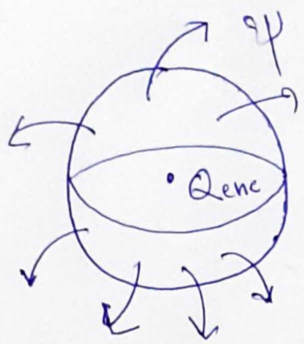
GAUSS'S LAW - MAXWELL'S EQUATION

Gauss's law states that the total electric flux ψ through any closed surface is equal to the total charge enclosed.



ψ = Electric flux

Q_{enc} = charge enclosed

$$\psi = Q_{enc} \rightarrow (1)$$

Let $\vec{D}$ = Electric flux density

$$\psi = \oint_S \vec{D} \cdot \vec{ds} \rightarrow (2)$$

The total charge enclosed = Q_{enc}

$$Q_{enc} = \int_V \rho_v \, dv$$ where ρ_v = volume charge density

$$Q_{enc} = \int_V \rho_v \, dv \rightarrow (3)$$

$$\psi = Q_{enc}$$

$\therefore$

$$\oint_S \vec{D} \cdot \vec{ds} = \int_V \rho_v \, dv \rightarrow (4)$$

From Divergence theorem

$$\oint_D \vec{D} \cdot \vec{ds} = \int_V \nabla \cdot \vec{D} \, dv \rightarrow (5)$$

(4) = (5)

$$\therefore \boxed{\nabla \cdot \vec{D} = \rho_v} \rightarrow (6)$$

Equation (6) is the First Maxwell's Equation

~~*) Gauss's law can be stated from as $Q = \oint_S \vec{D} \cdot d\vec{s} = \int_V \rho_v dv$~~

*) Gauss's law in the integral form can be is

$$Q = \oint_S \vec{D} \cdot \vec{ds} = \int_V \rho_v dv$$

*) Gauss's law in differential form or point form is

$$\boxed{\rho_v = \nabla \cdot \vec{D}}$$

Note : (1) If $\vec{D} = D_x \hat{a}_x + D_y \hat{a}_y + D_z \hat{a}_z$

$$\nabla \cdot \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

(2) If $\vec{D} = D_\rho \hat{a}_\rho + D_\phi \hat{a}_\phi + D_z \hat{a}_z$

$$\nabla \cdot \vec{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} [\rho D_\rho] + \frac{1}{\rho} \frac{\partial}{\partial \phi} [D_\phi] + \frac{\partial D_z}{\partial z}$$

(3) If $\vec{D} = D_r \hat{a}_r + D_\theta \hat{a}_\theta + D_\phi \hat{a}_\phi$

$$\nabla \cdot \vec{D} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 D_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (D_\theta \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} (D_\phi)$$

Applications of GAUSS'S LAW

The procedure for applying Gauss's law to Calculate the electric field involves first knowing whether Symmetry exists. Once it has been found that Symmetric charge distribution exists, we Construct a mathematical closed surface which is known as Gaussian Surface. The surface is chosen Such that $\vec{D}$ is normal or tangential to the Gaussian Surface. when $\vec{D}$ is normal to the Surface

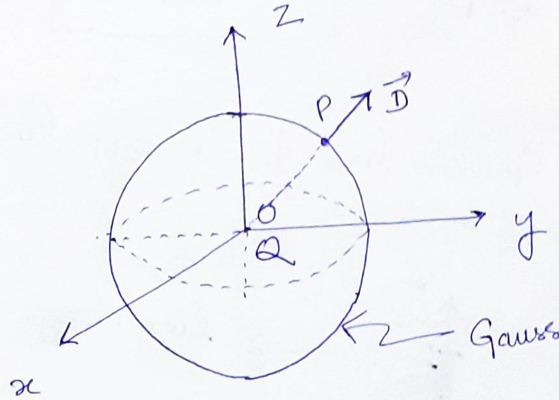
$$\vec{D}\,\hat{a}_n \cdot ds\,\hat{a}_n = D\,ds$$

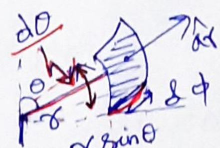
When $\vec{D}$ is tangential to the Surface

$$\vec{D} \cdot \vec{ds} = 0.$$

Gauss's law Can be applied to a point charge, Infinite line charge, Infinite sheet of charge, uniformly charged sphere. We will consider one by one

Application of Gauss's law to a Point charge





$\vec{ds} = r\,d\theta \cdot r\sin\theta\, d\phi\, \hat{a}_r$

Assume a point charge at the origen. To determine $\vec{D}$ at point P, it is easy to see that choosing a spherical surface containing "P" will satisfy symmetry conditions. Thus, a spherical surface centered at the origen is the Gaussian surface.

