DIGITAL IMAGE PROCESSING-1

Unit 2: Lecture 19-20

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Unit 2: Image Transforms

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Last Session

- Image basics cont..
- Desirable properties of image transforms
- 1 D and 2D DFT

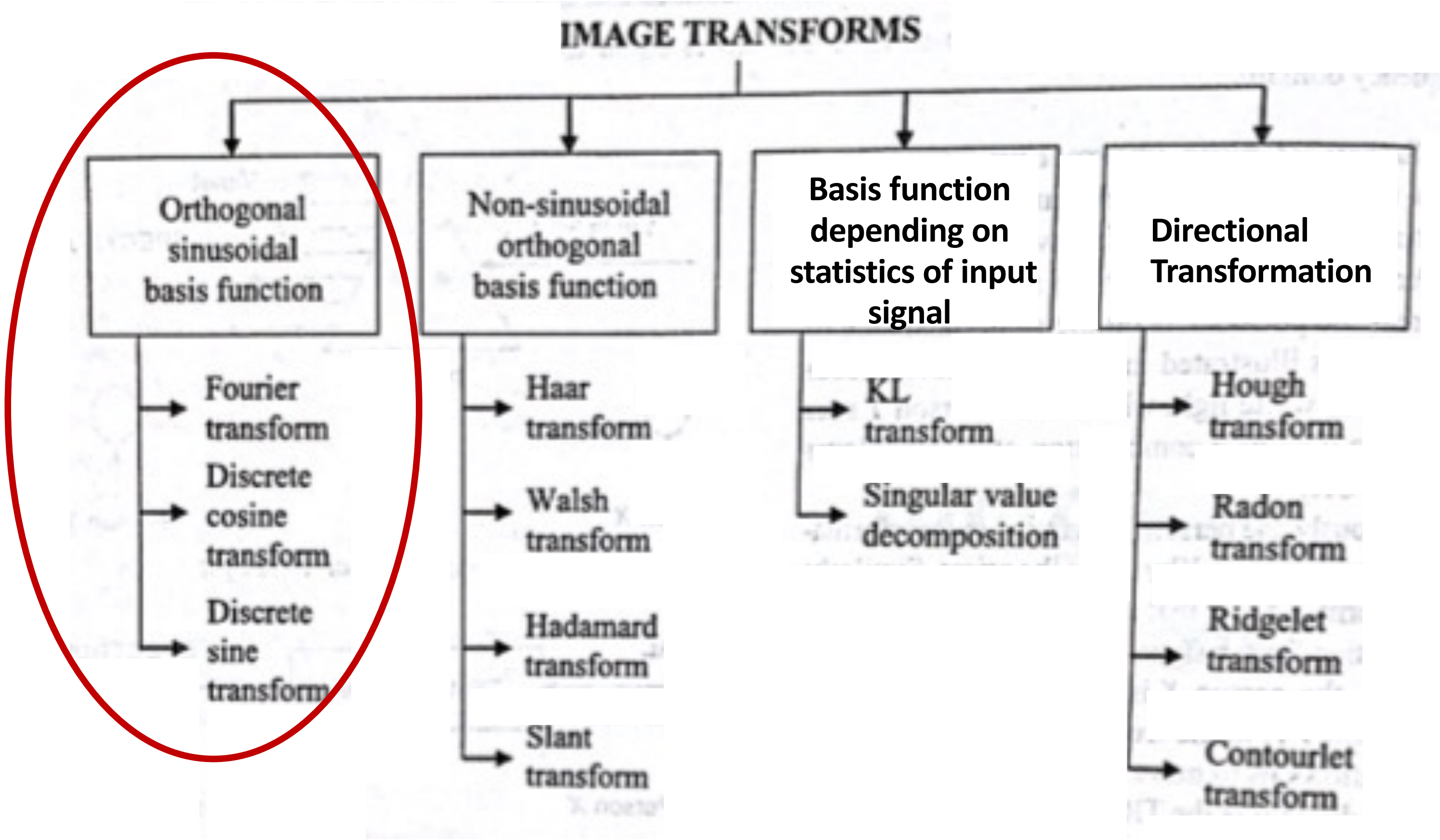
Today's Session

- 1 D and 2D DFT Cont..
- Discrete Cosine Transform (DCT)

Summary: 1D and 2D Unitary transform

- 1D unitary transform, $V=AU$
- 1D Inverse unitary transform, $U=A^{*T} V$
- 2D unitary transform, $V=AUA^{T}$
- 2D Inverse unitary transform, $U=A^{*T} V A^{*}$

Classification of Image Transforms



1D Discrete Fourier Transform (Unitary)

