DIGITAL COMMUNICATION

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BASEBAND PULSE SHAPING

Background on Random Process,
Power Spectrum of a Discrete PAM Signal

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Definition

A random variable can be considered to be a collection of real numbers, characterized by a probability description.

Similarly, a random process is an ensemble (collection) of real signals, characterized by a probability description.

It can also be viewed as a sequence/continuum of random variables.

It is denoted by $X(t)$ / $X(n)$.

Random Process

Definition

Examples:

Communications: message signal, noise, the received signal.

Statistics: Stock market index, rainfall, weather etc.

The result of sampling the random process $X(t)$ at $t = t_0$ is the random variable $X(t_0)$.

Any specific signal in the ensemble is called a “sample function” or realization, and is denoted by $x(t)$.

Definition

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A random process is said to be "deterministic" if the future values of any realization can be predicted from the past samples.

Ex:

1. $X(t) = A$, where $A \sim N(0, \sigma^2)$

2. $X(t) = A\cos(\omega_0 t + \phi)$, where A and ω_0 are constants, and $\phi \sim unf[-\pi, \pi]$

If the future values cannot be predicted from the past samples, then the process is called " non-deterministic".

Consider the r.p $X(t)$. At any $t = t_1$, we have the r.v $X_1 = X(t_1)$. Let its p.d.f be denoted by $f_X(x_1; t_1)$

we can define

$$\mu_X(t) = E[X(t)] \quad : \text{"Ensemble average" of } X(t).$$

$$\sigma_X^2(t) = E\left[\{X(t) - \mu_X(t)\}^2\right]$$

These are the "1st order statistics". They are, in general, functions of time.

[Also the mean squared value $E[X^2(t)]$]

Similarly, by considering the random variables

$X_1 = X(t_1)$ and $X_2 = X(t_2)$ at times t_1 and t_2,

we denote their joint p.d.f as $f_X(x_1, x_2; t_1, t_2)$

w.r.t this joint p.d.f, we can define

* "the autocorrelation function"

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

* "the autocovariance function".

$$C_X(t_1,t_2) = E\left[\{X(t_1) - \mu_X(t_1)\}\{X(t_2) - \mu_X(t_2)\}\right]$$

$$= R_X(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$$

* NOTE : $R_X(t_1,t_1) = E\left[X^2(t_1)\right]$

$C_X(t_1,t_1) = \sigma_X^2(t_1)$

In general, we can ~~define~~ denote the n^{th} order joint p.d.f as

$$f_X(x_1, x_2, \ldots, x_n ; t_1, t_2, \ldots, t_n)$$

Stationarity

A random process is said to be "first order stationary" if its first order density function does not change with time.

i.e, $f_X(x_1; t_1) = f_X(x_1; t_1 + \Delta)$

for any t_1 and any Δ.

$\Rightarrow$ $f_X(x_1; t_1)$ is independent of t_1, and can be expressed as $f_X(x_1)$.

Also

$$\left.\begin{aligned} \mu_X(t) &= \mu_X \\ \sigma_X^2(t) &= \sigma_X^2 \end{aligned}\right\} \text{ constants.}$$

Wide Sense Stationary (WSS)

The definitions of stationarity are "probabilistic" , and are difficult to either ensure or verify.

We instead go for a "statistical" notion of stationarity.

A process is said to be "wide sense stationary" (WSS),

if

i) $E[x(t)] = \mu_x$: a constant

ii) $R_x(t_1, t_2) = R_x(t_2 - t_1) = R_x(\tau)$

$$= E[x(t)\, x(t+\tau)]$$

Autocorrelation function

i.e., the autocorrelation is a function of only the "time lag" τ, and not of the actual time values t_1 and t_2.

Consider the autocorrelation function

$$R_X(\tau) = E[X(t)X(t+\tau)]$$

This function is a measure of the similarity between X(t) and X(t+τ).