$\vec{D}$ is normal to the Gaussian surface where $\vec{D} = D_r \hat{a}_r$

From Gauss's law: $Q = \oint_S \vec{D} \cdot \vec{ds}$

$$Q = \oint_S D_r \hat{a}_r \cdot r\sin\theta\, d\phi\, r\, d\theta\, \hat{a}_r$$

$$Q = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} D_r \left[r^2 \sin\theta\, d\phi\, d\theta \right]$$

$$Q = (D_r)(r^2) \int_{\theta=0}^{\pi} \sin\theta\, d\theta \left[\phi\right]\Big|_0^{2\pi}$$

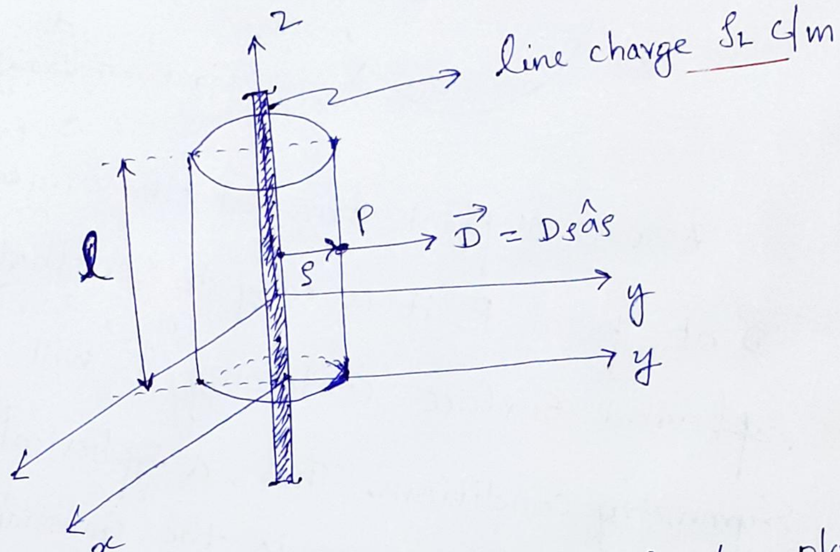
$$Q = (D_r)(r^2)(2\pi)\left[-\cos\theta\right]\Big|_{\theta=0}^{\theta=\pi}$$

$$Q = D_r r^2 2\pi \left[-(-1-1)\right] = (D_r) 4\pi r^2$$

$$Q = D_r 4\pi r^2$$

$$D_r = \frac{Q}{4\pi r^2} \Rightarrow \boxed{\vec{D} = \frac{Q}{4\pi r^2}\hat{a}_r}$$

Application of Gauss's law to infinite line charge



Consider an infinite line charge ρ_L C/m placed along the z-axis. Let the Guassian Surface be a cylinder with radius ρ, height "l" and Vector $\vec{D}$ is normal to the cylindrical Guassian Surface.

$$\therefore \vec{D} = D_\rho \hat{a}_\rho$$

From Gauss's law $\quad Q = \int_S \vec{D}\cdot\vec{ds}$

(40)

$$Q = \int_{\phi=0}^{2\pi} \int_{z=0}^{z=l} D_\rho \hat{a}_\rho \cdot \rho\, d\phi\, dz\, \hat{a}_\rho$$

$$Q = \rho\, 2\pi l\, D_\rho \rightarrow (1)$$

But the total charge enclosed

$$Q = \rho_L l \rightarrow (2)$$

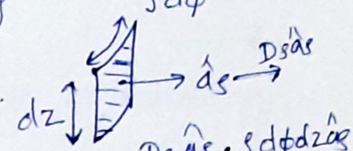
$$(1) = (2)$$

$$\rho_L \cancel{l} = 2\pi \rho \cancel{l} D_\rho$$

$$D_\rho = \frac{\rho_L}{2\pi\rho}$$

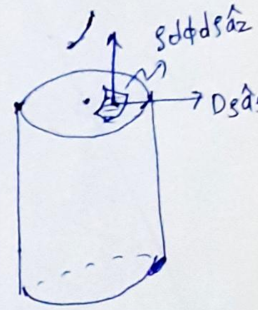
$$\vec{D} = \frac{\rho_L}{2\pi\rho} \hat{a}_\rho$$

