# FIR filters

(1) inherently stable
(2) linear phase
(3) need higher orders for similar magnitude response compared to IIR filters.

① Inherent stability of FIR filters.

for an FIR filter of length M and input $x(n)$ the o/p $y(n)$ is described by the difference equation

$$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \dots + b_{M-1} x(n-M+1) \quad ①$$

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$ where $b_k$ is the set of filter coefficients.

Alternatively, we can express the o/p sequence $y(n)$ as the convolution of the unit sample response $h(n)$ ~~and~~ of the system and the input signal.

$$y(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k)$$ where the lower & the upper limits reflect the causality & finite-duration characteristics of the filter

$b_k$ are related to $h(k)$ as

$$h(k) = \begin{cases} b_k & \text{for } 0 \le k \le M-1 \\ 0 & \text{otherwise} \end{cases}$$

[the BIBO stability states that if a system produces bounded output for every bounded input, then ~~the system~~ it is a stable system]

Here $h(k) = [b_0 \;\text{-----}\; b_{M-1}]^T$ (the FIR filter) is bounded (stable), ∴ for any bounded input $x(n)$ the o/p $y(n)$ is bounded.

∴ according to equation ① FIR filter produces bounded output for every bounded input. Hence they are inherently stable filters.

## Symmetric/Antisymmetric FIR filters and linear phase property

The symmetry and/antisymmetry of the unit sample response of FIR filters is related to their linear phase.

The unit sample response of FIR filters is symmetric if it satisfies the following condition.

$$h(n) = h(M-1-n) \qquad \text{——— (1)}$$

The unit sample response of FIR filters is anti-symmetric if it satisfies the following condition.

$$h(n) = -h(M-1-n) \qquad \text{——— (2)}$$

An FIR filter has linear phase if its unit sample response is symmetric or antisymmetric. This is proved separately for even and odd lengths of FIR filters.

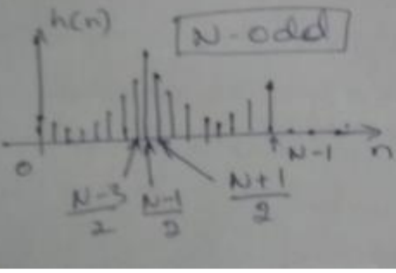
The fourier transform of unit sample response,

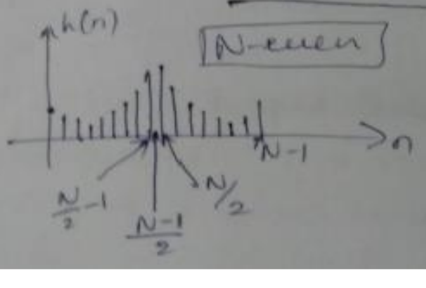
1 $$H(\omega) = \sum_{n=0}^{M-1} h(n)\, e^{-j\omega n}$$

Symmetric case

when the FIR filter length, M is odd

$$H(\omega) = \sum_{n=0}^{\frac{M-3}{2}} h(n)\, e^{-j\omega n} + h\left(\frac{M-1}{2}\right) e^{-j\omega\left(\frac{M-1}{2}\right)} + \sum_{n=\frac{M+1}{2}}^{M-1} h(n)\, e^{-j\omega n} \qquad \text{——— (3)}$$



Consider the last part of ③

i.e., $\sum_{n=\frac{M+1}{2}}^{M-1} h(n)\, e^{-j\omega n} = \sum_{n=\frac{M+1}{2}}^{M-1} h(M-1-n)\, e^{-j\omega n}$ ——— ④

for $\boxed{h(n) = h(M-1-n)}$ symmetric

~~where~~ let $M-1-n = k \Rightarrow n = M-1-k$

~~for~~ $n = \frac{M+1}{2}$ $\quad k = M-1-n = M-1-\left(\frac{M+1}{2}\right)$

$= \frac{2M-2-M-1}{2} = \frac{M-3}{2}$

$n = M-1 \quad k = M-1-n = M-1-(M-1) = 0$

$= \sum_{k=\frac{M-3}{2}}^{0} h(k)\, e^{-j\omega(M-1-k)}$

reversing the ~~summation~~ summation & changing k to n

$= \sum_{n=0}^{\frac{M-3}{2}} h(n)\, e^{-j\omega(M-1-n)}$ ——— ⑤

using ⑤ in ③ we get

$H(\omega) = \sum_{n=0}^{\frac{M-3}{2}} h(n)\, e^{-j\omega n} + h\left(\frac{M-1}{2}\right) e^{-j\omega\left(\frac{M-1}{2}\right)} + \sum_{\substack{\cancel{n=\frac{M+1}{2}}\\ n=0}}^{\frac{M-3}{2}} h(n)\, e^{-j\omega(M-1-n)}$

$H(\omega) = h\left(\frac{M-1}{2}\right) e^{-j\omega\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left[e^{-j\omega n} + e^{-j\omega(M-1-n)}\right]$ ——— ⑥

Consider the expression within square brackets

$\left[e^{-j\omega n} + e^{-j\omega(M-1-n)}\right]$ ——— ⑦

where $e^{-j\omega n} = e^{-j\omega n}\, e^{-j\omega\left(\frac{M-1}{2}\right)}\, e^{+j\omega\left(\frac{M-1}{2}\right)}$

$= e^{-j\omega\left(\frac{M-1}{2}\right)}\, e^{-j\omega\left(n-\left(\frac{M-1}{2}\right)\right)}$ ——— ⑦

$e^{-j\omega(M-1-n)} = e^{-j\omega(M-1)}\, e^{j\omega n}$

$= e^{-j\omega\left(\frac{M-1}{2}\right)}\, e^{-j\omega\left(\frac{M-1}{2}\right)}\, e^{j\omega n}$

$= e^{-j\omega\left(\frac{M-1}{2}\right)}\, e^{j\omega\left(n-\left(\frac{M-1}{2}\right)\right)}$ ——— ⑧

$\therefore \left[e^{-j\omega n}\, e^{-j\omega(M-1-n)}\right] = e^{-j\omega\left(\frac{M-1}{2}\right)}\, 2\cos\omega\left(n-\frac{M-1}{2}\right)$ —— (9)

using (9) in (6) we get

$$H(\omega) = h\left(\frac{M-1}{2}\right)e^{-j\omega\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\, e^{-j\omega\left(\frac{M-1}{2}\right)}\, 2\cos\omega\left(n-\frac{M-1}{2}\right)$$

$$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left(h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega\left(n-\frac{M-1}{2}\right)\right) \quad (10)$$

The polar form of $H(\omega)$ can be expressed as

$$H(\omega) = |H(\omega)|\, e^{j\angle H(\omega)}$$

where $|H(\omega)|$ is the magnitude.

and $\angle H(\omega)$ is the phase or angle of $H(\omega)$

$$|H(\omega)| = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega\left(n-\frac{M-1}{2}\right)$$

The magnitude can have negative values hence it is also called Pseudomagnitude

$$\angle H(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right) & \text{for } |H(\omega)| > 0 \\ -\omega\left(\frac{M-1}{2}\right) + \pi & \text{for } |H(\omega)| < 0 \end{cases}$$

$\left(\frac{M-1}{2}\right)$ is constant $\therefore$ $\angle H(\omega)$ is a linear function of '$\omega$'. ~~or the~~

Thus the phase is linearly proportional to the frequency.

When $|H(\omega)|$ changes sign, phase changes by '$\pi$'.

Hence the phase is said to be piecewise linear.

Thus, ~~the~~ FIR filters are linear phase filters.

Case 2: When the length of the FIR filter is even.

$$H(\omega) = \underbrace{e^{-j\omega\left(\frac{M-1}{2}\right)}}_{e^{j\angle-\omega\left(\frac{M-1}{2}\right)}}\underbrace{\left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)\cos\omega\left(n-\frac{M-1}{2}\right)\right]}_{|H(\omega)|}$$

$$\angle H(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right) & \text{for } |H(\omega)| > 0 \\ -\omega\left(\frac{M-1}{2}\right) + \pi & \text{for } |H(\omega)| < 0 \end{cases}$$

$\therefore$ the phase is Piecewise linear ~~and~~

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$H(z) = \sum_{n=0}^{N/2-1} h(n) z^{-n} + \sum_{n=N/2}^{N-1} h(n) z^{-n}$$

$$m = N-1-n \longrightarrow$$

$$H(z) = \sum_{n=0}^{N/2-1} h(n) z^{-n} + \sum_{m=\frac{N}{2}-1}^{0} h(N-1-m) z^{-(N-1-m)}$$

$$= \sum_{m=0}^{N/2-1} h(n) z^{-n} + \sum_{m=0}^{N/2-1} h(N-1-m) z^{-(N-1-m)}$$

$$= \sum_{n=0}^{N/2-1} h(n) z^{-n} + \sum_{n=0}^{N/2-1} \underbrace{h(N-1-n)}_{h(n)} z^{-(N-1-n)}$$

$$= \sum_{n=0}^{N/2-1} h(n) \left[ z^{-n} + z^{-(N-1-n)} \right]$$

$$z = e^{j\omega}$$

$$H(e^{j\omega}) = H(\omega) = \sum_{n=0}^{N/2-1} h(n) \left[ e^{-j\omega n} + e^{-j\omega(N-1-n)} \right]$$

$$= \sum_{n=0}^{N/2-1} h(n) e^{-j\omega\left(\frac{N-1}{2}\right)} \left[ e^{-j\omega\left[n-\left(\frac{N-1}{2}\right)\right]} + e^{j\omega\left(n-\left(\frac{N-1}{2}\right)\right)} \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N/2-1} 2h(n) \cos\omega\left(n-\left(\frac{N-1}{2}\right)\right) \quad \text{--- (1)}$$

Compare (1) with

$$H(e^{j\omega}) = H(\omega) = H_r(\omega)\, e^{j\theta(\omega)}$$

where $H_r(\omega)$ is a real function of $\omega$.

$$H_r(\omega) = \sum_{n=0}^{N/2-1} 2\,h(n) \cos\omega\left(n-\left(\frac{N-1}{2}\right)\right)$$

$$\theta(\omega) = \begin{cases} -\omega\left(\frac{N-1}{2}\right) & \text{if } H_r(\omega) > 0 \\ -\omega\left(\frac{N-1}{2}\right) + \pi & \text{if } H_r(\omega) < 0 \end{cases}$$

① A digital filter has frequency response $H(\omega)$ such that

$$0.95 \le |H(\omega)| \le 1.05 \quad \text{for } 0 \le \omega \le 0.3\pi$$
$$0 \le |H(\omega)| \le 0.005 \quad \text{for } 0.4\pi \le \omega \le \pi.$$

Let the sampling frequency be $F_s = 8KHz$.
Determine the passband & stopband frequencies in KHz, the passband ripple & the stopband attenuation in dB.

Solution

$$20\log_{10}\left(\frac{1}{0.95}\right) \quad \text{or} \quad 20\log_{10}(1.05) = 0.42\,dB$$
$$\delta_p = 0.42\,dB$$

$$-20\log_{10} 0.005 = 46\,dB = \delta_s.$$

$$\frac{2\pi F_p}{F_s} = \omega_p \qquad \frac{\overset{\omega_p}{0.3\pi}\, F_s}{2\pi} = F_p$$

$$\therefore \quad f_p = 1.2\,KHz$$

$$\frac{\omega_{st} F_s}{2\pi} = f_{st}$$

$$f_{st} = \frac{0.4\pi F_s}{2\pi}$$

0 to 0.3π

~~Anti~~ Antisymmetric.

i.e., when, $h(n) = -h(M-1-n)$.

the unit sample response is antisymmetric.

for M odd, the center point of the antisymmetric $h(n)$ is $n = \left(\frac{M-1}{2}\right)$.

consequently $h\left(\frac{M-1}{2}\right) = 0$.

However, if M is even, each term in $h(n)$ has a matching terms of opposite sign.

∴ the frequency response of an FIR filter with an antisymmetric unit sample response is

$$H(\omega) = H_r(\omega)\, e^{j\left(-\omega\left(\frac{M-1}{2}\right) + \pi/2\right)}$$

where $H_r(\omega) = 2\sum_{n=0}^{(M-3)/2} h(n) \sin\omega\left(\frac{M-1}{2} - n\right)$, M is odd

$$H_r(\omega) = 2\sum_{n=0}^{(M/2)-1} h(n) \sin\omega\left(\frac{M-1}{2} - n\right), \text{ M even.}$$

the phase characteristic of the filter for both M odd and M even is

$$\angle H(\omega) = \begin{cases} \frac{\pi}{2} - \omega\left(\frac{M-1}{2}\right), & \text{if } |H_r(\omega)| > 0 \\ \frac{3\pi}{2} - \omega\left(\frac{M-1}{2}\right), & \text{if } |H_r(\omega)| < 0. \end{cases}$$

* the frequency response formulas can be used to design linear-phase FIR filters with symmetric and antisymmetric unit sample responses.

