N(s) = \frac{1}{s^3+2s^2+2s+1}}$$ From Butterworth table

$\cancel{H_{HP}(s) = H_3(s)\big|_{s\to}}$

Cut-off frequency of Norm. LPF :- $\Omega_{cn} = \frac{\Omega_p}{\left[10^{-k_1/10}-1\right]^{1/2N}}$

$$\therefore \boxed{\Omega_{cn} = \frac{1}{\left[10^{0.1}-1\right]^{1/6}} = 1.2525 \text{ rad/sec}}$$

$$H_{HP}(s) = H_3(s)\Big|_{s \to \frac{\Omega_p'}{\Omega_{cn} s} = \frac{3.077}{1.25255} = \frac{2.456}{s}}$$

$$\therefore H_{HP}(s) = \frac{1}{\left[\frac{2.456}{s}\right]^3 + 2\left[\frac{2.456}{s}\right]^2 + 2\left[\frac{2.456}{s}\right] + 1}$$

$$\therefore \boxed{H_{HP}(s) = \frac{s^3}{s^3+4.912s^2+12.063s+14.814}}$$

$$H_{HP}(z) = H_{HP}(s)\Big|_{s \to \frac{2}{T}\frac{(z-1)}{(z+1)} = \frac{(z-1)}{(z+1)}} \text{ as } T = 2\text{sec.}$$

$$\therefore \boxed{H_{HP}(z) = \frac{0.034z^3 - 0.0914z^2 + 0.0914z - 0.034}{z^3+1.482z^2+0.929z+0.2032}}$$

(6) Design a LP-Butterworth digital filter using BLT for the following specifications:- Passband edge freq. $= f_p = 1KHZ$

Stopband edge freq. $= f_s = 3KHZ$

Passband ripple $= -2dB = K_1$

Stopband atth. $= K_2 = -20dB$

Sampling freq. $= F_s = 8KHZ$

Solution:- The angular band-edge frequencies are:-

$$\underline{\omega_p} = \frac{2\pi(1000)}{(8000)} = \boxed{0.25\pi \text{ rad}} \qquad \underline{\omega_s} = \frac{2\pi(3000)}{(8000)} = \boxed{0.75\pi \text{ rad}}$$

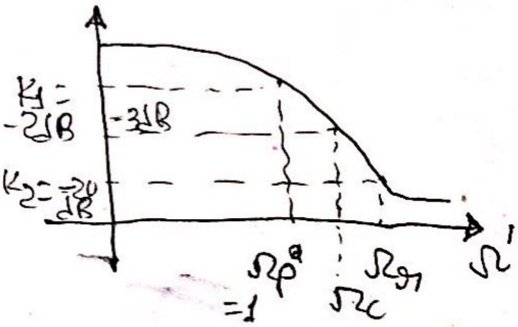
The pre-warped frequencies:- $\Omega_p' = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = \tan\left(\frac{0.25\pi}{2}\right)$

$\boxed{T = 2sec.}$ $\boxed{\Omega_p' = 0.41142 \text{ rad/sec.}}$

$$\underline{\Omega_s'} = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = \tan\left(\frac{0.75\pi}{2}\right) = \boxed{2.41142 \text{ rad/sec}}$$

For Normalized-LPF:-

$$\underline{\Omega_{s1}} = \frac{\Omega_s'}{\Omega_p'} = \frac{2.41142}{0.41142} = \boxed{5.8285 \text{ rad/sec.}}$$



$$\Omega_c \Big|_{norm\text{-}LPF} = \frac{\Omega_p}{[10^{-K_1/10} - 1]^{1/2N}}$$

$$N \geq \frac{\log_{10}\left[(10^{-K_1/10} - 1) / (10^{-K_2/10} - 1)\right]}{2\log_{10}(\Omega_p/\Omega_{s1})} = \frac{\log_{10}\left[(10^{0.2} - 1)/(10^2 - 1)\right]}{2\log_{10}(1/5.8285)}$$

$$N \geq \frac{\log_{10}[0.5849/99]}{2\log_{10}[0.1715]} = \frac{-2.2285}{-1.5311} = 1.455$$

$\boxed{N \simeq 2}$

$$\Omega_{cn} = \frac{1}{[10^{0.2}-1]^{1/4}} = \boxed{1.1435 \text{ rad/sec.}} \qquad \boxed{H_2(s) = \frac{1}{s^2+1.414s+1}}$$

