DIGITAL SIGNAL PROCESSING

ANALOG FILTERS

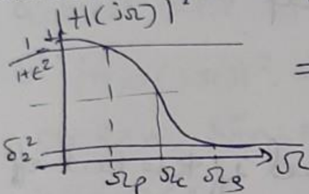
UNIT 3

PES UNIVERSITY

Analog filters

Introduction:

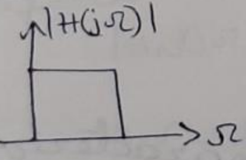
* An important approach to the design of digital filters is to apply a transformation to an existing analog filter.
* These analog procedures normally begin with a specification of the frequency response for the filter describing how the filter reacts to the sinusoidal inputs in the steady state.



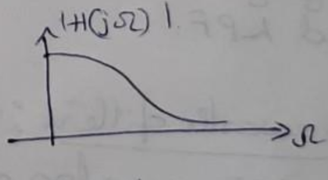
$\Rightarrow$ The frequency response of an analog filter is characterized by its passband frequency and ripple, stopband frequency and attenuation

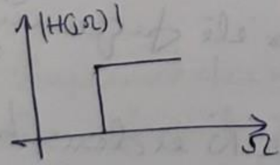
Basic types of frequency response

Ideal | **Non-Ideal**

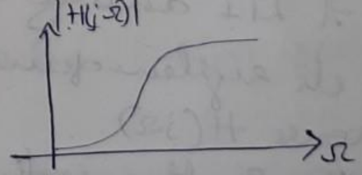


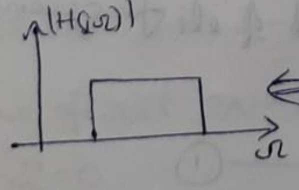
$\Leftarrow$ LPF $\Rightarrow$
Lowpass filter



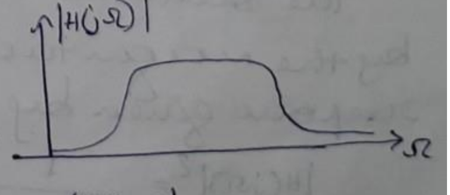


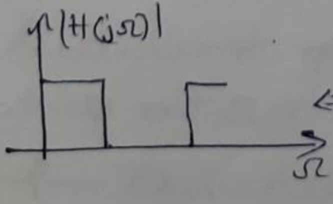
$\Leftarrow$ HPF $\Rightarrow$
High pass filter



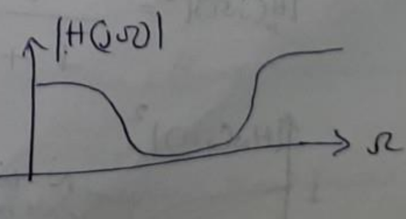


$\Leftarrow$ BPF $\Rightarrow$
Bandpass filter





$\Leftarrow$ BSF $\Rightarrow$
Bandstop filter



Although many analog filter design techniques are available in the literature, we will be studying Butterworth and Chebyshev filters

* Main feature of these filters is a compromise between a smooth passband and a sharp transition region.

Butterworth	Chebyshev
* A very smooth passband	* Not so smooth passband
* Relatively wide transition region	* Sharp transition region

Note:

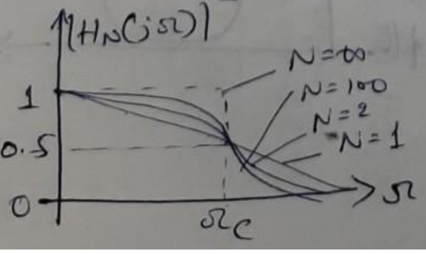
* A filter is said to be normalized if the cut-off frequency of the filter, $\Omega_c = 1$ rad/sec.
* LPF, HPF, BPF & BSFs can be designed by applying a specific transformation to a normalized LPF.

Butterworth filters:

A LTI analog filter can be characterized by its system function $H(s)$ or its frequency response $H(j\Omega)$

The Butterworth filter of order N is described by the magnitude squared of its frequency response given by

$$|H_N(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \quad \text{——— (1)}$$



Frequency response characteristics of a Butterworth filter

1. At $\Omega = 0$, $|H_N(j\Omega)|^2 = 1 \ \forall N$
2. At $\Omega = \Omega_c$, $|H_N(j\Omega)|^2 = \frac{1}{2} \ \forall$ finite N.

$$|H_N(j\Omega)| = \frac{1}{\sqrt{2}} = 0.707$$

$$20\log|H_N(j\Omega)| = -3\,dB$$

3. $|H_N(j\Omega)|^2$ is a monotonically decreasing function of Ω.
4. As 'N' increases, $|H_N(j\Omega)|^2$ approaches an ideal LPF frequency response
5. $|H_N(j\Omega)|^2$ is ~~called~~ maximally flat at the origin, since all order derivatives exist and are zero.

i.e., $\left.\frac{d^n|H(j\Omega)|}{d\Omega^n}\right|_{\Omega=0}$ for $n = 1, 2, \cdots 2N-1$

Consider the magnitude squared frequency response

$$|H_N(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$$|H_N(j\Omega)| = \left[\frac{1}{1+(\Omega/\Omega_c)^{2N}}\right]^{1/2}$$

Expand the above equation using power series about $\Omega = 0$ i.e.,

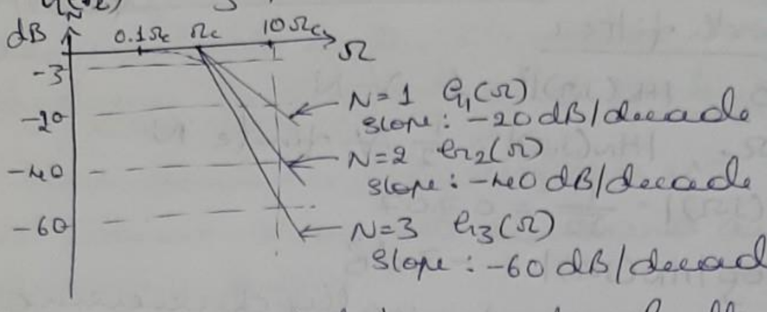
$$|H_N(j\Omega)| = 1 - \frac{1}{2}\left(\frac{\Omega}{\Omega_c}\right)^{2N} + \frac{3}{8}\left(\frac{\Omega}{\Omega_c}\right)^{4N} - \frac{5}{16}\left(\frac{\Omega}{\Omega_c}\right)^{6N} + \cdots$$

The first 2N-1 derivatives of $|H_N(j\Omega)|$ are zero at $\Omega = 0$.

$\therefore$ Butterworth approximation offers maximum flatness at $\Omega = 0$.

6. For convenience, the frequency response is expressed in decibels.

$G_1(\Omega) = 20\log|H_N(j\Omega)|$



filter gain plot for analog Butterworth filters of various orders 'N'

Consider,

$$G_N(\Omega) = 20\log|H_N(j\Omega)| = 10\log|H_N(j\Omega)|^2$$

$$= 10\log\left(\frac{1}{1+(\Omega/\Omega_c)^{2N}}\right) = -10\log\left(1+(\Omega/\Omega_c)^{2N}\right)$$

when $\Omega << \Omega_c$

$$G_N(\Omega) \approx -10\log(1) = 0\,dB$$

thus, for $\Omega << \Omega_c$, the magnitude $|H_N(j\Omega)|$ is almost zero.

for $\Omega >> \Omega_c$, the 1 in the summation becomes insignificant compared to $(\Omega/\Omega_c)^{2N}$

$$G_N(\Omega) = -10\log(\Omega/\Omega_c)^{2N} = -20N\log(\Omega/\Omega_c)$$

∴, the approximate gain for $\Omega << \Omega_c$ is 0dB while for $\Omega >> \Omega_c$ the slope of $G_N(\Omega)$ is $-20N$ dB/decade [per decade is ∵ the line drops 20N dBs everytime the frequency is incremented by 10]

Starting with the magnitude squared frequency response we find the system function H(S). ∴, determine the poles first.

for an ~~analog~~ analog system, the frequency response is obtained by using $s = j\Omega$ in the transfer function H(S) of the given system.

∴ with $\Omega = s/j$ in the frequency response we can determine the transfer function of the system

Since, $\Omega_c = 1$ rad/sec, the magnitude squared frequency response becomes:

$$|H_N(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} = \frac{1}{1+\Omega^{2N}}$$

At $s = j\Omega$, the magnitude of $H_N(s)$ and $H_N(-s)$ is same

i.e., $|H_N(j\Omega)|^2 = H_N(j\Omega)\,H_N(-j\Omega) = \dfrac{1}{1+\Omega^{2N}}$

$$\therefore\; H_N(s)\,H_N(-s) = \frac{1}{1+(s/j)^{2N}}$$

The poles are determined by the roots of the denominator.

i.e., $1 + (s/j)^{2N} = 0$

$$s^{2N} = (-1)\,(j)^{2N}$$

$$s = (-1)^{1/2N}\, j$$

$$s = e^{j\pi(2k+1)/2N}\, e^{j\pi/2} \quad k = 0 \cdots 2N-1$$

$$s = e^{j\pi(2k+1+N)/2N} \quad k = 0 \cdots 2N-1$$

or $\boxed{s = \pm\, e^{j\pi(2k+1+N)/2N} \quad k = 0 \cdots N-1}$

$e^{j\pi/2} = j$

$e^{j\pi(2k+1)} = -1$ $\quad k = 0 \cdots N-1$

for the filter, $H_N(s)$ to be stable and causal, the poles of $H_N(s)$ are on the left side of the s-plane are selected.

$$\therefore\; H_N(s) = \frac{1}{\prod\limits_{LH\,Poles}(s-s_k)} = \frac{1}{B_N(s)} = \frac{1}{a_N s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + 1}$$

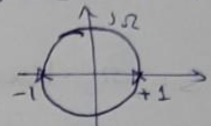
where $s_k \rightarrow$ left half plane poles of $H_N(s)\,H_N(-s)$

$B_N(s) \rightarrow$ Butterworth polynomial of order N

Example:

① for $N=1$, find the transfer function $H_1(s)$ of the normalized Butterworth filter.

$$s_k = \pm e^{j\pi(2k+1+N)/2N} \qquad k = 0 \ldots N-1$$



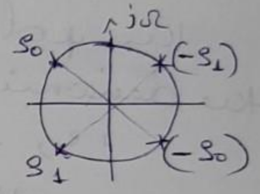
$k=0 \quad s_0 = \pm e^{j\pi} = \pm 1$

$$\therefore H_1(s) = \frac{1}{(s-(-1))} = \frac{1}{s+1}$$

poles of $H_N(s)\, H_N(-s)$ for $N=1$

② for $N=2$, find $H_2(s)$ of a normalized Butterworth filter

$$s_k = \pm e^{j\pi(2k+1+N)/2N} \qquad k=0,1$$



$s_0 = \pm e^{j3\pi/4}$ & $s_1 = \pm e^{j5\pi/4}$

$s_0 = -0.707 + j0.707$ & $0.707 - j0.707$

$s_1 = -0.707 - j0.707$ & $0.707 + j0.707$

$$H_2(s) = \frac{1}{(s-s_0)(s-s_1)} = \frac{1}{(s-(-0.707+j0.707)) \times (s-(-0.707-j0.707))}$$

$$= \frac{1}{s^2 + 1.414 s + 1} = \frac{1}{s^2 + \sqrt{2}\, s + 1}$$

③ for $N=3$

$s_0 = \pm e^{j4\pi/6} = \pm(0.5 - j0.866)$

$s_1 = \pm e^{j\pi} = \pm 1$

$s_2 = \pm e^{j8\pi/6} = \pm(0.5 + j0.866)$

$s_2 = s_0^*$

$$\therefore H_N(s) = \frac{1}{(s-s_1)(s-s_0)(s-s_0^*)}$$

$$= \frac{1}{(s-(-1))(s-(-0.5+j0.866))(s-(-0.5-j0.866))}$$

$$H_3(s) = \frac{1}{(s+1)(s^2+s+1)}$$

|||ly

(4) $N=4$

$$H_4(s) = \frac{1}{(s^2+0.765365s+1)(s^2+1.847765s+1)}$$

(5) $N=5$

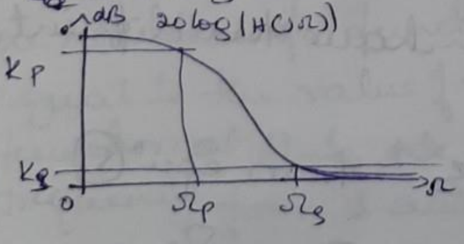
$$H_5(s) = \frac{1}{(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)}$$

first five Butterworth polynomials in a real factored form are given below

order N	Butterworth polynomial
1	$(s+1)$
2	$(s^2+\sqrt{2}s+1)$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.765365s+1)(s^2+1.847765s+1)$
5	$(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)$

