14/1/2? Subject Code : ~~[illegible]~~ (Theory)
~~[illegible]~~ (LAB)

16 (i)

Chapter 1 [Ch(7)] [10 hrs]
Discrete Fourier Transform (properties & applications)
(1) frequency domain sampling and reconstruction of discrete time signals.
(2) DFT as a linear transformation [Matrix representation]
(3) Relationship of DFT ~~to~~ with other transforms
(a) with aperiodic discrete-time signals FSC
(b) with Fourier transform of a .
(c) with Z-transform
(d) with ~~[illegible]~~ continuous time signals FSC
(4) properties of DFT
(5)
Chapter 2 [Ch(8)] [10 hrs]
~~(2)~~ Efficient computation of DFT : FFT
(1) Computation complexity of direct DFT
(2) Need for FFT
(3) Radix-2 FFT Algorithms & IFFT algorithms
(a) Decimation in time (DIT)
(b) Decimation in frequency (DIF)
algorithms
(4) Application of FFT in linear filtering
(a) Overlap Save
(b) Overlap Add
methods.

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Chapter 3
T1: ch(10)
T2: 3.0, 3.1, 3.2 (10 hrs)

Analog filters
(1) Introduction to filters
(2) Ideal Vs Non-Ideal filters
(3) Recursive Vs Non-Recursive filters.
(4) Design of analog filters & freq response characteristics.
  (a) Butterworth filters
  (b) Chebyshev filters
(5) Analog to Analog frequency transformation

Chapter 4
T1: ch(9) ch(10)
T2: 4.6 (10 hrs)

I FIR filters design
(1) Introduction
(2) Design of FIR filters using window functions.
  (a) Rectangular
  (b) ~~Hamming~~
  (c) Hanning
  (d) Blackman
  (e) Bartlett
  (f) Kaiser window
(3) Design of FIR filters using frequency sampling techniques
(4) FIR differentiator
(5) Hilbert ~~differentiator~~ Transformer

II Realization of FIR filters
(1) Direct form ~~I & II~~,
(2) Cascade
(3) Parallel
(4) Lattice realizations!

Chapter 5
T1: ch(9) ch(10)
T2: 5.1, 5.2, 5.3 (12 hrs)

I IIR filters from Analog filters
(1) Mapping of transfer function:-
  ~~(a) Approximation of derivatives~~
  (a) Backward Difference Method
  (b) Bilinear Transformation
  (c) Impulse Invariance transform
  (d) Matched Z - transform
(2) Verification for stability & linearity during mapping.

II Realization of IIR filters
(1) Direct form I & II
(2) Cascade
(3) Parallel
(4) Lattice realizations.

Text Books:
T1: DSP: Principles, Algorithms & applications
~~Author~~: Proakis and Manolakis
Edition: 4th (2007)

T2: Fundamentals of DSP (1986),
Author: L.C. Ludeman

References:

(1) DSP, S.K. Mitra, 2nd edition, 2004
(2) DSP, Oppenheim & Schafer, PHI, 2003

Chapter 1

The DFT: Its properties & applications

Frequency analysis of a discrete-time signal

$FT\{x(n)\} = X(\omega)$

* $X(\omega)$ is a continuous function of frequency
* $X(\omega)$ is not a computationally convenient representation of $x(n)$

∴ we take samples of the spectrum $X(\omega)$

$$\underbrace{x(n)}_{\text{time domain sequence}} \xrightarrow{FT} \underbrace{X(\omega) \xrightarrow{\text{Samples}} X(K)}_{\text{frequency domain representation}}$$

$X(k)$ or DFT is a powerful tool for performing the frequency analysis of the discrete time signals.

Frequency domain sampling: the DFT

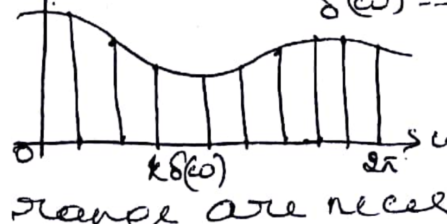
Objective: To establish a relationship b/w the sampled FT and the DFT

Frequency domain sampling and reconstruction of discrete-time signals.

Consider an aperiodic (finite energy) discrete-time signal $x(n)$ with FT

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \quad \text{—— ①}$$

∵ aperiodic finite energy signals have continuous spectra.



* $X(\omega)$ is periodic with 2π
* ∴ only the samples in the fundamental frequency range are necessary
* we take 'N' equidistant samples in the interval $0 \le \omega \le 2\pi$ with $\delta\omega = \frac{2\pi}{N}$

If we evaluate at $\omega = \frac{2\pi k}{N}$

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\left(\frac{2\pi k}{N}\right)n} \quad \text{--- (2)}$$

$$X\left(\frac{2\pi k}{N}\right) = \cdots + \sum_{n=-N}^{-1} x(n)\, e^{\frac{-j2\pi kn}{N}} + \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} + \sum_{n=N}^{2N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} + \cdots$$

$$= \sum_{l=-\infty}^{+\infty} \sum_{n=lN}^{lN+N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

Changing the limits from n to $n-lN$ & interchanging the summation we get,

$$X\left(\frac{2\pi k}{N}\right) = \sum_{l=-\infty}^{+\infty} \sum_{n=0}^{N-1} x(n-lN)\, e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} \underbrace{\sum_{l=-\infty}^{+\infty} x(n-lN)}_{x_p(n)}\, e^{\frac{-j2\pi kn}{N}}$$

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} x_p(n)\, e^{\frac{-j2\pi kn}{N}} \quad \text{--- (3)}$$

where $x_p(n) = \sum_{l=-\infty}^{+\infty} x(n-lN)$ --- (4)

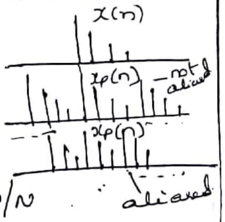
$x_p(n)$ is obtained by the periodic repetition of $x(n)$ every 'N' samples.

Expanding it in a fourier series

$$x_p(n) = \sum_{k=0}^{N-1} c_k\, e^{j2\pi kn/N} \quad \text{--- (5)} \quad n = 0 \cdots N-1$$

fourier coefficients

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi kn/N} \quad \text{--- (6)} \quad k = 0 \cdots N-1$$



Comparing (6) & (3) we get

$$c_k = \frac{1}{N} X\left(\frac{2\pi k}{N}\right) \quad \text{--- (7)}$$

Using (7) in (5) we get

$$x_p(n) = \sum_{k=0}^{N-1} \frac{1}{N} X\left(\frac{2\pi k}{N}\right) e^{j2\pi kn/N}$$

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) e^{j2\pi kn/N} \quad \text{--- (8)}$$

$n = 0 \cdots N-1$

Equation (8) provides the reconstruction of the periodic signal $x_p(n)$ from the samples of the spectrum $X(\omega)$.

However, it does not imply that we can recover $x(n)$ or $X(\omega)$ from the samples. For this we have to establish a relationship between $x(n)$ and $x_p(n)$

$x_p(n)$ is a periodic ~~repetition~~ extension of $x(n)$

$\therefore$ $x(n)$ can be recovered from $x_p(n)$, only if there is no aliasing in time domain.

i.e., if $x(n)$ is time limited to less than the period 'N' of $x_p(n)$.

we consider, $x(n)$, is non-zero in the interval $0 \le n \le L-1$

$\therefore$ when $N \ge L$

$$x(n) = x_p(n) \quad 0 \le n \le N-1$$

So, $x(n)$ can be recovered from $x_p(n)$ without ambiguity.

If $N < L$, there is time-domain aliasing, $\therefore$ it is not possible to recover $x(n)$ from $x_p(n)$.

$\therefore$ we conclude that the spectrum of an aperiodic discrete-time sequence of finite duration L, can be exactly recovered from its samples at frequency $\omega_k = \frac{2\pi k}{N}$ if $N \ge L$

$$x(n) = \begin{cases} x_p(n) & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{--- (9)}$$

finally, $X(\omega)$ can be computed using eqn (1)

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) e^{j2\pi kn/N} \quad \text{--- (10)}$$

$n = 0 \cdots N-1$

Since $x(n) = x_p(n)$

$$X(\omega) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) e^{j2\pi kn/N}\right] e^{-j\omega n} \quad (11)$$

$$X(\omega) = \sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right)\left[\frac{1}{N}\sum_{n=0}^{N-1} e^{-j\left(\omega-\frac{2\pi k}{N}\right)n}\right]$$

basic Interpolation function

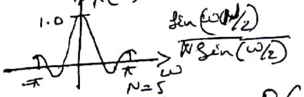
$$P(\omega) = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1}{N}\,\frac{1-e^{-j\omega N}}{1-e^{-j\omega}} = \frac{\sin\left(\frac{\omega N}{2}\right)}{N\sin\left(\frac{\omega}{2}\right)} e^{-j\omega(N-1)/2} \quad (12)$$

$\therefore$ (11) can be expressed as

$$X(\omega) = \sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) P\left(\omega - \frac{2\pi k}{N}\right) \quad N \geq L \quad (13)$$

The interpolation function $P(\omega)$ is not the familiar $\sin\theta/\theta$, but instead, it is a periodic counterpart of it, and it is due to the periodic nature of $X(\omega)$.

The phase shift in (12) reflects the fact that the signal $x(n)$ is a causal, finite-duration sequence of length N.



$P(\omega)$ has the property.

$$P\left(\frac{2\pi k}{N}\right) = \begin{cases} 1 & k=0 \\ 0 & k = 1 \cdots N-1 \end{cases} \quad (14)$$

Consequently, the function in (13) gives exactly the sample values $X\left(\frac{2\pi k}{N}\right)$ for $\omega = \frac{2\pi k}{N}$. At all other frequencies, the formula provides a properly weighted linear combination of the original ~~sequence~~ spectral samples.

The Discrete Fourier Transform

1. When $x(n)$ is infinite, the equally spaced frequency samples $X\left(\frac{2\pi}{N}k\right)$ do not represent the original signal $x(n)$.
   Instead $X\left(\frac{2\pi k}{N}\right)$ represents $x_p(n)$ an aliased version of $x(n)$
   $$x_p(n) = \sum_{l=-\infty}^{+\infty} x(n-lN)$$
2. When $x(n)$ is of finite duration i.e., $L \leq N$, then $x_p(n)$ is a periodic repetition of $x(n)$
   $\therefore X\left(\frac{2\pi k}{N}\right)$ represents $x(n)$,
   $$x_p(n) = \begin{cases} x(n) & 0 \leq n \leq L-1 \\ 0 & L \leq n \leq N-1 \end{cases}$$
3. $x(n) \equiv x_p(n)$ over a single period, padded with $N-L$ zeros.
   The original sequence can be obtained from the (8) equation by means of $\left\{X\left(\frac{2\pi k}{N}\right)\right\}$
4. The zero padding gives no additional information about the spectrum of the sequence $x(n)$.
5. 'L' # of equidistant samples of $X(\omega)$ are sufficient to reconstruct $X(\omega)$
6. However, zero padding the sequence $x(n)$ with $N-L$ zeros and finding N-point DFT gives a "better display" of FT $X(\omega)$.
   i.e., $X(\omega) = \sum_{n=0}^{L-1} x(n) e^{-j\omega n} \quad 0 \leq \omega \leq 2\pi$

$$X(k) \equiv X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{L-1} x(n) e^{-j\frac{2\pi kn}{N}}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad (1)$$

$k = 0 \cdots N-1$

this equation transforms a sequence $x(n)$ of length $L \le N$ into a sequence of frequency samples $\{X(K)\}$ of length N.

Why the name DFT?
Since the frequency samples are obtained by evaluating fourier transform $X(\omega)$ at a set of N equally spaced discrete frequencies. Equation ① is called the DFT.
$x(n)$ can be recovered from the frequency samples using

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{j2\pi kn/N} \quad n = 0 \,\dots\, N-1 \qquad (2)$$

This is called IDFT.

The DFT as a Linear Transformation

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad (3)$$

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, W_N^{-kn} \qquad (4)$$

$$W_N = e^{-j2\pi/N} \qquad (5)$$

Computation of DFT involves
N - complex multiplications
$N-1$ - complex additions
$\therefore$ an N-point DFT requires
N^2 — complex multiplications
$N(N-1)$ - complex additions

DFT and IDFT as linear transforms on sequences $x(n)$ and $X(K)$ respectively.
Consider an N-pt vector x_N of the signal $x(n)$ $n=0 \,.\, N-1$

Define an N-point vector X_N of the frequency samples $n = 0 \dots N-1$

and an $N \times N$ matrix W_N as

$$x_N = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} \quad X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} \quad W_N = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N & \cdots & W_N^{N-1} \\ \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \qquad (6)$$

So, the N-point DFT can be expressed in the matrix form as

$$X_N = W_N x_N \qquad (7)$$

where W_N is the matrix of the linear transformation. W_N is a symmetric matrix.
Assuming that inverse of W_N exists eqn ⑦ can be inverted by multiplying both sides by W_N^{-1}

$$x_N = W_N^{-1} X_N \qquad (8)$$

According to definition of IDFT

$$x_N = \frac{1}{N} W_N^{*} X_N \qquad (9)$$

Comparing ⑧ and ⑨ we get

$$W_N^{-1} = \frac{1}{N} W_N^{*} \qquad (10)$$

which in turn implies

$$W_N^{*} W_N = N I_N$$

where I_N is the identity matrix $(N \times N)$
$\therefore$ The matrix W_N in the transformation is an orthogonal (unitary) matrix.
& its inverse exists and is given as $\frac{W_N^{*}}{N}$.

Example
① Compute a 4-pt DFT of the sequence $x(n) = (0,1,2,3)$

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$$W_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = [6, -2+2j, -2, -2-2j]$$

$$X_4 = [6, -2+2j, -2, -2-2j]$$

② Compute 4-pt IDFT of ~~the~~
$X_4 = [6, -2+2j, -2, -2-2j]$

$$x_4 = \frac{1}{N} W_N^* X_4$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix} \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$x_0 = 6 - 2 + 2j - 2 - 2 - 2j = 0$

$x_1 = 1 \times 6 + j(-2+2j) + (-1)(-2) + (-j)(-2-2j)$

$= 6 - 2j + 2j^2 + 2 + 2j + 2j^2$

$= 8 - 2 - 2 = 4/4 = 1$

III^{ly} $x_4 = [0, 1, 2, 3]$

③ A finite duration sequence of length 'L' is given as $x(n) = \begin{cases} 1 & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$

find the N-point DFT of this sequence for $N \ge L$.

~~$X(K) = \sum_{n=0}$~~

Relationship of the DFT to other transforms

① To the fourier Series Co-efficients of a periodic sequence

② To Fourier transform of an aperiodic sequence.

③ to Z-transform

④ Relationship to fourier series co-efficients of a continuous time signal.

① Relationship to fourier Series Co-efficients of a periodic sequence.

A periodic sequence with a fundamental period N

$$x_p(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N} \quad -\infty \le n \le +\infty \quad \text{—①}$$

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N} \quad k = 0 \ldots N-1 \quad \text{—②}$$

Comparing ① & ② with DFT and IDFT equations.

Equⁿ ② is similar to the DFT equⁿ where $X(K) = N C_k$ for $x(n) = x_p(n)$ $0 \le n \le N-1$

Thus, the N-point DFT provides the exact line spectrum of a periodic sequence with fundamental period N.

② Relationship to the fourier transform of an aperiodic sequence

we know that the fourier transform of an aperiodic sequence when sampled at N equally spaced intervals gives

$$X(K) = X\left(\frac{2\pi K}{N}\right) = X(\omega)\Big|_{\omega = \frac{2\pi k}{N}}$$

$$= \sum_{n=-\infty}^{+\infty} x(n) e^{-j2\pi kn/N}$$

$k = 0 \ldots N-1$

there are the DFT coefficients of a periodic sequence of period N, given by

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

$x_p(n)$ is obtained by aliasing $x(n)$ over the interval $0 \le n \le N-1$

$$\hat{x}(n) = \begin{cases} x_p(n) & 0 \le n \le N-1 \\ 0 & \text{Otherwise} \end{cases}$$

$\hat{x}(n) = x(n)$ only when $L \le N$, $0 \le n \le N-1$

Only in this case IDFT of $\{X(k)\}$ yield the original sequence $x(n)$

③ Relationship to Z-transform

Consider $x(n)$ having the Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \text{——— ①}$$

If $X(z)$ is sampled at equally spaced intervals on a unit circle

$z_k = e^{j2\pi k/N} \quad k = 0 \cdots N-1$

$$X(k) \equiv X(z)\Big|_{z = e^{j2\pi k/N}} = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi kn/N} \quad \text{——— ②}$$

Expressing $X(z)$ as a function of $X(k)$ where $x(n)$ has finite duration of length N.

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}\right] z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left(e^{j2\pi k/N} z^{-1}\right)^n$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \frac{1-\left(e^{j2\pi k/N} z^{-1}\right)^N}{1 - e^{j2\pi k/N} z^{-1}}$$

$$= \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{j2\pi k/N} z^{-1}} \quad \text{——— ③}$$

When evaluated on a unit circle ③ yields the FT of the finite duration sequence in terms of its DFT.

$$X(\omega) = \frac{1 - e^{-j\left(\frac{2\pi k}{N}\right)N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{j(2\pi k/N)} e^{-j\frac{2\pi k}{N}}}$$

$$X(\omega) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{-j(\omega - 2\pi k/N)}} \quad \text{——— ④}$$

This expression for FT is a polynomial interpolation formula for $X(\omega)$ expressed in terms of values $\{X(k)\}$ of the polynomial at a set of equally spaced discrete frequencies $\omega_k = 2\pi k/N$, $k = 0 \cdots N-1$

④ Relationship ~~between~~ to fourier Series coefficients of a continuous time signal.

If $x_a(t)$ is a continuous time signal with a fundamental period $T_p = \frac{1}{F_0}$

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k t F_0} \quad \text{——— ①}$$

$$x(n) = x_a(nT) = \sum_{k=-\infty}^{+\infty} c_k e^{j2\pi k(nT)F_0}$$

If $x_a(t)$ is sampled at an uniform rate $F_s = \frac{N}{T_p} = \frac{1}{T}$

$$= \sum_{k=0}^{N-1} \left[\sum_{l=-\infty}^{+\infty} c_{(k-lN)}\right] e^{j2\pi kn/N} \quad \text{——— ②}$$

Equation ② is in the form of IDFT formula

where, $X(k) = N \sum_{l=-\infty}^{\infty} c_{k-lN} \equiv N\tilde{c}_k$

$$\tilde{c}_k = \sum_{l=-\infty}^{\infty} c_{k-lN}$$

Thus, the $\tilde{C}_k$ sequence is an aliased version of the sequence $\{C_k\}$.

Problems

① $x(n) = (1, 2, 1, 0)$ find DFT & from that IDFT of the sequence

Ans: $X(k) = (4, -j2, 0, j2)$

② compute the inverse DFT of the sequence $X(k) = (2, 1+j, 0, 1-j)$ $\quad W_N^{-kn} = [W_N^{kn}]^*$

Ans: $(1, 0, 0, 1)$

③ find 8-point DFT of the following sequence $x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$

$W_8^0 = 1$

$W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$W_8^2 = -j$

$W_8^3 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$W_8^4 = -W_8^0 = -1$

$W_8^5 = -W_8^1 = -\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$

$W_8^6 = -W_8^2 = -(-j)$

$W_8^7 = -(W_8^3) = -\left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$

$$X(k) = DFT\{x(n)\} = \sum_{n=0}^{7} x(n)\, W_8^{kn}$$

$X(k) = \{4, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$

Properties of DFT

DFT $\quad X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad k=0 \ldots N-1$ —①

$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn} \quad n=0 \ldots N-1$ —②

$W_N = e^{-j2\pi/N}$

The notation given below denotes the N-point DFT

$x(n) \xleftrightarrow[N]{DFT} X(k)$

Periodicity, Linearity and Symmetry

Periodicity: If $x(n)$ and $X(k)$ are an N-point DFT pair

then $x(n+N) = x(n) \;\forall n$ —③

$X(k+N) = X(k) \;\forall k$ —④

Linearity: If

$x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$

$x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$

Then for all real-valued or complex valued constants a_1 and a_2

$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[N]{DFT} a_1 X_1(k) + a_2 X_2(k)$ —⑤

Circular symmetries of a sequence

It is a known fact that the N-point DFT of a finite duration sequence $x(n)$ of length $L \le N$ is equivalent to the N-point DFT of a periodic sequence $x_p(n)$ of period N, which is a periodic ~~extentention~~ extension of $x(n)$

i.e., $x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$ —⑥

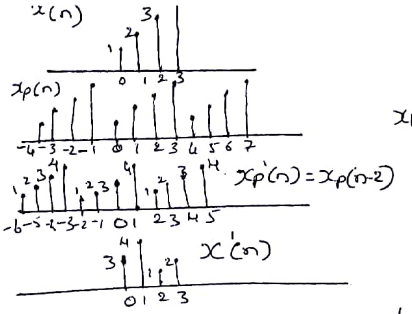
Now, shift $x_p(n)$ by k units to the right, then

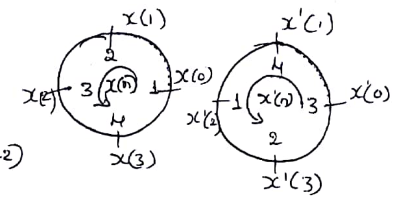
$x_p'(n) = x_p(n-k) = \sum_{l=-\infty}^{\infty} x(n-k-lN)$ —⑦

The finite duration sequence

$x'(n) = \begin{cases} x_p'(n) & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$ —⑧

$x'(n)$ is related to $x(n)$, the original signal by a circular shift as shown in the figure below





In general, the circular shift can be represented as the index modulo N.

i.e., $x'(n) = x(n-k, \text{modulo } N)$

$\equiv x((n-k))_N$

Example: If $k=2$ & $N=4$

$x'(n) = x((n-2))_4$

$x'(0) = x((-2))_4 = x(4-2) = x(2)$

$x'(1) = x((1-2))_4 = x((-1))_4 = x(4-1) = x(3)$

$x'(2) = x((2-2))_4 = x((0))_4 = x(0)$

$x'(3) = x((3-2))_4 = x((1))_4 = x(1)$

$\therefore$ $x'(n) = x(n)$ shifted ~~clock~~ by two units

The counter-clockwise direction has been chosen as the +ve direction.