The TF of LPF with $\boxed{\Omega_p' = 0.4142 \text{ rad/sec}}$ is

$$\therefore H_{LP}(s) = H_2(s)\Big|_{s \to \frac{s}{\Omega_{cn}\Omega_p'}} = \frac{s}{0.4736}$$

$$\therefore H_{LP}(s) = \frac{1}{\left[\frac{s}{0.4736}\right]^2 + 1.414\left[\frac{s}{0.4736}\right] + 1}$$

$$\therefore \boxed{H_{LP}(s) = \frac{0.2242}{s^2+0.6696s+0.2242}}$$

The digital TF is:- $H_{LP}(z) = H_{LP}(s)\Big|_{s \to \frac{2}{T}\frac{(z-1)}{(z+1)}} = \frac{(z-1)}{(z+1)}$

$$\therefore H_{LP}(z) = \frac{0.2242}{\frac{(z-1)^2}{(z+1)^2} + 0.6696\left(\frac{z-1}{z+1}\right) + 0.2242}$$

$$\therefore \boxed{H_{LP}(z) = \frac{0.1184z^2+0.2368z+0.1184}{z^2-0.8193z+0.2928}}$$

(7) Using BLT, design a digital BPF with the following specifications:-

Lower passband-edge freq. = f_l = 200Hz

Upper passband edge freq. = f_u = 400Hz

Lower stopband edge freq. = f_1 = 100Hz

Upper stopband edge freq. = f_2 = 500Hz

Passband ripple = K_1 = −3dB

stopband atth. = K_2 = −20dB

Sampling freq. = Fs = 2000Hz

<u>Solution</u>:- The angular bandedge frequencies are:-

$$\underline{\underline{\omega_l}} = \frac{2\pi(f_l)}{F_s} = \frac{2\pi(200)}{2000} = \boxed{0.2\pi \text{ rad}}$$

$$\underline{\underline{\omega_u}} = \frac{2\pi(f_u)}{F_s} = \frac{2\pi(400)}{2000} = \boxed{0.4\pi \text{ rad}}$$

$$\underline{\underline{\omega_1}} = \frac{2\pi(f_1)}{F_s} = \frac{2\pi(100)}{2000} = \boxed{0.1\pi \text{ rad}}$$

$$\underline{\underline{\omega_2}} = \frac{2\pi(f_2)}{F_s} = \frac{2\pi(500)}{2000} = \boxed{0.5\pi \text{ rad}}$$

<u>The pre-warped frequencies are</u>:- $\boxed{T = 2\text{sec}}$

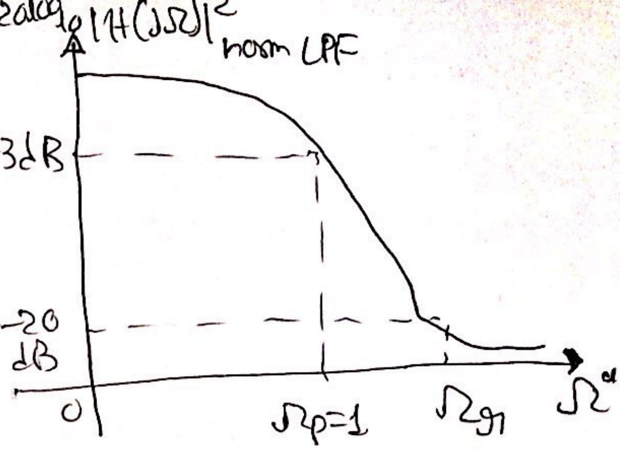
$$\underline{\underline{\Omega_l'}} = \frac{2}{T}\tan\left(\frac{\omega_l}{2}\right) = \boxed{0.325 \text{ rad/sec}}$$

$$\underline{\underline{\Omega_u'}} = \frac{2}{T}\tan\left(\frac{\omega_u}{2}\right) = \boxed{0.7265 \text{ rad/sec}}$$

$$\underline{\underline{\Omega_1'}} = \frac{2}{T}\tan\left(\frac{\omega_1}{2}\right) = \boxed{0.1584 \text{ rad/sec}}$$

$$\underline{\underline{\Omega_2'}} = \frac{2}{T}\tan\left(\frac{\omega_2}{2}\right) = \boxed{1 \text{ rad/sec}}$$

