DIGITAL IMAGE PROCESSING-1

Unit 2: Lecture 15-16

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Unit 2: Image Transforms

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Last Session

- Basic relationship between pixels cont..
- Regions and boundaries
- Linear/non linear operations on images
- Introduction to Image transforms

Today's Session

- Image transforms preliminaries
  - Orthogonal transforms
  - Unitary transforms
  - Separable transforms

Linear Transforms

- Linear transforms follow the superposition theorem (homogeneity and additive property)
- These transforms, decompose functions into weighted sums of orthogonal basis functions (?)
  - can be studied using the tools of linear algebra and functional analysis.
- **These transforms are the coefficients of linear expansions**

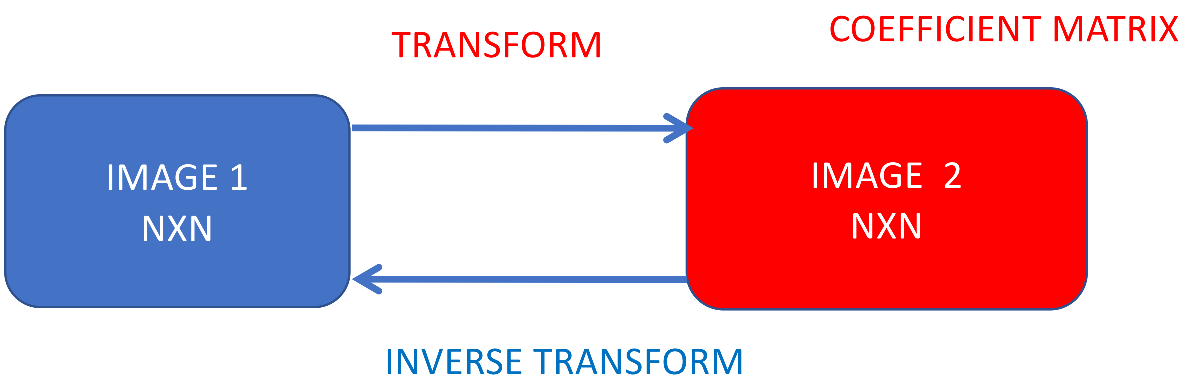
- **An arbitrary continuous signal x(t) can be represented by a series summation of set of orthogonal basis functions (?)**
- The series expansion is given as:

  $x(t) = \sum_{n=0}^{\infty} c_n a_n(t)$ --- infinite series expansion.

  $\hat{x}(t) = \sum_{n=0}^{N-1} c_n a_n(t)$ ----- finite series expansion

- **Finite expansion gives an approximate representation of x(t).**

- All the image processing approaches discussed thus far operate directly on the pixels of an input image
  - they work directly in the spatial domain.
- In some cases, image processing tasks are best formulated by transforming the input images, carrying the specified task in a transform domain, and applying the inverse transform to return to the spatial domain. Y



- An important class of 2-D linear transforms, denoted T(u, v), can be expressed in the general form

$$T(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) r(x,y,u,v)$$

where f(x, y) is an input image, r(x, y, u, v) is called a forward transformation kernel and evaluated for *u = 0,1,2,... , M− 1 and v = 0,1,2,...* , N− 1.

x and y are spatial variables, while M and N are the row and column dimensions of f.

Variables u and v are called the transform variables.

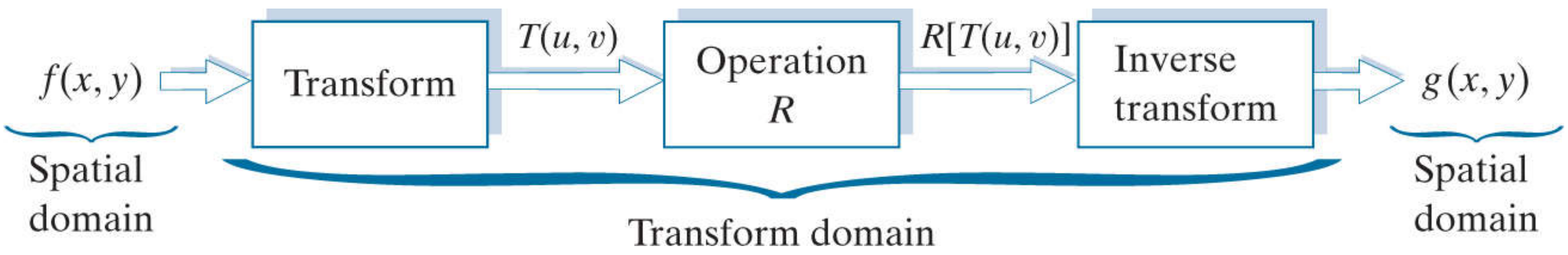
T(u, v) is called the forward transform of f(x, y)

- Given T(u, v), we can recover f(x, y) using the inverse transform of T(u, v):

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} T(u,v)s(x,y,u,v)$$

for x = 0,1,2,... , M− 1 and y = 0,1,2,... ,N − 1,where *s(x, y, u, v)* is called an inverse transformation kernel.

