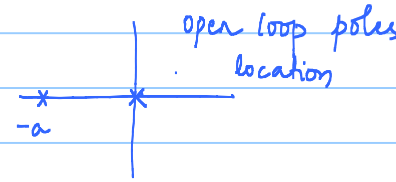
Root Locus

Note Title 28-10-2021

$$\boxed{G(s) = \frac{1}{s(s+a)}}$$. poles :



has two infinite zeros

Solving this would give closed loop poles $\rightarrow$ $1 + KG(s) = 0$

$$1 + K\frac{1}{s(s+a)} = 0$$

$$s^2 + as + k = 0$$

vary K from 0 to ∞ & find the poles of CLS.

When $k = 0$

$$s^2 + as = 0$$

$$s = 0, -a \qquad \text{(poles of CLS = poles of OLS)}$$

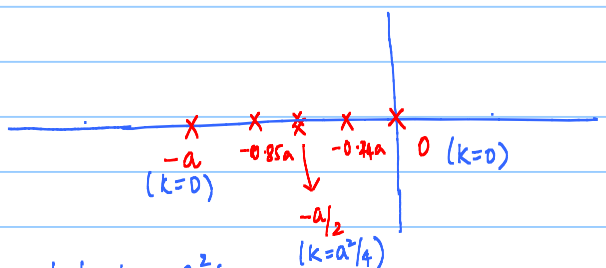
When $0 < K < a^2/4$

Let $K = a^2/8$

$$s^2 + as + a^2/8 = 0$$

$$s = \frac{-a \pm \sqrt{a^2 - 4a^2/8}}{2} = \frac{-a \pm a\sqrt{4/8}}{2} = -a\left(\frac{1 \mp \sqrt{1/2}}{2}\right)$$

$$= -a\,0.146,\ -a \times 0.85$$



Let $K = a^2/4$

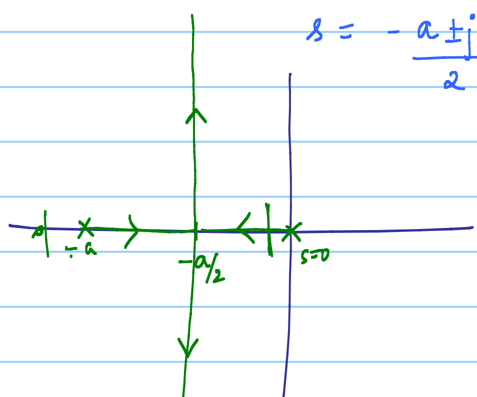
$$s^2 + as + a^2/4 = 0 \qquad s = -a/2,\ -a/2$$

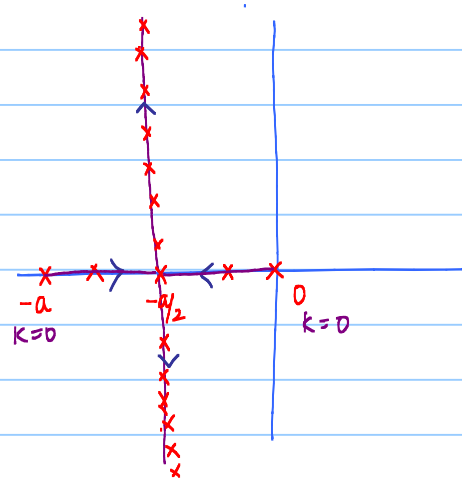
When $K > a^2/4$

Let $K = a^2/2$

$$s^2 + as + a^2/2 = 0$$

$$s = \frac{-a \pm ja}{2}$$



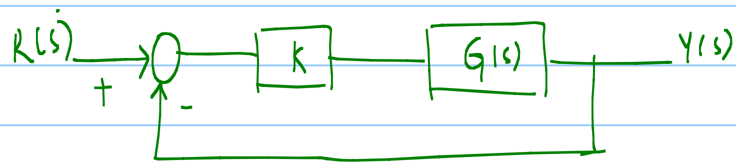


$0 < K < a^2/4$ - poles are real & distinct - Overdamped system

$K = a^2/4$ - poles are real & repeated - Critically damped System

$K > a^2/4$ - poles are complex - Underdamped System.

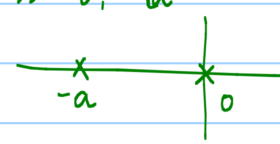
→ This system has poles entirely on LHP because RL lies entirely in left half s-plane. Hence the closed loop system is stable for all of K.



$$KG(s) = \frac{K}{s(s+a)}$$

poles OLTF $s = 0, -a$

Zeros - 2 at Infinity



Let $a < 0$

$$1 + KG(s) = 0$$

Solving this would give poles of closed loop system

$$1 + \frac{K}{s(s+a)} = 0$$

$$s^2 + as + K = 0$$

The range of K is 0 to ∞

When $K = 0$

$$s^2 + as + 0 = 0$$

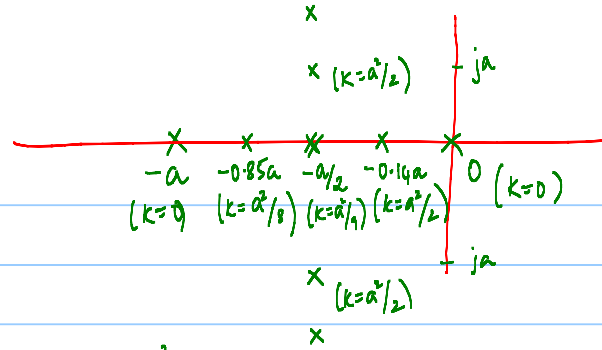
$$s = 0, -a$$ ⇒ The closed loop poles correspond to open loop poles

When $0 < K < a^2/4$

Let $K = a^2/8$

$$s^2 + as + a^2/8 = 0$$

$$s = \frac{-a \pm \sqrt{a^2 - 4a^2/8}}{2} = -a\left(\frac{1 \mp \sqrt{1/2}}{2}\right) = -0.852a, -0.14a$$



Let $K = a^2/4$

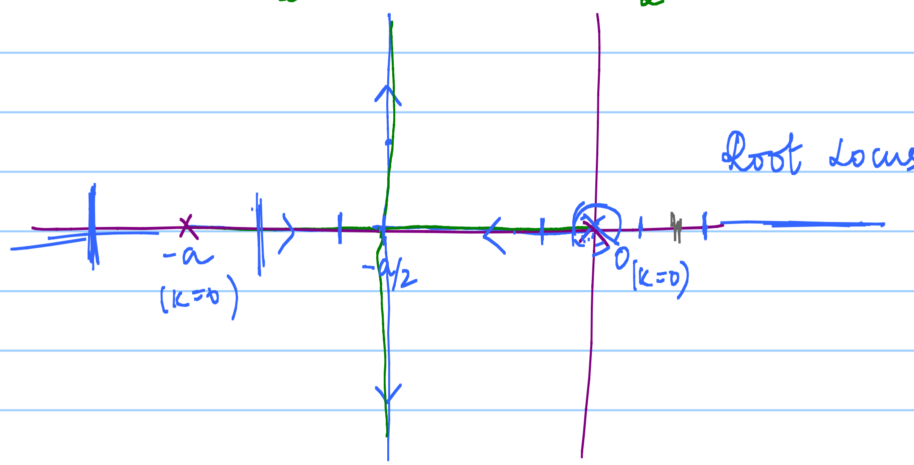
$$s^2 + as + a^2/4 = 0$$

$$s = -a/2, \; -a/2$$

Let $K = a^2/2$

$$s^2 + as + a^2/2 = 0$$

$$s = \frac{-a \pm \sqrt{a^2 - 4a^2/2}}{2} = \frac{-a \pm ja}{2}$$

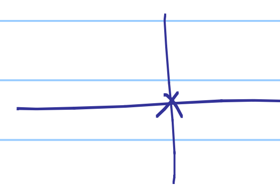


→ The root locus starts from open loop poles and terminate at open loop zeros.

