LINEAR ALGEBRA AND ITS APPLICATIONS
UE19MA251

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Transformations Represented by Matrices

Integration Matrix:

Consider the integration of a quadratic polynomial from 0 to 1. This transformation is linear which transforms P_2 to P_3 .

$$P_2 = \left\{ p(t) = a_0 + a_1 t + a_2 t^2, \quad a_0, a_1, a_2 \in R \right\}$$

$$\text{Basis} = \left\{ v_1 = 1, \ v_2 = t, \ v_3 = t^2 \right\}$$

$$P_3 = \left\{ q(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3, \quad b_i \in R \right\}$$

$$\text{Basis} = \left\{ u_1 = 1, \ u_2 = t, \ u_3 = t^2, \ u_4 = t^3 \right\}$$

$$A_{int} : P_2 \rightarrow P_3$$

Transformations Represented by Matrices

Images of v_i's are

$$\int_0^t v_1 \, dt = \int_0^t 1 \, dt = t = 0 \cdot u_1 + 1 \cdot u_2 + 0 \cdot u_3 + 0 \cdot u_4$$

$$\Rightarrow (0, 1, 0, 0)$$

$$\int_0^t v_2 \, dt = \int_0^1 t \, dt = \frac{t^2}{2} = 0 \cdot u_1 + 0 \cdot u_2 + \frac{1}{2} u_3 + 0 \cdot u_4$$

$$\Rightarrow (0, 0, \frac{1}{2}, 0)$$

$$\int_0^t v_3 \, dt = \int_0^1 t^2 \, dt = \frac{t^3}{3} = 0 \cdot u_1 + 0 \cdot u_2 + 0 \cdot u_3 + \frac{1}{3} u_4$$

$$\Rightarrow (0, 0, 0, \frac{1}{3})$$

Transformations Represented by Matrices

Integration matrix is

$$A_{int} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}_{4 \times 3}$$

$$A_{diff} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3 \times 4}$$

Transformations Represented by Matrices

Note:

→ $A_{diff} \cdot A_{int} = I_3$

→ Differentiation is a left inverse of integration.

→ Integration is a right inverse of differentiation.

→ Column space i.e. range of A_{int} is a subspace of P_3
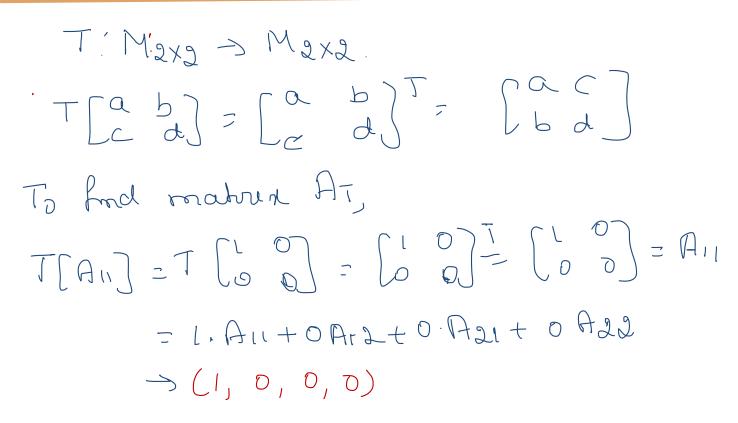
→ Kernel i.e Nullspace = $\{\vec{0} \in P_2\}$

Transformations Represented by Matrices

Problems:

1. For the space of all 2 x 2 matrices find the standard basis. For the linear transformation of transposing, find the matrix A with respect to this basis . Why is $A^2 = I$?

Solution : $M_{2\times2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, a,b,c,d \in R \right\}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ Basis $= \left\{ A_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right.$

$\left. A_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Transformations Represented by Matrices

$T: M_{2\times2} \to M_{2\times2}$

$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

To find matrix A_T,

$T[A_{11}] = T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A_{11}$

$= 1 \cdot A_{11} + 0 \, A_{12} + 0 \cdot A_{21} + 0 \, A_{22}$

$\to (1, 0, 0, 0)$

Transformations Represented by Matrices

$$T(A_{12}) = T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = A_{21}$$

$$= 0 \cdot A_{11} + 0 \cdot A_{12} + 1 \cdot A_{21} + 0 \cdot A_{22} \rightarrow [0, 0, 1, 0]$$

$$T(A_{21}) = T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = A_{12}$$

$$= 0 \cdot A_{11} + 1 \cdot A_{12} + 0 \cdot A_{21} + 0 \cdot A_{22} \rightarrow [0, 1, 0, 0]$$

$$T[A_{22}] = T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = A_{22}$$
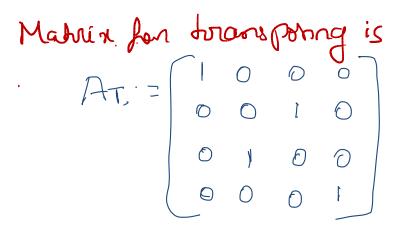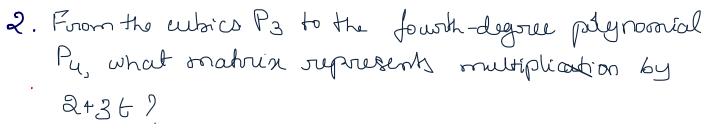
$$= 0 \cdot A_{11} + 0 \cdot A_{12} + 0 \cdot A_{21} + 1 \cdot A_{22} \rightarrow [0, 0, 0, 1]$$

Transformations Represented by Matrices

Matrix for transposing is

$$A_T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $A^2 = I$, because $(A^T)^T = A$.

Transformations Represented by Matrices

2. From the cubics P_3 to the fourth-degree polynomial P_4, what matrix represents multiplication by $2 + 3t$?

Solution: Basis for $P_3 = \{ v_1 = 1, v_2 = t, v_3 = t^2, v_4 = t^3 \}$

Basis for $P_4 = \{ u_1 = 1, u_2 = t, u_3 = t^2, u_4 = t^3, u_5 = t^4 \}$

To find matrix

$(2 + 3t) \, v_1 = (2 + 3t) \cdot 1 = 2 \cdot u_1 + 3 \cdot u_2 + 0 u_3 + 0 \cdot u_4 + 0 u_5$

$\rightarrow (2, 3, 0, 0, 0)$

Transformations Represented by Matrices

Similarly:

$(2+3t)v_2 = (2+3t)t = 2t + 3t^2 \rightarrow (0, 2, 3, 0, 0)$

$(2+3t)v_3 = (2+3t)t^2 = 2t^2 + 3t^3 \rightarrow (0, 0, 2, 3, 0)$

$(2+3t)v_4 = (2+3t)t^3 = 2t^3 + 3t^4 \rightarrow (0, 0, 0, 2, 3)$

The matrix A_T for this transformation is

$$A_T = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{5 \times 4}$$

THANK YOU