CONTROL SYSTEMS

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Stability of Control Systems

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Methods of Stability Analysis

- Time domain methods
  - Routh-Hurwitz Criterion
  - Root Locus Technique
- Frequency domain methods
  - Bode Analysis
  - Nyquist Stability Criterion

FREQUENCY DOMAIN METHODS

Stability of Control Systems: Bode Plot

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Advantages

- Frequency response tests are generally simple and can be determined accurately by use of sinusoidal signal generators and precise measurement equipment.
- Transfer function of complicated systems can be determined experimentally by frequency response tests
- System may be designed so that effects of undesirable noise are negligible and such analysis and design can be extended to non linear systems

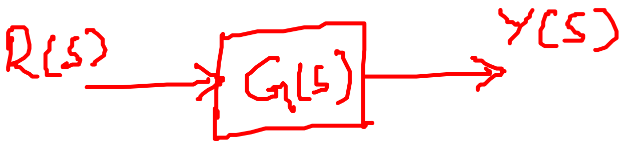
FREQUENCY RESPONSE

Advantages

- Modeling transfer functions from physical data.
- Designing compensators to meet steady state error and transient response requirements.
- Finding stability of systems

FREQUENCY RESPONSE

Sinusoidal Transfer Function



$$Y(s) = G(s)\,R(s)$$

$x(t) = A \sin wt \rightarrow$ Reference input

$$R(s) = \frac{Aw}{s^2+w^2}$$

$$Y(s) = G(s)\,R(s) = G(s)\frac{Aw}{s^2+w^2} = G(s)\frac{Aw}{(s+jw)(s-jw)} = \frac{A_1}{s+jw} + \frac{A_2}{s-jw}$$

$$A_1 = \left.\frac{G(s)\,Aw\,\cancel{(s+jw)}}{\cancel{(s+jw)}(s-jw)}\right|_{s=-jw} = \frac{G(-jw)\,A\cancel{w}}{-2j\cancel{w}} = \frac{A\,G(-jw)}{-2j}$$

FREQUENCY RESPONSE

Sinusoidal Transfer Function

$$A_2 = \frac{G(s)\,A\omega\,\cancel{(s-j\omega)}}{(s+j\omega)\,\cancel{(s-j\omega)}}\Bigg|_{s=j\omega} = \frac{G(j\omega)\,A\cancel{\omega}}{2j\cancel{\omega}} = \frac{A\,G(j\omega)}{2j}$$

$$Y(s) = \frac{1}{2j}\left[-\frac{A\,G(-j\omega)}{s+j\omega} + \frac{A\,G(j\omega)}{s-j\omega}\right]$$

$$G(j\omega) = |G(j\omega)|\,e^{j\phi}\,, \quad \phi = \angle G(j\omega)$$

$$G(-j\omega) = |G(j\omega)|\,e^{-j\phi}$$

$$Y(s) = \frac{A\,|G(j\omega)|}{2j}\left[-e^{-j\phi}\frac{1}{s+j\omega} + e^{+j\phi}\frac{1}{s-j\omega}\right]$$

ILT,
$$y(t) = \frac{A\,|G(j\omega)|}{2j}\left[-e^{-j\phi}\cdot e^{-j\omega t} + e^{j\phi}\cdot e^{j\omega t}\right] = \frac{A\,|G(j\omega)|}{2j}\left[-e^{-j(\phi+\omega t)} + e^{j(\phi+\omega t)}\right]$$

FREQUENCY RESPONSE

Sinusoidal Transfer Function

$$y(t) = A|G(j\omega)| \sin(\phi + \omega t)$$

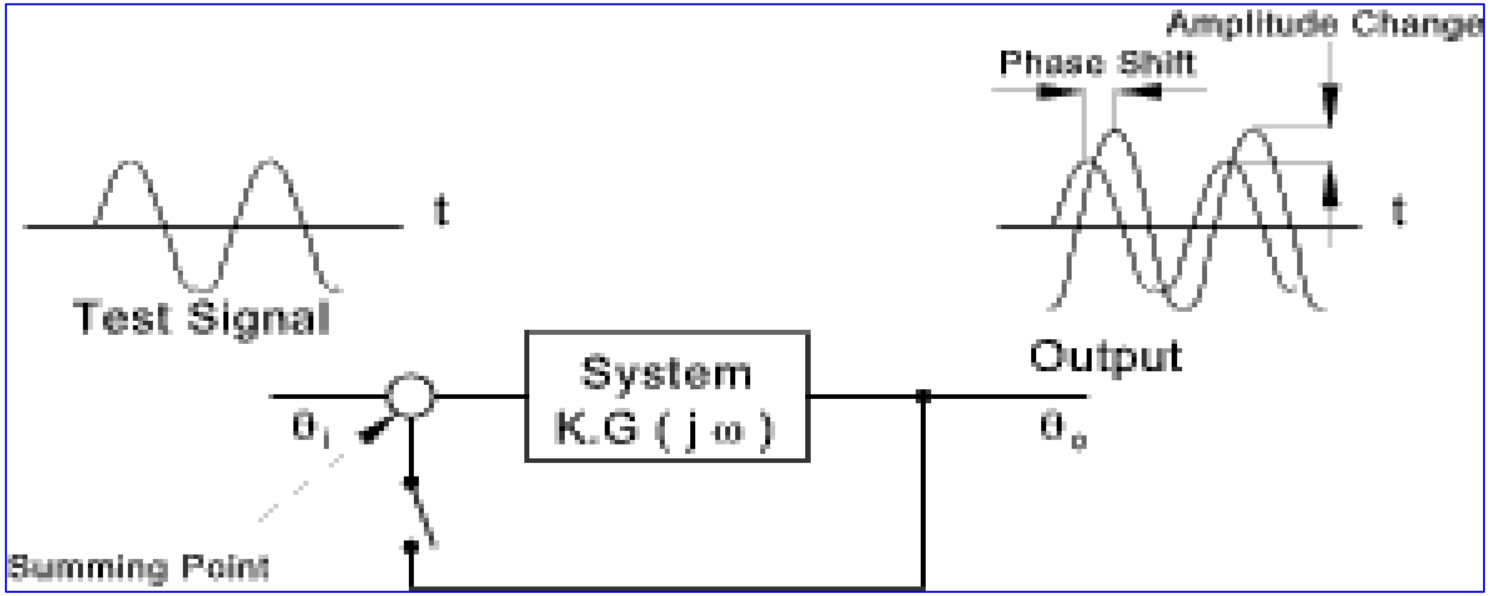
FREQUENCY RESPONSE

Sinusoidal Transfer Function

- **Sinusoidal transfer function:** Transfer function as a function of jω
- For a linear system sinusoidal input results in sinusoidal output
- The steady state output will have same frequency as the input
- Amplitude and phase of output will vary with frequency

$r(t) = A \sin \omega t$

$y(t) = A|G(j\omega)| \sin(\phi + \omega t)$



FREQUENCY RESPONSE

Sinusoidal Transfer Function

- In the steady state, sinusoidal inputs to a linear system generate sinusoidal responses of the same frequency, but Output differs in amplitudes and phase angle from the input
- Differences are function of frequency.
- Frequency response: Steady state response (output) of a system to sinusoidal inputs of varying frequency.
- It is plot of magnitude and phase as a function of frequency (as only magnitude and phase change with change in frequency).

Sinusoidal Transfer Function

Ex: For the sinusoidal transfer function G(s)=1/(s+1), find the magnitude and phase of G(s)

Procedure:

1. Put s= jω
2. G(j ω)=1/(j ω +1)

$\Rightarrow |G(j\omega)| = \frac{1}{\sqrt{\omega^2+1}}$

$\Rightarrow$ Magnitude in dB

$\underbrace{20\log_{10}|G(j\omega)|}_{M} = 20\log_{10}1 - 20\log_{10}\sqrt{\omega^2+1}$

$20\log_{10}1 \rightarrow 0$

$M = -20\log_{10}\sqrt{\omega^2+1}$

$\phi = \underline{|G(j\omega)} = 0 - \tan^{-1}\frac{\omega}{1}$

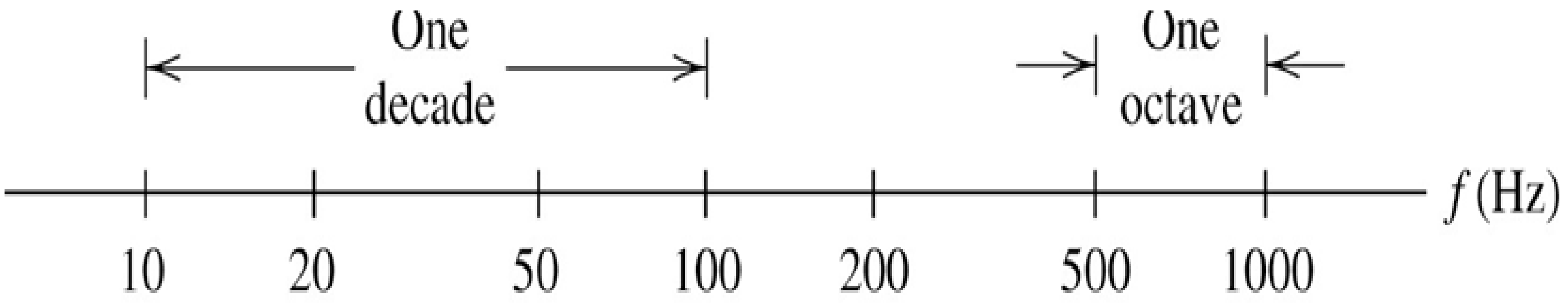
$$M = 20\log_{10}(1/\sqrt{1+\omega^2})\,dB$$

$$M = -20\log_{10}\sqrt{1+\omega^2}\,)\,dB$$

$$\varphi = -\tan^{-1}(\omega)$$

Sinusoidal Transfer Function

- On a logarithmic scale, the variable is multiplied by a given factor for equal increments of length along the axis.
- Decade Change –$\log_{10}$
- Octave change – $\log_2$



CONTROL SYSTEMS

Stability of Control Systems: Bode Plot

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- Bode Plot:

  Magnitude plot - magnitude vs frequency

  Phase plot – phase vs frequency

  Open loop transfer function

- Plotted in log scale(to cover large range)

  Magnitude in decibels(dB)

  Frequency in rad/sec (on log scale)

Bode Plot

- Consider an open loop transfer function

$$G(s)H(s) = \frac{k(s+z_1)(s+z_2)\ \dots.(s+z_m)}{(s+p_1)(s+p_2)\ \dots.(s+p_n)}$$

- Poles and Zeros are the singularities of this function
- Find the expressions of magnitude(M(ω) in dB) and phase(φ(ω)) in degrees

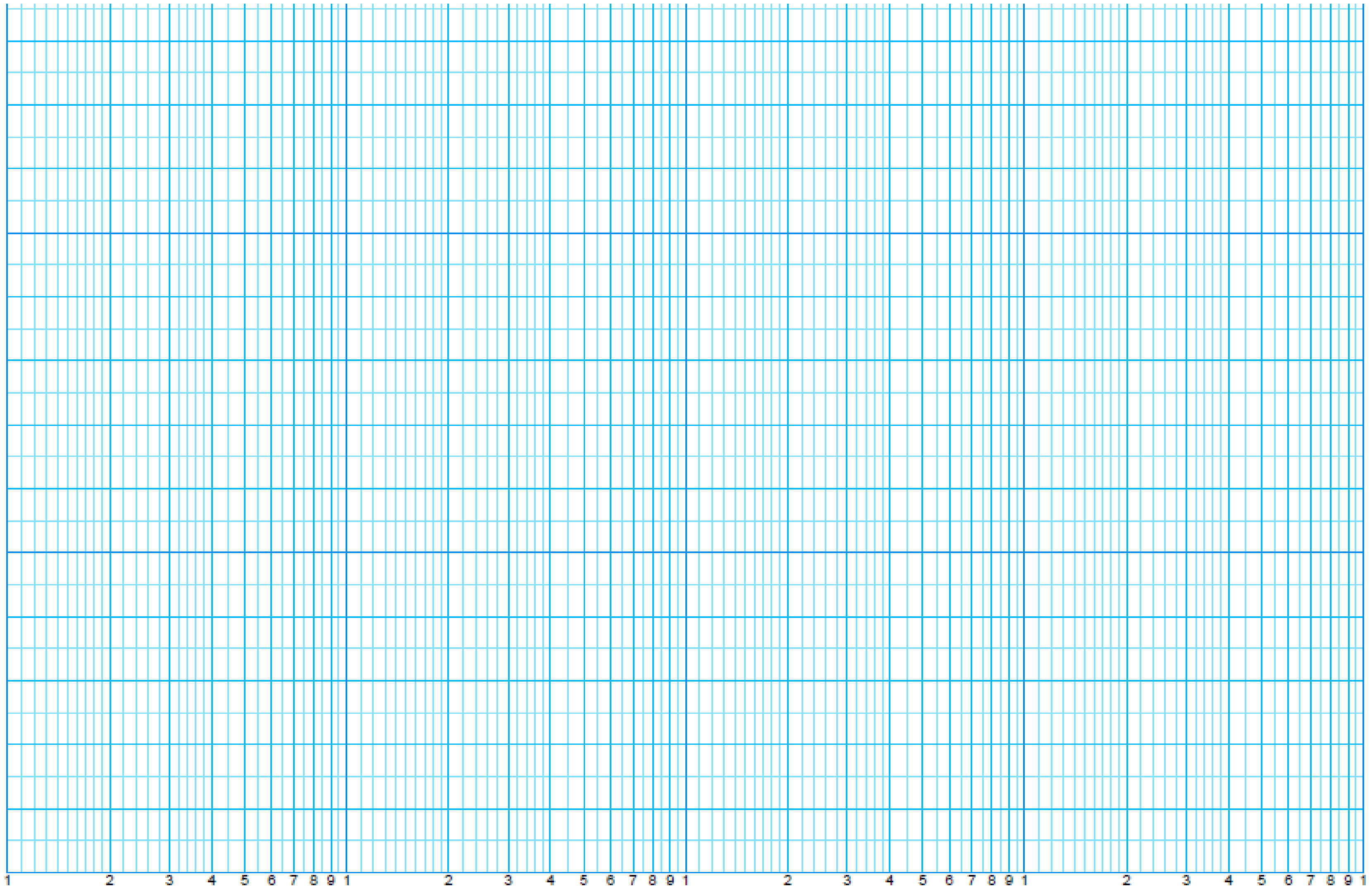
  of G(jω)H(jω)
- M and φ are plotted against log ω to give the magnitude and phase plots:

