CONTROL SYSTEMS

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UNIT 4: THE STABILITY OF LINEAR FEEDBACK SYSTEMS

THE ROOT LOCUS TECHNIQUE

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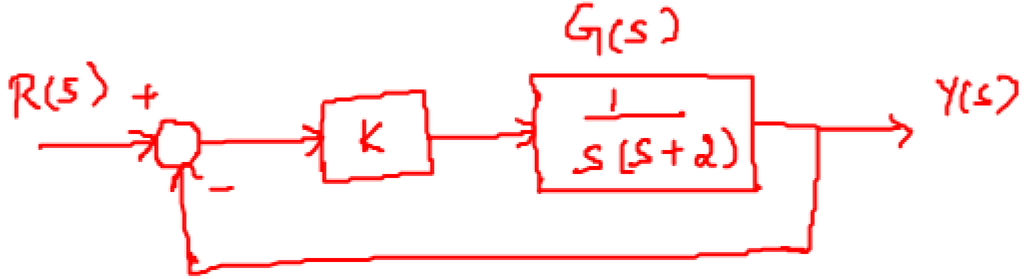
Root Locus Technique

- The Root Locus technique is a graphical method for sketching the locus of the roots in the s-plane as a parameter K is varied.

- This method provides a measure of the sensitivity of the roots of the systems to a variation in the parameter being considered. It is used to a great advantage in conjunction with the Routh-Hurwitz criterion.

- The root locus method provides graphical information and therefore an approximate sketch can be used to obtain qualitative information concerning the stability and performance of the system.

- The dynamic performance of a closed loop control system is described by the transfer function $T(s) = \frac{Y(s)}{R(s)} = \frac{p(s)}{q(s)}$ where p(s) and q(s) are polynomials .



- We have the characteristic equation

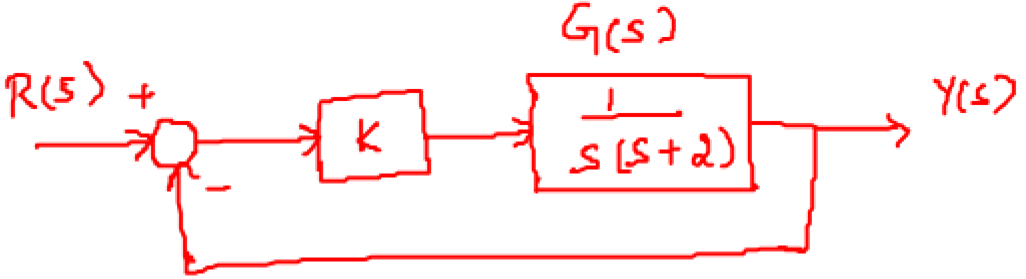
$$1 + K\,G(s) = 0$$

- where K is a variable parameter

Root Locus Technique

Order of deno = n
order of num = m

- The root locus is the path of the roots of the characteristic equation traced out in the s-plane as a system parameter is changed.



- The angle of *G(s)H(s)=-1* is

$$\angle G(s)H(s) = \angle -1$$

$$\angle G(s)H(s) = \pm 180^{\circ}(2k+1)$$

- Where *k*=1,2,3…

- The magnitude of *G(s)H(s)=-1* is

$$|G(s)H(s)| = |-1|$$

$$|G(s)H(s)| = 1$$

$$1 + K\,G(s) = 0$$

$$K\,G(s) = -1$$

$$|KG(s)| = 1$$

$$\angle KG(s) = 180^{\circ}(2q+1)$$

$$q = 0, 1, 2 \cdots n - m - 1$$

Root Locus Technique

- The angle criterion is used to determine the **departure angles** for the parts of the root locus near the open-loop poles and the **arrival angles** for the parts of the root locus near the open-loop zeros.
- When used with the magnitude criterion, the angle criterion can also be used to determine whether or not a point in the s-plane is on the root locus.
- Note that +180° could be used rather than -180°. The use of -180° is just a convention. Since +180° and -180° are the same angle, either produces the same result.

THE STABILITY OF LINEAR FEEDBACK SYSTEMS

Construction of Root Locus

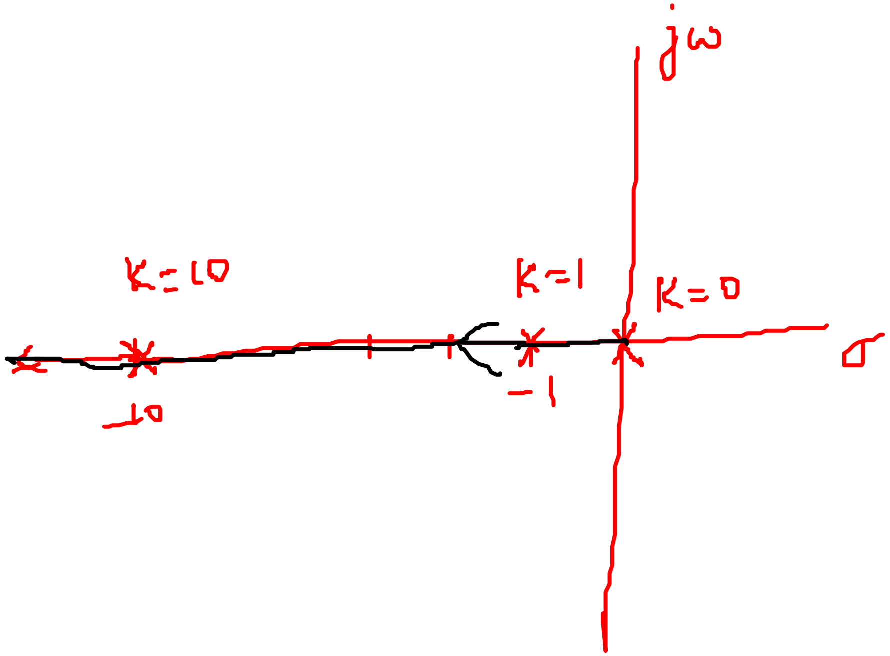
- The root locus is the path of the roots of the characteristic equation traced out in the s-plane as a system parameter is changed.
- For example, $G(s) = \frac{1}{s}$ and H(s) = 1

C.E $\quad 1 + K\,G(s) = 0$

$$1 + K\frac{1}{s} = 0$$

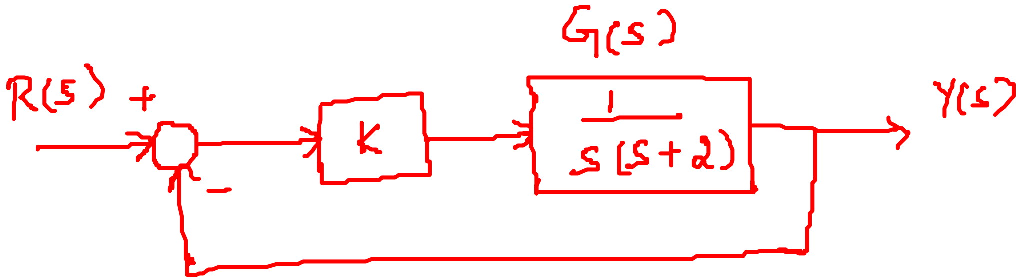
$$s + K = 0$$

K	S = -K
0	0
1	-1
10	-10
⋮	
∞	$-\infty$



Construction of Root Locus

- Ex 2, Draw the locus of poles of the closed loop system shown in figure by varying K from 0 to ∞.



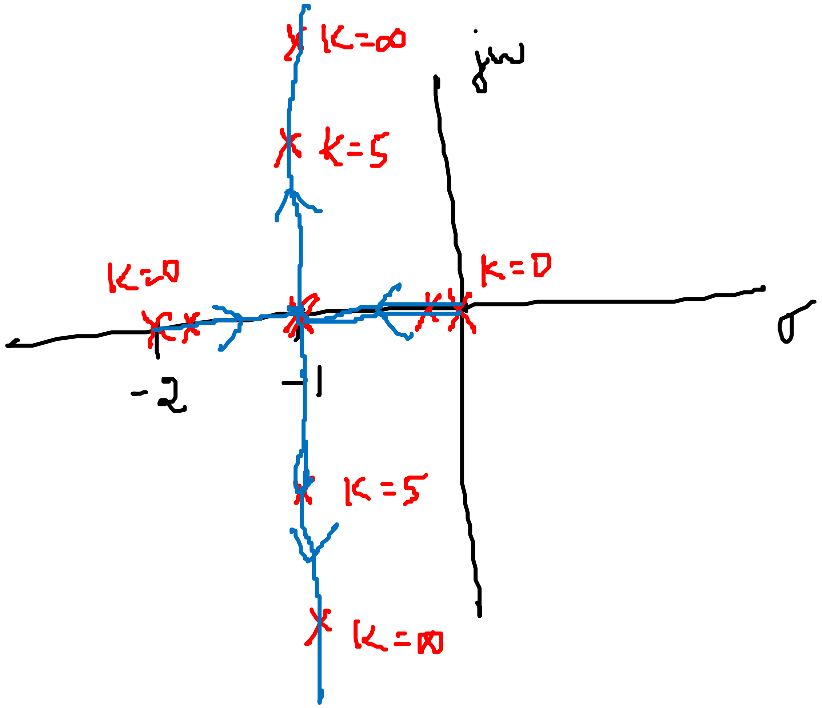
C.E, $1 + K\,G(s) = 0$

$$1 + K \frac{1}{s(s+2)} = 0$$

$$s^2 + 2s + K = 0$$

$$S_1 = -1 + \sqrt{1-K}, \quad S_2 = -1 - \sqrt{1-K}$$

K	S_1	S_2
0	0	−2
0.5	−0.29	−1.707
1	−1	−1
5	$-1+j2$	$-1-j2$
∞	$-1+j\infty$	$-1-j\infty$



THE STABILITY OF LINEAR FEEDBACK SYSTEMS

THE ROOT LOCUS PROCEDURE

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Root Locus Procedure

- To locate the roots of the characteristic equation in a graphical manner on the s-plane, we will define seven steps that facilitates the rapid sketching of the locus.