- ***f(x,y)* and *T(u,v)* are called a transform pair**



General approach for working in the linear transform domain

Image Transforms

- Tools that help us to move from one domain to other

  Ex.: Time(space) to frequency
- Change of co-ordinates
- No change in information content, only representation changes
- All of an image's transforms are equivalent in the sense that they contain the same information and total energy
- **They are reversible and differ only in the way that the information and energy is distributed among the transform's coefficients**

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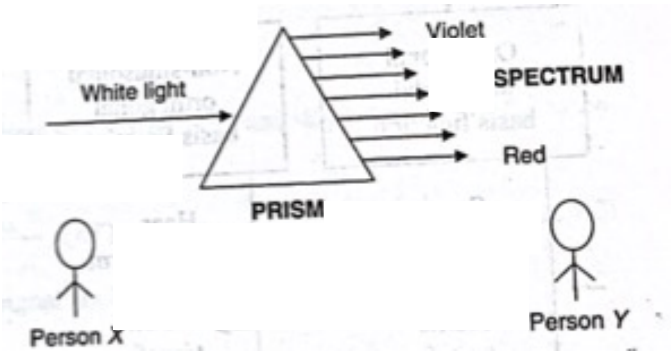
Need for Transforms

- **Fast computations/mathematical convenience-** Ex: convolutions

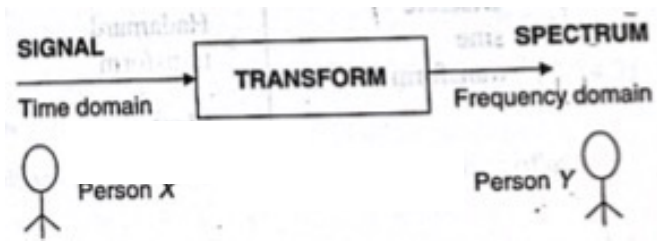
Convolution in time domain ⟺ Multiplication in the frequency domain

- **Better processing** - for smooth, moderate, fast changes in signals

- **Extracting more information-** Allow to extract more relevant information



Spectrum of white light



(Person Y gets more information)

Image Transforms

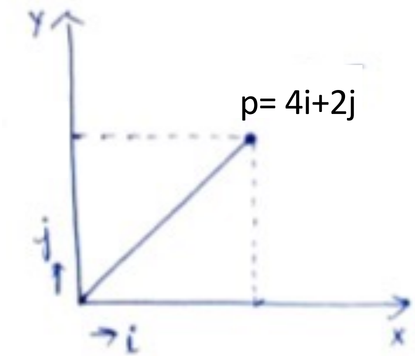
- **Two major reasons for transforming an image:**
  - The transformation may isolate critical components of the image pattern so that they are directly accessible for analysis
  - It may place image data in a more compact form so that they can be stored and transmitted efficiently

Image Transforms: Applications

- Preprocessing - filtering noise (HF), enhancement
- Restoration
- Data compression
- Feature extraction-edges, corners

Image Transforms Preliminaries

- **Basis functions** are used to describe any point in space
  - For ex. Euclidean basis (x,y) forms a basis using which one can describe a point by its coordinates on each axis
  - The vector from origin to point p is the weighted sum of all the unit vectors of the axis



- To analyse any signal it is easier to break the signal into smaller manageable parts called “basis functions”.
- **Any signal can be represented as a linear combination of simpler functions called basis functions**

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Orthogonal and Orthonormal Vectors

- Two vectors are orthogonal if they are perpendicular to each other. i.e. the dot product of the two vectors is zero.
- A set of vectors. $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ are mutually orthogonal if every pair of vectors is orthogonal. i.e.

$$\vec{v}_i.\vec{v}_j = 0,\ for\ all\ i \neq j.$$

- A set of vectors S is orthonormal if every vector in S has magnitude 1 and the set of vectors are mutually orthogonal.