→ The root locus is symmetric about real axis.

(2) $K\,G(s) = \dfrac{K}{S}$ open loop poles $s = 0$

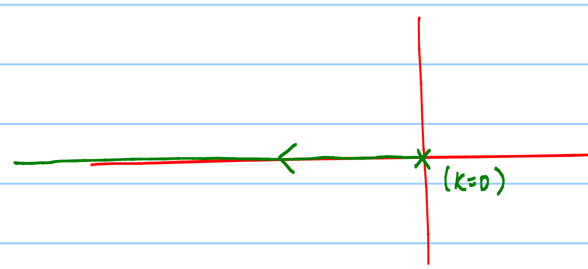
zeros – 1 at infinity



$$1 + \frac{K}{S} = 0$$

$$S + K = 0$$

$$s = -K$$



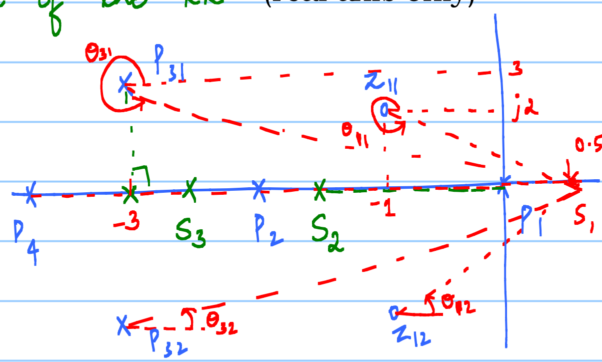
Root locus Procedure

1. Any point on the root locus must satisfy angle condition

$$\angle KG(s)H(s) = 180(2K+1)$$

2. RL is symmetric about real axis.

3. Count the no of poles and zeros (finite) of OL on the right of the test point, the number must be odd then the point is part of the RL (real axis only)



Total angle

$$\frac{s+2}{s+3} \qquad s = j\omega$$

$$\tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{3}\right)$$

angle contribution at S_1

P_1 - $0°$
P_2 - $0°$
P_{31} - θ_{31}
P_{32} - $360 - \theta_{31} = \theta_{32}$
P_4 - $0°$

Z_{11} - θ_{11}
Z_{12} - $360 - \theta_{11} = \theta_{12}$

$$\text{Total angle} = \phi_z - \phi_p$$

$$= \theta_{11} + 360 - \theta_{11} - (0° + 0° + \theta_{31} + 360 - \theta_{31} + 0°)$$

$$= 0$$

$\therefore S_1$ is not part of root locus.

4. Root locus will have as many branches as the number of open loop poles

no of OL poles

5. All the branches (n) starts from open loop poles

Let $$K\,G(s) = K \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)}$$

m no of zeros

n no of poles

CE of the CLS $$1 + K\,G(s) = 1 + K\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)} = 0$$

$\prod$ – product of sum

$\sum$ – sum of product

$$\prod_{i=1}^{n}(s+p_i) + K\prod_{i=1}^{m}\cancelto{0}{(s+z_i)} = 0$$

when $K=0$

$$\prod_{i=1}^{n}(s+p_i) = 0$$

6. All the m branches terminate at open loop zeros (finite)

$$q(s) = \prod_{i=1}^{n}(s+p_i) + K\prod_{i=1}^{m}(s+z_i) = 0$$

when $K = \infty$

$$K\left(\frac{1}{K}\prod_{i=1}^{n}(s+p_i) + \prod_{i=1}^{m}(s+z_i)\right) = 0$$

$$\Rightarrow \frac{1}{K}\prod_{i=1}^{n}(s+p_i) + \prod_{i=1}^{m}(s+z_i) = 0$$

$$\lim_{K\to\infty} \frac{1}{K}\prod_{i=1}^{n}(s+p_i) + \prod_{i=1}^{m}(s+z_i) = 0$$

$$\Rightarrow \prod_{i=1}^{m}(s+z_i) = 0$$

$\Rightarrow$ The m branches of RL would terminate at z_i $i \in [1, M]$

The remaining $n-m$ branches would terminate at infinity

7. **Angle of Asymptotes:**

$$1 + KG(s)H(s) = 1 + K\frac{s^m + b_{m-1}s^{m-1} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0} = 0$$

$$\lim_{s\to\infty}\left(KG(s)H(s)\right) = \lim_{s\to\infty}\left(K\frac{s^m + b_{m-1}s^{m-1} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}\right)$$

$$= K\lim_{s\to\infty}\left(\frac{s^m\left(1 + b_{m-1}/s + \dots + b_0/s^m\right)}{s^n\left(1 + b_{n-1}/s + \dots + a_0/s^m\right)}\right)$$

$\frac{(1+b_{m-1}/s+\dots)}{(1+a_{m-1}/s+\dots)}$ the terms doesnot contribute to RL, hence

$$KG(s)H(s) \approx \frac{K}{s^{n-m}}$$

$$1 + KG(s)H(s) = 0$$

Approximately

$$1 + \frac{K}{s^{n-m}} = 0$$

$$\frac{K}{s^{n-m}} = -1 \qquad\qquad \frac{1}{s^{n-m}} = M(e^{j\theta})^{n-m}$$

$$\angle KM(e^{j\theta})^{n-m} = \angle -1$$

$$(n-m)\theta = 180(2k_1+1)$$

Angle of asymptotes $\Rightarrow \boxed{\theta = \frac{180(2k_1+1)}{n-m}}$ $\qquad k_1 = 0, \pm1, \pm2, \pm3 \dots$

8. **Location of Centriod:**

$$1 + KG(s)H(s) = 0$$

$$G(s)H(s) = \frac{s^m + b_{m-1}s^{m-1} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}$$

$$\begin{array}{r|l}
 & s^{n-m} + (a_{n-1} - b_{m-1})s^{n-m-1} \\
\hline
s^m + b_{m-1}s^{m-1} + \dots + b_0 & s^n + a_{n-1}s^{n-1} + \dots \qquad + a_0 \\
 & s^n + b_{m-1}s^{n-1} + \dots + b_0 s^{n-m} \\
\hline
 & (a_{n-1} - b_{m-1})s^{n-1} + \dots \\
 & (a_{n-1} - b_{m-1})s^{n-1} + \dots \\
\hline
\end{array}$$

$$G(s)H(s) = \frac{1}{s^{n-m} + (a_{n-1} - b_{m-1})\, s^{n-m-1} + \dots\dots} \qquad — (1)$$

$$G(s)H(s) = \frac{1}{(s+\alpha)^{n-m}}$$

$$(s+\alpha)^{n-m} = s^{n-m} + (n-m)\,\alpha\, s^{n-m-1} + \dots$$

$$G(s)H(s) = \frac{1}{s^{n-m} + (n-m)\,\alpha\, s^{n-m-1} + \dots\dots} \qquad — (2)$$

