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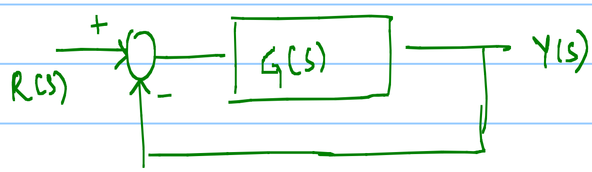
## Steady state error: $e_{ss}$




$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$E(s) = R(s) - Y(s)$$
$$= R(s)\left(1 - \frac{Y(s)}{R(s)}\right)$$

Let $\frac{Y(s)}{R(s)} = T(s)$ — closed loop transfer function

the steady state error,

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$e_{ss} = \lim_{s\to 0} sR(s)\left(1 - T(s)\right)$$

$$e_{ss} = \lim_{s\to 0} sR(s)\left(1 - \frac{G(s)}{1+G(s)}\right)$$

$$\boxed{e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}}$$

Step input $R(s) = 1/s$

$$e_{ss} = \lim_{s\to 0} \frac{s \times 1/s}{1+G(s)}$$

$$e_{ss} = \lim_{s\to 0} \frac{1}{1+G(s)}$$

$$e_{ss} = \frac{1}{1+\lim_{s\to 0} G(s)}$$

$$e_{ss} = \frac{1}{1+\lim_{s\to 0} G(s)} = \frac{1}{1+K_p}$$

where, $\boxed{K_p = \lim_{s\to 0} G(s)}$ — Static position error constant

Ramp input: $R(s) = 1/s^2$

$$e_{ss} = \lim_{s\to 0} \frac{s \times 1/s^2}{1+G(s)}$$

$$= \lim_{s\to 0} \frac{1}{s+sG(s)}$$

$$e_{ss} = \frac{1}{\lim_{s\to 0} sG(s)}$$

$$\boxed{e_{ss} = \frac{1}{\lim\limits_{s\to 0} s\,G(s)} = \frac{1}{K_v}}$$

$$\boxed{K_v = \lim_{s\to 0} s\,G(s)}$$ Static Velocity error Constant.

Parabolic input: $R(s) = 1/s^3$

$$e_{ss} = \lim_{s\to 0} \frac{s \times 1/s^3}{1 + G(s)}$$

G

$$e_{ss} = \frac{1}{\lim\limits_{s\to 0} s^2 G(s)}$$

$$\boxed{e_{ss} = \frac{1}{\lim\limits_{s\to 0} s^2 G(s)} = \frac{1}{K_a}}$$

$$\boxed{K_a = \lim_{s\to 0} s^2 G(s)}$$ Static acceleration error constant

General Structure of $G(s)$

$$G(s) = \frac{K(s+z_1)\ldots(s+z_m)}{s^N (s+p_1)\ldots\ldots(s+p_n)}$$

→ The term $s^N$ in the denominator representing a pole of multiplicity N at origin (no of integrators) denotes the 'TYPE' of the system.

type 0 – No poles at origin

type 1 – one pole at origin.

type 0:

$$\lim_{s\to 0} G(s) = \frac{K z_1 \times z_2 \times \ldots \times z_m}{p_1 \times p_2 \ldots \times p_n} = K'$$

$$\Rightarrow K_p = \lim_{s \to 0} G(s) = K'$$

Step input $\quad e_{ss} = \dfrac{1}{1+K_p} = K''$

$\Rightarrow$ type 0 systems will track step input with $e_{ss} = K''$

Ramp input: $\quad K_v = \lim_{s \to 0} s\,G(s) = 0$

$$\Rightarrow e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty$$

$\Rightarrow$ type 0 system can't track ramp input

parabolic input: $\quad K_a = \lim_{s \to 0} s^2 G(s) = 0$

$$e_{ss} = \frac{1}{K_a} = \infty$$

$\Rightarrow$ type 0 system can't track parabolic input.

type 1:
(1 pole at origin)

$K_p = \infty \Rightarrow e_{ss} = \dfrac{1}{1+K_p} = 0$ $\quad\Rightarrow$ type 1 systems can track step input

$K_v = \lim s\,G(s) = K'$

$e_{ss} = \dfrac{1}{K'}$ $\quad\Rightarrow$ type 1 system can track ramp input but with constant error $1/K'$

$K_a = \lim_{s \to 0} s^2 G(s) = 0 \Rightarrow e_{ss} = \dfrac{1}{K_a} = \infty$ $\quad\Rightarrow$ type 1 system can't track parabolic input

type 2:
(2 poles at origin)

$K_p = \infty \qquad e_{ss} = 0$

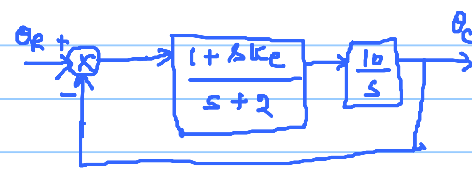
$K_v = \infty \qquad e_{ss} = 0$

$K_a = K' \qquad e_{ss} = \dfrac{1}{K'}$

$\Rightarrow$ type 2 system will follow step & ramp but not parabolic input.

| type | Step input $K_p$ | $e_{ss}$ | Ramp $K_v$ | $e_{ss}$ | Parabolic $K_a$ | $e_{ss}$ |
|---|---|---|---|---|---|---|
| 0 | $k'$ | $1/1+k'$ | 0 | $\infty$ | 0 | $\infty$ |
| 1 | $\infty$ | 0 | $k'$ | $1/k'$ | 0 | $\infty$ |
| 2 | $\infty$ | 0 | $\infty$ | 0 | $k'$ | $1/k'$ |

Determine the values of $e_{ss}$ (unit ramp input) with and without error rate control $k_e$. Given $\zeta = 0.6$ (with $k_e$)




Without $k_e$

$$G(s) = \frac{10}{s(s+2)}$$

With $k_e$

$$G(s) = \frac{10(1+sk_e)}{s(s+2)}$$

$$G(s) = \frac{(1+sk_e)10}{s(s+2)}$$

Without $k_e$: $k_e = 0$

$$G(s) = \frac{10}{s(s+2)}$$

$$k_p = \lim_{s\to 0} G(s) = \infty$$

Unit Step input $\boxed{e_{ss} = \frac{1}{\infty} = 0}$

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s \times \frac{10}{s(s+2)} = 5$$