$\therefore$ Circular shift of an N-point sequence is equivalent to a linear shift of its periodic extension & vice versa.

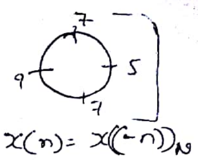
The inherent periodicity resulting from the arrangement of N-point sequence on the circumference of a circle gives a different definition of even and odd symmetry and time reversal of a sequence.

Circularly Even:

An N-point sequence is called circularly even if it is symmetric about the point zero on the circle.

$\Rightarrow x(N-n) = x(n) \quad 1 \le n \le N-1$

Example: if $x(n) = \{5, 7, 9, 7\}$



$x(4-1) = x(1)$ i.e., $x(3) = x(1)$

$x(4-2) = x(2)$ i.e., $x(2) = x(2)$

$x(4-3) = x(3)$ i.e., $x(1) = x(3)$

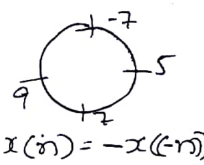
$x(n) = x((-n))_N$

Circularly Odd

An N-point sequence is called circularly odd if it is anti-symmetric about the point zero on the circle.

$\Rightarrow x(N-n) = -x(n) \quad 1 \le n \le N-1$

Example: if $x(n) = \{5, -7, 9, 7\}$



$x(4-1) = -x(1)$ i.e., $x(3) = -x(1)$

$x(4-2) = -x(2)$ i.e., $x(2) = -x(2)$

$x(4-3) = -x(3)$ i.e., $x(1) = -x(3)$

$x(n) = -x((-n))_N$

Time reversal of an N-point sequence

It is obtained by reversing its samples about the point zero on the circle.

$\therefore x((-n))_N = x(N-n) \quad 0 \le n \le N-1$

This is equivalent to plotting $x(n)$ in a clockwise direction on a circle if counter clockwise is considered as a positive direction.

Equivalent definition of even and odd sequences for the periodic sequence $x_p(n)$ is given as follows.

even: $x_p(n) = x_p(N-n) = x_p(-n)$

odd: $x_p(n) = -x_p(-n) = -x_p(N-n)$

If the periodic sequence is complex valued then,

Conjugate even: $x_p(n) = x_p^*(N-n)$

Conjugate odd: $x_p(n) = -x_p^*(N-n)$

∴ The periodic sequence $x(n)$ can be decomposed into

$$x_p(n) = x_{pe}(n) + x_{po}(n)$$

$$x_{pe}(n) = \frac{1}{2}\left[x_p(n) + x_p^*(N-n)\right]$$

$$x_{po}(n) = \frac{1}{2}\left[x_p(n) - x_p^*(N-n)\right]$$

Symmetry Properties of DFT

Let us assume that the N-point sequence $x(n)$ and its DFT are both complex valued.

Then -

$$x(n) = x_R(n) + j\,x_I(n) \quad 0 \le n \le N-1$$

$$X(k) = X_R(k) + j\,X_I(k) \quad 0 \le k \le N-1$$

$$\therefore \; X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} \left[x_R(n) + j x_I(n)\right]\left[\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$X_R(k) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\frac{2\pi kn}{N} + x_I(n)\sin\left(\frac{2\pi kn}{N}\right)\right] \quad \text{--- (A)}$$

$$X_I(k) = -\sum_{n=0}^{N-1}\left[x_R(n)\sin\left(\frac{2\pi kn}{N}\right) - x_I(n)\cos\left(\frac{2\pi kn}{N}\right)\right] \quad \text{--- (B)}$$

$|||^{ly}$ using $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{+j2\pi kn/N}$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k) + jX_I(k)\right]\left[\cos\left(\frac{2\pi kn}{N}\right) + j\sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\cos\left(\frac{2\pi kn}{N}\right) - X_I(k)\sin\left(\frac{2\pi kn}{N}\right)\right] \quad \text{--- (C)}$$

$$x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\sin\left(\frac{2\pi kn}{N}\right) + X_I(k)\cos\left(\frac{2\pi kn}{N}\right)\right] \quad \text{--- (D)}$$

Case(i): $x(n)$ is real ~~and even~~

$$X(N-k) = X(-k) = X^*(k)$$

Consequently, $|X(N-k)| = |X(k)|$ and $\angle X(N-k) = -\angle X(k)$

furthermore,

$$x_I(n) = 0$$

$x(n)$ is real and even $\quad x(n) = x_R(n)$

$$x(n) = x(N-n) \quad 0 \le n \le N-1$$

$X_I(k) = 0$,

$$\therefore \; X(k) = \sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi kn}{N}\right) \quad 0 \le k \le N-1$$

which is real valued and even.

Furthermore, since $X_I(k) = 0$, $\quad X(k) = X_R(k)$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\cos\left(\frac{2\pi kn}{N}\right)$$

$x(n)$ is real and odd $\quad x(n) = x_R(n)$

$$x(n) = -x(N-n) \quad 0 \le n \le N-1$$

$X_R(k) = 0$

$$\therefore \; X(k) = -j\sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi kn}{N}\right) \quad 0 \le k \le N-1$$

which is purely imaginary & odd.

Since $X_R(k) = 0$,

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\sin\left(\frac{2\pi kn}{N}\right) \quad 0 \le n \le N-1$$

$x(n)$ is purely imaginary

$$x(n) = j x_I(n)$$

$$X_R(k) = \sum_{n=0}^{N-1} x_I(n)\sin\frac{2\pi kn}{N}$$

$$X_I(k) = \sum_{n=0}^{N-1} x_I(n)\cos\frac{2\pi kn}{N}$$

$X_R(k)$ is odd and $X_I(k)$ is even.

$X_R(k)$ is odd and $X_I(k)$ is even & $X(k)$ is purely imaginary

if $x_I(n)$ is even, $X_R(k) = 0$ & $X(k)$ is purely real

if $x_I(n)$ is odd, $X_I(k) = 0$ & $X(k)$ is purely real

The symmetry properties discussed above can be summarized as follows:

$$x(n) = x_R^e(n) + x_R^o(n) + jx_I^e(n) + jx_I^o(n)$$

$$X(k) = X_R^e(k) + X_R^o(k) + jX_I^e(k) + jX_I^o(k)$$

for example: The DFT of the sequence

$$x_{pe}(n) = \frac{1}{2}\left[x_p(n) + x_p^*(N-n)\right]$$

is

$$X_R(k) = X_R^e(k) + X_R^o(k)$$

Symmetry Properties of DFT

N-Point sequence $x(n)$, $0 \le n \le N-1$	N-Point DFT
$x(n) \xleftrightarrow[N]{DFT}$	$X(k)$

$$x^*(n) \xleftrightarrow[N]{DFT} X^*(N-k)$$

$$x^*(N-n) \xleftrightarrow[N]{DFT} X^*(k)$$

$$x_R(n) \xleftrightarrow[N]{DFT} X_{ce}(k) = \frac{1}{2}\left[X(k) + X^*(N-k)\right]$$

$$j\,x_I(n) \xleftrightarrow[N]{DFT} X_{co}(k) = \frac{1}{2}\left[X(k) - X^*(N-k)\right]$$

$$x_{ce}(n) = \frac{1}{2}\left[x(n) + x^*(N-n)\right] \xleftrightarrow[N]{DFT} X_R(k)$$

$$x_{co}(n) = \frac{1}{2}\left[x(n) - x^*(N-n)\right] \xleftrightarrow[N]{DFT} jX_I(k)$$

Real Signals

Any real signal $x(n)$:

$$X(k) = X^*(N-k)$$
$$X_R(k) = X_R(N-k)$$
$$X_I(k) = -X_I(N-k)$$
$$|X(k)| = |X(N-k)|$$
$$\angle X(k) = -\angle X(N-k)$$

$$x_{ce}(n) = \frac{1}{2}\left[x(n) + x(N-n)\right] \xleftrightarrow[N]{DFT} X_R(k)$$

$$x_{co}(n) = \frac{1}{2}\left[x(n) - x(N-n)\right] \xleftrightarrow[N]{DFT} jX_I(k)$$

Multiplication of two DFTs and Circular Convolution

If $x_1(n)$ and $x_2(n)$ are two finite duration sequences of length N.

Their DFTs are;

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi kn/N} \quad k = 0 \cdots N-1$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n)\, e^{-j2\pi kn/N} \quad k = 0 \cdots N-1$$

Multiplying the two DFT's

$$X_3(k) = X_1(k)\, X_2(k) \quad k = 0 \cdots N-1$$

$$x_3(m) = \text{IDFT}\{X_3(k)\}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)\, e^{j2\pi km/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\, X_2(k)\, e^{j2\pi km/N}.$$

Substituting for $X_1(k)$ & $X_2(k)$ we get

$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi kn/N}\right]\left[\sum_{l=0}^{N-1} x_2(l)\, e^{-j2\pi kl/N}\right] e^{j2\pi km/N}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l)\left[\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right]$$

The inner sum in brackets has the form

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & a = 1 \\ \dfrac{1-a^N}{1-a} & a \neq 1 \end{cases}$$

where $a = e^{j2\pi(m-n-l)/N}$

We observe that $a = 1$, when $m-n-l$ is a multiple of N. On the other hand $a^N = 1$, for any value of $a \neq 0$.

$$\Rightarrow \sum_{k=0}^{N-1} a^k = \begin{cases} N & , l = m-n+pN = ((m-n))_N, \ p \text{ an integer} \\ 0 & \text{otherwise.} \end{cases}$$

$$\therefore\ x_3(m) = \frac{1}{\cancel{N}}\sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N\, (\cancel{N})$$

$$\boxed{x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N} \quad m = 0 \cdots N-1$$

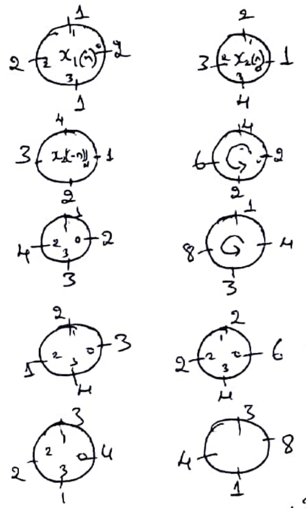
$\therefore$ Multiplication of the DFT's of two sequences is equivalent to the circular convolution of the two sequences in time domain.

Example

$x_1(n) = \{2, 1, 2, 1\}$

$x_2(n) = \{1, 2, 3, 4\}$

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N \qquad m = 0 \ldots N-1$$



$= 14$

$= 16$

$= 14$

$= 16$

$\therefore \; x_3(m) = \{14, 16, 14, 16\}$

Using DFT and IDFT

$X_1(k) = 2 + W_4^{k} + 2W_4^{2k} + W_4^{3k}$

$X_2(k) = 1 + 2W_4^{k} + 3W_4^{2k} + 4W_4^{3k}$

$X_3(k) = X_1(k) \cdot X_2(k)$

$= (6, 0, 2, 0)\,(10, -2+j2, -2, -2-j2)$

$= (60, 0, -4, 0)$

$$\text{IDFT}\{X_3(k)\} = x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)\, e^{j2\pi kn/N}$$

$= \{14, 16, 14, 16\}.$

using

$\text{IDFT}\{W_N^{kn_0}\} = \delta(n-n_0)$

Circular convolution using matrices.

$$\begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix} = \begin{bmatrix} x_2(0) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+4+6+2 \\ 4+1+8+3 \\ 6+2+2+4 \\ 8+3+4+1 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 14 \\ 16 \end{bmatrix}$$

① find the DFT of $x(n) = \delta(n-n_0)$

$$X(k) = \sum_{n=0}^{N-1} \delta(n-n_0)\, W_N^{kn}$$

Applying the sifting property

$X(k) = W_N^{kn}\big|_{n=n_0} = W_N^{kn_0}$

② find N-pt DFT of $x(n) = 1 \quad 0 \le n \le N-1$

$$X(k) = \sum_{n=0}^{N-1} W_N^{kn} = N\delta(k) = \begin{cases} N & k=0 \\ 0 & k \ne 0 \end{cases}$$

$$\sum_{n=0}^{N-1} a^n = \begin{cases} N & a = 1 \\ \dfrac{1-a^N}{1-a} & a \ne 1 \end{cases}$$

$$X(k) = \begin{cases} N & k=0 \\ 0 & k\ne 0 \end{cases} = N\delta(k)$$

③ find the DFT of $x(n) = \delta(n)$

$$X(k) = \sum_{n=0}^{N-1} \delta(n)\, W_N^{kn}$$

Applying the sifting property

$X(k) = W_N^{kn}\big|_{n=0} = W_N^{0} = 1.$

Additional Properties of DFT

① Time reversal of a sequence

$$x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} X((-k))_N = X(N-k)$$

Proof: $\text{DFT}\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)\, e^{-j2\pi kn/N}$

changing n to m & $N-n = m \;\therefore\; n = N-m$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi k(N-m)/N} \qquad \because\; e^{-j2\pi kN/N} = 1$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{j2\pi km/N}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi(N-k)m/N}$$

$= X(N-k) = X((-k))_N \qquad 0 \le k \le N-1$

② Circular time shift

$$x((n-l))_N \xleftrightarrow[N]{DFT} X(k)\, e^{-j2\pi kl/N}$$

Proof:

$$DFT\{x((n-l))_N\} = \sum_{n=0}^{N-1} x((n-l))_N W_N^{kn}$$

$$= \sum_{n=0}^{l-1} x(N+n-l) W_N^{kn} + \sum_{n=l}^{N-1} x(n-l) W_N^{kn}$$

Changing the limits from n to m

Let $N+n-l=m$	Let $n-l=m \Rightarrow n=m+l$
$n=m+l-N$	for $n=l$
for $n=0$;	$m=0$
$m=N-l$	for $n=N-1$
for $n=l-1$	$m=N-l-1$
$m=N-1$	

$$= \sum_{m=N-l}^{N-1} x(m) W_N^{k(m+l-N)} + \sum_{m=0}^{N-l-1} x(m) W_N^{k(m+l)}$$

Since $W_N^{kN}=1$

$$= X(k) W_N^{kl}$$

③ Circular frequency Shift

$$x(n) e^{+j2\pi ln/N} \xleftrightarrow[N]{DFT} X((k-l))_N$$

$$DFT\{x(n) e^{+j2\pi ln/N}\} = \sum_{n=0}^{N-1} x(n) e^{j2\pi ln/N} e^{-j2\pi kn/N}.$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi(k-l)n/N}$$

$$= X((k-l))_N$$

④ Circular Convolution

$$DFT\{h(n) ⓝ x(n)\} = H(k) X(k)$$

$$DFT\{x(n) ⓝ h(n)\} = DFT\left\{\sum_{l=0}^{N-1} x(l) h((n-l))_N\right\}$$

$$= \sum_{l=0}^{N-1} x(l)\, DFT\{h((n-l))_N\}$$

$$= \sum_{l=0}^{N-1} x(l) \underbrace{W_N^{kl}}\, H(k)$$

$$= X(k) H(k)$$

⑤ Multiplication in Time

$$x(n) y(n) \xleftrightarrow[N]{DFT} \frac{1}{N}[X(k) ⓝ Y(k)]$$

$$DFT\{x(n)y(n)\} = DFT\left\{\frac{1}{N}\sum_{l=0}^{N-1} X(l) e^{j2\pi ln/N} \cdot y(n)\right\}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} X(l) \left[DFT\{e^{j2\pi ln/N} \cdot y(n)\}\right]$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} X(l)\, Y((k-l))_N$$

$$= \frac{1}{N}[X(k) ⓝ Y(k)]$$

⑥ Innerproduct [Parseval's theorem]

$$\sum_{n=0}^{N-1} x^*(n) y(n) = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k) Y(k)$$

$$= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X^*(k) e^{-j2\pi kn/N}\right] y(n)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X^*(k) \underbrace{\sum_{n=0}^{N-1} y(n) e^{-j2\pi kn/N}}_{Y(k)}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X^*(k) Y(k)$$

$$x^*((-n))_N = x^*(N-n) \xleftrightarrow{DFT} X^*(k)$$

Proof: $IDFT\{X^*(k)\}$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X^*(k) e^{+j2\pi kn/N}$$

$$= \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{-j2\pi kn/N}\right]^* = \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j2\pi k(N-n)/N}\right]^*$$

$$= [x(N-n)]^* = x^*(N-n)$$

⑦ Complex Conjugate properties

$$x^*(n) \xleftrightarrow[N]{DFT} X^*((-k))_N = X^*(N-k)$$

$$DFT\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n) e^{-j2\pi kn/N}$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{j2\pi kn/N}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{-j2\pi(N-k)n/N}\right]^*$$

$$= [X(N-k)]^* = X^*(N-k) = X^*((-k))_N$$

Examples

① Let $x(n)$ be a finite length sequence with $X(k) = (0, 1+j, 1, 1-j)$. Using the properties of DFT, find DFTs of the following sequences

(a) $x_1(n) = e^{j\pi/2 n} x(n)$

(b) $x_2(n) = \cos(\pi/2\, n)\, x(n)$

(c) $x_3(n) = x((n-1))_4$

(d) $x_4(n) = (0, 0, 1, 0) \text{ (N)}_4\, x(n)$

(e) $x_5(n) = x((-n))_4$

Ans

(a) $\text{DFT}\{x_1(n)\} = \text{DFT}\{e^{j\pi/2 n} x(n)\} = \text{DFT}\{e^{j\frac{2\pi}{4}n} x(n)\}$

$= X((k-1))_4 = \{1-j, 0, 1+j, 1\}$

(b) $\text{DFT}\{x_2(n)\} = \text{DFT}\{\tfrac{1}{2}[e^{j\pi/2 n} + e^{-j\pi/2 n}]\, x(n)\}$

$= \text{DFT}\{\tfrac{1}{2}[x(n)\, e^{j2\pi/4 n} + x(n)\, e^{\frac{-j2\pi n}{4}}]\}$

$\boxed{\text{DFT}\{e^{j2\pi l n/N} x(n)\} = X((k-l))_N}$

$= \tfrac{1}{2}[X((k-1))_4 + X((k+1))_4]$

$= \tfrac{1}{2}[(1-j, 0, 1+j, 1) + (1+j, 1, 1-j, 0)]$

$= (1, 0.5, 1, 0.5)$

(c) $\text{DFT}\{x_3(n)\} = \text{DFT}\{x((n-1))_4\}$

$= X(k)\, W_N^{k(1)} = [0, 1+j, 1, 1-j]\, W_4^k$

$W_4^0 = 1, \quad W_4^1 = -j, \quad W_4^2 = -1, \quad W_4^3 = +j$

$= [0, 1+j, 1, 1-j]\,[1, -j, -1, +j]$

$= [0, (1+j)(-j), -1, (1-j)j]$

$X_3(k) = [0, 1-j, -1, 1+j]$

(d) $\text{DFT}\{(0, 0, 1, 0) \text{ (N)}_4\, x(n)\}$

$= \text{DFT}\{\delta(n-2) \text{ (N)}_4\, x(n)\}$

$= \text{DFT}\{x((n-2))_4\} = W_4^{2k} X(k) = (-1)^k X(k)$

$= [0, 1+j, 1, 1-j]\,[1, -1, 1, -1]$

$= [0, -1-j, 1, -1+j]$

(e) $\text{DFT}\{x((-n))_4\} = X((-k))_N = X(N-k)$

$= (0, 1-j, 1, 1+j)$

Example (2):

Compute DFT of the following sequences $x = [1, 1, 1, 1, 1, 1, 1, 1]$ & $h = \cos(0.25\pi n)$ and find DFT of $y(n) = x(n) \cdot h(n)$.