Compare the coefficient of s^{n-m-1} of (1) + (2)

$$(n-m)\,\alpha = a_{n-1} - b_{m-1}$$

$$\alpha = \frac{a_{n-1} - b_{m-1}}{n-m}$$

Centroid

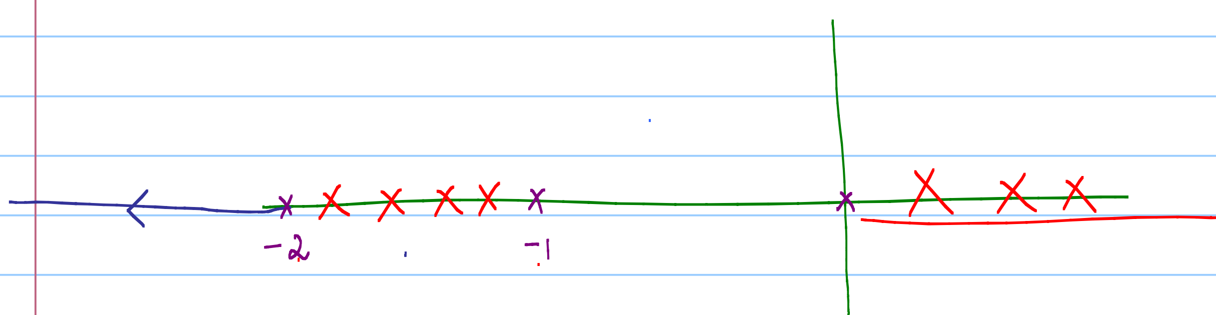
$$\boxed{\alpha = \frac{\sum_{i=1}^{n} P_i - \sum_{i=1}^{m} z_i}{n-m}}$$

1. $\cancel{K}$ $G(s) = \dfrac{K}{s(s+1)(s+2)}$

The poles of open loop $= 0, -1, -2$

no of OLP $= 3$

no of open loop zeros $= 0$ (3 at infinity)



Angle of asymptote.

$$\theta = \frac{180(2k+1)}{n-m} = \frac{180(2k+1)}{3-0}$$

$K=0$

$$\theta = \frac{180(0+1)}{3} = 60^\circ$$

$K=1$

$$\theta = \frac{180(2+1)}{3} = 180^\circ$$

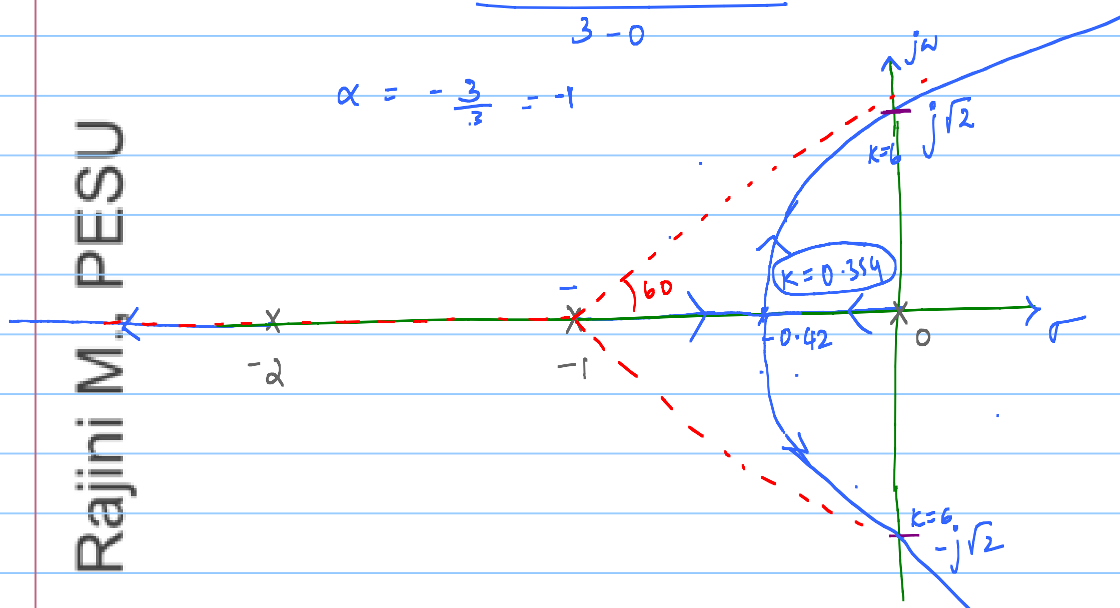
$K=2$ $\quad \theta = \frac{180(4+1)}{3} = 300^\circ$ $\qquad K=-1$ $\quad \theta = \frac{180(-2+1)}{3} = -60^\circ$

Centroid :

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m}$$

$$= \frac{-2-1+0 \;-\; 0}{3-0}$$

$$\alpha = -\frac{3}{3} = -1$$



9. Breakaway point :

The locus breaks away from the real axis when there are multiple roots (typically 2) of CLS

$$1 + KG(s) = (s+\sigma_i)^r f(s)$$

diff wrt s & evaluate at $s=-\sigma_i$

$$\frac{d}{ds}\left(1+KG(s)\right)\Big|_{s=-\sigma_i} = r(s+\sigma_i)^{r-1} f(s)\Big|_{s=-\sigma_i} + (s+\sigma_i)^r \frac{df(s)}{ds}\Big|_{s=-\sigma_i}$$

$$\boxed{K\frac{d}{ds}G(s)\Big|_{s=\sigma_i} = 0}$$

$$K \frac{d}{ds}\left(\frac{n(s)}{d(s)}\right) = \frac{K\left(d(s)\frac{d}{ds}n(s) - n(s)\frac{d}{ds}d(s)\right)}{(d(s))^2} = 0$$

$$\Rightarrow \quad d(s)\,n'(s) - n(s)\,d'(s) = 0 \quad \text{--- } (1)$$

Now consider

$$1 + K\,G(s) = 0$$

$$K\frac{n(s)}{d(s)} = -1$$

$$\boxed{K = -\frac{d(s)}{n(s)}}$$

$$\boxed{\frac{dK}{ds} = 0} = -\frac{d}{ds}\left(\frac{d(s)}{n(s)}\right) = -\left(\frac{n(s)\,d'(s) - d(s)\,n'(s)}{n(s)}\right) = 0$$

$$\Rightarrow \quad d(s)\,n'(s) - n(s)\,d'(s) = 0 \quad \text{--- } (2)$$

To find break away point

$$K = -\frac{d(s)}{n(s)}$$

$$\frac{dK}{ds} = 0 = -\frac{d}{ds}\left(\frac{d(s)}{n(s)}\right)$$

Solve for s

$$K = -\frac{(s(s+1)(s+2))}{1}$$

$$\frac{dK}{ds} = -\frac{d}{ds}\left(s(s+1)(s+2)\right) = 0$$

$$+\left[(s+1)(s+2) + s(s+2) + s(s+1)\right) = 0$$

$$s^2 + 3s + 2 + s^2 + 2s + s^2 + s = 0$$

$$3s^2 + 6s + 2 = 0$$

$$s = -0.42, \quad -1.578$$

$$\times$$

$$|K| = \left| -\frac{d(s)}{n(s)} \Big|_{s=-0.42} \right| = 0.384$$

jw- crossing :

$$1 + K\,G(s)\,H(s) = 0$$

$$1 + \frac{K}{s(s+1)(s+2)} = 0$$

$$s(s^2+3s+2) + K = 0$$

$$s^3 + 3s^2 + 2s + K = 0$$

s^3	1	2
s^2	3	K
s^1	$\frac{6-K}{3}$	0
s^0	K	

→ This row becomes zero when $K = 6$

Auxillary polynomial

$$3s^2 + 6 = 0$$

$$s = \pm j\sqrt{2}$$

The closed loop system is stable for $0 < K < 6$

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2. $$K\,G(s) = \frac{K(s+1)}{s(s+2)}$$