Design of low pass Butterworth filters:

Filter requirements are normally given in terms of a set of critical frequencies, e.g., $\Omega_s \to$ stopband frequency, $\Omega_p \to$ passband frequency, and gain K_p and K_s



$$0 \geq 20\log|H(j\Omega)| \geq K_p \quad \forall\ \Omega \leq \Omega_p \qquad (1)$$

$$20\log|H(j\Omega)| \leq K_s \quad \forall\ \Omega \geq \Omega_s \qquad (2)$$

The frequency response of the Butterworth filter is characterized by only two parameters, $N \to$ the order of the filter and $\Omega_c \to$ cut-off frequency

$$|H_N(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} \qquad (3)$$

Using (3) in (1)

$$20\log|H_N(j\Omega)| = 10\log|H_N(j\Omega)|^2$$

$$\Rightarrow 10\log\left(\frac{1}{1+(\Omega_p/\Omega_c)^{2N}}\right) = K_p$$

$$\therefore \left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = 10^{-K_p/10} - 1 \quad \text{——— (4)}$$

III^{ly} using (3) in (2)

$$10\log\left(\frac{1}{1+(\Omega_s/\Omega_c)^{2N}}\right) = K_s$$

$$\therefore \left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = 10^{-K_s/10} - 1 \quad \text{——— (5)}$$

Dividing (4) by (5) we get

$$\left(\frac{\Omega_p}{\Omega_s}\right)^{2N} = \frac{10^{-K_p/10} - 1}{10^{-K_s/10} - 1}$$

$$2N\log_{10}\left(\frac{\Omega_p}{\Omega_s}\right) = \log_{10}\left[\frac{10^{-K_p/10} - 1}{10^{-K_s/10} - 1}\right]$$

$$\boxed{N = \frac{\log_{10}\left[\left(10^{-K_p/10} - 1\right)/\left(10^{-K_s/10} - 1\right)\right]}{2\log_{10}\left(\frac{\Omega_p}{\Omega_s}\right)}} \quad \text{——— (6)}$$

Using this 'N', we can have two different selections for Ω_c.

from eqn (4) & from eqn (5)

$$\Omega_c = \frac{\Omega_p}{\left(10^{-K_p/10} - 1\right)^{1/2N}} \quad \text{——— (7)}$$

$$\Omega_c = \frac{\Omega_s}{\left(10^{-K_s/10} - 1\right)^{1/2N}} \quad \text{——— (8)}$$

Equation (7) is used when we desire to meet the passband requirement. (45)

Equation (8) is used when we want to meet the stopband requirement.

The third option is to take a value b/n the two values of Ω_c. This gives $H(j\Omega)$ that exceeds both requirements.

Analog to Analog Transformation :

The transformation methods is not limited in its application to Butterworth filters or normalized filters.

Here normalized low-pass filter is used as a prototype filter.

In the transfer function $H(S)$ of the normalized LPF, replace 'S' by S/Ω_u

$$H'(S) = H(S)\big|_{S \to \frac{S}{\Omega_u}} = H(S/\Omega_u)$$

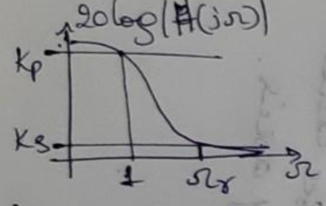
At $S = j\Omega$

$$H(S/\Omega_u) = H(j\Omega/\Omega_u)$$

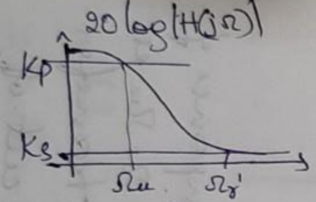
When $\Omega = \Omega_u$ we get $H'(S)\big|_{S=j\Omega_u} = H(j1)$

i.e., $|H'(j\Omega_u)| = |H(j1)|$

This means that the frequency response of the new transfer function evaluated at $\Omega = \Omega_u$ is equal to the value of the normalized transfer function at $\Omega = 1$. We have moved the cut-off frequency from 1 rad/sec to Ω_u rad/sec and hence have a scaling of the frequency axis.

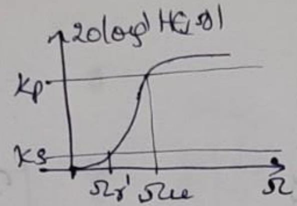


Normalized LPF



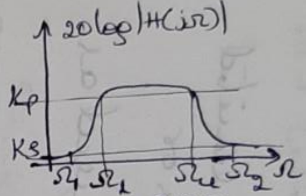
LPF

$s \to \dfrac{s}{\Omega_u}$



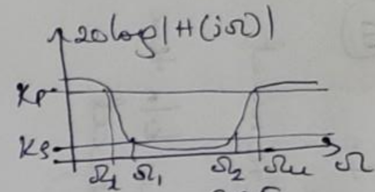
HPF

$s \to \dfrac{\Omega_u}{s}$



BPF

$s \to \dfrac{s^2 + \Omega_u \Omega_l}{s(\Omega_u - \Omega_l)}$



BSF

$s \to \dfrac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$

Backward Equations $\Rightarrow$

LPF: $\Omega_r = \dfrac{\Omega_r'}{\Omega_u}$

~~$\Omega_r' = \Omega_r \Omega_u$~~

HPF: $\Omega_r = \dfrac{\Omega_u}{\Omega_r'}$

~~$\Omega_r' = \dfrac{\Omega_u}{\Omega_r}$~~

BPF: $\Omega_r = \min\{|A|, |B|\}$

$A = \dfrac{-\Omega_1^2 + \Omega_l \Omega_u}{\Omega_1(\Omega_u - \Omega_l)}$

$B = \dfrac{\Omega_2^2 - \Omega_l \Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$

BSF: $\Omega_r = \min\{|A|, |B|\}$

$A = \dfrac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l \Omega_u}$

$B = \dfrac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_l \Omega_u}$

Forward Equations $\Rightarrow$

LPF: $\Omega_r' = \Omega_r \Omega_u$

HPF: $\Omega_r' = \dfrac{\Omega_u}{\Omega_r}$

BPF: $\Omega_{av} = (\Omega_u - \Omega_l)/2$

$\Omega_1 = \left[\Omega_r^2 \Omega_{av}^2 + \Omega_l \Omega_u\right]^{1/2} - \Omega_{av}\Omega_r$

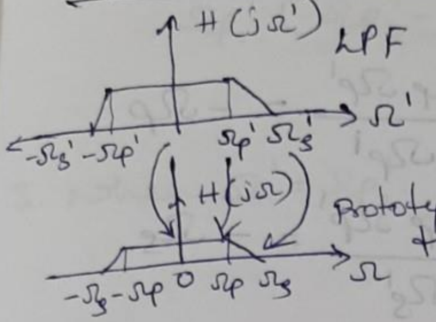
$\Omega_2 = \left(\Omega_r^2 \Omega_{av}^2 + \Omega_l \Omega_u\right)^{1/2} + \Omega_{av}\Omega_r$

BSF: $\Omega_{av} = (\Omega_u - \Omega_l)/2$

$\Omega_1 = \left((\Omega_{av}/\Omega_r)^2 + \Omega_l \Omega_u\right)^{1/2} - \Omega_{av}/\Omega_r$

$\Omega_2 = \left[(\Omega_{av}/\Omega_r)^2 + \Omega_l \Omega_u\right]^{1/2} + \Omega_{av}/\Omega_r$

Low-pass to low-pass transformation: (46)



$$s \rightarrow \frac{\Omega_p}{\Omega_p'} s$$

Ω_p' is the desired passband edge frequency

$$s = \frac{\Omega_p}{\Omega_p'} s$$

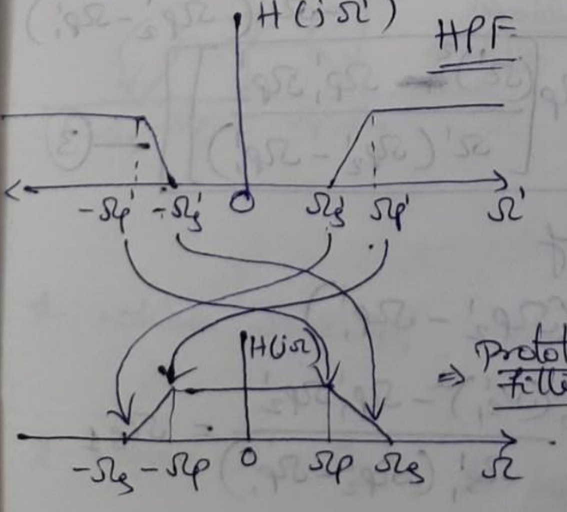
$$s = j\Omega \Rightarrow j\Omega = \frac{\Omega_p}{\Omega_p'}(j\Omega')$$

$$\boxed{\Omega = \frac{\Omega_p}{\Omega_p'}\Omega'} \quad \text{—(1)}$$

Equation (1) defines the relationship between the frequencies in the prototype analog LPF and those of the analog LPF desired.

1. when $\Omega' = 0$, $\Omega = 0$
2. when $\Omega' = \Omega_p'$, $\Omega = \frac{\Omega_p}{\Omega_p'}(\Omega_p') = \Omega_p$
3. when $\Omega' = \Omega_s'$, $\Omega = \frac{\Omega_p}{\Omega_p'}(\Omega_s') = \Omega_s$

Low-pass to High-pass transformation.



$$s \rightarrow \frac{\Omega_p \Omega_p'}{s}$$

$$j\Omega = \frac{\Omega_p \Omega_p'}{(j\Omega')}$$

$$\boxed{\Omega = -\frac{\Omega_p \Omega_p'}{\Omega'}} \quad \text{—(2)}$$

from equⁿ (2)

1. when $\Omega' = 0$, $\Omega = \infty$

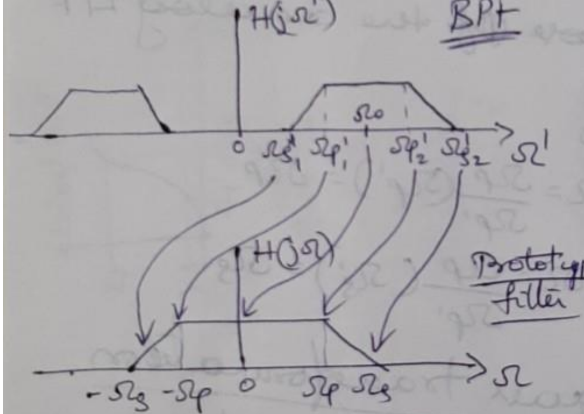
2. when $\Omega' = \Omega_p'$, $\Omega = -\dfrac{\Omega_p \Omega_p'}{\Omega_p'} = -\Omega_p$

3. when $\Omega' = \Omega_s'$, $\Omega = -\dfrac{\Omega_p \Omega_p'}{\Omega_s'} = -\Omega_s$

4. when $\Omega' = -\Omega_p'$, $\Omega = -\dfrac{\Omega_p \Omega_p'}{-\Omega_p'} = \Omega_p$

5. when $\Omega' = -\Omega_s'$, $\Omega' = \dfrac{-\Omega_p \Omega_p'}{-\Omega_s'} = \Omega_s$