Ans $X(k) = \sum_{n=0}^{7} (W_8^{k})^n = \begin{cases} 8 & k=0 \\ 0 & k \neq 0 \end{cases}$

$X(k) = \{8, 0, 0, 0, 0, 0, 0, 0\}$

$H(k) = \sum_{n=0}^{7} \cos(0.25\pi n)\, W_8^{nk}\, e^{-j\frac{2\pi}{8}nk}$

$= \tfrac{1}{2} \sum_{n=0}^{7} (e^{j0.25\pi n} + e^{-j0.25\pi n})\, e^{-j\frac{2\pi}{8}nk}$

$= \tfrac{1}{2} \sum_{n=0}^{7} [e^{j0.25\pi n} e^{-j\frac{2\pi}{8}nk} + e^{-j0.25\pi n} e^{-j\frac{2\pi}{8}nk}]$

$= \tfrac{1}{2} \sum_{n=0}^{7} [e^{-j(k-1)\pi/4\, n} + e^{-j(k+1)\pi/4\, n}]$

$= \tfrac{1}{2} \sum_{n=0}^{7} [W_8^{(k-1)n} + W_8^{(k+1)n}]$

$\boxed{\text{DFT: } \sum_{n=0}^{N-1} W_N^{(k-k_0)n} = N\,\delta(k-k_0)}$

$= \tfrac{1}{2}[8\,\delta(k-1) + 8\,\delta(k+1)]$

$= 4\,\delta(k-1) + 4\,\delta(k+1)$

$k = +1 \Rightarrow 4 \qquad k = -1 \Rightarrow (8-1) = 7$

$H(k) = (0, 4, 0, 0, 0, 0, 0, 4)$

$\text{DFT}\{y(n)\} = \text{DFT}\{x(n) \cdot h(n)\}$

$= \frac{1}{N}[X(k) \text{ (N)}\, H(k)]$

$= \frac{1}{8}\left[\sum_{l=0}^{N-1} X(l)\, H((k-l))_8\right]$

$= \frac{1}{8}[0, 32, 0, 0, 0, 0, 0, 32]$

$Y(k) = [0, 4, 0, 0, 0, 0, 0, 4]$

Example 3:

Two finite sequences 'h' & 'x' have the following DFTs

$X(k) = DFT\{x\} = [1, -2, 1, -2]$

$H(k) = DFT\{h\} = [1, j, 1, -j]$

Let 'y' be a 4-point sequence, defined as the circular convolution of x & h,

$y = x \, (N) \, h$.

find x(n) & h(n) & $y(n) = x \, (N) \, h$ in time domain & its equivalent in frequency domain.

$x(n) = (-0.5, 0, 1.5, 0)$

$h(n) = (0.5, -0.5, 0.5, 0.5)$

$y(n) = x(n) \, (N) \, h(n) = (0.5, 1, 0.5, -1)$

$ifft(H(k) \cdot X(k)) = ifft(1, -2j, 1, +2j)$

$= (0.5, 1, 0.5, -1)$

~~Example 4~~:

Example 4:

~~Prove~~ When an N-pt sequence x(n) is real, prove that

$X(k) = X^*(N-k)$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$X^*(k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-kn}$$

x(n) is real $\therefore x^*(n) = x(n)$

$$\therefore X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{-kn} W_N^{Nn}$$

$$X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{(N-k)n}$$

$X^*(k) = X(N-k)$

or

$X(k) = X^*(N-k)$

Example 5

Let X(k) be a 14-point DFT of a length 14 real sequence x(n). The first 8 samples of X(k) are given by $X(0)=12$; $X(1)=-1+j3$; $X(2)=3+j4$; $X(3)=1-j5$; $X(4)=-2+j2$; $X(5)=6+j3$; $X(6)=-2-j3$; $X(7)=10$. Determine the remaining samples of X(k).

$X(k) = X^*(N-k)$

$X(8) = X^*(14-8) = X^*(6) = -2+j3$

lly $X(9) = 6-j3$, $X(10) = -2-j2$, $X(11) = 1+j5$,

$X(12) = 3-j4$, $X(13) = -1-j3$.

Example 6

The even samples of the 11-point DFT of a length-11 real sequence are given by.

$X(0)=2$, $X(2) = -1-j3$, $X(4) = 1+j4$, $X(6) = 9+j3$,

$X(8) = 5$, $X(10) = 2+j2$.

Determine the missing odd samples of DFT.

$X(k) = X^*(N-k)$

$X(1) = X^*(11-1) = X^*(10) = 2-j2$

$X(3) = X^*(11-3) = X^*(8) = 5$

$X(5) = X^*(11-5) = X^*(6) = 9-j3$

$X(7) = X^*(11-7) = X^*(4) = 1-j4$

$X(9) = X^*(2) = -1+j3$

$X(11) = X^*(0) = 2$.

Example 7:

Compute the 5-point DFT of the sequence $x(n) = (1, 0, 1, 0, 1)$ & verify the symmetry property.

$W_5^0 = 1, \quad W_5^1 = e^{-j2\pi/5} = 0.309 - j0.951$

$W_5^2 = -0.809 - j0.587, \quad W_5^3 = -0.809 + j0.587$

$W_5^4 = e^{-j8\pi/5} = 0.309 + j0.951$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$= \{3, \; 0.5 + j0.364, \; 0.5 + j1.538, \; 0.5 - j1.538, \; 0.5 - j0.364\}$$

$X(k) = X^*(N-k)$

(Q) $x_p(n)$ is a periodic sequence

Let $x_p(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X_1(k)$

$x_p(n) \overset{DFT}{\underset{3N}{\longleftrightarrow}} X_3(k)$

(a) Write $X_3(k)$ in terms of $X_1(k)$

$$X_1(k) = \sum_{n=0}^{N-1} x_p(n) W_N^{kn}$$

$$X_3(k) = \sum_{n=0}^{3N-1} x_p(n) W_{3N}^{kn}$$

$$= \sum_{n=0}^{N-1} x_p(n) W_{3N}^{kn} + \sum_{n=N}^{2N-1} x_p(n) W_{3N}^{kn} + \sum_{n=2N}^{3N-1} x_p(n) W_{3N}^{kn}$$

$$= \sum_{n=0}^{N-1} x_p(n) W_N^{k/3\, n} + \sum_{n=0}^{N-1} x_p(n+N) W_{3N}^{k(n+N)} + \sum_{n=0}^{N-1} x_p(n+2N) W_{3N}^{k(n+2N)}$$

$$= \sum_{n=0}^{N-1} x_p(n) W_N^{k/3\, n} + \sum_{n=0}^{N-1} x_p(n) W_{3N}^{kN} W_{3N}^{kn} + \sum_{n=0}^{N-1} x_p(n) W_{3N}^{2kN} W_{3N}^{kn}$$

$$= \sum_{n=0}^{N-1} x_p(n) W_N^{k/3\, n} + \sum_{n=0}^{N-1} x_p(n) W_3^{k} W_N^{k/3\, n} + \sum_{n=0}^{N-1} x_p(n) W_3^{2k} W_N^{k/3\, n}$$

$$= \left[1 + W_3^k + W_3^{2k}\right] \sum_{n=0}^{N-1} x_p(n) W_N^{k/3\, n}$$

$$= \left[1 + W_3^k + W_3^{2k}\right] X_1(k/3)$$

Let $x_p(n) = \{\ldots \underset{\uparrow}{2}, 1, 2, 1, 2, 1 \ldots\}$

(b) $X_1(k) = 2 + W_2^k$

$$X_3(k) = 2 + W_{3\times2}^{k} + 2W_{3\times2}^{2k} + W_{3\times2}^{3k} + 2W_{3\times2}^{4k} + W_{3\times2}^{5k}$$

$$= (2 + W_2^{k/3}) + W_6^{2k}(2 + W_2^{k/3}) + W_6^{4k}(2 + W_2^{k/3})$$

$$= (1 + W_3^{k} + W_3^{2k}) \; X_1(k/3).$$

Verified.

Section 1 Problems

(1) $x(n) = \{1,1,1,1,0,0,0,0\}$.

$X(k) = \{4, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$

$|X(k)| = \sqrt{X_R^2(k) + X_I^2(k)} \qquad \angle X(k) = \tan^{-1}\left[\frac{X_I(k)}{X_R(k)}\right]$

$|X(k)| = \{4, 2.613, 0, 1.082, 0, 1.082, 0, 2.631\}$

$\angle X(k) = \{0, -67.5, 0, -22.5, 0, 22.5, 0, 67.5\}$.

$W_8^0 = 1, \quad W_8^1 = \frac{1}{\sqrt2} - \frac{j}{\sqrt2}, \quad W_8^2 = -j, \quad W_8^3 = \frac{-1}{\sqrt2} - \frac{j}{\sqrt2}$

(2) $x(n) = (1, -1, 1, -1) \qquad W_4^0 = 1, \; W_4^1 = -j, \; W_4^2 = -1, \; W_4^3 = +j$

$X(k) = (0, 0, 4, 0)$

(3) $x(n) = \cos 2\pi(50)t \qquad F_s = 200\text{Hz}$.

$= \cos \frac{\pi}{2} n = (1, 0, -1, 0)$

$X(k) = (0, 2, 0, 2)$

(4) Lemma

$$\sum_{n=0}^{N-1} W_N^{kn} = N\delta(k) = \begin{cases} N & k=0 \\ 0 & k \neq 0 \end{cases}$$

$$\sum_{n=0}^{N-1} W_N^{kn} = \frac{1 - W_N^{kN}}{1 - W_N^{k}} = \frac{1 - e^{-j2\pi k}}{1 - e^{-j\frac{2\pi}{N}k}} = \frac{1-1}{1 - e^{-j\frac{2\pi k}{N}}} = 0 \qquad k \neq 0$$

for $k = 0 \quad \sum_{n=0}^{N-1} (1)^n = N$

$\therefore \sum_{n=0}^{N-1} W_N^{kn} = N\delta(k)$

$\boxed{7.23}$ (a) $x(n) = \delta(n)$

$$X(k) = \sum_{n=0}^{N-1} \delta(n) W_N^{kn}$$

$$= W_N^{kn}\Big|_{n=0}$$

$$= 1$$

(b) $x(n) = \delta(n - n_0)$

$$X(k) = \sum_{n=0}^{N-1} \delta(n - n_0) W_N^{kn}$$

$$W_N^{kn}\Big|_{n=n_0} = W_N^{kn_0}$$

(c) $x(n) = \begin{cases} 1 & n \text{ is even} \\ 0 & n \text{ is odd.} \end{cases}$

$$X(k) = \sum_{n=0}^{N/2-1} x(2n) e^{-\frac{j2\pi k(2n)}{N}} + \sum_{n=0}^{N/2-1} x(2n+1) e^{-j\frac{2\pi k(2n+1)}{N}}$$

$$= \sum_{n=0}^{N/2-1} e^{-j\frac{2\pi k}{N/2} n} = \frac{1 - e^{-j\frac{2\pi k}{N/2}\cdot\frac{N}{2}}}{1 - e^{-j\frac{2\pi k}{N/2}}} = 0 \qquad k \neq 0.$$

$k=0 \quad \sum_{n=0}^{N/2-1} 1 = N/2$

$\therefore \quad X(k) = \begin{cases} N/2 & k=0 \\ 0 & \text{Otherwise} \end{cases}$

(d) $x(n) = \begin{cases} 1 & n=0 \cdots \frac{N}{2}-1 \\ 0 & n=N/2 \cdots N-1 \end{cases}$

$= \sum_{n=0}^{N/2-1} W_N^{kn} = \frac{1-W_N^{kN/2}}{1-W_N^k} = \frac{1-e^{-j2\pi k \frac{N}{2}/N}}{1-W_N^k}$

$= \frac{1-(-1)^k}{1-e^{-j2\pi k/N}} \qquad k \neq 0.$

$k=0 \quad = \sum_{n=0}^{N/2-1} 1 = N/2$

$X(k) = \begin{cases} N/2 & k=0 \\ \frac{1-(-1)^k}{1-e^{-j2\pi k/N}} & \text{Otherwise} \end{cases}$

(e) $x(n) = a^n$

$X(k) = \sum_{n=0}^{N-1} (aW_N^k)^n = \frac{1-a^N}{1-e^{j2\pi k/N}}$

(f) $x(n) = e^{+j2\pi k_0 n/N}$

$X(k) = \sum_{n=0}^{N-1} W_N^{-k_0 n} W_N^{kn} = \sum_{n=0}^{N-1} W_N^{(k-k_0)n}$

$= \frac{W_N^{(k-k_0)N}-1}{W_N^{(k-k_0)}-1} = 0 \qquad k \neq k_0$

$k=k_0 \quad = \sum_{n=0}^{N-1} 1 = N$

$X(k) = \begin{cases} N & k=k_0 \\ 0 & k \neq k_0 \end{cases} = N.\delta(k-k_0)$

(g) $x(n) = \cos\frac{2\pi k_0 n}{N}$

$= \sum_{n=0}^{N-1} \left[\frac{1}{2}\left(e^{\frac{j2\pi k_0 n}{N}} + e^{\frac{-j2\pi k_0 n}{N}}\right)\right] W_N^{kn}$

$= \frac{1}{2} \sum_{n=0}^{N-1} \left(W_N^{(k-k_0)n} + W_N^{(k+k_0)n}\right)$

$= \frac{N}{2}\left(\delta(k-k_0) + \delta(k+k_0)\right)$

$= \frac{N}{2}\left(\delta(k-k_0) + \delta(k+k_0-N)\right)$

$= N/2 \left(\delta(k-k_0) + \delta(k-(N-k_0))\right)$

(h) $x(n) = \sin\frac{2\pi k_0 n}{N} \quad n=0 \cdots N-1$

$X(k) = \sum_{n=0}^{N-1} \left(\frac{1}{2j}\left(e^{\frac{j2\pi k_0 n}{N}} - e^{\frac{-j2\pi k_0 n}{N}}\right)\right) W_N^{kn}$

$= \frac{1}{2j} \sum_{n=0}^{N-1} \left[W_N^{(k-k_0)n} - W_N^{(k+k_0)n}\right]$

$= \frac{N}{2j} \left[\delta(k-k_0) - \delta(k+k_0-N)\right]$

(8) Circular Correlation

when $x(n)$ & $y(n)$ are complex valued

$x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k) \qquad y(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} Y(k)$

$r_{xy}(l) \overset{DFT}{\underset{N}{\longleftrightarrow}} R_{xy}(k) = X(k)\, Y^*(k)$

where, $r_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N$

Proof:

$r_{xy}(l) = x(l) \circledN y^*(-l)$

~~Using symmetric~~ ~~complex~~ Using complex conjugate property

i.e., $x^*((-n))_N = x^*(N-n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X^*(k)$

$\boxed{R_{xy}(k) = X(k)\, Y^*(k)}$

Cross Correlation :

Cross Correlation of two real signals of finite energy $x(n)$ & $y(n)$ is

$r_{xy}(l) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-l) \quad \text{--- (1)} \quad l=0, \pm1, \pm2 \cdots$

or $r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\, y(n) \quad \text{--- (2)} \quad l=0, \pm1, \pm2 \cdots$

In equ^n (1) $y(n)$ is shifted to right by l units

l is +ve $\longrightarrow$ shift to right

l is −ve $\longrightarrow$ shift to left

In equ^n (2) $x(n)$ is shifted to left by l units

l is +ve $\longrightarrow$ shift to left

l is −ve $\longrightarrow$ shift to right

Shifting $x(n)$ to left by l units $\equiv$ Shifting $y(n)$ to right by l units.

Reversing the roles of x & y we get

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\, x(n-l) \quad \text{—③}$$

or $$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n+l)\, x(n) \quad \text{—④}$$

Comparing ① & ④ and ② & ③ we get

$$r_{xy}(l) = r_{yx}(-l)$$

i.e, r_{yx} is a folded version of r_{xy}, w.r.t $l=0$,

Example:

$x(n) = \{1, 2, 3\}$

$y(n) = \{3, 2, 1\}$

$$r_{xy} = \sum_{-\infty}^{+\infty} x(n+l)\, y(n)$$

$l=0$: 1 2 3 / 3 2 1 → $3+4+3 = 10$

$l=1$: 1 2 3 / 3 2 1 → $6+6 = 12$

$l=-1$: 1 2 3 / 3 2 1 → $2+2 = 4$

$l=2$: 1 2 3 / 3 2 1 → 9

$l=-2$: 1 2 3 / 3 2 1 → 1

$$[\ldots 1, 4, \underset{\substack{\uparrow \\ l=0}}{10}, 12, 9 \ldots]$$

(Q) How to perform correlation using convolution

conv $(x(n),$ fliplr $y(n))$.

Example:

$x_1(n) = \cos \frac{2\pi n}{N}$ $\quad x_2(n) = \sin \frac{2\pi n}{N}$ $\quad n=0 \ldots N-1$

(i) circular ~~convolution~~ convolution of $x_1(n)$ & $x_2(n)$

(ii) circular correlation of $x_1(n)$ & $x_2(n)$

(iii) circular auto correlation of $x_1(n)$ &

(iv) circular auto correlation of $x_2(n)$

$$X_1(k) = \frac{N}{2}[\delta(k-1) + \delta(k+1)] \quad \text{—①} \qquad X_2(k) = \frac{N}{2j}[\delta(k-1) - \delta(k+1)] \quad \text{—②}$$

$$X_3(k) = X_1(k)\, X_2(k)$$

$$= \frac{N^2}{4j}[\delta(k-1) - \delta(k-1)]$$

$$X_3(k) = \frac{N}{2}\left[\frac{N}{2j}\ \delta(k-1) - \delta(k+1)\right]$$

$$x_3(n) = \text{IDFT}[X_3(k)] = \frac{N}{2} \sin \frac{2\pi n}{N}$$

(ii) $r_{xy}(l) \overset{DFT}{\underset{N}{\longleftrightarrow}} R_{xy}(k) = X(k)\, Y^*(k)$

$$= \left[\frac{N}{2}[\delta(k-1) + \delta(k+1)]\right]^* \left[\frac{N}{2j}(\delta(k-1) - \delta(k+1))\right]$$

$$R_{xy}(k) = -\frac{N^2}{4j}[\delta(k-1) - \delta(k+1)]$$

$$r_{xy}(l) = -\frac{N}{2} \sin \frac{2\pi l}{N}.$$

(iii) $r_{xx}(l) \overset{DFT}{\underset{N}{\longleftrightarrow}} R_{xx}(k) = X(k)\, X^*(k)$

$$R_{xx}(k) = \left[\frac{N}{2}\ \delta(k-1) + \delta(k+1)\right]\left[\frac{N}{2}\ \delta(k+1) + \delta(k+1)\right].$$

$$= \frac{N^2}{4}[\delta(k-1) + \delta(k+1)].$$

$$r_{xx}(l) = \frac{N}{2} \cos \frac{2\pi l}{N} \qquad l = 0 \ldots N-1$$

(iv) $r_{yy}(l) \overset{DFT}{\underset{N}{\longleftrightarrow}} R_{yy}(k) = Y(k)\, Y^*(k)$

$$R_{yy}(k) = \left[\frac{N}{2j}[\delta(k-1) - \delta(k+1)]\right]^* \left[-\frac{N}{2j}(\delta(k-1) - \delta(k+1))\right]$$

$$= -\frac{N^2}{4j^2}[\delta(k-1) + \delta(k+1)]$$

$$R_{yy}(k) = \frac{N}{2}\left[\frac{N}{2}(\delta(k-1) + \delta(k+1))\right]$$

$$r_{yy}(l) = \frac{N}{2} \cos \frac{2\pi l}{N} \qquad l = 0 \ldots N-1.$$

Q) Let $X(k) = DFT\{x(n)\}$ for $n, k = 0 \ldots N-1$. Determine the relationship b/n $X(k)$ & the following DFTs

(a) $DFT\{x^*(n)\}$ (b) $DFT\{x[((-n))_N]\}$

(c) $DFT\{Re\{x(n)\}\}$ (d) $DFT\{Im\{x(n)\}\}$

(e) apply all the above properties to the sequence

$x(n) = IDFT\{[1, -j, 2, 3j]\}$

(a) $DFT\{x^*(n)\} \xleftrightarrow{DFT} X^*((-k))_N = (1, -3j, 2, j)$

(b) $DFT\{x((-n))_N\} \xleftrightarrow{DFT} X((-k))_N = (1, 3j, 2, -j)$

(c) $DFT\{Re\{x(n)\}\} \longleftrightarrow = \frac{1}{2}[X(k) + X^*(N-k)]$

$= [\frac{1}{2}(1, -j, 2, 3j) + (1, -3j, 2, j)]$

$= [1, -2j, 2, 2j]$

(c) $DFT\{Im\{x(n)\}\} = \frac{1}{2j}[X(k) - X^*(N-k)]$

$= \frac{1}{2j}((1, -j, 2, 3j) - (1, -3j, 2, j))$

$= (0, 1, 0, 1)$

(Q) Let $x(n) = IDFT\{X(k)\}$ for $n, k = 0 \ldots N-1$ Determine the relationship b/n $x(n)$ & the following IDFTs

(a) $IDFT\{X^*(k)\}$ (b) $IDFT\{X((-k))_N\}$

(c) $IDFT\{Re\{X(k)\}\}$ (d) $IDFT\{Im\{X(k)\}\}$

(e) apply all the above properties to the sequence

$X(k) = DFT\{[1, -2j, +j, -4j]\}$

(a) $IDFT\{X^*(k)\} = x^*((-n))_N = [1, 4j, -j, +2j]$

(b) $IDFT\{X((-k))_N\} = x((-n))_N = [1, -4j, j, -2j]$

(c) $IDFT\{Re\{X(k)\}\} = \frac{1}{2}[x(n) + x^*((-n))_N]$

$= \frac{1}{2}((1, -2j, +j, -4j) + (1, +4j, -j, 2j))$

$= \frac{1}{2}[2, 2j, 0, -2j]$

$= (1, j, 0, -j)$

(d) $IDFT\{Im\{X(k)\}\} = \frac{1}{2j}[x(n) - x^*((-n))_N]$ 20

$= \frac{1}{2j}[(1, -2j, +j, -4j) - (1, 4j, -j, 2j)]$

$= \frac{1}{2j}(0, -6j, +2j, -6j) = \underline{\underline{(0, -3, +1, -3)}}$

Q] Two finite sequences h and x have the following DFT's:

$X = DFT\{x\} = [1, -2, 1, -2]$

$H = DFT\{h\} = [1, j, 1, -j]$

Let $y = h \circledast x$ be the four point circular convolution of the two sequences. Using the properties of DFT.

