UNIT-1 - EMF

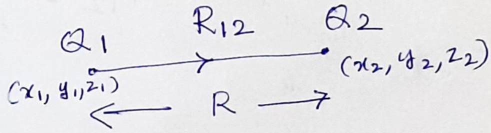
Electrostatic Fields

Electrostatics is a fascinating subject that has grown up in diverse areas of application.

Coulomb's Law and Field Intensity

Coulomb's law states that the force $\vec{F}$ between two point charges Q_1 and Q_2 is

1. Along the line joining them
2. Directly proportional to the product $Q_1 Q_2$ of the charges
3. Inversly proportional to the square of the distance R between them



$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2} \times \frac{\vec{R_{12}}}{|R_{12}|}$$

Since $\boxed{R_{12} = R}$

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\, \hat{a}_{R_{12}} = \vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0} \times \frac{\vec{R_{12}}}{R^3}$$

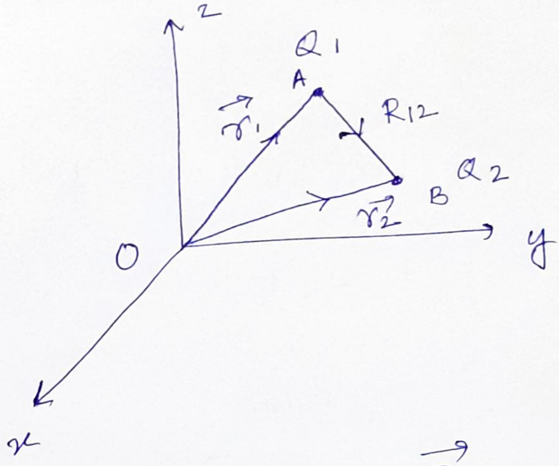
where $\varepsilon_0 = \frac{10^{-9}}{36\pi}$ Farad/meter

R = distance between Q_1 and Q_2

If Q_1 is at (x_1, y_1, z_1) and Q_2 is at (x_2, y_2, z_2)

$$R = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Consider Q_1 and Q_2 With position Vectors r_1 and r_2



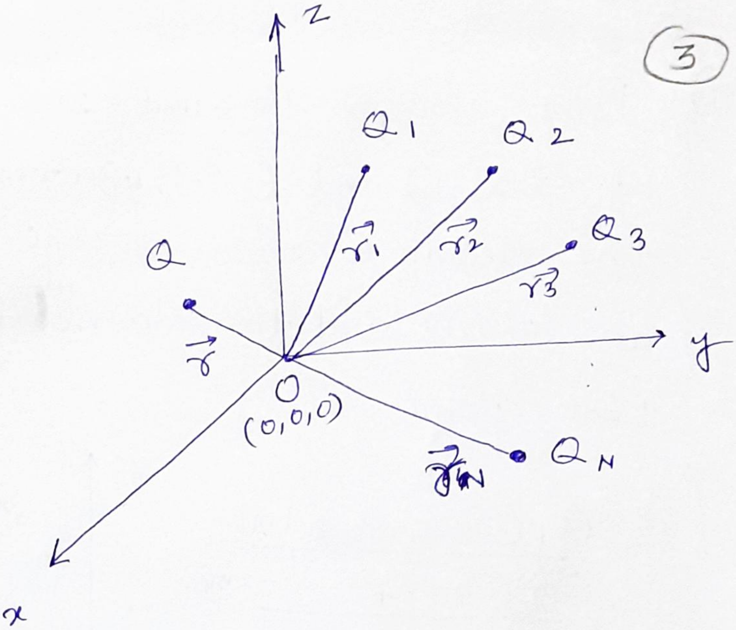
Note:
$\vec{r}_{PQ} = \vec{r}_Q - \vec{r}_P$
$\vec{r}_{AB} = \vec{r}_B - \vec{r}_A$
$\vec{R}_{12} = \vec{r}_2 - \vec{r}_1$

$$\therefore \vec{F}_{12} = \frac{Q_1 Q_2}{4\pi \varepsilon_0} \times \frac{\vec{R}_{12}}{R^3}$$ where $R = |R_{12}|$

$\vec{R}_{12} = \vec{r}_2 - \vec{r}_1$

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi \varepsilon_0} \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$ where $|\vec{r}_2 - \vec{r}_1| = R = |R_{12}|$

If we have more than two point charges, We can use the principle of superposition to determine the force on a particular charge. The principle states that if there are N-charges $Q_1, Q_2, \ldots Q_N$ located, respectively, at points With position vectors $\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N$, the resultant force $\vec{F}$ on a charge located at point "r" with position vectors $\vec{r}$ is the vector Sum of the forces exerted on "Q" by each of the Charges $Q_1, Q_2 \ldots Q_N$.

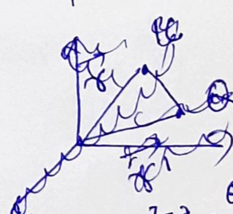


$$\vec{F} = \frac{QQ_1}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r_1})}{|\vec{r}-\vec{r_1}|^3} + \frac{QQ_2}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r_2})}{|\vec{r}-\vec{r_2}|^3} + \cdots$$

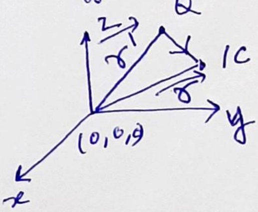
$$+ \frac{QQ_N}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r_N})}{|\vec{r}-\vec{r_N}|^3}$$

Electric Field Intensity

The electric field intensity "E" is the force that a unit positive charge experiences when placed in an eletric field.



$$\vec{E} = \frac{\vec{F}}{Q}$$



$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R = \frac{Q(\vec{r}-\vec{r'})}{4\pi\varepsilon_0|\vec{r}-\vec{r'}|^3}$$

$$= \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_R$$

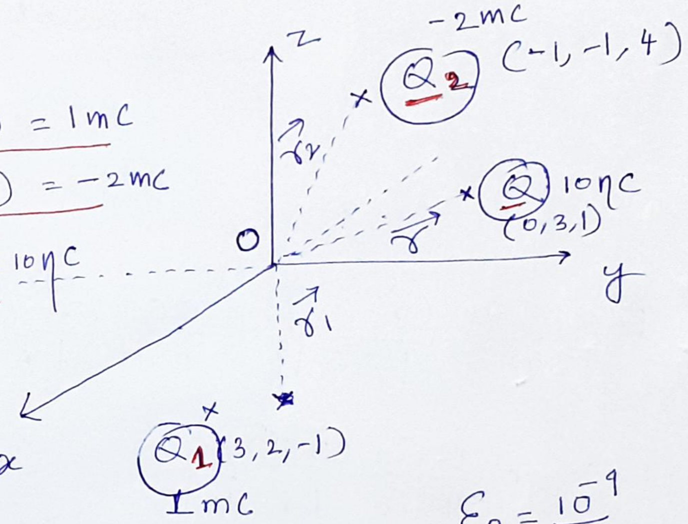
r = distance between unit +ve charge and the charge Q

1) Point charges 1mC and -2mC are located at (3, 2, -1) and (-1, -1, 4), respectively. Calculate the electric force on a 10nC charge located at (0, 3, 1) and the electric field intensity at that point. (4)

$Q_1 (3, 2, -1) = 1mC$

$Q_2 (-1, -1, 4) = -2mC$

$Q\ (0, 3, 1) = 10nC$



$\varepsilon_0 = \frac{10^{-9}}{36\pi}$

$$\vec{F} = \frac{QQ_1}{4\pi\varepsilon_0} \times \frac{\hat{a}_{R1}}{R_1^2} + \frac{QQ_2}{4\pi\varepsilon_0} \times \frac{\hat{a}_{R2}}{R_2^2}$$

