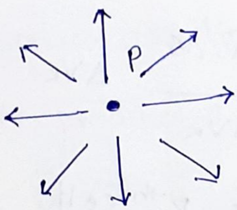
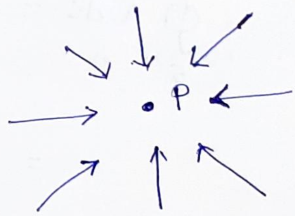
Divergence of a Vector and Divergence Theorem

The divergence of $\vec{A}$ at a given point 'P' is the outward flux per unit volume as the volume shrinks about P

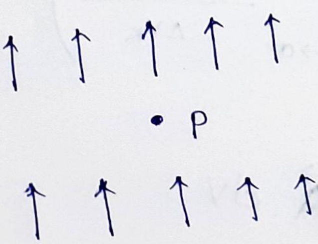
$$\text{div}\,\vec{A} = \nabla \cdot \vec{A} = \lim_{\Delta V \to 0} \frac{\oint \vec{A} \cdot ds}{\Delta V}$$



Positive divergence



Negative divergence



Zero divergence

ΔV = Volume enclosed by the closed surface "S" in which point "P" is located.

Divergence theorem

$$\oint_S \vec{A}\cdot d\vec{s} = \int_V \nabla\cdot\vec{A}\, dV$$

The divergence theorem states that the total outward flux of a vector field $\vec{A}$ through the closed surface S is same as the volume integral of the divergence of $\vec{A}$

$$\oint_S \vec{A}\cdot d\vec{s} = \sum_K \oint_{S_K} \vec{A}\cdot d\vec{s}$$

$$= \sum_K \frac{\oint_{S_K} \vec{A}\cdot ds}{\Delta V_K} \Delta V_K$$

ΔV_K = Volume of the K^{th} cell

S_K = K^{th} cell closed surface area

$$= \sum_K \left(\lim_{\Delta V_K \to 0} \frac{\oint_{S_K} \vec{A}\cdot ds}{\Delta V_K} \right) \Delta V_K$$

$$= \int_V \nabla\cdot\vec{A}\, dV$$

divergence of a vector in different coordinate systems

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$ [In Cartesian coordinate system]

$$\nabla \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$ [In cylindrical coordinate system]

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial (A_\phi)}{\partial \phi}$$

[In Spherical coordinate System]

Properties of the divergence of a vector field

1) It produces a scalar field

2) $\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$

3) $\nabla \cdot (V\vec{A}) = V\,\nabla \cdot \vec{A} + \vec{A} \cdot \nabla V$ where $\vec{A}$ is a vector, V is a scalar

1) Determine the divergence of the following vector fields

(a) $\vec{P} = x^2yz\,\hat{a}_x + xz\,\hat{a}_z$

(b) $\vec{Q} = \rho\sin\phi\,\hat{a}_\rho + \rho^2 z\,\hat{a}_\phi + z\cos\phi\,\hat{a}_z$

(c) $\vec{T} = \frac{1}{r^2}\cos\theta\,\hat{a}_r + r\sin\theta\cos\phi\,\hat{a}_\theta + \cos\theta\,\hat{a}_\phi$

Solution :

(a) $\vec{P} = x^2yz\,\hat{a}_x + xz\,\hat{a}_z$

$$\nabla\cdot\vec{P} = \frac{\partial}{\partial x}P_x + \frac{\partial}{\partial y}P_y + \frac{\partial}{\partial z}P_z$$

$$\nabla\cdot\vec{P} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(xz)$$

$$\nabla\cdot\vec{P} = 2xyz + x$$

(b) $\vec{Q} = \rho\sin\phi\,\hat{a}_\rho + \rho^2 z\,\hat{a}_\phi + z\cos\phi\,\hat{a}_z$

$$\nabla\cdot\vec{Q} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho Q_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(Q_\phi) + \frac{\partial Q_z}{\partial z}$$

$$\nabla\cdot\vec{Q} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho.\rho\sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\cos\phi)$$