STEP 1: Finding the number of loci and its characteristics

consider n = number of poles and m = number of zeros

❖When n>m

Number of loci= Number of poles=n

❖When m>n

Number of loci=Number of zeros=m

❖When m=n

Number of loci =m=n

Note:-

Root loci must be symmetrical with respect to the horizontal real axis because the complex roots must appear as pairs of complex conjugate roots.

Root Locus Procedure

- **STEP 2: Finding the starting and terminal points of root loci**

Write the characteristic equation.

$$1 + KG(s) = 0$$

Where G(s) is the open loop transfer function.

We are usually interested in determining the locus of roots as K varies from

$0 < K < \infty$.

- Write the polynomial in the form of poles and zeros as follows:

$$1 + K\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 0$$

Where z_i are zeros and p_j are poles.

Root Locus Procedure

- Re writing,

$\sum_{j=1}^{n}(s+p_j)$ + K($\sum_{i=1}^{n}(s+z_i)$)=0

→When K=0

$$\sum_{j=1}^{n} s+p_j=0$$

When solved, this yields the values of s that coincide with the poles of G(s).

→When K=∞

$$(\sum_{i=1}^{n} s+z_i)=0$$

When solved, this yields the values of s that coincide with the zeros of G(s).

The locus of the roots of the characteristic equation 1 + KG(s) = 0 begins at the poles of G(s) and ends at the zeros of G(s) as K increases from zero to infinity.

STEP 3: Existence of root locus on real axis

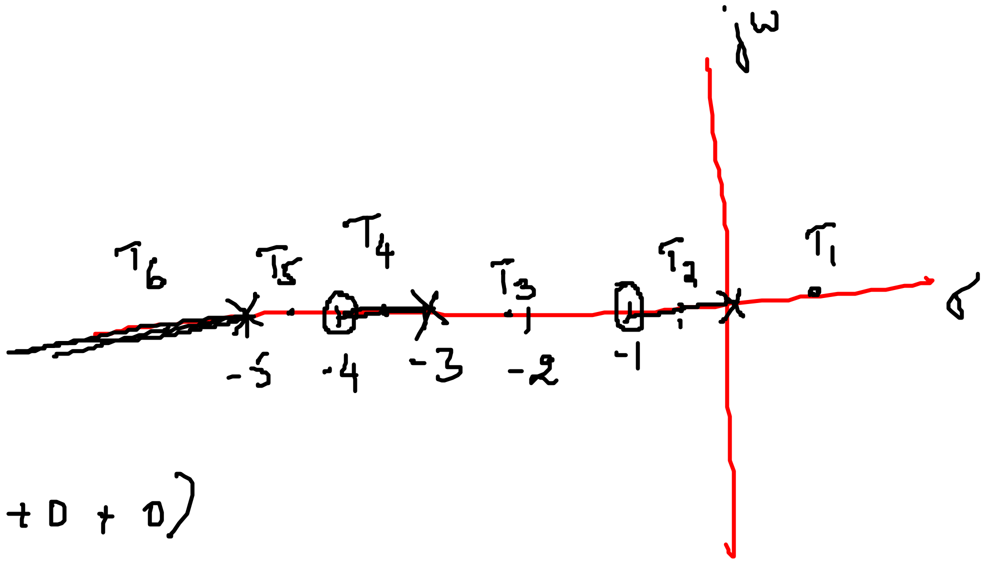
- If odd number of the open loop poles and zeros exist to the right side of a point on the real axis, then that point is on the root locus branch.
- Consider a point T on real axis, Any pole/zero on the right of this point T ,has angle contribution of 180 degrees and any pole/zero on the left of this point ,has angle contribution of 0 degrees.

THE STABILITY OF LINEAR FEEDBACK SYSTEMS

Root Locus Procedure

Ex, $G(s)H(s) = \dfrac{K(s+1)(s+4)}{s(s+3)(s+5)}$

$$\angle G(s)H(s) = (\angle K + \angle s+1 + \angle s+4) - (\angle s + \angle s+3 + \angle s+5)$$



Point T_2, $\angle G(s)H(s) = 0 + 0 + 0 - (180^\circ + 0 + 0)$

$= -180^\circ$

Root Locus Procedure

For point T ,

- Total angle = (angle contributions of zeros)-(angle contributions of poles) If the angle of the open loop transfer function at a point is an odd multiple of 180^0, then that point is on the root locus.
- Therefore, the branch of points which satisfies this condition is the real axis of the root locus branch.
- Complex conjugate poles and zeros have no effect on the location of root loci ,as their angle contribution is equal and opposite.

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Step 4: Determination of centroid and angle of asymptotes

- The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A,
- Where, ϕ_A is angle of asymptotes w.r.t real axis. Then,
- $\sigma_A = \dfrac{\sum poles\ of\ G(s) - \sum zeros\ of\ G(s)}{n-m}$ and
- $\phi_A = \dfrac{(2q+1) * \pi}{n-m}$ Where q=0,1, ….(n-m-1)

$\phi_A = \pi$

$q = \cancel{0,1,}(3-2-1)$

$q = 0$

$G(s) = \dfrac{K(s+1)(s+4)}{s(s+3)(s+5)}$

$\sigma_A = \dfrac{0+(-3)+(-5)-(-1-4)}{3-2}$

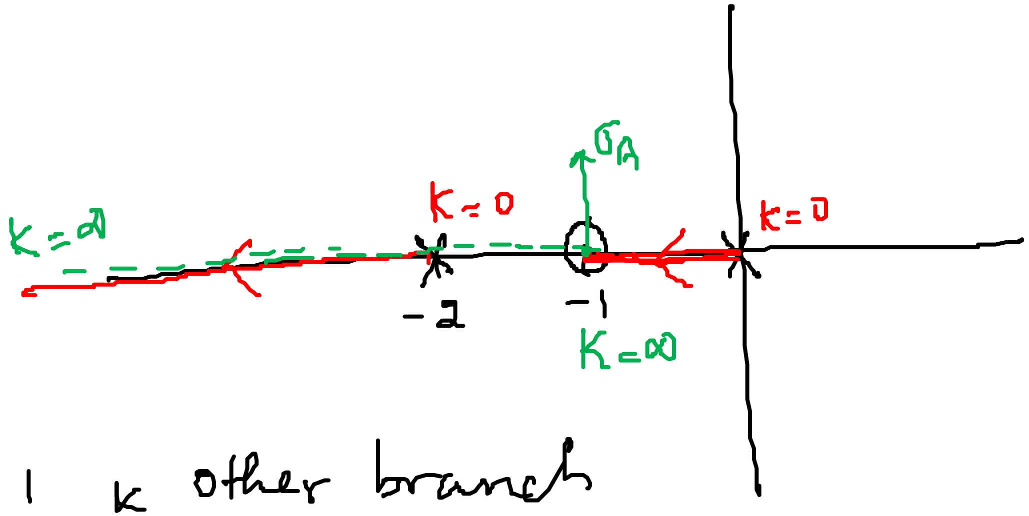
$= \dfrac{-8+5}{1} = -3$

Root Locus Procedure

Ex, Draw the root locus for the given OLTF

3

$$G(s) = \frac{s+1}{s(s+2)}$$



1. $n = 2$, $s = 0, -2$

   $m = 1$, $s = -1$

   no. of branches = 2

2. One branch terminates at -1 & other branch terminates at ∞

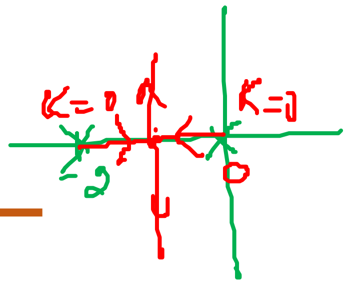
4. $\sigma_A = \frac{-2+1}{1} = -1$

   $\phi_A = \pi$

THE STABILITY OF LINEAR FEEDBACK SYSTEMS

Root Locus Procedure

$G(s) = \frac{1}{s(s+2)}$



- **STEP 5:Find Break-away and Break-in points**

- If there exists a real axis root locus branch between two open loop poles, then there will be a **break-away point** in between these two open loop poles.

- If there exists a real axis root locus branch between two open loop zeros, then there will be a **break-in point** in between these two open loop zeros.