$\vec{r} = (0, 3, 1)$

$\vec{r_1} = (3, 2, -1)$

$\vec{r_2} = (-1, -1, 4)$

$$\vec{F} = \frac{10\times10^{-9} \times 1\times10^{-3}}{4\pi \times \frac{10^{-9}}{36\pi}} \frac{(\vec{r} - \vec{r_1})}{|\vec{r} - \vec{r_1}|^3}$$

$$+ \frac{10\times10^{-9} \times (-2\times10^{-3})}{4\pi \times \frac{10^{-9}}{36\pi}} \times \frac{(\vec{r} - \vec{r_2})}{|\vec{r} - \vec{r_2}|^3}$$

$=$

$$\vec{F} = \frac{9\times 10\times 10^{-3}\left((0,3,1)-(3,2,-1)\right)}{\left|\sqrt{3^2+1^2+2^2}\right|^3}$$

$$- \frac{9\times 10\times 2\times 10^{-3}\left((0,3,1)-(-1,-1,4)\right)}{\left|\sqrt{1^2+4^2+3^2}\right|^3}$$

$$\vec{F} = 9\times 10^{-2}\left[\frac{(-3,1,2)}{14\sqrt{14}} + \frac{(-2)(1,4,-3)}{26\sqrt{26}}\right]$$

$$\vec{F} = -6.512\times 10^{-3}\,\hat{a}_x - 3.713\times 10^{-3}\,\hat{a}_y + 7.509\times 10^{-3}\,\hat{a}_z$$

$$\vec{F} = \left(-6.512\,\hat{a}_x - 3.713\,\hat{a}_y + 7.509\,\hat{a}_z\right)\times 10^{-3}\ \text{N}$$

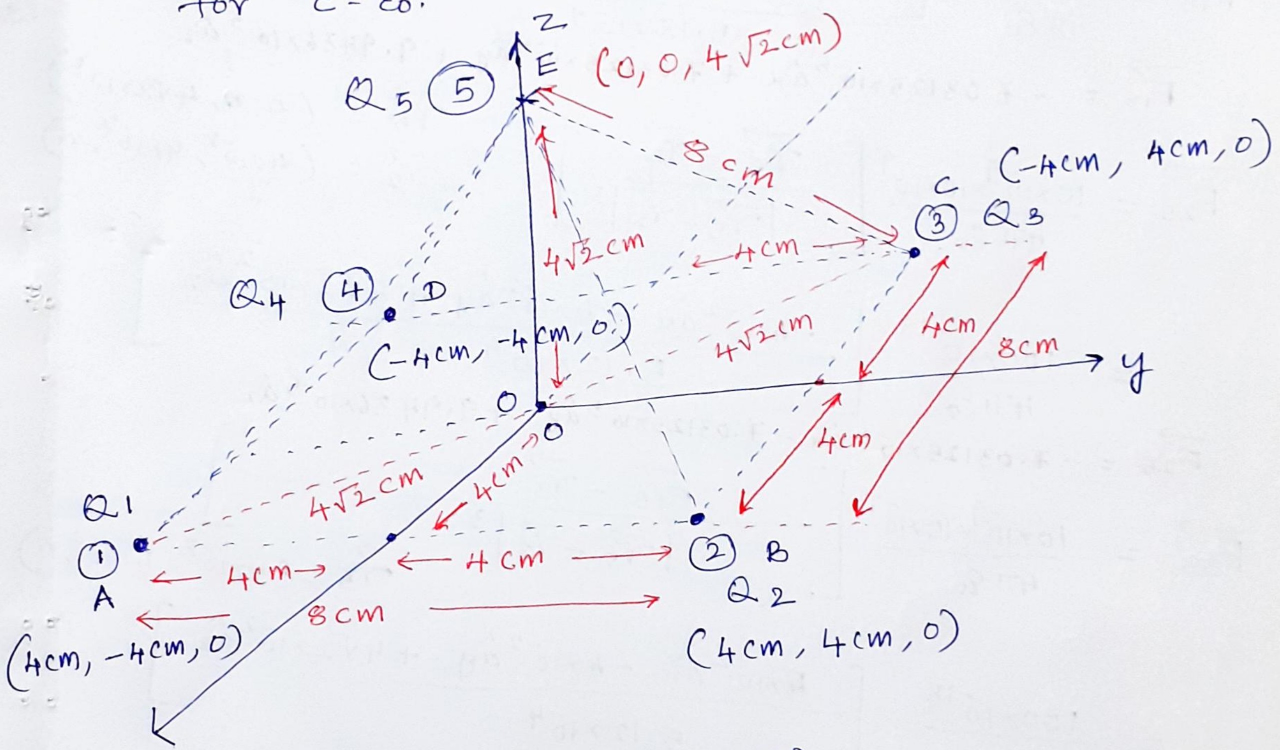
$$\vec{F} = \left(-6.512\,\hat{a}_x - 3.713\,\hat{a}_y + 7.509\,\hat{a}_z\right)\ \text{mN}$$

Electric Field Intensity

$$\vec{E} = \frac{\vec{F}}{Q} = \frac{(-6.512, -3.713, 7.509)\times 10^{-3}}{10\times 10^{-9}}$$

$$\vec{E} = \left(-651.2\,\hat{a}_x - 371.3\,\hat{a}_y + 750.9\,\hat{a}_z\right)\times 10^{3}\ \text{V/m}$$

1) Four 10nC positive charges are located in the z=0 plane at the corners of a square 8cm on a side. A fifth charge 10nC positive charge is located at a distance 8cm from the other charges. Calculate the magnitude of the total force on this fifth charge for $\varepsilon = \varepsilon_0$.



$$\vec{F} = \vec{F_{15}} + \vec{F_{25}} + \vec{F_{35}} + \vec{F_{45}}$$

$$\vec{F}_{15} = \frac{10\times10^{-9} \times 10\times10^{-9}}{4\pi\varepsilon_0} \times \frac{(\vec{r}_5 - \vec{r}_1)}{|\vec{r}_5 - \vec{r}_1|^3} \qquad \varepsilon_0 = \frac{10^{-9}}{36\pi}$$

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$$= \frac{100\times10^{-18}}{4\pi\varepsilon_0} \times \frac{\left[(0, 0, 4\sqrt{2}\times10^{-2}) - (4\times10^{-2}, -4\times10^{-2}, 0)\right]}{\left[\sqrt{(4\times10^{-2})^2 + (4\times10^{-2})^2 + (4\sqrt{2}\times10^{-2})^2}\right]^3}$$

$$\vec{F}_{15} = \frac{100\times10^{-18}}{4\pi\varepsilon_0}\left[\frac{-4\times10^{-2}\hat{a}_x + 4\times10^{-2}\hat{a}_y + 4\sqrt{2}\times10^{-2}\hat{a}_z}{5.12\times10^{-4}}\right]$$

$$\vec{F}_{15} = -7.03125\times10^{-5}\hat{a}_x + 7.03125\times10^{-5}\hat{a}_y + 9.9436\times10^{-5}\hat{a}_z$$

$$\vec{r}_5 = (0, 0, 4\sqrt{2}\times10^{-2})$$

$$\vec{r}_2 = (4\times10^{-2}, 4\times10^{-2}, 0)$$

$$\vec{F}_{25} = \frac{10\times10^{-9}\times10\times10^{-9}}{4\pi\varepsilon_0}\left[\frac{\vec{r}_5 - \vec{r}_2}{|\vec{r}_5 - \vec{r}_2|^3}\right]$$

$$= \frac{100\times10^{-18}}{4\pi\varepsilon_0}\left[\frac{-4\times10^{-2}\hat{a}_x - 4\times10^{-2}\hat{a}_y + 4\sqrt{2}\times10^{-2}\hat{a}_z}{5.12\times10^{-4}}\right]$$

$$\vec{F}_{25} = -7.03125\times10^{-5}\hat{a}_x - 7.03125\times10^{-5}\hat{a}_y + 9.9436\times10^{-5}\hat{a}_z$$