no of open loop poles $n = 2$

open pole locations $= 0, -2$

no of open loop zeros (finite) $m = 1$

open loop zero $= -1$

No of branches $= 2$

Angle of asymptotes

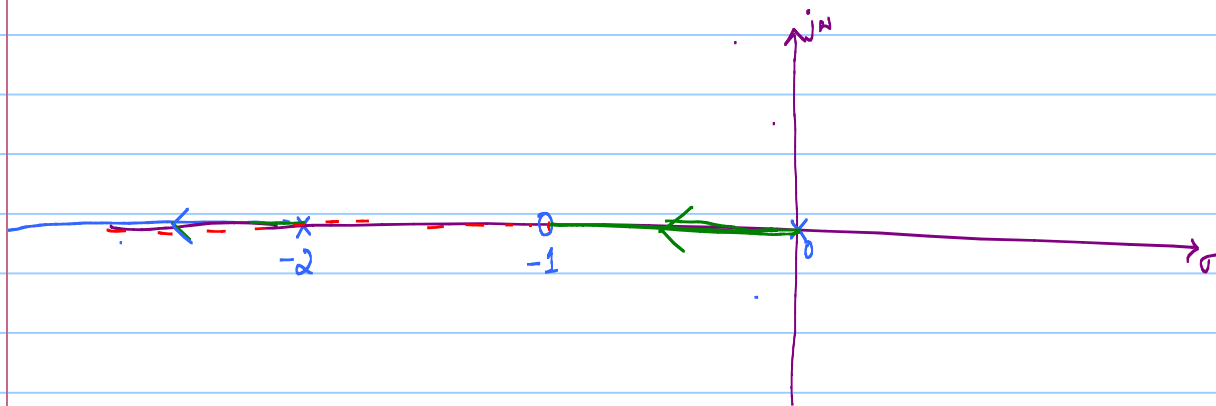
$$\theta = \frac{(2k+1)\pi}{n-m}$$

$k=0 \quad \theta = \frac{\pi}{2-1} = \pi$

$k=1 \quad \theta = 3\pi$

Location of centriod

$$\alpha = \frac{(0-2)-(-1)}{2-1} = -1$$



The closed loop system is stable for all values of k

3. $$KG(s)H(s) = \frac{K}{s^2(s+1)(s+2)}$$

no of open loop poles $= 4 \ (n)$

location $= 0, 0, -1, -2$

no of open loop zeros $= 0 \ (m)$

4 branches of RL must tend to infinity

Angle of asymptotes

$$\theta = \pm\frac{\pi}{4}, \ \pm\frac{3\pi}{4}$$

OR

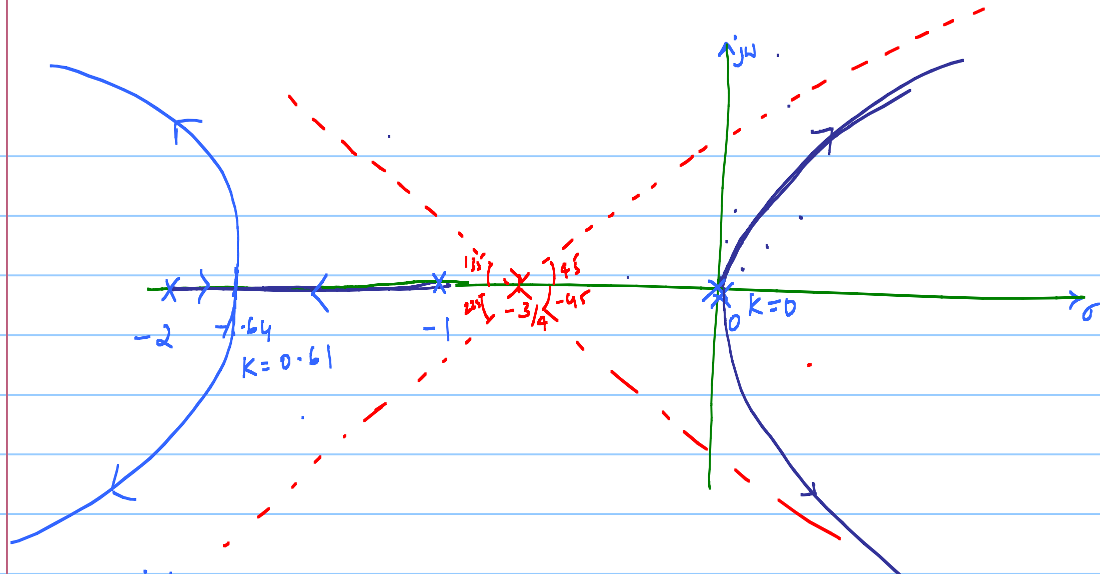
$$\frac{\pi}{4}, \quad \frac{3\pi}{4}, \quad \frac{5\pi}{4}, \quad \frac{7\pi}{4}$$

$k=0 \quad k=1 \quad k=2 \quad k=3$

Location of centriod

$$\alpha = \frac{(0+0-1-2)-0}{4}$$

$$= -3/4$$



Breakaway point

$$K = -\frac{s^2(s+1)(s+2)}{1}$$

$$\frac{dK}{ds} = 0 = -\left[2s(s^2+3s+2) + s^2(s+2) + s^2(s+1)\right] = 0$$

$$\Rightarrow 2s^3 + 6s^2 + 4s + s^3 + 2s^2 + s^3 + s^2 = 0$$

$$4s^3 + 9s^2 + 4s = 0$$

$s = 0, \; -0.609, \; -1.64$

$K=0$ X $K = 0.61$

$$|K| = \left|-\frac{d(s)}{n(s)}\right|_{s=-1.64} = 0.61$$

The closed loop system is unstable for all values of K

4. OLTF $K\,G(s)\,H(s) = \dfrac{s-1}{s(s^2+4s+4)}$

no of open loop poles = 3

location = 0, -2, -2

no of open loop zero = 1

location = +1

1 branch will terminate at $s=+1$ & 2 branches will tend to infinity

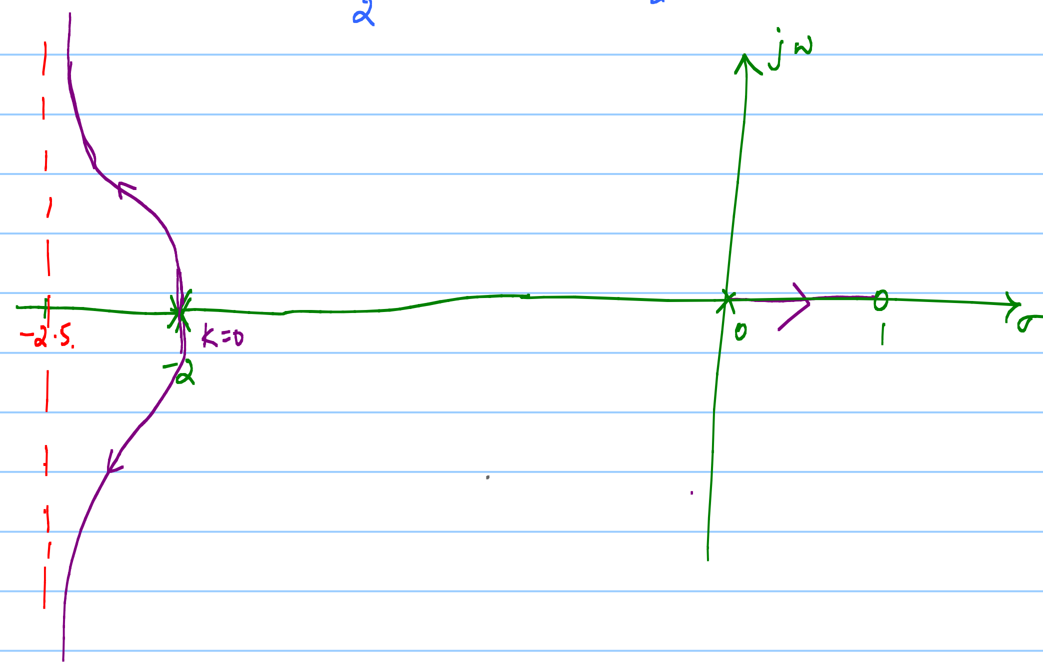
Angle of asymptote

$$\theta = \frac{(2k+1)\pi}{3-1}$$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

Centroid

$$\alpha = \frac{(0-2-2)-(1)}{2} = -\frac{5}{2}$$



The closed loop system is unstable for all of K.