For normalized prototype LPF:-



$$\boxed{\Omega_p = 1 \text{ rad/sec.}}$$

$$\boxed{\Omega_{r} = \min\{|A|, |B|\}}$$

$$A = \frac{-\Omega_1'^2 + \Omega_l'\Omega_u'}{\Omega_1'(\Omega_u' - \Omega_l')}$$

$$A = \frac{-(0.1584)^2 + (0.325)(0.7265)}{0.1584(0.4015)} = \frac{0.2110}{0.0636} = \boxed{3.318 \text{ rad/sec.}}$$

$$B = \frac{+\Omega_2'^2 - \Omega_l'\Omega_u'}{\Omega_2'(\Omega_u' - \Omega_l')} = \frac{1 - 0.2361}{0.4015} = \boxed{1.90258 \text{ rad/sec.}}$$

$$\boxed{\Omega_{r} = 1.90258 \text{ rad/sec}} \qquad \boxed{\Omega_{cn} = \frac{\Omega_p}{[10^{-K_1/10} - 1]^{1/2N}}}$$

$$N \geq \frac{\log_{10}[(10^{-K_1/10} - 1)/(10^{-K_2/10} - 1)]}{2\log_{10}[\Omega_p/\Omega_{r}]} = \frac{\log_{10}[0.9952/99]}{2\log_{10}[1/1.90258]}$$

$$\therefore N \geq \frac{\log_{10}[0.01005]}{2\log_{10}[0.5256]} = \frac{-1.99}{-0.5586} = \boxed{3.57}$$

$$\boxed{N \simeq 4}$$

$$\therefore \Omega_{cn} = \frac{1}{[10^{0.3} - 1]^{1/8}} = \boxed{1 \text{ rad/sec}}$$

$$\boxed{H_N(s) = H_4(s) = \frac{1}{s^4 + 2.6131s^3 + 3.4146s^2 + 2.6131s + 1}}$$

$$H_{BP}(s) = H_4(s)\Big|_{s \to \frac{s^2+\Omega_l'\Omega_u'}{[s(\Omega_u'-\Omega_l')]\Omega_{cn}}} = \frac{s^2+0.2361}{s(0.4015)}$$

$$\therefore \boxed{H_{BP}(s) = \frac{0.02602s^4}{s^8+1.049s^7+1.495s^6+0.9125s^5+0.6204s^4+0.2154s^3+0.08331s^2+0.01381s+0.003106}}$$

$$\therefore H_{BP}(z) = H_{BP}(s)\Big|_{s \to \frac{2}{T}\frac{(z-1)}{(z+1)} = \frac{(z-1)}{(z+1)}}$$

$$\therefore \boxed{H_{BP}(z) = \frac{0.004824z^8-0.0193z^6+0.02895z^4+0.0193z^2+0.004824}{z^8-3.937z^7+8.26z^6-11.22z^5+10.78z^4-7.391z^3+3.577z^2-1.115z+0.1874}}$$

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(8)(*) Design a 2nd order digital BPF with the following specifications:- Upper cut-off freq. = $f_u = 2.6\,KHz$

Lower cut-off freq. = $f_l = 2.4\,KHz$

Sampling freq. = $F_s = 8\,KHz$

Solution:- The angular-band edge freq.:-

$$\omega_u = \frac{2\pi f_u}{F_s} = \frac{2\pi(2.6K)}{8K} = \boxed{0.65\pi\ rad} \qquad \omega_l = \frac{2\pi f_l}{F_s} = \frac{2\pi(2.4K)}{8K}$$

$$\boxed{\omega_l = 0.6\pi\ rad}$$

Pre-warped freq.:- $\boxed{T = 2\,sec.}$

$$\Omega_l' = \frac{2}{T}\tan\left(\frac{\omega_l}{2}\right) = \tan\left(\frac{0.6\pi}{2}\right) = \boxed{1.376\ rad/sec.}$$

$$\Omega_u' = \frac{2}{T}\tan\left(\frac{\omega_u}{2}\right) = \tan\left(\frac{0.65\pi}{2}\right) = \boxed{1.6318\ rad/sec.}$$