  Bode plot
- Plotted on a Semilog graph sheet

  Frequency axis is logarithmic and magnitude axis is linear

FREQUENCY RESPONSE

Semi Log Sheet



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Bode Plot

- **Since it is a logarithmic plot, the product of terms are converted to additions**
- **Complicated functions can be visualized in terms of individual standard plots**
- **Since log scale is used, it is helpful in covering a large range of magnitude and phase**

$$G(s)H(s) = \frac{k(s+z_1)(s+z_2)\ldots.(s+z_m)}{(s+p_1)(s+p_2)\ldots.(s+p_n)}$$

$$G(j\omega) = \frac{K' \prod_{i=1}^{m} (1 + T_{z_i} j\omega)}{\prod_{j=1}^{n} (1 + T_{p_j} j\omega)}$$

$$|G(j\omega)| = \frac{K' \prod_{i=1}^{m} \sqrt{1 + T_{z_i}^2 \omega^2}}{\prod_{j=1}^{n} \sqrt{1 + T_{p_j}^2 \omega^2}}$$

$$\angle G(j\omega) = \sum_{i=1}^{m} \tan^{-1} \omega T_{z_i} - \sum_{j=1}^{n} \tan^{-1} \omega T_{p_j}$$

$$G(s) = \frac{K \prod_{i=1}^{m} (s + z_i)}{\prod_{j=1}^{n} (s + p_j)} \rightarrow \text{Pole - zero form}$$

$$= \frac{K' \prod_{i=1}^{m} (1 + T_{z_i} s)}{\prod_{j=1}^{n} (1 + T_{p_j} s)} \rightarrow \text{Time constant form}$$

$$20 \log_{10} |G(j\omega)| = 20 \log_{10} K' + \sum_{i=1}^{m} 20 \log_{10} \sqrt{1 + T_{z_i}^2 \omega^2} - \sum_{j=1}^{n} 20 \log_{10} \sqrt{1 + T_{p_j}^2 \omega^2}$$

FREQUENCY RESPONSE

Magnitude and Phase Response

- Magnitude in dB:

$$20\log|G(j\omega)H(j\omega)| = 20\log k + 20\log\sqrt{1+\omega^2 T_1{}^2} + 20\log\sqrt{1+\omega^2 T_2{}^2} --- 20\log\sqrt{1+\omega^2 T_m{}^2}$$
$$-20\log\sqrt{1+\omega^2 T_a{}^2} + 20\log\sqrt{1+\omega^2 T_b{}^2} ------ -20\log\sqrt{1+\omega^2 T_n{}^2}$$

- Phase:

$$\varphi(j\omega) = \tan^{-1}(\omega T_1) + \tan^{-1}(\omega T_2) + \ldots\ldots\ldots\ldots + \tan^{-1}(\omega T_m)$$
$$-\tan^{-1}(\omega T_a) - \tan^{-1}(\omega T_b) - \ldots\ldots\ldots - \tan^{-1}(\omega T_n)$$

BODE PLOT

Bode Magnitude Plots

$$G(s) = \frac{K(1+\tau s)}{s(1+\tau s)}$$

Simple function :

K , s^N , $\frac{1}{s^N}$, $(1+\tau s)$, $\frac{1}{1+\tau s}$

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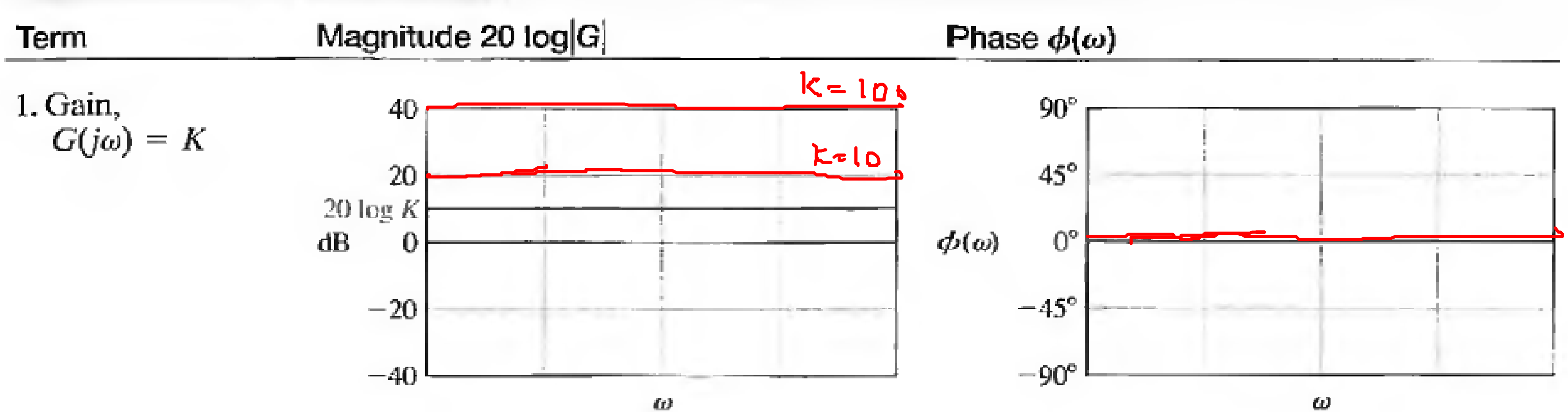
Bode Plot for Individual Singularities

- **Constant k:** Magnitude plot is a constant with magnitude 20logk dB

$|K| = 20 \log_{10} |K|$, $\angle K = 0^\circ$

$K = 10$, $M_{dB} = 20$ dB , $K = 100$, $M_{dB} = 40$ dB



Bode Plot for Individual Singularities

- **Simple zero at origin: $G(s) = s$**

$G(j\omega) = j\omega \Rightarrow |G(j\omega)| = \omega$

$20 \log_{10} |G(j\omega)| = 20 \log_{10} \omega$

  - Straight line with slope 20 dB/dec passing through 0dB line at ω =1

$$\boxed{20 \log|j\omega| = +20 \log \omega,}$$

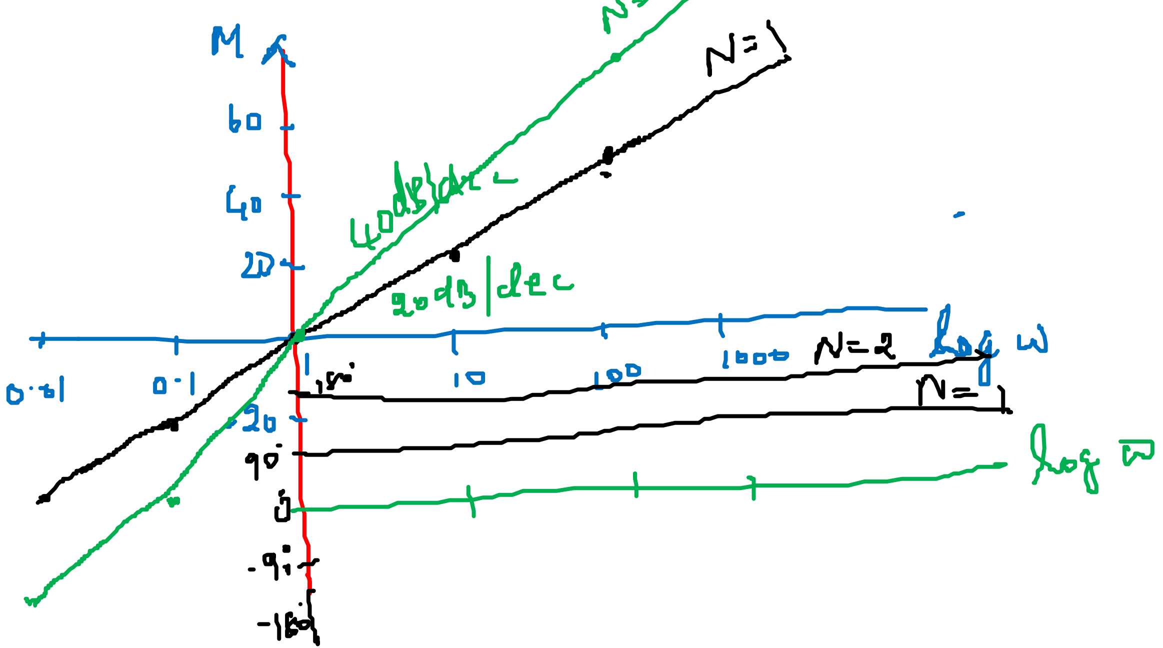
where the slope is +20 dB/decade and the phase angle is

$$\boxed{\phi(\omega) = +90°.}$$

$\omega = 1, \quad M = 0\text{ dB}$

$\omega = 10, \quad M = 20\text{ dB}$

$\omega = 100, \quad M = 40\text{ dB}$



Bode Plot for Individual Singularities

- **Multiple zeros at origin (N zeros):** $G(s) = s^N$, $G(j\omega) = (j\omega)^N$
  - $|G(j\omega)| = \omega^N$, $M = 20N \log_{10} \omega$
  - Straight line with slope 20N dB/dec passing through 0dB line at ω =1 $= 40 \log_{10} \omega$

$\angle G(j\omega) = N\pi/2$

$N = 2$,

ω	M (dB)	ϕ
0.01	-80	π
0.1	-40	π
1	0	π
10	40	π

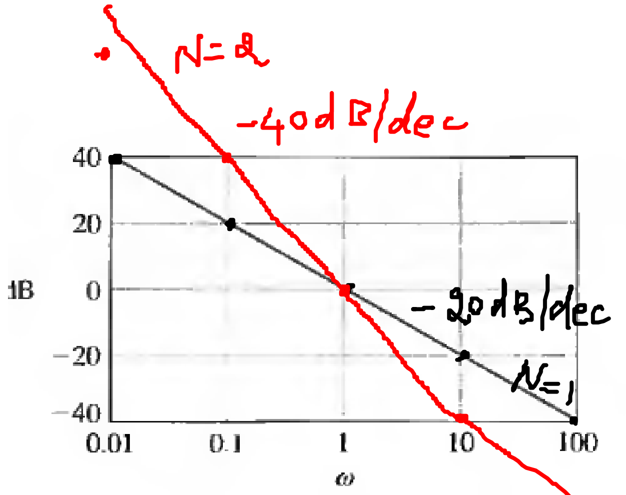
FREQUENCY RESPONSE

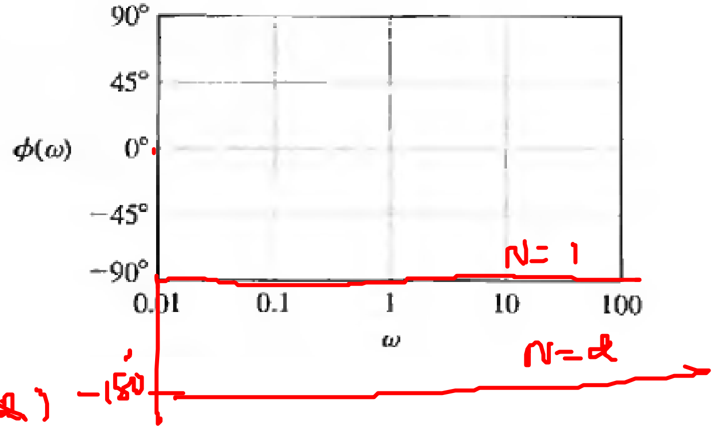
Bode Plot for Individual Singularities

- **Simple pole at origin:** $G(s) = \frac{1}{s}$, $G(j\omega) = \frac{1}{j\omega}$, $|G(j\omega)| = \frac{1}{\omega}$
  $M = -20\log_{10}\omega$
  - **Straight line with slope -20 dB/dec passing through 0dB line at ω =1** $\phi = 0 - \tan^{-1}\left(\frac{\omega}{0}\right)$ $= -\pi/2$

$$20\log\left|\frac{1}{j\omega}\right| = -20\log\omega \text{ dB}$$

$$\phi(\omega) = -90^\circ.$$





$G(s) = \frac{1}{s^N}$ $\Rightarrow$ $M = -20N\log_{10}\omega$ $\Rightarrow -40\log_{10}\omega$ $(N=2)$