5. $KG(s)H(s) = \frac{K(s+1)}{s^2+4s+13}$

no of open loop poles $= 2$

location $= -2 \pm j3$

no of open loop zero $= 1$

location $= -1$

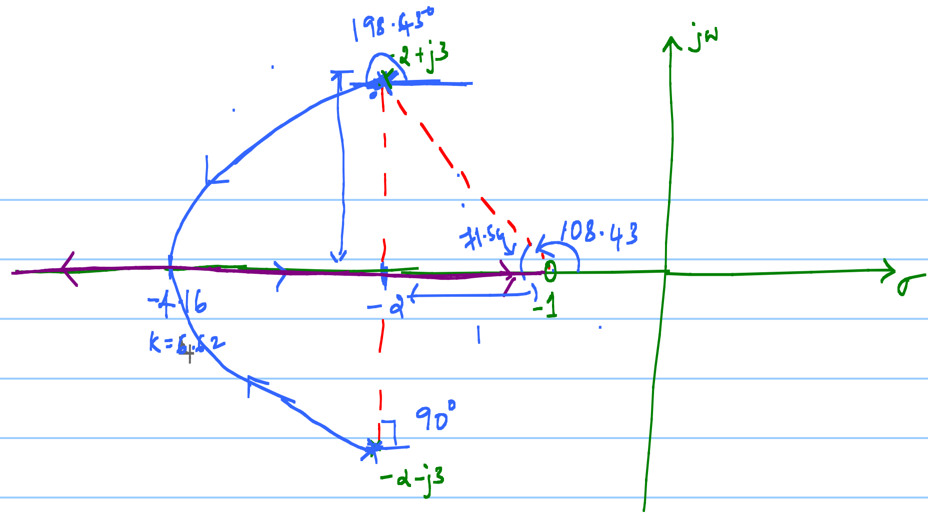
1 branch will terminate at -1 & other branch must tend to infinity

Angle of asymptotes

$$\theta = \frac{(2k+1)\pi}{1} = -\pi, \pi$$

Location of centroid

$$\alpha = -3$$



Break away point:

$$K = -\frac{(s^2+4s+13)}{s+1}$$

$$\frac{dK}{ds} = 0 = -\left[\frac{(s+1)(2s+4) - (s^2+4s+13)}{(s+1)^2}\right] = 0$$

$$s^2+2s-9=0$$

$$s = 2.16,\ -4.16 \checkmark$$

$$|K| = \left| -\frac{(s^2+4s+13)}{(s+1)} \right|_{s=-4.16}$$

Angle of departure

$$\phi_z - \phi_p = 180$$

Zero at -1 — Pole -2+j3 — Pole -2-j3

$$108.43 - (\phi_d + 90) = 180$$

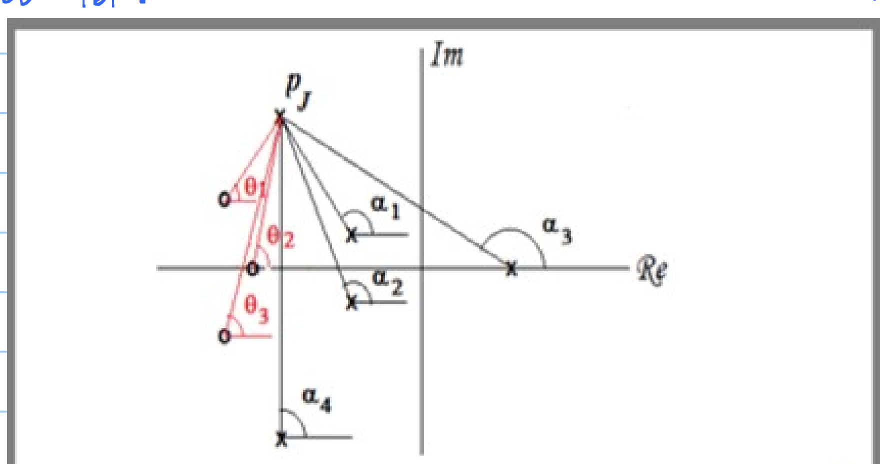
$\phi_d = -161.5$ $\qquad$ $\phi_d = 198.43°$

$\phi_d = 360 - 161.5$

$$KG(s)H(s) = \frac{s+1}{s^2+4s+13}$$

$s = -2+j3$

$$= \frac{(-2+j3+1)}{(s^2+4s+13)}\Bigg|_{s=-2+j\omega}$$



Angle of Departure from $P_J = (\theta_1+\theta_2+\theta_3) - (\alpha_1+\alpha_2+\alpha_3+\alpha_4) \pm 180°$

6. $$KG(s) = \frac{K(s+1)}{s(s+2)(s^2+2s+5)}$$

no of open loop poles = 4

location = $0, -2, -1 \pm j2$

no of open loop zero = 1

location = -1

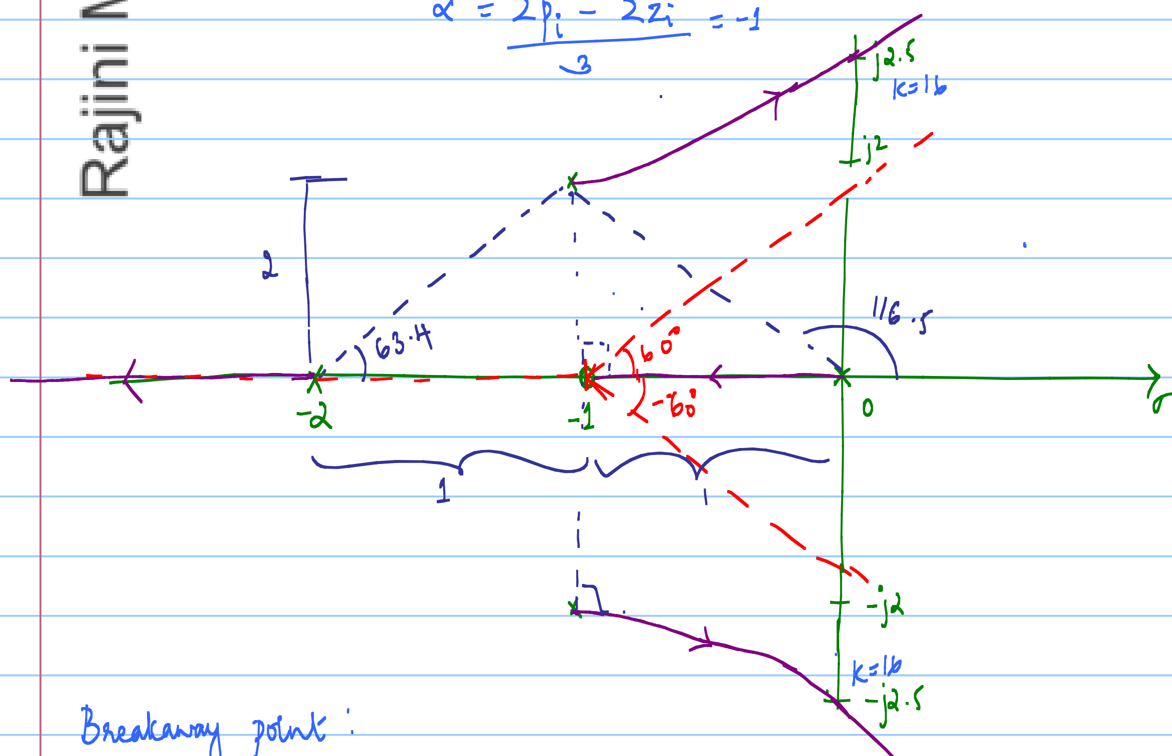
1 branch will terminate at -1 & other 3 branches must tend to infinity