- Unitary DFT of $u(n)$ having N elements is defined as

$$v(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} u(n) {W_N}^{kn} \quad k = 0,1, \dots N-1$$

Where $W_N \; = exp\left\{-j\,\frac{2\pi}{N}\right\}$

Inverse DFT is given by

$$u(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} v(k)\, {W_N}^{-kn} \quad n = 0,1, \dots N-1$$

- N x N unitary DFT matrix F is given by

$$F = \left\{\frac{1}{\sqrt{N}} W_N^{kn}\right\}, \qquad 0 \leq k, \quad n \leq N-1 \longrightarrow \boxed{\text{Basis Function of DFT}}$$

1D Discrete Fourier Transform (Unitary) matrix

$$F = \frac{1}{2}\begin{bmatrix} {W_4}^{00} & {W_4}^{01} & {W_4}^{02} & {W_4}^{03} \\ {W_4}^{10} & {W_4}^{11} & {W_4}^{12} & {W_4}^{13} \\ {W_4}^{20} & {W_4}^{21} & {W_4}^{22} & {W_4}^{23} \\ {W_4}^{30} & {W_4}^{31} & {W_4}^{32} & {W_4}^{33} \end{bmatrix}$$

$$\boldsymbol{F} = \frac{\mathbf{1}}{\mathbf{2}}\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\boldsymbol{j} & -\mathbf{1} & \boldsymbol{j} \\ \mathbf{1} & -\mathbf{1} & \mathbf{1} & -\mathbf{1} \\ \mathbf{1} & \boldsymbol{j} & -\mathbf{1} & -\boldsymbol{j} \end{bmatrix}$$

1D Discrete Fourier Transform (Unitary)

In matrix format:

1D forward DFT: $V = FU$

1D Inverse DFT : $U = F^{*T}V$

2D Discrete Fourier Transform (Unitary)

- Unitary DFT of u(m,n) having NxN elements is defined as

$$v(k,l) = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} u(m,n)\, {W_N}^{km} {W_N}^{ln}; 0 \leq k,l \leq N-1$$

Where $W_N \; = exp\left\{-j\,\frac{2\pi}{N}\right\}$

Inverse DFT is given by

$$\mathrm{u}(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} v(k,l) {W_N}^{-km} {W_N}^{-ln}; 0 \leq m,n \leq N-1$$

In matrix format:

2D forward DFT

$$V \;= FUF^T$$

2D Inverse DFT :

$$U \;= F^{*T} V F^*$$

1D Transform (DFT) generalized form

- Consider 1-D DFT where f(x) ↔ $T(u)$ are DFT pairs

Then $f(x) = \frac{1}{\sqrt{N}} \sum T(u) e^{j\frac{2\pi}{N}ux} = \sum T(u) h(x,u)$

where h(x,u) = $\frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}ux}$.

Inverse transform kernel

and T$(u) = \frac{1}{\sqrt{N}} \sum f(x) e^{-j\frac{2\pi}{N}ux} = \sum f(x) g(x,u)$

where g(x,u) = $\frac{1}{\sqrt{N}} e^{-j\frac{2\pi}{N}ux}$

Forward transform kernel

2D Transform (DFT) generalized form

- Consider 2D DFT:

  Any given transform in 2D can be written as

$$f(x,y) = \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} T(u,v)e^{j\frac{2\pi}{N}[ux+vy]} = \sum_{u=0}^{N-1}\sum_{v=0}^{N-1} T(u,v)\, h(x,y,u,v) \;\; and$$

Inverse transform kernel

$$T(u,v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)e^{-j\frac{2\pi}{N}[ux+vy]} = \sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)g(x,y,u,v)$$

Forward transform kernel

Property 1: Elements of $\boldsymbol{F}$ are complex valued $\boldsymbol{F} \neq \boldsymbol{F}^*$

Property 2: $\boldsymbol{F}$ is symmetric that is $\boldsymbol{F}^T = \boldsymbol{F}$

$$\boldsymbol{F}^T = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} = \boldsymbol{F}$$

Property 3 : $\boldsymbol{F}$ is unitary matrix $\boldsymbol{F}\boldsymbol{F}^{*T} = \boldsymbol{I}$