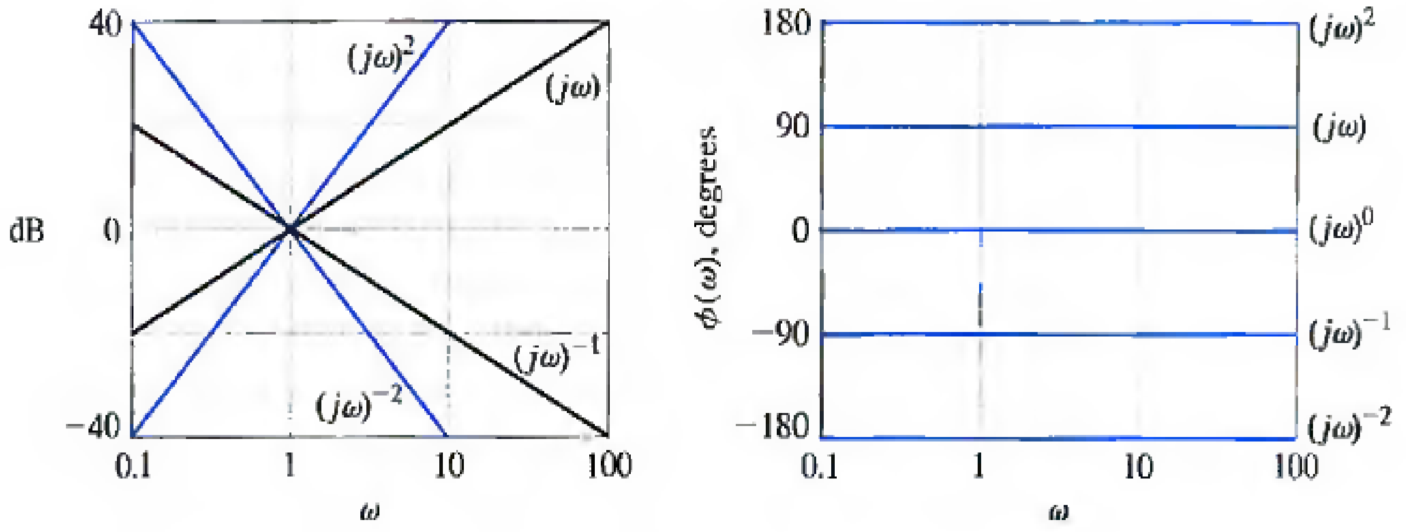
$\phi = -N\pi/2$

Bode Plot for Individual Singularities

- **Multiple poles at origin (N Poles):**
  - **Straight line with slope -20N dB/dec passing through 0dB line at ω =1**

$$20\log\left|\frac{1}{(j\omega)^N}\right| = -20N\log\omega,$$

$$\phi(\omega) = -90°N.$$



Bode Plot for Individual Singularities

(s+1) (s+3)

- Simple/multiple zero (zero on real axis)
  - Corner Frequency: Value of ω where the slope changes from 0 dB to 20dB/dec
  - Magnitude is 0 dB till corner/critical/break frequency and straight line with slope of 20 dB/dec or 20N dB/dec beyond corner frequency
  - Hence approximate (asymptotic) plot has 2 lines (0dB and 20dB/dec line) meeting at corner frequency $\omega=1/T$
  - Determine the true/actual plot

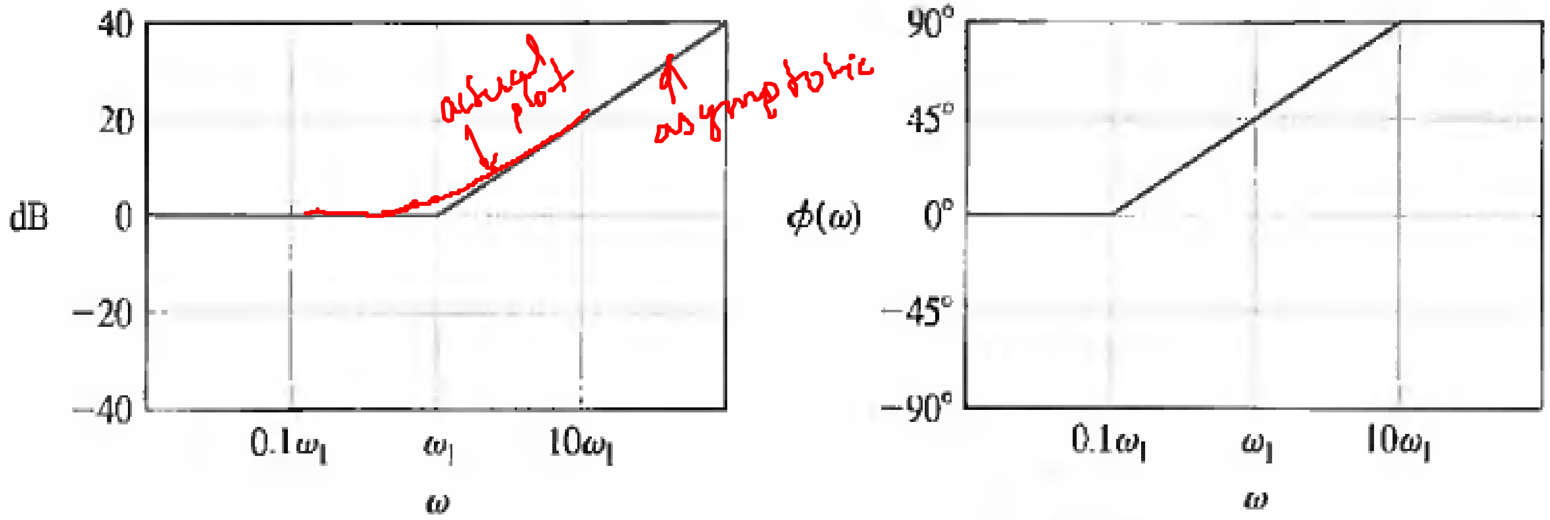
Bode Plot for Individual Singularities

Zero on the real axis: $(1 + \tau s)$

$G(j\omega) = 1 + j\omega\tau$

$|G(j\omega)| = \sqrt{1 + \omega^2\tau^2}$, $\angle G(j\omega) = \tan^{-1}\omega\tau$

$$20 \log |G(j\omega)| = 10 \log_{10}(1+\omega^2\tau^2) = \begin{cases} 0\ \text{dB}, & \omega\tau \ll 1 \\ 3\ \text{dB}, & \omega\tau = 1 \\ 20 \log_{10} \omega\tau, & \omega\tau \gg 1 \end{cases}$$



$(s + z_i)$

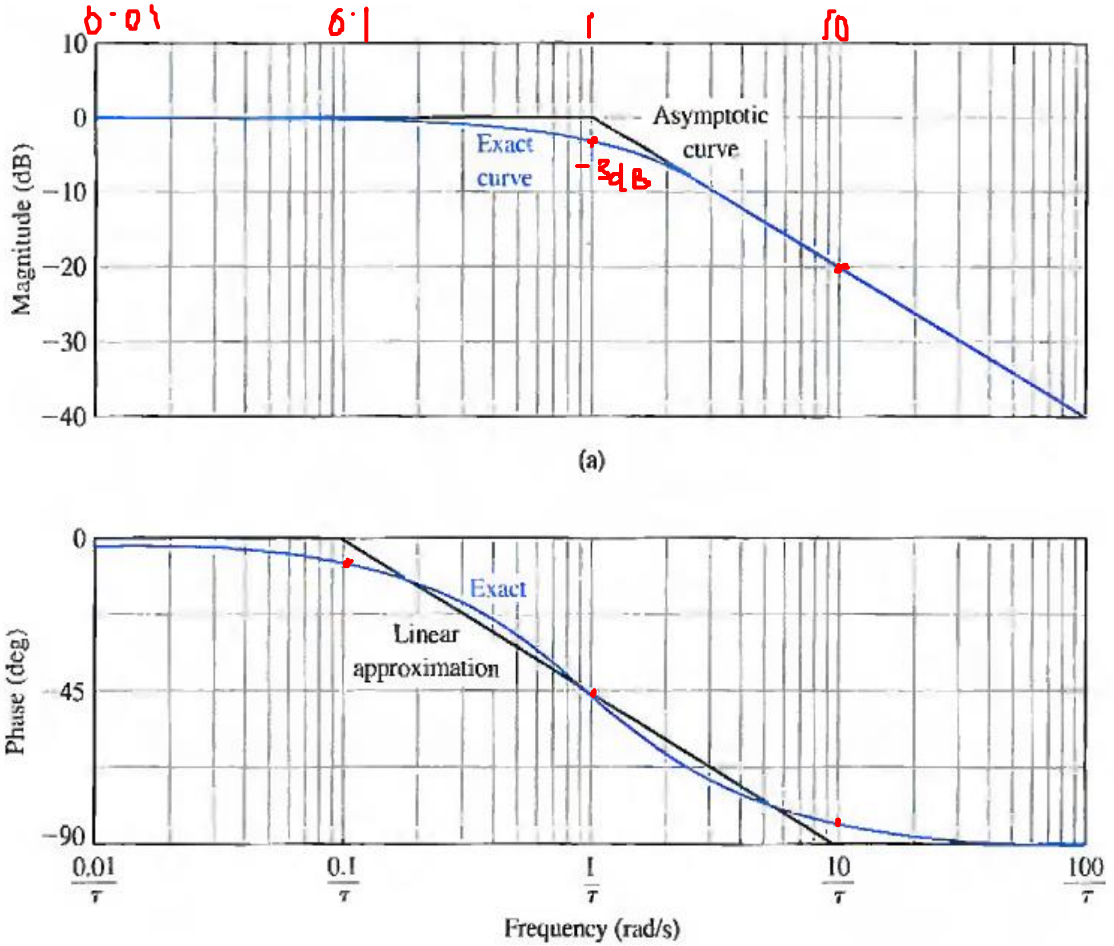
$= z_i\left(\frac{1}{z_i}s + 1\right)$

$\frac{1}{z_i} = \frac{1}{\tau} \Rightarrow$ corner freq.

FREQUENCY RESPONSE

Bode Plot for Individual Singularities

Pole on the real axis: $\frac{1}{(1+\tau s)}$



$$G(j\omega) = \frac{1}{1+j\omega\tau}$$

$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

$$|G(j\omega)|_{dB} = -10\log_{10}(1+\omega^2\tau^2)$$

$$= \begin{cases} 0\text{ dB} & \omega\tau \ll 1 \\ -3\text{ dB} & \omega\tau = 1 \\ -20\log_{10}(\omega\tau) & \omega\tau \gg 1 \end{cases}$$

$$\angle G(j\omega) = -\tan^{-1}\omega\tau$$

$$= \begin{cases} 0^\circ, & \omega\tau \ll 1 \\ -5^\circ, & \omega\tau = 0.1 \\ -45^\circ, & \omega\tau = 1 \\ -85^\circ, & \omega\tau = 10 \\ -90^\circ, & \omega\tau \gg 1 \end{cases}$$

Bode Plot for Individual Singularities

Complex conjugate poles or zeros: $\frac{1}{(s^2+2\zeta\omega_n s+\omega_n^2)} \rightarrow G(s) \Rightarrow$

$G(j\omega) = \frac{1}{-\omega^2 + 2\zeta\omega_n j\omega + \omega_n^2}$

$= \frac{1}{\omega_n^2\left(-\frac{\omega^2}{\omega_n^2} + 2j\zeta\frac{\omega}{\omega_n} + 1\right)}$

$= \frac{1}{\omega_n^2\left(-u^2 + 2j\zeta u + 1\right)}$

$= \frac{1}{(1-u^2) + j2\zeta u}$

Normalized form: $G(j\omega) = \frac{1}{(1-u^2)+j2\zeta u}$, $where\ u = \frac{\omega}{\omega_n}$

$|G(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2 + 4\zeta^2 u^2}} \rightarrow$ normalized form

$$\boxed{20\log|G(j\omega)| = -10\log((1-u^2)^2 + 4\zeta^2u^2),}$$

$$\boxed{\phi(\omega) = -\tan^{-1}\frac{2\zeta u}{1-u^2}.}$$

When $u \ll 1$, the magnitude is

$$20\log|G(j\omega)| = -10\log 1 = 0\ \text{dB},$$

When $u \gg 1$,

$$20\log|G(j\omega)| = -10\log u^4 = -40\log u,$$

FREQUENCY RESPONSE

Bode Plot for Individual Singularities

Complex conjugate poles or zeros: $\frac{1}{(s^2+2\zeta\omega_n s+\omega_n^2)}$

Normalized form: $G(j\omega) = \frac{1}{(1-u^2)+j2\zeta u}$, $where\ u = \frac{\omega}{\omega_n}$

$$\boxed{20 \log|G(j\omega)| = -10 \log((1 - u^2)^2 + 4\zeta^2 u^2),}$$

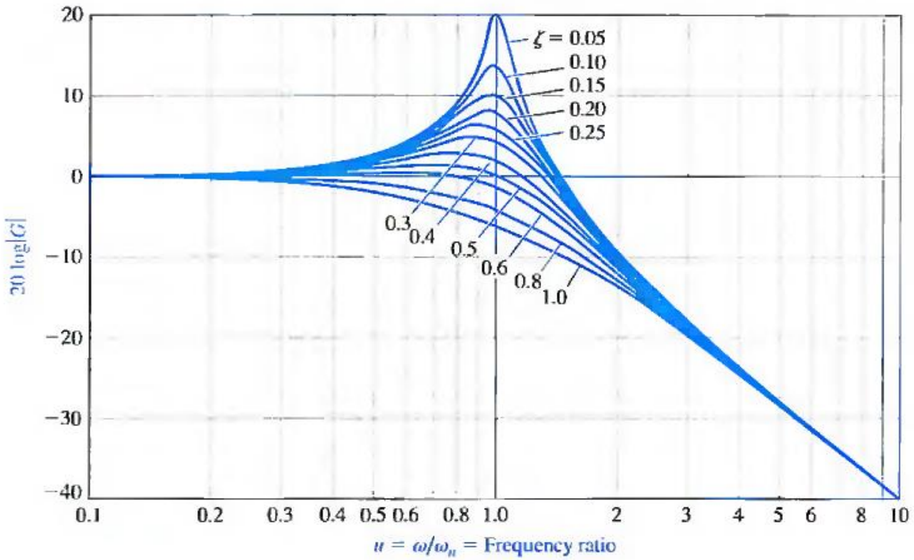
$$\boxed{\phi(\omega) = -\tan^{-1}\frac{2\zeta u}{1 - u^2}.}$$