on curved surface



$D_\rho \hat{a}_\rho \cdot \rho\, d\phi\, dz\, \hat{a}_\rho$

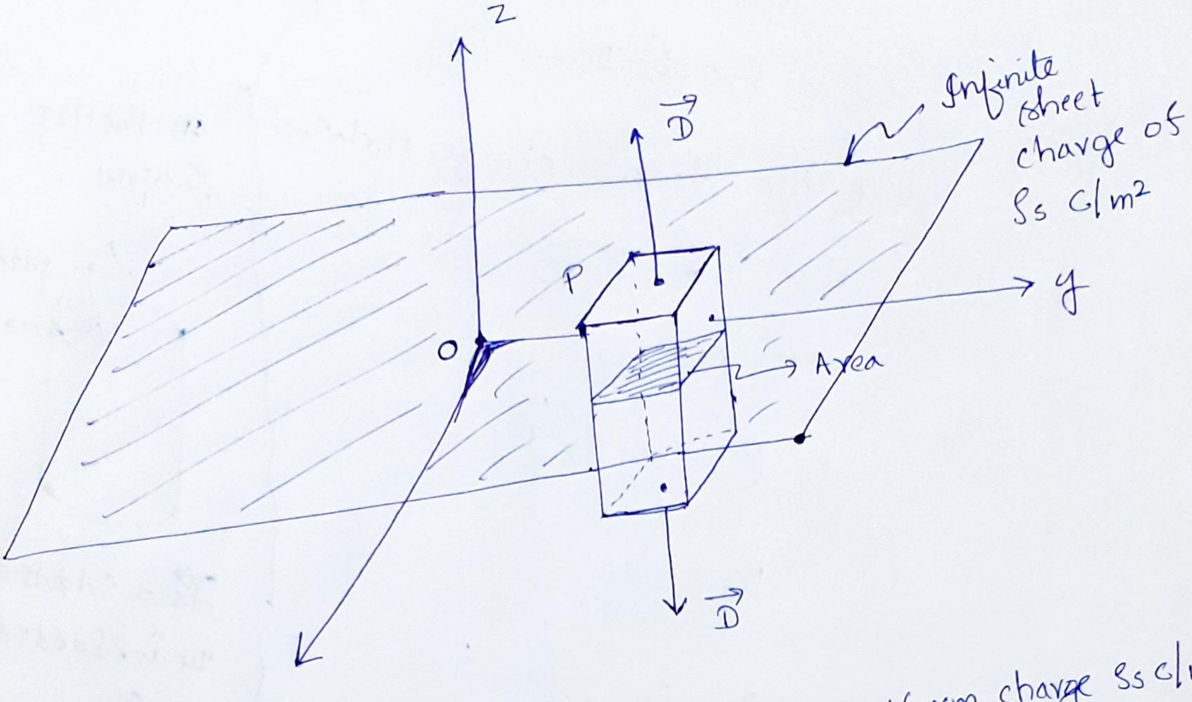
on the top surface.



$$\vec{ds} = \rho\, d\phi\, d\rho\, \hat{a}_z$$

$$D_\rho \hat{a}_\rho \cdot \rho\, d\phi\, d\rho\, \hat{a}_z = 0$$

Application of Gauss's law to Infinite sheet of charge



Consider an infinite sheet of uniform charge ρ_s C/m² lying on the z=0 plane.