Low-pass to Band-pass transformation



$$s = \Omega_p \frac{s^2 + \Omega_0^2}{w s}$$

$$= \Omega_p \frac{s^2 + \Omega_{p_1}' \Omega_{p_2}'}{s(\Omega_{p_2}' - \Omega_{p_1}')}$$

$$j\Omega = \Omega_p \frac{(j\Omega')^2 + \Omega_{p_1}' \Omega_{p_2}'}{j\Omega'(\Omega_{p_2}' - \Omega_{p_1}')}$$

$$\Omega = \Omega_p \frac{(-\Omega'^2) + \Omega_{p_1}' \Omega_{p_2}'}{-\Omega'(\Omega_{p_2}' - \Omega_{p_1}')}$$

$$\therefore \boxed{\Omega = \Omega_p \left[\frac{(\Omega')^2 - \Omega_{p_1}' \Omega_{p_2}'}{\Omega'(\Omega_{p_2}' - \Omega_{p_1}')}\right]} \quad \text{— (3)}$$

$\Omega_0 \rightarrow$ Center frequency

$\Omega_0^2 = \Omega_{p_1}' \Omega_{p_2}'$ & $w = (\Omega_{p_2}' - \Omega_{p_1}')$

1. when $\Omega' = \Omega_{s_1}'$, $\Omega = \Omega_p \dfrac{(\Omega_{s_1}')^2 - \Omega_{p_1}' \Omega_{p_2}'}{\Omega_{s_1}'(\Omega_{p_2}' - \Omega_{p_1}')} = \Omega_{s1}$

2. when $\Omega' = \Omega_{p1}'$, $\Omega = \Omega_p \dfrac{(\Omega_{p_1}')^2 - \Omega_{p_1}' \Omega_{p_2}'}{\Omega_{p_1}'(\Omega_{p_2}' - \Omega_{p_1}')}$

$$= \frac{\Omega_p\left(-\left(\Omega p_1' \Omega p_2' - (\Omega p_1')^2\right)\right)}{\Omega p_1' \left(\Omega p_2' - \Omega p_1'\right)} = \frac{\Omega p\left(-\Omega p_1'\left(\Omega p_2' - \Omega p_1'\right)\right)}{\Omega p_1'\left(\Omega p_2' - \Omega p_1'\right)} \quad (47)$$

$$= -\Omega p$$

3. when $\Omega' = \Omega p_2'$, $\Omega = \Omega p\left[\frac{(\Omega p_2')^2 - \Omega p_1' \Omega p_2'}{\Omega p_2'\left(\Omega p_2' - \Omega p_1'\right)}\right] = \Omega p$

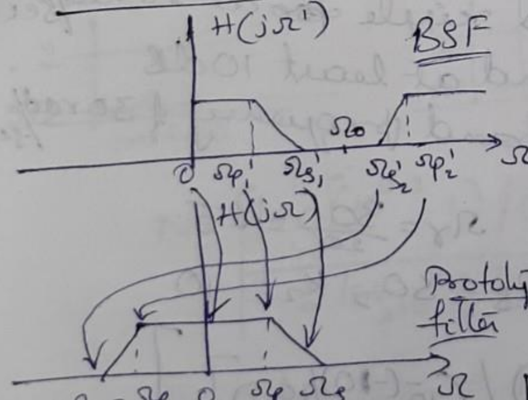
4. when $\Omega' = \Omega s_2'$, $\Omega = \Omega p\left[\frac{(\Omega s_2')^2 - \Omega p_1' \Omega p_2'}{\Omega s_2'\left(\Omega p_2' - \Omega p_1'\right)}\right] = \Omega s_2$

5. when $\Omega' = \Omega_0$, $\Omega = \Omega p\left[\frac{\Omega_0^2 - \Omega_0^2}{w\,\Omega_0}\right] = 0$

6. when $\Omega' = 0$, $\Omega = \Omega p\left[\frac{0^2 - \Omega p_1' \Omega p_2'}{0\left(\Omega p_2' - \Omega p_1'\right)}\right] = \infty$

7. $\Omega s = \min\left(|\Omega s_1|, |\Omega s_2|\right)$

Low-pass to Band stop transformation:



$$s \to \Omega p \frac{w s}{s^2 + \Omega_0^2}$$

$$\Omega_0^2 = \Omega p_1' \Omega p_2'$$

$$w = \Omega p_2' - \Omega p_1'$$

$$j\Omega = \Omega p \frac{\left(\Omega p_2' - \Omega p_1'\right)(j\Omega')}{(j\Omega')^2 + \Omega p_1' \Omega p_2'}$$

$$\boxed{\Omega = \Omega p\left[\frac{\Omega'\left(\Omega p_2' - \Omega p_1'\right)}{\Omega p_1' \Omega p_2' - (\Omega')^2}\right]} \quad (4)$$

1. when $\Omega' = \Omega p_1'$, $\Omega = \Omega p\left[\frac{\Omega p_1'\left(\Omega p_2' - \Omega p_1'\right)}{\Omega p_1' \Omega p_2' - (\Omega p_1')^2}\right] = \Omega p$

2. when $\Omega' = \Omega s_1'$, $\Omega = \Omega p\left[\frac{\Omega s_1'\left(\Omega p_2' - \Omega p_1'\right)}{\Omega p_1' \Omega p_2' - (\Omega s_1')^2}\right] = \Omega s_1$

3. when $\Omega' = \Omega s_2'$; $\Omega = \Omega p \left[\dfrac{\Omega s_2' (\Omega p_2' - \Omega p_1')}{\Omega p_1' \Omega p_2' - (\Omega s_2')^2} \right]$

$= \Omega s_2$

4. when $\Omega' = \Omega p_2'$; $\Omega = \Omega p \left[\dfrac{\Omega p_2' (\Omega p_2' - \Omega p_1')}{\Omega p_1' \Omega p_2' - (\Omega p_2')^2} \right]$

$= -\Omega p$

5. when $\Omega' = \Omega_0$, $\Omega = \Omega p \left[\dfrac{\Omega_0 (\Omega p_2' - \Omega p_1')}{\Omega p_1' \Omega p_2' - \Omega_0^2} \right] = \infty$

6. when $\Omega' = 0$, $\Omega = 0$

7. $\Omega s = \min \left(|\Omega s_1|, |\Omega s_2| \right)$

Butterworth filters Design

1) Design an analog Butterworth filter that has a 2dB passband ripple ~~for~~ at 20 rad/sec passband frequency and at least 10dB attenuation at a stopband frequency of 30 rad/sec

Solution

$\Omega_r = \dfrac{30}{20} = 1.5$

$\Omega p = 20$, $K_p = -2$, $\Omega s = 30$, $K_2 = -10$

$$N = \left\lceil \frac{\log_{10}\left[\left(10^{-(-2)/10} - 1\right) / \left(10^{-(-10)/10} - 1\right) \right]}{2 \log_{10} \left(\frac{1}{1.5}\right)} \right\rceil$$

$= \lceil 3.37 \rceil = 4$

$\Omega_c = \dfrac{1}{(10^{0.2} - 1)^{1/8}} = 1.0693$

$\Omega_{cp} = 20 \times 1.0693 = 21.3868$

$$H_4(s) = \frac{1}{(s^2+0.76536s+1)(s^2+1.84776s+1)}$$

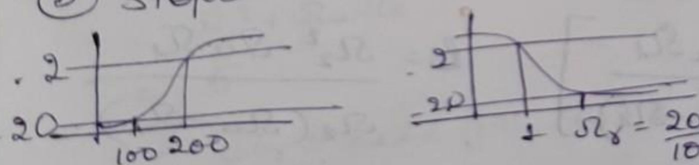
$$H_{LP}(s) = H_4(s)\Big|_{s \to \frac{s}{\Omega_c}} = \frac{s}{21.3868}$$

$$H_{LP}(s) = \frac{1}{\left(\left(\frac{s}{21.3868}\right)^2 + 0.76536\left(\frac{s}{21.3868}\right)+1\right) \times \left[\left(\frac{s}{21.3868}\right)^2 + 1.84776\left(\frac{s}{21.3868}\right)+1\right]}$$

2) Design a Butterworth analog high pass filter that meets the following requirements.

(a) Maximum passband ripple is 2dB and passband edge frequency is 200 rad/sec

(b) Stopband attenuation is 20dB at 100 rad/sec



$$\Omega_s = \frac{200}{100} = 2$$

$$N = \left\lceil \frac{\log_{10}\left[(10^{0.2}-1)/(10^2-1)\right]}{2\log_{10}\left(\frac{1}{2}\right)} \right\rceil = 4$$

$$\Omega_c = \frac{1}{(10^{0.2}-1)^{1/8}} = 1.0693$$

$$\Omega_{cp} = \frac{200}{1.0693} = 187.031$$

$$H_4(s) = \frac{1}{(s^2+0.76536s+1)(s^2+1.84776s+1)}$$

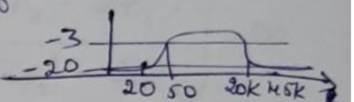
$$H_{HP}(s) = H_4(s)\Big|_{s \to \frac{\Omega_c}{s}} = \frac{187.031}{s}$$

$$H_{HP}(s) = \frac{s^4}{(s^2+143.1464s+34980.7521)(s^2+345.5s+34980.75)}$$

③ Design an analog Bandpass Butterworth filter with the following characteristics.

ⓐ a -3dB upper and lower cut-off frequency of 50Hz & 20KHz

ⓑ a stopband attenuation of at least 20dB at 20Hz and 45KHz



$\Omega_1 = 2\pi(20) = 125.663$ rad/sec

$\Omega_2 = 2\pi(45)\times10^3 = 2.827\times10^5$ rad/sec

$\Omega_u = 2\pi(20)\times10^3 = 1.256\times10^5$ rad/sec

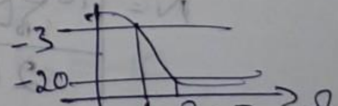
$\Omega_l = 2\pi(50) = 314.159$ rad/sec

$\Omega_r = \min(|A|, |B|)$

$$A = \left[\frac{-\Omega_1^2+\Omega_u\Omega_l}{\Omega_1(\Omega_u-\Omega_l)}\right] \qquad B = \frac{\Omega_2^2-\Omega_u\Omega_l}{\Omega_2(\Omega_u-\Omega_l)}$$

$= 2.505 \qquad\qquad = 2.25$

$\therefore\ \Omega_r = 2.25$



$$N = \frac{\log_{10}\left[(10^{0.3}-1)/(10^2-1)\right]}{2\log_{10}\left(\frac{1}{2.25}\right)} = 3$$

-1.99

-0.70

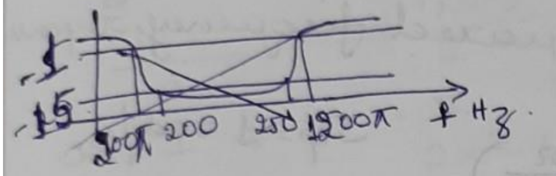
$$H_3(s) = \frac{1}{(s+1)(s^2+s+1)}$$

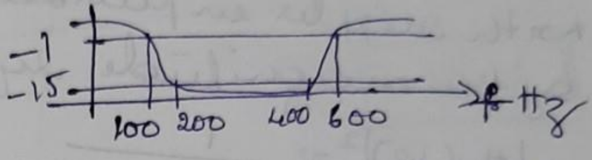
$$H_{BP}(s) = H_3(s)\Big|_{s\to\frac{s^2+\Omega_u\Omega_l}{s(\Omega_u-\Omega_l)}} = \frac{s^2+3.94\times10^7}{s(1.25\times10^5)}$$

$$H_{BP}(s) = \frac{1}{\left[\frac{s^2+3.9\times10^7}{s(1.25\times10^5)}+1\right]\left[\left[\frac{s^2+3.9\times10^7}{s(1.25\times10^5)}\right]^2+\left[\frac{s^2+3.9\times10^7}{s(1.25\times10^5)}\right]+1\right]}$$

(ii) Design an analog Bandstop Butterworth filter with the following characteristics.

