DIGITAL COMMUNICATION

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BASEBAND SHAPING

Background on Random Processes
Power Spectra of Discrete PAM Signals

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- Recall that a random variable $X: \Omega \rightarrow \mathbb{R}$ such that some constraints are satisfied
- A random process $X(t, \omega)$ is an indexed set of random variables
- At a high level, it is a collection of continuous-time or discrete-time signals, for every ω
- For a given t, $X(t, \omega)$ is a random variable

- Examples: Stock market index, rainfall, weather forecasting
- Examples: Message signal in communications, receiver noise, received signal etc.

- A random process is said to be "deterministic", if future values can be predicted from past samples. Example: $X(t) = A\cos(\omega t + \phi)$, where $\phi \sim U(0, 2\pi)$
- A "non-deterministic" random process is not deterministic

Statistical Parameters

- Recall that for a given $t, X(t, \omega)$ is a random variable
- Therefore, moments on $X(t)$ can be defined for a given value of t
- Let $\mu_X(t) = \mathbb{E}(X(t))$ -- This is the ensemble average
- Let $\sigma_X^2(t) = \mathbb{E}([X(t) - \mu_X(t)]^2)$ -- These two constitute the first order statististics
- Let $m_X^2(t) = \mathbb{E}(X^2(t))$ -- This is the mean squared value

- Consider two time instances t_1 and t_2. Then for all $t_1, t_2 \in \mathbb{R}$
- $R_{XX}(t_1, t_2) = \mathbb{E}(X(t_1)\, X(t_2))$ – This is called as the autocorrelation function
- $C_{XX}(t_1, t_2) = \mathbb{E}([X(t_1) - \mu_X(t_1)][X(t_2) - \mu_X(t_2)])$ -- Autocovariance function
- Note that $R_{XX}(t_1, t_1) = m_X^2(t_1)$
- Note that $C_{XX}(t_1, t_1) = \sigma_X^2(t_1)$

Strict Sense and Wide Sense Stationarity

- Recall that for a given t, $X(t,\omega)$ is a random variable
- A random process $X(t,\omega)$ is said to be first order stationary, if its PDF at any given time instances t and $t+\Delta t$ are equal, i.e., $f_X(x;t) = f_X(x;t+\Delta t)$
- A random process $X(t,\omega)$ is said to be stationary in strict sense if

$f(x_1,\dots,x_n;t_1,\dots,t_n) = f(x_1,\dots,x_n;t_1+\Delta t,\dots,t_n+\Delta t)$, for all $t_1,\dots,t_n,\Delta t \in \mathbb{R}, n \in \mathbb{N}$

- A random process $X(t,\omega)$ is said to be stationary in wide sense (WSS) if
  - $\mu_X(t) = \mu$, for all $t \in \mathbb{R}$
  - $R_{XX}(t_1,t_2) = \mathrm{R_{XX}}(\mathrm{t_2}-\mathrm{t_1}) = \mathrm{R_{XX}}(\tau)$, for all $t_1,t_2 \in \mathbb{R}$
- For continuous-time process, $R_{XX}(\tau) = R_{XX}(t,t+\tau)$
- For discrete-time process, $R_{XX}(k) = R_{XX}(n,n+k)$
- The PSD of a WSS process is the FT of its ACF: $S_X(f) = FT\{R_{XX}(\tau)\}$

POWER SPECTRAL DENSITY (PSD)

Power Spectrum of Discrete PAM Signals

- Let $v(t)$ denote a basic pulse of duration T_b seconds
- The discrete PAM signal is a random process, defined as

$$X(t) = \sum_{k=-\infty}^{\infty} A_k v(t - kT_b)$$

- Here, A_k is a discrete random variable, whose value depends on the k^{th} bit and the chosen format (such as unipolar, polar, bipolar or Manchester)

$$Unipolar : \quad A_k = \begin{cases} a & symbol\ 1 \\ 0 & symbol\ 0 \end{cases}$$

$$Polar/Manchester : \quad A_k = \begin{cases} a & symbol\ 1 \\ -a & symbol\ 0 \end{cases}$$

$$Bipolar : \quad A_k = \begin{cases} a, -a & alternating\ 1s \\ 0 & symbol\ 0 \end{cases}$$

POWER SPECTRAL DENSITY (PSD)

Power Spectrum of Discrete PAM Signals

- We can rewrite $X(t)$ as follows

$$
\begin{aligned}
X(t) &= \sum_{k=-\infty}^{\infty} A_k \left\{\delta(t - kT_b) * v(t)\right\} \\
&= \left\{ \sum_{k=-\infty}^{\infty} A_k \delta(t - kT_b) \right\} * v(t) \\
&= A(t) * v(t)
\end{aligned}
$$