$$\vec{F}_{35} = \frac{10\times10^{-9}\times10\times10^{-9}}{4\pi\varepsilon_0}\left[\frac{\vec{r}_5 - \vec{r}_3}{|\vec{r}_5 - \vec{r}_3|^3}\right]$$

$$\vec{r}_3 = (4\times10^{-2}, 4\times10^{-2}, 0)$$

$$= \frac{100\times10^{-18}}{4\pi\varepsilon_0}\left[\frac{4\times10^{-2}\hat{a}_x - 4\times10^{-2}\hat{a}_y + 4\sqrt{2}\times10^{-2}\hat{a}_z}{5.12\times10^{-4}}\right]$$

$$\vec{F}_{35} = +7.03125\times10^{-5}\hat{a}_x - 7.03125\times10^{-5}\hat{a}_y + 9.9436\times10^{-5}\hat{a}_z$$

$$\vec{F}_{45} = \frac{100\times10^{-18}}{4\pi\varepsilon_0}\left[\frac{\vec{r}_5 - \vec{r}_4}{|\vec{r}_5 - \vec{r}_4|^3}\right]$$

$$\vec{F}_{45} = \frac{100\times10^{-18}}{4\pi\varepsilon_0} \times \left[\frac{4\times10^{-2}\hat{a}_x + 4\times10^{-2}\hat{a}_y + 4\sqrt{2}\times10^{-2}\hat{a}_z}{5.12\times10^{-4}}\right]$$

$$\vec{F}_{45} = +7.03125\times10^{-5}\hat{a}_x + 7.03125\times10^{-5}\hat{a}_y + 9.9436\times10^{-5}\hat{a}_z$$

$$\vec{F} = \vec{F}_{15} + \vec{F}_{25} + \vec{F}_{35} + \vec{F}_{45}$$

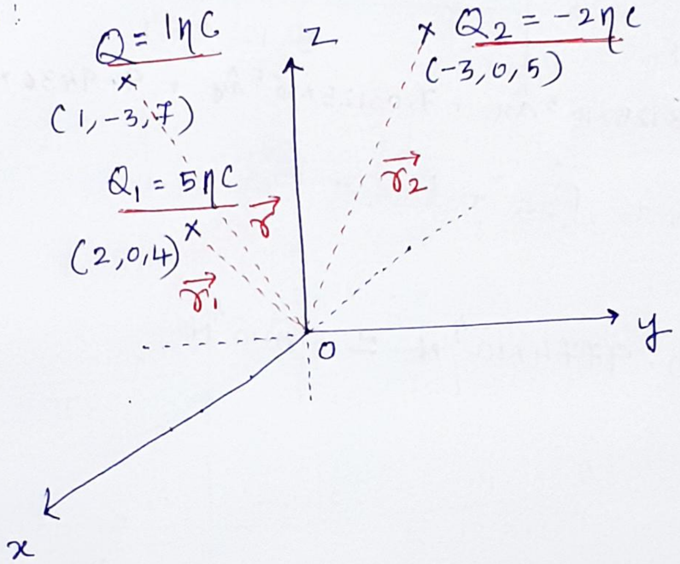
$$\vec{F} \cong 3.9774\times10^{-4}\,N \simeq 4\times10^{-4}\,N$$

2) Point charges $5\,nC$ and $-2\,nC$ are located at (2,0,4) and (−3,0,5) respectively.

(a) Determine the force on a $1\,nC$ point charge at (1,−3,7)

(b) Find the Electric field $\vec{E}$ at (1,−3,7)

Solution:



$$\vec{F} = \frac{Q_1 Q}{4\pi\varepsilon_0} \times \frac{(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3} + \frac{Q_2 Q}{4\pi\varepsilon_0} \times \frac{(\vec{r}-\vec{r}_2)}{|\vec{r}-\vec{r}_2|^3}$$

$$\vec{F} = \frac{5\times10^{-9}\times1\times10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}} \times \frac{[(1,-3,7)-(2,0,4)]}{\left[\sqrt{(1-2)^2+(-3-0)^2+(7-4)^2}\right]^3}$$

$$+ \frac{(-2\times10^{-9}\times1\times10^{-9})}{4\pi\times\frac{10^{-9}}{36\pi}} \times \frac{[(1,-3,7)-(-3,0,5)]}{\left[\sqrt{(1+3)^2+(-3-0)^2+(7-5)^2}\right]^3}$$

$$\vec{F} = 4.5\times10^{-8}\frac{[(-1,-3,3)]}{(\sqrt{19})^3} + \frac{(-1.8\times10^{-8})[(4,-3,2)]}{(\sqrt{29})^3}$$

$$= -0.5433\times10^{-9}\,\hat{a}_x - 1.63\times10^{-9}\,\hat{a}_y + 1.63\times10^{-9}\,\hat{a}_z +$$

$$-0.4610\times10^{-9}\,\hat{a}_x + 0.3457\times10^{-9}\,\hat{a}_y - 0.2305\times10^{-9}\,\hat{a}_z$$

$$\vec{F} = -1.0043\times10^{-9}\,\hat{a}_x - 1.2843\times10^{-9}\,\hat{a}_y + 1.3995\times10^{-9}\,\hat{a}_z \text{ N}$$

or

$$\vec{F} = -1.0043\,\hat{a}_x - 1.2843\,\hat{a}_y + 1.3995\,\hat{a}_z \text{ nN}$$

$$\vec{E} = \frac{Q_1}{4\pi\varepsilon_0} \times \frac{(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3} + \frac{Q_2}{4\pi\varepsilon_0} \times \frac{(\vec{r}-\vec{r}_2)}{|\vec{r}-\vec{r}_2|^3}$$

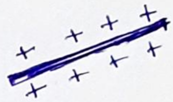
$$\vec{E} = \frac{5\times10^{-9}}{\frac{4\pi\times10^{-9}}{36\pi}} \times \frac{[(1,-3,7)-(2,0,4)]}{|(1,-3,7)-(2,0,4)|^3}$$

$$+ \frac{(-2\times10^{-9})}{\frac{4\pi\times10^{-9}}{36\pi}} \times \frac{[(-3,0,5)-(2,0,4)]}{|(-3,0,5)-(2,0,4)|^3}$$

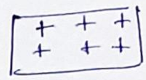
$$\vec{E} = -1.0043\,\hat{a}_x - 1.2843\,\hat{a}_y + 1.3995\,\hat{a}_z \text{ N/C or V/m}$$

Electric Fields due to Continuous Charge distributions

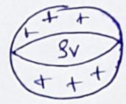
*) Line charge density - ρ_L C/m



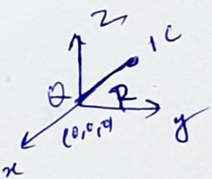
*) Surface charge density - ρ_S C/m²



*) Volume charge density - ρ_V C/m³

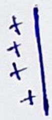


*) Point charge - +Q •

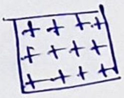


Electric field intensity due to a point charge Q $= \vec{E} = \dfrac{Q \, \hat{a}_R}{4\pi\varepsilon_0 R^2}$

Electric field intensity due to a line charge ρ_L C/m $= \vec{E} = \displaystyle\int_L \frac{\rho_L \, dl}{4\pi\varepsilon_0 R^2} \hat{a}_R$