- **Note** –

$G(s) = \frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}$

Break-away and break-in points exist only on the real axis root locus branches.

Root Locus Procedure

Steps to find break-away and break-in points are:

1. Write K in terms of s from the characteristic equation , $1+G(s)H(s)=0$
2. Differentiate K with respect to s and make it equal to zero. Substitute these values of s in the above equation.
3. The values of s for which the K value is positive are the **break points**.

or

$1 + K\,G(s) = 0$

$1 + K \frac{N(s)}{D(s)} = 0$

$K = \frac{-D(s)}{N(s)}$

$\frac{dK}{ds} = 0$

THE STABILITY OF LINEAR FEEDBACK SYSTEMS

Root Locus Procedure

Ex, Draw the root locus for the given OLTF

$$G(s) = \frac{1}{s(s+1)(s+2)}$$

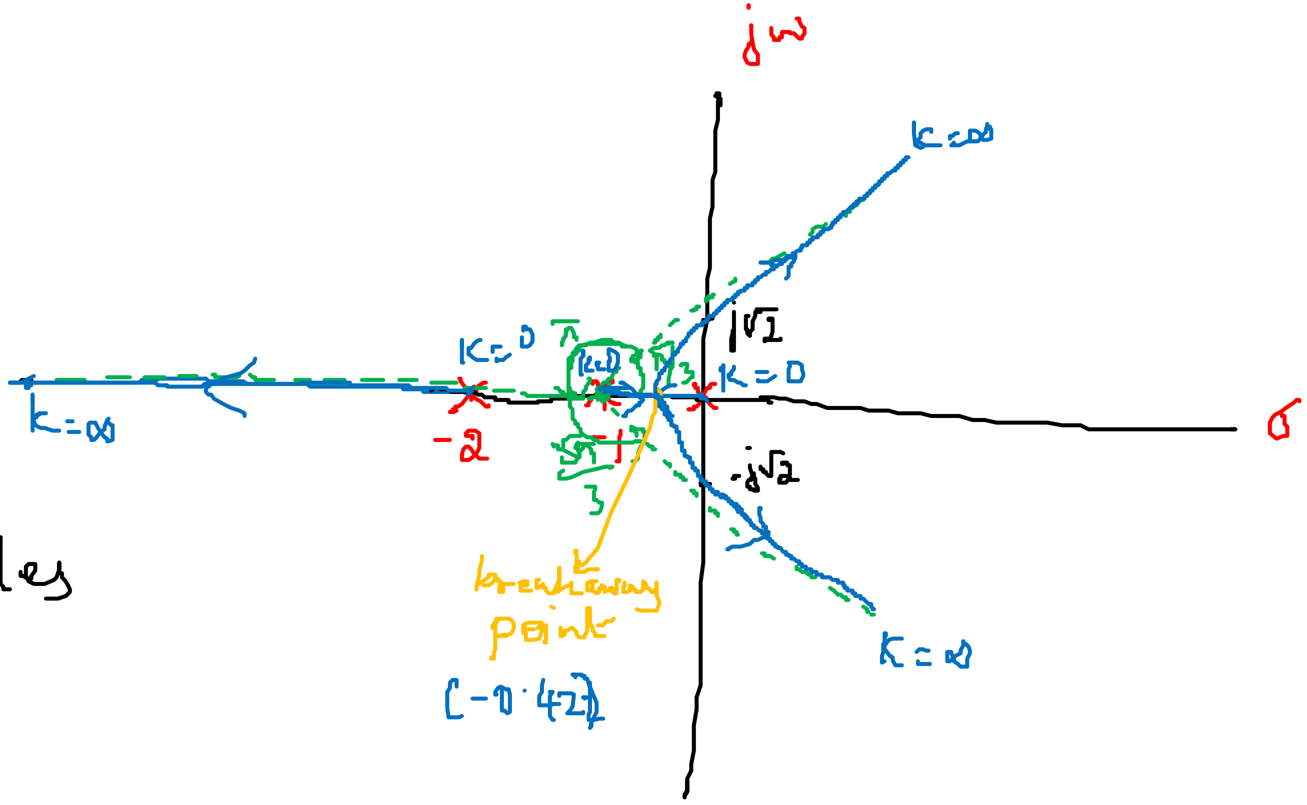
Sol:

1. $n = 3$, $s = 0, -1, -2$

$m = 0$

no. of branches $= n = 3$

2. The branches starts at OL poles & terminates at OL zeros



$$\sigma_A = \frac{0-1-2-0}{3-0} = \frac{-3}{3} = -1$$

, $q = 0, 1, 2$

$$\phi_A = \frac{(2q+1)\pi}{n-m} = \pi/3, \pi, \frac{5\pi}{3}$$

THE STABILITY OF LINEAR FEEDBACK SYSTEMS

Root Locus Procedure

$$K = -[s(s+1)(s+2)] = -(s^3 + 3s^2 + 2s)$$

$$\frac{dK}{ds} = -(3s^2 + 6s + 2) = 0$$

$$s = -0.4229, \quad -1.57$$

breakaway point

Root Locus Procedure

Step 6: Find the intersection points of root locus branches with an imaginary axis

We can calculate the point at which the root locus branch intersects the imaginary axis and the value of **K** at that point by using the RH method.

- If all elements of any row of the Routh array are zero, then the root locus branch intersects the imaginary axis and vice-versa.
- Identify the row in such a way that if we make the first element as zero, then the elements of the entire row are zero. Find the value of **K** for this combination.
- Substitute this **K** value in the auxiliary equation. You will get the intersection point of the root locus branch with an imaginary axis.

Root Locus Procedure

$1 + K\,G(s) = 0$

$1 + K \dfrac{1}{s^3 + 3s^2 + 2s} = 0$

$s^3 + 3s^2 + 2s + K = 0$

$$
\begin{array}{c|cc}
s^3 & 1 & 2 \\
s^2 & 3 & K \\
s^1 & \frac{6-K}{3} & \\
s^0 & K &
\end{array}
$$

for the s/m to be stable

$\dfrac{6-K}{3} > 0 \qquad K > 0$

$K < 6$

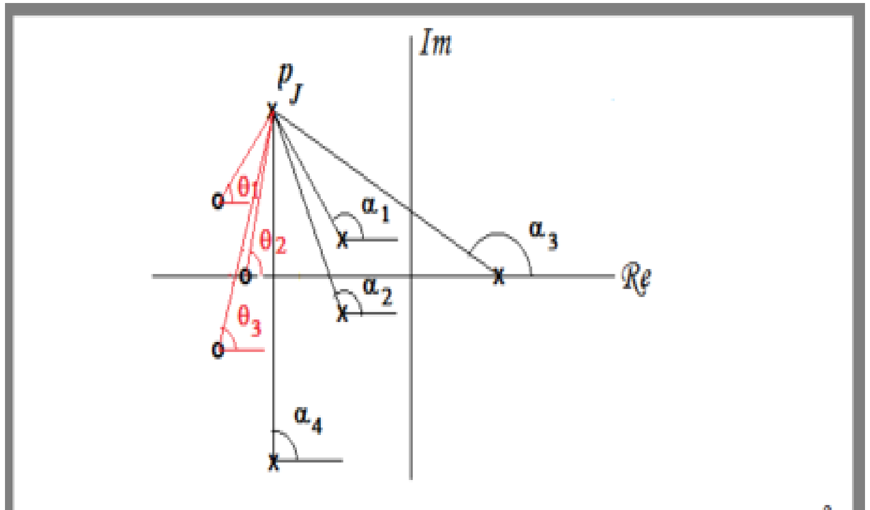
when $K = 6$, $A(s) = 3s^2 + 6 = 0$

$s^2 = -2$

$s = \pm j\sqrt{2}$

STEP 7: Angle of departure and angle of arrival

- The angle of departure and the angle of arrival can be calculated at complex conjugate open loop poles and complex conjugate open loop zeros respectively.
- The angle of departure is the angle at which the locus leaves a pole in the s-plane. The angle of arrival is the angle at which the locus arrives at a zero in the s-plane.
- Both arrival and departure angles are found using the angle criterion.