(a) Determine $DFT\{x((n-1))_4\}$ and $DFT\{h((n+2))_4\}$

(b) Determine $y(0)$ and $y(1)$

(a) $DFT\{x((n-1))_4\} = W_4^{k} X(k)$

$= [1, -j, -1, -j][1, -2, 1, -2]$

$= [1, 2j, -1, -2j]$

$DFT\{h((n+2))_4\} = W_4^{-2k} X(k)$

$= (-1)^k [1, j, 1, -j] = [1, -j, 1, j]$

(b) $Y(k) = X(k) H(k)$

$= (1, -2j, 1, +2j)$

$y(0) = \frac{1}{4}\sum_{k=0}^{N-1} Y(k) = \frac{1}{4}(1 - \cancel{2} + 1 + \cancel{2}) = \frac{2}{4} = \underline{\underline{0.5}}$

$y(1) = \frac{1}{4}\sum_{k=0}^{3} Y(k) W_4^{-k} = \frac{1}{4}(1, -2j, 1, +2j)(1, +j, -1, -j)$

$= \frac{1}{4}(1, +2, -1, 2) = 4/4 = \underline{\underline{1}}$

Q] find N-point DFT of the non-causal sequence

$x(n) = \{\frac{1}{2}, \underset{\uparrow}{\frac{1}{2}}, \frac{1}{2}\}$

$\Rightarrow x(n) = \{\underset{\underset{n=0}{\uparrow}}{\frac{1}{2}}, \frac{1}{2}, 0, 0, \ldots, \underset{\underset{n=N-1}{\uparrow}}{\frac{1}{2}}\}$

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$\therefore X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$

$= \frac{1}{2} + \frac{1}{2} W_N^{k} + \frac{1}{2} W_N^{(N-1)k}$

$= \frac{1}{2}\left(1 + (W_N^{k} + W_N^{-k})\right) = \frac{1}{2}\left(1 + 2\cos\frac{2\pi k}{N}\right)$

$= \frac{1}{2} + \cos\frac{2\pi k}{N}.$

Problems: Proakis Text Book → Chapter-7

$\boxed{7.1}$ The first five points of the 8-pt DFT of a real-valued sequence are $\{0.25, 0.125 - j0.3018, 0, 0.125 - j0.0518, 0\}$. Find the other three points.

Ans: Since, $x(n)$ is real, the real part of the DFT is even, imaginary part odd. Thus the remaining points are $\{0.125 + j0.0518, 0, 0.125 + j0.3018\}$

$\boxed{7.2}$ Compute the 8-pt circular convolution for the following sequences

(a) $x_1(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$ $\quad x_2(n) = \sin\frac{3\pi n}{8}$ $\quad n = 0 \quad 7$

$X_1(k) = \sum_{n=0}^{7} x_1(n) W_8^{kn}$

$X_1(k) = 1 + W_8^{k} + W_8^{2k} + W_8^{3k}$

$X_2(k) = \sum_{n=0}^{7} x_2(n) W_8^{kn} = \sum_{n=0}^{7} \sin\frac{3\pi n}{8} W_8^{kn}$

$= \sum_{n=0}^{7} [0, 0.92, 0.707, -0.38, -1, -0.38, 0.707, 0.92] W_8^{kn}$

$X_2(k) = 0 + 0.92 W_8^{k} + 0.707 W_8^{2k} - 0.38 W_8^{3k} - W_8^{4k} - 0.38 W_8^{5k} + 0.707 W_8^{6k} + 0.92 W_8^{7k}$

$X_3(k) = X_1(k) X_2(k)$

$= [1 + W_8^{k} + W_8^{2k} + W_8^{3k}][0 + 0.92 W_8^{k} + 0.707 W_8^{2k} - 0.38 W_8^{3k} - W_8^{4k} - 0.38 W_8^{5k} + 0.707 W_8^{6k} + 0.92 W_8^{7k}]$

$= \cancel{0 + 0.02 W_8^{k} + 0.04 W_8^{2k} + 0.06 W_8^{3k} + 0.08 W_8^{4k} + 0.10 W_8^{5k} + 0.12 W_8^{6k} + 0.14 W_8^{7k} + 0 + 0.02 W_8^{2k} + 0.04 W_8^{3k} + 0.06 W_8^{4k} + 0.08 W_8^{5k} + 0.10 W_8^{6k} + 0.12 W_8^{7k} + 0.14 W_8^{0}}$

$X_3(k) = 1.25 + 2.54 W_8^{k} + 2.54 W_8^{2k} + 1.24 W_8^{3k} + 0.24 W_8^{4k} - 1.05 W_8^{5k} - 1.05 W_8^{6k} + 0.25 W_8^{7k}$

Since $\text{DFT}\{\delta(n-n_0)\} = W_N^{kn_0}$

$x_3(n) = 1.25 + 2.54\,\delta(n-1) + 2.54\,\delta(n-2) + 1.24\,\delta(n-3) + 0.24\,\delta(n-4) - 1.05\,\delta(n-5) - 1.05\,\delta(n-6) + 0.25\,\delta(n-7)$

$x_3(n) = (1.25, 2.54, 2.54, 1.24, 0.24, -1.05, -1.05, 0.25)$

~~7.4 for the sequences $x_1(n) = \cos\frac{2\pi n}{N}$, $x_2(n) = \sin\frac{2\pi n}{N}$ find the N-point: (a) circular~~

$\boxed{7.5}$ Compute the quantity $\sum_{n=0}^{N-1} x_1(n) x_2^*(n)$ for the following pairs of sequences

(a) $x_1(n) = x_2(n) = \cos\frac{2\pi}{N} n$

$\sum_{n=0}^{N-1} x_1(n) x_2^*(n) = \frac{1}{4} \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi n}{N}} + e^{-j\frac{2\pi n}{N}}\right)^2$

$= \frac{1}{4} \sum_{n=0}^{N-1} \left(e^{j\frac{4\pi n}{N}} + e^{-j\frac{4\pi n}{N}} + 2\right)$

$= \frac{1}{4}\left[\sum_{n=0}^{N-1} (W_N^{-2})^n + \sum_{n=0}^{N-1} (W_N^{2})^n + \sum_{n=0}^{N-1} 2\right]$

$= \frac{1}{4}\left[\frac{1 - (W_N^{-2})^N}{1 - W_N^{-2}} + \frac{1 - (W_N^{2})^N}{1 - W_N^{2}} + 2N\right]$

$= \frac{1}{4}\left[\frac{1-1}{1 - W_N^{-2}} + \frac{1-1}{1 - W_N^{2}} + 2N\right]$

$= \frac{1}{4}[2N] = \frac{N}{2}$

(b) $x_1(n) = \cos\frac{2\pi n}{N}$, $x_2(n) = \sin\frac{2\pi n}{N}$

$\sum_{n=0}^{N-1} x_1(n) x_2^*(n) = \frac{1}{4j} \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi n}{N}} + e^{-j\frac{2\pi n}{N}}\right)\left(e^{j\frac{2\pi n}{N}} - e^{-j\frac{2\pi n}{N}}\right)$

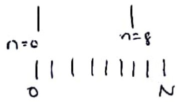
$= \frac{1}{4j} \sum_{n=0}^{N-1} \left(e^{j\frac{4\pi n}{N}} - e^{-j\frac{4\pi n}{N}}\right)$

$= 0$

(c) $\sum_{n=0}^{N-1} x_1(n)\, x_2^*(n)$.

$x_1(n) = \delta(n) + \delta(n-8)$ $\quad N>8$

$x_2(n) = u(n) - u(n-N)$



$= 1 + 1 = 2$

7.7 If X(k) is the DFT of the sequence x(n), determine the N-point DFTs of the sequences

(a) $x_c(n) = x(n)\cos 2\pi k_0 n/N$

(b) $x_s(n) = x(n)\sin \frac{2\pi k_0 n}{N}$ in terms of X(k)

$$X_c(k) = \sum_{n=0}^{N-1} \frac{1}{2} x(n)\left[e^{\frac{j2\pi k_0 n}{N}} + e^{\frac{-j2\pi k_0 n}{N}}\right] e^{\frac{-j2\pi kn}{N}}$$

$$= \frac{1}{2}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi(k-k_0)n}{N}} + \frac{1}{2}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi(k+k_0)n}{N}}$$

$$= \frac{1}{2} X((k-k_0))_N + \frac{1}{2} X((k+k_0))_N$$

(b) $$X_s(k) = \sum_{n=0}^{N-1} \frac{1}{2j} x(n)\left[e^{\frac{j2\pi k_0 n}{N}} - e^{\frac{-j2\pi k_0 n}{N}}\right] e^{\frac{-j2\pi kn}{N}}$$

$$= \frac{1}{2j}\sum_{n=0}^{N-1}\left[x(n)\, e^{\frac{-j2\pi(k-k_0)n}{N}} - x(n)\, e^{\frac{-j2\pi(k+k_0)n}{N}}\right]$$

$$= \frac{1}{2j}\left[X((k-k_0))_N - X((k+k_0))_N\right]$$

7.8 Determine the circular convolution of $x_1(n) = \{1,2,3,1\}$, $x_2(2) = \{4,3,2,2\}$ in time domain

$y(n) = x_1(n) \circledN_N x_2(n)$

$$= \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 1 & 1 & 2 \\ 2 & 3 & 1 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \\ 22 \\ 19 \end{bmatrix}$$

7.9 Repeat (7.8) in frequency domain

$X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, W_N^{kn}$

$= 1 + 2W_4^k + 3W_4^{2k} + W_4^{3k}$

$X_2(k) = \sum_{n=0} x_2(n)\, W_N^{kn}$

$= 4 + 3W_4^k + 2W_4^{2k} + 2W_4^{3k}$

$X_3(k) = X_1(k)\cdot X_2(k)$

$= [1 + 2W_4^k + 3W_4^{2k} + W_4^{3k}][4 + 3W_4^k + 2W_4^{2k} + 2W_4^{3k}]$

$= 17 + 19W_N^k + 22W_N^{2k} + 19W_N^{3k}$

$\text{IDFT}\{W_N^{kn_0}\} = \delta(n-n_0)$

$\therefore x_3(n) = 17 + 19\delta(n-1) + 22\delta(n-2) + 19\delta(n-3)$

$= [17, 19, 22, 19]$

7.10 Compute the energy of the N-pt sequence

$x(n) = \cos\frac{2\pi k_0 n}{N} \quad n = 0 \ldots N-1$

$x(n) = \frac{1}{2}\left(e^{\frac{j2\pi kn}{N}} + e^{\frac{-j2\pi kn}{N}}\right)$

$x(n)\, x^*(n) = \frac{1}{4}\left(e^{\frac{j4\pi kn}{N}} + e^{\frac{-j4\pi kn}{N}} + 2\right)$

$E = \sum_{n=0}^{N-1} x(n)\, x^*(n)$

$= \frac{1}{4}\sum_{n=0}^{N-1}\left(2 + e^{\frac{j4\pi kn}{N}} + e^{\frac{-j4\pi kn}{N}}\right)$

$= \frac{1}{4}\, 2N = \frac{N}{2}$

7.11 Given the eight-pt DFT of the sequence

$x(n) = \begin{cases} 1 & 0 \le n \le 3 \\ 0 & 4 \le n \le 7 \end{cases}$ Compute the DFT of the sequence.

(a) $x_1(n) = \begin{cases} 1 & n = 0 \\ 0 & n = 1 \text{ to } 4 \\ 1 & n = 5 \text{ to } 7 \end{cases}$

$x(n) = (1,1,1,1,0,0,0,0)$

$x_1(n) = (1,0,0,0,0,1,1,1) = x((n-5))_8$

$\Rightarrow X_1(k) = \text{DFT}\{x((n-5))_8\}$

$= W_8^{5k}\, X(k)$

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(b) $x_2(n) = \begin{cases} 0 & n = 0 \text{ to } 1 \\ 1 & n = 2 \text{ to } 5 \\ 0 & n = 6 \text{ to } 7 \end{cases}$

$x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$

$x_2(n) = \{0, 0, 1, 1, 1, 1, 0, 0\} = x((n-2))_8$

$X_2(k) = DFT\{x((n-2))_8\}$

$= W_8^{2k} X(k)$

$\boxed{7.12}$ Consider a finite-duration sequence

$x(n) = \{0, 1, 2, 3, 4\} \xrightarrow{6\text{-pt}} x(n) = \{0, 1, 2, 3, 4, 0\}$

(a) Sketch the sequence $s(n)$ with six-point DFT

$S(k) = W_2^k X(k) \quad k = 0 \cdots 5$

$S(k) = W_2^k X(k)$

$S(k) = W_6^{3k} X(k)$

$\Downarrow$ IDFT

$s(n) = x((n-3))_6$

$s(n) = \{3, 4, 0, 0, 1, 2\}$

$e^{-j\frac{2\pi(3k)}{2\times 3}} = W_6^{3k}$

(b) Determine the sequence $y(n)$ with six-point DFT $Y(k) = Re\{X(k)\}$

$y(n) = IDFT\left\{\frac{X(k) + X^*(k)}{2}\right\}$

$= \frac{1}{2}\left[IDFT\{X(k)\} + IDFT\{X^*(k)\}\right]$

$= \frac{1}{2}\left[x(n) + x^*((-n))_N\right]$

$= \frac{1}{2}\left[(0, 1, 2, 3, 4, 0) + (0, 0, 4, 3, 2, 1)\right]$

$y(n) = (0, \frac{1}{2}, 3, 3, 3, \frac{1}{2})$

(c) Determine the sequence $v(n)$ with 6-point DFT $V(k) = Im\{X(k)\}$

$v(n) = IDFT\left\{\frac{X(k) - X^*(k)}{2j}\right\}$

$v(n) = \frac{1}{2j}\left(x(n) - x^*((-n))_N\right)$

$= \frac{1}{2j}\left((0, 1, 2, 3, 4, 0) - (0, 0, 4, 3, 2, 1)\right)$

$= (0, -\frac{1}{2}j, j, 0, -j, \frac{1}{2}j)$

$\boxed{7.14}$ Consider the sequences

$x_1(n) = \{\underset{\uparrow}{0}, 1, 2, 3, 4\} \quad x_2(n) = \{\underset{\uparrow}{0}, 1, 0, 0, 0\}$

$s(n) = \{\underset{\uparrow}{1}, 0, 0, 0, 0\}$ and their five point DFTs

(a) Determine a sequence $y(n)$ so that $Y(k) = X_1(k) X_2(k)$

$\Rightarrow y(n) = x_1(n) \circledast_5 x_2(n)$

$= x_1(n) \circledast_5 \delta(n-1)$

$= x_1((n-1))_5$

$= (4, 0, 1, 2, 3)$

$x_2(n) = (0, 1, 0, 0, 0)$

$\Rightarrow \delta(n-1)$.

(b) Is there a sequence $x_3(n)$ such that $S(k) = X_1(k) X_3(k)$?

Let $x_3(n) = \{x_0, x_1, x_2, x_3, x_4\}$

Then,

$$\begin{bmatrix} 0 & 4 & 3 & 2 & 1 \\ 1 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this:

$x_0 = -0.18, \; x_1 = 0.22, \; x_2 = 0.02, \; x_3 = 0.02, \; x_4 = 0.02$

$\Rightarrow x_3(n) = \{-0.18, 0.22, 0.02, 0.02, 0.02\}$

7.17 Determine the eight-point DFT of the signal $x(n) = \{\underset{\uparrow}{1}, 1, 1, 1, 1, 1, 0, 0\}$ and find the magnitude and phase.

$$X(k) = \sum_{n=0}^{7} x(n)\, e^{\frac{-j2\pi kn}{8}}$$

$$X(k) = 1 + W_8^{k} + W_8^{2k} + W_8^{3k} + W_8^{4k} + W_8^{5k}$$

$$X(k) = \{\underset{\uparrow}{6}, (-0.7071 - j1.7071), (1 - j0), (0.7071 + j0.2929), 0, (0.7071 - j0.2929), (1+j), (-0.7071 + j1.7071)\}$$

$$|X(k)| = \{\underset{\uparrow}{6}, 1.85, 1.41, 0.77, 0, 0.77, 1.41, 1.85\}$$

$$\angle X(k) = \{\underset{\uparrow}{0}, -1.96, \frac{-\pi}{4}, 0.392, 0, -0.392, \frac{\pi}{4}, 1.96\}$$

7.24 Consider the finite-duration signal

$x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$

(a) Compute its 4-pt DFT by solving explicitly the 4-by-4 system of linear equations defined by the inverse DFT formula

$$x_N = \frac{1}{N} W_N^{*} X_N \qquad \text{as } W_N^{-1} = \frac{W_N^{*}}{N}$$

$$N x_N = W_N^{*} X_N$$

$$X_N = (N x_N)[W_N^{*}]^{-1} = N x_N \frac{[W_N^{*}]^{*}}{N}$$

$$\Rightarrow X_N = \cancel{4}\begin{bmatrix}1\\2\\3\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{bmatrix}\frac{1}{\cancel{4}}$$

$$= \begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{bmatrix}\begin{bmatrix}1\\2\\3\\1\end{bmatrix} = \begin{bmatrix}1+2+3+1\\ 1-2j-3+j\\ 1-2+3-1\\ 1+2j-3-j\end{bmatrix} = \begin{bmatrix}7\\ -2-j\\ 1\\ -2+j\end{bmatrix}$$

$$X(k) = [7, -2-j, 1, -2+j]$$

(b) Check the answer in part (a) by computing the 4-pt DFT, using its definition.

$$X(k) = \sum_{n=0}^{3} x(n)\, W_4^{kn}$$

$$= 1 + 2W_4^{k} + 3W_4^{2k} + W_4^{3k}$$

$W_4^0 = 1$

$W_4^1 = -j$

$W_4^2 = -1$

$W_4^3 = +j$

$$X(0) = 1 + 2 + 3 + 1 = 7$$

$$X(1) = 1 + 2W_4^1 + 3W_4^2 + W_4^3$$

$$= 1 - 2j - 3 + j = -2 - j$$

$$X(2) = 1 + 2W_4^2 + 3W_4^0 + W_4^2$$

$$= 1 - 2 + 3 - 1 = 1$$

$$X(3) = 1 + 2W_4^3 + 3W_4^2 + W_4^1$$

$$= 1 + 2(j) + 3(-1) + (-j) = -2 + j$$

$$\therefore X(k) = [7, -2-j, 1, -2+j]$$

3rd Chapter end Problems on DFT & properties – MDSP by Roberto Cristi.