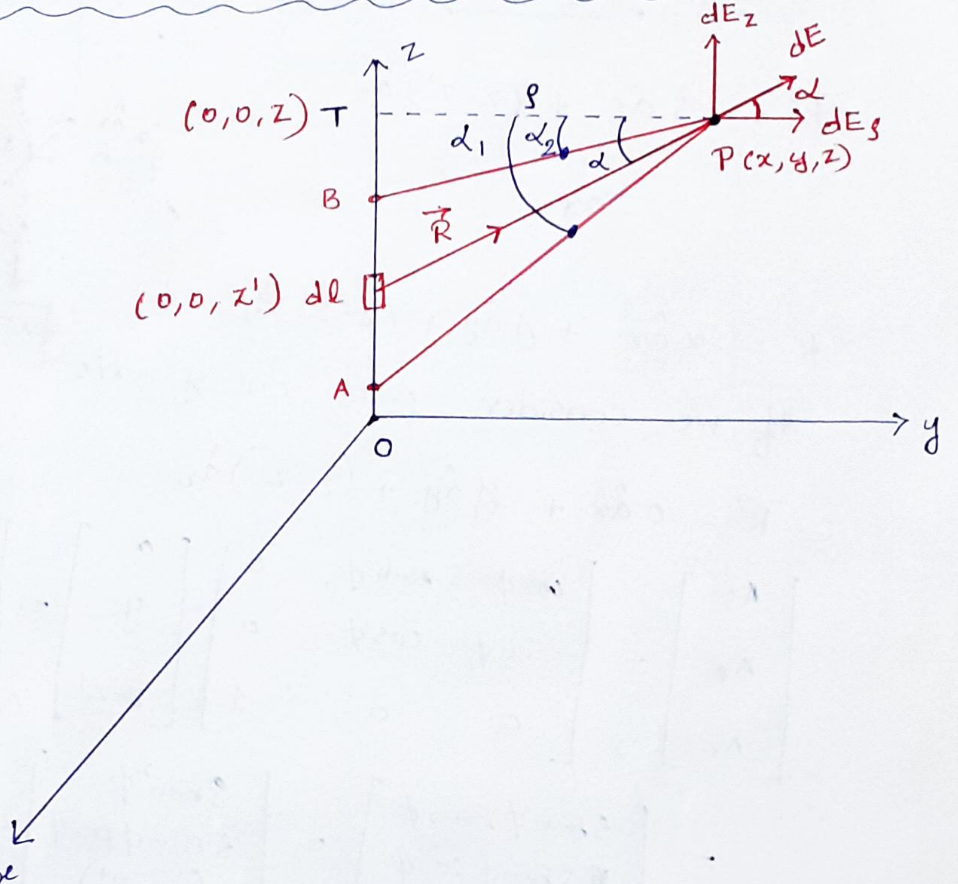


Electric field intensity due to Surface charge ρ_S C/m² $= \vec{E} = \displaystyle\int_S \frac{\rho_S \, dS}{4\pi\varepsilon_0 R^2} \hat{a}_R$



Electric field intensity due to volume charge ρ_V C/m³ $= \vec{E} = \displaystyle\int_V \frac{\rho_V \, dV}{4\pi\varepsilon_0 R^2} \hat{a}_R$

Electric Field intensity due to continuous line charge



Consider a line charge with uniform charge density "ρ_L" extending from "A" to "B" along the z-axis as shown in figure. The charge element dQ is associated with element $dl = dz$ of the line.

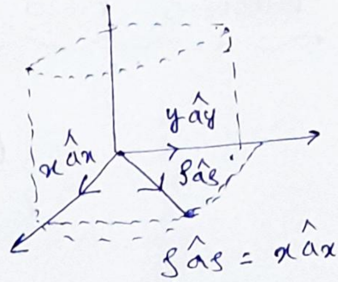
The electric field intensity at a point $P(x,y,z)$ is to be evaluated.

$$\vec{E} = \int_L \frac{\rho_L \, dl}{4\pi\varepsilon_0 R^2} \hat{a}_R \longrightarrow (1)$$

$$\vec{E} = \int_L \frac{\rho_L \, dl}{4\pi\varepsilon_0 R^2} \times \frac{\vec{R}}{|\vec{R}|} \rightarrow (2)$$

$$\vec{R} = \underbrace{x\,\hat{a}_x + y\,\hat{a}_y}\, + (z-z')\,\hat{a}_z$$

$$\vec{R} = \rho\,\hat{a}_\rho + (z-z')\,\hat{a}_z$$



$$\rho\,\hat{a}_\rho = x\,\hat{a}_x + y\,\hat{a}_y$$

$$\downarrow$$

$$\sqrt{x^2+y^2}$$

or

$$\vec{R} = x\,\hat{a}_x + y\,\hat{a}_y + (z-z')\,\hat{a}_z$$

If we consider point on y-axis

$$\therefore \vec{R} = 0\,\hat{a}_x + y\,\hat{a}_y + (z-z')\,\hat{a}_z$$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ y \\ z-z' \end{bmatrix} = \begin{bmatrix} y\sin\phi \\ y\cos\phi \\ (z-z') \end{bmatrix}$$

$$= \begin{bmatrix} \rho\sin\phi\sin\phi \\ \rho\sin\phi\cos\phi \\ (z-z') \end{bmatrix} = \begin{bmatrix} \rho\sin^2\phi \\ \rho\sin\phi\cos\phi \\ (z-z') \end{bmatrix}$$

But it is at $\phi = \pi/2$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \rho \\ 0 \\ (z-z') \end{bmatrix}$$

$$\vec{R} = \rho\,\hat{a}_\rho + 0\,\hat{a}_\phi + (z-z')\,\hat{a}_z$$

$$|\vec{R}| = \sqrt{\rho^2 + (z-z')^2} = R^{1/2} \quad (2)$$

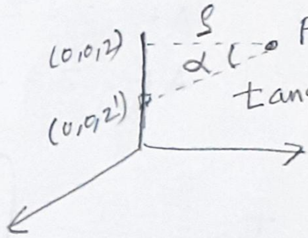
$$\vec{E} = \int_{z_A}^{z_B} \frac{\rho_L}{4\pi\varepsilon_0 R^2} \frac{\vec{R}\,dz'}{|\vec{R}|} = \int_{z_A}^{z_B} \frac{\rho_L}{4\pi\varepsilon_0} \times \frac{\vec{R}}{R \cdot R^{1/2}}\,dz'$$

$$\vec{E} = \int_{z_A}^{z_B} \frac{\rho_L}{4\pi\varepsilon_0} \times \frac{\left(\rho\,\hat{a}_\rho + (z-z')\,\hat{a}_z\right)\,dz'}{\left[\rho^2 + (z-z')^2\right]^{3/2}}$$

$$\vec{E} = \frac{\rho_L}{4\pi\varepsilon_0}\int_{z_A}^{z_B} \frac{\rho\, \hat{a}_\rho\, dz'}{\left[\rho^2 + (z-z')^2\right]^{3/2}} + \frac{\rho_L}{4\pi\varepsilon_0}\int_{z_A}^{z_B} \frac{(z-z')\, \hat{a}_z\, dz'}{\left[\rho^2 + (z-z')^2\right]^{3/2}}$$

$$\vec{E} = I_1 + I_2$$

$$I_1 = \frac{\rho_L}{4\pi\varepsilon_0}\int_{z_A}^{z_B} \frac{\rho\, \hat{a}_\rho\, dz'}{\left[\rho^2 + (z-z')^2\right]^{3/2}}$$



$\tan\alpha = \dfrac{(z-z')}{\rho}$

put $(z-z') = \rho \tan\alpha$

$-dz' = \rho \sec^2\alpha\, d\alpha$

At $z' = z_A$, $z - z_A = \rho \tan\alpha_1 \Rightarrow \alpha_1 = \tan^{-1}\left(\dfrac{z-z_A}{\rho}\right)$