Unit Ramp input $\boxed{e_{ss} = \frac{1}{K_v} = 1/5}$

With $k_e$: $k_e = 0.18$

$$G(s) = \frac{10(1+0.18s)}{s(s+2)}$$

$$k_p = \lim_{s\to 0} G(s) = \infty$$

$e_{ss} = 0$

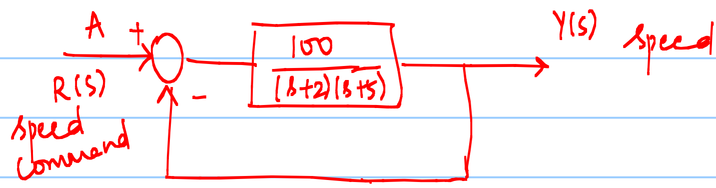
$$K_v = \lim_{s\to 0} s\,G(s) = \lim_{s\to 0} \cancel{s} \frac{10(1+0.18s)}{\cancel{s}(s+2)}$$

$$K_v = 5$$

$$e_{ss} = 1/5$$

$\Rightarrow$ PD controller can't reduce steady state error but reduces peak overshoot & also settling time.

The following is the model of a racing vehicle that affect the acceleration & speed attainable. The speed of the car is represented by the model shown




a) Calculate the steady state error of the car to step command in speed

b) Calculate overshoot of the speed to a step command

a) $$G(s) = \frac{100}{(s+2)(s+5)} \qquad \text{type } 0$$

$$K_p = \lim_{s\to 0} G(s) = \frac{100}{2\times 5} = 10$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{11}$$

b) CLTF $$T(s) = \frac{G(s)}{1+G(s)} = \frac{100}{s^2+7s+110}$$

$$\omega_n^2 = 110$$

$$\boxed{\omega_n = \sqrt{110}}$$

$$2\zeta\omega_n = 7$$

$$\Rightarrow \zeta = \frac{7}{2\sqrt{110}} = 0.334$$

$$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \qquad y(\infty) = 1$$

$$R(s) = A/s$$

$$C_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty}\left(r(t) - y(t)\right)$$

$$Y(s) = \frac{A\ 100}{s(s^2+7s+110)}$$

$$A/11 = r(\infty) - y(\infty)$$

$$y(\infty) = r(\infty) - A/11$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\,Y(s)$$

$$y(\infty) = \frac{A\,100}{110} = 0.909A$$

Since $y(\infty) = 0.909A$

$$M_p = 0.909A\left(e^{-\pi\zeta/\sqrt{1-\zeta^2}}\right) \rightarrow y(\infty) = 0.909$$

$$= 0.909A \times e^{-\pi\times 0.334/\sqrt{1-0.334^2}}$$

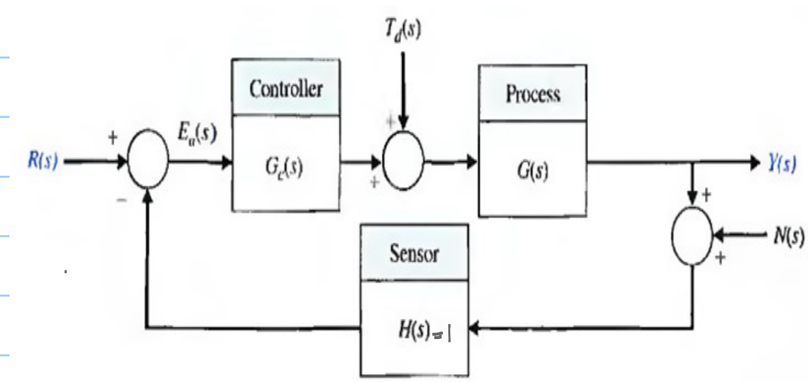
$$M_p = 0.2985A$$

$$\%\ M_p = 29.85 \times A\ \%$$

if $A = 1$

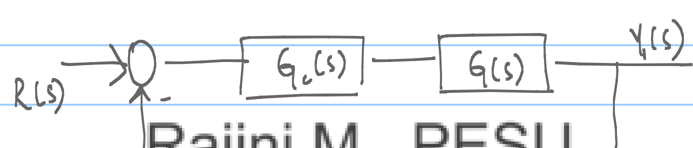
$$\%\ M_p = 29.85\ \%$$

Error Signal Analysis



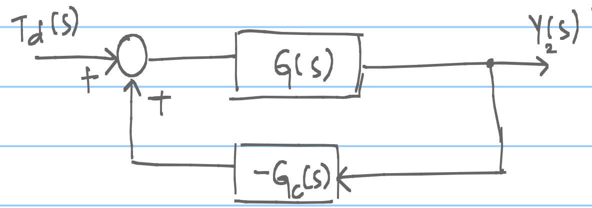

$$E(s) = Y(s) - R(s) = R(s) - Y(s)$$

Case (i) : $T_d(s) = 0 = N(s)$



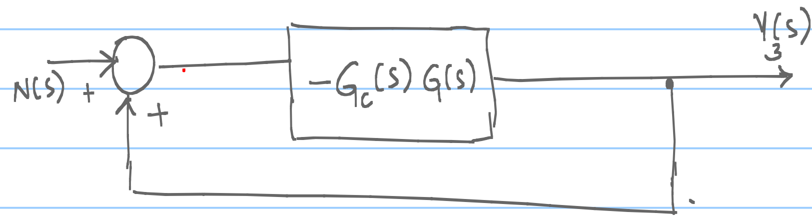
$$Y_1(s) = \frac{G_c(s)\,G(s)}{1 + G_c(s)\,G(s)} R(s)$$

