DIGITAL IMAGE PROCESSING-1

Unit 2: Lecture 25

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Unit 2: Image Transforms

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Last Session

- Walsh transforms
- Walsh - Hadamard transforms
- Slant Transform

Today's Session

- Haar Transform

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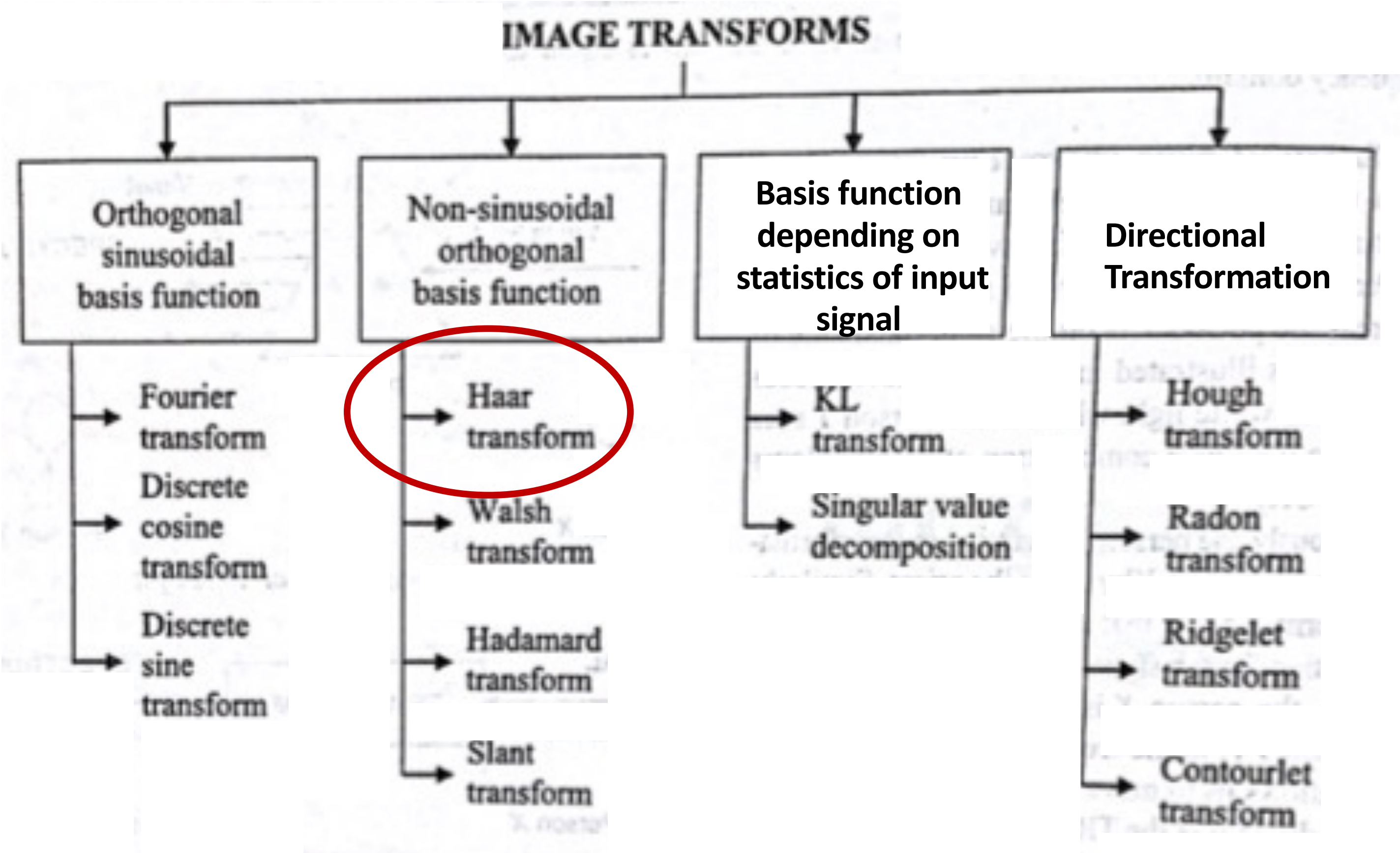
Classification of Image Transforms



Haar Transform

- Discovered in 1910, the basis functions of the Haar transform were later recognized to be the oldest and simplest orthonormal wavelets (to be studied later)
- **Here Haar transform is considered as another matrix-based transformation that employs a set of rectangular-shaped basis functions**
- The Haar transform is based on Haar functions, $h_u(x)$, that are defined over the continuous, half-open interval $x \in [0, 1)$
- Variable u is an integer that can be decomposed uniquely as **$u = 2^p + q$ for $u > 0$**

  where p is the largest power of 2 contained in u and q is the remainder: that is, $q = 2^p - u$

Haar Transform

- The Haar basis functions are given by

$$h_u(x) = \begin{cases} 1 & u = 0 \text{ and } 0 \le x < 1 \\ 2^{p/2} & u > 0 \text{ and } q/2^p \le x < (q+0.5)/2^p \\ -2^{p/2} & u > 0 \text{ and } (q+0.5)/2^p \le x < (q+1)/2^p \\ 0 & \text{otherwise} \end{cases}$$

When u is 0, $h_0(x) = 1$ for all x; the first Haar function is independent of continuous variable x.