At $z' = z_B$, $z - z_B = \rho \tan\alpha_2 \Rightarrow \alpha_2 = \tan^{-1}\left(\dfrac{z-z_B}{\rho}\right)$

$$\rho^2 + (z-z')^2 = \rho^2 + \rho^2\tan^2\alpha = \rho^2(1+\tan^2\alpha) = \rho^2\sec^2\alpha$$

$$I_1 = \frac{\rho_L}{4\pi\varepsilon_0}\int_{\alpha_1}^{\alpha_2} \frac{\rho\,(-\rho\sec^2\alpha\, d\alpha)}{(\rho^2\sec^2\alpha)^{3/2}}\,\hat{a}_\rho$$

$$= \frac{\rho_L}{4\pi\varepsilon_0}\int_{\alpha_1}^{\alpha_2} \frac{(-\rho^2)\sec^2\alpha\, \hat{a}_\rho}{\rho^3\sec^3\alpha}\, d\alpha = \frac{-\rho_L}{4\pi\varepsilon_0\rho}\int_{\alpha_1}^{\alpha_2} \cos\alpha\, d\alpha\, \hat{a}_\rho$$

$$= \frac{-\rho_L}{4\pi\varepsilon_0\rho}\left[+\sin\alpha\Big|_{\alpha_1}^{\alpha_2} \hat{a}_\rho\right]$$

$$= \frac{-\rho_L}{4\pi\varepsilon_0\rho}\left[\sin\alpha_2 - \sin\alpha_1\right]\hat{a}_\rho = \frac{\rho_L}{4\pi\varepsilon_0\rho}\left[\sin\alpha_1 - \sin\alpha_2\right]\hat{a}_\rho$$

$$\alpha_1 = \frac{\pi}{2} \qquad \alpha_2 = -\frac{\pi}{2}$$

$$I_1 = \frac{\rho_L}{4\pi\varepsilon_0\rho}\left[1 - (-1)\right]\hat{a}_\rho = \frac{\rho_L}{2\pi\varepsilon_0\rho}\hat{a}_\rho \rightarrow (3)$$

$$I_2 = \frac{\rho_L}{4\pi\varepsilon_0} \int_{z_A}^{z_B} \frac{(z-z')\,\hat{a}_z\, dz'}{\left[\rho^2 + (z-z')^2\right]^{3/2}}$$

put $z - z' = \rho \tan\alpha$

At $z = z_A \Rightarrow \alpha_1$

At $z = z_B \Rightarrow \alpha_2$

$$\rho^2 + (z-z')^2 = \rho^2 + \rho^2 \tan^2\alpha = \rho^2 \sec^2\alpha$$

$$I_2 = \frac{\rho_L}{4\pi\varepsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{\rho \tan\alpha \left(-\rho \sec^2\alpha \, d\alpha\right) \hat{a}_z}{\left(\rho^2 \sec^2\alpha\right)^{3/2}}$$

$$I_2 = \frac{-\rho_L \hat{a}_z}{4\pi\varepsilon_0 \rho} \int_{\alpha_1}^{\alpha_2} \frac{\cancel{\rho^2} \tan\alpha \sec^2\alpha}{\cancel{\rho^3} \sec^3\alpha}$$

$$I_2 = -\frac{\rho_L \hat{a}_z}{4\pi\varepsilon_0 \rho} \int_{\alpha_1}^{\alpha_2} \frac{\sin\alpha}{\cos\alpha} \times \frac{1}{\sec\alpha} \, d\alpha$$

$$I_2 = \frac{-\rho_L \hat{a}_z}{4\pi\varepsilon_0 \rho} \left[-\cos\alpha \Big|_{\alpha_1}^{\alpha_2} \right]$$

$$I_2 = \frac{\rho_L}{4\pi\varepsilon_0 \rho} \left[\cos\alpha \Big|_{\pi/2}^{-\pi/2} \right]$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 \rho} \left[\cos(-\pi/2) - \cos\pi/2 \right]$$

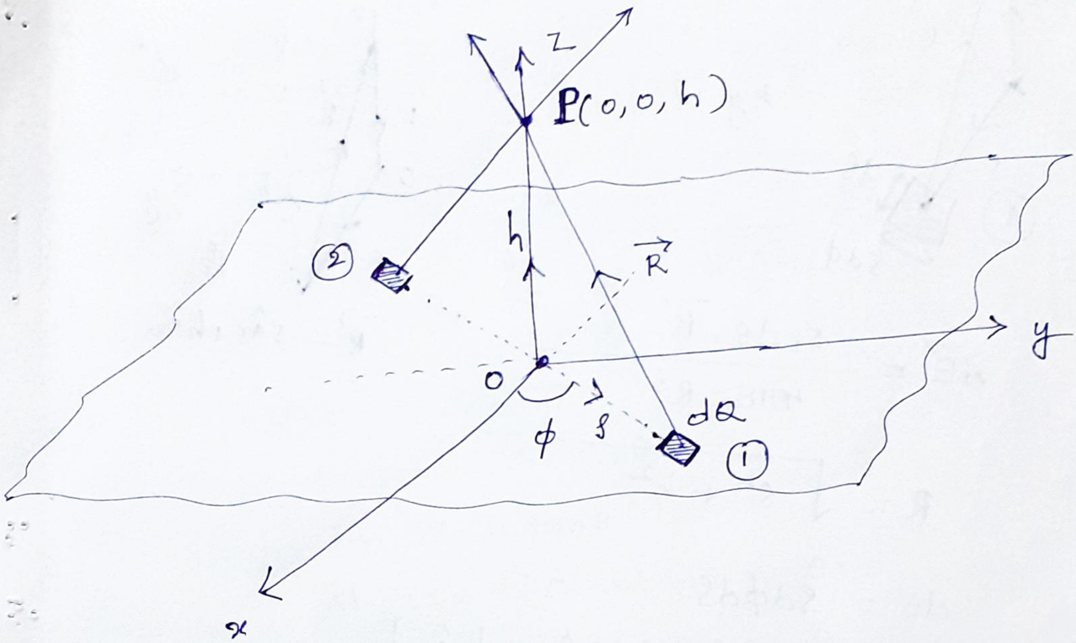
$$I_2 = 0$$

$$\therefore \vec{E} = I_1 + I_2$$

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0 \rho} \hat{a}_\rho}$$

Note: If the line is not along z-axis, ρ is the perpendicular distance from the line to the point of interest and $\hat{a}_\rho$ is a unit vector along <u>that</u> distance directed from the line charge to the field point.

Electric Field Intensity due to Infinite Sheet of Charge



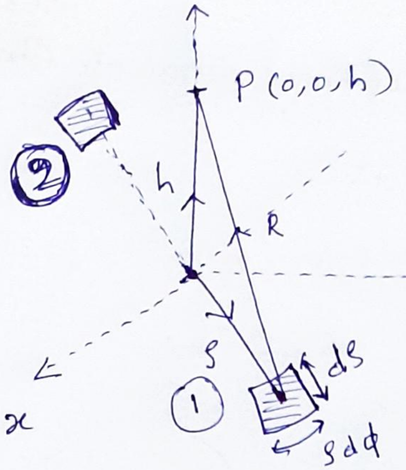
consider an infinite sheet of charge placed in the xy-plane with uniform charge density ρ_s. The charge associated with elemental area ds is given by

$$dQ = \rho_s ds$$

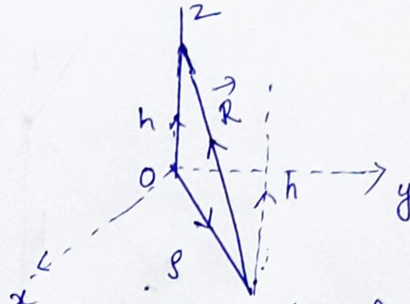
The electric field intensity at point "P" (0,0,h) by the charge "dQ" on the elemental surface-1, shown in figure

$$d\vec{E} = \frac{dQ}{4\pi\epsilon_0 R^2}\hat{a}_R$$

$$\vec{dE} = \frac{\rho_s ds}{4\pi\epsilon_0 R^2}\frac{\vec{R}}{|\vec{R}|} = \frac{\rho_s ds}{4\pi\epsilon_0}\frac{\vec{R}}{R^3} \rightarrow (1)$$



$\vec{R} = (-\rho)\hat{a}_\rho + h\,\hat{a}_z$



$\vec{R} = -\rho\hat{a}_\rho + h\hat{a}_z$

$$d\vec{E} = \frac{\rho_s\, ds\, \vec{R}}{4\pi\varepsilon_0\, R^3}$$

$$R = \sqrt{\rho^2 + h^2}$$

$$ds = \rho\, d\phi\, d\rho$$

$$d\vec{E} = \frac{\rho_s\, \rho\, d\phi\, d\rho\, [-\rho\,\hat{a}_\rho + h\,\hat{a}_z]}{4\pi\varepsilon_0\, [\rho^2+h^2]^{3/2}} \rightarrow (2)$$

Because of symmetry of the charge distribution, for every element (1), there is corresponding element (2) whose contribution along $\hat{a}_\rho$ cancels that of element (1).