Case (ii) : $R(s) = 0 = N(s)$




$$Y_2(s) = \frac{G(s)}{1 + G_c(s)G(s)} T_d(s)$$

Case (iii) : $R(s) = 0 = T_d(s)$




$$Y_3(s) = \frac{-G_c(s)\,G(s)}{1 - (-G_c(s)G(s))} N(s)$$

$$Y(s) = Y_1(s) + Y_2(s) + Y_3(s)$$

$$Y(s) = \frac{G_c G}{1 + G_c G} R(s) + \frac{G}{1 + G_c G} T_d(s) - \frac{G_c G}{1 + G_c G} N(s)$$

$$E(s) = R(s) - Y(s)$$

$$= \left(1 - \frac{G_c G}{1 + G_c G}\right) R(s) - \frac{G}{1 + G_c G} T_d(s) + \frac{G_c G}{1 + G_c G} N(s)$$

$$E(s) = \frac{1}{1 + G_c G} R(s) - \frac{G}{1 + G_c G} T_d(s) + \frac{G_c G}{1 + G_c G} N(s)$$

Let $L(s) = G_c(s)\,G(s)$ — Loop gain

$$E(s) = \overbrace{\frac{1}{1 + L(s)}}^{S(s)} R(s) - \frac{G}{1 + L(s)} T_d(s) + \overbrace{\frac{L(s)}{1 + L(s)}}^{C(s)} N(s)$$

Let sensitivity function

$$S(s) \triangleq \frac{1}{1+L(s)}$$

Complementary function

$$C(s) = \frac{L(s)}{1+L(s)}$$

$$S(s) + C(s) = \frac{1}{1+L(s)} + \frac{L(s)}{1+L(s)}$$

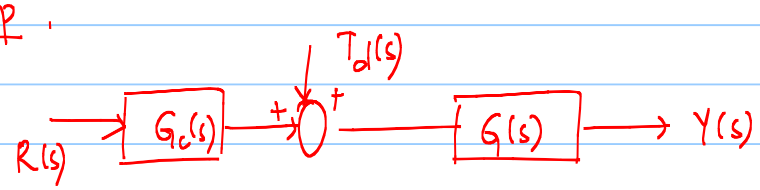
$$= 1$$

$$E(s) = S(s)R(s) - G(s)S(s)T_d(s) + C(s)N(s)$$

→ $\uparrow L(s) \Rightarrow$ effect of $T_d \downarrow$ in low freq

$\downarrow L(s) \Rightarrow$ effect of $N \downarrow$ in high frequency range

Open loop :




$$E(s) = R(s) - Y(s)$$

$$Y(s) = G_c(s)G(s)R(s) + G(s)T_d(s)$$

$$E(s) = R(s) - G_c(s)G(s)R(s) - G(s)T_d(s)$$

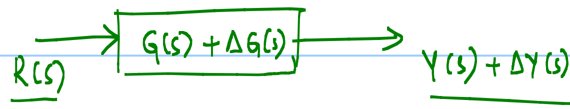
$$E(s) = \left(1 - G_c(s)G(s)\right)R(s) - G(s)T_d(s)$$

$\Rightarrow$ No way we can reduce the effect of disturbance on the open loop error signal

## Sensitivity to parameter variations:

$$G(s) \longrightarrow G(s) + \underbrace{\Delta G(s)}_{\text{Small perturbation}}$$

**Open loop system:** Let $T_d(s) = 0$




$$Y(s) + \Delta Y(s) = \underbrace{\left(G(s) + \Delta G(s)\right)R(s)}$$

$$E(s) + \Delta E(s) = R(s) - \left(Y(s) + \Delta Y(s)\right)$$

$$\Delta E(s) = R(s) - \left(G(s) + \Delta G(s)\right)R(s) - E(s)$$

$$= R(s) - \left(G(s) + \Delta G(s)\right)R(s) - \left(R(s) - \overset{G(s)\,R(s)}{Y(s)}\right)$$

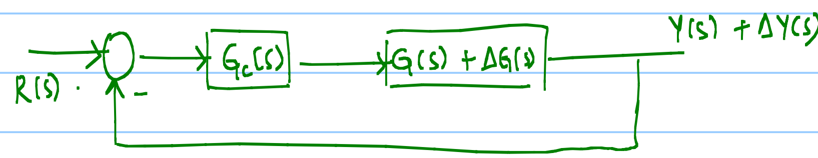
$$\Delta E(s) = \cancel{R(s)} - \left(\cancel{G(s)} + \Delta G(s)\right)R(s) - \cancel{R(s)} + \cancel{G(s)}R(s)$$

$$\Delta E(s) = -\Delta G(s)R(s)$$

$\Rightarrow$ $\Delta E(s)$ is directly proportional to change in the parameter

$\Rightarrow$ Open loop systems are sensitive to parameter variations

**Close loop system:** $T_d(s) = 0, \quad N(s) = 0$




$$Y(s) + \Delta Y(s) = \frac{G_c(s)\left(G(s) + \Delta G(s)\right)}{1 + G_c(s)\left(G(s) + \Delta G(s)\right)} R(s)$$

$$E(s) + \Delta E(s) = R(s) - \left(Y(s) + \Delta Y(s)\right)$$

$$E(s) + \Delta E(s) = \left(1 - \frac{G_c(G + \Delta G)}{1 + G_c(G + \Delta G)}\right) R(s)$$

$$E(s) + \Delta E(s) = \frac{1}{1 + G_c(G + \Delta G)} R(s)$$

$$\Delta E(s) = \frac{1}{1 + G_c(G + \Delta G)} R(s) - \underbrace{E(s)}_{R(s) - Y(s)}$$

$$\Delta E(s) = \frac{1}{1+G_c(G+\Delta G)} R(s) - R(s) + \frac{G_c G}{1+G_c G} R(s)$$

$$\Delta E(s) = \frac{-\Delta G(s)\, G_c(s)}{(1+G_c(G+\Delta G))(1+G_c G)} R(s)$$

WKT $\quad G_c(s)\,\Delta G(s) \ll G_c(s)\, G(s)$

$$\Delta E(s) \simeq \frac{-\Delta G(s)\, G_c(s)}{(1+G_c G)^2} R(s)$$

→ The change in the error signal is reduced by a factor $1+\underbrace{G_c(s)\,G(s)}_{L(s)}$ over the range of frequencies