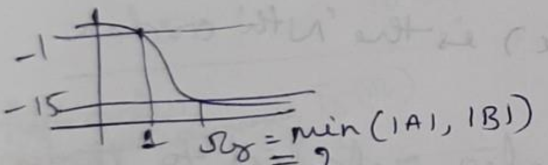
~~(a) α = 5dB passband ripple~~





$\Omega_l = 2\pi(100) = 200\pi$ $\quad \Omega_u = 2\pi(600) = 1200\pi$

$\Omega_1 = 2\pi(200) = 400\pi$ $\quad \Omega_2 = 2\pi(400) = 800\pi$



$\Omega_r = \min(|A|, |B|) = 2$

$A = 5 \qquad B = -2$

$\therefore \Omega_r = 2$

$$A = \frac{\Omega_1(\Omega_u-\Omega_l)}{-\Omega_1^2+\Omega_u\Omega_l} = 5 \qquad B = \frac{\Omega_2(\Omega_u-\Omega_l)}{-\Omega_2^2+\Omega_u\Omega_l} = -2$$

$\therefore \Omega_r = \min(|A|, |B|) = 2$

$$N = \frac{\log_{10}\left[(10^{0.1}-1)/(10^{1.5}-1)\right]}{2\log_{10}\left(\frac{1}{2}\right)} =$$

$N = 4$

$$H_n(s) = \frac{1}{(s^2+0.76536s+1)(s^2+1.84776s+1)}$$

$$H_{BS}(s) = H_n(s)\Big|_{s \to \frac{s(\Omega_u-\Omega_l)}{s^2+\Omega_u\Omega_l}} = \frac{s(1000\pi)}{s^2+(200\pi)(1200\pi)}$$

Chebyshev filters

There are two types of chebyshev filters

1. Type I → ripples in passband
2. Type II → ripples in stopband.

The normalized Chebyshev Type I filter with ripples in passband is characterized by the magnitude squared frequency response

$$|H_N(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)} \qquad \Omega_p = 1 \text{ rad/sec}$$

$$|H_N(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega)} \quad \text{——— (1)}$$

where $T_N(\Omega)$ or $T_N(x)$ is the 'Nth' order chebyshev polynomial.

ϵ is ~~the~~ a parameter related to the ripples in passband.

The chebyshev polynomials are defined by the following expression:

$$T_N(x) = \cos(Nt)\big|_{x=\cos t} \quad \text{——— (2)}$$

Both x and $T_N(x)$ are within the interval $[-1 \;\&\; +1]$

for $N=0$, $T_N(x) = \cos 0 = 1$

$N=1$, $T_N(x) = \cos t = x$

$N=2$, $T_N(x) = \cos 2t = 2\cos^2 t - 1 = 2x^2 - 1$

for $N>2$,

$$T_{N+1}(x) = \cos(N+1)t = \cos Nt \cos t - \sin Nt \sin t$$

$$T_{N-1}(x) = \cos(N-1)t = \cos Nt \cos t + \sin Nt \sin t$$

$$T_{N+1}(x) + T_{N-1}(x) = 2\cos Nt \cos t$$

$$T_{N+1}(x) = 2\cos Nt \cos t - T_{N-1}(x)$$

$$T_{N+1}(x) = 2x\,T_N(x) - T_{N-1}(x)$$

$\therefore \boxed{T_N(x) = 2x\,T_{N-1}(x) - T_{N-2}(x)}$ —— (3) (50)

$N=3 \quad T_3(x) = 2x(2x^2-1) - x = 4x^3 - 3x$

$T_4(x) = 2x(4x^3-3x) - (2x^2-1)$

$= 8x^4 - 8x^2 + 1$

$T_5(x) = 16x^5 - 20x^3 + 5x$

The first six Chebyshev polynomials

N	$T_N(x)$
0	1
1	x
2	$2x^2-1$
3	$4x^3-3x$
4	$8x^4-8x^2+1$
5	$16x^5-20x^3+5x$

=> Table 1

fifth order chebyshev polynomial $T_5(\Omega)$

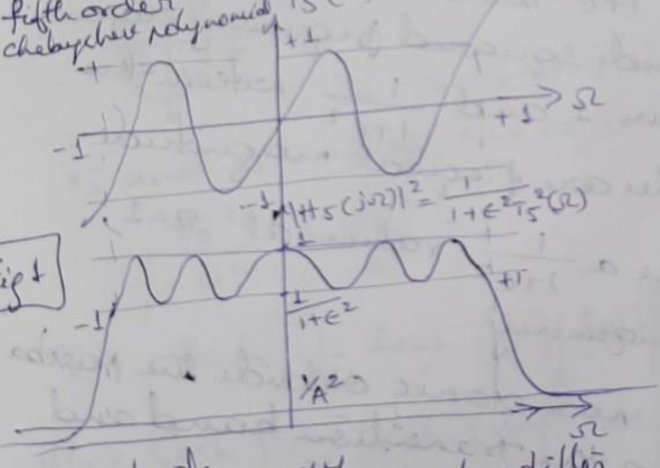
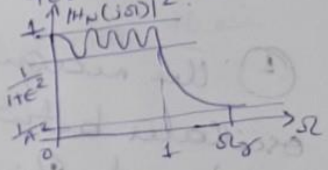


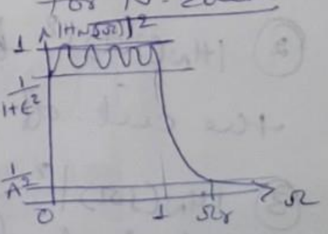
Fig 1

Fig. 2

for N-odd



for N-Even



Magnitude squared frequency response of Type I chebyshev filter

Chebyshev Polynomial $T_N(\Omega)$ or $T_N(x)$	Magnitude squared frequency response $\lvert H_N(j\Omega)\rvert^2$ of type I Chebyshev filter
1. It oscillates between +1 and −1 in the interval $x=-1$ to +1	1) These oscillations cause equal magnitude ripples in $\lvert H_N(j\Omega)\rvert^2$ 2) The amplitude of $\lvert H_N(j\Omega)\rvert^2$ oscillates between +1 & $\frac{1}{1+\epsilon^2}$ as Ω goes from −1 to +1 the periods of oscillations 3) The periods of oscillations are unequal.

2) It grows towards $+\infty$ and $-\infty$, outside the interval, $x[-1, +1]$

[Refer: Fig 1]

3) for $N=0$ & $N=$ even #s at $x=0$, $T_N(x)=1$

for $N=$ odd #s at $x=0$, $T_N(x)=0$

[i.e, at $\Omega=0$]

[Table - 1]

2) $|H_N(j\Omega)|^2$ heads toward zero outside -1 and $+1$ as the polynomial heads toward $+\infty$ & $-\infty$

3) $N=0$ & even & $\Omega=0$

$$|H_N(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega)}$$

$$= \frac{1}{1+\epsilon^2}$$

$N=$ odd & $\Omega=0$

$$|H_N(j\Omega)|^2 = \frac{1}{1+\epsilon^2(0)} = 1$$

[Refer; Fig(2)]

The following properties are observed from figure ② & equation

$$|H_N(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega)}$$

① The magnitude squared frequency response oscillates between 1 and $\frac{1}{1+\epsilon^2}$ within the passband. Ripples are of equal magnitude.

② $|H_N(j\Omega)|^2$ has a $\frac{1}{1+\epsilon^2}$ value at $\Omega=1$, the cut-off frequency.

③ $|H_N(j\Omega)|^2$ is monotonic outside the passband, including the transition band and the stopband.

④ The stopband begins at Ω_r, where the $|H_N(j\Omega)|^2$ is $\frac{1}{A^2}$

Design of a causal stable transfer function $H_N(s)$ (51)

The chebyshev magnitude squared frequency response at $\Omega_p = 1$ rad/sec, (normalized) is

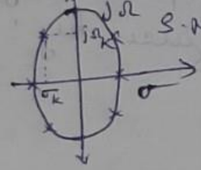
$$|H_N(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega)} \quad \text{---} (1)$$

find poles of $H_N(s)\, H_N(-s)$ and consider the poles on the left half of s-plane

$$H_N(s)\; H_N(-s) = 1 + \epsilon^2 T_N^2\left(\frac{s}{j}\right) = 0 \quad \text{---} (2)$$

$$s = j\Omega \quad \therefore \Omega = s/j$$

The poles of $H_N(s)\cdot H_N(-s)$ are found to lie on an ellipse in the s-plane as shown in figure.



If $s_k = \sigma_k + j\Omega_k$ is a pole, then σ_k and Ω_k satisfy the equation

$$\frac{\sigma_k^2}{a^2} + \frac{\Omega_k^2}{b^2} = 1$$

$$\sigma_k = -\sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right] \sin\left[\left(\frac{2k-1}{2N}\right)\pi\right]$$

$$\Omega_k = \cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right] \cos\left[\left(\frac{2k-1}{2N}\right)\pi\right]$$

$k = 1$ to N

Consider only the left half of s-plane, the transfer function of the normalized stable, low pass Chebyshev-I filter is:

$$H_N(s) = \frac{K}{\prod_{\substack{LHP\\ only}} (s - s_k)} = \frac{K}{V_N(s)} = \frac{K}{s^N + b_{N-1}s^{N-1} + \dots + b_1 s + b_0}$$

where 'K' is a normalizing factor whose value makes $|H(j0)| = 1$ for 'N' odd and $|H(j0)| = \frac{1}{\sqrt{1+\epsilon^2}}$ for 'N' even.

$V_N(s)$ is a polynomial

$$V_N(s) = s^N + b_{N-1} s^{N-1} + \dots + b_1 s + b_0$$

$$K = V_N(0) = b_0 \longrightarrow \text{'N' odd}$$

$$K = \frac{V_N(0)}{(1+\epsilon^2)^{1/2}}$$

$$= \frac{b_0}{(1+\epsilon^2)^{1/2}} \longrightarrow \text{'N' even}$$

Note: The poles are not equally spaced on the ellipse, but they do exhibit symmetry with respect to the 'σ' axis.

Selection of N:

The 'N' that satisfies a specified ripple characterized by ϵ and a stopband gain of '$\frac{1}{A}$' at a particular Ω_r is given by:

$$N = \left\lceil \frac{\log_{10}\left(g + \sqrt{g^2-1}\right)}{\log_{10}\left(\Omega_r + \sqrt{\Omega_r^2-1}\right)} \right\rceil$$

where $g = \left[\frac{(A^2-1)}{\epsilon^2}\right]^{1/2}$ & $A = \frac{1}{|H_N(j\Omega_r)|}$

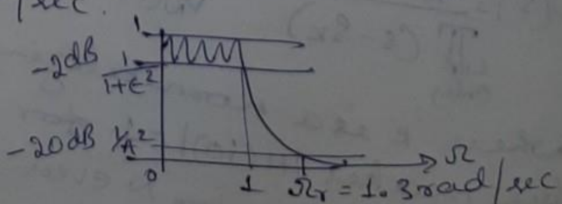
Examples:

1) Design a low-pass 1-rad/sec bandwidth Chebyshev-I filter with the following characteristics.

ⓐ Acceptable passband ripple of 2dB

ⓑ Cut-off radian frequency of 1 rad/sec

ⓒ Stopband attenuation of 20dB or greater beyond 1.3 rad/sec.