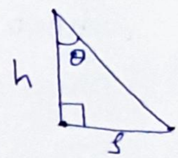
Thus the contributions to E_ρ add up to zero so that $\vec{E}$ has only z-component

$$\therefore\ d\vec{E} = \frac{\rho_s\, \rho\, d\phi\, d\rho\, h\,\hat{a}_z}{4\pi\varepsilon_0\, (\rho^2+h^2)^{3/2}}$$

$$\vec{E} = \int_S d\vec{E} = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\infty} \frac{\rho_s h \, \rho \, d\phi \, d\rho}{4\pi\varepsilon_0 [\rho^2 + h^2]^{3/2}} \hat{a}_z$$

$$\vec{E} = \frac{\rho_s h}{4\pi\varepsilon_0} \int_{\rho=0}^{\rho=\infty} \phi \Big|_0^{2\pi} \cdot \frac{\rho \, d\rho}{[\rho^2 + h^2]^{3/2}} \hat{a}_z$$

$$\vec{E} = \frac{\rho_s h}{4\pi\varepsilon_0} \int_{\rho=0}^{\rho=\infty} (2\pi) \cdot \frac{\rho \, d\rho}{(\rho^2 + h^2)^{3/2}} \hat{a}_z$$



put $\rho = h \tan\theta$

$d\rho = h \sec^2\theta \, d\theta$

$\tan\theta = \frac{\rho}{h}$

$\rho = h \tan\theta$

consider $\rho = h . \tan\theta$

At $\rho = 0$ $\quad 0 = h\tan\theta \Rightarrow \theta = 0$

At $\rho = \infty$ $\quad \infty = h\tan\theta \Rightarrow \theta = \pi/2$

$$\vec{E} = \frac{\rho_s h}{4\pi\varepsilon_0} \int_{\theta=0}^{\theta=\pi/2} \frac{2\pi (h\tan\theta)(h\sec^2\theta \, d\theta)}{[h^2\tan^2\theta + h^2]^{3/2}} \hat{a}_z$$

$$\vec{E} = \frac{\rho_s \cancel{h^3} \times \cancel{2\pi}}{2 \, \cancel{4\pi}\varepsilon_0} \int_{\theta=0}^{\theta=\pi/2} \frac{\tan\theta \sec^2\theta}{\cancel{h^3} \sec^3\theta} d\theta \, \hat{a}_z = \frac{\rho_s}{2\varepsilon_0} \int_{\theta=0}^{\theta=\pi/2} \frac{\tan\theta}{\sec\theta} d\theta \, \hat{a}_z$$

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0} \int_{\theta=0}^{\theta=\pi/2} \sin\theta \, d\theta \, \hat{a}_z = -\frac{\rho_s}{2\varepsilon_0} \left[\cos\theta \Big|_{\theta=0}^{\theta=\pi/2} \right] \hat{a}_z$$

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0} \hat{a}_z \rightarrow (1)$$

*) $\vec{E}$ has only z-component if the charge is in the x-y plane. Equation (1) is valid for $h > 0$

*) If $h < 0$, then $\vec{E} = \frac{\rho_s}{2\varepsilon_0} (-\hat{a}_z)$

*) In general for an infinite sheet of charge

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}\,\hat{a}_n$$

where $\hat{a}_n$ = unit vector normal to the sheet

*) $\vec{E}$ is independent of the distance between the sheet and the point of observation "P"

1) A circular ring of radius a carries a uniform charge ρ_L C/m and is placed on the xy-plane with axis the same as the z-axis

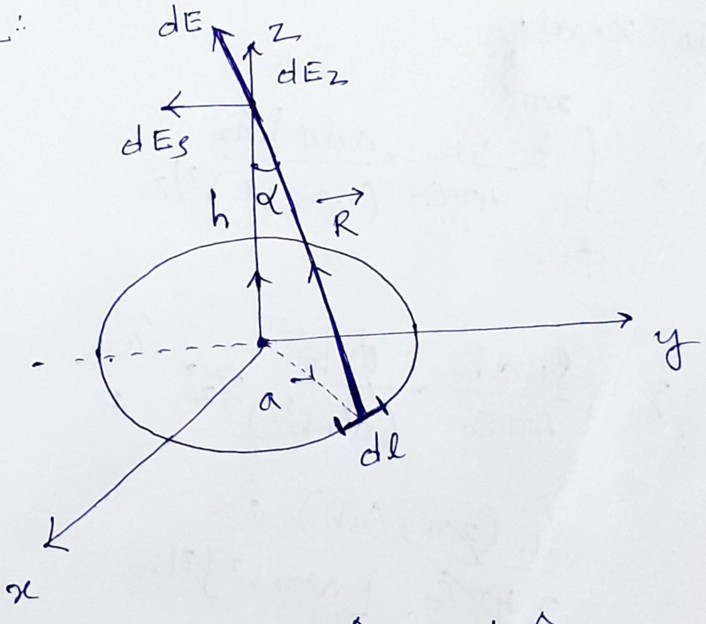
(a) show that

$$E(0,0,h) = \frac{\rho_L a h}{2\varepsilon_0 [h^2+a^2]^{3/2}} \hat{a}_z$$

(b) What values of "h" give the maximum value of $\vec{E}$?

(c) If the total charge on the ring is "Q", find $\vec{E}$ as $a \to 0$.

<u>Solution</u>:



$$\vec{R} = a(-\hat{a}_\rho) + h\hat{a}_z$$

$$R = |\vec{R}| = \sqrt{a^2+h^2}$$

$$\hat{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{-a\hat{a}_\rho + h\hat{a}_z}{\sqrt{a^2+h^2}}$$

(17a)

$$\vec{E} = \int_{\phi=0}^{\phi=2\pi} \frac{\rho_L}{4\pi\varepsilon_0} \frac{dl}{R^2} \hat{a}_R$$

$$dl = a\,d\phi$$

$$\hat{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{-a\hat{a}_\rho + h\hat{a}_z}{\sqrt{a^2+h^2}}$$

$$\vec{E} = \int_{\phi=0}^{\phi=2\pi} \frac{\rho_L}{4\pi\varepsilon_0} \times \frac{a\,d\phi\,(-a\hat{a}_\rho + h\hat{a}_z)}{(a^2+h^2)^{3/2}}$$

By ~~the~~ symmetry, the contributions along $\hat{a}_\rho$ add upto zero.

$$\therefore \vec{E} = \int_{\phi=0}^{2\pi} \frac{\rho_L}{4\pi\varepsilon_0} \times \frac{a\,d\phi\, h\hat{a}_z}{(a^2+h^2)^{3/2}}$$

$$= \frac{\rho_L a h}{4\pi\varepsilon_0} \times \frac{\phi\,\Big|_0^{2\pi}}{(a^2+h^2)^{3/2}} \times \frac{\hat{a}_z}{1}$$

$$= \frac{\rho_L (\cancel{2\pi})(ah)\,\hat{a}_z}{2\,\cancel{4\pi}\varepsilon_0\,[a^2+h^2]^{3/2}}$$

$$\vec{E} = \frac{\rho_L a h\,\hat{a}_z}{2\varepsilon_0\,[h^2+a^2]^{3/2}}$$

(18)

b) $$\vec{E} = \frac{\rho_L a h \hat{a}_z}{2\varepsilon_0 [h^2+a^2]^{3/2}}$$

$$\frac{d|\vec{E}|}{dh} = \frac{\rho_L a}{2\varepsilon_0}\left[\frac{(h^2+a^2)^{3/2}(1) - (h)\left(\frac{3}{2}(h^2+a^2)^{\frac{3}{2}-1} 2h\right)}{[(h^2+a^2)^{3/2}]^2}\right]$$