→ For large $L(s)$, $(1+L(s)) \simeq L(s)$

$$\Delta E(s) \simeq \frac{-\Delta G(s)\, \cancel{G_c(s)}}{L(s)\,(\cancel{G_c}(s)\, G(s))} R(s)$$

$$\Delta E(s) \simeq \frac{-\Delta G(s)}{L(s)\, G(s)} R(s)$$

⇒ The closed loop system has reduced sensitivity to parameter variations

⇒ CLOSED LOOP SYSTEMS ARE BETTER

**Sensitivity of T(s) :** $T(s)$ – closed loop transfer function

It is the ratio of percentage change in system transfer function $(T(s))$ to the percentage change in the process transfer function $(G(s))$

$$S_G^T = \frac{\Delta T(s)/T(s)}{\Delta G(s)/G(s)}$$

for small incremental changes $\Delta \to 0$ $\qquad T(s) = \dfrac{G}{1+GH}$

$$S_G^T = \frac{\partial T(s)}{\partial G(s)} \times \frac{G(s)}{T(s)}$$

Closed loop transfer

$$T(s) = \frac{G(s)\,G_c(s)}{1+G(s)H(s)G_c(s)}$$

$$S_G^T = \frac{\partial T(s)}{\partial G(s)} \times \frac{\cancel{G(s)}}{\frac{\cancel{G(s)}\,G_c(s)}{1+G(s)H(s)G_c(s)}}$$

$$\frac{\partial T(s)}{\partial G(s)} = \frac{(1+G(s)H(s)G_c(s))\,G_c(s) - G(s)G_c(s)\,G_c(s)H(s)}{(1+G(s)H(s)G_c(s))^2}$$

$$= \frac{G_c(s)}{(1+G(s)H(s)G_c(s))^2}$$

$$S_G^T = \frac{\cancel{G_c(s)}}{(1+G(s)H(s)G_c(s))^{\cancel{2}}} \times \frac{1+\cancel{G(s)H(s)G_c(s)}}{\cancel{G_c(s)}}$$

$$S_G^T = \frac{1}{1+G(s)H(s)G_c(s)}$$

To find the sensitivity wrt '$\alpha$' parameter

$\underline{T} \rightarrow G \rightarrow \underline{\alpha}$

$$S_\alpha^T = \frac{\partial T}{\partial \alpha}\,\frac{\alpha}{T}$$

$$= \frac{\partial T}{\partial G}\,\frac{\partial G}{\partial \alpha}\,\frac{\alpha}{T} \times \frac{G}{G}$$

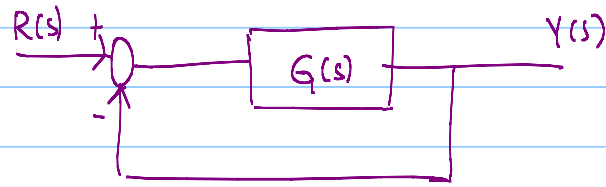
$$= \frac{\partial T}{\partial G} \times \frac{G}{T}\;\frac{\partial G}{\partial \alpha}\,\frac{\alpha}{G}$$

$$\boxed{S_\alpha^T = S_G^T\,S_\alpha^G}$$

## Open loop

$$S_G^G = \frac{\partial G}{\partial G} \frac{G}{G} = 1$$

- A closed loop system is used to track the sun to obtain maximum power from a photovoltaic array. The tracking system may be represented by $G(s) = \frac{100}{\tau s+1}$ where $\tau = 3$ *seconds* nominally
- a) Calculate the sensitivity of this system for a small change in $\tau$ $\longrightarrow$ $S_\tau^T$
- b) Calculate the time constant of the closed loop system response




$$T(s) = \frac{G}{1+G}$$

$$S_\tau^T = S_G^T \, S_\tau^G$$

$$S_G^T = \frac{1}{1+G(s)}$$

$$S_\tau^G = \frac{\partial}{\partial \tau}\left(\frac{\cancel{100}}{\tau s+1}\right) \times \frac{\tau}{\frac{\cancel{100}}{\tau s+1}}$$

$$S_\tau^G = \frac{-\tau s}{\tau s+1}$$

$$S_\tau^T = \frac{1}{1+\frac{100}{\tau s+1}} \times \frac{-\tau s}{\tau s+1}$$

$$S_\tau^T = \frac{-\tau s}{\tau s+101}$$

$\tau = 3$ sec

$$\boxed{S_\tau^T = \frac{-3s}{3s+101}}$$

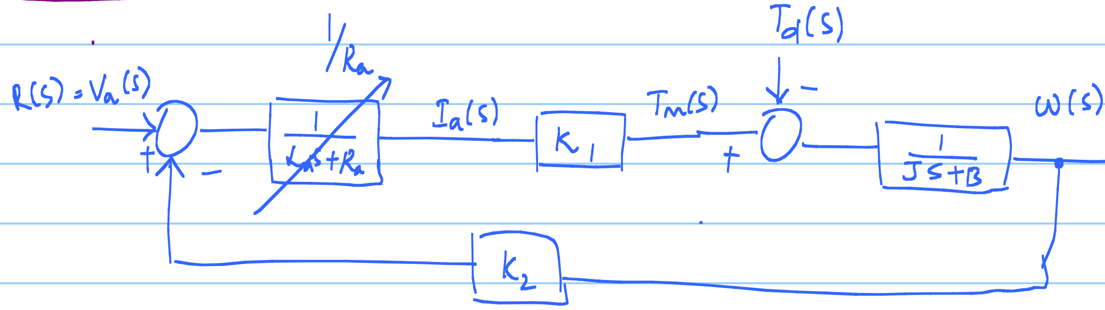
b) $$T(s) = \frac{G}{1+G} = \frac{100}{3s+101}$$

$$T(s) = \frac{100/101}{\frac{3}{101}s + 1}$$

Time constant $= 3/101$

## Disturbance signals in feedback control system

**DC Motor :**

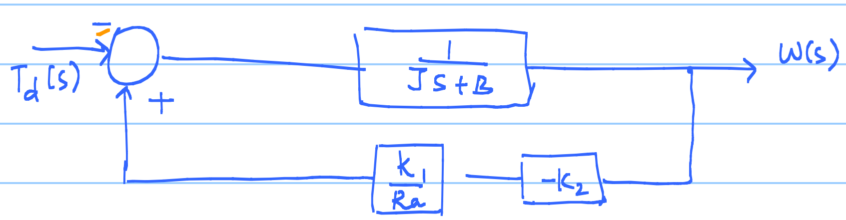



$L_a$ is negligable compared to $R_a \Rightarrow \frac{1}{L_a s + R} \approx \frac{1}{R_a}$