Angle of asymptotes

$$\theta = \frac{(2k+1)\pi}{3} = \underbrace{\frac{\pi}{3}}_{k=0};\ \underbrace{\pi}_{k=1},\ \underbrace{\frac{5\pi}{3}}_{k=2},\ \underbrace{\frac{7\pi}{3}}_{k=3}$$

Location of centroid

$$\alpha = \frac{\Sigma p_i - \Sigma z_i}{3} = -1$$



Breakaway point:

$s = -1.5721 \pm j0.909$, $-0.427 \pm j0.909$;

Since the branches originating from the poles $-1 \pm j2$ are not converging each other. Hence there is no need of breakaway point jω crossing:

$$1 + KG(s) = 0$$

$$1 + \frac{K(s+1)}{(s^2+2s)(s^2+2s+5)} = 0$$

$$s^4 + 4s^3 + 9s^2 + (K+10)s + K = 0$$

s^4	1	9	K
s^3	4	$K+10$	
s^2	$\cancel{\frac{36-K-10}{4}}$ $\frac{26-K}{4}$	K	
s^1	C_1	0	
s^0	K		

$$C_1 = \frac{\frac{26-K}{4}(K+10) - 4K}{\frac{26-K}{4}}$$

$$= \frac{260 - K^2}{26 - K}$$

To have a zero row, $C_1 = 0$

$$K^2 = 260$$

$$\Rightarrow K = \sqrt{260} \approx 16$$

$$\frac{26-16}{4} s^2 + 16 = 0$$

$$\frac{10}{4} s^2 + 16 = 0$$

$$s = \pm j2.5$$

$0 < K < 16$ – stable range

Angle of departure:

$$180 = \phi_z - \phi_p$$

$$180^\circ = \underbrace{90}_{\text{zero at } -1} - (\underbrace{63.4}_{\text{pole at } -2} + \underbrace{117.5}_{\text{pole at } 0} + \underbrace{90}_{\text{pole at } -1-j2} + \phi_d)$$

pole at $-1+j2$

$$\phi_d = 0^\circ$$

The closed loop system is stable for $0 < K < 16$

OLTF

7. $KG(s) = \dfrac{K(s^2-2s+2)}{(s+2)(s+3)(s+4)}$

no of open loop poles = 3

location = −2, −3, −4

no of open loop zero = 2

location = $1 \pm j$

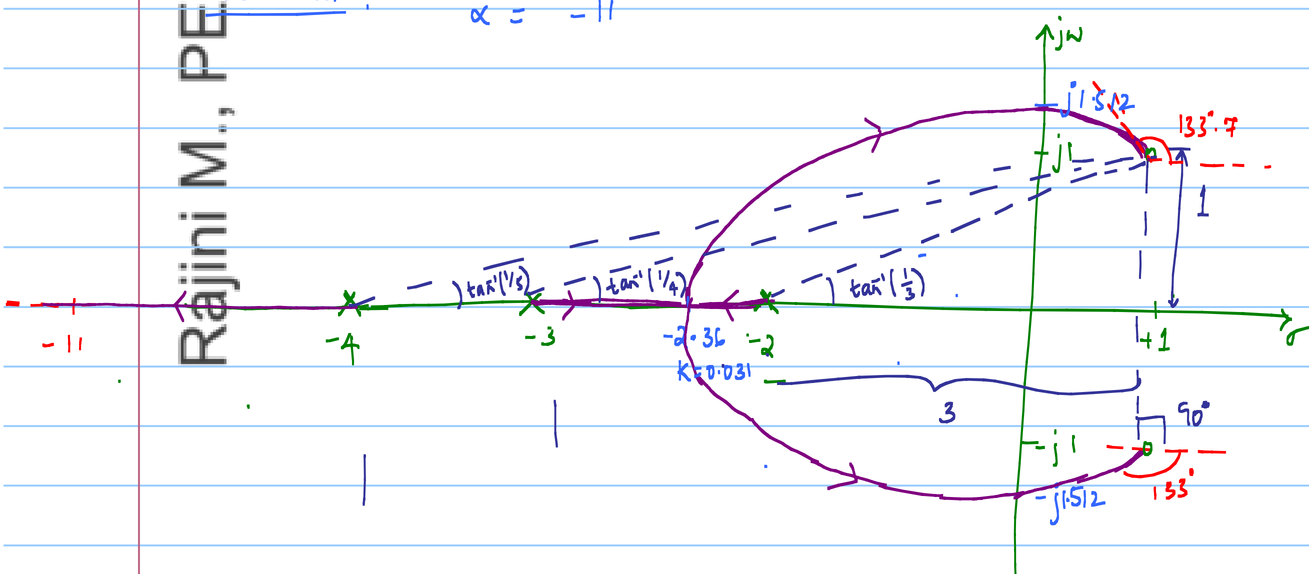
2 branches will terminate at $1 \pm j$ & other branch must tend to infinity

Angle of asymptote:

$$\theta = \pi$$

Centroid:

$$\alpha = -11$$



Breakaway point

$$K = -\frac{(s+2)(s+3)(s+4)}{s^2-2s+2}$$

$$(s^2+5s+6)(s+4) = s^3+5s^2+6s+4s^2+20s+24$$
$$= s^3+9s^2+26s+24$$

$$\frac{dK}{ds} = 0 = -\left[\frac{(s^2-2s+2)(3s^2+18s+26) - (2s-2)(s^3+9s^2+26s+24)}{(s^2-2s+2)^2}\right]$$

$$\Rightarrow s^4 - 4s^3 - 38s^2 - 12s + 100 = 0$$

$$s = -2.36,\ 1.412,\ -3.529,\ 8.42$$

$$|K| = \left| -\left(\frac{(s+2)(s+3)(s+4)}{s^2-2s+2}\right)\right|_{s=-2.35}$$

$K = 0.031$

<u>jw-crossing</u> :

$$1 + KG(s) = 0$$

$$s^3 + (9+K)s^2 + (26-2K)s + 24 + 2K = 0$$

s^3	1	$26-2K$
s^2	$9+K$	$24+2K$
s^1	b_1	0
s^0	$24+2K$	

$$b_1 = \frac{(9+K)(26-2K) - 24 - 2K}{9+K}$$

$b_1 = 0$

$$(9+K)(26-2K) - 24 - 2K = 0$$

$$K^2 - 3K - 105 = 0$$

$$K = 11.856 \checkmark, \ -8.8$$

AE : $(9+11.86)s^2 + (24 + 2\times 11.86) = 0$

$$s = \pm j1.512$$

stable range $0 < K < 11.856$

<u>Angle of arrival</u> :

