Digital Signal Processing

Ms. Ashwini

Department of Electronics and Communication.

Digital Signal Processing

Properties of DFT

Ms. Ashwini
Department of Electronics and Communication.

Properties of DFT

Periodicity, Linearity and Symmetry

Periodicity:

x(n) and X(k) are an N-point DFT pair

$$x(n+N) = x(n) \quad \text{for all } n$$

$$X(k+N) = X(k) \quad \text{for all } k$$

Linearity:

$$x_1(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)$$

$$x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_2(k)$$

For any real-valued or complex-valued constants a1 and a2

$$a_1x_1(n) + a_2x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} a_1X_1(k) + a_2X_2(k)$$

Properties of DFT

Circular Symmetries of a sequence

N-point DFT of x(n) = N-point DFT of $x_p(n)$ For L≤N

x p (n) obtained by periodically extending x(n)

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

Shift x p (n) by k units to the right

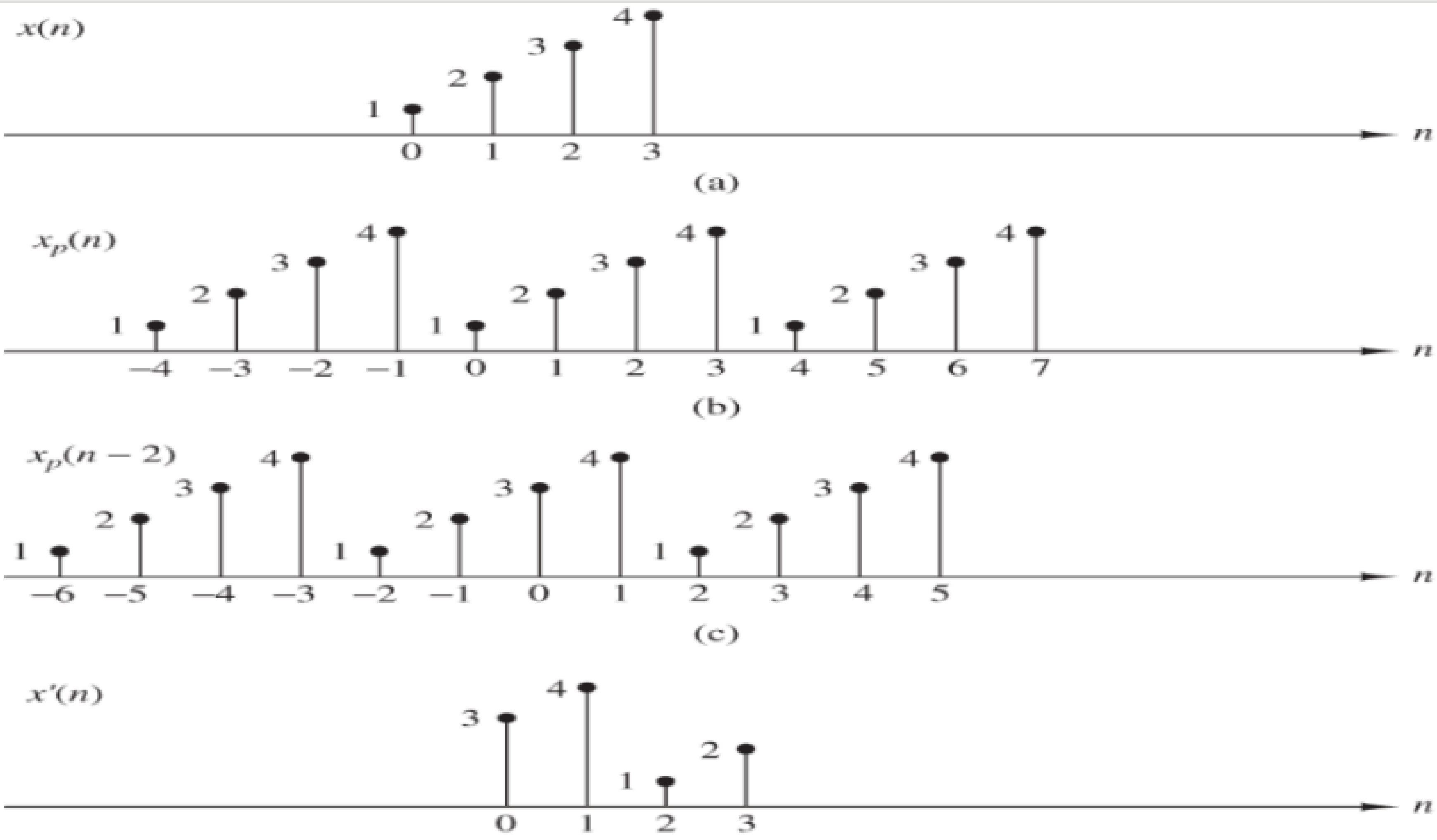
$$x'_p(n) = x_p(n-k) = \sum_{l=-\infty}^{\infty} x(n-k-lN)$$