For maximum $\frac{d|\vec{E}|}{dh} = 0$

$$(h^2+a^2)^{\frac{3}{2}} - \left(\frac{h^2 3}{\cancel{2}}\right)\cancel{2}(h^2+a^2)^{1/2} = 0$$

$$(h^2+a^2)^{1/2}\left[h^2+a^2 - 3h^2\right] = 0$$

$$h^2+a^2-3h^2 = 0$$

$$a^2 - 2h^2 = 0$$

$$a^2 = 2h^2$$

$$h^2 = \frac{a^2}{2} \qquad h = \pm\sqrt{\frac{a^2}{2}} = \pm\frac{a}{\sqrt{2}}$$

$$h = \pm\frac{a}{\sqrt{2}}$$

c) $$\vec{E} = \frac{\rho_L a h \hat{a}_z}{2\varepsilon_0 (a^2+h^2)^{3/2}}$$

$$\rho_L = \frac{Q}{2\pi a}$$

$$\vec{E} = \frac{\frac{Q}{2\pi \cancel{a}} \times \cancel{a} \times h \hat{a}_z}{2\varepsilon_0 (a^2+h^2)^{3/2}}$$

$$= \frac{Q h \hat{a}_z}{4\pi\varepsilon_0 (a^2+h^2)^{3/2}}$$

$$\lim_{a\to 0} \vec{E} = \frac{Qh}{4\pi\varepsilon_0 h^3}\hat{a}_z = \frac{Q}{4\pi\varepsilon_0 h^2}\hat{a}_z$$

2) A circular disk of radius "a" uniformly charged with ρ_s C/m². The disk lies on the z=0 plane with its axis along the z-axis

(a) show that at point (0, 0, h),

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}\left[1 - \frac{h}{[h^2+a^2]^{1/2}}\right]\hat{a}_z$$

(b) From this, derive the $\vec{E}$ field due to an infinite sheet of charge on the z=0 plane.

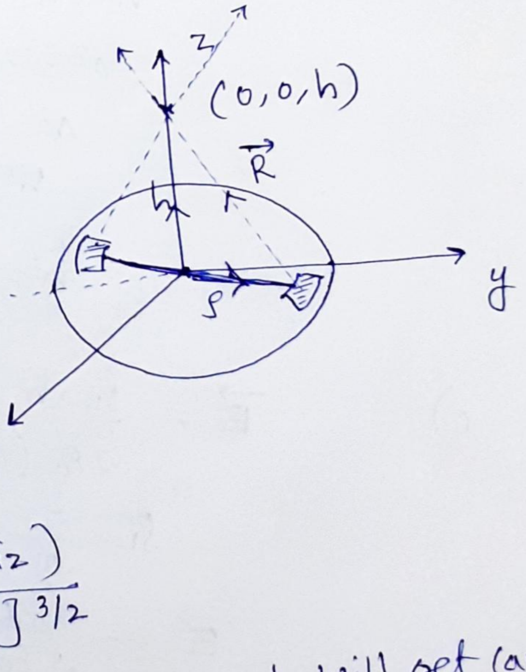
(c) If $a \ll h$, show that $\vec{E}$ is similar to the field due to a point charge.

Solution:

$$\vec{R} = -\rho\hat{a}_\rho + h\hat{a}_z$$

$$|\vec{R}| = \sqrt{\rho^2+h^2}$$

$$d\vec{E} = \frac{dQ}{4\pi\varepsilon_0} \times \frac{1}{R^2} \times \frac{\vec{R}}{|\vec{R}|}$$



$$d\vec{E} = \frac{dQ}{4\pi\varepsilon_0} \times \frac{\vec{R}}{R^3}$$

$$d\vec{E} = \frac{dQ}{4\pi\varepsilon_0} \times \frac{(-\rho\hat{a}_\rho + h\hat{a}_z)}{[\rho^2+h^2]^{3/2}}$$

Due to symmetry $\hat{a}_\rho$ component will get cancel

$$d\vec{E} = \frac{dQ}{4\pi\varepsilon_0} \times \frac{(h\hat{a}_z)}{(\rho^2+h^2)^{3/2}}$$

But $dQ = \rho_s \rho \, d\phi \, d\rho$

$$d\vec{E} = \frac{\rho_s [\rho \, d\phi \, d\rho]}{4\pi \varepsilon_0} \times \frac{h \, \hat{a}_z}{(\rho^2 + h^2)^{3/2}}$$

$$\vec{E} = \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} \frac{\rho_s [d\phi \, \rho \, d\rho]}{4\pi \varepsilon_0} \times \frac{h \, \hat{a}_z}{[\rho^2 + h^2]^{3/2}}$$

Integrating w.r.to ϕ

$$\vec{E} = \frac{\rho_s h}{4\pi \varepsilon_0} \int_{\rho=0}^{\rho=a} \frac{2\pi \, \rho \, d\rho \, \hat{a}_z}{[\rho^2 + h^2]^{3/2}}$$

At $\rho = 0$ $\theta = \tan^{-1}\left(\frac{\rho}{h}\right) = 0^\circ$

put $\rho = h \tan\theta$

$d\rho = h \sec^2\theta \, d\theta$

$\rho = a$ $\theta = \tan^{-1}\left(\frac{a}{h}\right)$

$$\vec{E} = \frac{\rho_s h \times \cancel{2\pi}}{\cancel{4\pi}_{2} \, \varepsilon_0} \int_{\theta=0}^{\theta=\tan^{-1}(a/h)} \frac{h \tan\theta \, h \sec^2\theta \cdot d\theta}{[h^2 \tan^2\theta + h^2]^{3/2}} \hat{a}_z$$

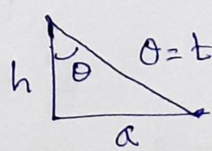
$$\vec{E} = \frac{\rho_s \cancel{h^3}}{2 \varepsilon_0} \int_{\theta=0}^{\theta=\tan^{-1}(a/h)} \frac{\tan\theta \sec^2\theta}{\cancel{h^3} \sec^3\theta} d\theta \, \hat{a}_z$$

$$= \frac{\rho_s}{2\varepsilon_0} \int_{\theta=0}^{\theta=\tan^{-1}(a/h)} \tan\theta \cdot \cos\theta \, d\theta \, \hat{a}_z$$

$$= \frac{\rho_s}{2\varepsilon_0} \int_{\theta=0}^{\theta=\tan^{-1}(a/h)} \sin\theta \, d\theta \, \hat{a}_z = \frac{\rho_s}{2\varepsilon_0} \left[-\cos\theta\right]_{\theta=0}^{\theta=\tan^{-1}\left(\frac{a}{h}\right)} \hat{a}_z$$

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0} \left[1 - \cos\left(\tan^{-1}\left(\frac{a}{h}\right)\right)\right] \hat{a}_z$$

θ



$\theta = \tan^{-1}\left(\frac{a}{h}\right)$

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0} \left[1 - \frac{h}{\sqrt{h^2 + a^2}}\right] \hat{a}_z$$

$\therefore \cos\theta = \frac{h}{\sqrt{a^2 + h^2}}$

b) Given $\vec{E} = \frac{\rho_s}{2\varepsilon_0}\left[1 - \frac{h}{(h^2+a^2)^{1/2}}\right]\hat{a}_z$