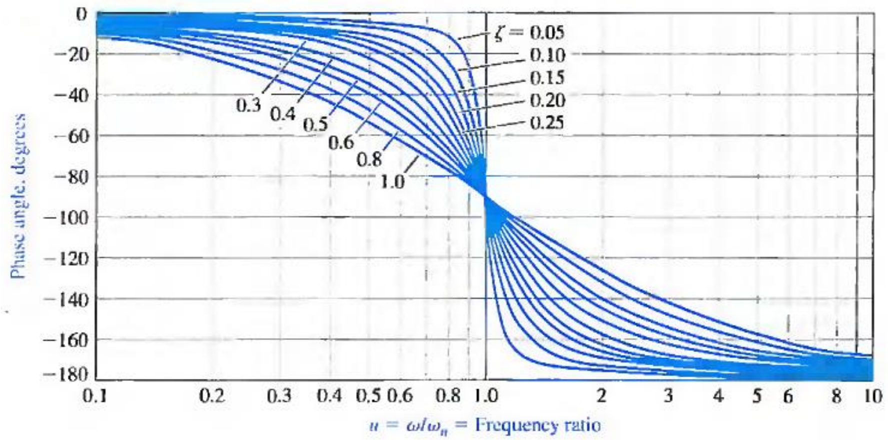
When $u \ll 1$, the magnitude is

$$20 \log|G(j\omega)| = -10 \log 1 = 0 \text{ dB},$$

When $u \gg 1$,

$$20 \log|G(j\omega)| = -10 \log u^4 = -40 \log u,$$





C.E $\Rightarrow$ $1+GH=0$
$GH=-1$
$|GH|=1$ $\rightarrow$
$20\log|GH|=0\,dB$

➢ We know that GH= -1; Gain= 0 dB, Phase=-180^0

- Gain Crossover Frequency (GCF)
  - There must be some frequency at which the Magnitude of G(jw)H(jw) =1
  - Frequency at which gain is 0dB or the magnitude plot crosses 0dB line
- Phase crossover frequency(PCF)

Phase $=-180^\circ$

  - There must be some frequency at which the phase of G(jw)H(jw) = -180
  - Frequency where angle is -180^0 or the phase plot crosses - 180^0 line.

Bode Plot for Individual Singularities

- **Gain Margin:** Gain measured w.r.t 0 dB axis at PCF
  - Difference between zero and gain at PCF
- **Phase Margin:** Phase measured w.r.t -180^0 at GCF
  - Difference between phase and -180^0 at GCF
- **Stability:** GM and PM should be positive (GCF < PCF)
- **Marginal stability:** both GM and PM are zero or GM = PM (GCF = PCF)
- **Unstable:** GM and PM are negative (GCF >PCF)

GM = 0 - ( )

PM = -180° +

Gain Margin and Phase Margin

- **GM:**

is +ve: if the intersection of line extended form PCF intersects the Magnitude plot below 0dB line

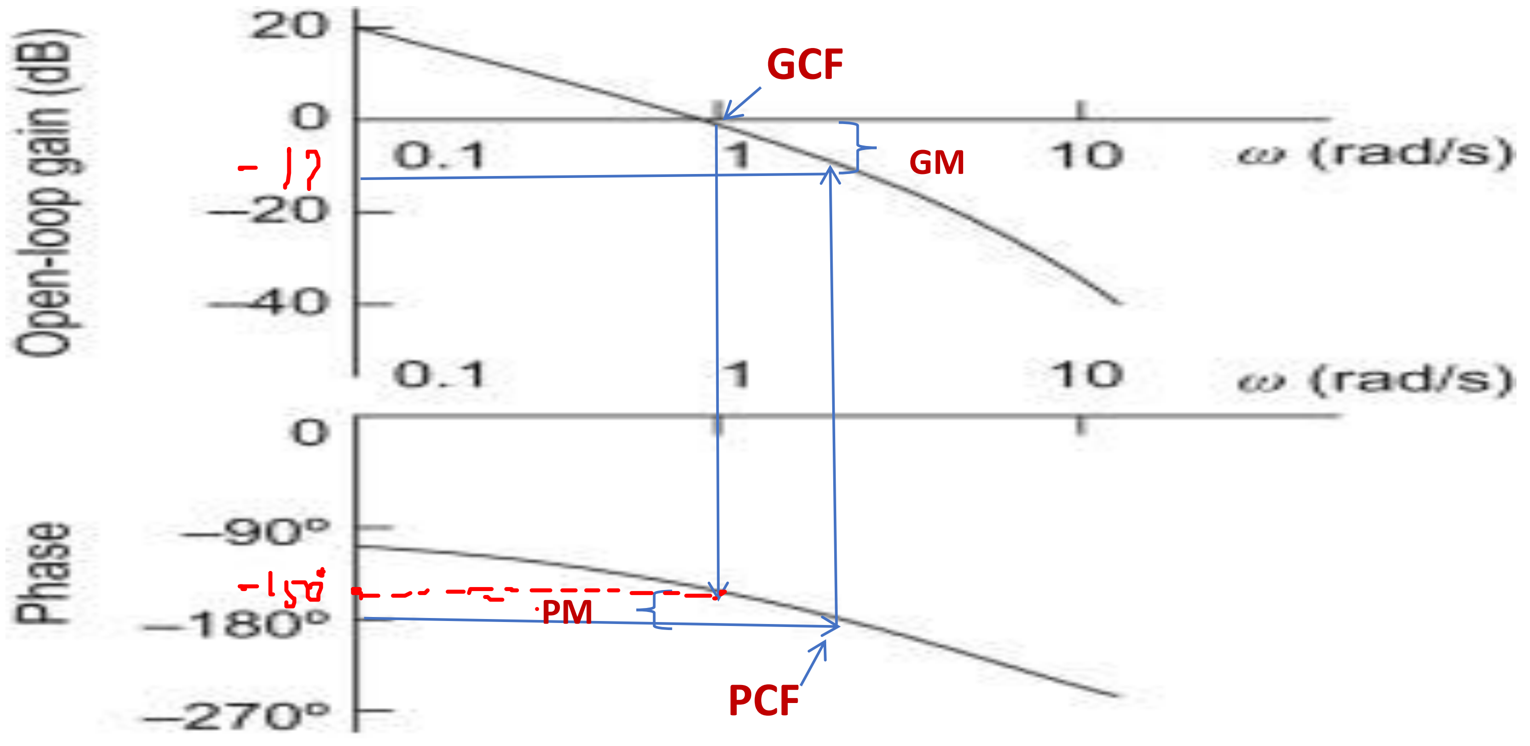
is –ve: if the line intersects above the 0dB line

- **PM:**

is +ve: if the intersection of line extended form GCF intersects the phase plot above - 180^0 line

is –ve: if the line intersects, above the - 180^0 line

- **Example**
- For the Bode plot shown in Figure, determine
- (a) whether the system is stable, (b) Justify the answer



$GM = 0 - (\text{Gain}|_{PCF})$

$GM = 0 - (-17)$

$= 17\ dB$

$PM = 180 + \angle(\phi)$ $\quad \phi|_{GCF}$

$PM = 180^\circ - 150^\circ$

$= 30^\circ$

Since GM & PM +ve, s/m is stable

FREQUENCY RESPONSE

Gain Margin and Phase Margin

$$G(s) = \frac{200(s+10)}{s(s+5)(s+20)} = \frac{\cancel{200} \times \cancel{10}\left(\frac{1}{10}s+1\right)}{s \times \cancel{5}\left(\frac{1}{5}s+1\right) 20\left(\frac{1}{20}s+1\right)} = \frac{20(0.1s+1)}{s(0.2s+1)(0.05s+1)}$$

Factor	Corner frequency	Slope contributed by each factor	Net Slope	Freq. range
20/S	NIL	-20dB/dec	-20dB	Start ω to ω=5 rad/sec
1/(1+0.2s)	Wc = 1/T= 5	0dB till the corner frequency and -20dB/dec after corner frequency	-40 dB	ω=5 rad/sec to ω=10 rad/sec
(1+0.1s)	Wc = 1/T= 10	0dB till the corner frequency and +20dB/dec after corner frequency	-20 dB	ω=10 rad/sec to ω=20 rad/sec
1/(1+0.05s)	Wc = 1/T= 20	0dB till the corner frequency and -20dB/dec after corner frequency	-40 dB	ω=20 rad/sec onwards

$$G(j\omega) = \frac{20[0.1j\omega+1]}{j\omega(0.2j\omega+1)(0.05j\omega+1)}$$

$$\angle G(j\omega) = \tan^{-1}0.1\omega - 90^\circ - \tan^{-1}0.2\omega - \tan^{-1}0.05\omega$$

$$G(s) = \frac{200(s+10)}{s(s+5)(s+20)}$$

$$A = 20\log\left|20/j\omega\right| = 20\log 20/\omega = 20\log 20 - 20\log\omega = 26 - 20\log\omega.$$

$\omega = 0.1 \quad A = 26 + 20 = 46.$

$\vdots$

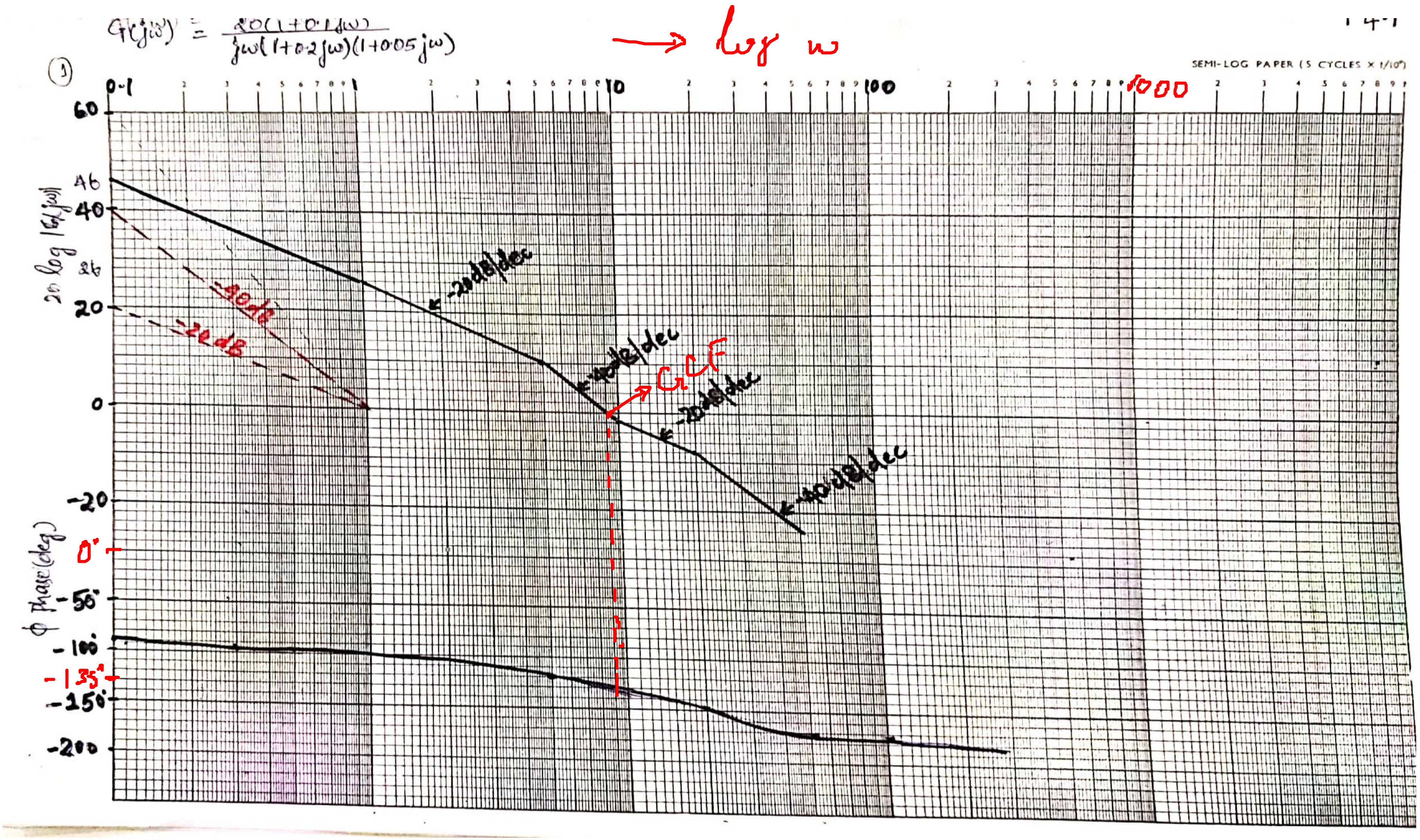
$\omega = 1 \quad A = 26 + 0 = 26.$ (Start from $\omega=1$ with 26 dB).

$$\angle G(j\omega) = -90^\circ - \tan^{-1}(0.2\omega/1) + \tan^{-1}(0.1\omega/1) - \tan^{-1}(0.05\omega/1)$$

ω	0	0.1	0.2	0.4	0.5	0.8	1	2	5	10	20	50	100	200	∞
$\angle G(j\omega)$	-90°	-90.86°	-91.71	-93.43	-94.28	-96.8	-98.4	-106.2	-122	-135	-147.5	-163.79	-171		-180

FREQUENCY RESPONSE

Gain Margin and Phase Margin



GCF = 9 rad/sec

PM = 180° + (-135°)