Solution: (52)

$$20\log|H(j1)| = 20\log\left[\frac{1}{1+\epsilon^2}\right]^{1/2} = 10\log\left[\frac{1}{1+\epsilon^2}\right] = -2$$

$$\therefore \quad \epsilon = \sqrt{10^{0.2}-1} = 0.76478$$

$$20\log|H_N(j\Omega_s)| = 20\log\left[\frac{1}{A^2}\right]^{1/2} = 20\log\left(\frac{1}{A}\right) = -20$$

$$A = 10^{20/20} = 10$$

$$g = \left[\frac{A^2-1}{\epsilon^2}\right]^{1/2} = 13.01$$

$$N = \left\lceil \frac{\log_{10}\left(13.01+\sqrt{13.01^2-1}\right)}{\log_{10}\left(1.3+\sqrt{1.3^2-1}\right)} \right\rceil = \lceil 4.3 \rceil = 5$$

$s_k = \sigma_k + j\Omega_k$ for $k = 1$ to N

$k=1, \quad s_1 = -0.0675 + 0.9735j$

$k=2, \quad s_2 = -0.1766 + 0.6016j$

$k=3, \quad s_3 = -0.2183$

$k=4, \quad s_4 = -0.1766 - 0.6016j$

$k=5, \quad s_5 = -0.0675 - 0.9735j$

$$H_5(s) = \frac{K}{(s+0.2183)(s+0.0675-0.97j)(s+0.067+0.97j)\times(s+0.17-0.6016j)(s+0.17+0.6016j)}$$

$K = b_0$ since N is odd.

$$H_5(s) = \frac{0.08172}{s^5+0.7064s^4+1.4995s^3+0.6934s^2+0.459s+0.08172}$$

Poles lie on an ellipse;

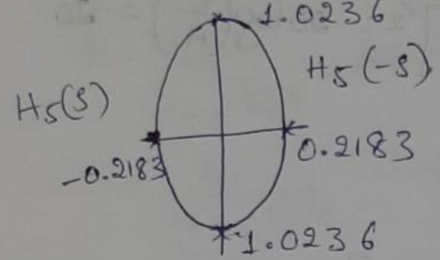
$$\therefore \quad \beta = \left[\frac{\sqrt{1+\epsilon^2}+1}{\epsilon}\right]^{1/N}$$

Major axis: $r_1 = \Omega_p\left[\frac{\beta^2+1}{2\beta}\right]$ Minor axis: $r_2 = \Omega_p\left[\frac{\beta^2-1}{2\beta}\right]$

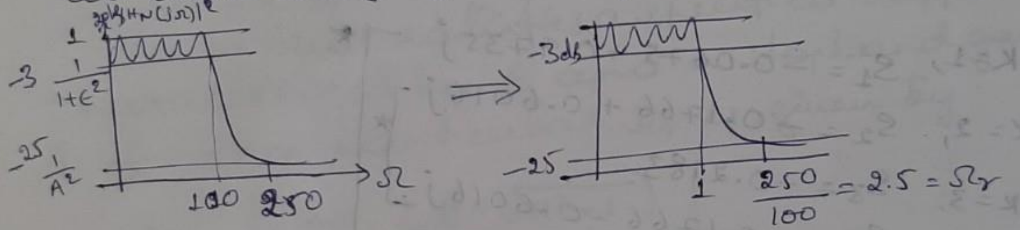
Ω_p is the passband frequency.

$\therefore\ \beta = 1.2419 \qquad r_1 = 1.0236$

$\Omega_p = 1\ \text{rad/sec} \qquad r_2 = 0.2183$



2] Design a chebyshev analog LPF that has a -3 dB cutoff frequency of 100 rad/sec and a stopband attenuation of 25 dB or greater for all radian frequencies past 250 rad/sec.



$\epsilon = \sqrt{10^{3/10} - 1} = \sqrt{10^{0.3} - 1} = 0.9976$

$A = 10^{25/20}, \quad g = \left[\frac{(10^{25/20})^2 - 1}{0.9976^2}\right]^{1/2}$

$\boxed{N = 3}$

Since, N is odd
$K = b_0$

$s_k = \sigma_k + j\Omega_k \quad k = 1 \text{ to } N$

$\therefore\ k=1, \quad s_1 = -0.1493 + j0.9038$

$k=2, \quad s_2 = -0.2986$

$k=3, \quad s_3 = -0.1493 - 0.9038j$

$$H_3(s) = \frac{K}{(s+0.2986)(s+0.1493-j0.9038)(s+0.1493+j0.9038)}$$

$$= \frac{0.25059}{(s+0.2986)(s+0.1493-j0.9038)(s+0.1493+j0.9038)}$$

(53)

$$H_{LP}(s) = H_3(s)\Big|_{s \to \frac{s}{100}}$$

$$H_{LP}(s) = \frac{0.25059}{\left(\frac{s}{100}+0.2986\right)\left(\frac{s}{100}+0.1493-j0.9038\right)\left(\frac{s}{100}+0.14-j0.9\right.}$$

3] Determine the order and magnitude squared ~~frequency~~ function of a normalized Chebyshev-I filter for the following specification
(i) Maximum passband ripple is 1 dB
(ii) Stopband attenuation is 40dB for $\Omega \geq 4$ rad/sec

Solution:

$\Omega_p = 1$ rad/sec

$$\epsilon = \sqrt{10^{0.1}-1} = 0.508$$

$\Omega_r = \Omega_s = 4$ rad/sec,

$$A = 10^{40/20} = 100, \quad g = \left[\frac{100^2-1}{0.508^2}\right]^{1/2} = 196.84$$

$$N = \left\lceil \frac{\log_{10}\left(196.84+\sqrt{(196.84)^2-1}\right)}{\log_{10}\left(4+\sqrt{4^2-1}\right)} \right\rceil = 3$$

for $N=3$, $T_N(\Omega) = 4\Omega^3 - 3\Omega$

$$\therefore |H_a(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega)} = \frac{1}{1+(0.508)^2(4\Omega^3-3\Omega)^2}$$

4] Design a fourth ~~second~~ order bandpass Chebyshev-I filter with the passband of 200Hz to 300Hz at an allowed ripple of 0.5 dB. for $N_{normzd} = 2$.

Solution: Given: $N=2$, $f_l = 200$Hz, $f_u = 300$Hz

$\Omega_l = 2\pi(200)$ $\quad \Omega_u = 2\pi(300)$

$\Omega_l = 1256.6$ $\quad \Omega_u = 1885$

$$\epsilon = \sqrt{10^{0.5/10}-1} = 0.349$$

$$S_k = \sigma_k + j\Omega_k \qquad k = 1 \text{ to } N$$

$S_1 = -0.71 + j1.004$ & $S_2 = -0.71 - j1.004$

$N=2, \Rightarrow$ even $\therefore K = \frac{b_0}{\sqrt{1+\epsilon^2}} = 1.431$

$$H_2(s) = \frac{K}{\prod(s-s_k)} = \frac{1.431}{s^2+1.425s+1.516}$$

$$H_{BP}(s) = H_2(s)\Big|_{s \to \frac{s^2+(1256.6\times1885)}{s(1885-1256.6)}} = \frac{s^2+2.36\times10^6}{s(628.4)}$$

$$\therefore H_{BP}(s) = \frac{1.431}{\left[\left(\frac{s^2+2.3\times10^6}{628.4s}\right)^2 + 1.425\left(\frac{s^2+2.3\times10^6}{628.4s}\right) + 1.516\right]}$$

5] Design a second order chebyshev highpass filter for the following specification
(i) Ripples in passband, 1 dB
(ii) Passband edge frequency, 1 rad/sec = $\Omega_{normalized}$ $\Omega_{desired}$

Solution:

$\epsilon = \sqrt{10^{0.1}-1} = 0.508$ | $\Omega_p = 1$ rad/sec

$N=2$

$s_k = \sigma_k + j\Omega_k \quad k=1,2$ | $s_1 = -0.54+j0.89$, $s_2 = -0.54-j0.89$

$$H_2(s) = \frac{K}{\prod(s-s_k)} = \frac{\frac{b_0}{\sqrt{1+\epsilon^2}}}{s^2+1.0977s+1.1025}$$

$$H_2(s) = \frac{0.9829}{s^2+1.0977s+1.1025}$$

$$H_{HP}(s) = H_2(s)\Big|_{s\to\frac{1}{s}}$$

$$= \frac{0.9829}{\left(\frac{1}{s}\right)^2+1.0977\left(\frac{1}{s}\right)+1.1025}$$

$$= \frac{0.9829s^2}{1+1.0977s+1.1025s^2}$$

6] Design a chebyshev filter to meet the following (52) specifications. **LPF**

$0.707 \le |H(j\Omega)| \le 1$ $\quad\quad \Omega_p = 0.325$ rad/sec

$|H(j\Omega)| \le 0.1$ $\quad\quad \Omega_s = 1$ rad/sec

Solution:

A_p in dB $= -20\log(0.707) = 3dB$

A_s in dB $= -20\log(0.1) = 20dB$

$\epsilon = \sqrt{10^{3/10}-1} = 0.997$ $\quad\quad \Omega_r = \frac{1}{0.325} = 3.076$

$A = 10^{20/20} = 10$

$g = \left[\frac{10^2-1}{(0.997)^2}\right]^{1/2} = 10.05$

$N = \left\lceil \frac{\log_{10}(10.05+\sqrt{10.05^2-1})}{\log_{10}(3.076+\sqrt{3.076^2-1})} \right\rceil = 2$

$\frac{1}{\sqrt{1+\epsilon^2}}$ $\to 0$ -3

$s_k = \sigma_k + j\Omega_k \quad k = 1, 2$ $\quad\quad K = \frac{b_0}{\sqrt{1+\epsilon^2}} = 0.5$

$s_1 = -0.32 + j0.777$

$s_2 = -0.32 - j0.777$

$H_N(s) = \frac{0.5}{s^2 + 0.644s + 0.707}$

$H_{LP}(s) = H_N(s)\Big|_{s \to \frac{s}{0.325}}$

$H_{LP}(s) = \frac{0.5}{\left(\frac{s}{0.325}\right)^2 + 0.644\left(\frac{s}{0.325}\right) + 0.707}$

7] An analog low pass filter is to be designed that has a passband cut-off frequency of 0.668 rad/sec with $\delta_p = 0.01$ and a stopband cut-off frequency $\Omega_s = 1$ rad/sec with $\delta_s = 0.01$ what orders of Butterworth, Chebyshev filters are necessary to meet the design specification.

$$A_p = 1 - \delta_p = 1 - 0.01 = 0.99$$
$$A_s = \delta_s = 0.01$$

$$\Omega_p = 0.668$$
$$\Omega_s = 1$$

$$20\log(0.99) = -0.08\ dB$$
$$20\log(0.01) = -40\ dB$$

$$\Omega_r = \frac{\Omega_s}{\Omega_p} = \frac{1}{0.668}$$
$$\Omega_r = 1.5$$

$$N_{Butterworth} = \left\lceil \frac{\log_{10}\left[\frac{(10^{\frac{0.08}{10}} - 1)}{(10^{40/10} - 1)}\right]}{2\log_{10}\left(\frac{1}{1.5}\right)} \right\rceil$$

$$= \lceil 16.27 \rceil = 17$$

$$N_{chebyshev} = \left\lceil \frac{\log_{10}[g + \sqrt{g^2 - 1}]}{\log_{10}[\Omega_r + \sqrt{\Omega_r - 1}]} \right\rceil$$

$$\epsilon = \sqrt{10^{\frac{0.08}{10}} - 1} = 0.136$$
$$A = 10^{40/20} = 100$$
$$g = \left[\frac{100^2 - 1}{0.136^2}\right]^{1/2} = 735.25$$

$$N = 7.57 = 8$$

The order of Chebyshev filter is lower than that of Butterworth filter

11 Discuss the four types of analog to analog frequency transformations.

Low-pass to low-pass

Suppose that we have a low-pass filter with passband edge frequency Ω_p and we wish to convert it into another lowpass filter with passband edge frequency Ω_p'.

The transformation that accomplishes this is

$$s \longrightarrow \frac{\Omega_p}{\Omega_p'} s.$$

Low-pass to high-pass

Suppose that we have a low-pass filter with passband edge frequency Ω_p and we wish to convert it into a high-pass filter with passband edge frequency Ω_p'.

The transformation that accomplishes this is

$$s \longrightarrow \frac{\Omega_p \Omega_p'}{s}$$

low-pass to band-pass

Suppose that we have a low pass filter with passband edge frequency Ω_p and we wish to convert it to a bandpass filter having lower and upper band edge frequencies Ω_l and Ω_u respectively,

The transformation that accomplishes this is

$$s \longrightarrow \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

low-pass to band-reject filter

Suppose that we have a lowpass filter with passband edge frequency Ω_p and we wish to convert it to a band-reject filter having lower and upper band edge frequencies Ω_l and Ω_u, respectively.

