## Unit II: Vector Spaces

1. Define column space and Null space of the matrix with example
2. Explain the special solution of the system of linear equations.
3. Describe ( geometrically ) the column space and null space of the following matrices:

i) $\begin{bmatrix} 0 \end{bmatrix}$, ii) $\begin{bmatrix} 0 & 1 \end{bmatrix}$, iii) $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, iv) $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

**Ans**: i) C(A)= z , N(A) = $R^1$ ; ii) C(A)= $R^1$ , N(A) = line in $R^2$

iii) C(A)= $R^2$ , N(A) = line in $R^3$ iv) C(A)= $R^2$ , N(A) = z in $R^2$

4. Write all the subspaces of $R^3$

5. Find the column space and null space of $A = \begin{bmatrix} 1 & 0 \\ 2 & 7 \\ 5 & 3 \end{bmatrix}$.

Give an example of a matrix whose column space is the same as that of A but null space is different.

**Answer**: C (A) is a 2d plane in $R^3$ and N(A) is the origin in $R^2$. The matrix

$A=\begin{bmatrix}1&0&1\\2&7&9\\5&3&8\end{bmatrix}$ has C(A) but its N(A) is a line in $R^3$ passing through ( 1, 1, -1 )

**6.** For which vectors $b=(b_1,b_2,b_3)$ does the following system $Ax=b$ have a solution?

i) $A=\begin{bmatrix}1&1&1\\0&2&2\\0&0&3\end{bmatrix}$ ii) $A=\begin{bmatrix}1&1&1\\0&2&2\\0&0&0\end{bmatrix}$

iii) $A=\begin{bmatrix}1&2&3\\2&4&6\\-1&-2&-3\end{bmatrix}$ iv) $A=\begin{bmatrix}1&2\\2&5\\-1&-2\end{bmatrix}$

**Answers: i) all b ii) $b_3=0$ iii) $b_2=2b_1$ iv) $b_3=-b_1$**

**7.** Decide the dependence or independence of the vectors

i) (1,3,2), (2,1,3), (3,2,1)

ii) (4,2,2), (2,4,2), (4,8,2)

iii) (0,0,0), (1,2,5), (-1,2,3)

Ans: i) dependent ii) independent iii) dependent

**8.** If $v_1, v_2, v_3$ are linearly independent, determine whether the vectors

$v_1 - v_2, v_2 - v_3, v_3 - v_1$ are linearly independent of dependent.

**9.** Reduce the following matrices to row echelon form and then reduced row echelon form.

i) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 6 & 13 & 20 & 27 \\ 9 & 26 & 43 & 62 \end{bmatrix}$ ii) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

**10.** Choose the number q so that ( if possible ) the ranks are (i) 1 (ii) 2 (iii) 3

i) $A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}$ ii) $A = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}$

**Ans:** For A, q = 3 gives rank 1 and every other q gives rank 2. For B , q = 6 gives rank 1 and every other q gives rank 2.

**11.** Define span of a set with example

**12.** Define Basis with example.

**13.**Find the row space and left null space of the matrix $A=\begin{bmatrix}1&1&1\\1&1&1\\2&2&2\end{bmatrix}$

Ans: Row Space: (1, 1, 1); Left null space: (-1, 1, 0), (-2, 0, 1)

**14.**Find the four fundamental subspaces, their dimensions and bases given

$$A=\begin{bmatrix}1&-1&2&-2&-3\\-2&0&0&1&2\\0&3&1&-1&6\\-1&-2&-3&3&9\end{bmatrix}$$

**Ans:** Answer : Basis for C(A) is columns 1, 2, 3 ; Basis for C($A^T$ ) is rows 1, 2, 3 Basis for N(A) is { ( 7, 1, 0, 3, 0 ) , ( -8, -5, -3, 0, 3 ) } ; N($A^T$ ) = { c ( 1, 0, 1, 1)}

**15.** Obtain the four fundamental subspaces of

$$A=\begin{bmatrix}1&1&-2&0&1\\1&-1&2&-3&1\\2&2&-1&0&1\\0&0&1&1&1\end{bmatrix}$$

**Ans:** Column space :( 1, 2, 3, 4); Row space: all the four rows;

Null space: (0, -1, -1, 0, 1) ; Left Null space : z

**16.** Construct a matrix whose column space contains the vectors (1,1,5), (0,3,1) whose null space contains (1, 1, 2)

Ans: $A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$

**17.** Find a left or right inverse for the following matrices

i) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ ii) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Ans: i) $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$ ii) $A = \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$