① Symmetric $h(n)$

the # of filter co-efficients that specify the frequency response is $(M+1)/2$ ~~when~~ for M odd

& $M/2$ when M is even

② Antisymmetric $h(n)$

The # of filter co-efficients that specify the frequency response is $\frac{(M-1)}{2}$ for M odd (since the mid point $h\left(\frac{M-1}{2}\right) = 0$)

& $M/2$ when M is even

The choice of symmetric or antisymmetric unit sample response depends on the application.

ⓐ. The symmetric condition $h(n) = h(M-1-n)$ yields a linear-phase FIR filter with a non-zero response at $\omega = 0$, if desired.

i.e., $H_r(0) = h\left(\frac{M-1}{2}\right) + 2 \sum_{n=0}^{\frac{(M-3)}{2}} h(n)$ M is odd

& $H_r(0) = 2 \sum_{n=0}^{\frac{(M-2)}{2}} h(n)$ M is even

ⓑ The antisymmetric condition $h(n) = -h(M-1-n)$ yields ~~a linear~~ using $\omega=0$ in the frequency response equations yields $H_r(0) = 0$ & $H_r(\pi) = 0$ when M is odd — not used in the design of LP or HP filter (not suitable).

IIIly $H_r(0) = 0$ when M is even.

∴ we would not use the antisymmetric condition in the design of a low-pass linear-phase FIR filter.

~~Design of linear phase FIR filters using windows.~~

Magnitude Characteristics and Order of FIR filter

The magnitude response ~~given in above figure~~ can be expressed as

$$1-\delta \le |H(\omega)| \le 1+\delta \quad \text{for } 0 \le \omega \le \omega_p$$

$$0 \le |H(\omega)| \le \delta_s \quad \text{for } \omega_s \le \omega \le \pi$$

The approximate empirical formula for order N is given as

$$N = \frac{-10\log_{10}(\delta \cdot \delta_s) - 15}{14\Delta f}$$

$\Delta f = \frac{\omega_s - \omega_p}{2\pi}$ is the transition band.

or $\Delta f = f_s - f_p$, where $\omega_s = 2\pi f_s$ & $\omega_p = 2\pi f_p$

the length of the filter i.e., $M = N$, the order of the filter

∴ FIR filters have higher order than IIR filters.

* The FIR filters have higher order ∵ it do not use feedback, hence they need long sequences for $h(n)$ (i.e., high order) to get sharp cut-off filters.
* ∵ of the need # of co-efficients, FIR filters require large processing time.
* Their processing time can be reduced using FFT algorithms.

## Design of linear phase FIR filters using Windows

* The problem of FIR filter design is simply to determine the M coefficients of $h(n) = 0 \ldots M-1$, from a specification of the desired frequency response $H_d(\omega)$ of the FIR filter.

The desired frequency response specification $H_d(\omega)$,

$$H_d(\omega) = \sum_{n=0}^{\infty} h_d(n)\, e^{-j\omega n} \qquad (1)$$

where, the corresponding unit sample response is determined using.

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega \qquad (2)$$

Thus, given $H_d(\omega)$ we can determine the unit sample response $h_d(n)$

The unit sample response $h_d(n)$ obtained from (1) is infinite in duration. For an FIR filter of length M the infinite $h_d(n)$ has to be truncated, say at $n = M-1$. This is equivalent to multiplying $h_d(n)$ by a "rectangular window" defined as

$$w(n) = \begin{cases} 1 & n = 0 \ldots M-1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the unit sample response of the FIR filter becomes $h(n) = h_d(n)\, w(n)$

$\Rightarrow \quad h(n) = \begin{cases} h_d(n) & n = 0, 1 \dots M-1 \\ 0 & \text{otherwise.} \end{cases}$

Effect of windowing ~~on the~~ or the window function on the desired frequency response $H_d(\omega)$.

Multiplication of the window function $w(n)$ with $h_d(n)$ is equivalent to convolution of $H_d(\omega)$ with $W(\omega)$ where $W(\omega)$ is the frequency-domain representation (fourier transform) of the window function

i.e., $\quad W(\omega) = \sum_{n=0}^{M-1} w(n)\, e^{-j\omega n}$

Thus the convolution of $H_d(\omega)$ and $W(\omega)$ yields a frequency response $H(\omega)$ of the FIR filter

i.e., $\quad H(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\nu)\, W(\omega - \nu)\, d\nu$

The fourier transform of the rectangular window is

$$W(\omega) = \sum_{n=0}^{M-1} (1)(e^{-j\omega})^n = \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}}.$$

$$= \frac{e^{-j\omega \frac{M}{2}} e^{+j\omega\frac{M}{2}} - e^{-j\omega\frac{M}{2}} e^{-j\omega\frac{M}{2}}}{e^{-j\omega/2} e^{+j\omega/2} - e^{-j\omega/2} e^{-j\omega/2}} = \frac{e^{-j\omega\frac{M}{2}}\left(e^{j\omega M/2} - e^{-j\omega\frac{M}{2}}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

$e^{j\theta} - e^{-j\theta} = 2\sin\theta$

$$= e^{-j\omega\left(\frac{M}{2}\right)} e^{\frac{j\omega}{2}} \frac{\sin\left(\frac{\omega M}{2}\right)}{\sin(\omega/2)} = \underbrace{e^{-j\omega\left(\frac{M-1}{2}\right)}}_{\text{Phase}} \underbrace{\frac{\sin\left(\frac{\omega M}{2}\right)}{\sin(\omega/2)}}_{\text{magnitude}}$$

This window function has a magnitude response.

$$|W(\omega)| = \frac{\left|\sin\left(\frac{\omega M}{2}\right)\right|}{\left|\sin\left(\frac{\omega}{2}\right)\right|}$$

and a piecewise linear phase.

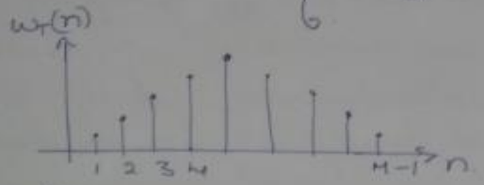
$$\underline{/W(\omega)} = \begin{cases} -\omega\left(\frac{M-1}{2}\right) & \text{when } \sin(\omega M/2) \geq 0 \\ -\omega\left(\frac{M-1}{2}\right) + \pi & \text{when } \sin(\omega M/2) < 0 \end{cases}$$

## Bartlett (Triangular window)

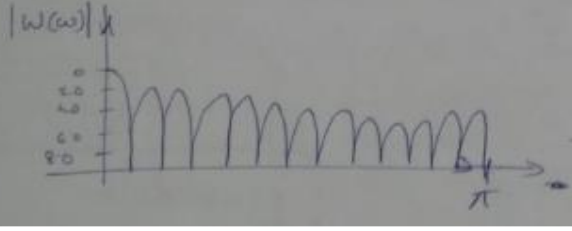
the sidelobes created ripples in $H(\omega)$.

is expressed as

$$w_T(n) = \begin{cases} 1 - \dfrac{2\left|n - \frac{M-1}{2}\right|}{M-1} & \text{for } n = 0, 1, \ldots M-1 \\ 0 & \rightarrow \text{otherwise.} \end{cases}$$



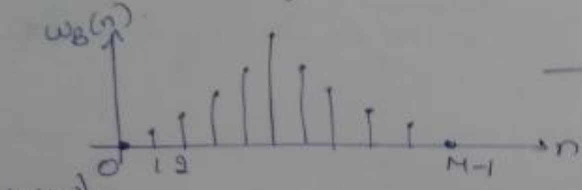

$\rightarrow$ time domain sketch



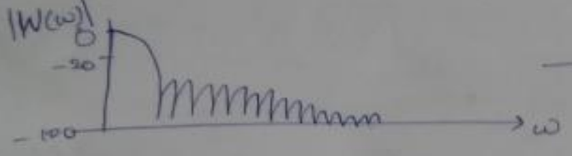

$\rightarrow$ magnitude sketch

## Blackman Window

$$w_B(n) = \begin{cases} 0.42 - 0.5\cos\frac{2\pi n}{M-1} + 0.08\cos\frac{4\pi n}{M-1} & \text{for } n = 0 \ldots M-1 \\ 0 & \text{otherwise} \end{cases}$$



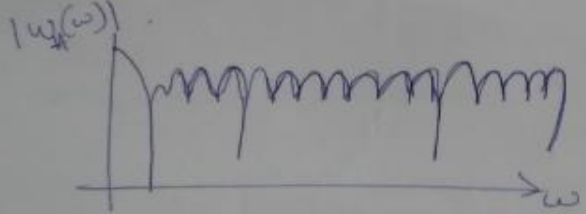

→ time domain sketch




→ Magnitude response

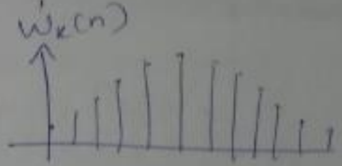
## Hamming Window

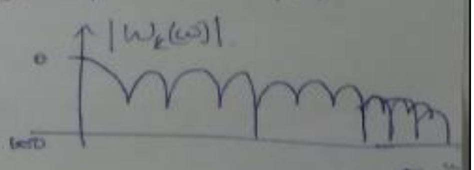
$$w_H(n) = 0.54 - 0.46\cos\frac{2\pi n}{M-1}$$

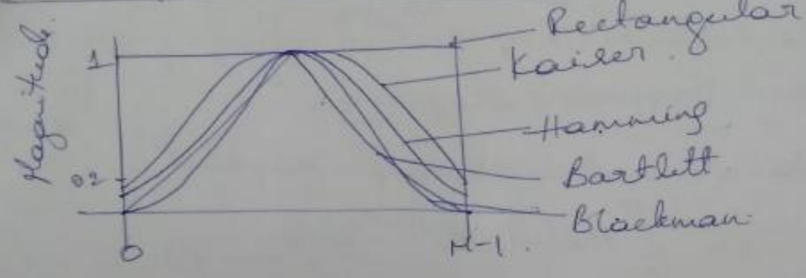

## Kaiser Window

$$w_K(n) = \frac{I_0\left[\alpha\sqrt{\left(\frac{M-1}{2}\right)^2 - \left(n - \frac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\frac{M-1}{2}\right)\right]}$$

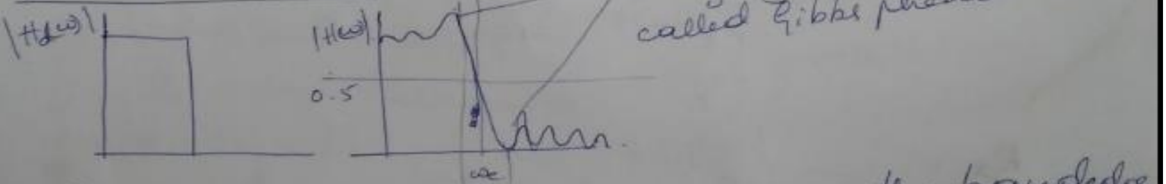


## Shapes of several window functions




## Gibbs phenomenon




* Ringing effect or oscillations ~~at~~ near the bandedge of filter is due to the sidelobes ~~of these~~ in the frequency response W(ω) of the window function
* Sidelobes are generated due to ~~too~~ abrupt discontinuity (in case of rectangular window) of the window function.
* Ringing effect is maximum in rectangular window since the sidelobes are larger in size.
* ∴ Different window functions are ~~generated~~ developed which contain taper and decays gradually to zero. This reduces side lobes & hence the ringing effect in H(ω).
* ~~Since~~ other window functions we might eliminate the

Example 4.7 (L.C. [illegible])

Design a LP digital filter to be used in an A/D - H(z) - D/A structure that will have a -3dB cut-off of $30\pi$ rad/sec and an attenuation of 50dB at $45\pi$ rad/sec. The filter is required to have linear phase and the system will use a sampling rate of 100 samples/sec.

$$\omega_c = \Omega_c \cdot T = 30\pi(0.01) = 0.3\pi \text{ rad}, \quad K_c \geq -3\text{dB}$$

$$\omega_r = \Omega_c \cdot T = 45\pi(0.01) = 0.45\pi \text{ rad}, \quad K_r \leq -50\text{dB}$$

$$\Delta\omega = 0.45\pi - 0.3\pi = 0.15\pi$$

$$0.15\pi \geq \frac{8\pi}{M} \Rightarrow M \geq 8/0.15 \qquad \boxed{M = 53.33}$$

$\therefore\ M = 55$ (to obtain an integer delay)
the next odd # was 55

$\frac{27}{54/2}$

Here $\omega_c = \omega_1 = 0.3\pi$ & $\frac{M-1}{2} = 27$.

$$\therefore\ h(n) = \frac{\sin 0.3\pi(n-27)}{\pi(n-27)}\left[0.54 - 0.46\cos\left(\frac{2\pi n}{55-1}\right)\right]$$

$0 \leq n \leq 54$

## Kaiser window

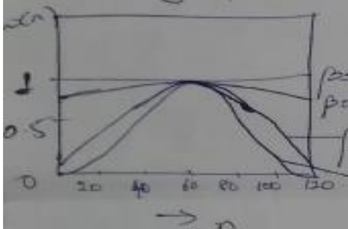
Using this we can control the width of the mainlobe and attenuation of sidelobes independently.

$$w_k(n) = k(\beta, N) = \begin{cases} \dfrac{I_0\left[\beta\sqrt{1-\left(\frac{n-\alpha}{\alpha}\right)^2}\right]}{I_0(\beta)} & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$$

$\beta$ → attenuation of sidelobes; $N$ → width of the mainlobe

$\alpha = \frac{M-1}{2}$

$I_0$ is the 0th order bessel function of the first kind. The value of $\beta$ controls the degree of taper towards the edge of the window.