3.3 Compute the DFT of the following sequences

(a) $x(n) = (1, 0, -1, 0)$

$X(k) = \sum_{n=0}^{3} x(n) W_4^{kn}$

$\boxed{X(k) = 1 - W_4^{2k}}$

$k=0, X(0) = 1-1=0$; $k=1, X(1) = 1 - W_4^2 = 1-(-1) = 2$

$k=2, X(2) = 1 - W_4^{2\times 2} = 1 - W_4^0 = 1-1=0$

$k=3, X(3) = 1 - W_4^{2\times 3} = 1 - W_4^2 = 1-(-1) = 2$

$\therefore \boxed{X(k) = (0, 2, 0, 2)}$

(b) $x(n) = (j, 0, j, 1)$

$X(k) = \sum_{n=0}^{3} x(n) W_4^{kn}$

$\boxed{X(k) = j + jW_4^{2k} + W_4^{3k}}$

$k=0, X(0) = j+j+1 = 1+2j$

$k=1, X(1) = j + jW_4^2 + W_4^3 = j-j+j = j$

$k=2, X(2) = j+j+(-1) = -1+2j$

$k=3, X(3) = j + j(-1) + (-j) = -j$

$\boxed{X(k) = [1+2j, j, -1+2j, -j]}$

(c) $x(n) = (1, 1, 1, 1, 1, 1, 1, 1)$ $k = 0 \dots N-1$

$X(k) = \sum_{n=0}^{7} x(n) W_N^{kn}$

$= N\delta(k)$

i.e., $X(k) = \begin{cases} N & k=0 \\ 0 & k \neq 0 \end{cases}$

(d) $x(n) = \cos(0.25\pi n)$ $n = 0 \dots 7$

$W_8 = e^{-j\frac{2\pi}{8}} = e^{-j\frac{\pi}{4}}$

$X(k) = \frac{1}{2} \sum_{n=0}^{7} \left[e^{j\pi/4 n} + e^{-j\pi/4 n}\right] W_8^{kn}$

$= \frac{1}{2} \sum_{n=0}^{7} \left[e^{j\pi/4 n} e^{-j\pi/4 nk} + e^{-j\pi/4 n} e^{-j\pi/4 nk}\right]$

$= \frac{1}{2} \sum_{n=0}^{7} \left[e^{-j\pi/4(k-1)n} + e^{-j\pi/4(k+1)n}\right]$

$= \frac{1}{2}\left[8\delta(k-1) + 8\delta(k+1)\right]$

$X(k) = 4\delta(k-1) + 4\delta(k-(N-1))$ $\quad 8-1=7$

$X(k) = 4\delta(k-1) + 4\delta(k-7)$

$\therefore X(k) = (0, 4, 0, 0, 0, 0, 0, 4)$

(e) $x(n) = 0.9^n$ $n = 0 \dots N-1$

$X(k) = \sum_{n=0}^{7} 0.9^n W_N^{kn} = \sum_{n=0}^{7} (0.9 W_8^k)^n = \frac{1-(0.9W_8^k)^8}{1-0.9W_8^k}$

$X(k) = \frac{1-(0.9)^8}{1-0.9W_8^k}$ for $k = 0 \dots 7$

3.4 Let $X(k) = (1, j, -1, -j)$ and $H(k) = (0, 1, -1, 1)$ be the DFTs of two sequences $x(n)$ and $h(n)$, respectively. Using the properties of DFT, determine the DFT's of the following:

Solution:

(a) $x[((n-1))_4]$

$DFT\{x((n-1))_4\} = W_4^{k(1)} X(k)$

$= [W_4^0, W_4^1, W_4^2, W_4^3] \cdot [1, j, -1, -j]$

$= [1, -j, -1, +j] \cdot [1, j, -1, -j]$

$= (1, 1, 1, 1)$

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(b) $DFT\{x((n+3))_4\}$

$= W_4^{-3k}X(k) = [W_4^0, W_4^{-3}, W_4^{-2}, W_4^{-1}][1, j, -1, -j]$

$= [1, -j, -1, +j][1, j, -1, -j]$

$= [1, 1, 1, 1]$

(c) $Y(k) = H(k)\,X(k)$

$= [0, 1, -1, 1][1, j, -1, -j]$

$= [0, j, 1, -j]$

(d) $DFT\{(-1)^n x(n)\} = \sum_{n=0}^{3} x(n)\left(e^{-j\frac{2\pi}{4}}\right)^{kn}$

$= \sum_{n=0}^{3} (-1)^n x(n) \left(e^{-j2\pi/4}\right)^{kn}$

$= \sum_{n=0}^{3} \left(e^{-j\pi}\right)^n x(n) \left(e^{-j\pi/2}\right)^{kn} = \sum_{n=0}^{3} x(n)\, e^{-j\pi/2(k+2)n}$

$= X((k+2))_2$

$X((k+2))_2 = [-1, -j, 1, j]$

(e) $DFT\{j^n x(n)\} = \sum_{n=0}^{3} \left(e^{-j\frac{3\pi}{2}}\right)^n x(n)\, W_4^{kn}$

$= \sum_{n=0}^{3} W_4^{3n}\, x(n)\, W_4^{kn}$

$= X((k+3))_4$

$X((k+3))_4 = (-j, 1, j, -1)$

(f) $x((-n))_4$

$DFT\{x((-n))_4\} = X((-k))_4$

$= (1, -j, -1, j)$

(g) $DFT\{x((2-n))_4\} = DFT\{y((n-2))_4\}$ provided $\boxed{y(n) = x((-n))_4}$

$\therefore$ $DFT\{x((2-n))_4\} = DFT\{y((n-2))_4\}$

$= W_4^{2k}\, Y(k)$ | since, $y(n) = x((-n))_4$

$= W_4^{2k}\, X((-k))_4$

$= [W_4^0, W_4^2, W_4^0, W_4^2][1, -j, -1, j]$

$= [1, -1, 1, -1][1, -j, -1, j]$

$= [1, j, -1, -j]$

3.5 Let $x(n)$, $n = 0 \dots 7$ be an 8-point sequence with DFT $X(k) = [1, 1-j, 1, 0, 1, 0, 1, 1+j]$. Using properties of DFT, determine the DFT of the following sequences:

(a) $DFT\{x(n)\, e^{j\frac{2\pi}{8}n}\} = X((k-1))_8$

$X((k-1))_8 = [1+j, 1, 1-j, 1, 0, 1, 0, 1]$

(b) $DFT\{x(n) \circledast \delta((n-2))_8\}$

$DFT\{x((n-2))_8\} = W_8^{2k}\, X(k)$

$= [1, -j, -1, j, 1, -j, -1, j][1, 1-j, 1, 0, 1, 0, 1, 1+j]$

$= (1, -1-j, -1, 0, 1, 0, -1, -1+j)$

3.6 A four point sequence $x(n)$ has DFT $X(k) = [1, j, 1, -j]$. Using properties of DFT, determine the DFT of the following sequences:

(a) $DFT\{(-1)^n x(n)\} = DFT\left\{\left(e^{+j\frac{2\pi}{4}}\right)^{2n} x(n)\right\}$

$= X((k-2))_4 = (1, -j, 1, j)$

(b) $DFT\{x((n+1))_4\} = W_4^{-k(1)}\, X(k)$

$= [W_4^0, W_4^{-1}, W_4^{-2}, W_4^{-3}][1, j, 1, -j]$

$= [1, +j, -1, -j][1, j, 1, -j]$

$= [1, -1, -1, -1]$

(c) $DFT\{x(n) \circledast \delta((n-2))_4\}$

$= DFT\{x((n-2))_4\} = W_4^{2k} X(k)$

$= (W_4^0, W_4^2, W_4^0, W_4^2)(1, j, 1, -j)$

$= (1, -1, 1, -1)(1, j, 1, -j)$

$= (1, -j, 1, +j)$

(d) $DFT\{x((-n))_4\} = X((-k))_4$

$= (1, -j, 1, j)$

3.7 Two finite sequences $x(n) = [x(0), x(1), x(2), x(3)]$ and $h(n) = [h(0), h(1), h(2), h(3)]$ have DFT's given by

$X(k) = DFT\{x(n)\} = [1, j, -1, -j]$

$H(k) = DFT\{h(n)\} = [0, 1+j, 1, 1-j]$

Using the properties of the DFT, compute the following:

(a) $DFT\{[x(3), x(0), x(1), x(2)]\}$

$DFT\{x((n-1))_4\} = W_4^k X(k)$

$= [1, -j, -1, j][1, j, -1, -j]$

$= [1, 1, 1, 1]$

(b) $DFT\{[h(0), -h(1), h(2), -h(3)]\}$

$DFT\{(-1)^n h(n)\} = DFT\{(e^{\frac{j2\pi}{4}})^{2n} h(n)\}$

$= H((k-2))_4$

$= [1, 1-j, 0, 1+j]$

(c) $DFT\{h(n) \circledast x(n)\} = H(k)X(k)$

$= \underset{X(k)}{[1, j, -1, -j]}\ \underset{H(k)}{[0, 1+j, 1, 1-j]}$

$= [0, -1+j, -1, -1-j]$

(d) $DFT\{x(0), h(0), x(1), h(1), x(2), h(2), x(3), h(3)\}$

$y(n) = [x(0), h(0), x(1), h(1), x(2), h(2), x(3), h(3)]$

$$DFT\{y(n)\} = \sum_{n=0}^{7} y(n) W_8^{nk}$$

$$= \sum_{m=0}^{3} y(2m) W_8^{2mk} + \sum_{m=0}^{3} y(2m+1) W_8^{(2m+1)k}$$

$Y(k) = X(k) + W_8^k H(k)$ for $k = 0 \dots 7$

$X(k) = [1, j, -1, -j]$ & $H(k) = [0, 1+j, 1, 1-j]$

$\Rightarrow Y(k) = \underbrace{[1, j, -1, -j]}_{N/2} + W_8^k \underbrace{[0, 1+j, 1, 1-j]}_{N/2}$ $k = 0 \dots 7$

$Y(k) = [1 + W_8^0(0),\ j + (1+j)W_8^1,\ -1 + 1(W_8^2),\ -j + (1-j)W_8^3,$
$1 + W_8^4(0),\ j + (1+j)W_8^5,\ -1 + 1(W_8^6),\ -j + (1-j)(W_8^7)]$

$W_8^0 = 1$	$W_8^4 = (-W_8^0) = -1$
$W_8^1 = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$	$W_8^5 = (-W_8^1) = -\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$
$W_8^2 = -j$	$W_8^6 = -W_8^2 = -(-j) = +j$
$W_8^3 = -\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$	$W_8^7 = -W_8^3 = +\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$

$Y(k) = [1,\ j + (1+j)e^{-j\pi/4},\ -1 - j,\ -j + (1-j)e^{-j3\pi/4},\ 1,$
$j + (1+j)e^{-j5\pi/4},\ -1 + j,\ -j + (1-j)e^{-j7\pi/4}]$

or

$= [1,\ j + (1+j)e^{-j\pi/4},\ -1 - j,\ -j + (1-j)e^{-j3\pi/4},\ 1,$
$j + (1+j)e^{+j3\pi/4},\ -1 + j,\ -j + (1-j)e^{+j\pi/4}]$

$DFT\{\delta(n-n_0)\} = W_N^{kn_0}$

~~7.2(a) Contd.~~

~~$X_3(k) = X_1(k) X_2(k)$~~

~~$= (1 + W_8^k + W_8^{2k} + W_8^{3k})[0 + 0.92W_8^k + 0.707W_8^{2k} + 0.38W_8^{3k} + W_8^{4k} - 0.38W_8^{5k} + 0.707W_8^{6k} + 0.92W_8^{7k}]$~~

~~$X_3(k) = 1.25 + 2.54W_8^k + 2.54W_8^{2k} + 1.24W_8^{3k} + 0.24W_8^{4k} - 1.05W_8^{5k} - 1.053W_8^{6k} + 0.25W_8^{7k}$~~

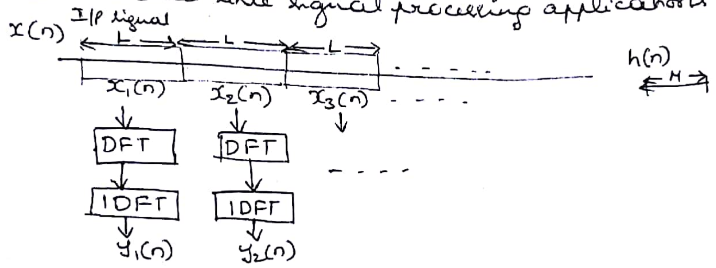
~~$\therefore x_3(n) = (1.25, 2.54, 2.54, 1.24, 0.24, -1.05, -1.05, 0.25)$~~

Unit-IV : Fast-fourier T

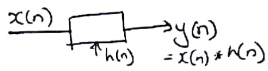
filtering of long data sequences :

* In practical applications, the input sequence $x(n)$ is often a very long sequence.

Eg: In real-time signal processing applications



$y(n) = y_1(n) + y_2(n) - - - -$



There are two methods

1. Overlap-add
2. Overlap-save

I] Overlap-add Method :

* Size of data blocks $\longrightarrow$ L points
* append $(M-1)$ zeros to each data block
* M is the length of $h(n)$, append $(L-1)$ zeros to $h(n)$
* Compute N-point DFT of each i/p data block $x(n)$
* Compute N-point DFT of $h(n)$
* find $Y(k) = H(k)\,X(k) \qquad k = 0 \cdots N-1$
* Compute IDFT of each block to find $y(n)$
* O/p blocks are fitted together to get $y(n)$.

Example (1):

Perform $x(n) * h(n)$ for the sequences $x(n)$ & $h(n)$ given below, using overlap-add method.

$h(n) = (1, 1, 1)$

$x(n) = (1, 2, 0, -3, 4, 2, -1, 1, -2, 3, 2, 1, -3)$

$M = 3, \quad \therefore N = 2^M = 8, \quad \therefore L = 6$

$N = M + L - 1$

$8 = 3 + L - 1$

$\underline{L = 6}$

$x_1(n) = (\underbrace{1, 2, 0, -3, 4, 2}_{L \text{ values}}, \underbrace{0, 0}_{(M-1) \text{ zeros}})$

$x_2(n) = (\underbrace{-1, 1, -2, 3, 2, 1}_{L \text{ values}}, \underbrace{0, 0}_{(M-1) \text{ zeros}})$

$h(n) = (\underbrace{1, 1, 1}_{M \text{ values}}, \underbrace{0, 0, 0, 0, 0}_{(L-1) \text{ zeros}})$

$y_1(n) = x_1(n) \; Ⓝ \; h(n)$

$\rightarrow y_1(n) = \text{IDFT}\{Y_1(k)\} = \text{IDFT}\{X_1(k) \cdot H(k)\}$

$y_1(n) = \text{IDFT}\left\{(1 + 2W_8^k - 3W_8^{3k} + 4W_8^{4k} + 2W_8^{5k})(1 + W_8^k + W_8^{2k})\right\}$

$y_1(n) = \text{IDFT}\left\{1 + 3W_8^k + 3W_8^{2k} - W_8^{3k} + W_8^{4k} + 3W_8^{5k} + 6W_8^{6k} + 2W_8^{7k}\right\}$

Since $\text{IDFT}\{W_N^{kn_0}\} = \delta(n - n_0)$.

$y_1(n) = 1 + 3\delta(n-1) + 3\delta(n-2) - \delta(n-3) + \delta(n-4) + 3\delta(n-5) + 6\delta(n-6) + 2\delta(n-7)$

$y_1(n) = (1, 3, 3, -1, 1, 3, 6, 2)$ ——— (1)

$y_2(n) = x_2(n) \; Ⓝ \; h(n)$

$y_2(n) = \text{IDFT}\{Y_2(k)\} = \text{IDFT}\{X_2(k) \cdot H(k)\}$

$y_2(n) = \text{IDFT}\left\{(-1 + W_8^k - 2W_8^{2k} + 3W_8^{3k} + 2W_8^{4k} + W_8^{5k})(1 + W_8^k + W_8^{2k})\right\}$

$y_2(n) = \text{IDFT}\left\{-1 - 2W_8^{2k} + 2W_8^{3k} + 3W_8^{4k} + 6W_8^{5k} + 3W_8^{6k} + W_8^{7k}\right\}$.

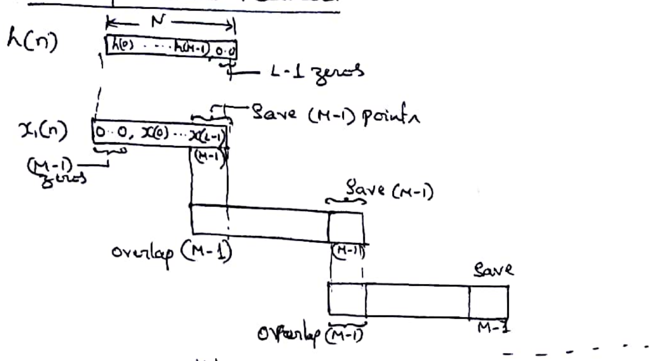
$\text{IDFT}\{W_N^{kn_0}\} = \delta(n - n_0)$

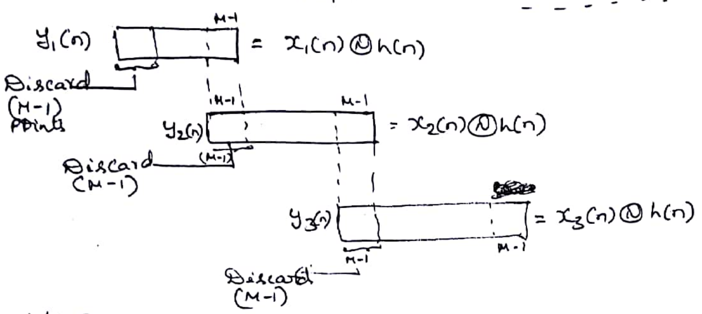
$y_2(n) = (-1, 0, -2, 2, 3, 6, 3, 1)$

(M-1) pts overlap

$y_1(n) =$	1	3	3	-1	1	3	6	2						
$y_2(n) =$							-1	0	-2	2	3	6	3	1
$y(n) =$	1,	3,	3,	-1,	1,	3	5,	2	-2,	2,	3,	6,	3,	1

Overlap - Save Method





$y(n) = y_1(n), y_2(n), y_3(n) \ldots$

Example :(1)

Using overlap-save method compute $y(n) = x(n) * h(n)$ Perform circular convolution, where $h(n) = (1,1,1)$ & $x(n) = (1, 2, 0, -3, 4, 2, -1, 1, -2, 3, 2, 1)$ (8)

Solution: length of $h(n) = 3 = M$

$N = 2^3 = 8$, $N = L + M - 1$

$8 = L + 3 - 1$

$L = 6$

$x_1(n) = (\underbrace{0, 0}_{(M-1)\text{ zeros}}, 1, 2, 0, -3, \overbrace{4, 2}^{(M-1)})$

$x_2(n) = (\underbrace{4, 2}_{(M-1)\text{ overlap}}, -1, 1, -2, 3, 2, 1)$

$h(n) = (1, 1, 1, 0, 0, 0, 0, 0)$

$y_1(n) = x_1(n) \circledast_8 h(n)$

$y_1(n) = \text{IDFT}\{Y_1(K)\} = \text{IDFT}\{X_1(K)\ H(K)\}$

$y_1(n) = \text{IDFT}\{(W_8^{2K} + 2W_8^{3K} - 3W_8^{5K} + 4W_8^{6K} + 2W_8^{7K})(1 + W_8^{K} + W_8^{2K})\}$

$y_1(n) = \text{IDFT}\{6 + 2W_8^{K} + W_8^{2K} + 3W_8^{3K} + 3W_8^{4K} - W_8^{5K} + W_8^{6K} + 3W_8^{7K}\}$

$\text{DFT}\{\delta(n-n_0)\} = W_N^{kn_0}$

$\therefore \hat{y_1}(n) = \{6 + 2\delta(n-1) + \delta(n-2) + 3\delta(n-3) + 3\delta(n-4) - \delta(n-5) + \delta(n-6) + 3\delta(n-7)\}$

$\hat{y_1}(n) = (\underbrace{6, 2}_{M-1\text{ points Discarded}}, 1, 3, 3, -1, 1, 3)$

$y_1(n) = (1, 3, 3, -1, 1, 3)$ —— (1)

$y_2(n) = x_2(n) \circledast_8 h(n)$

$y_2(n) = \text{IDFT}\{Y_2(K)\} = \text{IDFT}\{X_2(K) \cdot H(K)\}$

$y_2(n) = \text{IDFT}\{(4 + 2W_8^{K} - W_8^{2K} + W_8^{3K} - 2W_8^{4K} + 3W_8^{5K} + 2W_8^{6K} + W_8^{7K}) \times (1 + W_8^{K} + W_8^{2K})\}$

$y_2(n) = \text{IDFT}\{7 + 7W_8^{K} + 5W_8^{2K} + 2W_8^{3K} - 2W_8^{4K} + 2W_8^{5K} + 3W_8^{6K} + 6W_8^{7K}\}$

$\text{DFT}\{\delta(n-n_0)\} = W_N^{kn_0}$

$\hat{y_2}(n) = (\underbrace{7, 7}_{M-1\text{ points discarded}}, 5, 2, -2, 2, 3, 6)$

$y_2(n) = (5, 2, -2, 2, 3, 6)$

$y(n) = (y_1(n), y_2(n))$

$= (1, 3, 3, -1, 1, 3, 5, 2, -2, 2, 3, 6 \ldots)$

Example:(2)

Let $x(n) = (1, 1, -1, -1, 2, -1, 3, -3, 5, 0, 2, 1)$

& $h(n) = (1, 2, 1)$

Using over-lap-save method, find $y(n) = x(n) * h(n)$

Perform only 5-pt circular convolution.