= 45°

no -180° crossing

GM = ∞

PM & GM are +ve

∴ system is stable

Second order closed loop system

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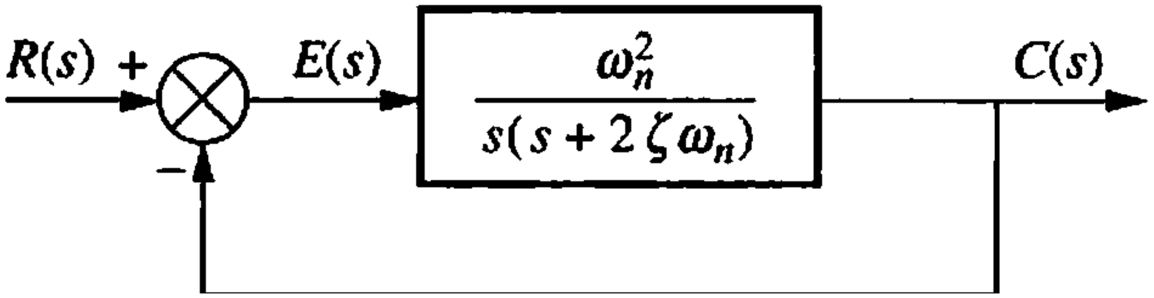
Consider second order closed loop system:



Closed loop transfer function

$$\frac{C(s)}{R(s)} = T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Magnitude sub $s = j\omega$, $T(j\omega) =$

$$M = |T(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}$$

Second order closed loop system

Magnitude:

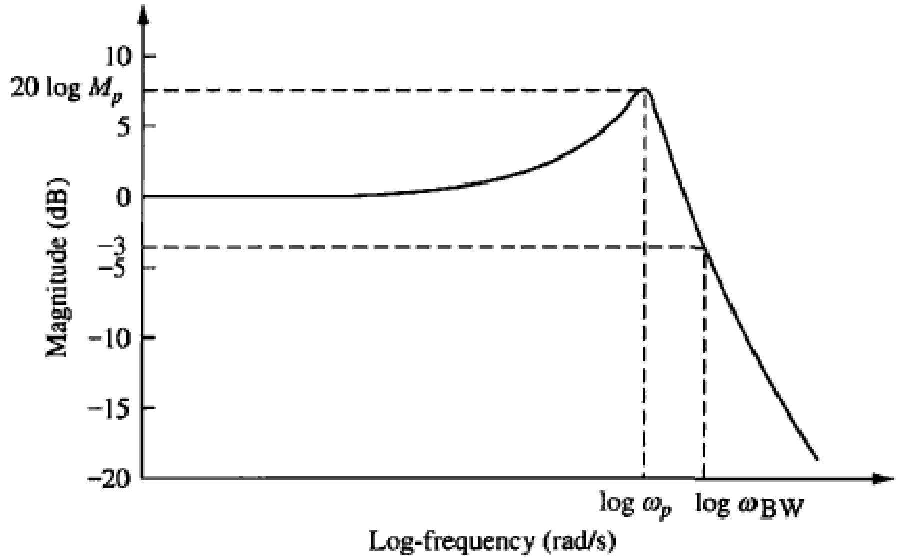
$$M = |T(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}$$

Diff. M w.r.to ω^2 and equating to zero yields maximum value M_p

$$\boxed{M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}}$$

Where the frequency is

$$\boxed{\omega_p = \omega_n\sqrt{1-2\zeta^2}}$$



FREQUENCY RESPONSE

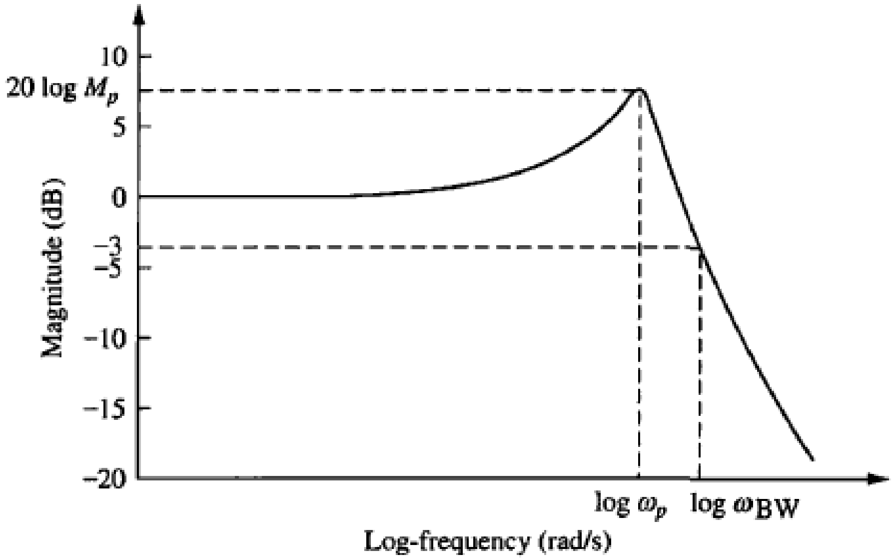
Second order closed loop system

Cutoff frequency:

The frequency at which the magnitude of the closed loop frequency response is -3dB below its zero frequency is called cutoff frequency

Bandwidth:

Bandwidth is the frequency at which the frequency has declined 3dB from its low frequency value.



$$M = \frac{1}{\sqrt{2}}$$

$$|G(j\omega)| = \left|\frac{1}{(1-u^2)+j2\zeta u}\right| = \frac{1}{\sqrt{2}}$$

Solving for u

$$u = \frac{\omega_B}{\omega_n} \Rightarrow \omega_B = \omega_n\sqrt{[(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}]}$$

Example 2

$$G(s) = \frac{K(1+0.2s)(1+0.025s)}{s^3(1+0.001s)(1+0.005s)} \Rightarrow G(j\omega) = \frac{K(1+0.2j\omega)(1+0.025j\omega)}{(j\omega)^3(1+0.001j\omega)(1+0.005j\omega)}$$

$1/0.2 = 5$ $\quad \frac{1}{0.025} = 40$ $\quad \frac{1}{0.001} = 1000$ $\quad 1/0.005 = 200$

Let $K = 1$

factor	C.F	slope (dB)	Net slope (dB)	freq range
$1/s^3$	—	-60	-60	0.1 to 5
$1+0.2j\omega$	$\frac{1}{0.2} = 5$	20	-40	5 to 40
$1+0.025j\omega$	$40 = \frac{1}{0.025}$	20	-20	40 to 200
$1/(1+0.005j\omega)$	200	-20	-40	200 to 1000
$1/(1+0.001j\omega)$	1000	-20	-60	1000 to ∞

$$A = 20\log\left|\frac{1}{(j\omega)^3}\right| = 20\log\frac{1}{\omega^3}$$

$$= 20\log_{10}1 - 3\times 20\log\omega$$

$$= -60\log_{10}\omega$$

$\omega = 0.1$

$A = 60\,dB$

$$\phi = -270 + \tan^{-1}0.2\omega + \tan^{-1}0.025\omega - \tan^{-1}0.005\omega - \tan^{-1}0.001\omega$$

ω	0.1	0.8	1	3	10	15	30	60	100	300	400	∞
$\angle G(j\omega)$	-268	-260	-257	-236	-198	-183	-163	-148	-147	-172	-182	-270

$20\log K = 60$

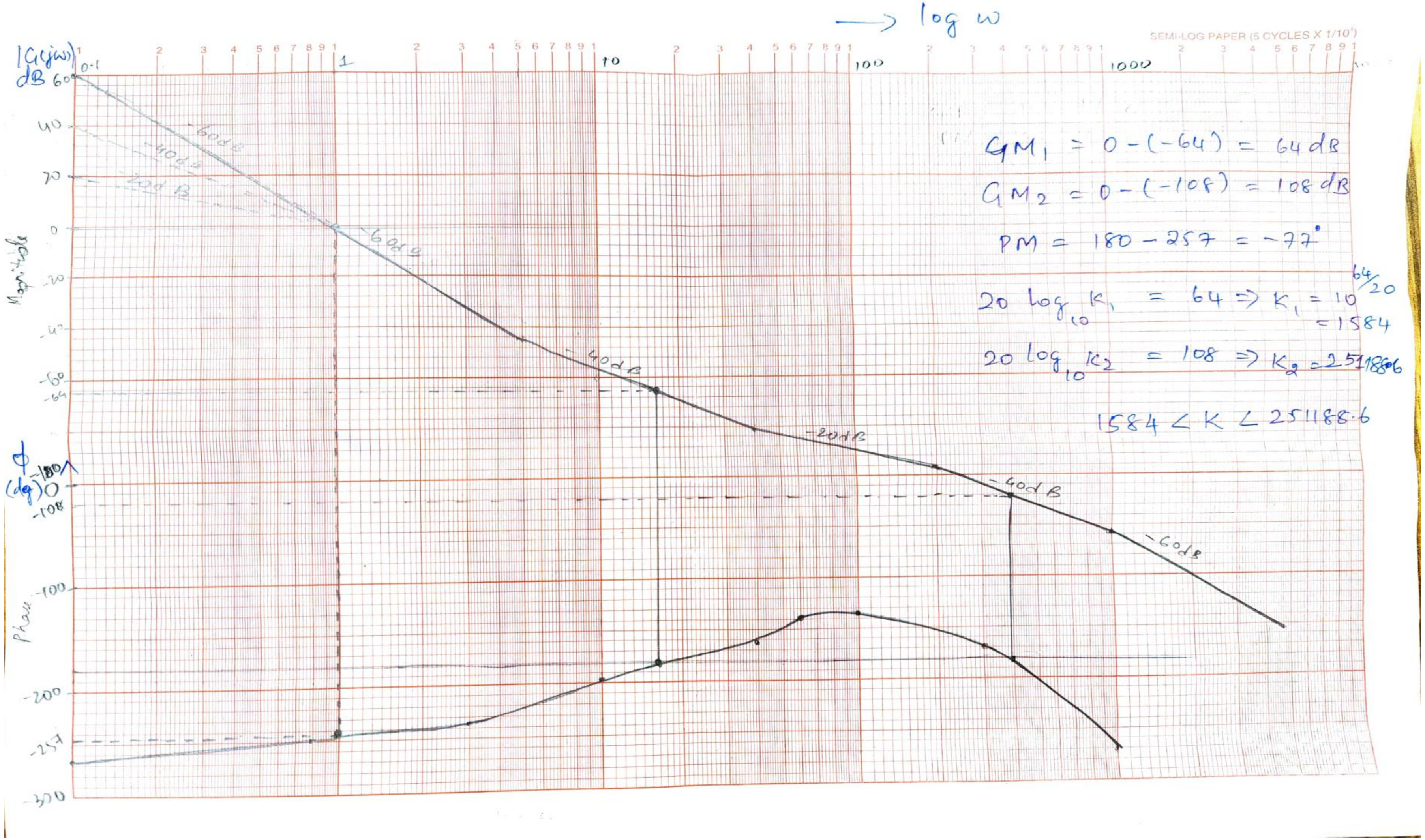
$20\log K = 104$

$1122 < K < 167880$

Stability

FREQUENCY RESPONSE

Example 2



FREQUENCY RESPONSE

Example 3

$$G(s) = \frac{1}{(s+3)^3} \qquad G(s) = \frac{1}{27(s+3)^3} = \frac{0.037}{(0.33s+1)^3}$$

$$G(j\omega) = G(s)\Big|_{s=j\omega} = \frac{0.037}{(1+0.33j\omega)^3}$$

$$20\log(0.037) = -28.63$$

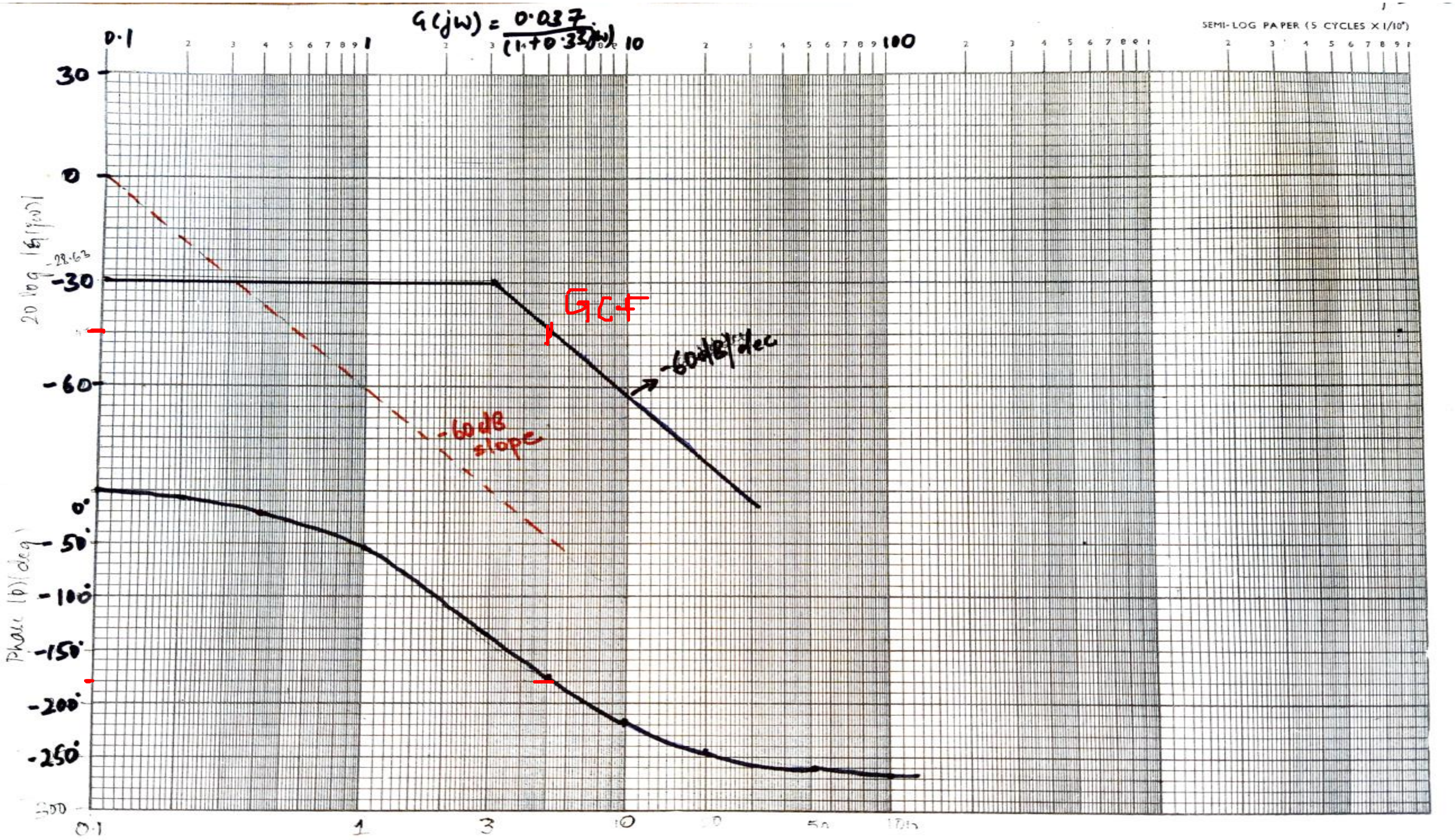
Factor	corner freq	Contributed slope	Net slope	Freq range
0.037	none	0	0	0.1 to 3
$1/(0.33j\omega+1)^3$	$1/0.33 = 3$	$-20\times 3 = -60$	-60	3 to ∞

$$\angle G(j\omega) = -3\tan^{-1}(0.33\omega)$$

ω	0.1	0.3	0.4	1	3	5	10	20	30	50	100	∞
$\angle G(j\omega)$	-5.7	-5.65	-22	-55	-44.7	-177	-220	-245	-252.69	-260	-265	-270