- Therefore, the generation of $X(t)$ can be viewed as an output of an LTI system, with input $A(t)$ and impulse response $v(t)$

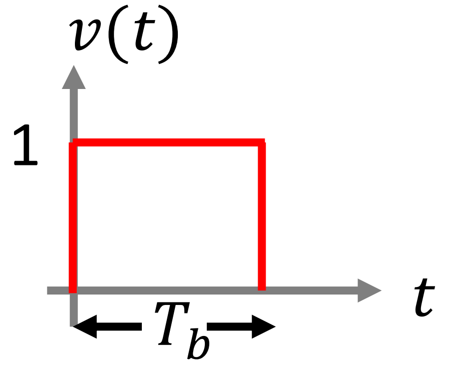
POWER SPECTRAL DENSITY (PSD)

Power Spectrum of Discrete PAM Signals

- Assume that $v(t)$ is an output of an LTI system with response $h(t)$ and input $w(t)$
- When $w(t)$ is a WSS process, $v(t)$ is also WSS and $S_v(f) = |H(f)|^2 S_w(f)$
- A discrete PAM signal is a cyclo-stationary process (periodic and stationary). Hence,

$$S_X(f) = \frac{|V(f)|^2}{T_b} S_A(f)$$

- Therefore, to evaluate $S_X(f)$, we need $|V(f)|^2$
- Assume that $v(t)$ is a rectangular function with unit amplitude



POWER SPECTRAL DENSITY (PSD)

Power Spectrum of Discrete PAM Signals

$$V(f) = \int_0^{T_b} v(t)e^{-j2\pi ft}$$

$$= \frac{-1}{j2\pi f} e^{-j2\pi ft}\Big|_0^{T_b} = \frac{1}{j2\pi f}\left[1 - e^{-j2\pi fT_b}\right]$$

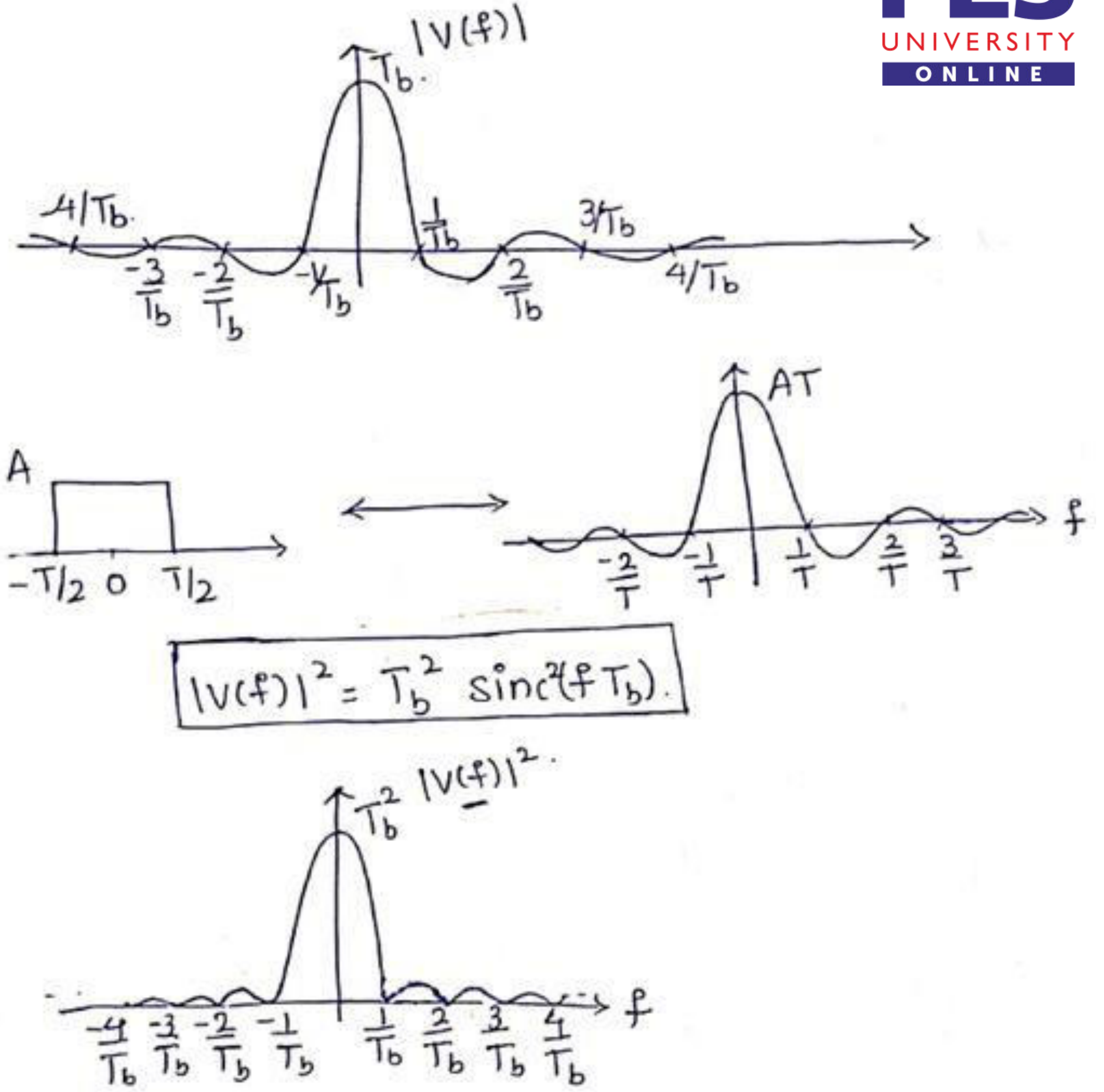
$$= \frac{e^{-j\pi fT_b}}{\pi f}\left[\frac{e^{j\pi fT_b} - e^{-j\pi fT_b}}{2j}\right]$$