Error: $E(s) = R(s) - \omega(s)$

Let $R(s) = 0$, we will analyse the effect of disturbace

$$E(s) = -\omega(s)$$




$$\frac{\omega(s)}{T_d(s)} = -\frac{R_a}{(Js+B)R_a + K_1K_2}$$

$$-\frac{1/Js+B}{1 - \left(\frac{1}{Js+B}\right)\left(-\frac{K_2K_1}{R_a}\right)}$$

$$E(s) = -\omega(s)$$

$$= \frac{R_a}{(Js+B)R_a + K_1K_2} T_d(s)$$

Let $T_d(s) = \frac{D}{S}$  $T_d(t) = D \quad t \geq 0$ — disturbance signal

$$E(s) = \frac{DR_a}{(Js+B)R_a + K_1K_2} \times \frac{1}{s}$$

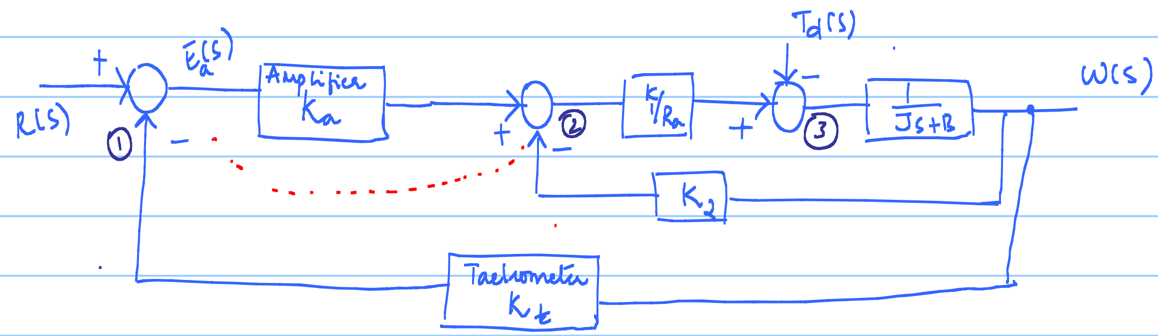
Steady state error

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

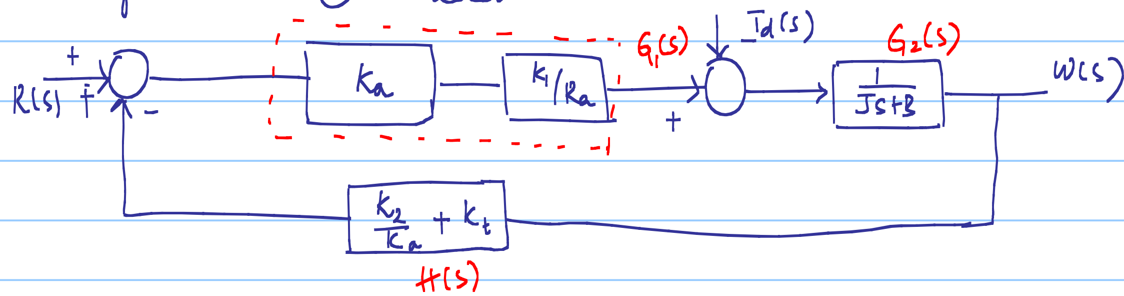
$$e_{ss} = \frac{DR_a}{K_1K_2 + BR_a} = -\omega(\infty)$$

The steady state value of the error cannot be reduced as the values of $R_a$, $B$, $K_1, K_2$ are fixed for the DC motor

$\Rightarrow$ Disturbance upon the open loop system cannot be reduced.



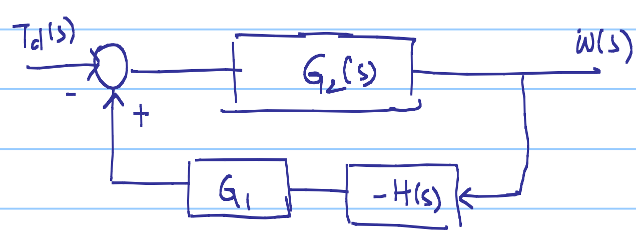

Moving summer 2 ahead the block $K_1$




$$G_1(s) = \frac{K_aK_1}{R_a} \qquad G_2(s) = \frac{1}{Js+B} \qquad H(s) = \frac{K_2 + K_tK_a}{K_a}$$

$$E(s) = R(s) - W(s) \qquad R(s) = 0$$

$$= -W(s)$$




$$W(s) = \frac{G_2(s)}{1 - G_1G_2(-H)} \times T_d(s)$$

$$W(s) = -\frac{G_2(s)}{\underbrace{1+G_1G_2H}_{\approx G_1G_2H}} T_d(s)$$

$G_1G_2H >> 1$

$$E(s) = -W(s) = \frac{\cancel{G_2}}{G_1\cancel{G_2}H} T_d(s)$$

$$E(s) = \frac{1}{G_1H} T_d(s)$$

Let $T_d(s) = \frac{D}{s}$

$$E(s) = \frac{T_d(s)}{\frac{\cancel{K_a}k_1}{R_a} \times \left(\frac{k_2 + k_t k_a}{\cancel{k_a}}\right)} = \frac{T_d(s)}{\frac{k_1k_2}{R_a} + \frac{k_1k_t k_a}{R_a}}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \cancel{s} \times \frac{D/\cancel{s}}{\frac{k_1k_2}{R_a} + \frac{k_1k_tk_a}{R_a}}$$

$$e_{ss} = \frac{D}{\frac{k_1k_2}{R_a} + \frac{k_1k_tk_a}{R_a}}$$

Choose $k_a$ & $k_t$ such that the steady state error reduces

∴ Closed loop systems have better disturbance rejection.