$$x'(n) = \begin{cases} x'_p(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

x'(n) related to x(n) by a circular shift

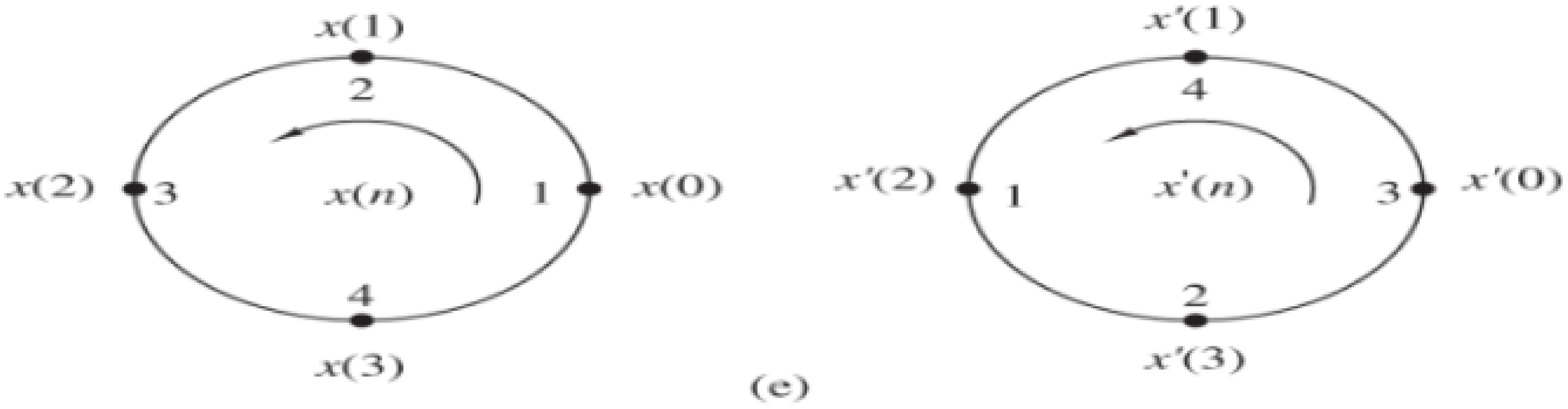
Properties of DFT

Circular Symmetries of a sequence



linear shift

Properties of DFT
Circular Symmetries of a sequence



Circular shift

Circular shift represented as the index modulo N

$$x'(n) = x(n-k, \text{modulo } N)$$
$$\equiv x((n-k))_N$$

Circular shift of an N-point sequence is equivalent to a line is shift of its periodic extension.
Counter-clockwise direction is considered the positive direction

Properties of DFT

Circular Symmetries of a sequence

Circularly even:

$$x(N-n) = x(n) \qquad 1 \leq n \leq N-1$$

The N-point sequence is circularly even if it is symmetric about the point zero on the circle

Circularly odd:

$$x(N-n) = -x(n) \qquad 1 \leq n \leq N-1$$

The N-point sequence is circularly odd if it is anti-symmetric about the point zero on the circle

Time reversal:

The time reversal of an N-point sequence is attained by r eversing its samples about the point zero on the circle Thus the seq,

$$x((-n))_N = x(N-n) \qquad 0 \leq n \leq N-1$$

It is equivalent to plotting x(n) in a clockwise direction on a circle.

Properties of DFT

Circular Symmetries of a sequence

Example: circularly even sequence

Circularly even sequence A sequence is said to be circularly even if it is symmetric about the point zero on the circle i.e.,

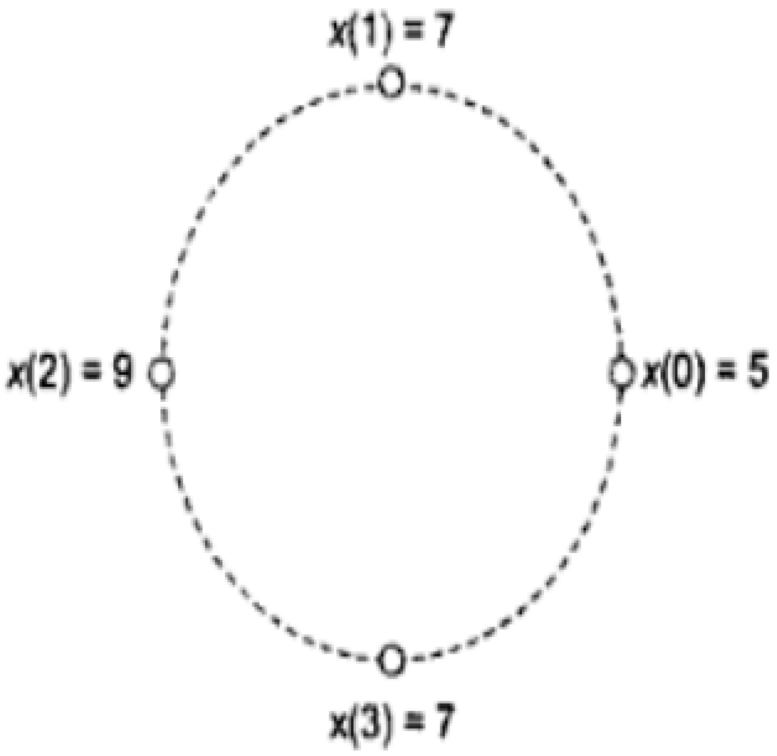
$$x(N-n) = x(n) \qquad 1 \le n \le N-1$$