Angle of Departure from $p_J = (\theta_1+\theta_2+\theta_3) - (\alpha_1+\alpha_2+\alpha_3+\alpha_4) + q\,180^\circ$

Sum of zero angles − Sum of pole angles = 180°

$(\theta_1+\theta_2+\theta_3) - (\alpha_1+\alpha_2+\alpha_3+\underbrace{\alpha_4}_{\theta_d}) = 180^\circ \Rightarrow -\theta_d = 180^\circ - (\theta_1+\theta_2+\theta_3) + (\alpha_1+\alpha_2+\alpha_3+\alpha_4)$

Root Locus Procedure

- The formula for the **angle of departure** ϕ_d is

$$\phi_d = 180 - \varphi$$

- The formula for the **angle of arrival** φa is

$$\phi_a = 180 + \varphi$$

- Where,

$$\varphi = \sum\phi_p - \sum\phi_z$$

= sum of angles of pole – sum of angles of zeros

THE STABILITY OF LINEAR FEEDBACK SYSTEMS

Example 1

$-\theta_d = 162^\circ$

$\theta_d = 198^\circ$

Draw the root locus of the control system having open loop transfer function, G(s)H(s)= $\frac{K(s+1)}{s^2+4s+13}$

Sol:

1. $n = 2$, $s = -2 \pm j3$

   $m = 1$, $s = -1$

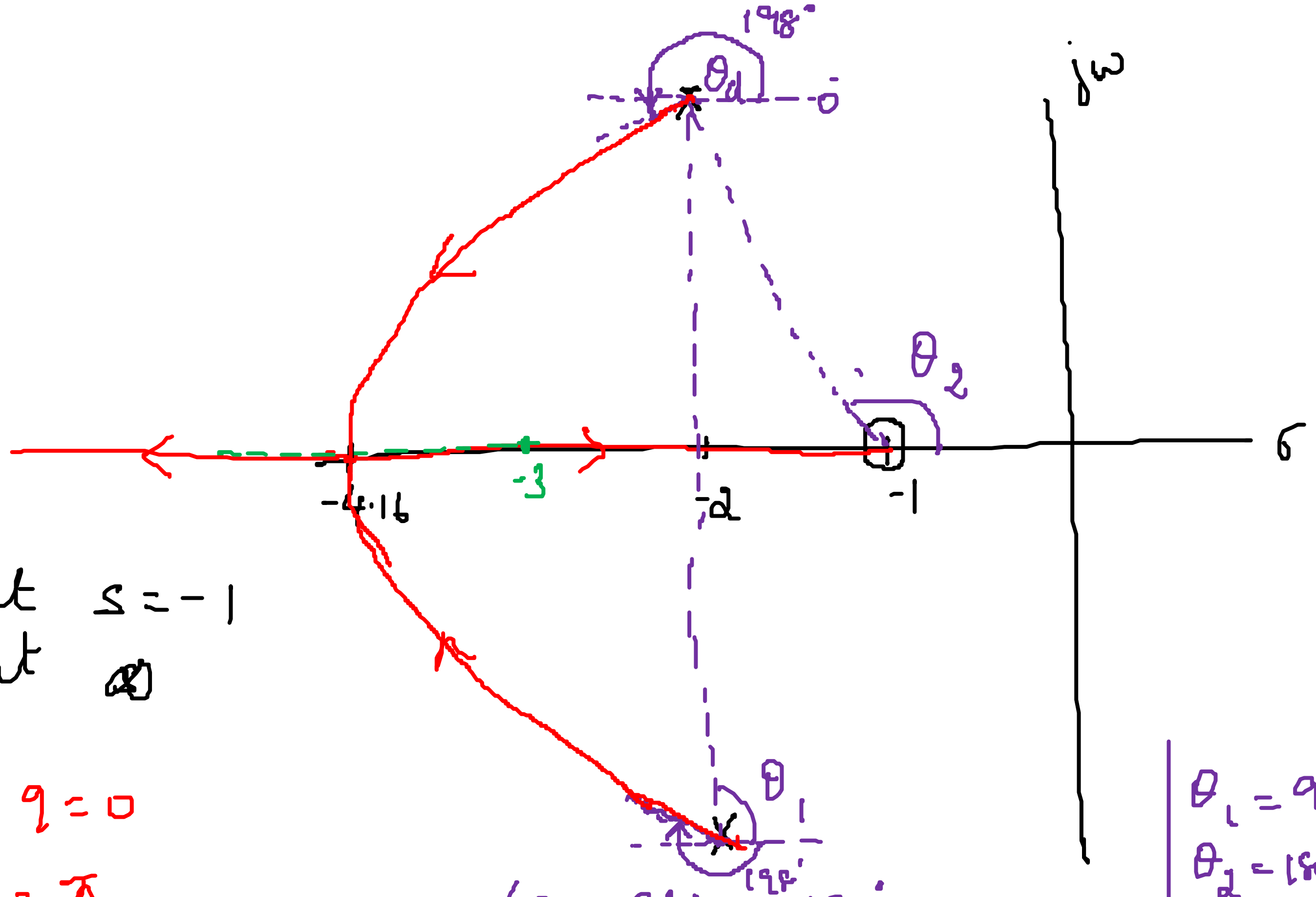
2. no. of branches $= n = 2$

   One branch will terminate at $s = -1$

   other " " at ∞

3. $\sigma_A = \frac{-2-2-(-1)}{2-1} = -3$ $\quad q = 0$

   $\phi_A = \pi$



$\theta_2 - (\theta_1 + \theta_d) = 180^\circ$

$-\theta_d = 180^\circ - \theta_2 + \theta_1 = 162^\circ$

$\theta_1 = 90^\circ$

$\theta_2 = 180^\circ - \tan^{-1}\left(\frac{3}{1}\right) = 108.43^\circ$

THE STABILITY OF LINEAR FEEDBACK SYSTEMS

Example 1

5. break-in points

$$K = \frac{-(s^2+4s+13)}{(s+1)}$$

$$\frac{dK}{ds} = \frac{(2s+4)(s+1) - (s^2+4s+13)}{(s+1)^2} = 0$$

$$s^2 + 2s - 9 = 0$$

$$s = -4.16,\ 2.16$$

break in point

Example 1

Draw the root locus of the control system having open loop transfer function,

$$G(s) = \frac{s^2 - 2s + 2}{(s+2)(s+3)(s+4)}$$

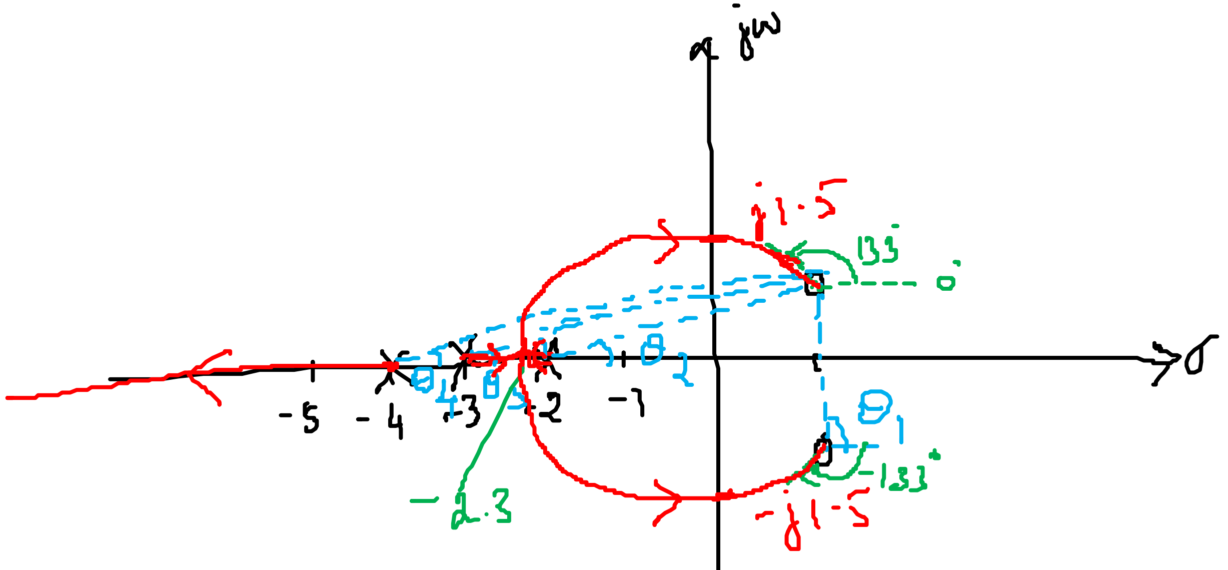
Sol: $n = 3, \quad s = -2, -3, -4$

$m = 2, \quad s = 1 \pm j$

no. of branches, $n = 3$

$n - m = 1$ branch will terminate at ∞.