- For all other values of u, $h_u(x) = 0$ except in the half-open intervals

$[q/2^p, (q+0.5)/2^p)$ and $[(q+0.5)/2^p, (q+1)/2^p)$

where it is a rectangular wave of magnitude $2^{p/2}$ and $-2^{p/2}$, respectively.

Haar Transform

- Parameter p determines the amplitude and width of both rectangular waves, while q determines their position along x.
- **As u increases, the rectangular waves become narrower and the number of functions that can be represented as linear combinations of the Haar functions increases**
- **Haar transform is based on orthogonal matrices whose elements are either 1 or -1 or multiplied by powers of $\sqrt{2}$**
  - Basis functions of Haar Transform are non sinusoidal functions.
- It is computationally efficient transform as the transform of N-point vector requires only 2(N-1) additions and N multiplications

$$h_u(x) = \begin{cases} 1 & u = 0 \text{ and } 0 \le x < 1 \\ 2^{p/2} & u > 0 \text{ and } q/2^p \le x < (q+0.5)/2^p \\ -2^{p/2} & u > 0 \text{ and } (q+0.5)/2^p \le x < (q+1)/2^p \\ 0 & \text{otherwise} \end{cases}$$

$u = 2^p + q$ for $u > 0$

Haar Transform

When u is 0, $h_u(x)$ is independent of p and q. as $h_0(x) = 1$ for all x

for u> 0 ; u= $2^p + q$

u	*p*	*q*
1	0	0
2	1	0
3	1	1

Haar Transform

- The transformation matrix of the discrete Haar transform can be obtained by substituting the inverse transformation kernel

$$s(x,u) = \frac{1}{\sqrt{N}} h_u(x/N) \quad \text{f or} x = 0, 1, ..., N-1$$

for u = 0, 1, ... , N − 1, where N = 2^n

- The resulting transformation matrix, denoted A_H , can be written as a function of the N×N Haar matrix

$$\mathbf{H}_N = \begin{bmatrix} h_0(0/N) & h_0(1/N) & \cdots & h_0(N-1/N) \\ h_1(0/N) & h_1(1/N) & & \vdots \\ \vdots & & \ddots & \\ h_{N-1}(0/N) & \cdots & & h_{N-1}(N-1/N) \end{bmatrix} \qquad \mathbf{A}_{\mathrm{H}} = \frac{1}{\sqrt{N}} \mathbf{H}_N$$

Haar Transform Matrix

Example 1: for N= 2, determine the Haar transformation matrix A_H

$$h_u(x) = \begin{cases} 1 & u = 0 \text{ and } 0 \le x < 1 \\ 2^{p/2} & u > 0 \text{ and } q/2^p \le x < (q+0.5)/2^p \\ -2^{p/2} & u > 0 \text{ and } (q+0.5)/2^p \le x < (q+1)/2^p \\ 0 & \text{otherwise} \end{cases}$$

Soln:

$$s(x,u) = \frac{1}{\sqrt{N}} h_u(x/N) \quad \text{f or} x = 0, 1, \ldots, N-1$$

for u = 0, 1, … , N− 1, where N = 2^n

$$\mathbf{H}_N = \begin{bmatrix} h_0(0/N) & h_0(1/N) & \ldots & h_0(N-1/N) \\ h_1(0/N) & h_1(1/N) & & \vdots \\ \vdots & & \ddots & \\ h_{N-1}(0/N) & \ldots & & h_{N-1}(N-1/N) \end{bmatrix}$$

$$\mathbf{A}_\mathrm{H} = \frac{1}{\sqrt{N}} \mathbf{H}_N$$

Here u = 0,1 and x = 0,1.

$$H_2 = \begin{bmatrix} h_0(\frac{0}{N}) & h_0(\frac{1}{N}) \\ h_1(\frac{0}{N}) & h_1(\frac{1}{N}) \end{bmatrix}.$$

For u=0 , $h_0(x) = 1$

u= 2^p + q. ; u>0

u=1, p = 0, q=0 $h_1(0/2)$= 1 ; $h_1(1/2)$= -1

$$\mathbf{A}_\mathrm{H} = \frac{1}{\sqrt{2}} \begin{bmatrix} h_0(0) & h_0(1/2) \\ h_1(0) & h_1(1/2) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Haar Transform