$$180 = \phi_z - \phi_p$$

$$180 = (\phi_A + 90^\circ) - (\underset{\text{pole at } -2}{18.43} + \underset{\text{pole } -3}{14.03} + \underset{\text{pole at } -4}{11.3})$$

$$\phi_A = 133.7^\circ$$

---

8. $$KG(s) = \frac{K(s+0.5)}{s^2(s+4.5)}$$

no of open loop poles = 3

location = 0, 0, -4.5

no of open loop zero = 1

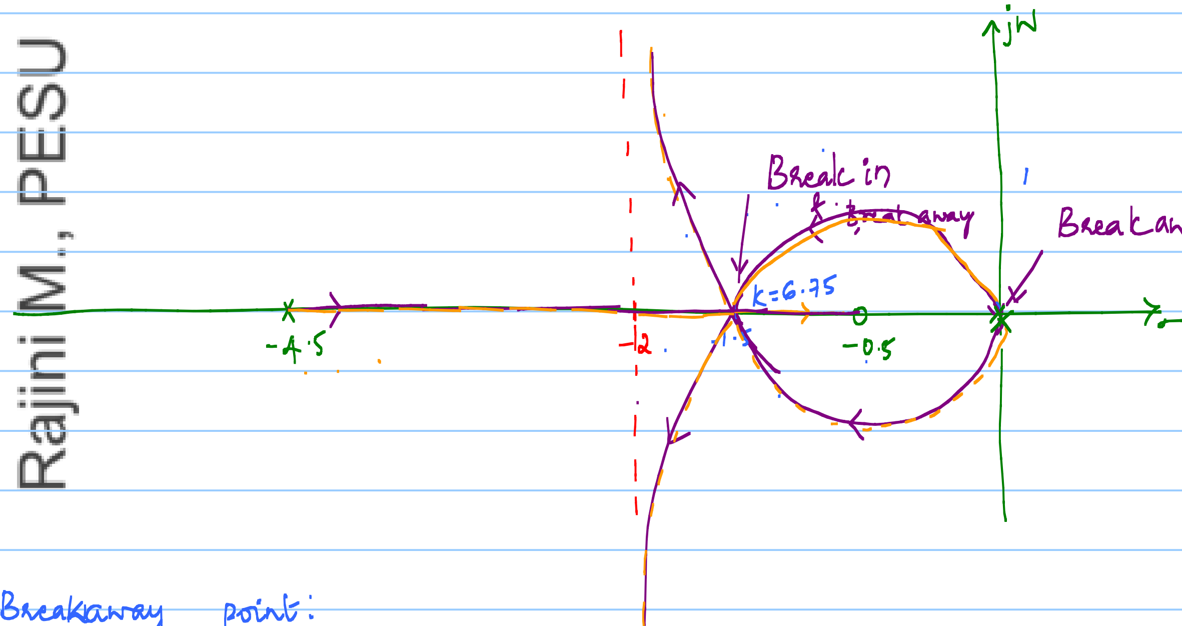
location = -0.5

1 branches will terminate at -0.5 & other branches must tend to infinity

<u>Angle of asymptote</u>:

$\theta = \pm \pi/2$

<u>centroid</u>: $\alpha = -2$



<u>Breakaway point</u>:

$$K = -\frac{(s^3 + 4.5s^2)}{(s+0.5)}$$

$$\frac{dk}{ds} = 0 = -\left[\frac{(s+0.5)(3s^2+9s) - s^3 - 4.5s^2}{(s+0.5)^2}\right]$$

$2s^3 + 6s^2 + 4.5s = 0$

$s(2s^2 + 6s + 4.5) = 0$

$s = 0, -1.5, -1.5$

Breakaway Break in Break away

$$|K| = \left| -\frac{s^2(s+4.5)}{(s+0.5)} \right|_{s=-1.5}$$

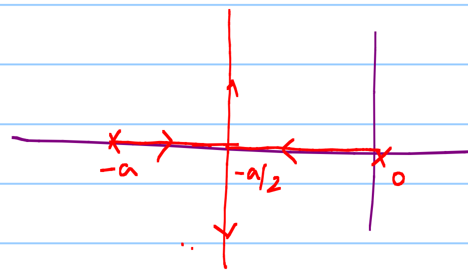
$|K| = 6.75$

The CLS is stable for $0 < K < \infty$

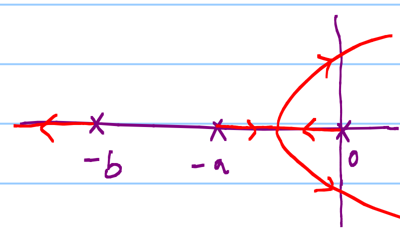
Effect of Adding poles & zeros to $KG(s)H(s)$

→ An addition of a pole will move the root locus to Right half s-plane

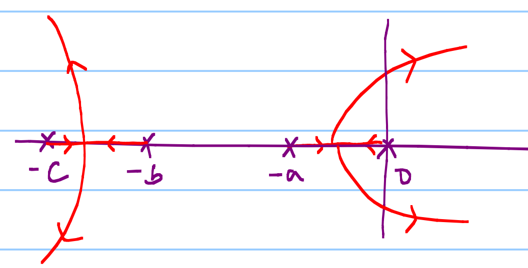
1. $KG(s)H(s) = \dfrac{K}{s(s+a)}$



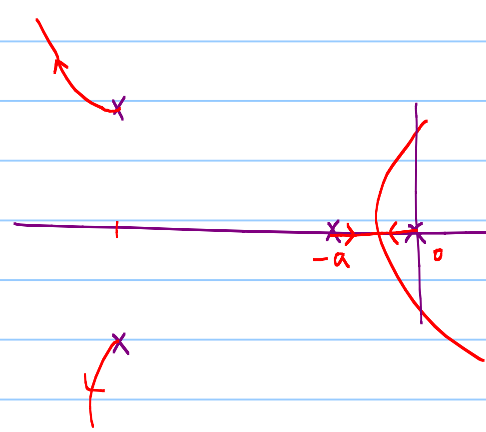
$KG(s)H(s) = \dfrac{K}{s(s+a)(s+b)}$



$KG(s)H(s) = \dfrac{K}{s(s+a)(s+b)(s+c)}$

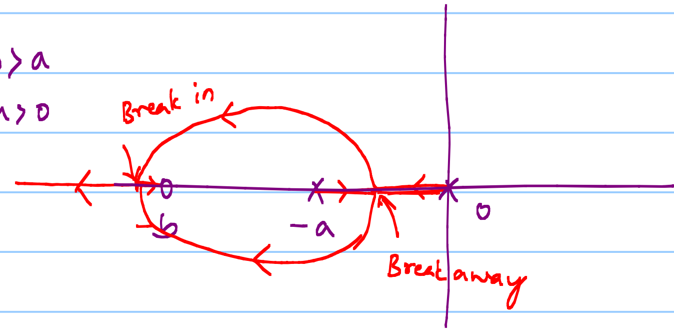


$KG(s)H(s) = \dfrac{K}{s(s+a)(s^2+bs+c)}$

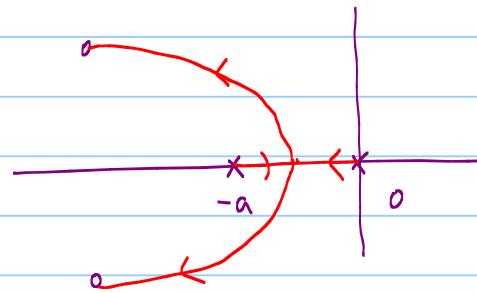


→ Adding left half plane zeros to $KG(s)H(s)$ generally shift root locus to left half s-plane