$N=5 \Rightarrow$ given

$M=3 \rightarrow$ length of $h(n)$

$\therefore \; N = L+M-1$

$5 = L+3-1$

$L = 3$

$x_1(n)=\{0,0,1,1,-1\}$

$x_2(n)=\{1,-1,-1,2,-1\}$

$x_3(n)=\{2,-1,3,-3,5\}$

$x_4(n)=\{-3,5,0,2,1\}$

$h(n)=\{1,2,1,0,0\}$

$y_1(n)=x_1(n) ⓝ h(n)$

$=\{\cancel{-1},\cancel{-1},1,3,2\}$

$y_2(n)=x_2(n) ⓝ h(n)$

$=\{\cancel{1},\cancel{0},-2,-1,2\}$

$y_3(n)=x_3(n) ⓝ h(n)$

$=\{\cancel{9},\cancel{8},3,2,2\}$

$y_4(n)=x_4(n) ⓝ h(n)$

$=\{\cancel{1},\cancel{0},7,7,5\}$

$y(n)=\{1,3,2,-2,-1,2,3,2,2,7,7,5\}$

Unit-II - Fast Fourier Transform

* for the efficient implementation of DFT

$$X(k)=\sum_{n=0}^{N-1} x(n)\,W_N^{kn} \qquad k=0\cdots N-1$$

Complex Multiplications : N^2
for N-pt DFT

Complex Additions : $N(N-1)$
for N-pt DFT

when $x(n)$ ~~is complex~~ and $X(k)$ are complex valued.

$$X_R(k)=\sum_{n=0}^{N-1}\left[x_R(n)\cos\frac{2\pi kn}{N}+x_I(n)\sin\frac{2\pi kn}{N}\right]$$

$$X_I(k)=-\sum_{n=0}^{N-1}\left[x_R(n)\sin\frac{2\pi kn}{N}-x_I(n)\cos\frac{2\pi kn}{N}\right]$$

for: N-pt DFT

# of trigonometric functions – $2N^2$

# of real multiplications – $4N^2$

# of real additions – $4N(N-1)$

Based on Divide-and-Conquer, there are two approaches

1. Decimation in time (DIT)
2. Decimation in frequency (DIF)

} Fast fourier Transform

I Decimation in time fast fourier transform

$$X(k)=\sum_{n=0}^{N-1} x(n)\,W_N^{kn} \qquad k=0\cdots N-1 \quad \text{---(1)}$$

$$=\sum_{\text{even}} x(n)W_N^{kn}+\sum_{\text{odd}} x(n)W_N^{kn}$$

$$=\sum_{m=0}^{N/2-1} x(2m)\,W_N^{2km}+\sum_{m=0}^{N/2-1} x(2m+1)\,W_N^{(2m+1)k}$$

$$=\sum_{m=0}^{N/2-1} f_1(m)\,W_{N/2}^{km}+W_N^{k}\sum_{m=0}^{N/2-1} f_2(m)\,W_{N/2}^{km}$$

$$X(k)=F_1(k)+W_N^{k}F_2(k) \quad \text{---(2)} \quad k=0\cdots N/2-1$$

Periodic with N/2

$$X(k+N/2)=F_1(k+N/2)+W_N^{(k+N/2)}F_2(k+N/2)$$

$$=F_1(k)-W_N^{k}F_2(k) \quad \text{---(3)}$$

$k=0\cdots N/2-1$

|||ly

$$F_1(k)=V_{11}(k)+W_{N/2}^{k}V_{12}(k) \quad \text{---(4)}$$

$$F_1(k+N/4)=V_{11}(k)-W_{N/2}^{k}V_{12}(k) \quad \text{---(5)}$$

$k=0\cdots N/4-1$

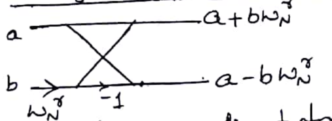
$$F_2(K) = V_{21}(K) + W_{N/2}^{K} V_{22}(K)$$
$$F_2(K+N/4) = V_{21}(K) - W_{N/2}^{K} V_{22}(K)$$
$$K = 0 \cdots \tfrac{N}{4}-1$$

where
$$V_{11}(K) = DFT\{v_{11}(n)\} = DFT\{f_1(2n)\}$$
$$V_{12}(K) = DFT\{v_{12}(n)\} = DFT\{f_1(2n+1)\}$$
$$V_{21}(K) = DFT\{v_{21}(n)\} = DFT\{f_2(2n)\}$$
$$V_{22}(K) = DFT\{v_{22}(n)\} = DFT\{f_2(2n+1)\}$$
$$K = 0 \cdots N/4 - 1$$

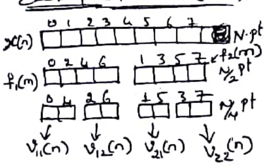
& are periodic with $N/4$

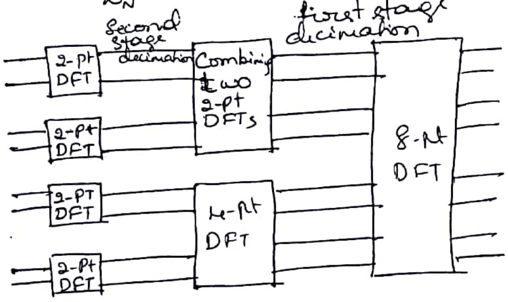
Butterfly diagram-radix-2

∴



Example for DIT (N=8)





~~For performing~~

To compute DFT,

$N = 2^v$, $v = \log_2 N$ times of decimation can be done.

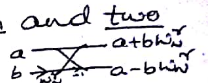
To compute an N-pt DFT using FFT we require

$\log_2 N$ # of stages

$N/2 \rightarrow$ butterfly diagrams per stage

there in total $(N/2 \log_2 N)$ # of butterfly diagrams are needed.

for one butterfly diagram, we have to perform one complex multiplication and two complex additions.



∴ for $(N/2 \log_2 N)$ # of butterfly diagrams we need $(N/2 \log_2 N)$ complex multiplications and $(N \log_2 N)$ complex additions.

Bit-reversed I/P & O/P

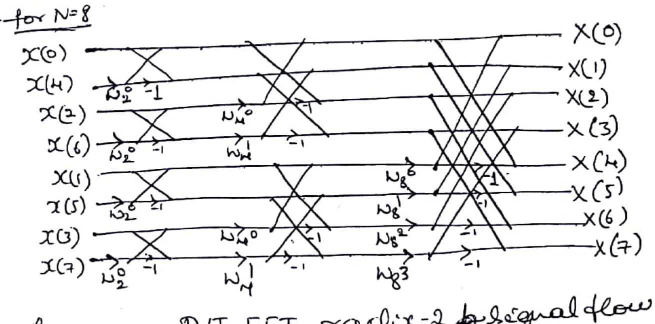
In order	$n_2 n_1 n_0$	$n_1 n_0 n_2$	$n_0 n_1 n_2$ (Bit reversed order)
0 →	0 0 0	0 0 0	0 0 0 → 0
1 →	0 0 1	0 1 0	1 0 0 → 4
2 →	0 1 0	1 0 0	0 1 0 → 2
3 →	0 1 1	1 1 0	1 1 0 → 6
4 →	1 0 0	0 0 1	0 0 1 → 1
5 →	1 0 1	0 1 1	1 0 1 → 5
6 →	1 1 0	1 0 1	0 1 1 → 3
7 →	1 1 1	1 1 1	1 1 1 → 7

for N-pt DFT, if the input is given in order we get output in bit-reversed order.

& when the input is given in bit-reversed order output will be in order.

※ for, Decimation-in-time, input is given in bit-reversed order and output will be in order.

Eg:- for N=8



for N=8, DIT-FFT-radix-2, signal flow graph

The above diagram for $N=8$ has

$\log_2 N = \log_2 8 = 3 \Rightarrow$ Stages

$N/2 = 8/2 = 4 \Rightarrow$ Butterfly diagrams per stage.

Total # of complex multiplications:

$$1\left(\frac{N}{2}\log_2 N\right) = \frac{N}{2}\log_2 N = \underline{12}$$

Total # of complex additions

$$2\left(\frac{N}{2}\log_2 N\right) = N\log_2 N = \underline{24}.$$

(Q) What is in-place computation?

For computing N-point DFT using FFT algorithms, 2N registers are enough to hold the values, ~~as~~ As output of the previous stage ~~need not be~~ need not be stored.

Examples:

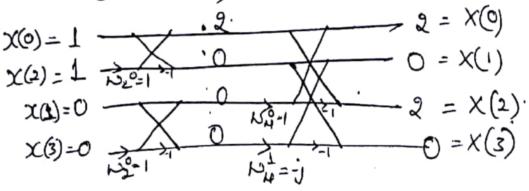
Using radix-2, DIT-FFT algorithm, determine the DFT of the following sequences

(a) $x(n) = (1, 0, 1, 0)$

(b) $x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$

(c) $x(n) = (1, 2, 3, 4, 4, 3, 2, 1)$

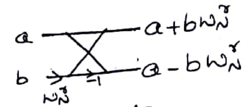
(a) $x(n) = (1, 0, 1, 0)$



$W_4^0 = \left(e^{-j\frac{2\pi}{4}}\right)^0 = 1$, $W_4^1 = \left(e^{-j\frac{2\pi}{4}}\right)^1 = -j$

$W_4^2 = \left(e^{-j\frac{2\pi}{4}}\right)^2 = -1$, $W_4^3 = \left(e^{-j\frac{2\pi}{4}}\right)^3 = +j$

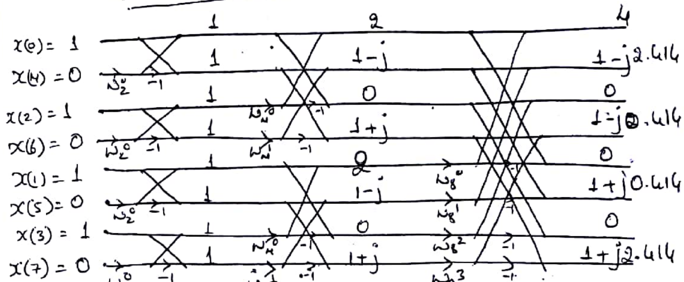
(b) $x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$



Scaling factors

$W_8^0 = 1$	$W_8^4 = -1 = (-W_8^0)$
$W_8^1 = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$	$W_8^5 = -\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}} = (-W_8^1)$
$W_8^2 = -j$	$W_8^6 = +j = (-W_8^2)$
$W_8^3 = -\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$	$W_8^7 = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}} = (-W_8^3)$

$W_2^0 = W_4^0 = W_8^0 = 1$, $W_4^1 = W_8^2 = -j$

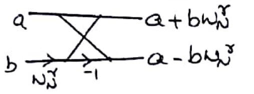


Output at first stage	II Stage o/p	III Stage o/p
$X_1(0) = 1 + 0 \times W_2^0$	$X_2(0) = 1 + W_4^0 \times 1 = 2$	$X_3(0) = 2 + 2W_8^0 = 4$
$X_1(1) = 1 - 0 \times W_2^0$	$X_2(1) = 1 + 1 \times W_4^1 = 1 - j$	$X_3(1) = 1 - j + (1-j)W_8^1 = 1 - j2.414$
$X_1(2) = 1 + 0 \times W_2^0$	$X_2(2) = 1 - 1 \times W_4^0 = 0$	$X_3(2) = 0 + 0 \times W_8^2 = 0$
$X_1(3) = 1 - 0 \times W_2^0$	$X_2(3) = 1 - 1 \times W_4^1 = 1 + j$	$X_3(3) = (1+j) + (1+j)W_8^3 = 1 - j0.414$
$X_1(4) = 1 + 0 \times W_2^0$	$X_2(4) = 1 + 1 \times W_4^0 = 2$	$X_3(4) = 2 - 2W_8^0 = 0$
$X_1(5) = 1 - 0 \times W_2^0$	$X_2(5) = 1 + 1 \times W_4^1 = 1 - j$	$X_3(5) = (1-j) - (1-j)W_8^1 = 1 + j0.414$
$X_1(6) = 1 + 0 \times W_2^0$	$X_2(6) = 1 - 1 \times W_4^0 = 0$	$X_3(6) = 0 - 0 \times W_8^2 = 0$
$X_1(7) = 1 - 0 \times W_2^0$	$X_2(7) = 1 - 1 \times W_4^1 = 1 + j$	$X_3(7) = (1+j) - (1+j)W_8^3 = 1 + j2.414$

$\therefore X(k) = [4,\ 1-j2.414,\ 0,\ 1-j0.414,\ 0,\ 1+j0.414,\ 0,\ 1+j2.414]$

Since, $x(n)$ is real, $X(k) = X^*(N-k)$

(c) $x(n) = (1, 2, 3, 4, 4, 3, 2, 1)$



$W_8^0 = W_4^0 = W_2^0 = 1$

$W_8^1 = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

$W_4^1 = W_8^2 = -j$

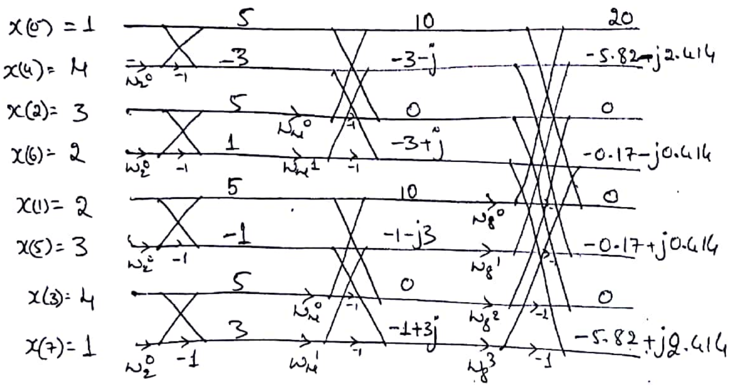
$W_8^3 = -\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

$W_8^4 = -(W_8^0)$

$W_8^5 = -(W_8^1)$

$W_8^6 = -(W_8^2)$

$W_8^7 = -(W_8^3)$



I Stage o/p	II Stage o/p	III Stage o/p
$X_1(0) = 1+4\times W_2^0 = 5$	$X_2(0) = 5+5\times W_4^0 = 10$	$X_3(0) = 10+10\times W_8^0 = 20$
$X_1(1) = 1-4\times W_2^0 = -3$	$X_2(1) = -3+1\times W_4^1 = -3-j$	$X_3(1) = -3-j+(-1-3j)W_8^1 = -5.82-j2.414$
$X_1(2) = 3+2\times W_2^0 = 5$	$X_2(2) = 5-5\times W_4^0 = 0$	$X_3(2) = 0+0\times W_8^2 = 0$
$X_1(3) = 3-2\times W_2^0 = 1$	$X_2(3) = -3-1\times W_4^1 = -3+j$	$X_3(3) = -3+j+(-1+3j)W_8^3 = -0.171-j0.414$
$X_1(4) = 2+3\times W_2^0 = 5$	$X_2(4) = 5+5\times W_4^0 = 10$	$X_3(4) = 10-10\times W_8^0 = 0$
$X_1(5) = 2-3\times W_2^0 = -1$	$X_2(5) = -1+3\times W_4^1 = -1-j3$	$X_3(5) = -3-j-(-1-3j)W_8^1 = -0.171+j0.414$
$X_1(6) = 4+1\times W_2^0 = 5$	$X_2(6) = 5-5\times W_4^0 = 0$	$X_3(6) = 0-0\times W_8^2 = 0$
$X_1(7) = 4-1\times W_2^0 = 3$	$X_2(7) = -1-3\times W_4^1 = -1+3j$	$X_3(7) = -3+j-(-1+3j)W_8^3 = -5.82+j2.414$

$\therefore X(k) = [20,\ -5.82-j2.414,\ 0,\ -0.17-j0.414,\ 0,\ -0.17+j0.414,\ 0,\ -5.82+j2.414]$

II Decimation in frequency (DIF) FFT

The output, $X(k)$ is divided into smaller and smaller subsequences. N is considered to be a power of 2, $N = 2^v$ — (1)

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad k = 0 \dots N-1$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=N/2}^{N-1} x(n) W_N^{kn}$$

$m = n - N/2$, $n = m + N/2$ & $n = N-1$, $m = \frac{N}{2} - 1$

$n = N/2$, $m = 0$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{m=0}^{\frac{N}{2}-1} x\left(m+\frac{N}{2}\right) W_N^{k\left(m+\frac{N}{2}\right)}$$

Changing 'm' to 'n' ~~n~~ we get.

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + W_N^{N/2\,k} \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{kn}$$

$$\cancel{= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left(n+\frac{N}{2}\right)\right]}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^k x(n+N/2)\right] W_N^{kn} \quad \text{— (2)}$$

$X(k)$ is decimated taking even and odd samples.

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2k} x(n+N/2)\right] W_N^{2kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \underbrace{\left[x(n) + x(n+N/2)\right]}_{e(n)} W_{N/2}^{kn} \quad \text{— (3)}$$

III^{ly}

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{(2k+1)} x(n+N/2)\right] W_N^{(2k+1)n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \underbrace{\left\{\left[x(n) - x(n+N/2)\right] W_N^{n}\right\}}_{o(n)} \cdot W_{N/2}^{kn} \quad \text{— (4)}$$

$\therefore$ $e(n) = x(n) + x(n + N/2)$ —— (5)

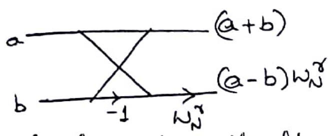
$o(n) = [x(n) - x(n + N/2)] W_N^n$ —— (6)

Using (5) & (6) in (3) & (4), respectively we get

$$X(2k) = \sum_{n=0}^{N/2 - 1} e(n)\, W_{N/2}^{kn}$$

$$X(2k+1) = \sum_{n=0}^{N/2 - 1} o(n)\, W_{N/2}^{kn} \qquad k = 0 \cdots \frac{N}{2} - 1$$

This decimation is continued → 2-pt DFTs,



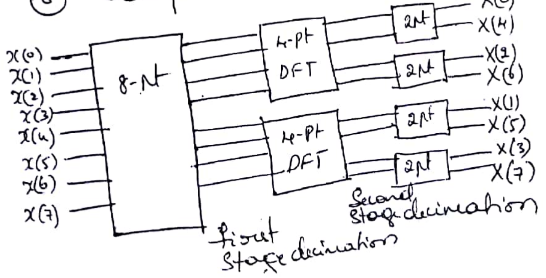
→ 1 → complex multiplication

→ 2 → complex additions

radix-2 Butterfly diagram for DIF-FFT

✗. for N-point DFT using radix-2, DIT-FFT

1. # of stages —— $\log_2 N$
2. # of Butterfly diagrams per stage —— $N/2$
3. I/p is given in order
4. O/p will be in bit-reversed order
5. Complex multiplications → $\frac{N}{2} \log_2 N$
6. Complex additions ——→ $N \log_2 N$



Example:

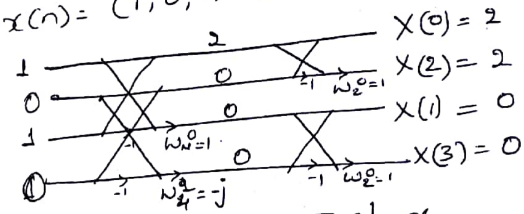
find DFT of the following sequences using DIF-FFT ~~algorithm~~ method.