Note : From the above three properties we can rearrange the matrix notation as follows

$$\boxed{\begin{aligned} V &= FUF \\ U &= F^* V F^* \end{aligned}}$$

$$\boldsymbol{F} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$V = FU$$

$$U = F^{*T} V$$

$$V = FUF^T$$

$$U = F^{*T} V F^*$$

Property 4 : Sequency is not in order

Sequency is the number of sign changes in row of transformation kernel

$$\begin{array}{cc} \boldsymbol{F} = \frac{\mathbf{1}}{\mathbf{2}}\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\boldsymbol{j} & -\mathbf{1} & \boldsymbol{j} \\ \mathbf{1} & -\mathbf{1} & \mathbf{1} & -\mathbf{1} \\ \mathbf{1} & \boldsymbol{j} & -\mathbf{1} & -\boldsymbol{j} \end{bmatrix} & \begin{matrix} \rightarrow & 0 \\ \rightarrow & 2 \\ \rightarrow & 3 \\ \rightarrow & 1 \end{matrix} \end{array}$$

$$\boldsymbol{F} = \frac{\mathbf{1}}{\mathbf{2}}\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\boldsymbol{j} & -\mathbf{1} & \boldsymbol{j} \\ \mathbf{1} & -\mathbf{1} & \mathbf{1} & -\mathbf{1} \\ \mathbf{1} & \boldsymbol{j} & -\mathbf{1} & -\boldsymbol{j} \end{bmatrix}$$

$$V = FUF^T$$

$$U = F^{*T}VF^*$$

Property 5: Separable property

Now given $T(u,v) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) g(x,y,u,v)$.

- For 2D DFT g(x,y,u,v) = $\frac{1}{N} e^{j\frac{2\pi}{N}[ux+vy]}$
- Also, g(x,y,u,v) = $\frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}[ux]} \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}[vy]}$

= g_1(x,u) g_2(y,v) $\rightarrow$ Separable

Here = g_1(x,u) g_1(y,v)

- Hence 2D DFT is separable and symmetric

Property 5: Separable property implication

- **A 2D transform can be computed using successive 1D operation on rows and columns.**

WKT

$$v(k,l) = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} u(m,n) \cdot W_N^{km} \; W_N^{ln} \qquad 0 \le k,l \le N-1$$

$$= \frac{1}{\sqrt{N}} \cdot \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} u(m,n) \cdot W_N^{km} \cdot W_N^{ln}$$

$$= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \underbrace{\left[\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} u(m,n)\, W_N^{ln} \right]}_{\text{1D DFT of } u(m,n) \text{ ;(Row operation)}} W_N^{km}$$

$$= \underbrace{\frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \left[v(m,l) \right] W_N^{km}}_{\text{1D DFT of } v(m,l) \text{ ;(column operation)}}$$

$$u(m,n) \xrightarrow{\text{Row Transform}} v(m,l) \xrightarrow{\text{Column Transform}} v(k,l)$$

$$\boldsymbol{F} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$V = FUF^T$$

$$U = F^{*T} V F^*$$

Property 6 : Periodic property

Fourier transform of a discrete signal (DFT) is periodic

$$v(k+N, l+N) = v(k,l) \quad \forall\, k,l$$
$$u(m+N, n+N) = u(m,n) \quad \forall\, m,n$$

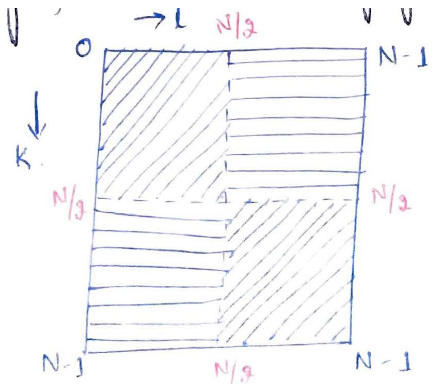
$$\boldsymbol{F} = \frac{\mathbf{1}}{\mathbf{2}}\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\boldsymbol{j} & -\mathbf{1} & \boldsymbol{j} \\ \mathbf{1} & -\mathbf{1} & \mathbf{1} & -\mathbf{1} \\ \mathbf{1} & \boldsymbol{j} & -\mathbf{1} & -\boldsymbol{j} \end{bmatrix}$$

$$V = FUF^T$$
$$U = F^{*T}VF^*$$

Property 7 : **Conjugate symmetry**

For real images, DFT exhibits conjugate symmetry

$$v(k,l) = v^*(N-k, N-l)$$



$$v\left(\frac{N}{2}+1, \frac{N}{2}+1\right) = v^*\left(\frac{N}{2}-1, \frac{N}{2}-1\right)$$

* One quadrant of the image is same as the other quadrant of the image as shown in shaded part in the above fig.

* Thus only half of the image transform is useful & the other half is redundant & can be reproduced.

* ∴ Instead of storing/transmitting N*N image, we can transmit/store $\frac{N}{2} * \frac{N}{2}$ image.