$\beta = 0$ ——— ||| to rectangular window.
$\beta = 7$
$\beta = 5$ → Similar to Hamming window.
$\beta = 10$

for $M = 128$

The parameter '$\alpha$' in the expression for $w(n)$, determines the size of the window ($N$).

where $\alpha = \frac{M-1}{2}$

① $\delta = \min(\underset{(\delta_p)}{\delta_1}, \underset{(\delta_s)}{\delta_2})$

the attenuation in dB is then expressed in dB

② $A = -20 \log \delta$.

③ $\Delta\omega = \omega_s - \omega_p$

④ The value $\beta$ can be obtained by

$$\beta = \begin{cases} 0.1102\,(A-8.7) & \text{if } A > 50 \\ 0.5842\,(A-21)^{0.4} + 0.07886\,(A-21) & \text{if } 21 \le A \le 50 \\ 0 & \text{if } A < 21 \end{cases}$$

(5) $M = \dfrac{A-8}{2.285\,\Delta\omega}$

normally this results in a fractional value

∴ is rounded up to the nearest odd integer.

(6) Substitute for $\beta$ & $\alpha$ in ~~the~~ $\omega(n)$

$$I_0\{x\} = 1 + \sum_{n=1}^{\infty}\left[\left(\frac{x}{2}\right)^n \frac{1}{n!}\right]^2$$

Zero order Bessel function

(or)

$$I_0(x) = 1 + \frac{0.25x^2}{(1!)^2} + \frac{(0.25x^2)^2}{(2!)^2} + \frac{(0.25x^2)^3}{(3!)^2} + \cdots$$

The series normally converges satisfactorily if more than about ten terms are included.

Example: Design a LPF with a cut-off frequency $\omega_c = \frac{\pi}{4}$, $\Delta\omega = 0.02\pi$, stopband ripple $\delta_s = 0.01$. Use Kaiser window.

(1) $A = -20\log\delta_s = -20\log(0.01) = 40\,dB$

(2) $M \geq \dfrac{A-8}{2.285\Delta\omega} = \dfrac{40-8}{2.285 * 0.02\pi}$

$M \geq 223.189$

$\therefore M = 225 \quad \therefore \alpha = \dfrac{M-1}{2} = \dfrac{225-1}{2} = 112$

$$\omega(n) = \frac{I_0\left\{\beta\sqrt{1-\left(\frac{n-\alpha}{\alpha}\right)^2}\right\}}{I_0\{\beta\}} \quad 0 \leq n \leq N-1$$

(3) $\beta = 0.5842(A-21)^{0.4} + 0.07886(A-21)$
$= 3.4$

13/10/11 ③

$$\therefore w(n) = \frac{I_0\left\{3.4\sqrt{1-\left(\frac{n-112}{112}\right)^2}\right.}{I_0\{3.4\}} \quad 0 \le n \le 224$$

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha} & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

$\alpha = 112 \qquad \omega_c = \frac{\pi}{4}$

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha} e^{j\omega n}\, d\omega$$

$$= \frac{\sin(\omega_c(n-\alpha))}{\pi(n-\alpha)} \quad n \neq \alpha$$

$$h_d(n) = \frac{\omega_c}{\pi} \quad n = \alpha$$

due to the smearing effect of the window function.

$$\omega_c' = \omega_c + \frac{\Delta\omega}{2}$$

$$= \frac{\pi}{4} + \frac{0.02\pi}{2} = 0.26\pi.$$

$$h_d(n) = \begin{cases} \frac{\sin(0.26\pi(n-112))}{\pi(n-112)} & n \neq 112 \\ 0.26 & n = 112 \end{cases}$$

$h(n) = h_d(n)w(n)$

⑤ Suppose that we want to design a LPF of order $N = 63$ with a ~~cutoff frequency~~ $\omega_p = 0.3\pi$ & stopband frequency $\omega_s = 0.32\pi$. What will be the approximate stopband & ripple that would be obtained if the filter were designed using a Kaiser window?

$\omega_p = 0.3\pi$ rad

$\omega_s = 0.32\pi$ rad

$\Delta f = \frac{\omega_s - \omega_p}{2\pi} = 0.01$ or $\Delta\omega$

$N = 63$.

$$\therefore 63 = \frac{-20\log\delta_s - 8}{x}$$

$\delta_s = \underline{0.1413}$.

Design an FIR linear phase filter using Kaiser window to meet the following specifications.

$$0.99 \le |H(e^{j\omega})| \le 1.01 \qquad 0 \le |\omega| \le 0.19\pi$$

$$|H(e^{j\omega})| \le 0.01 \qquad 0.21\pi \le |\omega| \le \pi$$

$1-\delta_1 = 0.99 \qquad \delta_2 = 0.01$

$\delta_1 = 0.01$

$\therefore \delta = \min(\delta_1, \delta_2) = 0.01,$

$A = -20\log_{10}\delta = -20\log_{10} 0.01 = 40$

$\omega_c = 0.2\pi, \quad \beta = 3.395, \quad M = 223$

$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)} & \text{for } -\omega_c \le \omega \le +\omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$h_d(n) = \begin{cases} \dfrac{\sin 0.2\pi(n-111)}{\pi(n-111)} \times w_k(n) \\ \dfrac{0.2\pi}{\pi} \times w_k(n). \end{cases}$$

Design of linear phase FIR filters using frequency sampling method

(b) $H_d(\omega)$, is the desired frequency response of the FIR filter we want to design.

$H_d(\omega)$ is sampled uniformly at 'M' points.

i.e., frequency samples at

$$\omega = \frac{2\pi}{M} k, \qquad k = 0 \dots M-1$$

The sampled desired frequency response is DFT.

$$\Rightarrow H(k) = H_d(\omega)\Big|_{\omega=\omega_k}$$

$$H(k) = H_d\left(\frac{2\pi}{M} k\right) \qquad k = 0 \dots M-1$$

M-point DFT

we can get $h(n)$ by taking the IDFT of $H(k)$

$$h(n) = \frac{1}{M} \sum_{k=0}^{M-1} H(k)\, e^{j2\pi kn/M} \qquad n = 0 \dots M-1$$

$h(n)$ is the unit sample response of FIR filter of length 'M' obtained using the frequency sampling method

In order to realize the FIR filter, the coefficients $h(n)$ should be all real.

for this all complex terms must appear in complex conjugate pairs

from (2): $H(M-K)\, e^{j2\pi n(M-K)/M}$, consider this term.

$$H(M-K)\, e^{j2\pi n(M-K)/M} = H(M-K)\, e^{j2\pi n}\, e^{-j2\pi nk/M}$$

$$e^{j2\pi n} = \cos(2\pi n) + j\sin(2\pi n) = 1 \text{ always.}$$

$$\therefore H(M-K)\, e^{j2\pi n(M-K)/M} = H(M-K)\, e^{-j2\pi nK/M}$$

Normally $|H(M-K)| = |H(K)|$,

This relation is based on the fact that

magnitude of DFT from ~~0 to π is~~ 0 to π is same as that from π to 2π.

Hence

$$H(M-K)\, e^{j2\pi n(M-K)/M} = H(K)\, e^{-j2\pi kn/M}$$

∴ the term $H(K)\, e^{-j2\pi kn/M}$ is complex conjugate of $H(K)\, e^{+j2\pi kn/M}$

Hence $H(M-K)\, e^{j2\pi n(M-K)/M}$ is complex conjugate of $H(K)\, e^{\pm j2\pi kn/M}$

i.e. $H(M-K) = H^*(K)$

Using this relation of complex conjugate terms,

$$h(n) = \frac{1}{M}\left\{H(0) + 2\sum_{K=1}^{P} \mathrm{Re}\left[H(K)\, e^{j2\pi kn/M}\right]\right\}$$

where, $P = \begin{cases} \frac{M-1}{2} & \text{if } M \text{ is odd} \\ \frac{M}{2}-1 & \text{if } M \text{ is even} \end{cases}$

$\frac{7-1}{2} = 6/2 = 3$

$6/2 - 1 = 3 - 1 = 2$

This equation can be used to compute the coefficients of FIR filter.

$M=6$
$3-1$
$=2$

**Ex: (1)** Design a LP FIR filter using frequency ~~response~~ sampling technique having cut off frequency $\pi/2$ rad/sample. The filter should have linear phase & length of 17.

**Solution:**

$$H_d(\omega) = \begin{cases} e^{-j\omega\frac{M-1}{2}} & \text{for } |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases} \quad \text{—— (1)}$$

in (1) the range of ω is $-\pi$ to $+\pi$.

We want $H_d(\omega)$ only for the values of ω.

$$\therefore\ H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)} & 0 \le \omega \le \omega_c \\ 0 & \omega_c \le \omega \le \pi \end{cases}$$

(2)

$M = 17 \qquad \omega_c = \pi/2$

$$\therefore \; H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{17-1}{2}\right)} & 0 \le \omega \le \pi/2 \\ 0 & \pi/2 \le \omega \le \pi \end{cases}$$

This is the desired frequency response of the required LPF.

(2) Sample $H_d(\omega)$

$\omega = \frac{2\pi k}{M} \qquad k = 0 \cdots M-1$

$k = 0 \cdots 16.$

$\omega = \frac{2\pi}{17} k$

$$H(k) = H_d(\omega)\Big|_{\omega = \frac{2\pi k}{17}}$$

$$= \begin{cases} e^{-j\left(\frac{2\pi k}{17}\right)8} & 0 \le \frac{2\pi k}{17} \le \pi/2 \\ 0 & \pi/2 \le \frac{2\pi k}{17} \le \pi \end{cases}$$

$$= \begin{cases} e^{-j \frac{16\pi k}{17}} & 0 \le k \le \frac{17}{4} \\ 0 & \frac{17}{4} \le k \le \frac{17}{2}. \end{cases}$$

$0 \le k \le 17/4 \Rightarrow 0 \le k \le 4.25$

$\therefore$

$\boxed{0 \le k \le 4}$

$\text{lly} \quad 4.25 \le k \le 8.5$

$5 \le k \le 8$

$$H(k) = \begin{cases} e^{-j \frac{16\pi k}{17}} & 0 \le k \le 4. \\ 0 & 5 \le k \le 8. \end{cases} \qquad \text{--- (2)}$$

③ $h(n) = \frac{1}{M}\left\{ H(0) + 2\sum_{K=1}^{\frac{M-1}{2}} Re\left[H(K) e^{j2\pi kn/M}\right]\right\}$.

put $K = 0$ in ②   $H(0) = 1$ & $M = 17$

$h(n) = \frac{1}{17}\left\{1 + 2\sum_{K=1}^{8} Re\left[H(K) e^{j2\pi Kn/17}\right]\right\}$.