Since, we need a 2nd-order BPF, we select $\boxed{N=1}$, from the Butterworth tables:-

$$\boxed{H_N(s) = H_1(s) = \frac{1}{s+1}}$$

$$H_{BP}(s) = H_1(s)\Big|_{s \to \frac{s^2 + \Omega_l' \Omega_u'}{s(\Omega_u' - \Omega_l')}} = \frac{s^2 + 2.245}{(s)(0.2558)}$$

$$\therefore H_{BP}(s) = \frac{1}{\frac{s^2 + 2.245}{0.2558s} + 1} \quad ; \quad \boxed{H_{BP}(s) = \frac{0.2558s}{s^2 + 0.2558s + 2.245}}$$

$$H_{BP}(z) = H_{BP}(s)\Big|_{s \to \frac{2}{T}\frac{(z-1)}{(z+1)} = \frac{(z-1)}{(z+1)}}$$

$$\therefore H_{BP}(z) = \frac{0.2558\frac{(z-1)}{(z+1)}}{\frac{(z-1)^2}{(z+1)^2} + 0.2558\left(\frac{z-1}{z+1}\right) + 2.245}$$

$$\therefore H_{BP}(z) = \frac{0.2558(z-1)}{z^2 - 2z + 1 + 0.2558(z^2 - 1) + 2.245(z+1)^2}$$

$$\therefore H_{BP}(z) = \frac{0.2558z - 0.2558}{z^2 - 2z + 1 + 0.2558z^2 - 0.2558 + 2.245z^2 + 4.49z + 2.245}$$

$$H_{BP}(z) = \frac{0.2558z - 0.2558}{3.5z^2 + 2.49z + 2.9892}$$

$$\therefore \boxed{H_{BP}(z) = \frac{0.0732 - 0.073}{z^2 + 0.711z + 0.8541}}$$

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(9) Using BLT design a digital BSF with the following specifications:-

Lower passband edge freq. = $f_l = 35$ Hz

Upper passband edge freq. = $f_u = 215$ Hz

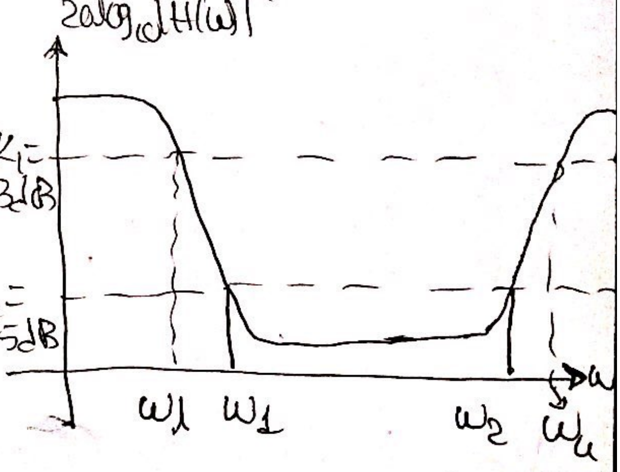
Lower stopband edge freq. = $f_1 = 100$ Hz

~~Stop~~ Upper stopband edge freq. = $f_2 = 150$ Hz

Passband ripple = $K_1 = -3$ dB

Stopband ripple = $K_2 = -15$ dB

Sampling freq. = $F_S = 500$ Hz



The angular-bandedge freq. are:-

$$\omega_l = \frac{2\pi f_l}{F_S} = \frac{2\pi(35)}{500} = \boxed{0.14\pi \text{ rad}}$$

$$\omega_u = \frac{2\pi f_u}{F_S} = \frac{2\pi(215)}{500} = \boxed{0.86\pi \text{ rad}}$$

$$\omega_1 = \frac{2\pi(f_1)}{F_S} = \frac{2\pi(100)}{500} = \boxed{0.4\pi \text{ rad}}$$

$$\omega_2 = \frac{2\pi(f_2)}{F_S} = \frac{2\pi(150)}{500} = \boxed{0.6\pi \text{ rad}}$$