Consider the sequence $x(n) = \{5, 7, 9, 7\}$

Here $\quad x(4-1) = x(1) \qquad$ i.e., $x(3) = x(1)$

$\quad x(4-2) = x(2) \qquad$ i.e., $x(2) = x(2)$

$\quad x(4-3) = x(3) \qquad$ i.e., $x(1) = x(3)$



Circularly even sequence

Properties of DFT

Circular Symmetries of a sequence

Example: circularly odd sequence

Circularly odd sequence A sequence is circularly odd if it is not symmetric about $x(0)$ on the circle i.e.,

$$x(N-n) = -x(n) \qquad 1 \le n \le N-1$$

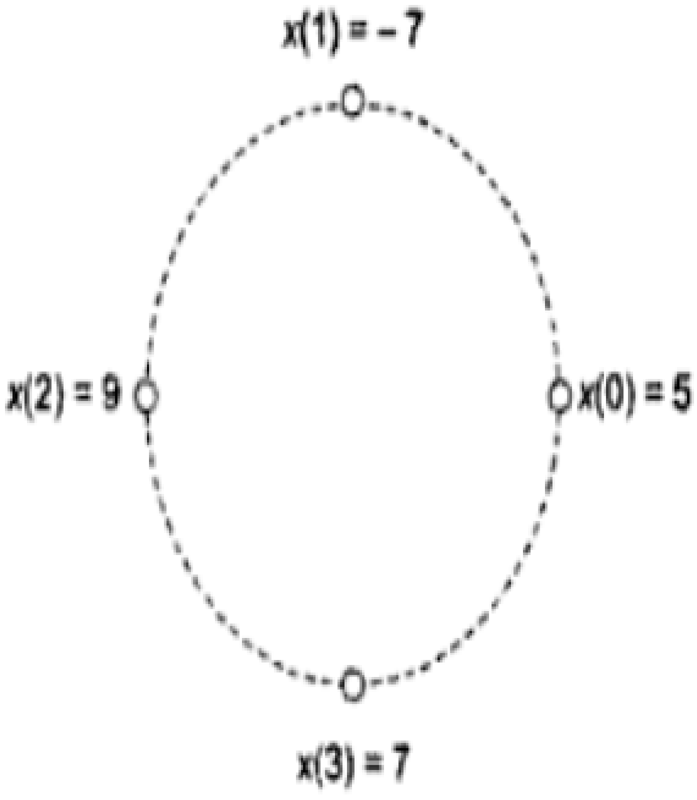
Consider the sequence $x(n) = \{5, -7, 9, 7\}$

Here $\quad x(4-1) = -x(1) \qquad$ i.e., $x(3) = -x(1)$

$\quad x(4-2) = -x(2) \qquad$ i.e. $x(2) = -x(2)$

$\quad x(4-3) = -x(3) \qquad$ i.e. $x(1) = -x(3)$

Thus the given sequence is circularly odd.



Circularly odd sequence

Properties of DFT
Circular Symmetries of a sequence

An equivalent definition of even and odd sequences for the associated periodic sequence x p (n):

$$\text{even:} \quad x_p(n) = x_p(-n) = x_p(N-n)$$

$$\text{odd:} \quad x_p(n) = -x_p(-n) = -x_p(N-n)$$

If the periodic sequence is complex valued, then

$$\text{conjugate even:} \quad x_p(n) = x_p^*(N-n)$$

$$\text{conjugate odd:} \quad x_p(n) = -x_p^*(N-n)$$

Properties of DFT

Circular Symmetries of a sequence

Hence, we decompose the sequence x p (n) as

$$x_p(n) = x_{pe}(n) + x_{po}(n)$$

$$x_{pe}(n) = \frac{1}{2}[x_p(n) + x_p^*(N-n)]$$

$$x_{po}(n) = \frac{1}{2}[x_p(n) - x_p^*(N-n)]$$

Properties of DFT
Symmetry properties of DFT

Assume that the N-point sequence x(n) and its DFT are bot h complex valued, then,