$h(n) = \frac{1}{17}\left[1 + 2\sum_{K=1}^{4} Re\left[e^{-j16\pi K/17}\, e^{j2\pi kn/17}\right]\right]$

$= \frac{1}{17}\left\{1 + 2\sum_{K=1}^{4} Re\left[e^{-j2\pi K(8-n)/17}\right]\right\}$.

$e^{j\theta} = \cos\theta + j\sin\theta$   real part in $e^{j\theta} = \cos\theta$

$\therefore h(n) = \frac{1}{17}\left\{1 + 2\sum_{K=1}^{4} \cos\left[\frac{2\pi K(8-n)}{17}\right]\right\}$.

$n = 0 \ldots 16$.

unit sample response of the FIR filter.

② $h(n)$

$H_d(e^{j\omega}) = \begin{cases} e^{-j(N-1)\omega/2} & 0 \le |\omega| \le \pi/2 \\ 0 & \pi/2 \le |\omega| \le \pi \end{cases}$

$N = 7$.

$\frac{1}{17}\left(1 + 2\left[\cos\frac{2\pi(3-n)}{7} + \cos\frac{4\pi(3-n)}{7}\right]\right)$   $n = 0 \ldots 6$

0.2934   $\cos 0.8976 + \cos(1.795$

n 0   $+ 0.995$

| | | | | | | |
|---|---|---|---|---|---|---|
| n | 1 | 2 | 3 | 4 | 5 | 6 |
| | 0.2938 | 0.2940 | 0.294117 | 0.29404 | 0.29382 | 0.29346 |

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$= \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} H(k) e^{j\frac{2\pi nk}{N}} z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} H(k) \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi k}{N}} z^{-1}\right)^{n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} H(k) \frac{1-\left(e^{j\frac{2\pi k}{N}} z^{-1}\right)^{N}}{1-e^{j\frac{2\pi k}{N}} z^{-1}} \quad a \neq 1$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} H(k) \left[\frac{1-z^{-N}}{1-e^{j\frac{2\pi k}{N}} z^{-1}}\right].$$

$1-e^{j\frac{2\pi k}{N}} z^{-1}$

a pole-zero cancellation takes place. $H(z)$ has only zeros.

$$\boxed{z = e^{j\omega}}$$

$$H(e^{j\omega}) = H(\omega) = \frac{1}{N} \sum_{k=0}^{N-1} H(k) \left[\frac{1-e^{-j\omega N}}{1-e^{j\frac{2\pi k}{N}} e^{-j\omega}}\right]$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} H(k) \frac{e^{-j\omega N/2} e^{j\omega N/2} - e^{-j\omega N/2} e^{-j\omega N/2}}{\cancel{e^{-j\omega N/2} e^{j\omega N/2}}}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} H(k) \frac{e^{-j\omega N/2}\left[e^{j\omega N/2} - e^{-j\omega N/2}\right]}{e^{j\frac{2\pi k}{N}\times\frac{1}{2}} e^{-j\omega/2}\left[e^{-j\frac{2\pi k}{N}\cdot\frac{1}{2}} e^{+\frac{j\omega}{2}} - e^{j\frac{2\pi k}{N}\frac{1}{2}} e^{-\frac{j\omega}{2}}\right]}.$$

$$e^{-j\omega\frac{(N-1)}{2}} \sum_{k=0}^{N-1} \frac{H(k)}{N} e^{-j\frac{\pi k}{N}} \frac{\sin\left(\omega N/2\right)}{\sin\left(\frac{\omega}{2} - \frac{2\pi k}{N}\right)}$$

(5B)
(5a)-

(2) Design a low pass FIR filter using frequency sampling technique having cutoff frequency of $\pi/2$ rad/sample. The filter should have linear phase and length of N.

$$H_d(e^{j\omega}) = \begin{cases} e^{-j(N-1)\omega/2} & 0 \le |\omega| \le \pi/2 \\ 0 & \pi/2 \le |\omega| \le \pi . \end{cases}$$

$$H(k) = \begin{cases} e^{-j\left(\frac{2\pi k}{7}\right).3} & 0 \le \frac{2\pi k}{7} \le \pi/2 \\ 0 & \pi/2 \le \frac{2\pi k}{7} \le \pi \end{cases} \quad 0 \ldots 6.$$

$$= \begin{cases} e^{-j\frac{6\pi k}{7}} & 0 \le k \le 7/4 \\ 0 & 7/4 \le k \le 7/2 . \end{cases}$$

$$\frac{17}{4} \quad 1.8 \quad \frac{7}{4}$$

$0 \le k \le 2.$

$2 < k \le 4.$

(iii) $$h(n) = \frac{1}{M}\left\{H(0) + 2\sum_{k=1}^{\frac{M-1}{2}} Re\left[H(k)\, e^{j2\pi kn/M}\right]\right\}$$

$$= \frac{1}{7}\left\{1 + 2\sum_{k=1}^{2} Re\left[e^{-j6\pi k/7} \cdot e^{j2\pi kn/7}\right]\right\}$$

$$= \frac{1}{7}\left\{1 + 2\sum_{k=1}^{2} Re\left[e^{-j2\pi k(3-n)/7}\right]\right.$$

$$= \frac{1}{7}\left\{1 + 2\sum_{k=1}^{2} \cos\left[\frac{2\pi k(3-n)}{7}\right]\right\} \quad n = 0 \ldots 6$$

$-0.11456,\ 0.07928,\ 0.320997,\ 0.42857,$
$0.320997,\ 0.07928,\ -0.11456.$

h(n) for 1st N=17 problem

0.0398, −0.0488, −0.034559, 0.06598, 0.03154, −0.10747, −0.0299, 0.31876, 0.5294, 0.31876 - - - - - - 0.0398.

(Q) Derive an expression for system function if the h(n) is obtained using frequency S.T

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$= \sum_{n=0}^{N-1} \left[ \frac{1}{N} \sum_{k=0}^{N-1} H(k) e^{\frac{j2\pi nk}{N}} \right] z^{-n}$$

$$H(z) = \frac{1}{N} \sum_{k=0}^{N-1} H(k) \sum_{n=0}^{N-1} e^{\frac{j2\pi kn}{N}} z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} H(k) \sum_{n=0}^{N-1} \left( e^{j\frac{2\pi k}{N}} z^{-1} \right)^{n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} H(k) \frac{1 - \left[ e^{j\frac{2\pi k}{N}} z^{-1} \right]^{N}}{1 - e^{j\frac{2\pi k}{N}} z^{-1}}$$

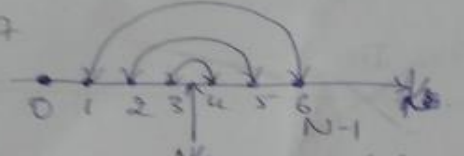
$$= \frac{1}{N} \sum_{k=0}^{N-1} H(k) \left[ \frac{1 - z^{-N}}{1 - e^{j\frac{2\pi k}{N}} z^{-1}} \right].$$

$1 - z^{-N}$ has a factor

$1 - e^{j\frac{2\pi k}{N}} z^{-1}$

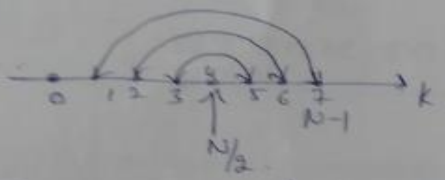
Pole-zero cancel.

N=7




N/2 unfolding index

N=8




$x(n)$

This conjugate symmetry of $X(k)$ exists only when $x(n)$ is real.

$$X(N-k) = X^*(k) \quad \text{or} \quad X(k) = X^*(N-k)$$

When we sample $H_d(\omega)\big|_{\omega=\frac{2\pi k}{M}}$

$$H(k) = H_d(\omega)\big|_{\omega=2\pi k/M}$$

$$H\left(\frac{2\pi k}{M}\right) \ \forall \ k=0 \ldots M-1$$

will not be complex conjugate of

When $x(n)$ is finite length then DFT $X(k)$ is an exact match to its DTFT $X(\omega)$

Advantages:

(i) Unlike window method, this technique can be used for any given magnitude response.

(ii) This method is useful for the design of non prototype filters where the desired magnitude response can take any irregular shape.

Disadvantages: The frequency response obtained by interpolation is equal to the desired frequency response only at the sampled points. At other points, there will be a finite error present.

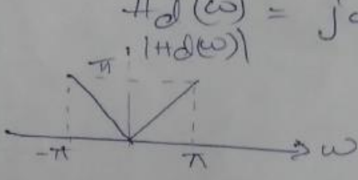
# Design of FIR Differentiators 8/11/11 ①

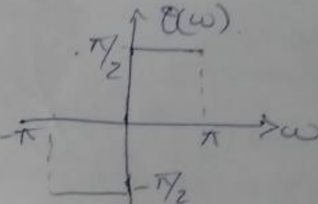
⑥ Differentiators are used to take derivatives of the i/p signal. It can be used to find the instantaneous rate of change or slope.

We are going to look at non-recursive techniques using the window function.

The frequency response of the ideal digital differentiator is linearly proportional to frequency.

$$H_d(\omega) = j\omega, \quad -\pi \le \omega \le \pi$$




Magnitude response
Even symmetry

Phase response
Odd symmetry.

with respect to $\omega = 0$

The unit sample response $h_d(n)$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} j\omega\, e^{j\omega n}\, d\omega.$$

$$= \begin{cases} \dfrac{\cos n\pi}{n} & -\infty < n < \infty \quad n \ne 0 \\ 0 & \text{for } n = 0. \end{cases}$$

* $h_d(n)$ is a two-sided infinite-length impulse response. ∴ the ideal differentiator is an unrealizable system.

* $h_d(n)$ is antisymmetric w.r.t $n = 0$
  [∵ $h_d(n) = -h_d(-n)$]

* ~~Shifting~~ translating $h_d(n)$ to right by $\alpha = \frac{N-1}{2}$, we get antisymmetry about $n = \alpha$

$$hd'(n) = h_d(n-\cancel{\alpha}\alpha) = \frac{\cos(n-\alpha)\pi}{(n-\alpha)} \qquad n \neq \alpha$$

$$= \quad 0 \qquad n = \alpha$$

[L' Hospital's rule]

h(n), the impulse response of an FIR differentiator will have linear phase if it exhibits symmetry or antisymmetry about its mid-point.

for antisymmetry,

$$h(n) = -h(N-1-n)$$

Since hd'(n) is antisymmetric about $n=\alpha$ we multiply hd'(n) with a window function w(n) i.e., symmetric about $n=\alpha$.

$$h(n) = hd'(n)\,\omega(n) \qquad 0 \le n \le N-1$$

$$h(n) = \begin{cases} \dfrac{\cos[\pi(n-\alpha)]}{n-\alpha} \times \omega(n) & 0 \le n \le N-1,\ n \neq \alpha \\ 0 & n = \alpha \end{cases}$$

Since h(n) is antisymmetric about $n=\alpha$ & zero at $n=\alpha$, N should be an odd integer only.

the magnitude frequency response for N odd & $h(n) = -h(N-1-n)$ is

$$|H(\omega)| = |H_r(\omega)|$$

$$= \left| 2 \sum_{n=0}^{\left(\frac{N-3}{2}\right)} h(n) \sin\left[\omega\left(\left(\frac{N-1}{2}\right) - n\right)\right] \right|$$

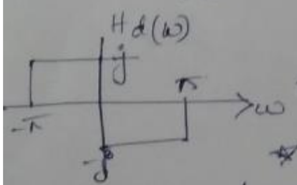
## Design of Hilbert transformers

8/11/11 (2)

* The ideal Hilbert transformer is an allpass filter.
* It produces an o/p signal that is phase shifted by 90° with respect to the input signal. ∴, called as 90° phase shifter.
* The frequency response of an ideal digital Hilbert transformer over one period is ~~given as~~,

$$H_d(\omega) = \begin{cases} -j & 0 \le \omega \le \pi \\ j & -\pi \le \omega \le 0 \end{cases}$$

* the frequency response is similar to that of an ideal LPF both having discontinuities separated by $\pi$.




$$H_d(\omega + 2\pi) = H_d(\omega)$$

* $|H_d(\omega)| = 1$ for all frequencies & 90° phase shift.
* The unit sample response of an ideal Hilbert transformer ~~is given as~~ obtained by computing the inverse DTFT.