If it is an infinite sheet of charge then $a = \infty$

$$\therefore \lim_{a\to\infty} \vec{E} = \lim_{a\to\infty} \frac{\rho_s}{2\varepsilon_0}\left[1 - \frac{h}{(h^2+a^2)^{1/2}}\right]\hat{a}_z$$

$$= \lim_{a\to\infty} \frac{\rho_s}{2\varepsilon_0}\left[1 - \cancel{\frac{\cancel{h}\times 1}{\cancel{h}\left[1+\left(\frac{a}{h}\right)^2\right]^{1/2}}}^{\,0}\right]\hat{a}_z$$

$$= \frac{\rho_s}{2\varepsilon_0}\hat{a}_z$$

c) Given $\vec{E} = \frac{\rho_s}{2\varepsilon_0}\left[1 - \frac{h}{(h^2+a^2)^{1/2}}\right]\hat{a}_z$

If $a \ll h$

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}\left[1 - \frac{\cancel{h}}{\cancel{h}\left[1+\left(\frac{a}{h}\right)^2\right]^{1/2}}\right]\hat{a}_z$$

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}\left[1 - \left(1+\left(\frac{a}{h}\right)^2\right)^{-1/2}\right]\hat{a}_z$$

Apply Binomial theorem $\Rightarrow (1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \dots$

Neglecting higher order

$$(1+x)^n \cong 1 + nx$$

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}\left[1 - \left(1 - \frac{1}{2}\left(\frac{a}{h}\right)^2\right)\right]\hat{a}_z$$

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}\left[\cancel{1} - \cancel{1} + \frac{1}{2}\left(\frac{a}{h}\right)^2\right]\hat{a}_z = \frac{\rho_s a^2}{4\varepsilon_0 h^2}\hat{a}_z$$

$$\vec{E} = \frac{\rho_s \pi a^2}{4\pi\varepsilon_0 h^2}\hat{a}_z \quad [\text{Multiply and divide by } \pi]$$

$$\vec{E} = \frac{[\rho_s (\pi a^2)]}{4\pi \varepsilon_0 h^2} \hat{a}_z$$

Q = Total charge

= $\rho_s \times$ Area of the circle

$= \rho_s \times \pi a^2$

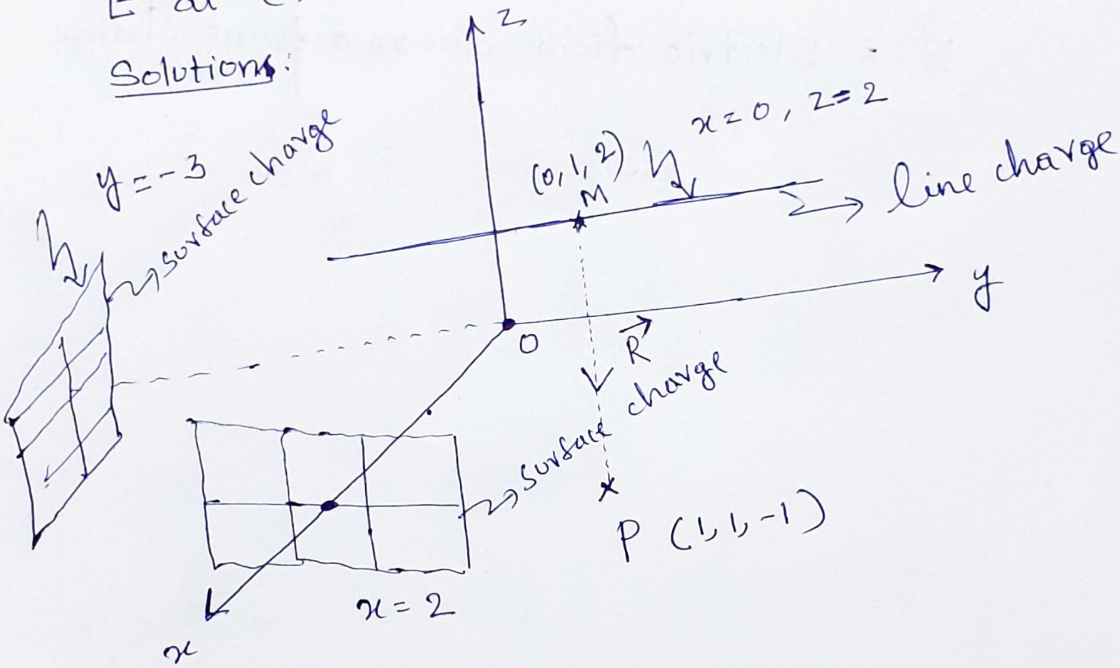
22

$$\vec{E} = \frac{Q}{4\pi \varepsilon_0} \times \frac{1}{h^2} \hat{a}_z$$

$\vec{E}$ = Electric field due to a point charge

3) Planes $x = 2$ and $y = -3$ respectively, carry charges $10\,\eta C/m^2$ and $15\,\eta C/m^2$. If the line $x = 0,\ z = 2$ carries charge $10\pi\ nC/m$, calculate $\vec{E}$ at $(1, 1, -1)$ due to the three charge distributions.

Solutions:



$$\vec{E_1} = \frac{\rho_{s1}}{2\varepsilon_0}(-\hat{a}_x) = 5\,\frac{\cancel{10}\times10^{-9}}{\cancel{2}\times\frac{10^{-9}}{36\pi}} \times (-\hat{a}_x) = -180\pi\,\hat{a}_x$$
$$= -565.4866\,\hat{a}_x$$

$$\vec{E_2} = \frac{\rho_{s2}}{2\varepsilon_0}\hat{a}_y$$

$$\vec{E_2} = \frac{15\times10^{-9}}{\cancel{2}\times\frac{10^{-9}}{18\ 36\pi}}\hat{a}_y = 270\pi\,\hat{a}_y = 848.23\,\hat{a}_y$$

$$\vec{E_3} = \frac{\rho_L}{2\pi\varepsilon_0}\frac{\hat{a}_\rho}{\rho}$$

$$\hat{a}_\rho = \frac{\vec{R}}{|\vec{R}|} \qquad \vec{R} = (1,1,-1) - (0,1,2)$$
$$\vec{R} = \hat{a}_x + 0\,\hat{a}_y - 3\,\hat{a}_z$$

$$\rho = R = |\vec{R}| = \sqrt{1+3^2} = \sqrt{10}$$

$$\vec{E_3} = \frac{10\pi \times 10^{-9}}{2\pi\varepsilon_0} \times \frac{1}{(\sqrt{10})} \times \frac{[\hat{a}_x - 3\hat{a}_z]}{(\sqrt{10})}$$

$$= \frac{10\pi \times 10^{-9}}{2\pi\varepsilon_0} \times \left(\frac{\hat{a}_x - 3\hat{a}_z}{10}\right)$$

$$= \frac{\cancel{10}\,\pi \times \cancel{10^{-9}}}{\cancel{2\pi} \times \frac{\cancel{10^{-9}}}{\cancel{36\pi}\,18}} \times \frac{(\hat{a}_x - 3\hat{a}_z)}{\cancel{10}}$$

$$\vec{E_3} = 18\pi\,\hat{a}_x - 54\pi\,\hat{a}_z = 56.5486\,\hat{a}_x - 169.64\,\hat{a}_z$$

$$\vec{E} = \vec{E_1} + \vec{E_2} + \vec{E_3}$$

$$\vec{E} = -162\pi\,\hat{a}_x + 270\pi\,\hat{a}_y - 54\pi\,\hat{a}_z$$

$$\vec{E} = -508.9380\,\hat{a}_x + 848.23\,\hat{a}_y - 169.6460\,\hat{a}_z$$

4) In the above example if the line $x=0$ and $z=2$ is rotated through 90° about the point $(0,2,2)$ so that it becomes $x=0$, $y=2$, find $\vec{E}$ at $(1,1,-1)$

[Ans: $-282.7\,\hat{a}_x + 565.5\,\hat{a}_y$ V/m]