(a) $x(n) = (1, 0, 1, 0)$

(b) $x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$

(c) $x(n) = (1, 2, 3, 4, 4, 3, 2, 1)$

(a) $x(n) = (1, 0, 1, 0)$

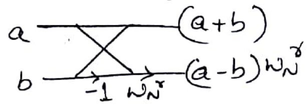


O/p I stage	O/p II stage
$1 + 1 = 2$	$2 + 0 = 2$
$0 + 0 = 0$	$(2 - 0)1 = 2$
$(1 - 1) W_N^0 = 0$	$0 + 0 = 0$
$(0 - 0)\cdot -j = 0$	$(0 - 0)1 = 0$

Since, the O/p is in bit-reversed order

$X(k) = (2, 0, 2, 0)$

(b) $x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$

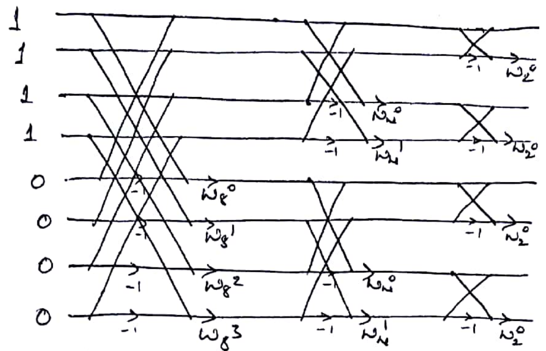


$W_8^0 = W_4^0 = W_2^0 = 1$

$W_8^1 = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

$W_8^2 = -j = W_4^1$

$W_8^3 = -\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$



I Stage o/p

$1+0=1$

$1+0=1$

$1+0=1$

$1+0=1$

$(1-0)1 = 1 = w_8^0$

$(1-0)\frac{1}{\sqrt2}-\frac{j}{\sqrt2} = 0.707-j0.707 = w_8^1$

$(1-0)-j = -j = w_8^2$

$(1-0)-\frac{1}{\sqrt2}-\frac{j}{\sqrt2} = -0.707-j0.707 = w_8^3$

II Stage o/p

$1+1=2$

$1+1=2$

$(1-1)1=0$

$(1-1)j=0$

$1+(-j)=1-j$

$w_8^1+w_8^3$

$(1-(-j))1 = 1+j$

$(w_8^1-w_8^3)w_4^1$

III Stage o/p

$X(0) = 2+2 = 4$

$X(4) = (2-2)1 = 0$

$X(2) = 0+0 = 0$

$X(6) = (0-0)1 = 0$

$X(1) = (1-j)+(w_8^1+w_8^3) = 1-j+\frac{1}{\sqrt2}-\frac{j}{\sqrt2}-\frac{1}{\sqrt2}-\frac{j}{\sqrt2} = 1-2.414j$

$X(5) = [(1-j)-(w_8^1+w_8^3)]1 = 1+j0.414$

$X(3) = [(1+j)+(w_8^1-w_8^3)w_4^1] = 1-j0.414$

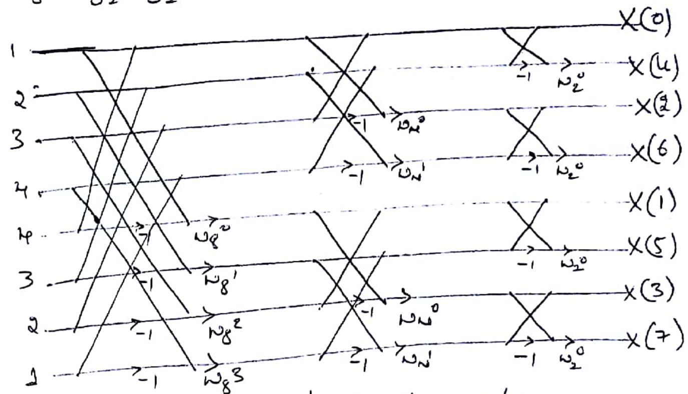
$X(7) = [(1+j)-(w_8^1-w_8^3)w_4^1]1 = 1+j2.414$

$X(k) = \{4, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$

(c) $x(n) = (1,2,3,4,4,3,2,1)$

$w_8^0 = w_4^0 = w_2^0 = 1$ $\quad w_8^2 = w_4^1 = -j$

$w_8^1 = \frac{1}{\sqrt2}-\frac{j}{\sqrt2}$ $\quad w_8^3 = \frac{-1}{\sqrt2}-\frac{j}{\sqrt2}$



I Stage o/p

$1+4=5$

$2+3=5$

$3+2=5$

$4+1=5$

$(1-4)w_8^0 = -3$

$(2-3)w_8^1 = -w_8^1$

$(3-2)w_8^2 = w_8^2$

$(4-1)w_8^3 = 3w_8^3$

II Stage o/p

10

10

0

0

$(-3+w_8^2)$

$(-w_8^1+3w_8^3)$

$(-3-w_8^2)w_4^0$

$(-w_8^1-3w_8^3)w_4^1$

III Stage o/p

$10+10=20$

$10-10=0$

$0+0=0$

$0-0=0$

$(-3+w_8^2)+(-w_8^1+3w_8^3) = -5.82-j2.414$

$(-3+w_8^2)-(-w_8^1+3w_8^3) = -0.171+j0.414$

$(-3-w_8^2)w_4^0+(-w_8^1-3w_8^3)w_4^1 = -0.171-j0.414$

$(-3-w_8^2)w_4^0-(-w_8^1-3w_8^3)w_4^1 = -5.82+j2.414$

$\therefore X(k) = [20, -5.82-j2.414, 0, -0.171-j0.414, 0, -0.171+j0.414, 0, -5.82+j2.414]$

Inverse Decimation in-time (IDIT-FFT)

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \quad \text{--- (1)}$$

$$= \frac{1}{N}\left[\sum_{k=0}^{N/2-1} X(k)\, W_N^{-kn} + \sum_{k=N/2}^{N-1} X(k)\, W_N^{-kn}\right]$$

changing limits

$l = k - N/2$ $\quad k = N/2 \quad l = 0$

$k = l + N/2$ $\quad k = N-1 \quad l = \frac{N}{2} - 1$

$$= \frac{1}{N}\left[\sum_{k=0}^{N/2-1} X(k)\, W_N^{-kn} + \sum_{l=0}^{N/2-1} X(l+N/2)\, W_N^{-(N/2+l)n}\right]$$

$$= \frac{1}{N}\left[\sum_{k=0}^{N/2-1} X(k)\, W_N^{-kn} + \sum_{k=0}^{N/2-1} X(k+N/2)\, W_N^{-(k+N/2)n}\right]$$

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N/2-1}\left(X(k) + (-1)^n X(k+N/2)\right) W_N^{-kn}\right] \quad \text{--- (2)}$$

for $n = 2r$

$$x(2r) = \frac{1}{2}\left[\frac{1}{N/2}\left\{\sum_{k=0}^{N/2-1}\underbrace{\left[X(k) + X(k+N/2)\right]}\, W_{N/2}^{-kr}\right\}\right] \quad \text{--- (3)}$$

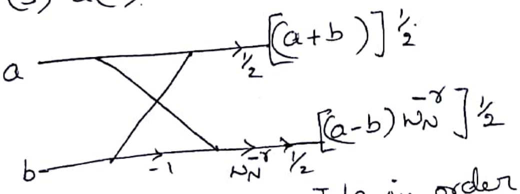
for $n = 2r+1$

$$x(2r+1) = \frac{1}{2}\left[\frac{1}{N/2}\left\{\sum_{k=0}^{N/2-1}\left[X(k) - X(k+N/2)\right] W_N^{-k(2r+1)}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{N/2}\left\{\sum_{k=0}^{N/2-1}\left\{X(k) - X(k+N/2)\right\} W_{N/2}^{-kr}\, W_N^{-k}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{N/2}\left\{\sum_{k=0}^{N/2-1}\underbrace{\left[\left[X(k) - X(k+N/2)\right] W_N^{-k}\right]}\, W_{N/2}^{-kr}\right\}\right] \quad \text{--- (4)}$$

from (3) & (4)

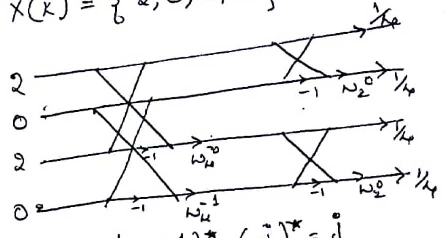


I/p in order
o/p in bit-reversed order

Examples

1] find inverse DFT of the following $X(k)$ using inverse DIT-FFT method.

(a) $X(k) = \{2, 0, 2, 0\}$



$W_4^0 = 1, \quad W_4^{-1} = (W_4^1)^* = (-j)^* = j$

I Stage o/p

$2 + 2 = 4$

$0 + 0 = 0$

$(2-2)1 = 0$

$(0-0)j = 0$

II Stage o/p

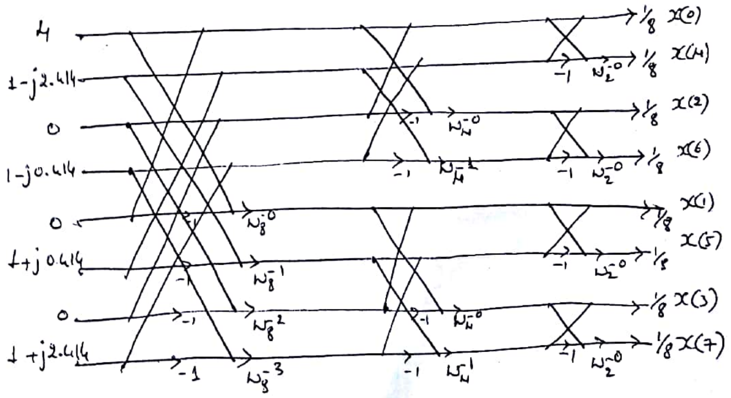
$4 + 0$	$= 4 \times 1/4$	$= 1$		$x(0)$
$(4-0)1$	$= 4 \times 1/4$	$= 1$		$x(2)$
0	$= 0$	$= 0$		$x(1)$
0	$= 0$	$= 0.$		$x(3).$

$x(n) = (1, 0, 1, 0)$

(b) $X(k) = [4, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414]$

$W_8^0 = W_4^0 = W_2^0 = 1$ $\quad W_8^{-2} = W_4^{-1} = +j$

$W_8^{-1} = (W_8^1)^* = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$ $\quad W_8^{-3} = (W_8^3)^* = -\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}.$



I Stage o/p

$4+0 = 4$

$1 - 2.414j + 1 + 0.414j = 2-2j$

$0+0 = 0$

$1 - j0.414 + 1 + j2.414 = 2+2j$

$(4-0)W_8^0 = 4$

$[(1-j2.414) - (1+j0.414)]W_8^{-1} = (-2.828j)W_8^{-1}$

$(0-0)W_8^{-2} = 0$

$[(1-j0.414) - (1+j2.414)]W_8^{-3}$
$= (-2.828j)W_8^{-3}$

II Stage o/p

$4+0 = 4$

$(2-2j) + (2+2j) = 4$

$(4-0)1 = 4$

$[(2-2j) - (2+2j)]\, j = 4$

$4+0 = 4$

$(-2.828j)W_8^{-1} - (2.828j)W_8^{-3} = 4$

$(4-0)1 = 4$

$[(-2.828j)W_8^{-1} + 2.828j\,W_8^{-3}]\, j$
$= 4$

III Stage o/p

$4+4 = 8 \times 1/8 = 1 = x(0)$

$(4-4)1 = 0 \times 1/8 = 0 = x(4)$

$4+4 = 8 \times 1/8 = 1 = x(2)$

$(4-4)1 = 0 \times 1/8 = 0 = x(6)$

$4+4 = 8 \times 1/8 = 1 = x(1)$

$(4-4)1 = 0 \times 1/8 = 0 = x(5)$

$4+4 = 8 \times 1/8 = 1 = x(3)$

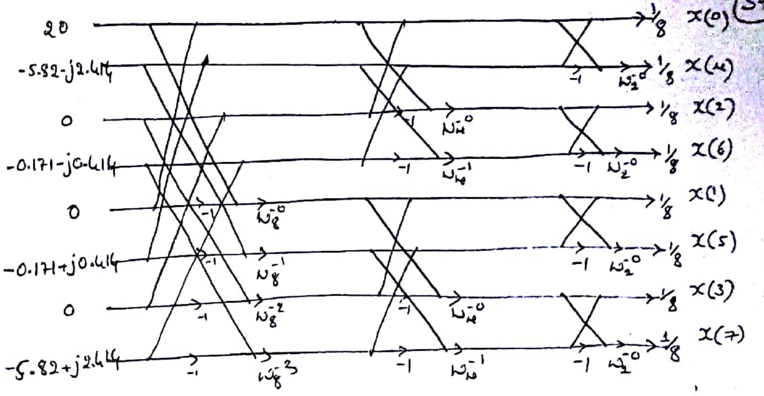
$(4-4)1 = 0 \times 1/8 = 0 = x(7)$

$x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$

(c) $X(K) = [20, -5.82-j2.414, 0, -0.171-j0.414, 0, -0.171+j0.414, 0, -5.82+j2.414]$

$W_8^0 = W_N^{-0} = W_2^{-0} = 1$ $\quad W_8^{-2} = +j$

$W_8^{-1} = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$ $\quad W_8^{-3} = \frac{-1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$



I Stage o/p

1) $20+0 = 20$

2) $(-5.82-j2.414) + (-0.171+j0.414)$
$= -6-2j$

3) $0+0 = 0$

4) $(-0.171-j0.414) + (-5.82+j2.414)$
$= -6+2j$

5) $(20-0)W_8^0 = 20$

6) $[(-5.82-j2.414) - (-0.171+j0.414)]W_8^{-1}$
$= -2-6j$

7) $(0-0)W_8^{-2} = 0$

8) $[(-0.171-j0.414) - (-5.82-j2.414)]W_8^{-3}$
$= -2+6j$

II Stage o/p

1) $20+0 = 20$

2) $(-6-2j) + (-6+2j) = -12$

3) $(20-0)W_4^{-0} = 20$

4) $[(-6-2j) - (-6+2j)]W_4^{-1}$
$= 4$

5) $(20+0) = 20$

6) $(-2-6j) + (-2+6j) = -4$

7) $(20-0)W_4^{-0} = 20$

8) $[(-2-6j) - (-2+6j)]W_4^{-1} = 12$

III Stage o/p

1) $20 + (-12) = 8 \times 1/8 = 1 = x(0)$

2) $(20-(-12))1 = 32 \times 1/8 = 4 = x(4)$

3) $(20+4) = 24 \times 1/8 = 3 = x(2)$

4) $(20-4)1 = 16 \times 1/8 = 2 = x(6)$

5) $20 + (-4) = 16 \times 1/8 = 2 = x(1)$

6) $(20-(-4))1 = 24 \times 1/8 = 3 = x(5)$

7) $20+12 = 32 \times 1/8 = 4 = x(3)$

8) $(20-12)1 = 8 \times 1/8 = 1 = x(7)$

$\therefore x(n) = (1, 2, 3, 4, 4, 3, 2, 1)$

Inverse Decimation in frequency (IDIF-FFT)

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \qquad n = 0 \ldots N-1 \quad \text{— (1)}$$

$$= \frac{1}{N}\left[\sum_{l=0}^{N/2-1} X(2l)\, W_N^{-2ln} + \sum_{l=0}^{N/2-1} X(2l+1)\, W_N^{-(2l+1)n}\right]$$

$$= \frac{1}{N}\left[\sum_{l=0}^{N/2-1} X(2l)\, W_{N/2}^{-ln} + \sum_{l=0}^{N/2-1} X(2l+1)\, W_{N/2}^{-ln}\, W_N^{-n}\right]$$

$$= \frac{1}{2}\,\frac{1}{N/2}\left[\sum_{l=0}^{N/2-1} F_1(l)\, W_{N/2}^{-ln} + \sum_{l=0}^{N/2-1} F_2(l)\, W_{N/2}^{-ln}\; W_N^{-n}\right]$$

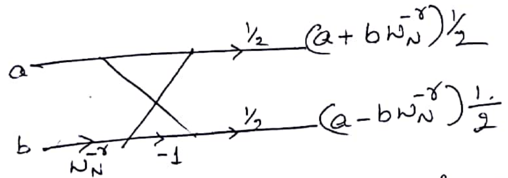
$$x(n) = \frac{1}{2}\left[f_1(n) + W_N^{-n} f_2(n)\right] \qquad n = 0 \ldots N/2 - 1$$

$$x(n) = \frac{1}{2}\left[\left(f_1(n) + W_N^{-n} f_2(n)\right)\right] \quad \text{— (2)}$$

$$x(n+N/2) = \frac{1}{2}\left[\left(f_1(n+N/2) + W_N^{-(n+N/2)} f_2(n+N/2)\right)\right]$$

$$= \frac{1}{2}\left[\left(f_1(n) - W_N^{-n} f_2(n)\right)\right] \quad \text{— (3)}$$

from (2) & (3)



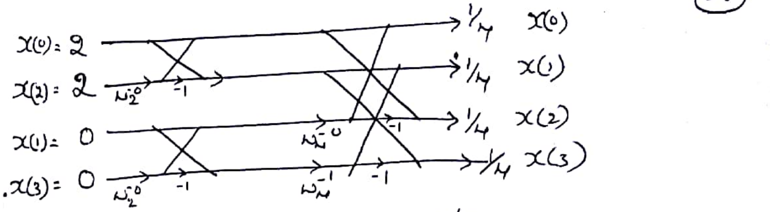
※ I/p is given in bit-reversed order and o/p will be in order.

Example

(Q) find $x(n)$ for the following $X(k)$ using inverse decimation-in-frequency FFT method.