Pre-warped frequencies are:- $\boxed{T = 2\,\text{sec.}}$

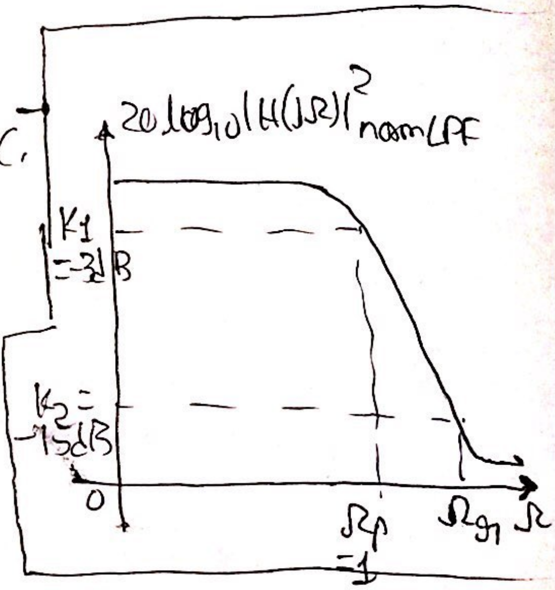
$$\Omega_l' = \frac{2}{T}\tan\left(\frac{\omega_l}{2}\right) = \boxed{0.2235 \text{ rad/sec}}$$

$$\Omega_u' = \frac{2}{T}\tan\left(\frac{\omega_u}{2}\right) = \boxed{4.4737 \text{ rad/sec.}}$$

$$\Omega_1' = \frac{2}{T}\tan\left(\frac{\omega_1}{2}\right) = \boxed{0.7265 \text{ rad/sec.}}$$

$$\Omega_2' = \frac{2}{T}\tan\left(\frac{\omega_2}{2}\right) = \boxed{1.3764 \text{ rad/sec}}$$

For normalized LPF:- $\Omega_p = 1$ rad/sec.



$$\boxed{\Omega_s = \min\{|A|, |B|\}}$$

$$A = \frac{\Omega_1'(\Omega_u' - \Omega_l')}{-\Omega_1'^2 + \Omega_l'\Omega_u'}$$

$$\therefore A = \frac{0.7265 \times 4.25}{-0.5278 + 0.9998} = \frac{3.0876}{0.472}$$

$$\boxed{A = 6.541 \text{ rad/sec.}}$$

$$B = \frac{\Omega_2'(\Omega_u' - \Omega_l')}{-\Omega_2'^2 + \Omega_l'\Omega_u'}$$

$$B = \frac{1.3764 \times 4.25}{-1.8944 + 0.9998} = \frac{5.8497}{-0.8946} = \boxed{-6.5389 \text{ rad/sec}}$$

$$\boxed{B = 6.5389 \text{ rad/sec}}$$

$$\boxed{\Omega_s = B = 6.5389 \text{ rad/sec.}}$$

$$N \geq \frac{\log_{10}\left[(10^{-K_1/10} - 1)/(10^{-K_2/10} - 1)\right]}{2\log_{10}\left[\Omega_p/\Omega_s\right]} = \frac{\log_{10}\left[0.9952/30.6227\right]}{2\log_{10}\left[1/6.5389\right]}$$

$$N \geq \frac{-1.4881}{-1.631} = 0.912 \quad \boxed{N \simeq 1}$$

$$\Omega_{cn} = \frac{\Omega_p}{[10^{-0.3} - 1]^{1/2}} = \frac{1}{0.9976} = \boxed{1\ \text{rad/sec}}$$

$$\boxed{H_N(s) = H_1(s) = \frac{1}{s+1}}$$ From Butterworth tables

$$H_{BS}(s) = H_1(s)\Big|_{s \to \frac{s(\Omega_u' - \Omega_l')}{[s^2 + \Omega_l'\Omega_u']\Omega_{cn}}} = \boxed{\frac{(s)(4.25)}{s^2+1}}$$

$$H_{BS}(s) = \frac{1}{\frac{4.25s}{s^2+1} + 1} = \boxed{\frac{s^2+1}{s^2+4.25s+1}}$$

$$H_{BS}(z) = H_{BS}(s)\Big|_{s \to \frac{2}{T}\frac{(z-1)}{(z+1)} = \frac{(z-1)}{(z+1)}}$$

$$\therefore H_{BS}(z) = \frac{\frac{(z-1)^2}{(z+1)^2} + 1}{\frac{(z-1)^2}{(z+1)^2} + 4.25\frac{(z-1)}{(z+1)} + 1} = \frac{(z-1)^2 + (z+1)^2}{(z-1)^2 + 4.25(z^2-1) + (z+1)^2}$$

$$\therefore H_{BS}(z) = \frac{2z^2+2}{2z^2+2+4.25z^2-4.25}$$

$$H_{BS}(z) = \frac{2z^2+2}{6.25z^2-2.25} = \boxed{\frac{0.32z^2+0.32}{z^2-0.36}}$$