The transformation that accomplishes this is

$$s \longrightarrow \Omega_p \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}$$

12 $H(s) = \dfrac{1}{s^2 + \sqrt{2}\, s + 1}$ represents the transfer function of a low pass filter with ~~pa~~ cut-off frequency of 1 rad/s. Using frequency transformation, find the transfer function of the

following analog filters.

a) a low-pass filter with cutoff frequency 10 rad/s.

$\Omega_c = 1$ rad/s

$\Omega_c' = 10$ rad/s

Transformation : $s \longrightarrow \frac{\Omega_c}{\Omega_c'} s$

ie, $s \longrightarrow \frac{s}{10}$

$\therefore$ Required $H(s) = \dfrac{1}{\left(\frac{s}{10}\right)^2 + \sqrt{2}\left(\frac{s}{10}\right) + 1}$

$$= \frac{1}{\frac{s^2}{100} + \frac{\sqrt{2}}{10}s + 1}$$

$$= \frac{100}{s^2 + 10\sqrt{2}s + 100}$$

b) a highpass filter with cut-off frequency of 100 rad/s

Transformation: $s \rightarrow \dfrac{\Omega_c \Omega_c'}{s} = \dfrac{100}{s}$

$\Omega_c = 1$ rad/s, $\Omega_c' = 100$ rad/s

$\therefore$ Required $H(s) = \dfrac{1}{\left(\frac{100}{s}\right)^2 + \sqrt{2}\left(\frac{100}{s}\right) + 1}$

$$= \frac{1}{\frac{100^2}{s^2} + \sqrt{2}\,\frac{100}{s} + 1}$$

$$= \frac{s^2}{100^2 + \sqrt{2}\,100s + s^2}$$

c) a bandpass filter with centre frequency $\Omega_0 = 100$ rad/s and bandwidth $B_0 = 10$ rad/s

$\Omega_l = 95, \quad \Omega_u = 105$

Tansformation : $s \to \Omega_c \dfrac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$

$$= \frac{s^2 + 9975}{s \cdot 10}$$

$$\therefore \text{Required } H(s) = \frac{1}{\left(\frac{s^2+9975}{10s}\right)^2 + \sqrt{2}\left(\frac{s^2+9975}{10s}\right) + 1}$$

$$= \frac{1}{\frac{s^4 + 9975^2 + 19950s}{100s^2} + \sqrt{2}\,\frac{s^2+9975}{10s} + 1}$$

$$= \frac{100s^2}{s^4 + 19950s + 9975^2 + \sqrt{2}\,(s^2+9975)\,10s + 100s^2}$$

$$= \frac{100s^2}{s^4 + 100s^2 + 10\sqrt{2}s^3 + (19950 + 99750\sqrt{2})s + 9975^2}$$

Compare Butterworth and Chebyshev filters.

1. Low pass Butterworth filters are all pole filters

Lowpass Chebyshev Type-I filters are all pole filters, but Type-II Chebyshev filters contain both poles and zeros.

2. Frequency response of Butterworth filters is monotonic in both passband and stopband.

Type-I Chebyshev filters exhibit equiripple behaviour in the passband and a monotonic characteristic in the stopband.

Type-II Chebyshev filters exhibit a monotonic behavior in the passband and an equiripple behavior in the stopband.

3. For the given frequency response characteristics, order of the Chebyshev filter is less than the order of the Butterworth filter.

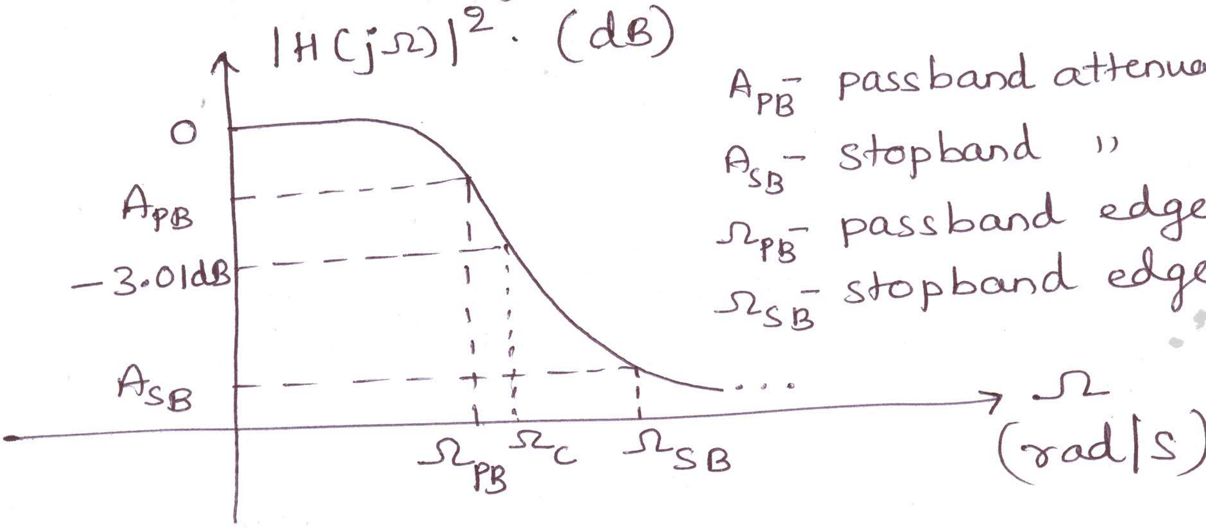
4. Poles of a Butterworth filter lie on a circle.

Poles of a Chebyshev filter lie on an ellipse.

19 Derive the expression for order and cut-off frequency of Butterworth lowpass filter.

The magnitude squared response of low

pass Butterworth filter is as shown below:



A_{PB} - passband attenuation
A_{SB} - stopband "
Ω_{PB} - passband edge
Ω_{SB} - stopband edge

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \quad \text{---} \quad (1)$$

i) To find the order of the filter.

$$A_{PB} = -10\log_{10}\left(|H(j\Omega)|^2\right)\Big|_{\Omega=\Omega_{PB}}$$

$$= -10\log_{10}\left(\frac{1}{1+\left(\frac{\Omega_{PB}}{\Omega_c}\right)^{2N}}\right)$$

$$= 10\log_{10}\left(1+\left(\frac{\Omega_{PB}}{\Omega_c}\right)^{2N}\right)$$

$$\therefore \quad 1+\left(\frac{\Omega_{PB}}{\Omega_c}\right)^{2N} = 10^{\frac{A_{PB}}{10}}$$

$$\left(\frac{\Omega_{PB}}{\Omega_c}\right)^{2N} = 10^{\frac{A_{PB}}{10}} - 1 \quad \text{----} \quad (2)$$

$$A_{SB} = -10\log_{10}\left(|H(j\Omega)|^2\right)\Big|_{\Omega=\Omega_{SB}}$$

$$= -10\log_{10}\left(|H(j\Omega)|^2\right)\Big|_{\Omega=\Omega_{SB}}$$

$$= -10\log_{10}\left(\frac{1}{1+\left(\frac{\Omega_{SB}}{\Omega_c}\right)^{2N}}\right)$$

$$= 10\log_{10}\left(1+\left(\frac{\Omega_{SB}}{\Omega_c}\right)^{2N}\right)$$

$$1+\left(\frac{\Omega_{SB}}{\Omega_c}\right)^{2N} = 10^{\frac{A_{SB}}{10}}$$

$$\therefore \left(\frac{\Omega_{SB}}{\Omega_c}\right)^{2N} = 10^{\frac{A_{SB}}{10}} - 1 \quad \text{----} \quad (3)$$

(2) ÷ (3) gives,

$$\left(\frac{\Omega_{PB}}{\Omega_{SB}}\right)^{2N} = \frac{10^{\frac{A_{PB}}{10}} - 1}{10^{\frac{A_{SB}}{10}} - 1}$$

$$\therefore 2N\log_{10}\left(\frac{\Omega_{PB}}{\Omega_{SB}}\right) = \log_{10}\left(\frac{10^{\frac{A_{PB}}{10}} - 1}{10^{\frac{A_{SB}}{10}} - 1}\right)$$

$$\therefore N = \frac{\log_{10}\left(\frac{10^{\frac{A_{PB}}{10}} - 1}{10^{\frac{A_{SB}}{10}} - 1}\right)}{2\log_{10}\left(\frac{\Omega_{PB}}{\Omega_{SB}}\right)}$$

$$\therefore N = \frac{\log_{10}\left(\sqrt{\frac{10^{\frac{A_{PB}}{10}}-1}{10^{\frac{A_{SB}}{10}}-1}}\right)}{\log_{10}\left(\frac{\Omega_{PB}}{\Omega_{SB}}\right)} \quad \text{--- (4)}$$

$$N = \lceil N \rceil$$

(Note: To get the order of Chebyshev Type I filter, replace 'log' with '$\cosh^{-1}$')

ii) To find the cut-off frequency

From (2),

$$\left(\frac{\Omega_{PB}}{\Omega_C}\right)^{2N} = 10^{\frac{A_{PB}}{10}} - 1$$

$$\therefore \frac{\Omega_{PB}}{\Omega_C} = \left[10^{\frac{A_{PB}}{10}} - 1\right]^{\frac{1}{2N}}$$

$$\therefore \Omega_C = \frac{\Omega_{PB}}{\left[10^{\frac{A_{PB}}{10}} - 1\right]^{\frac{1}{2N}}} \quad \text{--- (5)}$$

iii) To find the location of poles.

From (1), $|H(j\Omega)|^2 = \dfrac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$

$$\therefore H(j\Omega)\,H^*(j\Omega) = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$$\therefore H(j\Omega)\,H(-j\Omega) = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$$\therefore H(s)\,H(-s) = \frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}} \quad \text{--- (6)}$$

To find location of poles,

$$1+\left(\frac{s}{j\Omega_c}\right)^{2N} = 0$$

$$\left(\frac{s}{j\Omega_c}\right)^{2N} = -1 = e^{j(2k+1)\pi} \qquad k = 0, 1, 2, \ldots 2N-1$$

$$\therefore \frac{s_k}{j\Omega_c} = e^{j(2k+1)\frac{\pi}{2N}}, \quad k = 0, 1, 2, \ldots 2N-1$$

$$s_k = j\Omega_c\, e^{j(2k+1)\frac{\pi}{2N}} \quad \ldots \quad (7)$$

$$k = 0, 1, \ldots 2N-1$$

There are '2N' roots. out of which N-roots correspond to $H(s)$ and N correspond to $H(-s)$.