Example 2: for N= 4, determine the Haar transformation matrix A_H

Soln: $$s(x,u) = \frac{1}{\sqrt{N}} h_u(x/N) \quad \text{for} \, x = 0, 1, \ldots, N-1$$

for u = 0, 1,2 ... , N− 1, where N = 2^n

u= $2^p + q$. ; u>0

Here u = 0,1,2,3 and x = 0,1,2,3

$$H_4 = \begin{bmatrix} h_0(\frac{0}{4}) & h_0(\frac{1}{4}) & h_0(\frac{2}{4}) & h_0(\frac{3}{4}) \\ h_1(\frac{0}{4}) & h_1(\frac{1}{4}) & h_1(\frac{2}{4}). & h_1(\frac{3}{4}) \\ h_2(\frac{0}{4}) & h_2(\frac{1}{4}) & h_2(\frac{2}{4}) & h_2(\frac{3}{4}) \\ h_3(\frac{0}{4}) & h_3(\frac{1}{4}) & h_3(\frac{2}{4}) & h_3(\frac{3}{4}). \end{bmatrix}$$

$$h_u(x) = \begin{cases} 1 & u = 0 \text{ and } 0 \le x < 1 \\ 2^{p/2} & u > 0 \text{ and } q/2^p \le x < (q+0.5)/2^p \\ -2^{p/2} & u > 0 \text{ and } (q+0.5)/2^p \le x < (q+1)/2^p \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{H}_N = \begin{bmatrix} h_0(0/N) & h_0(1/N) & \cdots & h_0(N-1/N) \\ h_1(0/N) & h_1(1/N) & & \vdots \\ \vdots & & \ddots & \\ h_{N-1}(0/N) & \cdots & & h_{N-1}(N-1/N) \end{bmatrix}$$

$$\mathbf{A}_\mathrm{H} = \frac{1}{\sqrt{N}} \mathbf{H}_N$$

Haar Transform

$u = 2^p + q$. ; $u>0$

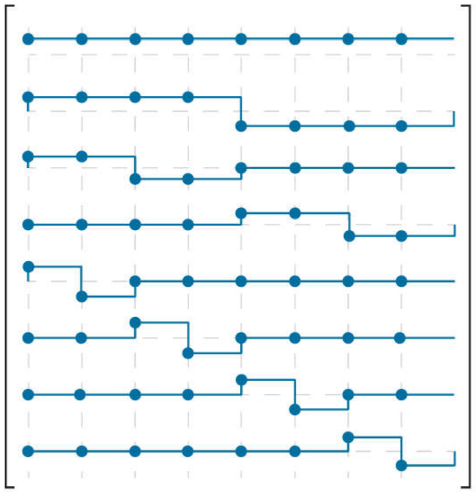
u	*p*	*q*
1	0	0
2	1	0
3	1	1

$$\mathbf{A}_{\mathrm{H}} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

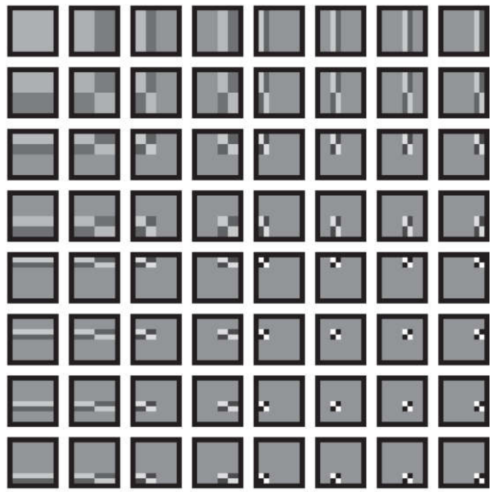
A_H is real, orthogonal, and sequency ordered

Haar Transform

Haar transform for N= 8 .



$$\begin{bmatrix} 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 \\ 0.35 & 0.35 & 0.35 & 0.35 & -0.35 & -0.35 & -0.35 & -0.35 \\ 0.50 & 0.50 & -0.50 & -0.50 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.50 & 0.50 & -0.50 & -0.50 \\ 0.71 & -0.71 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.71 & -0.71 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.71 & -0.71 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.71 & -0.71 \end{bmatrix}$$



a b c

The transformation matrix and basis images of the discrete Haar transform for N= 8 .
(a) Graphical representation of orthogonal transformation matrix H,
(b) H rounded to two decimal places, and (c) basis images.

Haar Transform

- An important property of the Haar transformation matrix is that it can be decomposed into products of matrices with fewer nonzero entries than the original matrix.
- This is true of all of the transforms we have discussed so far
  - They can be implemented in FFT-like alogrithms of complexity O(N $\log_2$N) .
- The Haar transformation matrix, however, has fewer nonzero entries before the decomposition process begins, making less complex algorithms on the order of O(N) possible.
- The basis images of the separable 2-D Haar transform for images of size 8 × 8 also have few nonzero entries.

Next Session

- KL transform
- SVD

THANK YOU

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