$$= e^{-j\pi fT_b}\frac{\sin(\pi fT_b)}{\pi f}$$

$$\therefore |V(f)| = \frac{\sin(\pi fT_b)}{\pi f} = \frac{T_b \sin(\pi fT_b)}{\pi fT_b}$$

$$\therefore |V(f)| = T_b \operatorname{sinc}(fT_b)$$

$$\boxed{\therefore |V(f)|^2 = T_b{}^2 \operatorname{sinc}^2(fT_b)}$$



POWER SPECTRAL DENSITY (PSD)

Power Spectrum of Discrete PAM Signals

- Now that we have evaluated $V(f)$, we next need to evaluate $S_A(f)$
- Recall that the power spectrum is the Fourier transform of the autocorrelation
- A sequence of samples x_k can be represented either as a discrete-time sequence $x(n)$ or a continuous-time signal $x(t)$

$$x(n) = \sum_{k=-\infty}^{\infty} x_k \delta(n-k) \qquad x(t) = \sum_{k=-\infty}^{\infty} x_k \delta(t-kT)$$

- Now, in the Fourier representation

$$x(n) \overset{F.T}{\longleftrightarrow} X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x_k e^{-j\omega k}$$

$$x(t) \overset{F.T}{\longleftrightarrow} X(f) = \int_{t=-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$X(f) = \int_{t=-\infty}^{\infty} \sum_k x_k \delta(t-KT) e^{-j2\pi ft} dt$$

$$= \sum_k x_k \left\{ \int_{t=-\infty}^{\infty} \delta(t-KT) e^{-j2\pi ft} dt \right\}$$

POWER SPECTRAL DENSITY (PSD)

Power Spectrum of Discrete PAM Signals

- Recall that the power spectrum is the Fourier transform of the autocorrelation
- Therefore for us, the sequence of interest $x_k = R_A(k)$
- From the previously shown development, we can write

$$X(f) = \int_{t=-\infty}^{\infty} \sum_k x_k \delta(t - KT) e^{-j2\pi f t} dt$$

$$= \sum_k x_k \left\{ \int_{t=-\infty}^{\infty} \delta(t - KT) e^{-j2\pi f t} dt \right\}$$

$$\boxed{X(f) = \sum_k x_k e^{-j2\pi f K T}}$$

$$\boxed{S_A(f) = \sum_{n=-\infty}^{\infty} R_A(n) e^{-j2\pi f n T_b}}$$

$$\text{where} \quad \boxed{R_A(n) = E[A_k A_{k-n}]}$$

POWER SPECTRUM OF PAM SIGNALS

Power Spectrum of Discrete PAM Signals

- Next, we intend to calculate the power spectra for the following cases
  - Unipolar NRZ
  - Bipolar NRZ
  - Polar NRZ
  - Manchester coding
- In each case, we first evaluate $S_A(f)$ from $R_A(n)$ and then evaluate $S_X(f)$ using

$$S_X(f) = \frac{|V(f)|^2}{T_b} S_A(f)$$

- Note that we have shown

$$|V(f)|^2 = T_b^2 \operatorname{sinc}^2(f T_b)$$

THANK YOU

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