$$\sigma_A = \frac{-2-3-4-(1+1)}{3-2} = -11, \quad \phi_A = \pi$$



$\theta_1 = 90°$

$\theta_2 = \tan^{-1}\frac{1}{3} = 18.43°$

$\theta_3 = \tan^{-1} 1/4 = 14°$

$\theta_4 = \tan^{-1} 1/5 = 11$

$\theta_a + \theta_1 - (\theta_2 + \theta_3 + \theta_4) = 180°$

$\theta_a = 180° - \theta_1 + [\theta_2 + \theta_3 + \theta_4]$

$= 133.4°$

Example 1

To find breakaway points

$$K = \frac{-(s^3 + 9s^2 + 26s + 24)}{s^2 - 2s + 2}$$

$$\frac{dK}{ds} = s^4 - 4s^3 - 38s^2 - 12s + 100 = 0$$

$$s = 8.48,\ 1.41,\ -2.365,\ -3.52$$

-2.365 ← breakaway point

To find $j\omega$-axis crossing

$$1 + KG(s) = 0$$

$$s^3 + (9+K)s^2 + (26-2K)s + 24 + 2K = 0$$

s^3	1	$26-2K$
s^2	$9+K$	$24+2K$
s^1	$\frac{(9+K)(26-2K)-(24+2K)}{9+K}$	
s^0	$24+2K$	

$$(9+K)(26-2K) - (24+2K) > 0$$

$$K < 11.8$$

when $K = 11.8$

$$A(s) = (9 + 11.8)s^2 + (24 + 2 \times 11.8) = 0$$

$$s = \pm j1.512$$

THE STABILITY OF LINEAR FEEDBACK SYSTEMS

Example 1

Ex, $G(s)H(s) = \dfrac{K}{s(s+4)(s^2+4s+20)}$

sol: $n = 4$, $s = 0, -4, -2 \pm j4$

$m = 0$

no. of branches $= n = 4$

The branches will start at OL poles and terminates at ∞

$$\sigma_A = \frac{(0-4-2-2)-0}{4-0} = -2$$

$q = 0, 1, 2, 3$, $\phi_A = \dfrac{(2q+1)\pi}{n-m} \Rightarrow \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