$$KG(s)H(s) = \frac{K(s+b)}{s(s+a)} \qquad b > a,\ a > 0$$



$$K\,G(s)H(s) = \frac{K(s^2+bs+c)}{s(s+a)}$$



$$KG(s)H(s) = \frac{K(s+1)}{s(s-1)(s^2+4s+16)}$$

Ex, $G(s)H(s) = \dfrac{K}{s(s+4)(s^2+4s+20)}$

sol: $n = 4$, $s = 0, -4, -2 \pm j4$

$m = 0$

no. of branches $= n = 4$

The branches will start at OL poles and terminates at ∞

$$\phi_A = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \qquad \sigma_A = \frac{(0-4-2-2)-0}{4-0} = -2$$

$$K = \frac{-D(s)}{N(s)} = \frac{-(s^4+8s^3+36s^2+80s)}{1}$$

$$\frac{dK}{ds} = -(4s^3+24s^2+72s+80) = 0$$

$$s = -2,\ -2 \pm j2.44$$

find $j\omega$ axis crossing

$$1+KG(s)H(s) = 0$$

$$1+\frac{K}{s(s+4)(s^2+4s+20)} = 0$$

$$s^4+8s^3+36s^2+80s+K = 0$$

s^4	1	36	K
s^3	8	80	
s^2	26	K	
s^1	$\frac{2080-8K}{26}$		
s^0	K		

for the system to be stable

$K>0$ & $\frac{2080-8K}{26}>0$

$$0<K<260$$

When $K=260$, $A(s) = 26s^2+260 = 0$

$s^2 = -260/26 \Rightarrow s = \pm j\sqrt{10} = \pm j3.16$

$$0-(\theta_d+\theta_1+\theta_2+\theta_3) = 180^\circ$$

$$-\theta_d = 180^\circ+\theta_1+\theta_2+\theta_3$$

$$\theta_d = -450^\circ$$

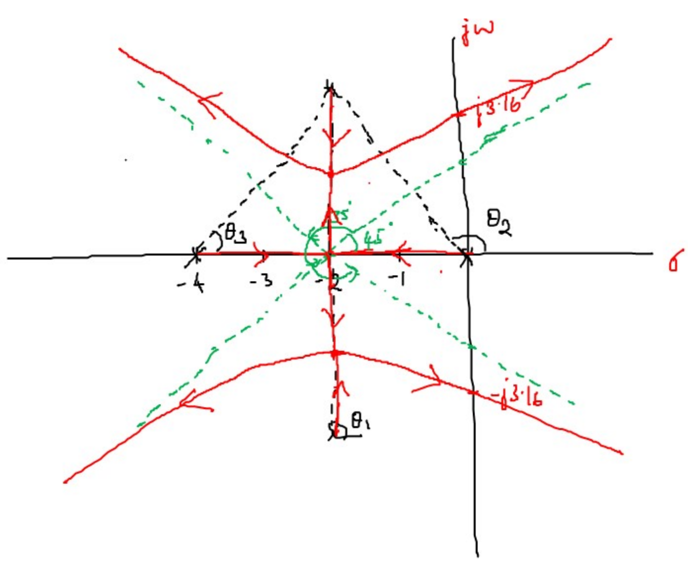
$$= 90^\circ$$

$\theta_1 = 90^\circ$

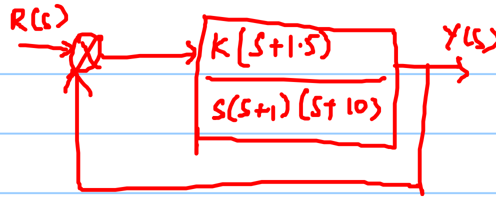
$\theta_2 = 180^\circ-\tan^{-1}\frac{4}{2}$

$= 116.56^\circ$

$\theta_3 = 63.4^\circ$



Ex, Design the value of gain k to yield 1.52% of overshoot



Sol: $G(s) = \dfrac{k(s+1.5)}{s(s+1)(s+10)}$

$n = 3, \quad s = 0, -1, -10$

$m = 1, \quad s = -1.5$

no. of branches $n = 3$

$n - m = 3 - 1 = 2$ branches will terminate at ∞

$\sigma_A = \dfrac{0 - 1 - 10 - (-1.5)}{3 - 1} = -4.75, \quad q = 0, 1, \quad \phi_A = \pi/2, 3\pi/2$

$\dfrac{dk}{ds} = 2s^3 + 15.5s^2 + 33s + 15 = 0$

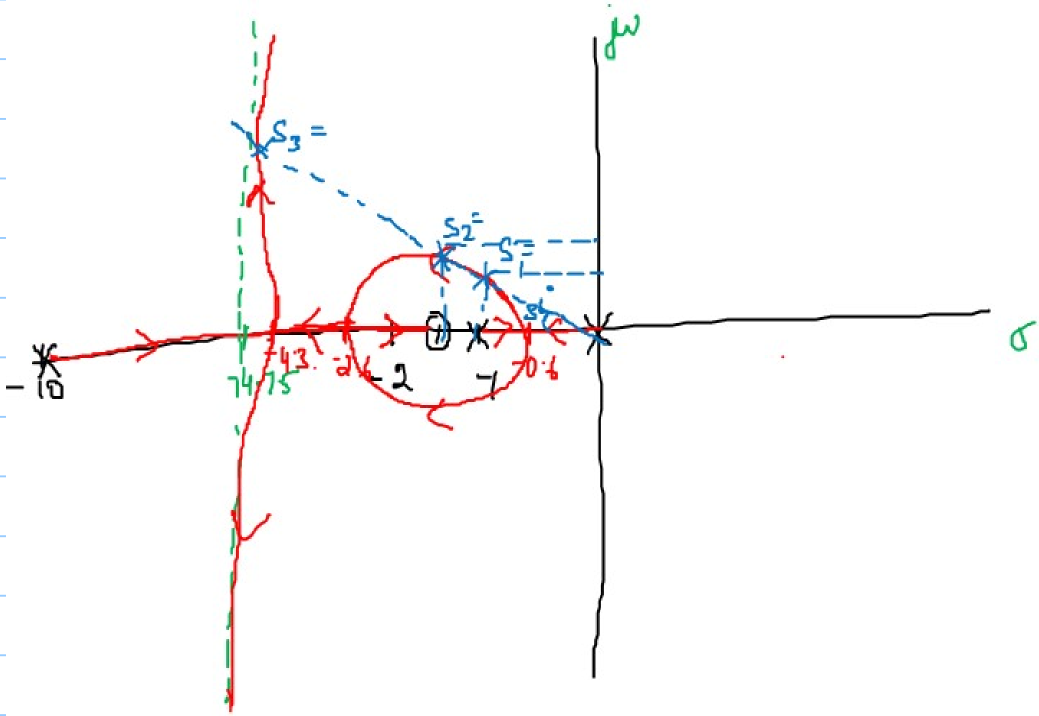
$s = -2.768, -4.36, -0.621$

$e^{-\zeta\pi/\sqrt{1-\zeta^2}} \rightarrow M_p = 1.52\%$

$\Rightarrow \zeta = 0.8$

$\zeta = \cos\theta$

$\theta = \cos^{-1}\zeta = 36.86°$



$\zeta\omega_n$

$s_1 = -0.67 + j0.66$

$s_2 = -1.19 + j0.90$

$s_3 = -4.6 + j3.45$

$t_s = \dfrac{4}{\zeta\omega_n}$

As angle line cuts the Root Locus at 3 points, there will be 3 values for K. Final K value is chosen based on settling time.

Since settling time for s_3 is smaller, K value for s_3 is chosen as final value

$K = 39.36$

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