Property 7 : Conjugate symmetry extension

For

Real and even sequence

If $x(n) = x(N - n)$

If a signal is *even* in addition to being real, then its DFT is also real and even

Real and odd sequence

If $x(n) = -x(N - n)$

Similarly, if a signal is *odd* and real, then its DFT is odd and *purely imaginary*.

Properties of DFT

Property 8 : Circular convolution of two sequences is equal to product of their DFTs.

$$x \otimes h = DFT(x).DFT(h)$$

Discrete Cosine transform (DCT)

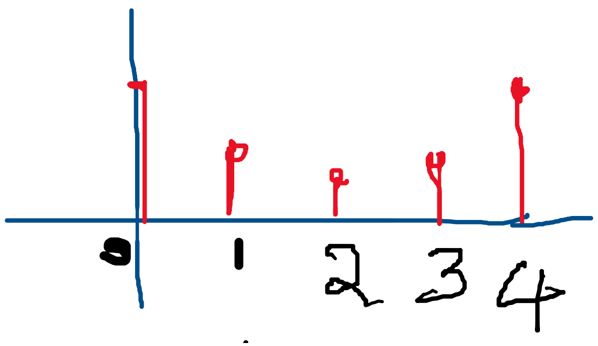
- **Derived from DFT using its symmetry property**
- The DFT represents N-point sequence u(n) as a linear combination of complex exponentials (basis functions)
- **The transformed sequence, v(k) is generally complex even if u(n) is real**

Objective of DCT:

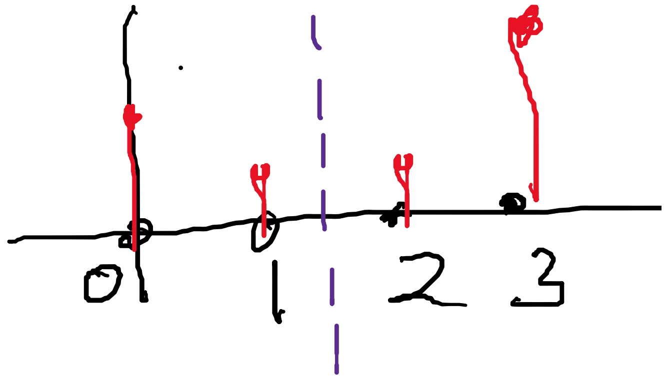
- **To have orthogonal transforms that represent a real time-domain sequence u(n) by a real transform domain sequence v(k)**
- **This is achieved by incorporating symmetry in the input signal u(n)**

Discrete Cosine transform (DCT)

- **Types of symmetry (even):** $x(n) = x(N - n)$

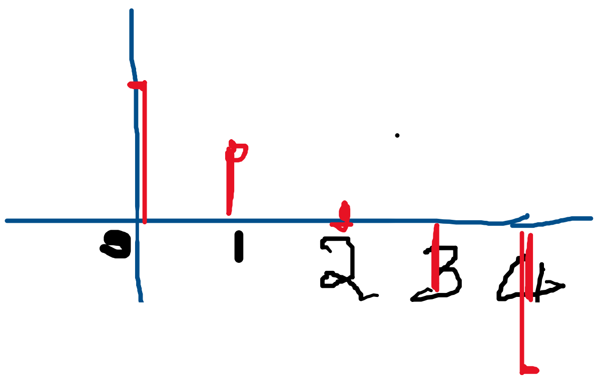


Whole sample symmetry

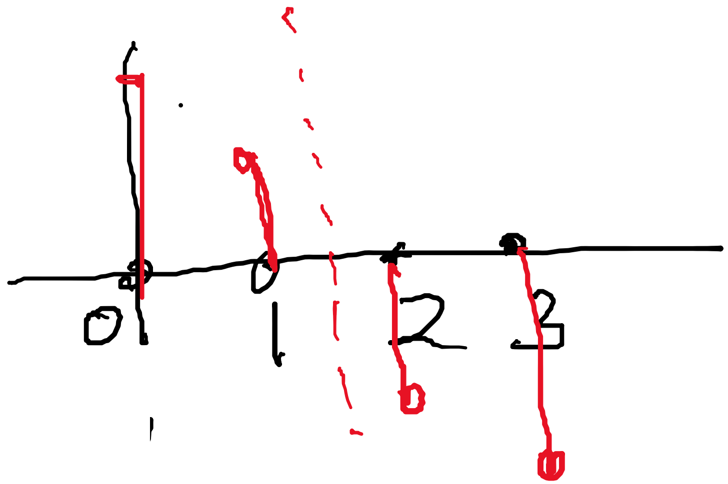


Half sample symmetry

- **Types of symmetry:**



Whole sample anti- symmetry



Half sample anti-symmetry

Discrete Cosine transform (DCT)