FREQUENCY RESPONSE

Example 3



GM = 0 - (-45)

= 45

PM = ∞

∴ system is stable

FREQUENCY RESPONSE

Example 4

$$G(s) = \frac{5(1+0.1s)}{s(1+0.5s)\left(1+\frac{0.6}{50}s+\left(\frac{s}{50}\right)^2\right)}$$

$\frac{1}{0.1} = 10$

$\frac{1}{0.5} = 2$

$\omega_n = 50$

$$G(j\omega) = \frac{5(1+j0.1\omega)}{j\omega(1+j0.5\omega)(1+j0.6(\omega/50)+(j\omega/50)^2)}$$

$A = 20 \log(5/\omega) = 20 \log 5 - 20 \log \omega = 14 - 20 \log \omega$.

$A = 20 \log \left|\frac{5}{j\omega}\right| = 20 \log \frac{5}{\omega}$

At $\omega = 0.1$, $A = 14 - 20 \log_{10} 0.1$

$= 34$ dB

Factor	Corner freq	Slope contributed by each factor	Net slope	Freq range
$5/j\omega$	none	-20 dB/dec	-20	0.1 to 2
$1/(1+0.5j\omega)$	2	-20 dB/dec	-40	2 to 10
$1+j0.1\omega$	10	20 dB/dec	-20	10 to 50
$1/\left(1+j0.6\left(\frac{\omega}{50}\right)+\left(\frac{j\omega}{50}\right)^2\right)$	50	-40 dB/dec	-60	50 to ∞

ω	0.1	0.4	0.8	1	2	5	10	20	50	100	∞
$\angle G(j\omega)$	-92	-99	-108	-112	-125	-135	-131	-126	-189		

$$\angle G(j\omega) = -90 + \tan^{-1}(0.1\omega) - \tan^{-1}(0.5\omega) - \tan^{-1}\left(\frac{0.6\omega/50}{1-\omega^2/50^2}\right)$$

$\frac{50 \times 0.6\omega}{2500 - \omega^2}$

$\tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right)$

$-\left(\pi - \tan^{-1}\left(\frac{0.6\omega/50}{\omega^2/50^2 - 1}\right)\right) \quad \forall\ \omega > 50^2$

$2\zeta = 0.6 \Rightarrow \zeta = \frac{0.6}{2} = 0.3$

$u = \frac{\omega}{50}$.

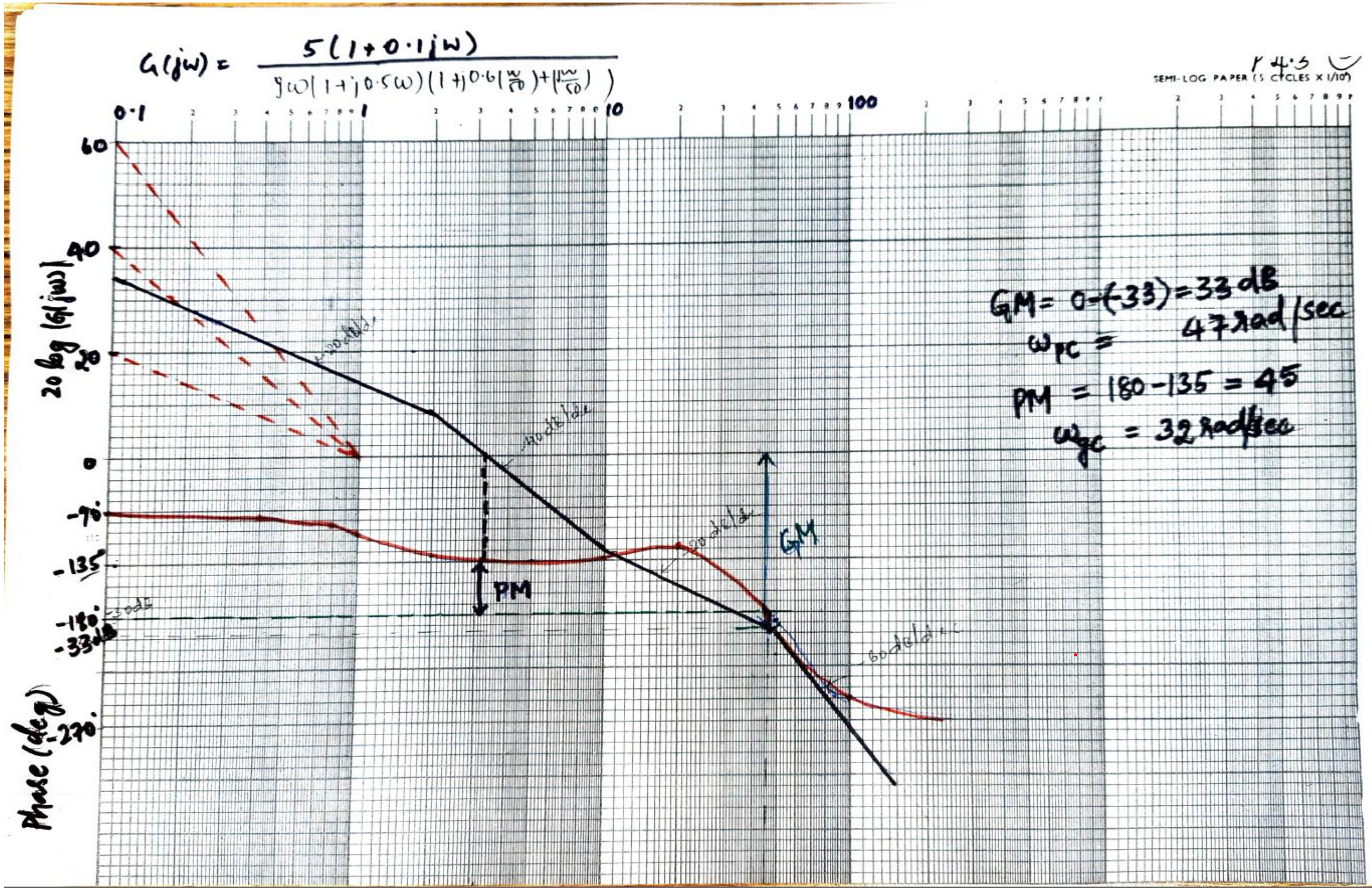
error $= -10 \log(4\zeta^2)$

$= -10 \log(4 \times 0.3^2)$

$= 4.4$ dB

FREQUENCY RESPONSE

Example 4



GM = 33 dB

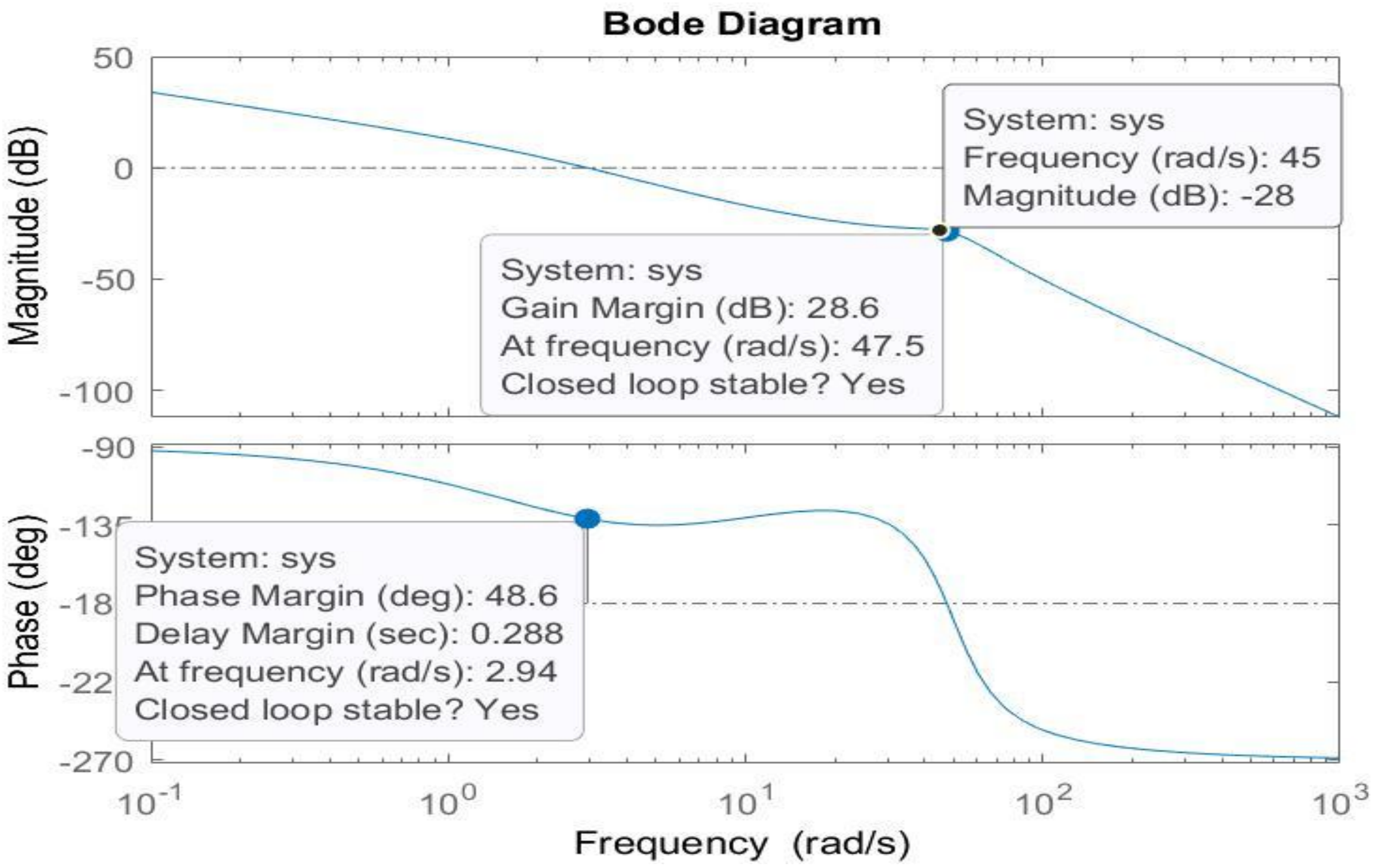
error

= 33 - 4.4

= 28.6 dB

FREQUENCY RESPONSE

Example 4



FREQUENCY RESPONSE

Example 5

$\omega_n^2 = 100 \Rightarrow \omega_n = 10$

$2\zeta\omega_n = 4 \Rightarrow \zeta = \frac{2}{10} = 0.2$

The open loop transfer function of an unity feedback system is given by $G(s) = \frac{10(s+3)}{s(s+2)(s^2+4s+100)}$. Draw the Bode plot and hence find the gain margin & phase margin.