$$K = \frac{-D(s)}{N(s)} = -(s^4 + 8s^3 + 36s^2 + 80s)$$

$$\frac{dK}{ds} = -(4s^3 + 24s^2 + 72s + 80) = 0$$

$$s = -2, -2 \pm j2.44$$

THE STABILITY OF LINEAR FEEDBACK SYSTEMS

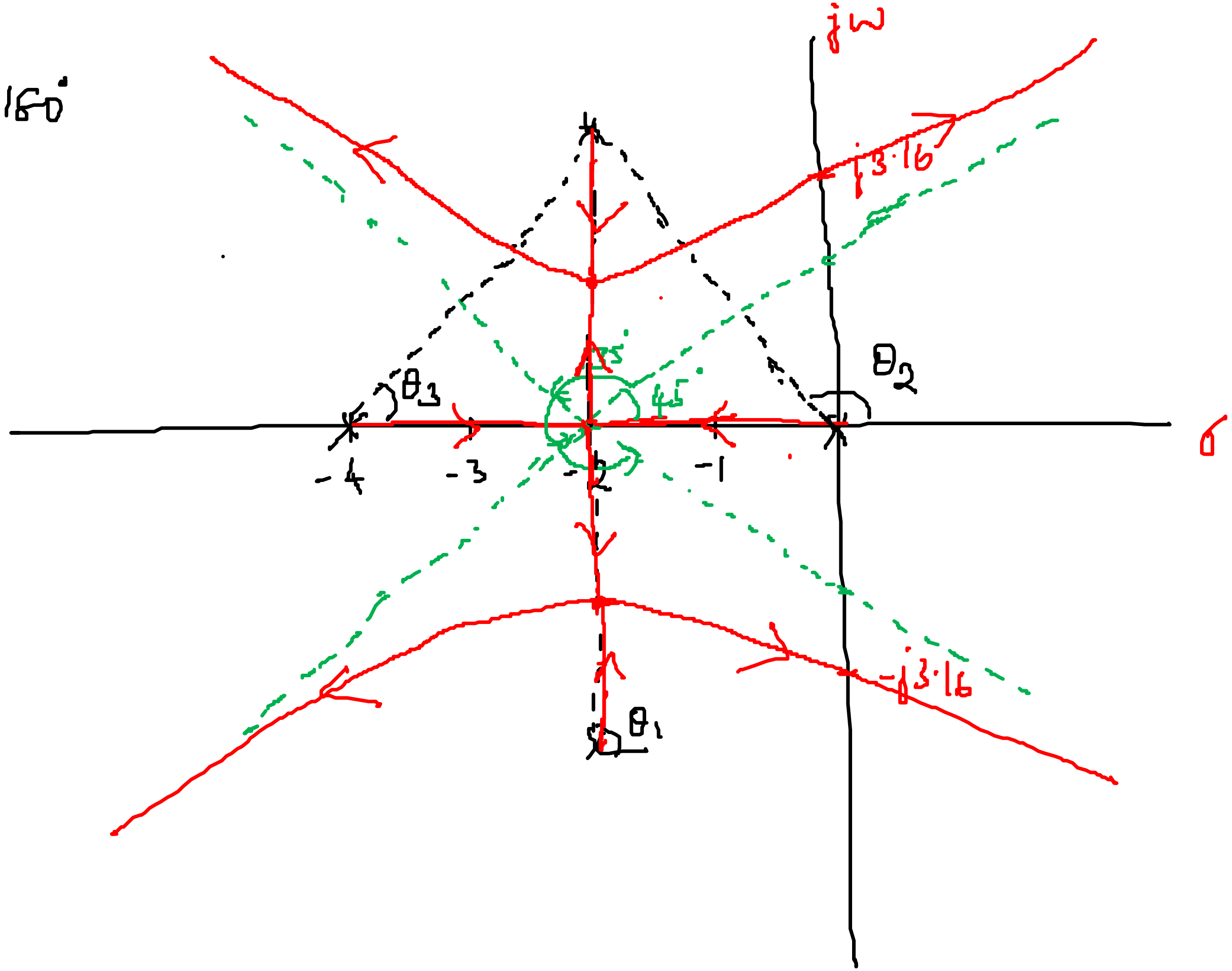
Example 1

$$0 - (\theta_d + \theta_1 + \theta_2 + \theta_3) = 180^\circ$$

$$-\theta_d = 180^\circ + \theta_1 + \theta_2 + \theta_3$$

$$\theta_d = -450^\circ$$

$$= 90^\circ$$



$$\theta_1 = 90^\circ$$

$$\theta_2 = 180^\circ - \tan^{-1}\frac{4}{2}$$

$$= 116.56^\circ$$

$$\theta_3 = 63.4^\circ$$

THE STABILITY OF LINEAR FEEDBACK SYSTEMS

Example 1

find $j\omega$ axis crossing

$$1 + KG(s)H(s) = 0$$

$$1 + \frac{K}{s(s+4)(s^2+4s+20)} = 0$$

$$s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

s^4	1	36	K
s^3	8	80	
s^2	26	K	
s^1	$\frac{2080-8K}{26}$		
s^0	K		

for the system to be stable

$K > 0$ & $\frac{2080-8K}{26} > 0$

$$0 < K < 260$$

when $K = 260$, $A(s) = 26s^2 + 260 = 0$

$s^2 = -260/26 \Rightarrow s = \pm j\sqrt{10} = \pm j3.16$

Example 1

Ex, $G(s) = \dfrac{K(s+2)}{s^2+2s+13}$

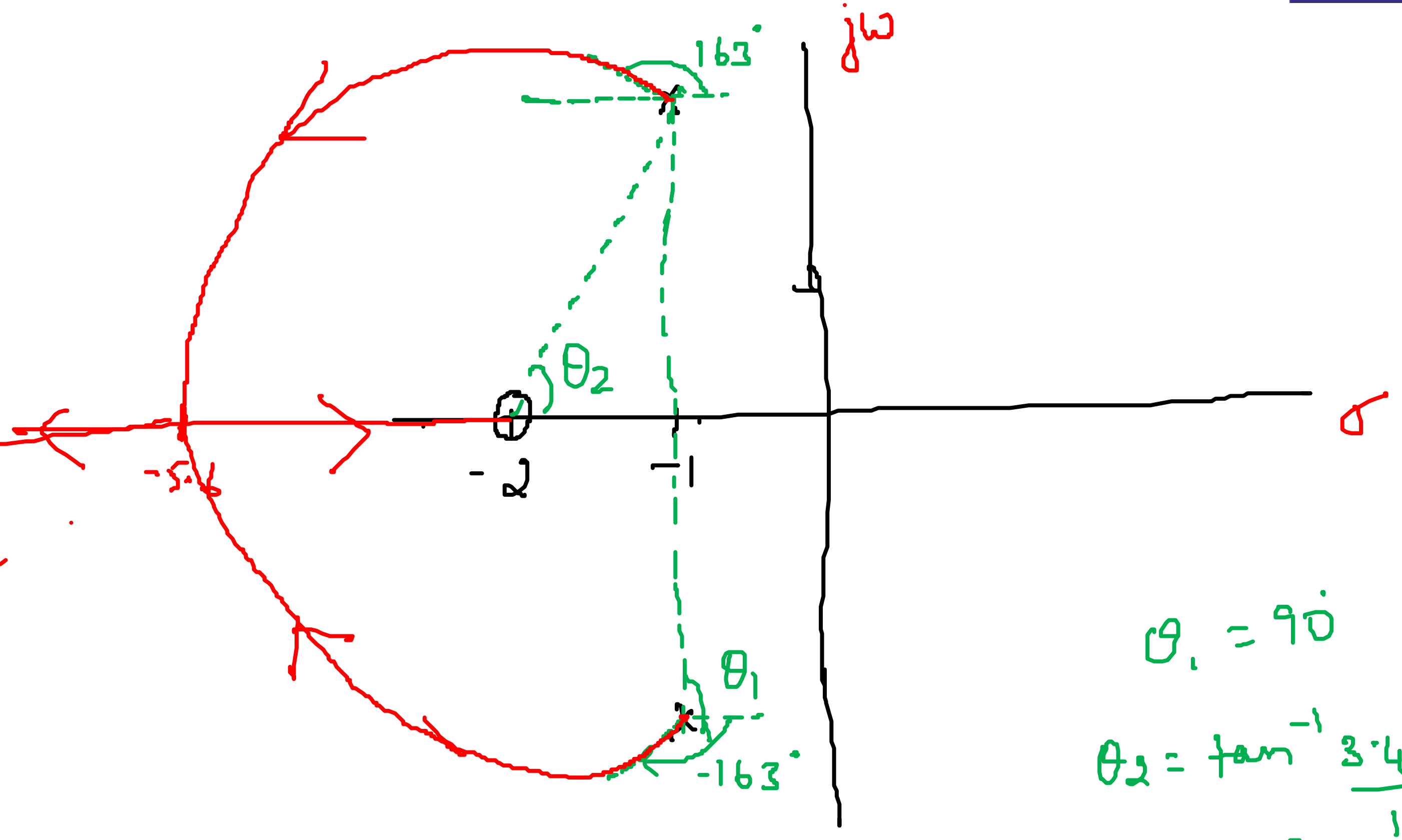
Sol: $n=2$, $s = -1 \pm j\sqrt{12}$

$m = 1$, $s = -2$

no. of branches $n = 2$

one branch will terminate at -2 & other branch will terminate at ∞

$\sigma_A = 0$, $\phi_A = \pi$



$\theta_1 = 90°$

$\theta_2 = \tan^{-1} 3.46$

$= 73.88°$

$\theta_2 - (\theta_d + \theta_1) = 180°$

$+\theta_d = 180° - \theta_2 + \theta_1 = -196°$

$= 163°$

THE STABILITY OF LINEAR FEEDBACK SYSTEMS

Example 1

$$K = \frac{-(s^2+2s+13)}{s+2}$$

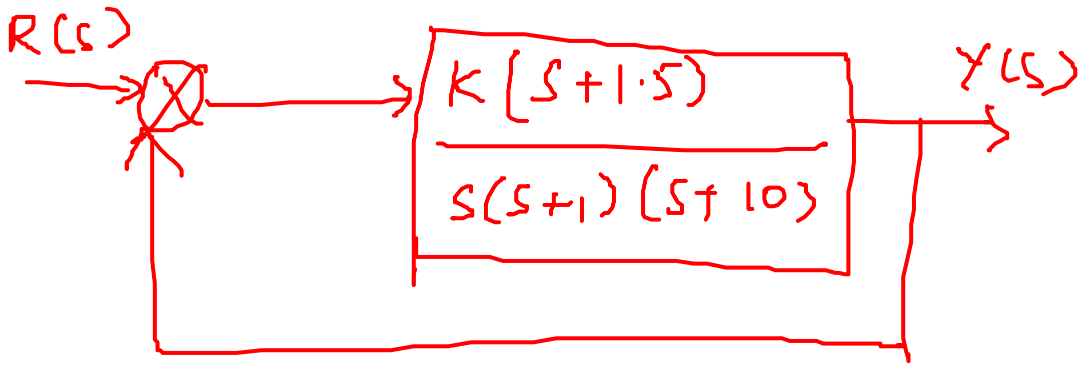
$$\frac{dK}{ds} = 0 \Rightarrow -(s^2+4s-9) = 0$$

$$s = 1.60, \quad -5.60$$

THE STABILITY OF LINEAR FEEDBACK SYSTEMS

Example 1

Ex, Design the value of gain K to yield 1.52 % of overshoot



Sol: $G(s) = \dfrac{K(s+1.5)}{s(s+1)(s+10)}$

$n = 3, \quad s = 0, -1, -10$

$m = 1, \quad s = -1.5$

no. of branches $n = 3$

$n - m = 3 - 1 = 2$ branches will terminate at ∞

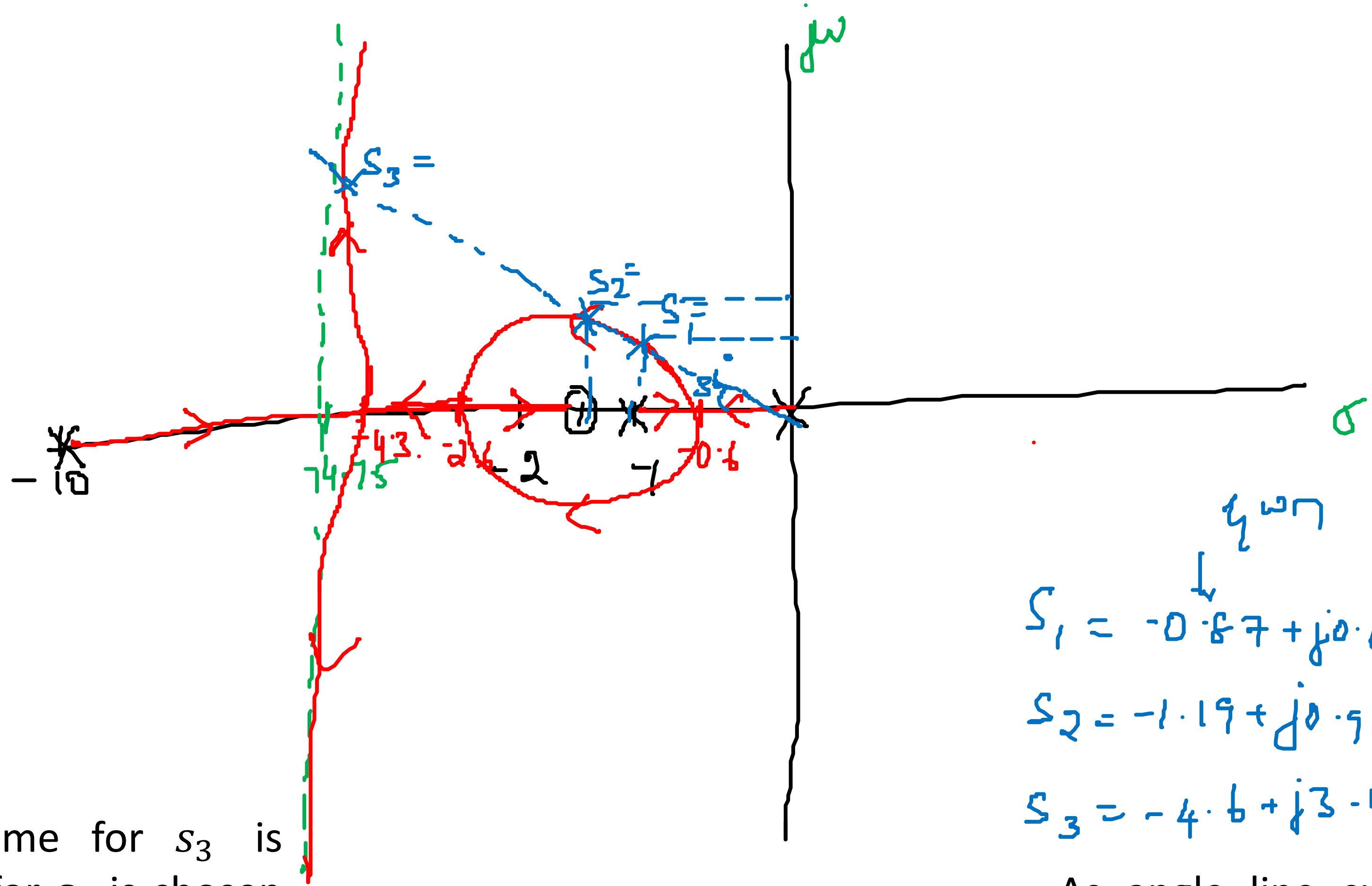
$\sigma_A = \dfrac{0 - 1 - 10 - (-1.5)}{3 - 1} = -4.75$, $\quad q = 0, 1$

$\phi_A = \pi/2, \; 3\pi/2$

THE STABILITY OF LINEAR FEEDBACK SYSTEMS

Example 1

$t_s = \frac{4}{\zeta\omega_n}$



$\frac{dK}{ds} = 2s^3 + 15.5s^2 + 33s + 15 = 0$

$s = -2.788, -4.36, -0.621$

$e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

$M_p = 1.52\%$

$\Rightarrow \zeta = 0.8$

$\zeta = \cos\theta$

$\theta = \cos^{-1}\zeta = 36.86^\circ$

$\zeta\omega_n$

$s_1 = -0.87 + j0.66$

$s_2 = -1.19 + j0.90$

$s_3 = -4.6 + j3.45$

Since settling time for s_3 is smaller, K value for s_3 is chosen as final value

K = 39.36

As angle line cuts the Root Locus at 3 points, there will be 3 values for K. Final K value is chosen based on settling time.