Control of Transient response: $T_d(s) = N(s) = 0$

Let us consider the DC motor.

$$\frac{\theta(s)}{V_a(s)} = \frac{k_1}{s(Js+B)(L_as+R_a)+k_1k_2s}$$

$L_a$ is negligible

$$\frac{\theta(s)}{V_a(s)} = \frac{k_1}{s((Js+B)R_a+k_1k_2)} \Rightarrow \frac{s\,\theta(s)}{V_a(s)} = \frac{k_1}{(Js+B)R_a+k_1k_2}$$

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step response : $V_a(s) = \frac{V_a}{s}$  $V_a$ - amplitude of the voltage

$$\omega(s) = \frac{k_1 V_a}{s(JR_a s + BR_a + k_1k_2)}$$

$$\omega(s) = \frac{A}{s} + \frac{C}{JR_a s + \underbrace{(BR_a + k_1k_2)}_{K'}}$$

$$\omega(s) = \frac{k_1 V_a}{k'} \times \frac{1}{s} - \frac{k_1 V_a \times JR_a / k'}{JR_a s + k'}$$

$$\omega(s) = \frac{k_1 V_a}{k'} \frac{1}{s} - \frac{k_1 V_a / k'}{s + k'/JR_a}$$

ILT $\longrightarrow$

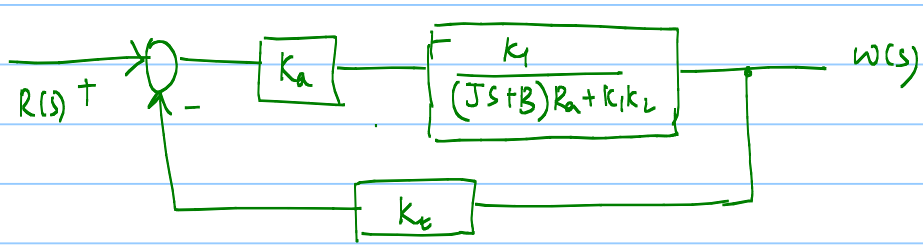
$$\omega(t) = \frac{k_1 V_a}{k'}\left(1 - e^{-k'/JR_a}\right), \quad t \geq 0$$

$$\omega(t) = \frac{k_1 V_a}{BR_a + k_1k_2}\left(1 - e^{-\frac{BR_a + k_1k_2}{JR_a}t}\right) \quad t \geq 0$$

Contributes to transient response

$\Rightarrow$ Transient response ($t_r, t_p, M_p$) cannot be changed as $B$, $R_a$, $k_1$, $k_2$ & $J$ are fixed for DC motor.

closed loop system



$$\frac{W(s)}{R(s)} = \frac{K_a K_1}{((Js+B)R_a + K_1K_2 + K_1K_aK_t)}$$

$$R(s) = 1/s$$

$$W(s) = \frac{K_a K_1}{JR_a s + \underbrace{BR_a + K_1K_2 + K_1K_aK_t}_{K'}} \times \frac{1}{s}$$

$$W(s) = \frac{A}{s} + \frac{C}{JR_a s + K'}$$

$$A = sW(s)\Big|_{s=0} = \frac{K_a K_1}{JR_a s + K'}\Big|_{s=0}$$

$$A = \frac{K_a K_1}{K'}$$

$$C = (JR_a s + K')\,W(s)\Big|_{s=-K'/JR_a} = \frac{K_a K_1}{-K'/JR_a}$$

$$W(s) = \frac{K_a K_1}{K'}\frac{1}{s} - \frac{\cancel{JR_a} K_a K_1}{K'}\frac{1}{\cancel{JR_a}(s + K'/JR_a)}$$

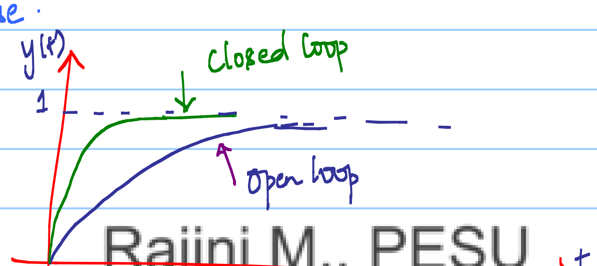
ILT $\longrightarrow$

$$w(t) = \frac{K_a K_1}{K'}\left(1 - e^{-tK'/JR_a}\right), \quad t \geq 0$$

$$= \frac{K_a K_1}{K'}\left(1 - e^{-t\left(\frac{BR_a + K_1K_2}{JR_a}\right)} e^{-t\frac{K_1K_aK_t}{JR_a}}\right) \quad t \geq 0$$

$K_a$ & $K_t$ is designed in such a way that the rise time, $t_p$ & $M_p$ can be controlled.

$\therefore$ Closed loop systems have better control over transient response.

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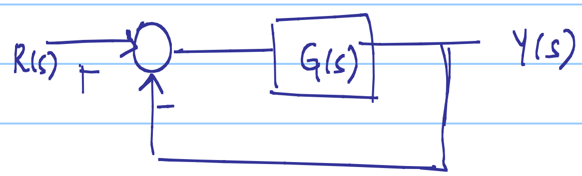
The transfer function of the process is given by

$$G(s) = \frac{k}{(s+4)^2} \qquad \text{OLTF}$$

Calculate the steady state error for step command 'A'




Method 1 :

$$E(s) = R(s) - Y(s) \qquad Y(s) = \frac{G}{1+G} R(s)$$

$$= R(s) - \frac{G}{1+G} R(s) \qquad =$$

$$= \left(1 - \frac{G}{1+G}\right) R(s)$$

$$= \frac{R(s)}{1+G(s)}$$

$$E(s) = \frac{R(s)}{1 + k/(s+4)^2} = \frac{A}{s\left(1 + k/(s+4)^2\right)}$$

$$e_{ss} = \lim_{s \to 0} s E(s) \quad \text{— Final value Theorem}$$

$$e_{ss} = \lim_{s \to 0} \cancel{s} \frac{A}{\cancel{s}\left(1 + k/(s+4)^2\right)}$$

$$e_{ss} = \frac{A}{1 + k/16}$$

Method 2 ;