$\therefore$ Poles of $H(s)$ are,

$$s_k = j\Omega_c\, e^{j(2k+1)\frac{\pi}{2N}}, \quad k = 0, 1, 2, \ldots N-1$$

$$= j\Omega_c\left[\cos\left\{(2k+1)\frac{\pi}{2N}\right\} + j\sin\left\{(2k+1)\frac{\pi}{2N}\right\}\right]$$

$$= -\Omega_c \sin\left\{\frac{(2K+1)\pi}{2N}\right\} + j\Omega_c \cos\left\{\frac{(2K+1)\pi}{2N}\right\} \quad \text{---} (8)$$

$K = 0, 1, 2, 3, \ldots N-1$

Equation (8) gives location of poles.

$$H(s) = \frac{(-1)^N \prod_{K=0}^{N-1} S_K}{\prod_{K=0}^{N-1} (s - s_K)} \quad \text{---} \quad (9)$$

20 Define Chebyshev polynomial and list its properties.

Chebyshev polynomial:

$$C_N(\Omega) = \cos\left[N \cos^{-1}(\Omega)\right] \quad \text{----} \quad (1)$$

$$C_{N+1}(\Omega) = \cos\left[(N+1)\cos^{-1}(\Omega)\right]$$

$$= \cos\left[N\cos^{-1}(\Omega) + \cos^{-1}(\Omega)\right]$$

$$= \cos\left[N\cos^{-1}(\Omega)\right]\Omega - \sin\left[N\cos^{-1}(\Omega)\right] \sin\left[\cos^{-1}(\Omega)\right] \quad \text{---} (2)$$

$$C_{N-1}(\Omega) = \cos\left[(N-1)\cos^{-1}(\Omega)\right]$$

$$= \cos\left[N\cos^{-1}(\Omega)\right]\Omega + \sin\left[N\cos^{-1}(\Omega)\right] \sin\left[\cos^{-1}(\Omega)\right] \quad \text{---} (3)$$

(2)+(3) gives,

$$C_{N+1}(\Omega)+C_{N-1}(\Omega) = 2\Omega C_N(\Omega)$$

$$\therefore C_{N+1}(\Omega) = 2\Omega C_N(\Omega) - C_{N-1}(\Omega) \quad \cdots (4)$$

From (1)

$$C_0(\Omega) = 1.$$

$$C_1(\Omega) = \Omega$$

$$C_2(\Omega) = 2C_1(\Omega)\Omega - C_0(\Omega)$$
$$= 2\Omega^2 - 1$$

$$C_3(\Omega) = 2C_2(\Omega)\Omega - C_1(\Omega)$$
$$= 4\Omega^3 - 3\Omega$$

$$C_4(\Omega) = 8\Omega^4 - 8\Omega^2 + 1$$

Properties of Chebyshev polynomial

1. $C_N(\Omega) = -C_N(-\Omega)$ for odd N

   $C_N(\Omega) = C_N(-\Omega)$ for even N

2. $C_N(0) = (-1)^{\frac{N}{2}}$ for even N

   $C_N(0) = 0$ for odd N

3. $C_N(1) = 1$ for all N

4. $C_N(\Omega)$ oscillates with equal ripple between ± 1 for $|\Omega| < 1$

$$0 \leq |C_N(\Omega)| \leq 1 \quad \text{for} \quad |\Omega| \leq 1$$

5. $C_N(\Omega)$ monotonically increases for $|\Omega|>1$

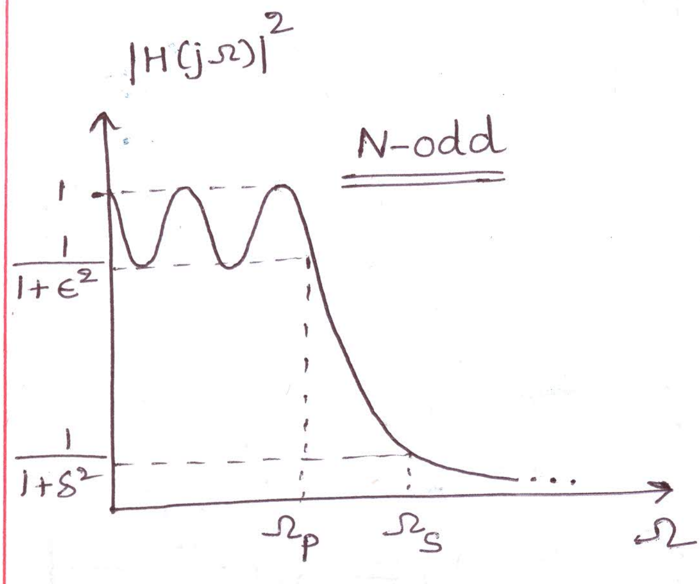
$$|C_N(\Omega)| > 1 \quad \text{for} \quad |\Omega| > 1$$

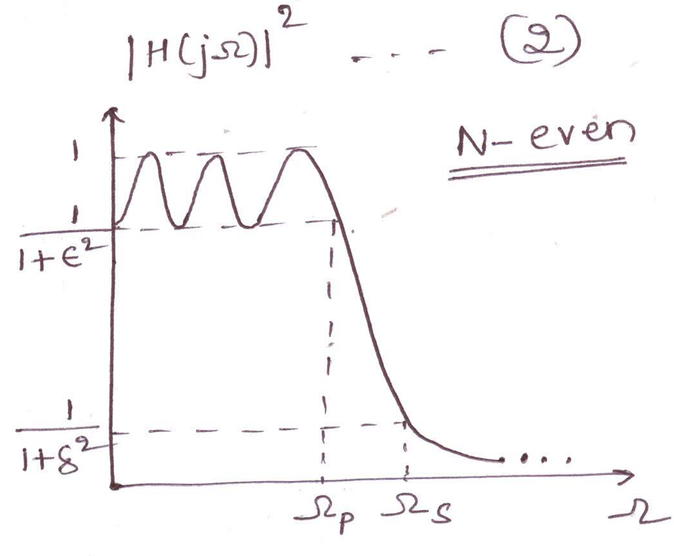
2| Derive the expression for order and cut-off frequency of Chebyshev lowpass filter (Type I)

The magnitude squared response of Type-I low pass chebyshev filter is given by

$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 C_N^2\left(\frac{\Omega}{\Omega_{PB}}\right)} \quad \text{---} \quad (1)$$

$$C_N\left(\frac{\Omega}{\Omega_{PB}}\right) = \begin{cases} \cos\left(N \cos^{-1}\left(\frac{\Omega}{\Omega_{PB}}\right)\right), & \text{for } \Omega \leq \Omega_{PB} \\ \cosh\left(N \cosh^{-1}\left(\frac{\Omega}{\Omega_{PB}}\right)\right), & \text{for } \Omega > \Omega_{PB} \end{cases}$$





--- (2)

$$A_{PB} = -10\log_{10}\left(|H(j\Omega)|^2\right)\Big|_{\Omega=\Omega_{PB}}$$

$$= -10\log_{10}\left(\frac{1}{1+\epsilon^2}\right)$$

$$= 10\log_{10}\left(1+\epsilon^2\right)$$

$$\therefore 1+\epsilon^2 = 10^{\frac{A_{PB}}{10}}$$

$$\therefore \epsilon = \sqrt{10^{\frac{A_{PB}}{10}} - 1} \quad \text{---} (3)$$

i) To find the order of the filter,

$$A_{SB} = -10\log_{10}\left(|H(j\Omega)|^2\right)\Big|_{\Omega=\Omega_{SB}}$$

$$= -10\log_{10}\left(\frac{1}{1+\epsilon^2 C_N^2\left(\frac{\Omega_{SB}}{\Omega_{PB}}\right)}\right)$$

$$= 10\log_{10}\left(1+\epsilon^2 C_N^2\left(\frac{\Omega_{SB}}{\Omega_{PB}}\right)\right)$$

$$= 10\log_{10}\left(1+\epsilon^2\cosh^2\left(N\cosh^{-1}\left(\frac{\Omega_{SB}}{\Omega_{PB}}\right)\right)\right)$$

$$\therefore 1+\epsilon^2\cosh^2\left(N\cosh^{-1}\left(\frac{\Omega_{SB}}{\Omega_{PB}}\right)\right) = 10^{\frac{A_{SB}}{10}}$$

$$\therefore \epsilon\cosh\left(N\cosh^{-1}\left(\frac{\Omega_{SB}}{\Omega_{PB}}\right)\right) = \sqrt{10^{\frac{A_{SB}}{10}} - 1}$$

$$\cosh\left(N\cosh^{-1}\left(\frac{\Omega_{SB}}{\Omega_{PB}}\right)\right) = \frac{\sqrt{10^{\frac{A_{SB}}{10}}-1}}{\epsilon}$$

$$= \sqrt{\frac{10^{\frac{A_{SB}}{10}}-1}{10^{\frac{A_{PB}}{10}}-1}}$$

$$\therefore N\cosh^{-1}\left(\frac{\Omega_{SB}}{\Omega_{PB}}\right) = \cosh^{-1}\left(\sqrt{\frac{10^{\frac{A_{SB}}{10}}-1}{10^{\frac{A_{PB}}{10}}-1}}\right)$$

$$\therefore N = \frac{\cosh^{-1}\left(\sqrt{\frac{10^{\frac{A_{SB}}{10}}-1}{10^{\frac{A_{PB}}{10}}-1}}\right)}{\cosh^{-1}\left(\frac{\Omega_{SB}}{\Omega_{PB}}\right)} \quad \text{---} (4)$$

$$N = \lceil N \rceil \quad \text{---} (5)$$

ii) To find Ω_c,

$$\left.|H(j\Omega)|^2\right|_{\Omega=\Omega_c} = \frac{1}{2}$$

$\therefore$ from (1), we can write,

$$\epsilon^2 C_N^2\left(\frac{\Omega_c}{\Omega_{PB}}\right) = 1$$

$$\epsilon^2 \cosh^2\left(N\cosh^{-1}\left(\frac{\Omega_c}{\Omega_{PB}}\right)\right) = 1$$

$$\therefore \quad \epsilon \cosh\left(N \cosh^{-1}\left(\frac{\Omega_c}{\Omega_{PB}}\right)\right) = 1$$

$$N \cosh^{-1}\left(\frac{\Omega_c}{\Omega_{PB}}\right) = \cosh^{-1}\left(\frac{1}{\epsilon}\right)$$

$$\therefore \quad \Omega_c = \Omega_{PB} \cosh\left[\frac{1}{N} \cosh^{-1}\left(\frac{1}{\epsilon}\right)\right] \quad \text{---(6)}$$

22 Design an analog Butterworth filter that has a passband ripple of 2 dB at 1 rad/s and atleast 30 dB attenuation at 3 rad/s.

$A_{PB} = 2\ dB$ $\qquad \Omega_{PB} = 1\ rad/s$

$A_{SB} = 30\ dB$ $\qquad \Omega_{SB} = 3\ rad/s.$

1. To find the order of the filter,

$$N = \frac{\log_{10}\left[\frac{10^{0.1A_{PB}} - 1}{10^{0.1A_{SB}} - 1}\right]}{2\log\left[\frac{\Omega_{PB}}{\Omega_{SB}}\right]}.$$

$$= \left[\frac{-3.53}{2 \times -0.477}\right]$$

$$= 4.$$

2. To find the cut-off frequency

$$\Omega_c = \frac{\Omega_{PB}}{\left[10^{0.1A_{PB}} - 1\right]^{\frac{1}{2N}}} = \frac{1}{\left(10^{0.2} - 1\right)^{\frac{1}{8}}}$$

$$= 1.0693\ rad$$

3. To find pole-locations:

$$S_k = \Omega_c \left[-\sin\left(\frac{(2k+1)\pi}{2N}\right) + j\cos\left(\frac{(2k+1)\pi}{2N}\right) \right]$$

$k = 0, 1, \ldots N-1$

$S_0 = -0.4092 + j0.9879$

$S_1 = -0.9879 + j0.4092$

$S_2 = -0.9879 - j0.4092$

$S_3 = -0.4092 - j0.9879$

4. Transfer function,

$$H(s) = \frac{(-1)^N \prod_{k=0}^{N-1} S_k}{\prod_{k=0}^{N-1} (S - S_k)}$$

$$= \frac{\Omega_c^N}{\prod_{k=0}^{N-1} (S - S_k)}$$

$$= \frac{1.3074}{(S-S_0)(S-S_3)(S-S_1)(S-S_2)}$$

$$= \frac{1.3074}{(S^2 + 0.8184S + 1.1435)(S^2 + 1.9759S + 1.1435)}$$

23 For the specifications given below, design an analog Butterworth filter.

$$0.9 \le |H(j\Omega)| \le 1 \quad \text{for} \quad 0 \le \Omega \le 0.2\pi$$

$$|H(j\Omega)| \le 0.2 \quad \text{for} \quad \Omega \ge 0.4\pi$$

Given: $A_{PB} = -20\log_{10}(0.9) = 0.915$ dB

$A_{SB} = -20\log_{10}(0.2) = 14$ dB.

$\Omega_{PB} = 0.2\pi$ rad/s

$\Omega_{SB} = 0.4\pi$ rad/s.

Note: If magnitude squared response is given, ~~then~~ for example,

$0.9 \le |H(j\Omega)|^2 \le 1$, then,

$A_{PB} = -10\log(0.9) = 0.458$ dB.