Orthogonal and Orthonormal Vectors

Let $\mathcal{V}$ be an inner-product space. A set of vectors

$$\{u_1, u_2, \ldots, u_n, \ldots\} \in \mathcal{V}$$

is called **orthonormal** if and only if

$$\forall i, j : \langle u_i, u_j \rangle = \delta_{ij}$$

where δ_{ij} is the Kronecker delta and $\langle \cdot, \cdot \rangle$ is the inner product defined over $\mathcal{V}$.

Orthogonal Vectors

- In linear algebra, two **vectors** in an inner product space are **orthonormal** if they are **orthogonal** (perpendicular) and are **unit vectors**.
- An **orthonormal** set which forms a basis is called an **orthonormal** basis.

- So vectors being **orthogonal** puts a restriction on the angle **between** the vectors whereas vectors being **orthonormal** puts restriction on both the angle **between** them as well as **the length of those vectors.**

Unitary and Hermitian Matrix

- **Unitary Matrix:** A Matrix ‘A’ is a unitary matrix if **$A^{-1} = A^{*T}$**

  where A* is conjugate of A

- Condition for orthogonality of matrices: **$A A^{-1} = I$**
- Condition for orthogonality of unitary matrices: **$A A^{-1} = A A^{*T} = I$**
- Condition for orthogonality of unitary real matrices: **$A A^{-1} = A A^{T} = I$**
- **Or $A^{-1} = A^{T}$**
- **Hermitian Matrix: $A^{H} = A^{*T}$**
  - **For a real matrix $A^{H} = A^{T}$**
- **So for a Unitary Matrix: $A^{-1} = A^{H}$**

Examples

1. What is the complex conjugate and Hermitian of T?

$$T = \begin{pmatrix} 2+j & 3 \\ -2 & 7-j \end{pmatrix}$$

Sol: Complex conjugate is

$$T = \begin{pmatrix} 2-j & 3 \\ -2 & 7+j \end{pmatrix}$$

Hermitian of T = T^{*T}

$$T^H = \begin{pmatrix} 2-j & -2 \\ 3 & 7+j \end{pmatrix}$$

Examples

2. Show that unitary matrix A is orthogonal

$$A=\frac{1}{3}\begin{pmatrix}-1 & 2 & -2\\ -2 & 1 & 2\\ 2 & 2 & 1\end{pmatrix}$$

Sol: Condition for orthogonality is

$$A^{-1}=A^{T}$$

$$A\times A^{T}=I,$$

$$A^{-1}=A^{T}=\frac{1}{3}\begin{pmatrix}-1 & -2 & 2\\ 2 & 1 & 2\\ -2 & 2 & 1\end{pmatrix}$$

$$A\times A^{T}=\frac{1}{3}\begin{pmatrix}-1 & 2 & -2\\ -2 & 1 & 2\\ 2 & 2 & 1\end{pmatrix}\times\frac{1}{3}\begin{pmatrix}-1 & -2 & 2\\ 2 & 1 & 2\\ -2 & 2 & 1\end{pmatrix}$$

Since $A\times A^{-1}=A\times A^{T}=I$, Hence given matrix is orthogonal

- **Unitary transform:** A discrete linear transform is unitary if its transform matrix conforms to the unitary condition $A\,A^{*T} = A \times A^{H} = I$

  where A = transformation matrix and A^{H} = **Hermitian matrix and** $A^{H} = A^{*T}$
- **When the transform matrix A is unitary, the defined transform is called unitary transform**
  - **Determinant and eigen value of a unitary matrix have unity magnitude**
- **Image Transforms represent the given image as a series summation of a set of Unitary Matrices (basis matrices)**

1D Unitary Transform

- Let u(n) be a one dimensional sequence where $0 \le n \le N - 1$ consisting of N elements.
- Unitary transform is defined as follows :

$$\boldsymbol{v(k) = \sum_{n=0}^{N-1} a(k,n).u(n) \qquad 0 \le k \le N-1}$$

Where

$v\ (k)$ = Transformed sequence

$a(k,\ n)$ = Forward Transform **kernel**

- The original signal can be recovered back by inverse transform

$$\boldsymbol{u(n) = \sum_{k=0}^{N-1} v(k).a^{*}(k,n) \qquad 0 \le n \le N-1}$$

Where $a^*(k,\ n)$ =conjugate of $a(k,\ n)$, called inverse transformation kernel