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} j e^{j\omega n} d\omega - \int_{0}^{\pi} j e^{j\omega n} d\omega\right] = \begin{cases} \frac{2\sin^2(\pi n/2)}{\pi n}, & n \ne 0 \\ 0 & n = 0 \end{cases}$$

$$= \begin{cases} 0 & , n = 0 \\ \frac{1}{\pi n}\left[1 - (-1)^n\right], & \text{for } n \ne 0. \end{cases}$$

(or)

$$h_d(n) = \begin{cases} \frac{2\sin^2(\pi n/2)}{\pi n} & n \ne 0 \\ 0 & n = 0. \end{cases}$$

* $h_d(n)$ is infinite and non-causal. unrealizable system; ∴, it is
* The ideal HT is an infinite-length impulse response. a two-sided
* Since $h_d(n) = -h_d(-n)$, $h_d(n)$ is antisymmetric.
* ∴ we have to design linear-phase FIR Hilbert transformers with an antisymmetric impulse response:

$$h(n) = -h(N-1-n)$$

* Translate $h_d(n)$ to right by an amount $\alpha = \frac{N-1}{2}$.
* Now, $h_d'(n) = h_d(n-\alpha)$ is antisymmetric about $n=\alpha$.

$$h_d'(n) = h_d(n-\alpha) = \begin{cases} \dfrac{2\sin^2\left[\frac{\pi}{2}(n-\alpha)\right]}{\pi(n-\alpha)}, & n \neq \alpha \\ 0, & n=\alpha \end{cases}$$

* The finite impulse response $h(n)$ of a Hilbert transformer ~~is obtained by truncating $h_d'(n)$ by~~ will have linear phase if the impulse response exhibits either symmetry or antisymmetry about the midpoint, $n = \frac{N-1}{2}$.
* Here, $h(n)$ is designed to have antisymmetry since, $h_d'(n)$ is antisymmetric about $n=\alpha$.
* The finite impulse response $h(n)$ is obtained by multiplying $h_d'(n)$ with a window function that is symmetric about $n=\alpha$.

$$h(n) = h_d'(n)\,\omega(n), \quad 0 \le n \le N-1$$

$$h(n) = \begin{cases} \dfrac{2\sin^2\left[\frac{\pi}{2}(n-\alpha)\right]}{\pi(n-\alpha)} \times \omega(n) & 0 \le n \le N-1 \quad n \neq \alpha \\ 0 & n=\alpha. \end{cases}$$

* Since $h(n)$ is antisymmetric about $n=\alpha$ and is zero at $n=\alpha$, $N$ has to be an odd integer only.
* The magnitude response for $N$ odd & $h(n) = -h(N-1-n)$ is

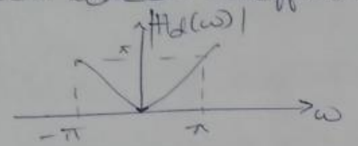
$$|H(\omega)| = |H_r(\omega)|$$

$$= \left| 2 \sum_{n=0}^{\left(\frac{N-3}{2}\right)} h(n) \sin\left[\omega\left(\left(\frac{N-1}{2}\right) - n\right)\right] \right|$$

Applications:

* in the generation of analytic signal
* Generation of single side band modulated signals
* Radar & speech signal processing

Example 1: Using the Hamming window design a 21-point differentiator. The magnitude response of an ideal differentiator is shown below.




Solution

$$h_d'(n) = \begin{cases} \dfrac{\cos[\pi(n-\alpha)]}{(n-\alpha)} & n \neq \alpha \\ 0 & n = \alpha \end{cases} \qquad \& \quad \alpha = 10$$

$\therefore h(n) = h_d'(n)\, w(n) \qquad 0 \le n \le N-1$

$w(n) = 0.54 - 0.46 \cos\left[\dfrac{2\pi n}{N-1}\right] \qquad 0 \le n \le N-1$

$$\therefore h(n) = \begin{cases} \dfrac{\cos \pi(n-10)}{(n-10)} \times 0.54 - 0.46 \cos\left[\dfrac{2\pi n}{N-1}\right] & 0 \le n \le N-1,\ n \neq \alpha \\ 0 & n = \alpha. \end{cases}$$

Using the above equation we can determine the co-efficients of an FIR differentiator.

Example 2:

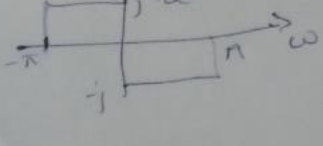
Using a rectangular window design a 11-tap Hilbert transformer. The magnitude response of an ideal HT is shown below. Determine

a. the transfer function of the FIR HT.

b. the difference equation realization for the FIR HT

c. expression for the magnitude frequency response.

$w(n) = \begin{cases} 1 & 0 \le n \le 10 \\ 0 & \text{otherwise} \end{cases}$




$\therefore h(n) = h_d'(n)\, w(n) \qquad 0 \le n \le N-1$

$$h(n) = \begin{cases} \dfrac{2 \sin^2\left(\frac{\pi(n-\alpha)}{2}\right)}{\pi (n-\alpha)} * w(n) & 0 \le n \le 10 \\ 0 & \text{otherwise.} \end{cases}$$

| N | $h(n) = hd(n)\,\omega(n)$ |
|---|---|
| 0 | $-0.1273 * 1$ |
| 1 | $0 * 1$ |
| 2 | $-0.2122 * 1$ |
| 3 | $0$ |
| 4 | $-0.6366 * 1$ |
| 5 | $0$ |
| 6 | $0.6366 * 1$ |
| 7 | $0$ |
| 8 | $0.2122 * 1$ |
| 9 | $0$ |
| 10 | $0.1273 * 1$ |

(a) $H(z) = \sum_{n=0}^{10} h(n) z^{-n}$

$$= -0.1273 - 0.2122 z^{-2} - 0.6366 z^{-4} + 0.6366 z^{-6} + 0.2122 z^{-8} + 0.1273 z^{-10}$$

$$= 0.1273 \left(z^{-10} - 1\right) + 0.2122\left(z^{-8} - z^{-2}\right) + 0.6366\left(z^{-6} - z^{-4}\right)$$

(b) $H(z) = Y(z) / X(z)$

$$\frac{Y(z)}{X(z)} = 0.1273 \left(z^{-10} - 1\right) + 0.2122\left(z^{-8} - z^{-2}\right) + 0.6366\left(z^{-6} - z^{-4}\right)$$

taking inverse z-transform

$$y(n) = -0.1273\left[x(n) - x(n-10)\right] - 0.2122\left[x(n-2) - x(n-8)\right] - 0.6366\left[x(n-4) - x(n-6)\right]$$

(c) $|H(\omega)| = |H_r(\omega)|$

$$= \left| 2 \sum_{n=0}^{\frac{N-3}{2}} h(n) \sin\left[\omega\left(\left(\frac{N-1}{2}\right) - n\right)\right]\right|$$

$$= \left| 2 \sum_{n=0}^{4} h(n) \sin\left[\omega(5-n)\right]\right|$$

$$= \left| -0.2546 \sin 5\omega - 0.4244 \sin 3\omega - 1.2732 \sin\omega \right|$$

## Example (3) MDP 6.13



A Hilbert transform is a filter with frequency response $H_d(\omega) = -j\,\text{sign}(\omega)$ where $\text{sign}(\omega) = \pm 1$ for $\omega \neq 0$ being the signum function.

ⓐ plot the magnitude and phase of the filter.

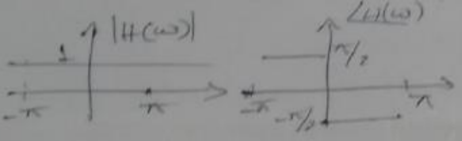
ⓑ Determine the impulse response $h_d[n]$;

ⓒ Determine the causal approximation $h(n)$ for $n = 0 \ldots N-1$ using a rectangular window $N = 40$

**Solution**

ⓐ Since $|H_d(\omega)| = |-j\,\text{sign}(\omega)| = 1$ for all $\omega$, i.e.

$$\text{Phase}\,(H(\omega)) = \begin{cases} -\pi/2 & \omega > 0 \\ +\pi/2 & \omega < 0 \end{cases}$$




ⓑ the impulse response

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} -j\,\text{sign}(\omega)\, e^{j\omega n}\, d\omega = \begin{cases} \dfrac{2\sin^2\left(\frac{n\pi}{2}\right)}{n\pi} & n \neq 0 \\ 0 & n = 0 \end{cases}$$

ⓒ $h(n) = h_d(n)\,\omega(n)$ where $\omega(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \end{cases}$

Example FIR differentiator

Use the window method with a Hamming window to design a 7-tap differentiator. The magnitude response of an ideal differentiator is shown.
Compute ~~and plot~~ the mag response of the resulting FIR differentiator.

~~$h_d'(n)$=~~ ~~$h(0)$~~

~~$h(n)$~~ = ~~0.0267~~

| $h_d'(n)$ | $w_H(n)$ | $h(n)$ |
|---|---|---|
| $\frac{1}{3}$ | 0.08 | 0.0267 |
| $-\frac{1}{2}$ | 0.31 | $-0.155$ |
| 1 | 0.77 | 0.77 |
| 0 | 1 | 0 |
| $-1$ | 0.77 | $-0.77$ |
| $\frac{1}{2}$ | 0.31 | 0.155 |
| $-\frac{1}{3}$ | 0.08 | $-0.0267$ |

$$H(\omega) = |0.0534 \sin 3\omega - 0.31 \sin 2\omega + 1.54 \sin \omega|$$

Example FIR differentiator

Use the window method with a Hamming window to design a 7-tap differentiator. The magnitude response of an ideal differentiator is shown. Compute ~~and plot~~ the mag response of the resulting FIR differentiator.

~~$h_d'(n) =$~~ ~~$h$~~

~~$h(n) = 0.0267$~~

| $h_d'(n)$ | $w_H(n)$ | $h(n)$ |
|---|---|---|
| $\frac{1}{3}$ | 0.08 | 0.0267 |
| $-\frac{1}{2}$ | 0.31 | −0.155 |
| 1 | 0.77 | 0.77 |
| 0 | 1 | 0 |
| −1 | 0.77 | −0.77 |
| $\frac{1}{2}$ | 0.31 | 0.155 |
| $-\frac{1}{3}$ | 0.08 | −0.0267 |

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Use the window method with a Hamming window to design a 7-tap differentiator. The magnitude response of an ideal differentiator is shown. Compute ~~and plot~~ the mag response of the resulting FIR differentiator.

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$$H(\omega) = |0.0534 \sin 3\omega - 0.31 \sin 2\omega + 1.54 \sin\omega|$$

37 * We are going to discuss the implementation of digital (1)
7) filters in a real-time application.

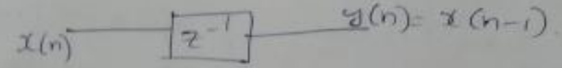
* Until now we had seen the design of digital filters.
* These digital filters are LTI discrete-time systems
* These systems are described by difference equations.
* The difference equations can ~~by~~ be implement on hardware or software.
* There are several ways to implement a difference equations.
* These different ways are called ~~diff~~ digital filter structures or realization structures
* The choice of ~~these~~ a specific realization is influenced by three major factors.
  (1) computational complexity
  (2) Memory requirements
  (3) finite-word-length effects or finite precision effects that refer to the quantization effects that are inherent in any digital implementation of the system, either in hardware or in software.

Apart from the aforementioned 3 factors, the factors that play a role in the selection of the specific implementation are,
(1) whether the structure or the realization lends itself to parallel processing.
(2) whether the ~~cs~~ computations can be pipelined

Elementary Operations:

The building block that form the basis of the implentation are

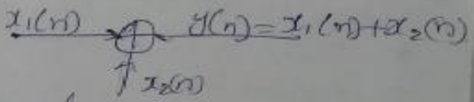
① Time Delay.




This requires one memory storage unit to store the signal for one time delay, corresponding to one sampling interval.

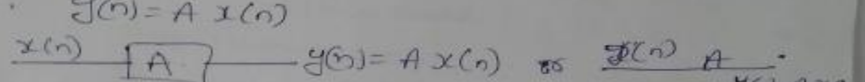
② Sum

$$y(n) = x_1(n) + x_2(n)$$




provides the sum of 2 i/p signals

③ Scaling: $y(n) = A\,x(n)$




Example 1:

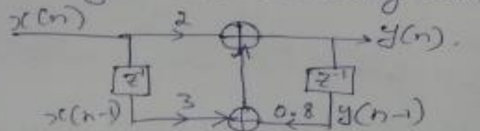
Consider the linear difference equation.

$$y(n) - 0.8\,y(n-1) = 2\,x(n) + 3\,x(n-1)$$

$x(n)$ – i/p signal & $y(n)$ – o/p signal

$$y(n) = 0.8\,y(n-1) + 2\,x(n) + 3\,x(n-1)$$

A realization is readily available by inspection




2-storage elements to store ~~for~~ $x(n)$ & $y(n)$ for one sampling interval.