(a) $X(k) = [2, 0, 2, 0]$

(b)



I Stage o/p

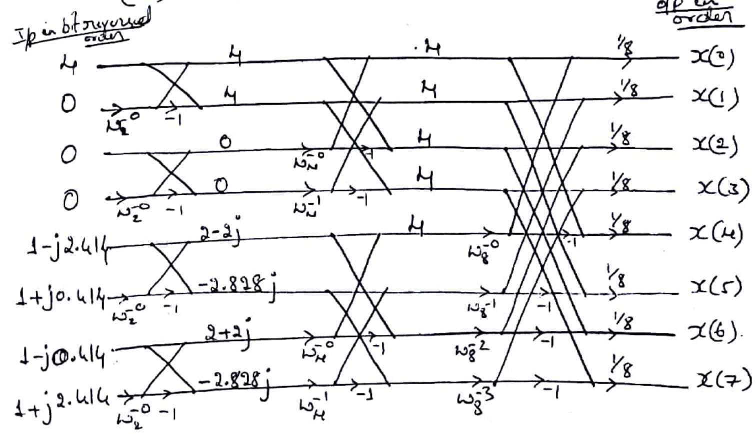
1. $2 + 2(1) = 4$
2. $2 - 2(1) = 0$
3. $0 = 0$
4. $0 = 0$

II Stage o/p

1. $4 + 0(1) = 4 \times 1/4 = 1 \quad x(0)$
2. $(0 + 0(j)) = 0 \times 1/4 = 0 \quad x(1)$
3. $4 - 0(1) = 4 \times 1/4 = 1 \quad x(2)$
4. $0 - 0(j) = 0 \times 1/4 = 0 \quad x(3)$

$\therefore x(n) = [1, 0, 1, 0]$

(b) $X(k) = [4,\ 1-j2.414,\ 0,\ 1-j0.414,\ 0,\ 1+j0.414,\ 0,\ 1+j2.414]$



I Stage o/p

1. $4 + (0)(1)$
2. $4 - (0)(1)$
3. $0 + 0$
4. $0 - 0$
5. $(1 - j2.414) + (1 + j0.414)(1) = 2 - 2j$
6. $(1 - j2.414) - (1 + j0.414)(1) = -2.828j$
7. $(1 - j0.414) + (1 + j2.414)(1) = 2 + 2j$
8. $(1 - j0.414) - (1 + j2.414)(1) = -2.828j$

II Stage o/p

① $4 + 0(1) = 4$

② $4 + 0(j) = 4$

③ $4 - 0(1) = 4$

④ $4 - 0(j) = 4$

⑤ $(2-2j)+(2+2j)(1) = 4$

⑥ $(-2.828j)+(-2.828j)(j) = 2.828 - 2.828j$

⑦ $(2-2j)-(2+2j)(1) = -4j$

⑧ $(-2.828j)-(-2.828j)j = -2.828 - 2.828j$

III Stage o/p

① $4+4 = 8 \times 1/8 = 1$

② $4+(2.828-2.828j)W_8^{-1} = 8 \times 1/8 = 1$

③ $4+(-4j)W_8^{-2} = 8 \times 1/8 = 1$

④ $4+(-2.828-2.828j)W_8^{-3} = 8 \times 1/8 = 1$

⑤ $4-4(1) = 0$

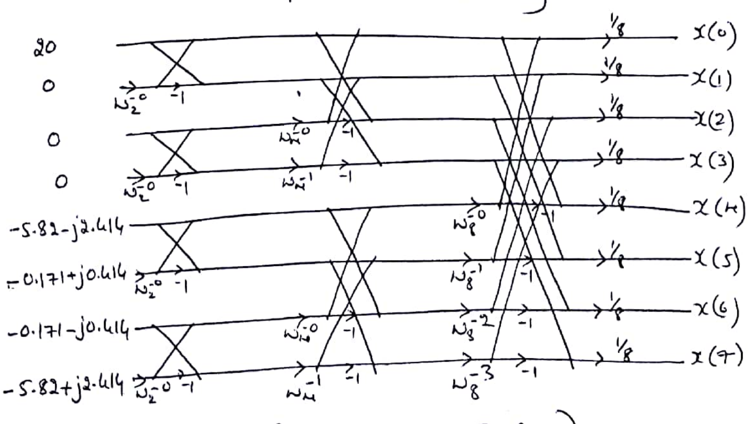
⑥ $4-(2.828-2.828j)W_8^{-1} = 0$

⑦ $4-(-4j)W_8^{-2} = 4+4j(j) = 0$

⑧ $4-(-2.828-2.828j)W_8^{-3} = 0$

$\therefore x(n) = [1, 1, 1, 1, 0, 0, 0, 0]$

(c) $X(k) = [20, -5.82-j2.414, 0, -0.171-j0.414, 0, -0.171+j0.414, 0, -5.82+j2.414]$



$x(n) = (1, 2, 3, 4, 4, 3, 2, 1)$

I Stage o/p

① $20 + 0(1)$

② $20 - 0(1)$

③ 0

④ 0

⑤ $(-5.82-j2.414) + (-0.171+j0.414)(1) = -6 - 2j$

⑥ $(-5.82-j2.414) - (-0.171+j0.414)(1) = -5.65 - 2.83j$

⑦ $(-0.171-j0.414) + (-5.82+j2.414)(1) = -6 + 2j$

⑧ $(-0.171-j0.414) - (-5.82+j2.414)(1) = +5.65 - 2.83j$

II Stage o/p

① $20+0(1) = 20$

② $20+0(j) = 20$

③ $20-0(1) = 20$

④ $20-0(j) = 20$

⑤ $(-6-2j)+(-6+2j)(1) = -12$

⑥ $(-5.65-j2.83)+(5.65-2.83j)j = -2.82 + 2.82j$

⑦ $(-6-2j) - (-6+2j)(1) = -4j$

⑧ $(-5.65-j2.83)-(5.65-2.83j)j = -5.65-j2.83 - 5.65j + 2.83j^2 = -8.48 - 8.48j$

III Stage o/p

① $20+(-12)(1) = 8 \times 1/8 = 1 = x(0)$

② $20+(-2.82+2.82j)W_8^{-1} = 16 \times 1/8 = 2 = x(1)$

③ $20+(-4j)W_8^{-2} = 24 \times 1/8 = 3 = x(2)$

④ $20+(-8.48-8.48j)W_8^{-3} = 32 \times 1/8 = 4 = x(3)$

⑤ $20-(-12)(1) = 32 \times 1/8 = 4 = x(4)$

⑥ $20-(-2.82+2.82j)W_8^{-1} = 24 \times 1/8 = 3 = x(5)$

⑦ $20-(-4j)W_8^{-2} = 16 \times 1/8 = 2 = x(6)$

⑧ $20-(-8.48-8.48j)W_8^{-3} = 8 \times 1/8 = 1 = x(7)$

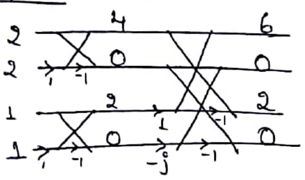
Aplications

① $x_1(n) = (2,1,2,1)$ & $x_2(n) = (1,2,3,4)$
Determine $y(n) = x_1(n) \text{ Ⓝ } x_2(n)$ using radix 2 DIT-FFT and IDIT-FFT method.

Solution:-

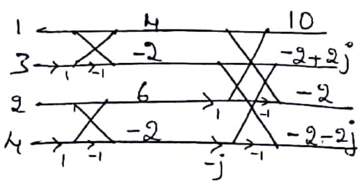
① find $X_1(k)$.
② find $X_2(k)$
③ $Y(k) = X_1(k) \cdot X_2(k)$
④ $y(n) = \text{IDFT}\{Y(k)\}$

Step 1: $x_1(n) = (2,1,2,1)$



$\therefore X_1(k) = [6,0,2,0]$

Step 2: $x_2(n) = (1,2,3,4)$



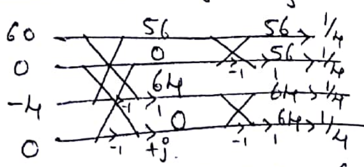
$X_2(k) = [10, -2+2j, -2, -2-2j]$

Step 3: $Y(k) = X_1(k) \cdot X_2(k)$

$= [6,0,2,0]\,[10, -2+2j, -2, -2-2j]$

$= [60, 0, -4, 0]$

Step 4: $\text{IDFT}\{Y(k)\}$



$= 14 = y(0)$
$= 14 = y(1)$
$= 16 = y(2)$
$= 16 = y(3)$

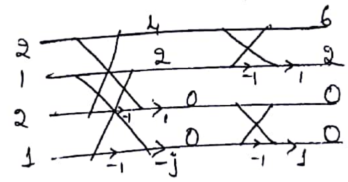
$\therefore y(n) = (14, 16, 14, 16)$

② Determine $y(n) = x_1(n) \text{ Ⓝ } x_2(n)$ using radix-2 DIF-FFT and IDIF-FFT method

$x_1(n) = (2,1,2,1)$ & $x_2(n) = (1,2,3,4)$

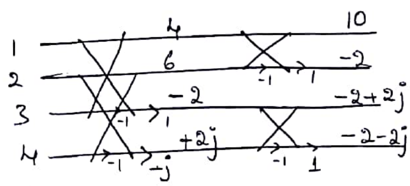
Solution

Step 1: $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$



$X_1(k) = [6,0,2,0]$

Step 2: $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$



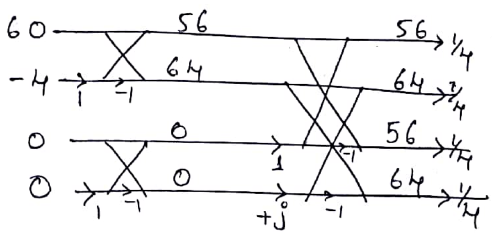
$X_2(k) = [10, -2+2j, -2, -2-2j]$

Step 3: $Y(k) = X_1(k)\, X_2(k)$

$= [6,0,2,0]\,[10, -2+2j, -2, -2-2j]$

$= [60, 0, -4, 0]$

Step 4: $\text{IDFT}\{Y(k)\} = y(n)$



$= 14 = y(0)$
$= 16 = y(1)$
$= 14 = y(2)$
$= 16 = y(3)$

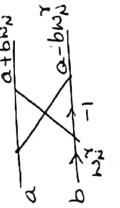
$\therefore y(n) = (14, 16, 14, 16)$

Fast fourier Transform (FFT)

Radix-2

FFT → DIT, DIF

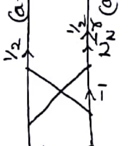
DIT
① I/p in Bit-Reversed order
② O/p in order
③



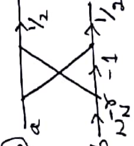
DIF
① I/p in order
② O/p in Bit-Reversed order
③

IFFT → IDIT, IDIF

IDIT
① I/p in order
② O/p in Bit-reversed order
③



IDIF
① I/p in Bit-reversed order
② O/p in order
③



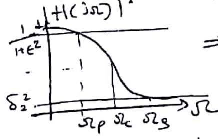
For N-pt DFT using FFT:-
Complex Multiplications: $\frac{N}{2}\log_2 N$
Complex Additions: $N\log_2 N$

$N/2$ # of Butterfly diagrams per stage
$\log_2 N$ # of stages

Analog filters

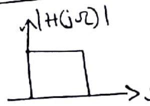
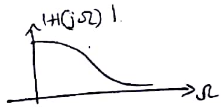
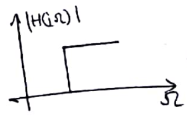
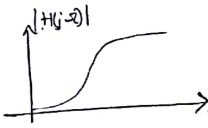
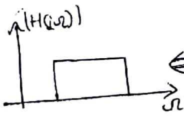
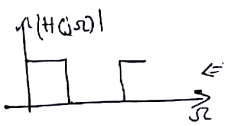
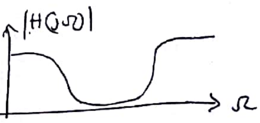
Introduction:

* An important approach to the design of digital filters is to apply a transformation to an existing analog filter.
* These analog procedures normally begin with a specification of the frequency response for the filter describing how the filter reacts to the sinusoidal inputs in the steady state.



⇒ The frequency response of an analog filter is characterized by its passband frequency and attenuation ripple, stopband frequency and attenuation.

Basic types of frequency response

Ideal		Non-Ideal
	⇐ LPF ⇒ Lowpass filter	
	⇐ HPF ⇒ High pass filter	
	⇐ BPF ⇒ Bandpass filter	
	⇐ BSF ⇒ Bandstop filter	

1) Determine 4-point DFT of the sequence $x(n) = (0.5)^n;\ 0 \le n \le 3$ by sampling DTFT at 4-points per cycle.

$$x(n) = (0.5)^n;\ 0 \le n \le 3$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{3} (0.5)^n e^{-j\omega n}$$

$$= \frac{1-(0.5)^4 e^{-j4\omega}}{1-0.5e^{-j\omega}}$$

sampling at 4-points/cycle

$$\omega = \frac{2\pi}{4}k$$

$$X(k) = X(\omega)\Big|_{\omega=\frac{2\pi}{4}k};\ k = 0,1,2,3$$

$$X(k) = \frac{1-(0.5)^4 e^{-j4\frac{2\pi}{4}k}}{1-0.5e^{-j\frac{2\pi}{4}k}}$$

$$= \frac{0.9375}{1-0.5\cos\left(\frac{\pi k}{2}\right)+0.5j\sin\left(\frac{\pi k}{2}\right)}$$

$X(0) = 1.875$

$X(1) = 0.75 - 0.375j$

$X(2) = 0.625$

$X(3) = 0.75 + 0.375j$

2) Using linearity property find DFT of $x(n) = (2,1,1,1,1,1,1,1)$

$$x(n) = (1,0,0,0,0,0,0,0) + (1,1,1,1,1,1,1,1)$$

$$= x_1(n) + x_2(n)$$

$x_1(n) = \delta(n)$ & $x_2(n) = 1^n;\ 0 \le n \le 7$

$$X_1(k) = \sum_{n=0}^{N-1} \delta(n) W_N^{kn} = 1;\ k = 0,1,\ldots 7$$

$$X_2(k) = \sum_{n=0}^{7} 1^n e^{-j\frac{2\pi}{N}kn} = 8;\ k=0$$

$$= 0;\ k \ne 0$$

$$X(k) = X_1(k) + X_2(k)$$

$$= (9,1,1,1,1,1,1,1)$$

3) Given $x(n) = \left(\frac{1}{2}\right)^n [u(n) - u(n-4)]$. Find the following without computing 4-point DFT $X(k)$

1) $\sum_{k=0}^{3} X[k] X^*[k]$ 2) $X[0] + X[2]$

1) $$\sum_{k=0}^{3} X(k) X^*(k) = 4 \sum_{n=0}^{3} x(n) x^*(n)$$

from parseval's Theorem.

$$= 4\left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64}\right)$$

$$= 5.3125$$

2) $$X(0) + X(2) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8}$$

$$= 2 + \frac{1}{2} = 2.5$$

4) Given $x(n) = \delta(n) - 2\delta(n-5)$. Find $X(k)$. Let $Y(k) = e^{j\frac{2\pi k}{5}} X(k)$. Find $y(n)$. Let $Z(k) = X(k)W(k)$ where $W(k)$ is the 10-point DFT of $w(n) = (1,1,1,1,1,1,1,0,0,0)$ find $z(n)$.

$N = 10$. So $x(n) = (1,0,0,0,0,-2,0,0,0,0)$

$$X(k) = \sum_{n=0}^{9} x(n) e^{-j\frac{2\pi}{10}kn}$$

$X(0) = -1$

$$X(1) = \sum_{n=0}^{9} x(n) e^{-j\frac{2\pi}{10}n} = 1 + x(5)e^{-j\frac{2\pi}{10}5}$$

$$= 1 + 2 = 3$$

$X(2) = x(0) + x(5)\, e^{-j\frac{2\pi}{10}\times 2\times 5} = -1$

$X(3) = x(0) + x(5)\, e^{-j\frac{2\pi}{10}\times 3\times 5} = 3$

$X(4) = -1,\ X(5) = 3,\ X(6) = -1$

$X(7) = 3,\ X(8) = -1\ \ X(9) = 3$

$X(k) = (-1, 3, -1, 3, -1, 3, -1, 3, -1, 3)$

$$Y(k) = e^{j\frac{2\pi k}{5}} X(k) = e^{j\frac{2\pi}{10}\times k\times 2} X(k)$$

$$y(n) = x((n+2))_{10}$$

$$= (0,0,0,-2,0,0,0,0,1,0)$$

$Z(k) = X(k)\, W(k)$

$z(n) = x(n) \textcircled{10}\, w(n)$

$z(n) = (-1,-1,1,1,1,-1,-1,-2,-2,-2)$

5) A 174-point DFT X(k) of a real-valued sequence x(n) has the following DFT samples $X[0]=11$, $X[9]=-3.4+J5.9$, $X[k_1]=7.1+J2.4$, $X[51]=5-J1.6$, $X[k_2]=8.7+J4.9$, $X[87]=4.5$, $X[113]=8.7-J4.9$, $X[k_3]=5+J1.6$, $X[162]=7.1-J2.4$ and $X[k_4]=-3.4-J5.9$. Remaining DFT values are of zero value.

1) Find k_1, k_2, k_3 & k_4

from conjugate symmetry property

$$X[k] = X^*[N-k] \;;\; N=174$$

$$X[k_1] = X^*[174-k_1] = X^*[162]$$

$$174-k_1 = 162 \Rightarrow k_1 = 12$$

$$X[k_2] = X^*[174-k_2] = X^*[113]$$

$$174-k_2 = 113 \Rightarrow k_2 = 61$$

$$X[k_3] = X^*[174-k_3] = X^*[51]$$

$$174-k_3 = 51 \Rightarrow k_3 = 123$$

$$X[k_4] = X^*[174-k_4] = X^*[9]$$

$$174-k_4 = 9 \Rightarrow k_4 = 165$$

6) compute 8-point DFT of the sequence $x(n)=(1,2,2,1,3,2,4,3)$ using single 4-point DFT.

$x(n)=(1,2,2,1,3,2,4,3)$

even samples $x_1(n)=(1,2,3,4)$

odd samples $x_2(n)=(2,1,2,3)$

$$z(n) = x_1(n) + J x_2(n)$$

$$= (1+2J,\ 2+J,\ 3+2J,\ 4+3J)$$

$$Z(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -J & -1 & J \\ 1 & -1 & 1 & -1 \\ 1 & J & -1 & -J \end{bmatrix} \begin{bmatrix} 1+2J \\ 2+J \\ 3+2J \\ 4+3J \end{bmatrix} = \begin{bmatrix} 10+8J \\ -4+2J \\ -2 \\ -2J \end{bmatrix}$$

$$x_1(n) = \frac{z(n)+z^*(n)}{2}$$

$$X_1(k) = \frac{Z(k) + Z^*((-k))_4}{2}$$

$$= (10,\ -2+2J,\ -2,\ -2-2J)$$

$$x_2(n) = \frac{z(n)-z^*(n)}{2J}$$

$$X_2(k) = \frac{Z(k) - Z^*((-k))_4}{2J}$$

$$= (8,\ 2J,\ 0,\ -2J)$$

$$X(k) = X_1(k) + W_8^k X_2(k);\ k=0,1,2,3$$

$$X(k+4) = X_1(k) - W_8^k X_2(k);\ k=0,1,2,3$$

$$X(k) = (18,\ -0.585+J3.414,\ -2,\ -3.414-J0.585,\ 2,\ -3.414+J0.585,\ -2,\ -0.585-J3.414)$$

7) Let $x_1(n)=(0,1,0,-1)$, $x_2(n)=(1,0,-3,0)$ and $x_3(n)=(1,2,2,1)$. Find corresponding DFTs using a single 4-point DFT.

let $y(n) = x_1(n) + x_2(n) = (1,1,-3,-1)$

circularly odd seq   circularly even seq

$$z(n) = y(n) + J x_3(n)$$

$$= (1+J,\ 1+2J,\ -3+2J,\ -1+J)$$

$$Z(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -J & -1 & J \\ 1 & -1 & 1 & -1 \\ 1 & J & -1 & -J \end{bmatrix} \begin{bmatrix} 1+J \\ 1+2J \\ -3+2J \\ -1+J \end{bmatrix} = \begin{bmatrix} -2+6J \\ 5-3J \\ -2 \\ 3+J \end{bmatrix}$$

$$Z^*((-k))_4 = (-2-6J,\ 3-J,\ -2,\ 5+3J)$$

$$y(n) = \frac{z(n)+z^*(n)}{2} \qquad x_3(n) = \frac{z(n)-z^*(n)}{2J}$$

$$Y(k) = \frac{Z(k)+Z^*((-k))_4}{2}$$

$$= (-2,\ 4-2J,\ -2,\ 4+2J)$$

but $y(n) = x_1(n) + x_2(n)$

Since $x_2(n)$ is even $\rightarrow$ DFT is purely real

$$X_2(k) = (-2,\ 4,\ -2,\ 4)$$

& $x_1(n)$ is odd $\rightarrow$ DFT is purely imaginary

$$X_1(k) = (0,\ -2J,\ 0,\ 2J)$$

$$X_3(k) = \frac{Z(k) - Z^*((-k))_4}{2J}$$

$$= (6,\ -1-J,\ 0,\ -1+J)$$

8) 5-point DFT of a complex sequence $x(n)$ is $X(k) = (J, 1-J, 1+2J, 2-J, 3-2J)$ Compute $Y(k)$ if $y(n) = x^*(n)$

$$y(n) = x^*(n)$$

$$Y(k) = X^*((-k))_5$$

$$= (-J, 3+2J, 2+J, 1-2J, 1+J)$$

9) For a real 4-point sequence if the first 3 DFT values of 4-point DFT are $(5, 2+3J, 3)$ determine the sequence.

4^{th} value is $X(3) = X^*(1)$
$= 2-3J$

$$x(n) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & J & -1 & -J \\ 1 & -1 & 1 & -1 \\ 1 & -J & -1 & J \end{bmatrix}\begin{bmatrix} 5 \\ 2+3J \\ 3 \\ 2-3J \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

10) let $x(n) = \delta(n) + 2\delta(n-1) + \delta(n-3)$ obtain $y(n)$ whose 5-point DFT $Y(k) = [X(k)]^2$ where $X(k)$ is 5-point DFT of $x(n)$.

$$x(n) = (1, 2, 0, 1, 0)$$

$$Y(k) = [X(k)]^2 = X(k)\,X(k)$$

$$y(n) = x(n) \text{ (5) } x(n)$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 4 \\ 2 \\ 4 \end{bmatrix}$$