$$\nabla\cdot\vec{Q} = \frac{1}{\cancel{\rho}}\times 2\cancel{\rho}\sin\phi + \frac{1}{\rho}(0) + \cos\phi$$

$$\nabla\cdot\vec{Q} = 2\sin\phi + \cos\phi$$

(c) $\vec{T} = \frac{1}{r^2}\cos\theta\,\hat{a}_r + r\sin\theta\cos\phi\,\hat{a}_\theta + \cos\theta\,\hat{a}_\phi$

$$\nabla\cdot\vec{T} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(T_\phi)$$

$$\nabla\cdot\vec{T} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2}\times\frac{1}{\cancel{r^2}}\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin\theta\cos\phi\sin\theta) +$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\cos\theta)$$

$$= 0 + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin^2\theta\cos\phi\right) + 0$$

$$= 0 + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin^2\theta\cos\phi\right) + 0$$

$$= 0 + \frac{1}{\cancel{r}\cancel{\sin\theta}} \times 2\cancel{r}\cancel{\sin\theta}\cos\theta\cos\phi$$

$$= 2\cos\theta\cos\phi$$

2) Determine the divergence of the following vector fields and evaluate them at specified points

(a) $\vec{A} = yz\,\hat{a}_x + 4xy\,\hat{a}_y + y\,\hat{a}_z$ at $(1, -2, 3)$

(b) $\vec{B} = \rho z\sin\phi\,\hat{a}_\rho + 3\rho z^2\cos\phi\,\hat{a}_\phi$ at $(5, \pi/2, 1)$

(c) $\vec{C} = 2r\cos\theta\cos\phi\,\hat{a}_r + r^{1/2}\,\hat{a}_\phi$ at $(1, \pi/6, \pi/3)$

Solution: (a) $\vec{A} = yz\,\hat{a}_x + 4xy\,\hat{a}_y + y\,\hat{a}_z$

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(4xy) + \frac{\partial}{\partial z}(y)$$

$$= 0 + 4x + 0 = 4x\Big|_{(1,-2,3)} = 4$$

(b) $\vec{B} = \rho z\sin\phi\,\hat{a}_\rho + 3\rho z^2\cos\phi\,\hat{a}_\phi$

$$\nabla\cdot\vec{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(B_\phi) + \frac{\partial}{\partial z}(B_z)$$

$$\nabla\cdot\vec{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho.\rho z\sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(3\rho z^2\cos\phi) + \frac{\partial}{\partial z}(0)$$

$$\nabla\cdot\vec{B} = \frac{1}{\cancel{\rho}}\,2\cancel{\rho}z\sin\phi + \frac{1}{\cancel{\rho}}\,3\cancel{\rho}z^2(-\sin\phi)$$

$$\nabla\cdot\vec{B} = 2z\sin\phi - 3z^2\sin\phi\Big|_{At\,(5,\pi/2,1)} = 2\times 1\times\sin\frac{\pi}{2} - 3\times 1^2\times\sin\pi/2$$

$$= 2 - 3 = -1$$

41

$$\vec{C} = 2r\cos\theta\cos\phi\,\hat{a}_r + r^{1/2}\,\hat{a}_\phi$$

$$\nabla\cdot\vec{C} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 C_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(C_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(C_\phi)$$

$$\nabla\cdot\vec{C} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \times 2r\cos\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(0\times\sin\theta)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^{1/2})$$

$$\nabla\cdot\vec{C} = \frac{1}{r^2}\frac{\partial}{\partial r}(2r^3\cos\theta\cos\phi) + 0 + 0$$

$$\nabla\cdot\vec{C} = \frac{1}{\cancel{r^2}} \times 6\cancel{r^2}\cos\theta\cos\phi$$

$$\nabla\cdot\vec{C} = 6\cos\theta\cos\phi$$

$$\nabla\cdot\vec{C}\Big|_{(1,\pi/6,\pi/3)} = 6\times\cos\frac{\pi}{6}\cos\frac{\pi}{3} = 6\times\frac{\sqrt{3}}{2}\times\frac{1}{2} = \frac{6\sqrt{3}}{4} = 1.5\sqrt{3}$$

$$= 2.5980$$