$$R(s) = A/s$$

$$e_{ss} = \frac{A}{1 + \lim_{s \to 0} G(s)} = \frac{A}{1+k_p}$$

$$K_p = \lim_{s \to 0} G(s) = \frac{K}{16}$$

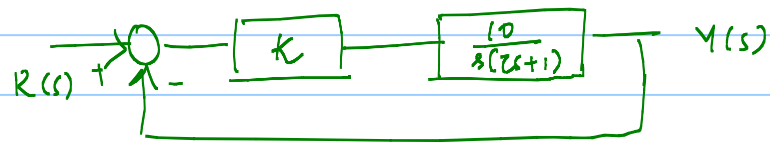
$$e_{ss} = \frac{A}{1 + K/16}$$

b) What should be the value of K such that error $e_{ss} = 0.01$. Let $A = 1$

$$0.01 = \frac{1}{1 + K/16}$$

$$K = 1584$$

2. Let $G(s) = \dfrac{10}{s(\tau s+1)}$ where $\tau = 0.001$ sec,




a) Calculate $e_{ss}$ (Step input is A)

b) Calculate the required K in order to maintain $e_{ss} = 0.1$ mm for a ramp input $\underbrace{10 \text{ cm/s}}_{0.1 \text{ m/s}}$ $R(s) = \dfrac{0.1}{s^2}$.

a) $$e_{ss} = \lim_{s \to 0} s\left(R(s) - Y(s)\right)$$

$$e_{ss} = \lim_{s \to 0} \frac{\cancel{s}\, A/\cancel{s}}{1 + K\,G(s)} = \lim_{s \to 0} \frac{A}{1 + K\dfrac{10}{s(\tau s+1)}} = 0$$

b) $$e_{ss} = \lim_{s \to 0} s \left[ \frac{0.1/s^2}{1 + \dfrac{10K}{s(\tau s+1)}} \right]$$

$E(s) = R(s) - Y(s)$

$= \left(1 - \dfrac{KG(s)}{1+KG(s)}\right) R(s)$

$E(s) = \dfrac{R(s)}{1+KG(s)}$

$$K_v = \lim_{s \to 0} s\,G(s)$$

$$= \lim_{s \to 0} \frac{s \times 10K}{s(\tau s+1)}$$

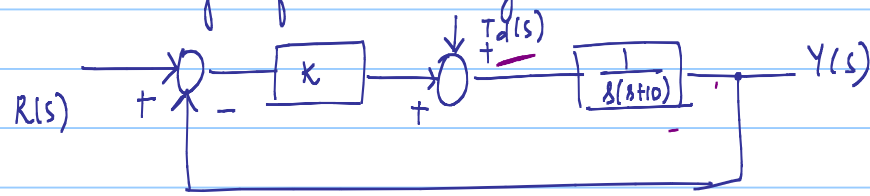
$K_v = 10K \Rightarrow e_{ss} = 0.1/K_v = 0.1/10K$

$$e_{ss} = 0.1/10K = 0.01/K$$

$$0.1 \times 10^{-3} = \frac{0.1}{10K}$$

$$\boxed{K = 100}$$

3. Consider a unity feedback system.




Let $K = 120$

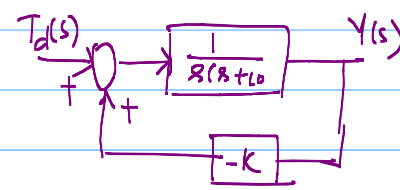
a) Calculate $e_{ss}$ of the closed loop due to unit step input $T_d(s) = 0$

b) Calculate steady state response $Y_{ss} = \lim_{t \to \infty} y(t)$ when $T_d(s) = 1/s$

& $R(s) = 0$

$$\boxed{T(s) = \left.\frac{Y(s)}{R(s)}\right|_{T_d(s)=0} = \frac{K}{s(s+10)+K}}$$

$$\left.\frac{Y(s)}{T_d(s)}\right|_{R(s)=0} = \frac{1}{s^2+10s+K}$$




a) $T_d(s) = 0$ $\quad e_{ss} = \lim_{s \to 0} s\,E(s) = \lim_{s \to 0} \cancel{s}\left(1 - T(s)\right)\cancel{R(s)}\ \cancel{1/s}$

$E(s) = R(s) - Y(s)$

$R(s) - T(s)R(s)$

$$e_{ss} = (1 - T(0)) = 1 - K/K = 0$$

b) $R(s) = 0$ $\quad Y_{ss} = \lim_{s \to 0} sY(s) = \lim_{s \to 0} s \times \frac{T_d(s)}{s^2+10s+K} = \lim_{s \to 0} s \times \frac{1/s}{s^2+10s+K}$

$$Y_{ss} = 1/K$$

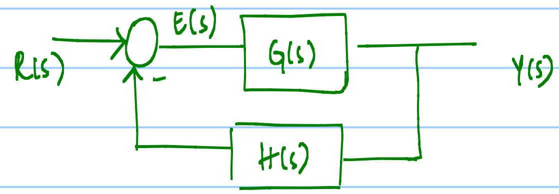
Error with respect to disturbance

$$E(s) = Y(s)\Big|_{R(s)=0}$$

$$e_{ss} = \lim_{s \to 0} s\,E(s) = \lim_{s \to 0} sY(s) = Y_{ss} = 1/K$$

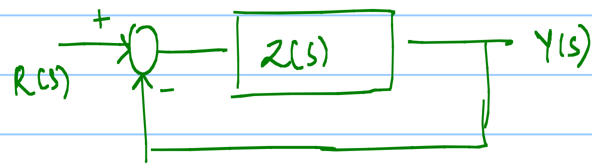
Increase 'K' sufficiently large such that error with respect to disturbance is small.

## Steady state error : $e_{ss}$




Non unity feedback system.