"Brute force" realization

Basic FIR filter structures: 

An FIR system does not have feedback. ∴ the $y(n-k)$ will be absent. It is described by the difference eqn

$$\therefore \quad y(n) = \sum_{K=0}^{M-1} b_K\, x(n-K) \qquad \text{———— (1)}$$

$$Y(Z) = \sum_{K=0}^{M-1} b_K\, Z^{-k} X(Z)$$

$$H(Z) = \frac{Y(Z)}{X(Z)} = \sum_{K=0}^{M-1} b_K Z^{-K} \qquad \text{———— (2)}$$

This is the system function of the FIR ~~filter~~ system. Taking Inverse Z-transform of the above equation we get the unit sample response of FIR system

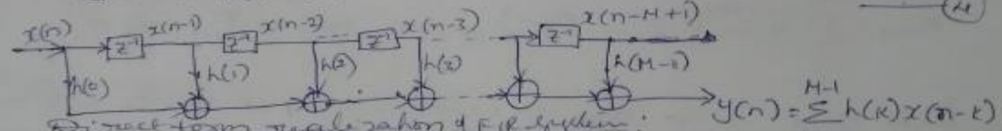
$$\text{i.e.,} \quad h(n) = \begin{cases} b_n & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{———— (3)}$$

## Direct form structure

This realization structure is obtained ~~dire~~ by implementing equation (1) directly.

$$y(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k)$$

$$\therefore \; y(n) = h(0)x(n) + h(1)x(n-1) + \cdots + h(M-1)\,x(n-M+1) \qquad \text{———— (4)}$$




Direct form realization of FIR system:

Number of storage elements required - $M-1$

—"— of multiplication — $M$

—"— of additions ———— $M-1$

$$H(z) = 1 + 2z^{-1} - 3z^{-2} - 4z^{-3} + 5z^{-4}$$

$$y(n) = x(n) + 2x(n-1) - 3x(n-2) - 4x(n-3) + 5x(n-4)$$

## Lattice Structure for FIR systems:

The lattice implementation of FIR filters is based on ~~the~~ a simple polynomial recursion.

let $A_m(z) = a(0) + a(1)z^{-1} + \dots + a(m)z^{-m}$ be a polynomial of degree m.

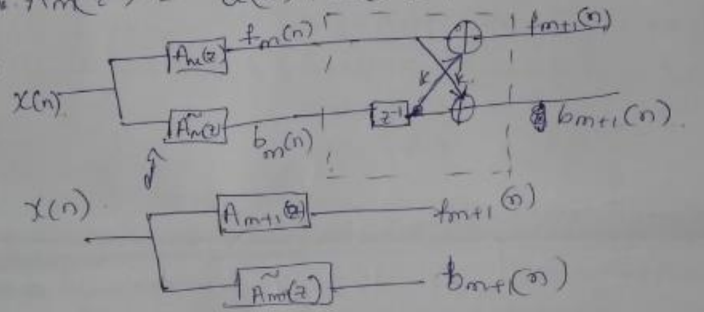
$$\tilde{A}_m(z) = a(m) + a(m-1)z^{-1} + \dots + a(0)z^{-m}.$$

Then,

a. $$\tilde{A}_m(z) = z^{-m} A_m(z^{-1})$$

where $A_m(z^{-1}) = a(0) + a(1)z + \dots + a(m)z^{m}$

b.




$$\therefore\ A_{m+1} = A_m(z) + K\tilde{A}_m(z)z^{-1}$$

$$\tilde{A}_{m+1} = \tilde{A}_m(z)z^{-1} + K A_m(z)$$

$$\tilde{A}_{m+1} = z^{-(m+1)} A_{m+1}(z^{-1}) = K A_m(z) + \tilde{A}_m(z) z^{-1}$$

The goal is to implement FIR filter with transfer function.

$$H(z) = 1 + h(1)z^{-1} + \dots + h(N)z^{-N}.$$

& represent it in terms of "reflection coefficients" $K_0, \dots K_{N-1}$.

$$\begin{bmatrix} A_{m+1}(z) \\ \tilde{A}_{m+1}(z) \end{bmatrix} = \begin{bmatrix} 1 & K \\ K & 1 \end{bmatrix} \begin{bmatrix} A_m(z) \\ z^{-1}\tilde{A}_m(z) \end{bmatrix}.$$

or
$$\begin{bmatrix} A_m(z) \\ z^{-1}\tilde{A}_m(z) \end{bmatrix} = \frac{1}{1-K^2} \begin{bmatrix} 1 & -K \\ -K & 1 \end{bmatrix} \begin{bmatrix} A_{m+1}(z) \\ \tilde{A}_{m+1}(z) \end{bmatrix}.$$

$$\begin{bmatrix} a_m(0) + a_m(1)z^{-1} + \cdots \\ 0 + a_m(m)z^{-1} + \cdots \end{bmatrix} = \frac{1}{1-K^2} \begin{bmatrix} 1 & -K \\ -K & 1 \end{bmatrix} \begin{bmatrix} a_{m+1}(0) + a_{m+1}(1)z^{-1} + \cdots \\ a_{m+1}(m+1) + a_{m+1}(m)z^{-1} + \cdots \end{bmatrix}$$

we determine K from the "zero" coefficient of the bottom row:

$$0 = -K\,a_{m+1}(0) + a_{m+1}(m)$$

$$K_m = \frac{a_{m+1}(m+1)}{a_{m+1}(0)}$$

$$A_N(z) = H(z) = 1 + h(1)z^{-1} + \cdots + h(N)z^{-N}$$

$$\tilde{A}_N(z) = \tilde{H}(z) = h(N) + h(N-1)z^{-1} + \cdots +$$

$$A_m(z) = \left(\frac{1}{1-K_m^2}\right)\left(A_{m+1}(z) - K\,\tilde{A}_{m+1}(z)\right)$$

Example

$$H(z) = 1 - 1.4z^{-1} + 0.26z^{-2} + 1.5444z^{-3} - 0.576z^{-4} - 0.4147z^{-5}$$

$$A_5(z) = H(z)$$

$$K_4 = \frac{a_5[5]}{a_5[0]} = -0.4147$$

$$A_4(z) = \frac{1}{1-K_4^2}\left(A_5(z) - K_4 \tilde{A}_5(z)\right)$$

$$= 1 - 1.9793 z^{-1} + 1.0873 z^{-2} + 1.9949 z^{-3} - 1.3969 z^{-4}$$

$K_3 = -1.3969$

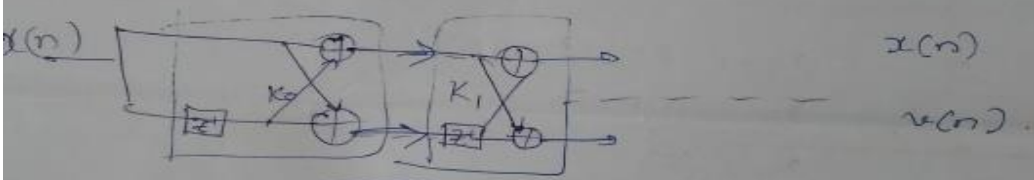
$A_3(z) = 1 - 0.8488 z^{-1} - 2.7399 z^{-2} + 0.8096 z^{-3}$

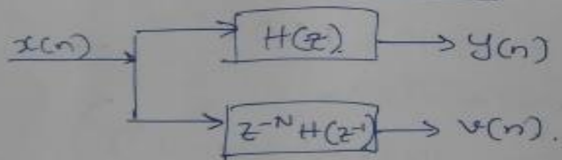
$K_2 = 0.8096$

$A_2(z) = 1 + 3.968 z^{-1} - 5.9514 z^{-2}$

$K_1 = -5.9514$, $A_1(z) = 1 - 0.8014 z^{-1}$

$K_0 = -0.8014$




(5.12) $A(z) = 1 - 1.8856 z^{-1} + 0.7728 z^{-2} + 0.8610 z^{-3} - 1.1221 z^{-4} + 0.5398 z^{-5} - 0.1296 z^{-6}$

$A_6(z) = A(z)$.

$\tilde{A}_6(z) = -0.1296 + 0.5398 z^{-1} - 1.1221 z^{-2} + 0.8610 z^{-3} + 0.7728 z^{-4} - 1.8856 z^{-5} + z^{-6}$

(4)

$$K_5 = \frac{a_6(6)}{a_6(0)} = -0.1296$$

$$A_5(z) = \frac{1}{1-K_5^2}\left(A_6(z) - K_5 \tilde{A}_6(z)\right)$$

$$= 1 - 1.8467z^{-1} + 0.6381z^{-2} + 0.9892z^{-3} - 1.0396z^{-4}$$
$$+ 0.3005z^{-5}$$

$K_4 = 0.3005$

$$A_4(z) = 1 - 1.8866z^{-1} + 0.3747z^{-2} + 0.8766z^{-3}$$
$$-0.5326z^{-4}$$

$K_3 = -0.5326$

$$A_3(z) = 1 - 1.7028z^{-1} + 0.8017z^{-2} - 0.0303z^{-3}$$

$K_2 = -0.0303$

$$A_2(z) = 1 - 1.6800z^{-1} + 0.7508z^{-2}$$

$K_1 = 0.7508$

$$A_1(z) = 1 - 0.9596z^{-1}$$

$K_0 = -0.9596$

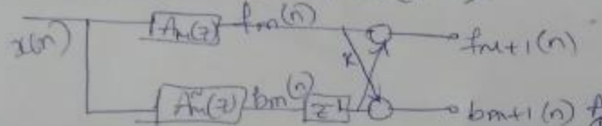
$A_0(z) = 1.$

$$A_m(z) = a_m(0) + a_m(1)z^{-1} - - - + a_m(m)z^{-m}$$

$$\tilde{A}_m(z) = a_m(m) + a_m(m-1)z^{-1} + \cdots + a_m(0)z^{-m}$$

$$A_m(z^{-1}) = a_m(0) + a_m(1)z + \cdots + a_m(m)z^{-m}.$$

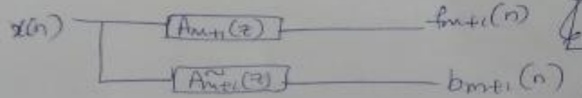
$$\tilde{A}_m(z) = z^{-m} A_m(z^{-1})$$




$$A_{m+1}(z) = A_m(z) + \tilde{A}_m(z)\, z^{-1} K$$

$$\tilde{A}_{m+1}(z) = A_m(z)K + z^{-1}\tilde{A}_m(z)$$

$$\tilde{A}_{m+1}(z) = z^{-m+1} A_{m+1}(z^{-1})$$




$$\begin{bmatrix} A_{m+1}(z) \\ \tilde{A}_{m+1}(z) \end{bmatrix} = \begin{bmatrix} 1 & K \\ K & 1 \end{bmatrix} \begin{bmatrix} A_m(z) \\ z^{-1}\tilde{A}_m(z) \end{bmatrix}.$$

$$H(z) = 1 + h(1)z^{-1} + h(2)z^{-2} + \cdots + h(N)z^{-N}$$

$k_0 \ldots k_{N-1}$

$$\begin{bmatrix} A_m(z) \\ z^{-1}\tilde{A}_m(z) \end{bmatrix} = \frac{1}{1-K^2} \begin{bmatrix} 1 & -K \\ -K & 1 \end{bmatrix} \begin{bmatrix} A_{m+1}(z) \\ \tilde{A}_{m+1}(z) \end{bmatrix}$$

$$\begin{bmatrix} \cancel{A_m(z)} \\ a_m(0) + a_m(1)z^{-1} - - - \\ 0 + a_m(m)z^{-1} + - - - - \end{bmatrix} = \frac{1}{1-K^2} \begin{bmatrix} 1 & -K \\ -K & 1 \end{bmatrix} \begin{bmatrix} a_{m+1}(0) + a_{m+1}(1)z^{-1} \\ a_{m+1}(m+1) + a_{m+1}(m)z^{-1} \end{bmatrix}$$

$$0 = -K a_{m+1}(0) + a_{m+1}(m+1).$$

$$K_m = \frac{a_{m+1}(m+1)}{a_{m+1}(0)}$$

$$A(z) = H(z) = 1 + h(1)z^{-1} + h(2)z^{-2} - - + h(N)z^{-N}$$

$$\tilde{A}(z) = \tilde{H}(z) = h(N) + h(N-1)z^{-1} + - - - -$$

$$A_m(z) = \left[\frac{1}{1-K_m^2}\right] \left[ A_{m+1}(z) - K \tilde{A}_{m+1}(z) \right]$$

