LINEAR ALGEBRA AND ITS APPLICATIONS
UE19MA251

Projections And Least Squares

The failure of Gaussian Elimination is almost certain when we have several equations in one unknown.

$$a_1 x = b_1$$
$$a_2 x = b_2$$
$$----$$
$$a_m x = b_m$$

This system is solvable if $b = (b_1, \ldots, b_m)$ is a multiple of $a = (a_1, \ldots., a_m)$.

Projections And Least Squares

If the system is inconsistent, then we choose that value of a that minimizes an average error E in the m equations. The most convenient average comes from the ***sum of squares***:

Squared Error $$E^2 = \sum_{i=1}^{m} (a_i x - b_i)^2$$

If there is an exact solution the minimum error is E = 0. If not, the minimum error occurs when $\frac{dE^2}{dx} = 0$

Solving for x, the least squares solution is $\hat{x} = \frac{a^T b}{a^T a}$

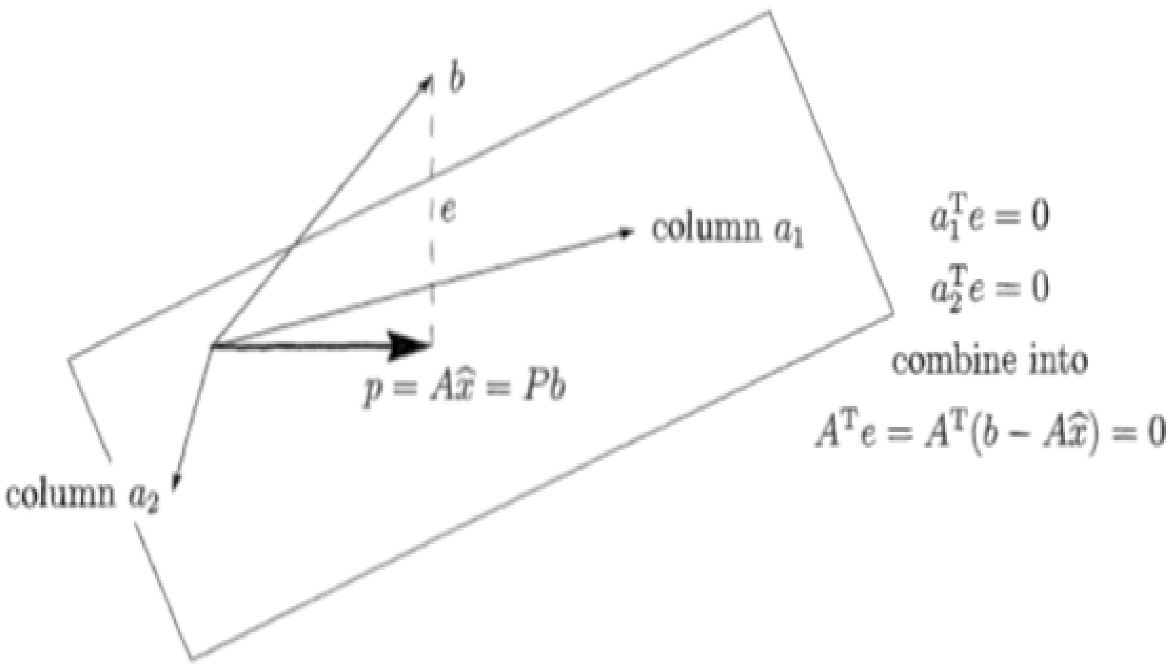
Least Squares Problem With Several Variables

Consider a system of equations Ax = b that is inconsistent.

The vector b lies outside C(A) and we need to project it onto C(A) to get the point p in C(A) that is closest to b. The problem here is the same as to minimize the error $E = \| Ax - b \|$ and this is exactly the distance from b to the point Ax in C(A).

Searching for the least squares solution $\hat{x}$ is the same as locating the point p that is closest to b.

Least Squares Problem With Several Variables



Least Squares Problem With Several Variables

The error vector $e = b - A\hat{x}$ must be perpendicular to C(A) and hence can be found in the left null space of A.

Thus, $A^T(b - A\hat{x}) = 0 \ or \ A^T A\hat{x} = A^T b$

These are called the *Normal Equations*.

Solving them, we get the optimal solution $\hat{x}$

Note :

If b is orthogonal to C(A) then its projection is the zero vector.

Problems on Projections and Least squares

Find the projection of $b = (1, 2, 7)$ onto the column space of A spanned by $(1, 1, -2)$ and $(1, -1, 4)$.
Split b into $p + q$ with p in $C(A)$ and q in $N(A^T)$.

Solution: Let p be the projection of b onto $C(A)$ which is spanned by $(1, 1, -2)$ and $(1, -1, 4)$

So $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}$ and $p = A\hat{x}$

Unit 3. Linear Transformations and Orthogonality

Problems on Projections and Least squares

Normal equation to find $\hat{x}$ is

$A^T A \hat{x} = A^T b \Rightarrow \begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix} \hat{x} = \begin{bmatrix} -11 \\ 27 \end{bmatrix}$

$\Rightarrow \hat{x} = \begin{bmatrix} 9/22 \\ 37/22 \end{bmatrix}$

$\therefore p = A\hat{x} = \begin{bmatrix} 23/11 \\ -14/11 \\ 65/11 \end{bmatrix}$ and $p + q = b$

$\Rightarrow q = b - p$

$q = \begin{bmatrix} -12/11 \\ 36/11 \\ 12/11 \end{bmatrix}$

and $q \in N(A^T)$

Problems on Projections and Least squares

2. Find a basis for the orthogonal complement of the row space of $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$. Split the vector $(3,3,3)$ into a row space component x_r and a null space component x_n.

Solution: x_r and x_n are projections of $x = (3,3,3)$ onto $C(A^T)$ and $N(A)$ respectively.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \underset{R_2 \to R_2 - R_1}{\sim} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow N(A) = \left\{ \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \right\}$$

Unit 3. Linear Transformations and Orthogonality

Problems on Projections and Least squares

Let $a = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$

Projection of b onto a line through 'a' is x_n.

$$x_n = \hat{x} a = \frac{a^T b}{a^T a} \cdot a = \frac{-9}{9} \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

We know that $x = x_r + x_n \Rightarrow x_r = x - x_n = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$

THANK YOU