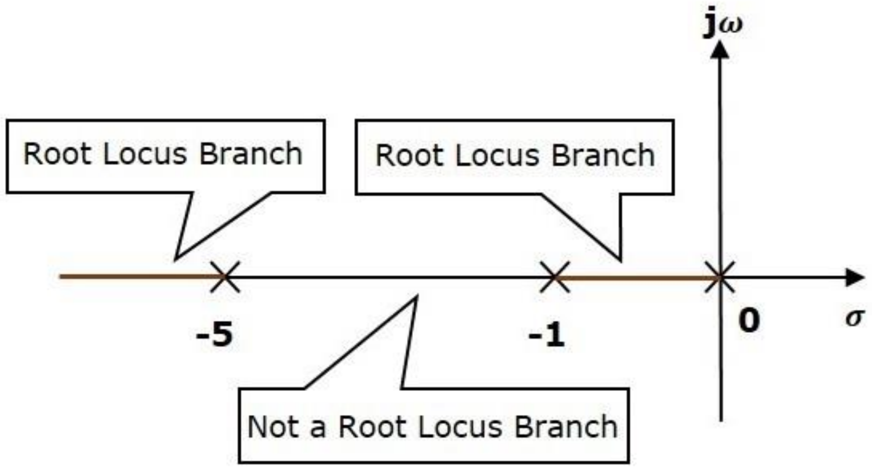
Example 1

Draw the root locus of the control system having open loop transfer function, $G(s)H(s)=K/s(s+1)(s+5)$

Step 1: The given open loop transfer function has three poles at $s=0, s=-1$ and $s=-5$.

It doesn't have any zero. Therefore, the number of root locus branches is equal to the number of poles of the open loop transfer function , is equal to 3.

The three poles are located are shown in the above figure. The line segment between $s=-1$ and $s=0$ is one branch of root locus on real axis. And the other branch of the root locus on the real axis is the lir



THE STABILITY OF LINEAR FEEDBACK SYSTEMS

Example 2

A–6–1. Sketch the root loci for the system shown in Figure 6–39(a). (The gain K is assumed to be positive.) Observe that for small or large values of K the system is overdamped and for medium values of K it is underdamped.

Solution. The procedure for plotting the root loci is as follows:

1. Locate the open-loop poles and zeros on the complex plane. Root loci exist on the negative real axis between 0 and -1 and between -2 and -3.
2. The number of open-loop poles and that of finite zeros are the same. This means that there are no asymptotes in the complex region of the s plane.

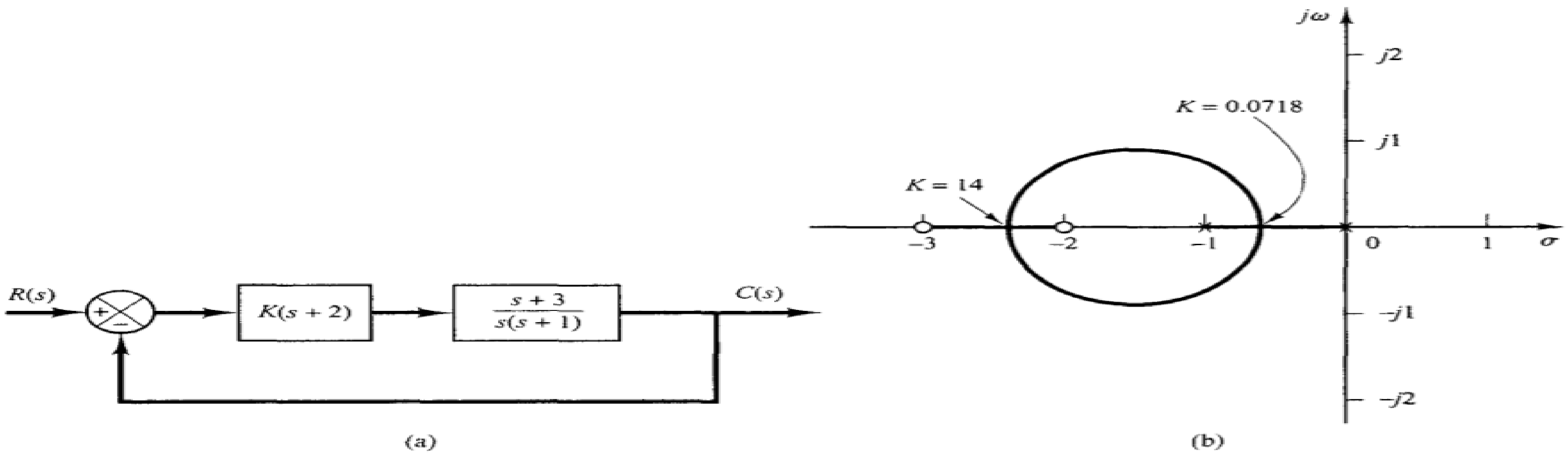


Figure 6–39
(a) Control system; (b) root-locus plot.

Example 2

3. Determine the breakaway and break-in points. The characteristic equation for the system is

$$1 + \frac{K(s+2)(s+3)}{s(s+1)} = 0$$

or

$$K = -\frac{s(s+1)}{(s+2)(s+3)}$$

The breakaway and break-in points are determined from

$$\frac{dK}{ds} = -\frac{(2s+1)(s+2)(s+3) - s(s+1)(2s+5)}{[(s+2)(s+3)]^2}$$

$$= -\frac{4(s+0.634)(s+2.366)}{[(s+2)(s+3)]^2}$$

$$= 0$$

as follows:

$$s = -0.634, \qquad s = -2.366$$

Notice that both points are on root loci. Therefore, they are actual breakaway or break-in points. At point $s = -0.634$, the value of K is

$$K = -\frac{(-0.634)(0.366)}{(1.366)(2.366)} = 0.0718$$

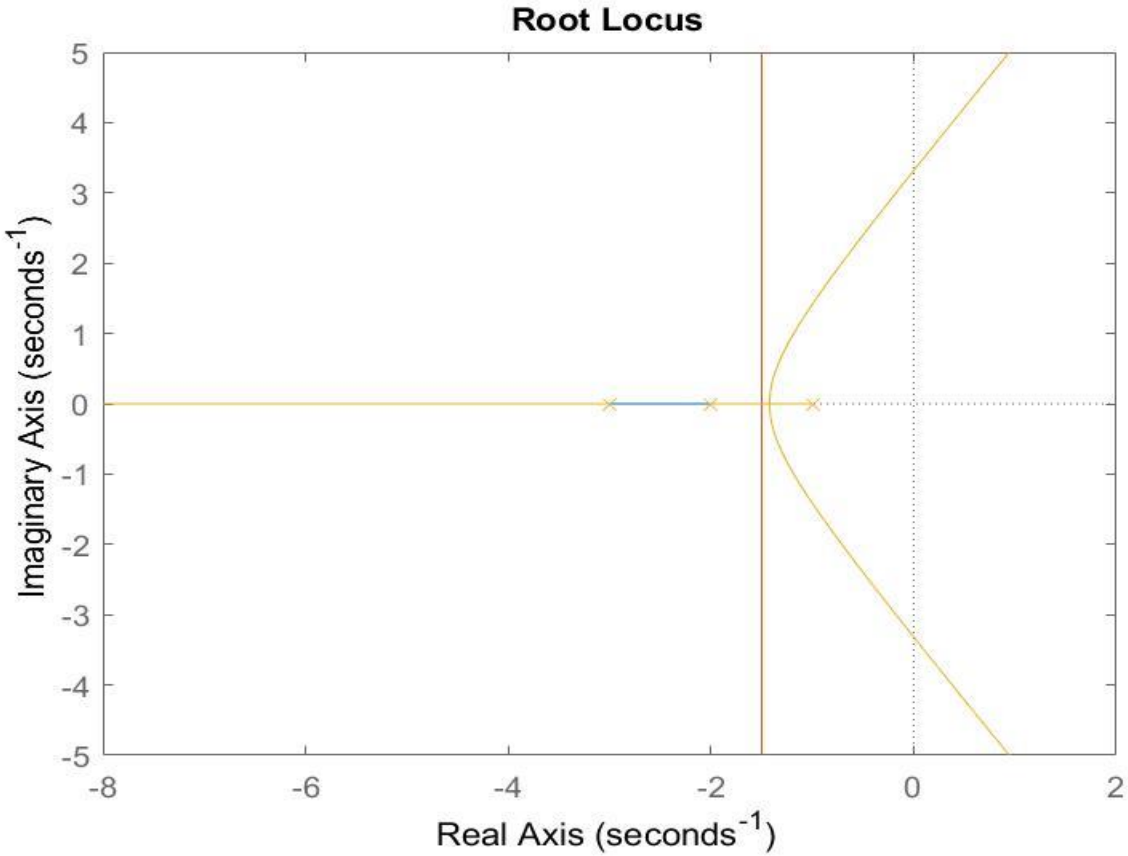
Similarly, at $s = -2.366$,

$$K = -\frac{(-2.366)(-1.366)}{(-0.366)(0.634)} = 14$$

(Because point $s = -0.634$ lies between two poles, it is a breakaway point, and because point $s = -2.366$ lies between two zeros, it is a break-in point.)