(1) Consider the FIR filter with system function

$$H(z) = 1 + 2.88z^{-1} + 3.4048z^{-2} + 1.74z^{-3} + 0.4z^{-4}$$

Sketch the direct-form and lattice realizations of the filter and determine in detail the corresponding i/p o/p equns.

$$A_4(z) = H(z)$$

$$K_3 = 0.4$$

$$A_3(z) = 1 + 2.6z^{-1} + 2.432z^{-2} + 0.7z^{-3}$$

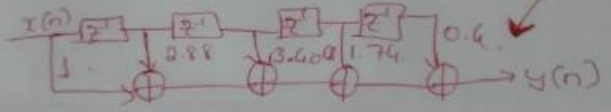
$$K_2 = 0.7$$

$$A_2(z) = 1 + 1.76z^{-1} + 1.2z^{-2}$$

$$K_1 = 1.2$$

$$A_1(z) = 1 + 0.8z^{-1} = \frac{1}{1-K_1^2}\left[A_2(z) - K_1\tilde{A}_2(z)\right].$$

$$K_0 = 0.8$$




$$\cancel{A_0(z) = \tilde{A}_0(z)}$$

(2) Determine all the FIR filters which are specified by the lattice parameters $K_0 = \frac{1}{2}$, $K_1 = 0.6$, $K_2 = 0.7$, $K_3 = \frac{1}{3}$

$$A_0(z) = \tilde{A}_0(z) = 1$$

$$A_1(z) = 1 + \underset{K_0}{\tfrac{1}{2}}z^{-1} = \frac{1}{1-(\frac{1}{2})^2}\left[A_2(z) - K_1\tilde{A}_2(z)\right]$$

$$A_1(z) = A_0(z) + K_0 z^{-1}\tilde{A}_0(z)$$

$$= 1 + \frac{1}{2}z^{-1}$$

$$A_2(z) = A_1(z) + K_1 z^{-1}\tilde{A}_1(z)$$

$$= \left(1 + \frac{1}{2}z^{-1}\right) + (0.6)z^{-1}\left(\frac{1}{2} + z^{-1}\right)$$

$$= 1 + \frac{1}{2}z^{-1} + \frac{0.6}{2}z^{-1} + 0.6z^{-2}$$

$$= 1 + 0.8z^{-1} + 0.6z^{-2}$$

$$A_3(z) = A_2(z) + K_2\tilde{A}_2(z)$$

$$= \{1 + 0.8z^{-1} + 0.6z^{-2}\} + 0.7z^{-1}\{0.6 + 0.8z^{-1} + z^{-2}$$

9.16) Consider ~~the~~ FIR lattice filter with co-efficients $K_0 = 0.65$, $K_1 = -0.34$, $K_2 = 0.8$.

ⓐ find the impulse response ~~by drawn~~ of the filter

ⓑ Draw the equivalent direct-form structure

$$A_0(z) = \tilde{A}_0(z) = 1$$

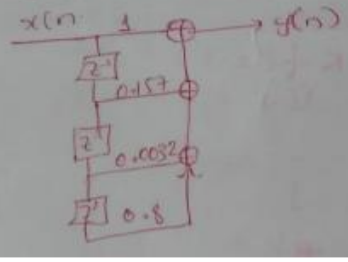
$$A_1(z) = A_0(z) + K_0 z^{-1} \tilde{A}_0(z)$$
$$= 1 + 0.65 z^{-1}$$

$$A_2(z) = A_1(z) + K_1 z^{-1} \tilde{A}_1(z) = 1 + 0.65z^{-1} + (-0.34)z^{-1}(0.65 + z^{-1})$$
$$= 1 + 0.65z^{-1} - 0.34 \times 0.65 z^{-1} - 0.34z^{-2}$$
$$= 1 + 0.43z^{-1} - 0.34z^{-2}$$

$$H(z) = A_3(z) = A_2(z) + K_2 z^{-1} \tilde{A}_2(z)$$
$$= 1 + 0.43z^{-1} - 0.34z^{-2} + 0.8z^{-1} + 0.8 \times 0.43 z^{-2} - 0.34 \times 0.8 \times z^{-3}$$

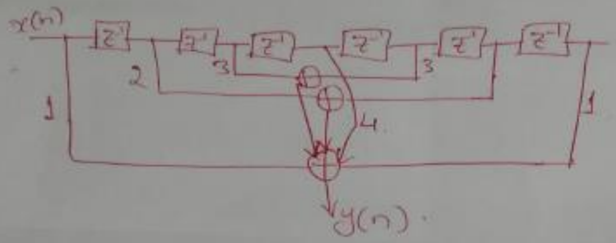
0.344 0.272

$$= 1 + 1.23z^{-1} + 0.004z^{-2} - 0.272z^{-3}$$
$$= 1 + 0.157z^{-1} + 0.0032z^{-2} + 0.8z^{-3}$$



$h(n) = \{1, 2, 3, 4, 3, 2, 1\}$

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 3z^{-4} + 2z^{-5} + z^{-6}$$




$$H(z) = 1 + 2.88z^{-1} + 3.4048z^{-2} + 1.74z^{-3} + 0.4z^{-4}$$

$A_4(z) = H(z)$

$\tilde{A}_4(z) = 0.4 + 1.74z^{-1} + 3.4048z^{-2} + 2.88z^{-3} + z^{-4}$

$\boxed{K_3 = \underline{0.4}}$

$$= \frac{1}{1-0.4^2}\Big[1 - 2.88z^{-1} + 3.4048z^{-2} + 1.74z^{-3} + 0.4z^{-4} - 0.4(0.4 + 1.74z^{-1} + 3.4048z^{-2} + 2.88z^{-3} + z^{-4})\Big]$$

$$= \frac{1}{1-0.16}\Big[1 - 2.88z^{-1} + 3.4048z^{-2} + 1.74z^{-3} + \cancel{0.4z^{-4}} - 0.16 - 0.696z^{-1} - 1.3619z^{-2} - 1.152z^{-3} \cancel{- 0.4z^{-4}}$$

$$= 0.84 - 3.576z^{-1} -$$

## Location of zeros of linear-phase FIR transfer functions

FIR filter length-M (filter order = M-1)

* (M-1) poles at the origin
* (M-1) zeros located in the Z-plane :

The impulse response of a linear-phase filter is either symmetric or antisymmetric

i.e., $h(n) = \pm h(M-1-n)$

$$h(n) \longleftrightarrow H(z)$$

$$h(n+M-1) \longleftrightarrow z^{M-1} H(z) \rightarrow \text{time shift property}$$

$$h(-n+M-1) \longleftrightarrow z^{-(M-1)} H(z^{-1}) \rightarrow \text{time reversal property}$$

$$h(n) = \pm h(M-1-n) \quad 0 \le n \le M-1$$

$$H(z) = \pm z^{-(M-1)} H(z^{-1}) = \pm z^{-(M-1)} H\left(\frac{1}{z}\right)$$

i.e., $H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$

$$= h(0) + h(1) z^{-1} + \cdots + h(N-1) z^{-(N-1)}$$

$$h(n) = h(M-1-n)$$

$$H(z) = \sum_{n=0}^{M-1} h(M-1-n) z^{-n}$$

$$= h(M-1) + h(M-2) z^{-1} + \cdots + h(0) z^{-(M-1)}$$

$$= z^{-(M-1)} \left[ h(0) + h(1) z^{1} + \cdots + h(M-1) z^{(M-1)} \right]$$

$$H(z) = z^{-(M-1)} H(z^{-1}) = z^{-(M-1)} H\left(\frac{1}{z}\right)$$

Zeros of $H(z)$ must appear in reciprocal pairs.

thus if $z_0$ is a zero of $H(z)$, $\frac{1}{z_0}$ is also a zero.

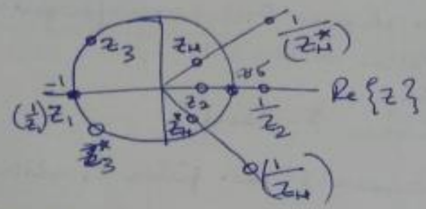
4 typical cases are

(1) $z_1 = -1$, then $z_1^{-1} = -1$, $\Rightarrow$ zero $z_1$ lies at $-1$

(2) $z_2$ is a real zero with $|z_2| < 1$, then $z_2^{-1}$ is also a zero.

(3) $z_3$ is a complex zero with $|z_3| = 1$, then $z_3^{-1} = z_3^*$.

(4) $z_4$ is a complex zero with $|z_4| \neq 1$, then we have 4 zeros: $z_4, z_4^{-1}, z_4^*, (z_4^*)^{-1}$




1) $z = z_1$, $z = \frac{1}{z_1}$

∵ $h(n) = \pm h(M-1-n)$

$H(z) = \pm z^{-(M-1)} H(z^{-1})$

$H(z) = \pm z^{-(M-1)} H(\frac{1}{z})$

2) $h(n)$ is real in most applications

$h(n) = h^*(n)$

$H(z) = H^*(z^*)$

Thus a zero at $z = z_3$ is associated with a zero at $z = z_3^*$.

3) $h(n)$ is real

$h(n) = \pm h(M-1-n) = h^*(n) = \pm h^*(M-1-n)$

$H(z) = \pm z^{-(M-1)} H(\frac{1}{z}) = H^*(z) = \pm (z^*)^{-(M-1)} H^*(\frac{1}{z^*})$

If there is zero at $z = z_4 = re^{j\theta}$, then there must be a zero at $z = \frac{1}{z_4} = \frac{1}{r}e^{-j\theta}$, $z = z_4^* = re^{-j\theta}$, $z = \frac{1}{z_4^*} = \frac{1}{r}e^{j\theta}$.

$z_4 = re^{j\theta}$, $\frac{1}{z_4} = \frac{1}{r}e^{-j\theta}$, $z_4^* = re^{-j\theta}$, $\frac{1}{z_4^*} = \frac{1}{r}e^{j\theta}$

4) If $r = 1$ the $\frac{1}{r} = 1$, ∴ zeros are on unit circle i.e., $z = z_3 = e^{j\theta}$, $|z_3| = 1$, then $z_3^{-1} = z_3^*$. There are 2 zeros in their group, namely,

$z_3 = e^{j\theta}$ & $z_3^{-1} = z_3^* = e^{-j\theta}$

If zeros are on the real line and occur in pairs, $\theta = 0$ or $\theta = \pi$, A real zero is paired with its reciprocal zero appearing at $z_2 = r$, $z_2^{-1} = \frac{1}{r}$

If $r = 1$, & $\theta = 0$ or $\theta = \pi$, zeros are either at $z = z_5 = 1$ or $z = z_1 = -1$. Note, at that a zero at $z = \pm 1$ is its own reciprocal.

# Window Summary

Comparison of all the windows:-

| WINDOW | $\Delta\omega$ | fn of window | Window first slobe $A_{dB}$ | LPF $A_s$ |
|---|---|---|---|---|
| Rectangular | $\frac{4\pi}{N}$ | $1$ | 13 | 21 |
| Bartlet | $\frac{8\pi}{N}$ | $1 - 2\frac{\left\| n - \frac{(N-1)}{2} \right\|}{N-1}$ | 27 | 25 |
| Hanning | $\frac{8\pi}{N}$ | $.5 - .5\cos\frac{2\pi n}{N-1}$ | 32 | 44 |
| Hamming | $\frac{8\pi}{N}$ | $.54 - .46\cos\frac{2\pi n}{N-1}$ | 43 | 53 |
| Blackman | $\frac{12\pi}{N}$ | $.42 - .5\cos\frac{2\pi n}{N-1} + .08\cos\frac{4\pi n}{N-1}$ | 58 | 74 |

# Practise problem-1

Design a LP digital filter to be used in an A/D → H(z) → D/A structure that will have a −3db cut off of 30π rad/sec and an attenuation of 50db at 45π rad/sec. Filter should have linear phase and s/m will use a sampling rate of 100 samples/s

$\omega_c = \Omega_c T = .3\pi$ rad. $\geq -3$db

$\omega_s = \Omega_s T = .45\pi$ rad $\leq -50$db.