$G(s) = \frac{0.15(0.33s+1)}{s(0.5s+1)(1+0.04s+0.01s^2)}$ $\qquad G(j\omega) = \frac{0.15(0.33j\omega+1)}{j\omega(0.5j\omega+1)(1+0.04j\omega+0.01(j\omega)^2)}$

$(1-0.01\omega^2)+0.04\omega j$

$\frac{0.4}{10}$ → 24

$G(s) = \frac{10(s+3)}{s(s+2)(s^2+4s+100)}$

$= \frac{10 \times 3\left(\frac{1}{3}s+1\right)}{s\,2\left(\frac{1}{2}s+1\right)100\left(\frac{1}{100}s^2+\frac{4}{100}s+1\right)}$

$= \frac{\frac{3}{20}(0.33s+1)}{s(0.5s+1)(0.01s^2+0.04s+1)}$

factor	corner freq	slope contributed	Net slope	Freq range
$\frac{0.15}{j\omega}$	none	-20	-20	0.1 to 2
$1/(0.5j\omega+1)$	2	-20	-40	2 to 3
$0.33j\omega+1$	3	20	-20	3 to 10
$\frac{1}{1+0.04j\omega+\left(\frac{j\omega}{10}\right)^2}$	10	-40	-60	10 to ∞

For quadratic factor
Total error at corner freq

$-10\log(4\zeta^2) = -10\log(4\times0.2^2)$

$= 7.95 \approx 8$

$\therefore$ GM $= 40 - 8 = 32$ dB

ω	0.1	1	5	10	20	100
$\angle G(j\omega)$	-91.2	-100	-114	-186	-258	-270

$-\left(\pi - \tan^{-1}\frac{0.04\omega}{0.01\omega^2-1}\right)$ $\quad \omega^2 > 100$

$\angle G(j\omega) = -90 - \tan^{-1}(0.5\omega) + \tan^{-1}(0.33\omega) - \tan^{-1}\left(\frac{0.04\omega}{1-0.01\omega^2}\right)$

$-0.01\omega^2+1 < 0$

$0.01\omega^2 > 1$

$\omega^2 > 1/0.01$

$\omega^2 > 100$

$A = 20\log\left(\frac{0.15}{\omega}\right) = 20\log 0.15 - 20\log 0.1 = 3.52$

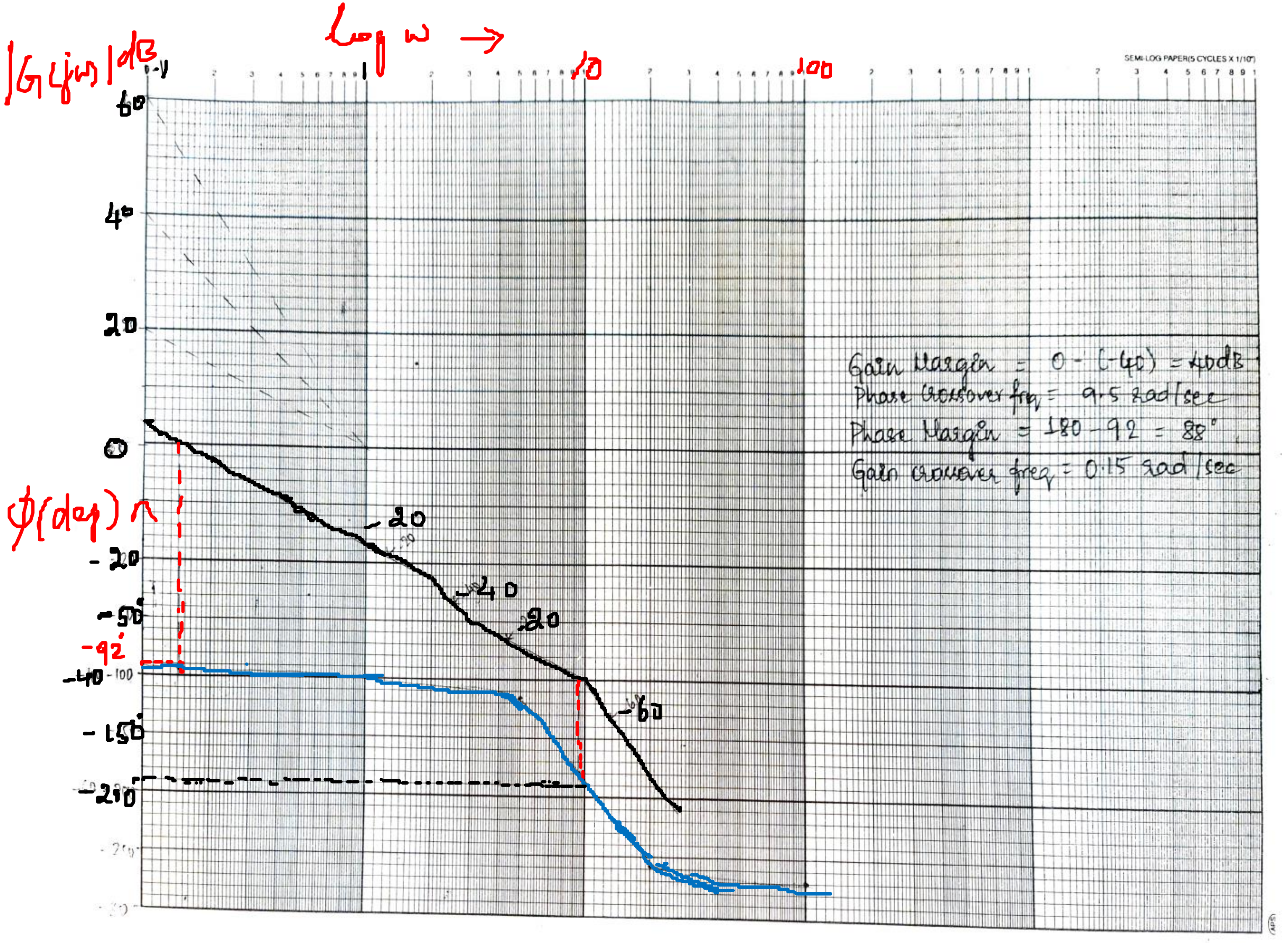
$20\log 0.15 - 20\log 1 = -16.478$

$A = 20\log_{10}\left|\frac{0.15}{j\omega}\right| = 20\log_{10} 0.15 - 20\log_{10}\omega$

at $\omega = 0.1$, $A = 3.52$ dB

FREQUENCY RESPONSE

Example 5



GCF = 0.18 rad/sec

PCF = 9 rad/sec

GM = 0 - (-40) = 40 dB

error = 8 dB

GM = 40 - 8 = 32 dB

PM = 180 - 92° = 88°

Example 6 – H.W

Draw the bode plot for the system $G(s) = \frac{10(s+1)}{(s+2)(s+5)}$

Example 7 – Manual calculations

Find the PCF and GM for the system with $G(s) = \dfrac{10}{s(1+0.5s)(1+0.1s)}$

Sol. To find PCF, (ω_{pc})

$$\angle G(j\omega)\Big|_{\omega=\omega_{pc}} = -180^\circ$$

$$-90^\circ - \tan^{-1} 0.5\omega - \tan^{-1} 0.1\omega = -180^\circ$$

$$-\tan^{-1} 0.5\omega_{pc} - \tan^{-1} 0.1\omega_{pc} = -90^\circ$$

$$\tan^{-1} \overset{A}{0.5\omega_{pc}} + \tan^{-1} \overset{B}{0.1\omega_{pc}} = 90^\circ$$

$$\tan^{-1}\left(\frac{0.5\omega_{pc} + 0.1\omega_{pc}}{1-(0.5\omega_{pc} \times 0.1\omega_{pc})}\right) = 90^\circ$$

$$G(j\omega) = \frac{10}{j\omega(1+0.5j\omega)(1+0.1j\omega)}$$

$$\frac{0.6\omega_{pc}}{1-0.05\omega_{pc}^2} = \infty = \frac{1}{0}$$

$$1 - 0.05\omega_{pc}^2 = 0$$

$$\omega_{pc} = 2\sqrt{5} = 4.47 \text{ rad/sec}$$

$$GM = \frac{1}{|G(j\omega)H(j\omega)|}\Bigg|_{\omega=\omega_{pc}}$$

GM in dB

$$M = -20\log_{10}|G(j\omega)H(j\omega)|_{\omega=\omega_{pc}}$$

FREQUENCY RESPONSE

Example 7 – Manual calculations

$$GM_{dB} = -20 \log_{10} \left(\frac{10}{\omega \sqrt{1+(0.5\omega)^2} \sqrt{1+(0.1\omega)^2}} \right) \Bigg|_{\omega = \omega_{pc}}$$

$$= 1.57 \text{ dB}$$

Example 8

Find the GCF, PCF, PM and GM for the system with $G(s) = \frac{10}{(s+20)}$

Sol:

$$G(s) = \frac{10/20}{(1 + s/20)} = \frac{0.5}{1 + s/20}$$

$$G(j\omega) = \frac{0.5}{1 + j\omega/20}$$

$$G.M = \frac{1}{M|_{\omega=\omega_{pc}}}, \quad PM = 180^\circ + \phi_{\omega=\omega_{gc}}$$

To find ω_{gc},

$$\left.\frac{0.5}{\sqrt{1+\frac{\omega^2}{400}}}\right|_{\omega=\omega_{gc}} = 1 \Rightarrow 0.5 = \sqrt{1+\frac{\omega_{gc}^2}{400}}$$

$\omega_{gc} = 17.32j \Rightarrow \because$ it is imaginary GCF does not exist

$\therefore$ PM is ∞

To find ω_{pc}, $\phi = -\tan^{-1}\omega/20 \,|_{\omega=\omega_{pc}} = -180^\circ$

$\omega_{pc} = 0$

$$GM = \frac{1}{M|_{\omega=\omega_{pc}}} = \frac{1}{0.5} = 2, \quad GM_{dB} = -20\log 2 = 6\,dB$$

BODE PLOT

Reverse Bode Analysis

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Department of Electronics and Communication Engineering

- Finding the transfer function for the given Bode plot
- Steps:
  1. Initial slope of magnitude plot represents poles or zeros at the origin, if slope is
     - -20dB => 1 pole at origin
     - -40dB => 2 poles at origin
     - 0dB => No pole
     - 20dB => 1 zero at origin
     - 40 dB => 2 zeros at origin

Step 2: Observe the shift in the magnitude plot at corner frequency which represents 20logK, from which K can be determined

$$20\log_{10}|G(j\omega)|_{at\ \omega=\omega_1} = 20\log_{10}(X)$$

Where X can be K or $\frac{K}{\omega}$ $or\ K\omega\ or\ \frac{K}{\omega^2}\ or\ K\omega^2$ and so on, depends on the initial slope

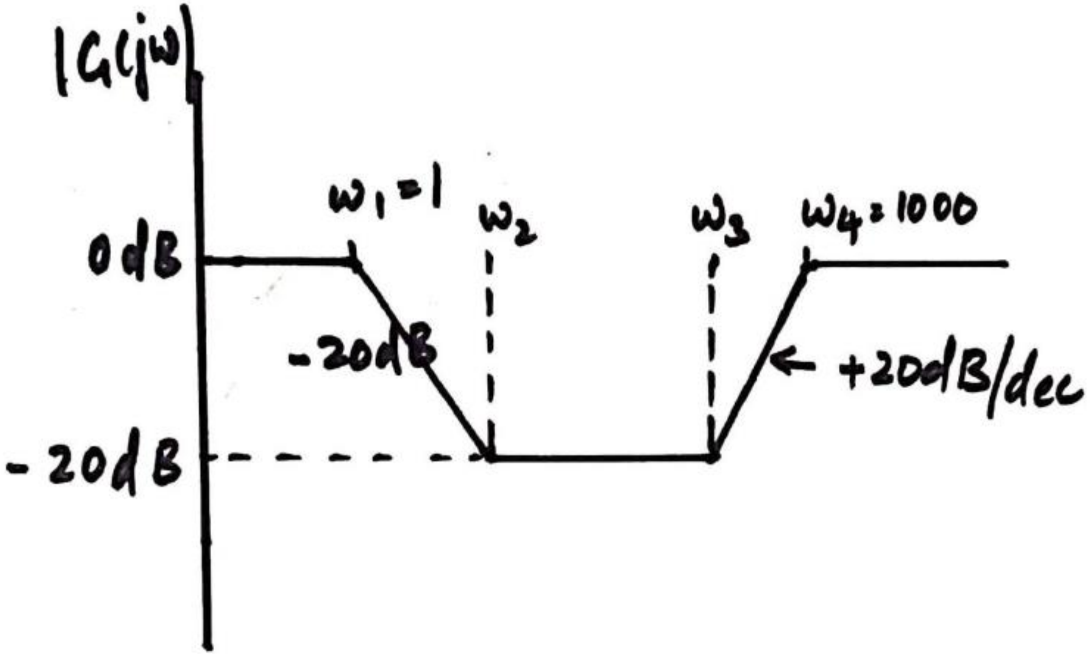
Step 3: Identify the corner frequencies

Step 4: If changes in slope is -20dB/decade, then the factor is a simple pole 1/(1+τs)

Step 5: If changes in slope is 20dB/decade, then the factor is a simple zero (1+τs), where τ = 1/ω

Reverse Bode Analysis – Example 1

Determine the transfer function for the given bode plot



$$Slope = \frac{y_2 - y_1}{\log(\omega_2) - \log(\omega_1)}$$

FREQUENCY RESPONSE

Reverse Bode Analysis – Example 1

- $20\log_{10}|G(j\omega)| = 20\log_{10}K - 20\log_{10}\omega$
  - $0dB = 20\log_{10}K \Rightarrow K = 1$
- Guess the factors , $G(s) = K\dfrac{(1+s/\omega_2(1+s/\omega_3)}{(1+s/\omega_1)(1+s/\omega_4)}$
- Where ω_1 and ω_4 are known but ω_2 and ω_3 are unknown
- So we need to determine ω_2 and ω_3
- $Slope = \dfrac{y_2-y_1}{\log(\omega_2)-\log(\omega_1)} \Rightarrow -20 = \dfrac{-20-0}{\log_{10}\omega_2-\log_{10}1}$
- ω_2=10, similarly, $\omega_3 = 100$