For discrete-time random process,

$$R_X(k) = E[X(n)X(n+k)]$$

Power Spectrum of a Discrete PAM Signal

Discrete PAM Signals

Let $v(t)$ indicate the basic pulse shape for NRZ schemes. (Figure)

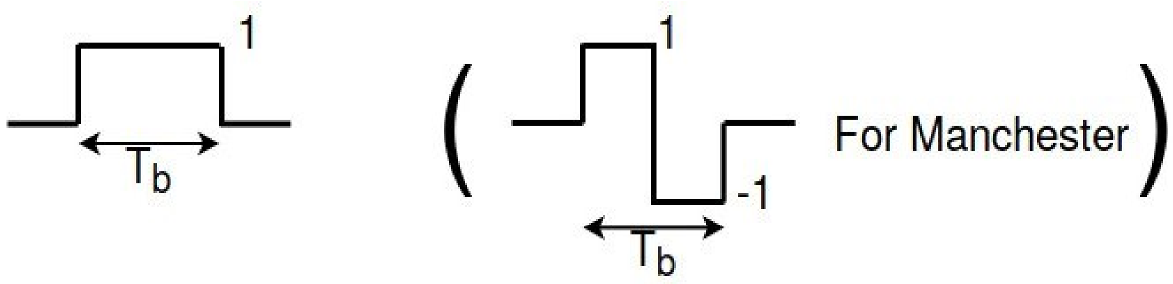


Figure :Basic Pulse Shape of NRZ

The different discrete PAM signals can be expressed as the realization of the random process given by:

$$X(t) = \sum_{k=-\infty}^{\infty} A_k v(t - kT_b) \qquad (1)$$

where A_k is a discrete random variable that determines the PAM format (Unipolar, Polar, etc..)

Discrete PAM Signals

A_k for the different pulse shaping formats can be expressed as:

$$\mathit{Unipolar}: \quad A_k = \begin{cases} a & \mathit{symbol}\ 1 \\ 0 & \mathit{symbol}\ 0 \end{cases} \tag{2}$$

$$\mathit{Polar/Manchester}: \quad A_k = \begin{cases} a & \mathit{symbol}\ 1 \\ -a & \mathit{symbol}\ 0 \end{cases} \tag{3}$$

$$\mathit{Bipolar}: \quad A_k = \begin{cases} a, -a & \mathit{alternating}\ 1s \\ 0 & \mathit{symbol}\ 0 \end{cases} \tag{4}$$

Power Spectrum of a Discrete PAM Signal

Discrete PAM Signals

We can rewrite $X(t)$ in eqn. (1) as:

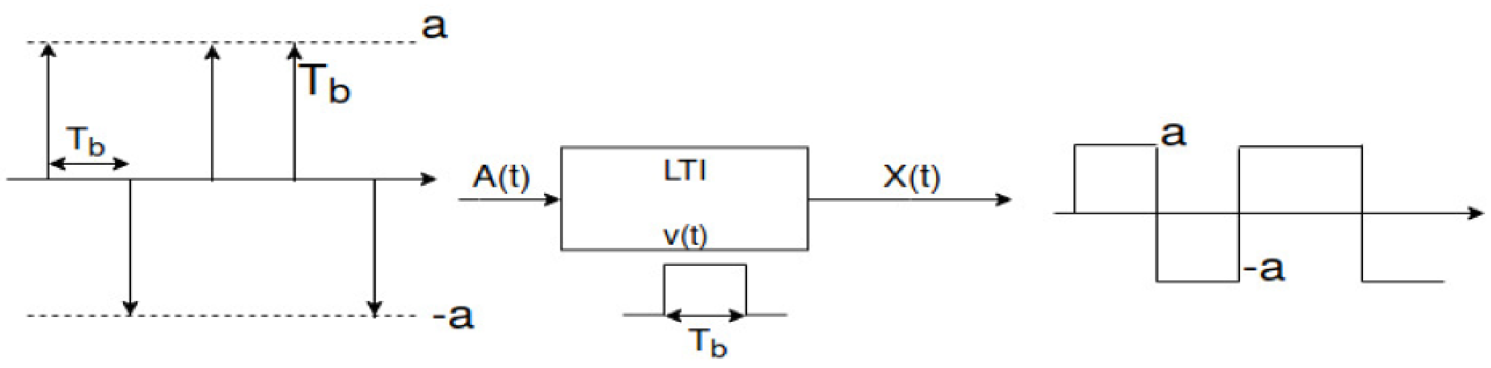
$$X(t) = \sum_{k=-\infty}^{\infty} A_k \left\{\delta(t - kT_b) * v(t)\right\}$$