We can use Hamming, Blackman, Kaiser window for −50db But use Hamming window as it has smallest transition band hence smallest N.

$$N \geq \frac{8\pi}{(.45-.3)\pi} = 53.3 \approx 55 \text{ (to get integer delay)}$$

Taking $\omega_c = \omega_1 = .3\pi$

$$\alpha = \frac{N-1}{2} = 27$$

For hamming window,

$$h(n) = \frac{\sin[.3\pi(n-27)]}{\pi[n-27]}\left[.54 - .46\cos\frac{2\pi n}{54}\right]$$

Practise problem-2

1) Obtain coeff of an FIR filter to meet the following specs. $F_s = 8 KH_z$ Pass band edge freq $1.5 KH_z$ Stop band edge freq $2 KH_z$ Mini. stop band attenuation 50 db.

$$\omega_p = \frac{2\pi f_p}{F_s} = \frac{2\pi \times 1.5 \times 10^3}{8 \times 10^3} = .375\pi \text{ rad.}$$
$(\Omega_p T)$

$$\omega_s = \frac{2\pi f_s}{F_s} = \frac{2\pi \times 2 \times 10^3}{8 \times 10^3} = .5\pi \text{ rad.}$$

Choosing Hamming window

$$N \geq \frac{8\pi}{\omega_s - \omega_p} \geq 64 \simeq 65 \quad (\alpha \text{ to be integer})$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega = \frac{\sin \omega_c (n-\alpha)}{\pi (n-\alpha)} \quad n \neq \alpha$$

$$= \frac{\omega_c}{\pi} \quad n = \alpha$$

$$\omega_c = \omega_p + \frac{\Delta\omega}{2} = .4375\pi \text{ rad.}$$

$$\alpha = \frac{N-1}{2} = 32$$

$$h(n) = h_d(n)\left[.54 - .46 \cos \frac{2\pi n}{N-1}\right]$$

Practise problem-3

Find expression for the impulse response $h(n)$ of a linear phase L.P.F. using Kaiser window to satisfy the following specs for the equivalent analog filter. Stop band attenuation 40 db

Pass band ripple .01 db.

Transition width $1000\pi$ r/s.

Ideal cut off freq. $2400\pi$ r/s.

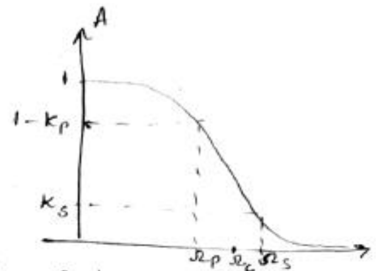
$F_s = 10 KH_z$

$A_s = -20\log K_s = 40\,db \qquad \therefore K_s = .01$

$\omega_c = \Omega_c T = 2400\pi \times \frac{1}{10k} = .24\pi \text{ rad}$

$\Delta\omega = \Delta\Omega T = 1000\pi \times \frac{1}{10k} = .1\pi \text{ rad}$

$\Delta f = \frac{.1\pi}{2\pi}$




$A_p = -20\log(1-\cancel{K_p}) = .01\,db$

$1 - K_p = -\frac{.01}{20}$

$\therefore K_p = .00115$

$K = \text{mini}[K_s, K_p] = .00115 \qquad \therefore -20\log(.00115) = 58.8db$

$\therefore N \geq \frac{A - 7.95}{14.36\Delta f} \geq \frac{58.8 - 7.95}{14.36 \times \frac{.1\pi}{2\pi}} \geq 70.82 \simeq 71$

$\therefore \alpha = \frac{N-1}{2} = 35.$

For 40db attenuation $N = \frac{40 - 7.95}{14.36 \times \frac{.1\pi}{2\pi}} = 44.8 \simeq 45$

Forcing $K_p = K_s$ leads to higher $N$

For $A > 50\,db \qquad \beta = .1102(58.8 - 8.7)$
$= 5.48$

$$w_k(n) = \frac{I_0\left\{\beta\sqrt{1-\left(\frac{n-\alpha}{\alpha}\right)^2}\right\}}{I_0\{\beta\}} = \frac{I_0\left\{5.48\sqrt{1-\left(\frac{n-35}{35}\right)^2}\right\}}{I_0\{5.48\}}$$

For LPF

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha} & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

$$h_d(n) = \frac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)} \quad n \neq \alpha \qquad = \frac{\omega_c}{\pi} \quad n = \alpha$$

$$\omega_c' = \omega_c + \frac{\Delta\omega}{2}$$
$$= .24\pi + \frac{.1\pi}{2} = .29\pi$$

To account for smearing effect of window fn.

$$h_d(n) = \frac{\sin .29\pi(n-35)}{\pi(n-35)} \qquad n \neq 35$$
$$= .29 \qquad n = 35$$

where $I_0(\beta) = 1 + \sum_{n=1}^{\infty}\left[\left(\frac{\beta}{2}\right)^n \frac{1}{n!}\right]^2$

Finally

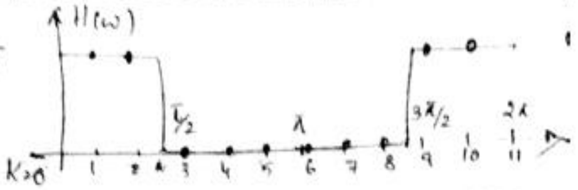
$h(n) = h_d(n)\, w(n)$

# Practise problem-4

1) Design a LPF based on freq. sampling technique

$$H_d(\omega) = \begin{cases} e^{-j\alpha\omega} & 0 < \omega < \pi/2 \\ 0 & \pi/2 < \omega < 3\pi/2 \end{cases} \qquad N = 11$$

$$\alpha = \frac{N-1}{2} = \frac{10}{2} = 5$$



$$\omega_k = \frac{2\pi k}{N} = \frac{2\pi k}{11}$$

$$\omega_k = [0, .18\pi, .36\pi, .55\pi, .73\pi, .91\pi, 1.09\pi, 1.27\pi, 1.45\pi, 1.64\pi, 1.82\pi]$$

$0 \le \omega \le .5\pi \qquad k = 0, 1, 2$

$0.5\pi \le \omega \le 1.5\pi \qquad k = 3, 4, 5, 6, 7, 8$

$1.5\pi \le \omega \le 2\pi \qquad k = 9, 10$

$$H(k) = H_d(e^{j\omega})\Big|_{\omega=\omega_k} = e^{-j\omega\alpha k} = e^{-j\frac{2\pi k \times 5}{11}} \quad \text{for } 0, 1, 2, 9, 10$$

$$= 0 \quad \text{for } 3, 4, 5, 6, 7, 8$$

$$h(n) = \frac{1}{N}\left\{H(0) + 2\sum_{k=0}^{(N-1)/2} Re\left[H(k)\, e^{j2\pi nk/N}\right]\right\}$$

$$= \frac{1}{11}\left\{1 + 2\sum_{k=0}^{5} Re\left[H(k)\, e^{j2\pi nk/N}\right]\right\}$$

$$= \frac{1}{11}\left\{1 + 2\sum_{k=1}^{2} Re\left[e^{-j5\,2\pi k/11}\; e^{j2\pi nk/11}\right]\right\}$$

$$= \frac{1}{11}\left\{1 + 2\sum_{k=1}^{2}\left[e^{j\frac{2\pi k}{11}(n-5)}\right]\right\}$$

$$h(n) = \frac{1}{11}\left\{1 + 2\cos\frac{2\pi(n-5)}{11} + 2\cos\frac{4\pi(n-5)}{11}\right\}$$

for $n = 0 \ldots 10$ Symmetry at $n = 5$

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} = \sum_{n=0}^{10} h(n)\, z^{-n}$$

Practise problem-5

2) Desired HPF freq. response

$H(K) = (0, 0, 0, 1, 1, 1, 1, 1, 0, 0)$

Find antisymmetric impulse response using freq. sampling technique.

w.k.t $H(K) = |H(K)| e^{j\theta_K}$ shifted sequence.

$$\theta_K = -\alpha \omega_K + \pi/2$$

$$= -\left(\frac{N-1}{2}\right)\frac{2\pi K}{N} + \frac{\pi}{2} \quad \text{for } 0 \leq K \leq 5$$

$$= .5\pi - .9\pi K$$

$H(K) = 0$ for $K = 0, 1, 2$

$H(K) = 1 \times e^{j(.5\pi - .9\pi K)}$ for $K = 3, 4, 5$

$H(6) = H^*(4) = e^{j3.1}$

$H(7) = H^*(3) = e^{j2.2\pi}$

$H(8) = H^*(2) = 0$

$H(9) = H^*(1) = 0.$

$$h(n) = \frac{1}{N}\sum_{K=0}^{N-1} H(K) e^{j\frac{2\pi K n}{N}} = \frac{1}{10}\sum_{K=0}^{9} H(K) e^{j\frac{2\pi K n}{10}}$$

$$= \frac{1}{10}\left[H(3) e^{j\frac{6\pi n}{10}} + H(4) e^{j\frac{8\pi n}{10}} + H(5) e^{j\frac{10\pi n}{10}} + H(6) e^{j\frac{12\pi n}{10}} + H(7) e^{j\frac{14\pi n}{10}}\right]$$

$$= \frac{1}{10}\left[\cos(.6\pi n - 2.2\pi) + \cos(.8\pi n - 3.1\pi) + \cos(\pi n - 4\pi) + \cos(1.2\pi n - 3.1\pi) + \cos(1.4\pi n + 2.2\pi)\right]$$

Practise problem-6

1) Determine coeff. $K_m$ of lattice FIR filter. $H(z) = 1 + 2z^{-1} + \frac{z^{-2}}{3}$

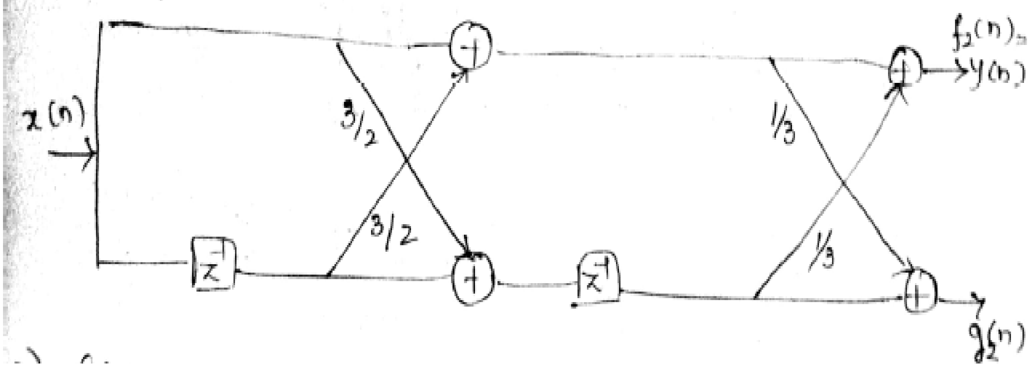
Here $\alpha_2(1) = 2$ ; $\alpha_2(2) = \frac{1}{3}$. & $M = 2$. For $m = 2$

$K_m = \alpha_m(m)$ $\qquad \alpha_{m-1}(i) = \dfrac{\alpha_m(i) - \alpha_m(m)\alpha_m(m-i)}{1 - K_m^2}$

$K_2 = \alpha_2(2) = \underline{\frac{1}{3}}$. $\qquad \alpha_1(1) = \dfrac{\alpha_2(1) - \alpha_2(2)\alpha_2(1)}{1 - K_2^2}$ $\qquad 1 \le i \le m-1$

For $m = 1$

ie $\quad K_1 = \dfrac{\alpha_2(1)[1 - \alpha_2(2)]}{1 - \alpha_2(2)^2} = \dfrac{\alpha_2(1)}{1 + \alpha_2(2)} = \underline{\frac{3}{2}}$



Practise problem-7

2) Given $y(n) = x(n) + 3.1x(n-1) + 5.5x(n-2) + 4.2x(n-3) + 2.3x(n-4)$

Here $\frac{Y(z)}{X(z)} = 1 + 3.1z^{-1} + 5.5z^{-2} + 4.2z^{-3} + 2.3z^{-4}$

Here $M = 4$ $\quad K_4 = 2.3$. For $m = 4$

$$\alpha_3(i) = \frac{\alpha_4(i) - \alpha_4(4)\alpha_4(4-i)}{1 - K_4^2} \qquad 1 \le i \le 3$$

$\alpha_3(1) = 1.529 \quad \alpha_3(2) = 1.667 \quad \alpha_3(3) = 0.683$.

For $m = 3$ $\qquad K_3 = 0.683$

$\alpha_2(1) = .732 \qquad \alpha_2(2) = 1.167$

$m = 2 \qquad K_2 = 1.167$

$\alpha_1(1) = 0.338 = K_1$