$-20dB \rightarrow \dfrac{1}{1+j\omega/\omega_1}$

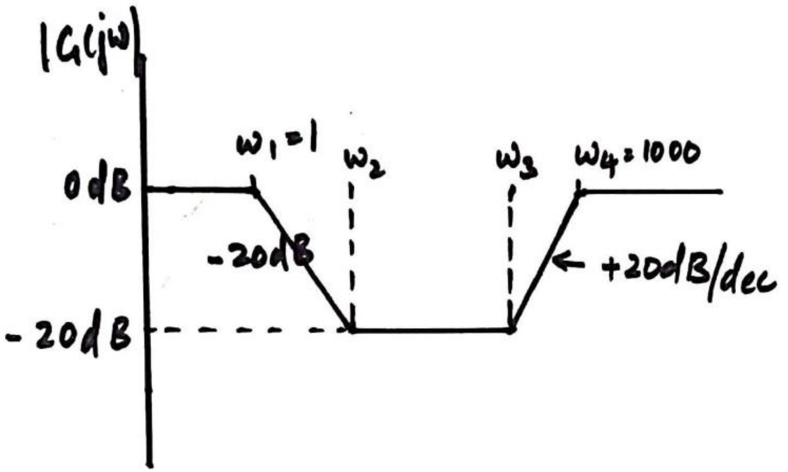
$+20dB \rightarrow 1+j\omega/\omega_2$

$20dB \rightarrow 1+j\omega/\omega_3$

$-20dB \rightarrow \dfrac{1}{1+j\omega/\omega_4}$

$\log_{10}\omega_2 = 1$

$\omega_2 = 10^1 = 10$



FREQUENCY RESPONSE

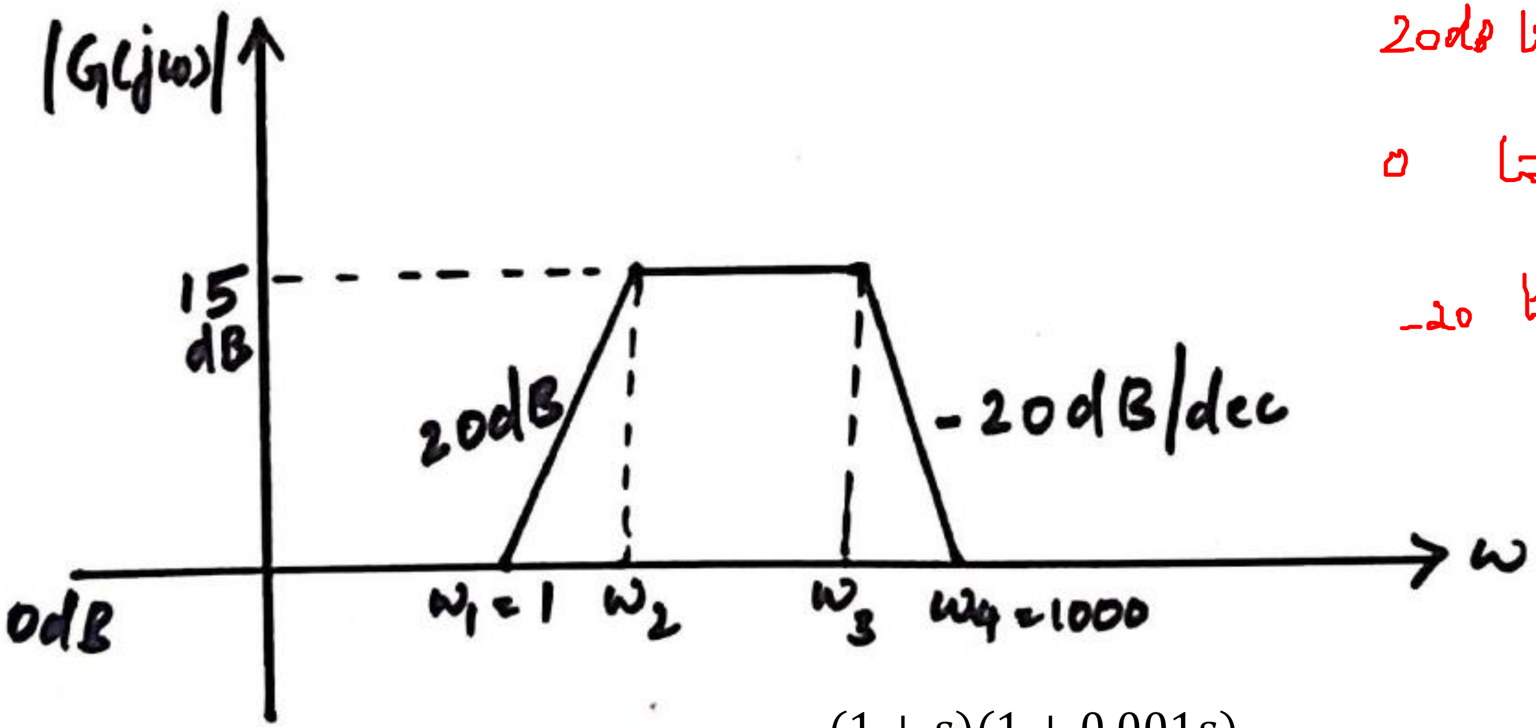
Reverse Bode Analysis – Example 1

- $G(s) = K \frac{(1+s/10(1+s/100)}{(1+s/1)(1+s/1000)}$

- $G(s) = \frac{(1+0.1s)(1+0.01s)}{(1+s)(1+0.001s)}$

Reverse Bode Analysis – Example 2

Determine the transfer function for the given bode plot



$$G(s) = \frac{(1+s)(1+0.001s)}{(1+0.177s)(1+0.0056s)}$$

0 to 20dB   20 dB $1 + j\omega/\omega_1$

20dB to 0dB   -20dB $\frac{1}{1 + j\omega/\omega_2}$

0 to -20   -20dB $\frac{1}{1 + j\omega/\omega_3}$

-20 to 0   20 dB $1 + j\omega/\omega_4$

$$20 = \frac{15 - 0}{\log_{10}\omega_2 - \log_{10}\omega_1}$$

$0 \because \omega_1 = 1$

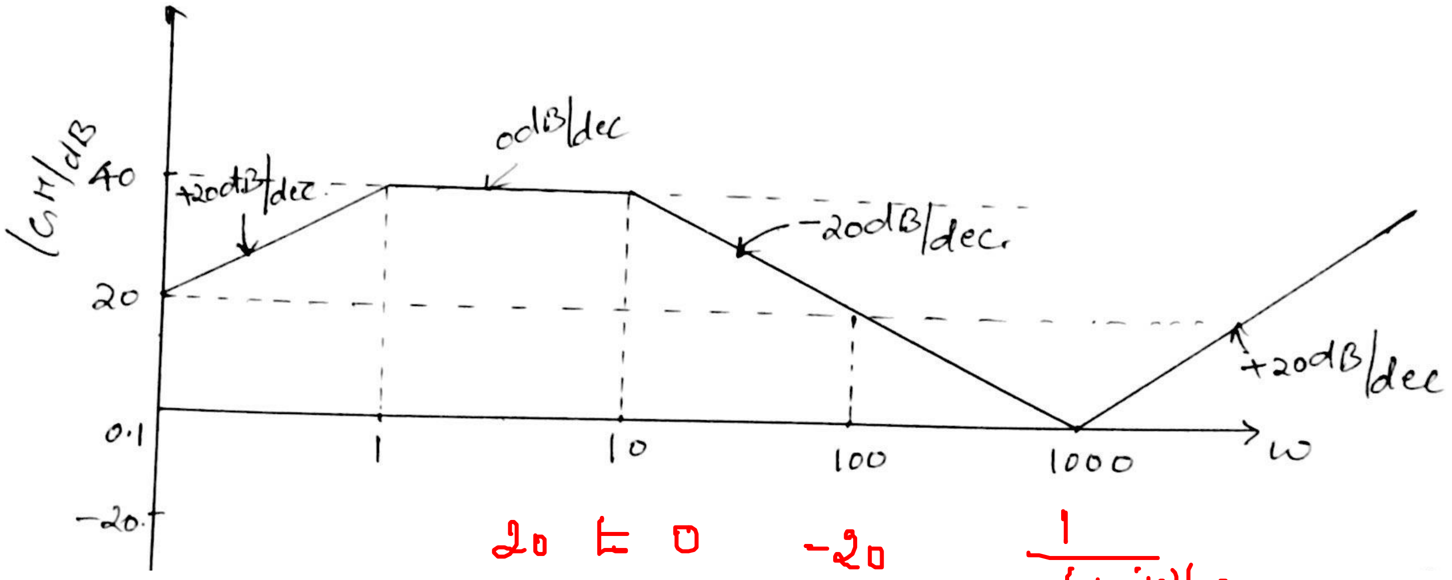
$$\log_{10}\omega_2 = 15/20$$

$$\omega_2 = 10^{15/20}$$

FREQUENCY RESPONSE

Reverse Bode Analysis – Example 3

Determine the transfer function of the system whose approximate plot is shown in figure below.



initial slope +20dB

$\Rightarrow$ zero at origin

$40\,dB = 20 \log_{10} K\omega_1$, $\omega_1 = 1$

$= 20 \log_{10} K + 20 \log_{10} \omega_1$

$\log_{10} K = 2$

$K = 10^2 = 100$

20 to 0 -20 $\dfrac{1}{1 + j\omega/\omega_1}$

0 to -20 -20 $1/(1 + j\omega/\omega_2)$

-20 to 20 40 $(1 + j\omega/\omega_3)^2$

$$G(s) = \frac{100\, s\,(1 + s/1000)^2}{(1 + s/1)(1 + s/10)}$$

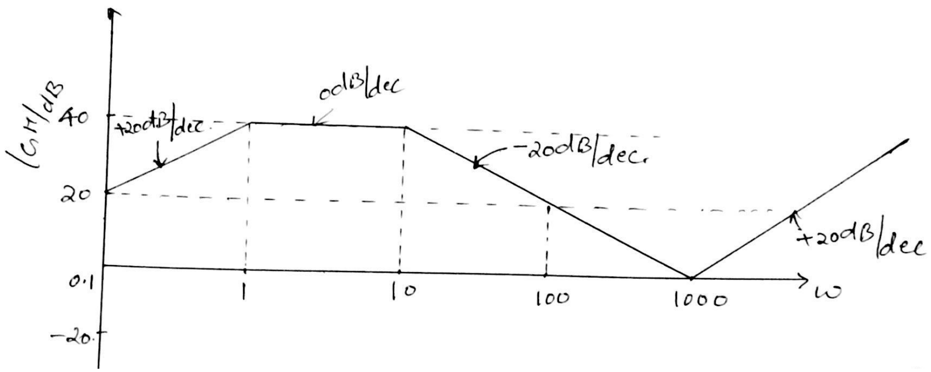
FREQUENCY RESPONSE

Reverse Bode Analysis – Example 3

- Initial slope is +20dB/dec indicating a simple zero at origin
- This line should have crossed zero dB line at ω=1

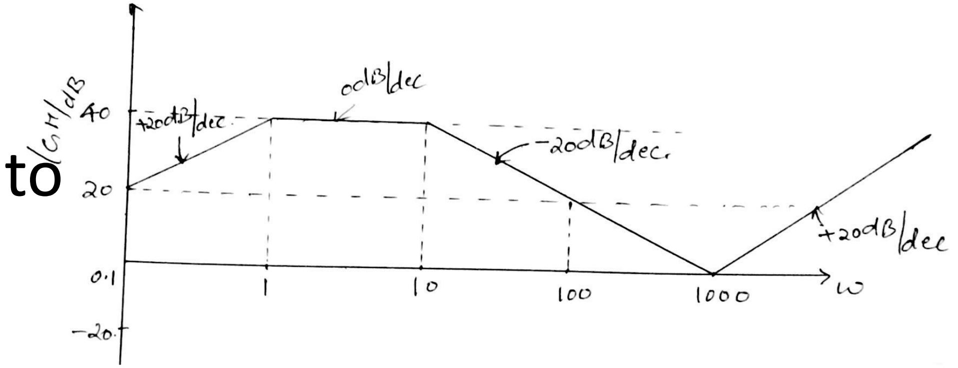
  but at ω=1 gain is 40dB
- This indicates a shift of 40dB due to constant
- $20\log_{10}|G(j\omega)| = 20\log_{10}(X)$
- => magnitude at $\omega_1 = 20\log_{10} K\omega_1$
- => 40 =$20\log_{10} k + 20\log_{10}\omega_1$

  => 20 log k = 40 or k=100
- Corner frequencies: 1,10,1000



Reverse Bode Analysis – Example 3

- Corner frequencies: 1,10,1000
- At ω=1,slope changes from 20dB/dec to 0dB/dec indicating that there is a pole with -20dB/dec line from corner frequency at ω=1 hence canceling the previous slope
- At ω=10,slope changes from 0dB/dec to -20dB/dec to indicating that there is a pole with -20dB/dec line from corner frequency at ω=10
- At ω=1000, slope changes from -20dB/dec to +20dB/dec indicating that there are two zeros with +20dB/dec line from this corner frequency(resulting in +40 dB/dec line) which gives a slope of +20 dB/dec

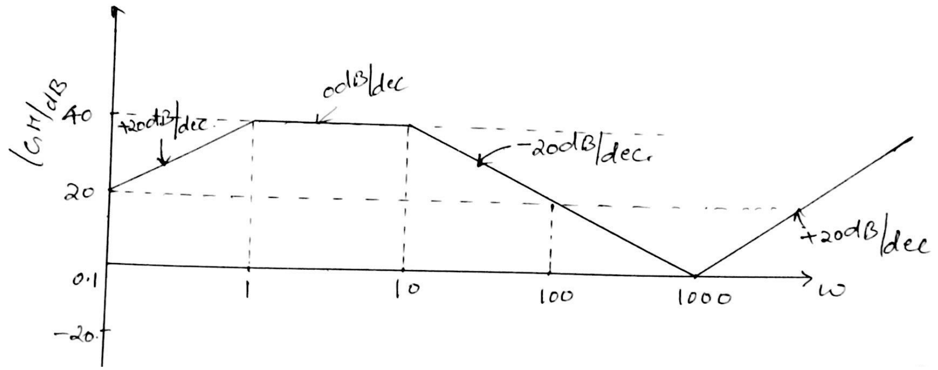


FREQUENCY RESPONSE

Reverse Bode Analysis – Example 3

- Hence the open loop transfer function would be

$$\backslash \quad G(s)H(s) = \frac{100s(1+0.001s)^2}{(1+s)(1+0.1s)}$$



THANK YOU

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