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$




$$\frac{Y(s)}{R(s)} = \frac{Z(s)}{1+Z(s)}$$

$$\frac{Z(s)}{1+Z(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$\frac{G(s)}{1+G(s)H(s)} = Z(s)\left(1 - \frac{G(s)}{1+G(s)H(s)}\right)$$

$$\Rightarrow \quad Z(s) = \frac{G(s)}{1+G(s)H(s) - G(s)}$$

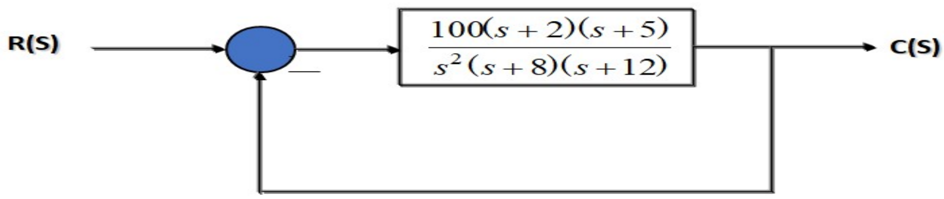
$$E(s) = R(s) - Y(s)$$
$$= R(s)\cdot\left(1 - \frac{Y(s)}{R(s)}\right)$$

Let $\frac{Y(s)}{R(s)} = T(s)$ — Closed loop transfer function

the steady state error,

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

- **For the system shown in figure below evaluate the static error constants and find the expected steady state errors for the standard step, ramp and parabolic inputs.**




$$K_p = \lim_{s \to 0} G(s)$$

$$K_p = \lim_{s \to 0} \left( \frac{100(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$K_p = \infty$$

$$K_v = \lim_{s \to 0} sG(s)$$

$$K_v = \infty$$

$$K_a = \lim_{s \to 0} s^2 G(s)$$

$$K_a = \lim_{s \to 0} \left( \frac{100s^2(s+2)(s+5)}{s^2(s+8)(s+12)} \right)$$

$$K_a = \left( \frac{100(0+2)(0+5)}{(0+8)(0+12)} \right) = 10.4$$

$$e_{ss} = \frac{1}{1 + K_p} = 0$$

$$e_{ss} = \frac{1}{K_v} = 0$$

$$e_{ss} = \frac{1}{K_a} = 0.09$$

Determine the static error constants of the system represented by the OLTF with unity feedback

$$G(s) = \frac{k(s+2)}{s(s^3+7s^2+12s)}$$

Also determine the type & order of the system. Find the $e_{ss}$ for a unit parabolic input.

Sol: type = 2, order = 4

$K_p = \lim_{s \to 0} G(s) = \infty$

$K_v = \lim_{s \to 0} sG(s) = \infty$

$K_a = \lim_{s \to 0} s^2 G(s) = \frac{2k}{12} = k/6$

$e_{ss} = \frac{1}{K_a} = \frac{1}{k/6} = \frac{6}{k}$

Find the static error constants for a system

$$G(s) = \frac{s+10}{s(s^3+5s^2+15s)}$$

assuming UFB, also calculate $e_{ss}$ when $r(t) = 2t\,u(t)$ & $r(t) = 4t^2 u(t)$

Sol: type = 2, order = 4

$L[t^n] = \frac{n!}{s^{n+1}}$

$K_p = \lim_{s \to 0} G(s) = \infty$

$K_v = \lim_{s \to 0} sG(s) = \infty$

$K_a = \lim_{s \to 0} s^2 G(s) = \frac{10}{15} = 2/3$

$R(s) = 2/s^2$

$e_{ss} = \frac{2}{K_v} = \frac{2}{\infty} = 0$

$R(s) = 8/s^3$

$e_{ss} = \frac{8}{K_a} = \frac{8}{2/3} = 12$

$e_{ss} = \lim_{s \to 0} s \cdot E(s) = \lim_{s \to 0} s \cdot \frac{R(s)}{1+G(s)}$

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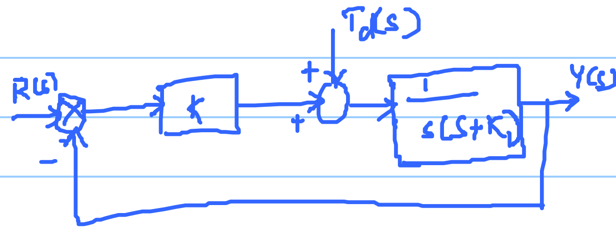
- Consider the unity feedback system shown. The system has two parameters, the controller gain K and the constant $K_1$ in the process.
- A) calculate the sensitivity of CLTF to changes in $K_1$




Sol: To find CLTF, $T_d(s) = 0$

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\frac{K}{s(s+K_1)}}{1 + \frac{K}{s(s+K_1)}}$$

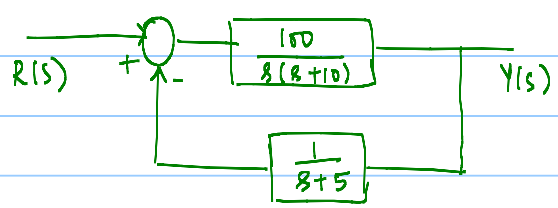
$$= \frac{K}{s^2 + K_1 s + K}$$

$$S^T_{K_1} = \frac{\partial T}{\partial K_1} \cdot \frac{K_1}{T}$$

$$= \frac{-\cancel{K} \cdot s}{(s^2+K_1 s+K)^{\cancel{2}}} \quad \frac{K_1}{\frac{\cancel{K}}{\cancel{s^2+K_1 s+K}}}$$

$$= \frac{-s K_1}{s^2 + K_1 s + K}$$

make K as large as possible, To lose $S^T_{K_1}$




Determine the type, order, error constants & $e_{ss}$ for unit step input

$$G(s) = \frac{100}{s(s+10)} \qquad H(s) = \frac{1}{s+5}$$

$$Z(s) = \frac{G}{1 + GH - G}$$

$$Z(s) = \frac{100/s(s+10)}{1 + \frac{100}{s(s+10)} \times \frac{1}{s+5} - \frac{100}{s(s+10)}}$$

OLTF $$Z(s) = \frac{100(s+5)}{s^3+15s^2-50s-400}$$

type - 0

CLTF $$T(s) = \frac{Z}{1+Z}$$

$$= \frac{100(s+5)}{(s^3+15s^2-50s-400+100s+500)}$$

Order - 3

$$K_p = \lim_{s \to 0} Z(s)$$

$$= -5/4$$

$$K_v = \lim_{s \to 0} s\,Z(s) = 0$$

$$K_a = \lim_{s \to 0} s^2 Z(s) = 0$$

for step $$e_{ss} = \frac{1}{1+K_p}$$

$$e_{ss} = \frac{4}{4-5} = -4$$