$$= \left\{ \sum_{k=-\infty}^{\infty} A_k \delta(t - kT_b) \right\} * v(t)$$

$$= A(t) * v(t)$$

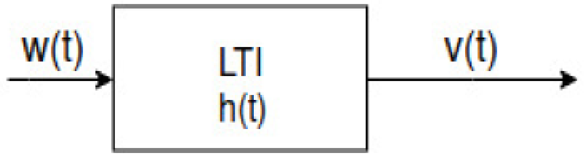
Power Spectrum of a Discrete PAM Signal
Discrete PAM Signals

The Generation of $X(t)$ can be modeled as:



Polar Example

If v(t) is a WSS process as:



Then,

$$S_v(f) = |H(f)|^2 S_w(f)$$

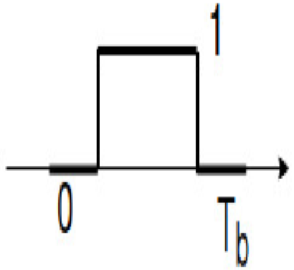
Discrete PAM Signals

In the present case, X(t) is a cyclostationary process with period T_b. Hence it follows that:

$$\boxed{S_X(f) = \frac{|V(f)|^2}{T_b} S_A(f)} \tag{5}$$

Finding $|V(f)|^2$:

$$w.k.t.\ v(t) = \begin{cases} 1 & 0 \leq t \leq T_b \\ 0 & Elsewhere \end{cases} \tag{6}$$



Discrete PAM Signals

$$V(f) = \int_{-\infty}^{\infty} v(t) e^{-j2\pi ft} \tag{7}$$

Substituting v(t) from (6) into (7), we get:

$$\begin{aligned}
V(f) &= \int_{0}^{T_b} v(t) e^{-j2\pi ft} \\
&= \frac{-1}{j2\pi f} e^{-j2\pi ft} \Big|_0^{T_b} = \frac{1}{j2\pi f}\left[1 - e^{-j2\pi f T_b}\right] \\
&= \frac{e^{-j\pi f T_b}}{\pi f}\left[\frac{e^{j\pi f T_b} - e^{-j\pi f T_b}}{2j}\right] \\
&= e^{-j\pi f T_b} \frac{sin(\pi f T_b)}{\pi f}
\end{aligned}$$

Discrete PAM Signals

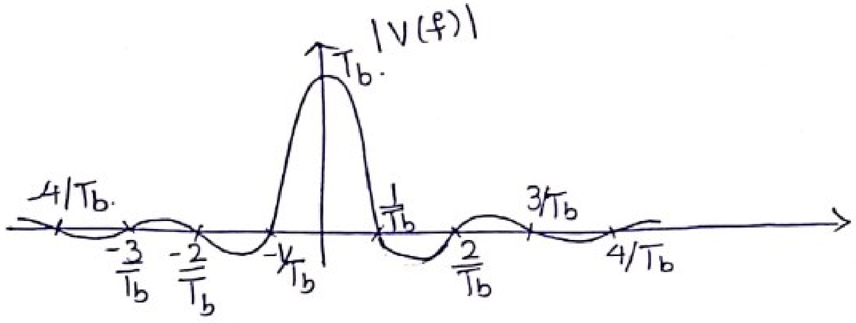
$$\therefore |V(f)| = \frac{sin(\pi f T_b)}{\pi f} = \frac{T_b sin(\pi f T_b)}{\pi f T_b}$$