Effect of adding poles

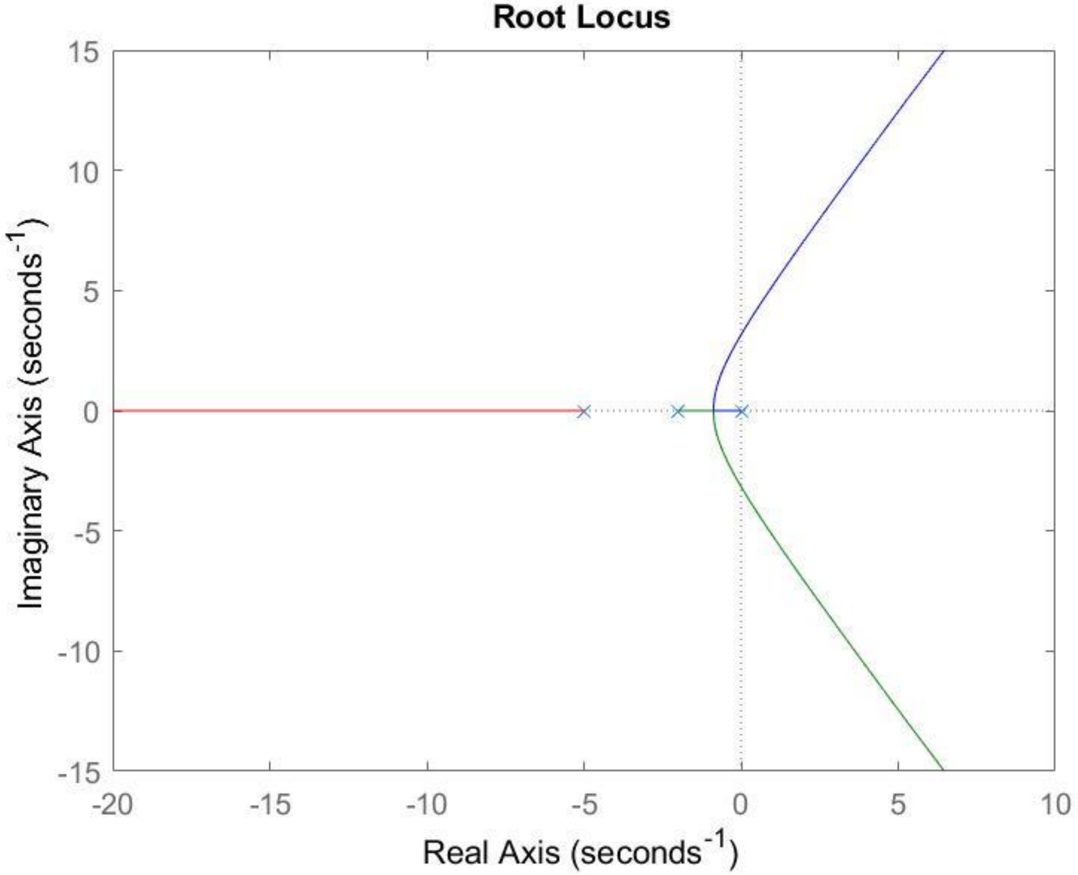
- Adding a pole to G(s)H(s) has the effect of pushing the root loci toward the right half
- $G_1(s) = \frac{1}{s+1}$(blue),
- $G_2(s) = \frac{1}{(s+2)(s+1)}$ (red),
- $G_2(s) = \frac{1}{(s+2)(s+1)(s+3)}$(yellow)

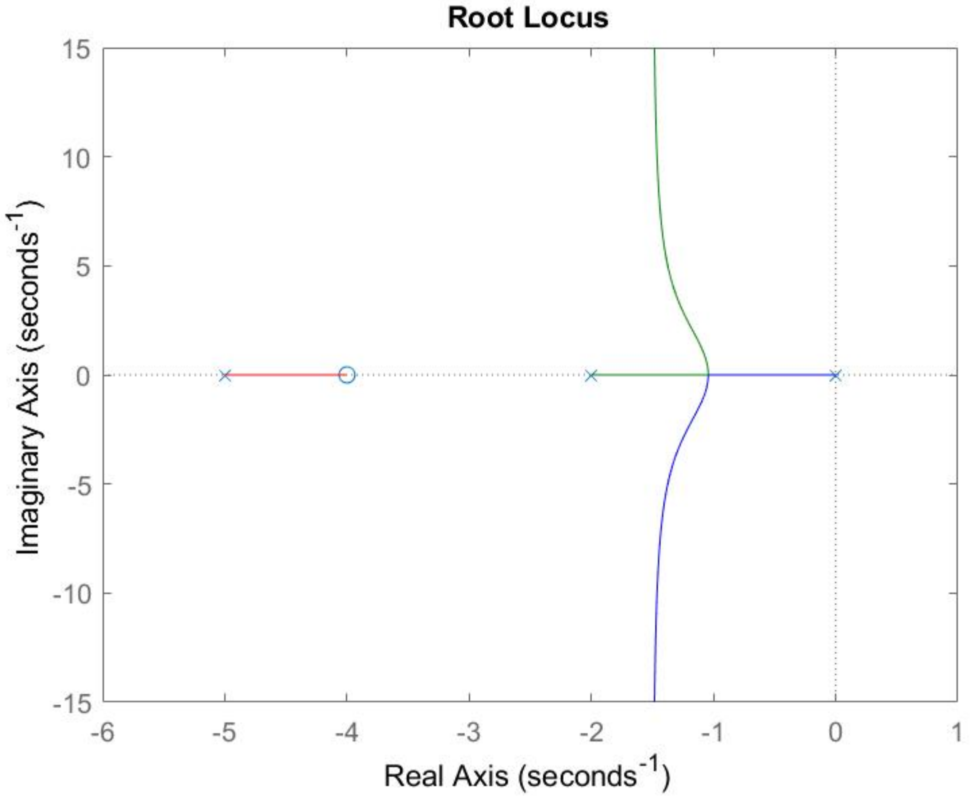


THE STABILITY OF LINEAR FEEDBACK SYSTEMS

Effect of adding zeros

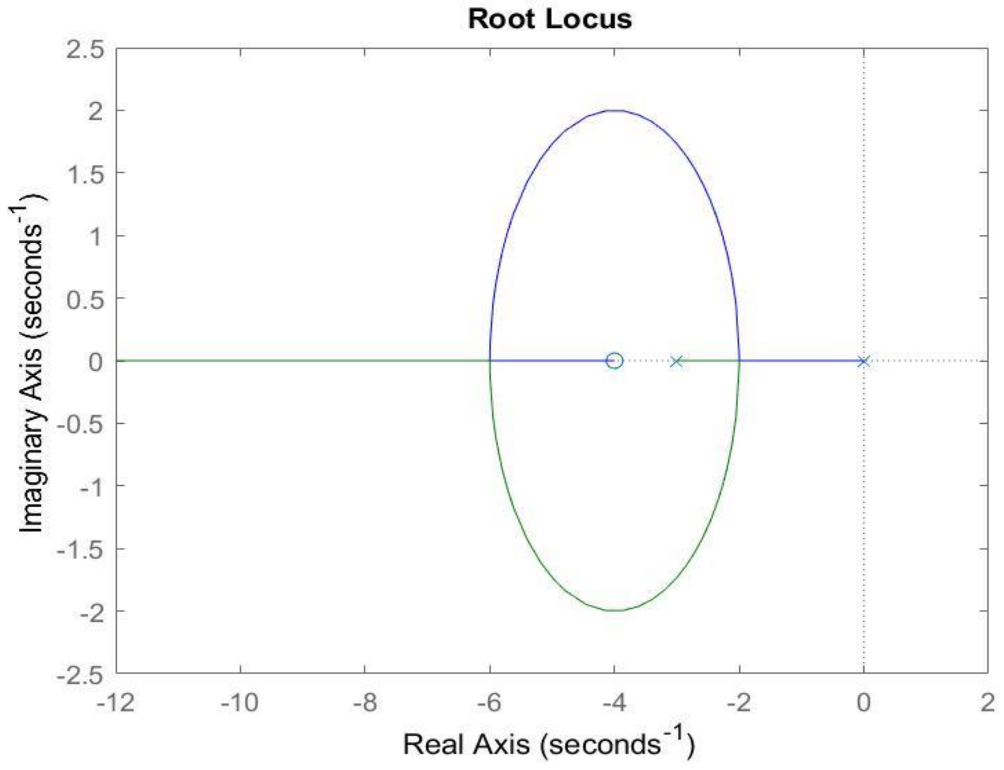
Adding a zero to G(s)H(s) has the effect of pushing the root loci toward the left half $G_1(s) = \frac{1}{s(s+a)(s+b)}$, $G_2(s) = \frac{K(s+c)}{s(s+a)(s+b)}$





Effect of adding zeros

Adding LHS zeros to G(s)H(s) generally has the effect of moving and bending the root loci toward the left half s-plane. For example $G(s) = \frac{K(s+b)}{s(s+a)}$, b>a, a>0



Applications of Root Locus Technique

i. So the root locus is a plot showing how the poles (and zeros) of the closed-loop transfer function move around the s plane, as some parameter (usually the proportional gain inside our controller) is varied.

ii. By looking at the root locus we get a feel for how the behavior of our closed loop system (i.e. plant controller with feedback) changes as we vary the parameter.

Applications of Root Locus Technique

i. We can use that information to decide how to set the gain of our controller to achieve the desired closed-loop response (e.g. stability & damping)- assuming our plant model is correct.

ii. Used to design feedback systems and also in the model for pacemaker control of oxygen saturation level , design and sensitivity analysis of closed-loop thermoacoustic engines etc.

THANK YOU

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