$$x(n) = x_R(n) + jx_I(n) \qquad 0 \le n \le N-1$$

$$X(k) = X_R(k) + jX_I(k) \qquad 0 \le k \le N-1$$

Substituting in the equation the expression for DFT we get

$$X_R(k) = \sum_{n=0}^{N-1} \left[x_R(n) \cos \frac{2\pi kn}{N} + x_I(n) \sin \frac{2\pi kn}{N} \right]$$

$$X_I(k) = -\sum_{n=0}^{N-1} \left[x_R(n) \sin \frac{2\pi kn}{N} - x_I(n) \cos \frac{2\pi kn}{N} \right]$$

Properties of DFT
Symmetry properties of DFT

Similarly, the expression for IDFT we get

$$x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\cos\frac{2\pi kn}{N} - X_I(k)\sin\frac{2\pi kn}{N}\right]$$

$$x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\sin\frac{2\pi kn}{N} + X_I(k)\cos\frac{2\pi kn}{N}\right]$$

Properties of DFT

Symmetry properties of DFT

Real-valued sequence

If the sequence x(n) is real, $X(N-k) = X^*(k) = X(-k)$

Real and even:

If x(n) is real and even, then $x(n) = x(N-n) \qquad 0 \le n \le N-1$

Hence the DFT reduces to
$$X(k) = \sum_{n=0}^{N-1} x(n) \cos \frac{2\pi kn}{N} \qquad 0 \le k \le N-1$$

Which is itself real valued and even . Furthermore, since X I (k) = 0, the IDFT reduces to,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cos \frac{2\pi kn}{N} \qquad 0 \le n \le N-1$$

Properties of DFT

Symmetry properties of DFT

Real and odd:

If x(n) is real and odd $$x(n) = -x(N-n) \qquad 0 \le n \le N-1$$

X R (k)=0. Hence $$X(k) = -j\sum_{n=0}^{N-1} x(n)\sin\frac{2\pi kn}{N} \qquad 0 \le k \le N-1$$

Which is purely imaginary and odd. Since X R (k)=0, the IDFT reduces to

$$x(n) = j\frac{1}{N}\sum_{k=0}^{N-1} X(k)\sin\frac{2\pi kn}{N} \qquad 0 \le n \le N-1$$

Properties of DFT

Symmetry properties of DFT

Purely Imaginary

x(n)=jx I (n). Consequently,

odd $\longrightarrow$ $$X_R(k) = \sum_{n=0}^{N-1} x_I(n) \sin \frac{2\pi kn}{N}$$

even $\longrightarrow$ $$X_I(k) = \sum_{n=0}^{N-1} x_I(n) \cos \frac{2\pi kn}{N}$$

If x I (n) is odd, X I (k) = 0 and hence X(k) is purely real.
If x I (n) is even, X R (k) = 0 and hence X(k) is purely imaginary.

Properties of DFT
Symmetry properties of DFT

Symmetry properties summarized as

$$x(n) = x_R^e(n) + x_R^o(n) + jx_I^e(n) + jx_I^o(n)$$

$$X(k) = X_R^e(k) + X_R^o(k) + jX_I^e(k) + jX_I^o(k)$$

Properties of DFT
Summary of symmetry properties

TABLE 7.1 Symmetry Properties of the DFT

N-Point Sequence $x(n)$, $0 \le n \le N-1$	N-Point DFT
$x(n)$	$X(k)$
$x^*(n)$	$X^*(N-k)$
$x^*(N-n)$	$X^*(k)$
$x_R(n)$	$X_{ce}(k) = \frac{1}{2}[X(k) + X^*(N-k)]$
$jX_I(n)$	$X_{co}(k) = \frac{1}{2}[X(k) - X^*(N-k)]$
$x_{ce}(n) = \frac{1}{2}[x(n) + x^*(N-n)]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}[x(n) - x^*(N-n)]$	$jX_I(k)$
Real Signals	
Any real signal	$X(k) = X^*(N-k)$
$x(n)$	$X_R(k) = X_R(N-k)$
	$X_I(k) = -X_I(N-k)$
	$\lvert X(k)\rvert = \lvert X(N-k)\rvert$
	$\angle X(k) = -\angle X(N-k)$
$x_{ce}(n) = \frac{1}{2}[x(n) + x(N-n)]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}[x(n) - x(N-n)]$	$jX_I(k)$

PES UNIVERSITY ONLINE

THANK YOU

Ms. Ashwini
Department of Electronics and Communication

ashwinib@pes.edu

+91 80 6666 3333
Ext 741