1D Unitary Transform

$$\boldsymbol{v(k)} = \sum_{\boldsymbol{n=0}}^{\boldsymbol{N-1}} \boldsymbol{a(k,n).u(n)} \qquad \boldsymbol{0 \le k \le N-1}$$

$$\boldsymbol{u(n)} = \sum_{\boldsymbol{k=0}}^{\boldsymbol{N-1}} \boldsymbol{v(k).a^*(k,n)} \qquad \boldsymbol{0 \le n \le N-1}$$

- **In matrix notation forward unitary transform is given by V = A U**
- **Inverse unitary transform is given by** $\mathrm{U} = A^{*T}\ \mathrm{V}$

Separable Transform

- A function of 2 independent variables is said to be **separable**, if it can be expressed as a product of 2 functions, each of them depending on only one variable

  a(m,n)=a(m).a(n)

- Hence for a 2D separable transform $g(x,y,u,v) = g_1(x,u).\ g_2(y,v)$
- If g_1 and g_2 are functionally same then RHS = $g_1(x,u).g_1(y,v)$ and then the function is called Symmetric
- Examples of class of transforms that are separable and symmetric $\rightarrow DFT, DCT$ … …

2D Separable and Unitary Transform

- The **forward kernel** is said to be separable if

$$\boldsymbol{a_{k,l}(m,n) = a_k(m).a_l(n) = a(k,m).b(l,n)}$$

Where

$\boldsymbol{a(k,m) = A}$ **and** $\boldsymbol{b(l,n) = B}$ are Unitary matrices

$AA^{*T} = I$. and $BB^{*T} = I$

2D Separable and Unitary Transform

- Let $u(m, n)$ be an image of size N x N, then forward transformation is given by

$$\boldsymbol{v(k,l)} = \sum_{\boldsymbol{m=0}}^{\boldsymbol{N-1}} \sum_{\boldsymbol{n=0}}^{\boldsymbol{N-1}} \boldsymbol{u(m,n)\ a_{k,l}(m,n)} \qquad \boldsymbol{0 \le k, l \le N-1}$$

- Inverse transformation is given by

$$\boldsymbol{u(m,n)} = \sum_{\boldsymbol{k=0}}^{\boldsymbol{N-1}} \sum_{\boldsymbol{l=0}}^{\boldsymbol{N-1}} \boldsymbol{v(k,l)\ a^{*}{}_{k,l}(m,n)} \qquad \boldsymbol{0 \le m, n \le N-1}$$

Where

$a_{k,l}(m,n)$ is forward transformation kernel

$a^{*}{}_{k,l}(m,n)$is reverse transformation kernel

2D Separable Unitary Transform

- **Separable unitary transforms:** reduces Computational Complexity
- If the forward kernel is separable and symmetric then the forward transform is given by

$$v(k,l) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} u(m,n)\ a_{k,l}(m,n) \quad 0 \le k,l \le N-1$$

$$v(k,l) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} a(k,m)u(m,n)\ a(l,n) \quad 0 \le k,l \le N-1$$

- In matrix

  Forward transform is given by $V = A\ U\ A^T$

Inverse transform is given by

$U = A^{*T} V\ A^*$

Where

U is input matrix

V is transformed output matrix A is transformation kernel

$AA^{*T} = I$. and $BB^{*T} = I$

- **If this is satisfied, then we say that the transform is a separable transform**

Summary: 1D and 2D Unitary transform

- 1D unitary transform, $V=AU$
- 1D Inverse unitary transform, $U=A^{*T} V$
- 2D unitary transform, $V=AUA^T$
- 2D Inverse unitary transform, $U=A^{*T} V A^*$

Basis Images

V=AU

- Any image can be represented as a linear combination of **basis images.**
- **Basis images :** Images can be expanded in terms of discrete set of basis arrays called "Basis images"
- The basis images can be generated by unitary matrices.
- Basis image is represented as

$$B^*_{k,l} = b^*_k \,.\, {b^*}^T_l$$

Where

${b^*}_k$ is k^{th} column of A^{*T}

${b^*}_l$ is l^{th} column of A^{*T}

Basis Images

- Using the transformed image v and these basis images, original image can be generated as follows:

$$u = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} v(k,l) B_{k,l}^{*}$$

Unitary Image Transforms Examples

3. Prove that the unitary transform works for the given image $F = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

by using the image transform kernel $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Sol: F is real. So forward transform is given as $g = T \times F \times T^T$

$$g = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Original image can be recovered as $f = T^T \times g \times T$

$$f = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \times \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

As original image is recovered this transformation works

- It can be observed that there is no information loss
- **This is the reason for popularity of unitary transforms**

Next Session

- Image basis cont..
- Desirable properties of image transforms
- 1 D and 2D DFT

THANK YOU

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