Aim: To develop a symmetric or antisymmetric periodic extension from a specified finite length sequence

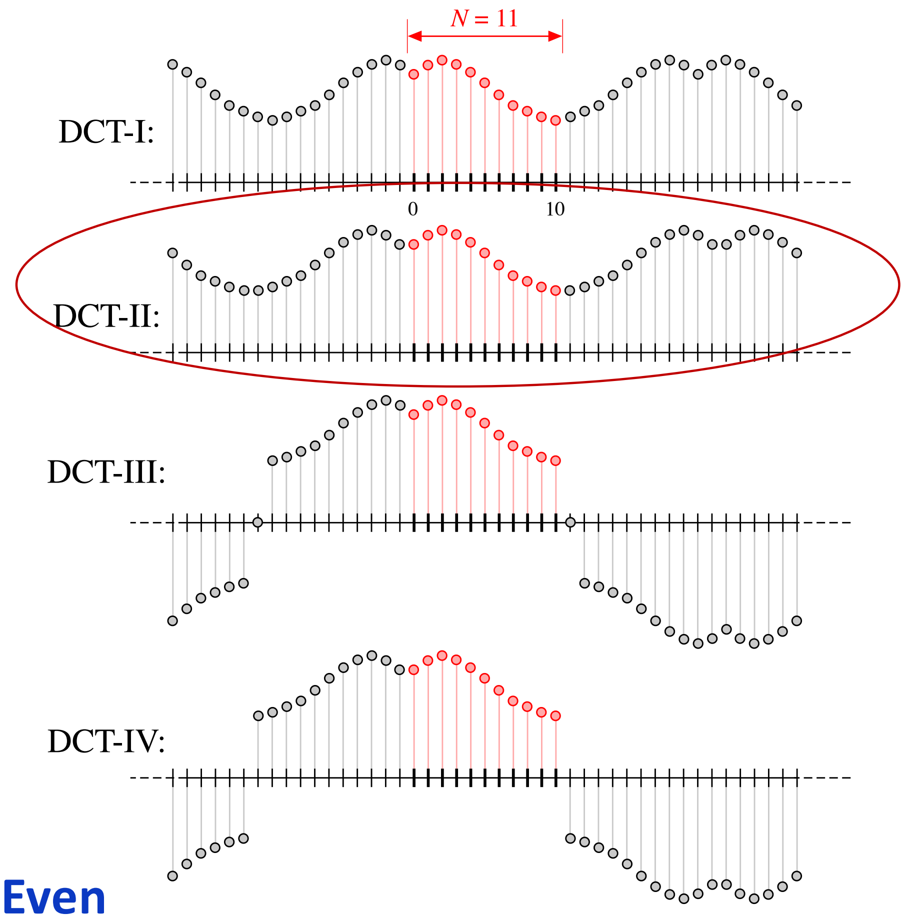
- Total combinations =

For even symmetry (4 for N even and 4 for N odd):

Type 1, Type 2, Type 3, Type 4, Type 5, Type 6, Type 7, Type 8

Even symmetric extension: Discrete Cosine Transform (DCT);Type 1 -4(most common)

Periodic extension process: N Odd



Similarly we get 4 combinations for N Even

Discrete Cosine Transform (DCT)

- In DCT the objective is to find N x N orthogonal transform that expresses a real sequence x(n) as a linear combination of cosine sequences
- **This is possible if N-point sequence is real and even that is x(n) = x(N-n), n = 0,1,2,….N-1. Then X(k) is also real and even**
- To get a DCT for any N-point real sequence, we can find 2N-point DFT of an even extension of the sequence x(n)
- **There are 8 different ways to perform this even extension giving rise to 8 definitions of DCT**
- **Also there are 8 ways to perform odd (anti symmetric) extensions leading to 8 types of DST**
- Type 2 DCT (DCT II) is widely used for speech and image processing applications as part of various standards (Rao & Huang, 1996). **DCT generally refers to DCT II**

Discrete Cosine Transform (DCT)

- In DFT basis sequences are complex sequencs. Hence X(k) is in general complex even if x(n) is real
- In DCT basis functions are cosine. Since cosines are both periodic and have even symmetry the extension of x(n) (outside the range 0 to N-1) in the synthesis will be both periodic and symmetric
- Just like DFT involved implicit assumption of periodicity, DCT involves implicit assumption of both periodicity and even symmetry
- It is a powerful signal decorrelator. Amplitude of autocorrelation after DCT is small
- It's a real valued function and thus can be effectively used in real time DSP operations

Next Session

- DCT and DST
- Discrete Walsh Transform

THANK YOU

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