1. Order of the filter,

$$N = \frac{\log_{10}\left[\dfrac{10^{0.1A_{PB}} - 1}{10^{0.1A_{SB}} - 1}\right]}{2\log_{10}\left(\dfrac{\Omega_{PB}}{\Omega_{SB}}\right)} = \frac{-2.0122}{2 \times -0.301} = 3.34$$

$\therefore N = 4$

2. Cut-off frequency

$$\Omega_c = \frac{\Omega_{PB}}{\left[10^{0.1A_{PB}} - 1\right]^{\frac{1}{2N}}} = 0.754 \text{ rad/s.}$$

3. Pole locations:

$$S_k = \Omega_c \left[-\sin\left(\frac{(2k+1)\pi}{2N}\right) + j\cos\left(\frac{(2k+1)\pi}{2N}\right) \right]$$

$$k = 0, 1, 2, 3$$

$$S_0 = -0.2885 + j0.6966$$

$$S_1 = -0.6966 + j0.2885$$

$$S_2 = -0.6966 - j0.2885$$

$$S_3 = -0.2885 - j0.6966$$

4. Transfer function:

$$H(S) = \frac{(-1)^N \prod_{k=0}^{N-1} S_k}{\prod_{k=0}^{N-1} (S - S_k)} = \frac{\Omega_c^N}{\prod_{k=0}^{N-1} (S - S_k)}$$

$$= \frac{0.3232}{(S-S_0)(S-S_3)(S-S_1)(S-S_2)}$$

$$= \frac{0.3232}{(S^2 + 0.577S + 0.5685)(S^2 + 1.3932S + 0.5684)}$$

(52)

26 Design an ~~dig~~ analog, low pass, Type-I Chebyshev filter to meet the following specifications.

Acceptable passband ripple = 2 dB

Stopband attenuation = 30 dB

Passband edge frequency = 1 rad/s

Stopband edge frequency = 2 rad/s

$A_{PB} = 2\,dB \qquad A_{SB} = 30\,dB$

$\Omega_{PB} = 1\,rad/s \qquad \Omega_{SB} = 2\,rad/s$

1. To find the order of the filter,

$$N = \frac{\cosh^{-1}\left(\sqrt{\dfrac{10^{0.1A_{SB}} - 1}{10^{0.1A_{PB}} - 1}}\right)}{\cosh^{-1}\left(\dfrac{\Omega_{SB}}{\Omega_{PB}}\right)} = \left[\frac{4.41}{\cancel{3.348}\ 1.316}\right] = 4$$

2. $$\varepsilon = \sqrt{10^{0.1A_{PB}} - 1} = 0.7648$$

3. $$k = \left[\frac{1 + \sqrt{1 + \epsilon^2}}{\epsilon}\right]^{\frac{1}{N}} = 1.311$$

4. $$\sigma_k = -\Omega_{PB}\left(\frac{k^2 - 1}{2k}\right)\sin\left(\frac{(2k+1)\pi}{2N}\right)$$

$k = 0, 1, \ldots N-1$

$\sigma_0 = -0.1049$

$\sigma_1 = -0.2532$

$\sigma_2 = -0.2532$

$\sigma_3 = -0.1049$

5. $$\Omega_k = \Omega_{PB}\left(\frac{R^2+1}{2R}\right)\cos\left(\frac{(2k+1)\pi}{2N}\right)$$

$k = 0, 1, \ldots N-1$

$\Omega_0 = 0.9580$

$\Omega_1 = 0.3968$

$\Omega_2 = -0.3968$

$\Omega_3 = -0.9580$

6. Pole locations:

$S_k = \sigma_k + j\Omega_k$

$S_0 = -0.1049 + 0.9580j$

$S_1 = -0.2532 + 0.3968j$

$S_2 = -0.2532 - 0.3968j$

$S_3 = -0.1049 - 0.9580j$

7. Transfer function:

$$H(S) = \frac{(-1)^N b_0 \prod_{k=0}^{N-1} S_k}{\prod_{k=0}^{N-1}(S - S_k)}$$

$$= \frac{b_0 \cdot \prod_{k=0}^{3} s_k}{(s-s_0)(s-s_3)(s-s_1)(s-s_2)}$$

$$= \frac{0.7943 \times 0.2058}{(s^2+0.2098s+0.9287)(s^2+0.5064s+0.2216)}$$

$$b_0 = \frac{1}{\sqrt{1+\epsilon^2}} = 0.7943$$

$\because$ N is even

$$\therefore H(s) = \frac{0.1634}{(s^2+0.2098s+0.9287)(s^2+0.5064s+0.2216)}$$

27 Obtain an analog Chebyshev filter of type I that meets the following requirements

$$\frac{1}{\sqrt{2}} \le |H(j\Omega)| \le 1, \quad 0 \le \Omega \le 2$$

$$|H(j\Omega)| < 0.1, \quad \Omega \ge 4$$

$$A_{PB} = -20\log\left(\frac{1}{\sqrt{2}}\right) = 3.010 \text{ dB}$$

$$A_{SB} = -20\log(0.1) = 20 \text{ dB}$$

$$\Omega_{PB} = 2 \text{ rad/s}$$

$$\Omega_{SB} = 4 \text{ rad/s}$$

1. Order of the filter

$$N = \frac{\cosh^{-1}\left(\sqrt{\dfrac{10^{0.1 A_{SB}} - 1}{10^{0.1 A_{PB}} - 1}}\right)}{\cosh^{-1}\left(\dfrac{\Omega_{SB}}{\Omega_{PB}}\right)}$$

$$= \frac{\cosh^{-1}(9.95)}{\cosh^{-1}(2)}$$

$$= \frac{2.988}{1.317}$$

$$= 2.269$$

$$\therefore N = 3$$

2. $\varepsilon = \sqrt{10^{0.1 A_{PB}} - 1} = 1$

3. $\beta = \left[\dfrac{1 + \sqrt{1+\epsilon^2}}{\epsilon}\right]^{\frac{1}{N}} = 1.3415$

4. $\sigma_k = -\Omega_{PB}\left(\dfrac{\beta^2 - 1}{2\beta}\right)\sin\left(\dfrac{(2k+1)\pi}{2N}\right)$ $\quad k = 0, 1, 2$

$$\sigma_0 = -2 \times 0.298 \times 0.5 = -0.298$$

$$\sigma_1 = -2 \times 0.298 \times 1 = -0.596$$

$$\sigma_2 = -2 \times 0.298 \times 0.5 = -0.298$$

5. $\Omega_k = \Omega_{PB}\left(\dfrac{\beta^2 + 1}{2\beta}\right)\cos\left(\dfrac{(2k+1)\pi}{2N}\right)$ $\quad k = 0, 1, 2$

$$\Omega_0 = 2 \times 1.0435 \times 0.866 = 1.807$$

$$\Omega_1 = 0$$

$\Omega_2 = -1.807$

6. Pole locations:

$s_0 = -0.298 + j1.807$

$s_1 = -0.596$

$s_2 = -0.298 - j1.807$

7. Transfer function,

$$H(s) = \frac{(-1)^N b_0 \prod_{k=0}^{2} s_k}{\prod_{k=0}^{2} (s - s_k)}$$

$b_0 = 1 \quad \because N \text{ is odd}$

$$\therefore H(s) = \frac{2}{(s + 0.596)(s^2 + 0.596s + 3.354)}$$

Transfer function of standard normalized Butterworth filter. (Cut-off frequency = 1 rad/sec)

1. $N=1$

$$H(s) = \frac{1}{s+1}$$

2. $N=2$

$$H(s) = \frac{1}{s^2+\sqrt{2}s+1}$$

3. $N=3$

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

4. $N=4$

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.8477s+1)}$$

For the following specifications, design a highpass, analog Butterworth filter.

$A_{PB} = 3\,dB$ $\Omega_{PB} = 1000\ rad/s$

$A_{SB} = 15\,dB$ $\Omega_{SB} = 500\ rad/s$

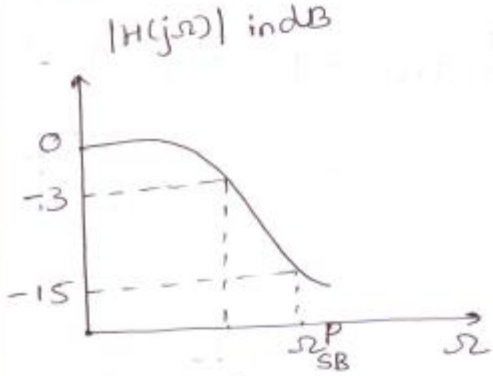


fig. (1)

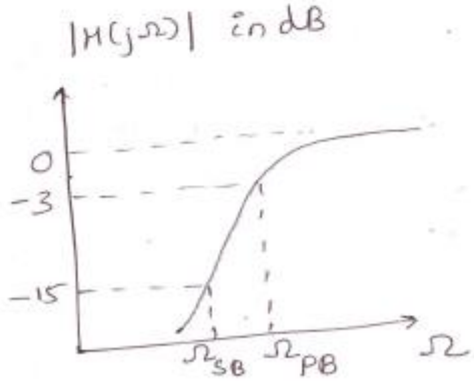


fig (2).

To design a high pass Butterworth filter, we first design a low pass prototype filter and then apply frequency transformation.

Frequency response of prototype filter is as shown in fig. (a).

Here, $\Omega^P_{PB} = 1$ rad/s.

To get Ω_{SB} of prototype filter, ie, Ω^P_{SB}

$$\cancel{\Omega^P_{SB} = \frac{\Omega_{SB}}{\Omega_{PB}}}$$

$$\Omega^P_{SB} = \frac{\Omega_{PB}}{\Omega_{SB}} = \frac{1000}{500} = 2$$

To design prototype lpf,

$$N = \frac{\log\left(\sqrt{\dfrac{10^{0.1A_{SB}} - 1}{10^{0.1A_{PB}} - 1}}\right)}{\log\left(\dfrac{\Omega^P_{SB}}{\Omega^P_{PB}}\right)} = \frac{0.744}{0.301} = 2.47$$

$\therefore N = 3$

for $N=3$, transfer function of prototype filter is

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

$\therefore$ Transfer function of the required high pass filter,

$$H^{HP}(s) = H(s)\Big|_{s \Rightarrow \frac{1000}{s}}$$

$$= \frac{1}{\left(\frac{1000}{s}+1\right)\left(\left(\frac{1000}{s}\right)^2 + \frac{1000}{s}+1\right)}$$

$$= \frac{s^3}{(s+1000)\left(s^2+1000s+1000^2\right)}$$

32 Given that $|H(j\Omega)|^2 = \frac{1}{1+16\Omega^4}$, determine the analog filter transfer function H(s). (70)

Magnitude squared response of Butterworth filter is given by.

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \quad \ldots (1)$$

Magnitude squared response given in the problem is,

$$|H(j\Omega)|^2 = \frac{1}{1+16\Omega^4}$$

$$= \frac{1}{1+\left(\frac{\Omega}{\frac{1}{2}}\right)^{2\times 2}} \quad \ldots (2)$$

Comparing (1) and (2), we get,

$$\Omega_c = \frac{1}{2} \text{ rad/s}$$

$$N = 2$$

$\therefore$ Pole locations are,

$$S_K = -\Omega_c\left[\sin\left(\frac{(2K+1)\pi}{2N}\right) + j\cos\left(\frac{(2K+1)\pi}{2N}\right)\right]$$

$K = 0, 1.$

$$S_0 = -\Omega_c\left[\sin\left(\frac{\pi}{4}\right) + j\cos\left(\frac{\pi}{4}\right)\right] \quad \left(\text{putting } K=0\right)$$

$$= -0.3536 + j\,0.3536$$

$$S_1 = S_0^*$$

$$= -0.3536 - j\,0.3536$$

$$\therefore H(S) = \frac{\Omega_c^N}{(S-S_0)(S-S_1)}$$

$$= \frac{0.25}{S^2 + 0.7072S + 0.25}$$

Note:

1. Maximally flat analog filter means Butterworth filter
2. Analog filter with monotonic frequency response means Butterworth filter
3. The filter which has ripples in passband and monotonic response in stopband is Chebyshev Type-I filter.