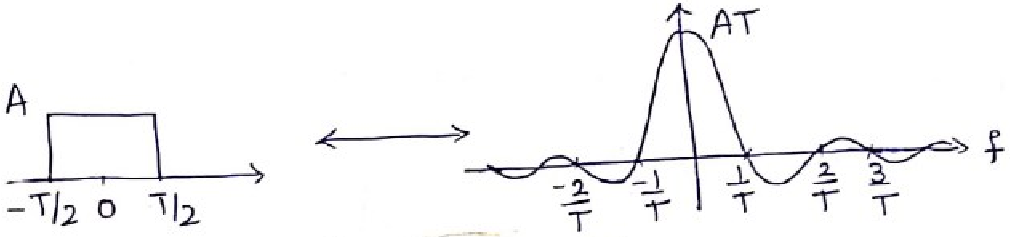
$$\therefore |V(f)| = T_b sinc(fT_b)$$

$$\boxed{\therefore |V(f)|^2 = {T_b}^2 sinc^2(fT_b)} \qquad (8)$$

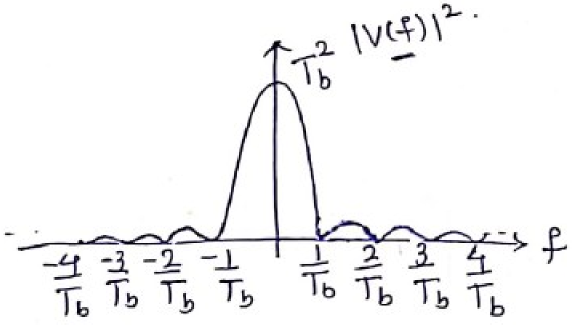
Power Spectrum of a Discrete PAM Signal

Discrete PAM Signals





$$\boxed{|V(f)|^2 = T_b^2 \, \text{sinc}^2(f T_b).}$$



Power Spectrum of a Discrete PAM Signal

Finding $S_A(f)$:

We know that power spectrum is the Fourier Transform of the Autocorrelation Function. Consider the sequence of samples x_k .

$$x(n) = \sum_{k=-\infty}^{\infty} x_k \delta(n-k) \qquad (9)$$

$$x(t) = \sum_{k=-\infty}^{\infty} x_k \delta(t-kT) \qquad (10)$$

$$x(n) \stackrel{\text{F.T}}{\longleftrightarrow} X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x_k e^{-j\omega k}$$

$$x(t) \stackrel{\text{F.T}}{\longleftrightarrow} X(f) = \int_{t=-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

Power Spectrum of a Discrete PAM Signal

Finding $S_A(f)$:

$$X(f) = \int_{t=-\infty}^{\infty} \sum_k x_k \delta(t - KT) e^{-j2\pi ft} dt$$

$$= \sum_k x_k \left\{ \int_{t=-\infty}^{\infty} \delta(t - KT) e^{-j2\pi ft} dt \right\}$$

$$WKT \quad \delta(t) \xleftrightarrow{F.T} 1 \text{and} \therefore \delta(t - KT) \xleftrightarrow{F.T} e^{-j2\pi fKT} \quad (Time\ Shift\ Property)$$

$$\boxed{X(f) = \sum_k x_k e^{-j2\pi fKT}} \qquad (11)$$

$$\boxed{S_A(f) = \sum_{n=-\infty}^{\infty} R_A(n) e^{-j2\pi fnT_b}} \quad where \quad \boxed{R_A(n) = E[A_k A_{k-n}]} \qquad (12)$$

Power Spectrum of a Discrete PAM Signal

Finding $S_A(f)$ for different schemes

We will find $S_X(f)$ for each of the following three cases:

i NRZ Unipolar

ii NRZ Polar

Iii NRZ Bipolar.

Iv Manchester Coding

Note: To obtain $S_X(f)$, we first find $S_A(f)$ from $R_A(n)$ and substitute in

$$S_X(f) = \frac{|V(f)|^2}{T_b} S_A(f)$$

THANK YOU

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