To determine $\vec{D}$ at point P, assume a rectangular box that is cut symmetrically by the sheet of charge and has two of its faces parallel to the sheet as shown in figure.

$$\vec{D} = D_z \hat{a}_z$$

From Gauss's law

$$Q = \oint_S \vec{D} \cdot \vec{ds} = \left[\int_{\text{top surface}} D_z \hat{a}_z \cdot dx\,dy\,\hat{a}_z + \int_{\text{botton surface}} D_z(-\hat{a}_z) \cdot dx\,dy(-\hat{a}_z) \right]$$

$$\int_{\text{top surface}} D_z \hat{a}_z \cdot dx\,dy\,\hat{a}_z = D_z \int_{x=0}^{a} \int_{y=0}^{a} dx\,dy = D_z (a \times a) = D_z A \quad \left[\text{where } A = a^2\right]$$

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$$\int_{\text{bottom surface}} D_z(-\hat{a}_z) \cdot dx\,dy\,(-\hat{a}_z) = \int_{x=0}^{a}\int_{y=0}^{a} D_z\,dx\,dy$$

$$= D_z\, a \times a = D_z a^2 = D_z A \quad \text{where } A = a \times a,\ A = a^2$$

$$\therefore\ Q = D_z [A + A] \rightarrow (1)$$

$\cancel{D_z = \frac{Q}{2A}}$

$\cancel{\vec{D} = \frac{Q}{2A}}$

$Q_1 = D_z A = D_z A$

$Q_2 = D_z A = D_z A$

Charge enclosed $\therefore$ $Q = \rho_s A \rightarrow (2)$

$Q = \rho_s A$

$(1) = (2)$

$\rho_s A = D_z A + D_z A$

$\rho_s A = D_z [2A]$

$D_z = \frac{\rho_s}{2}$

$$D_z = \frac{\rho_s \cancel{A}}{2\cancel{A}}$$

$D_z = \frac{\rho_s}{2}\hat{a}_n$

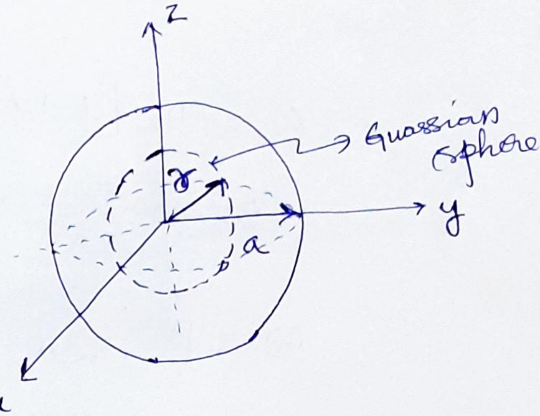
$$\boxed{\vec{D} = \frac{\rho_s}{2}\hat{a}_z}$$

$D_z = \frac{\rho_s}{2}\hat{a}_z$

Application of Gauss's law to a Uniformly Charged Sphere

There are two cases

Case (i): ~~...~~ $r \leq a$



consider a sphere of radius "a" with a uniform charge $\rho_0 \; C/m^3$.

Assume a Guassian Surface for $r \leq a$.

For $r \leq a$

$$Q_{enc} = \int_V \rho_v \, dv = \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

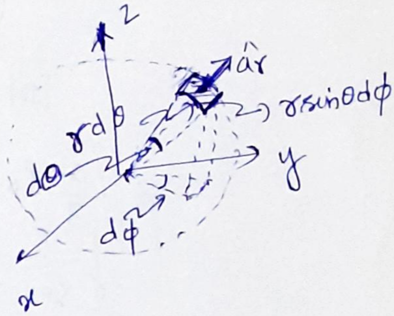
$$Q_{enc} = \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{r^3}{3}\Big|_0^{r} \sin\theta \, d\theta \, d\phi$$

$$Q_{enc} = \rho_0 \int_{\phi=0}^{2\pi} \frac{r^3}{3} \times \left[-\cos\theta \Big|_{\theta=0}^{\theta=\pi}\right] d\phi$$

$$= \rho_0 \frac{r^3}{3} \times [2] \left[\phi \Big|_0^{2\pi}\right] = \rho_0 \frac{r^3 \times 2 \times 2\pi}{3}$$

$$Q_{enc} = \rho_0 \frac{4}{3} \pi r^3 \rightarrow (1)$$

$$\psi = \oint_S \vec{D}\cdot\vec{ds} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} D_r \hat{a_r}\cdot r^2 \sin\theta \, d\theta \, d\phi \, \hat{a_r}$$



$$\psi = D_r \int_{\theta=0}^{\pi} r^2 [2\pi] \sin\theta \, d\theta$$

$$\psi = D_r [2\pi][2] r^2 \rightarrow (2)$$

$$\psi = D_r (4\pi r^2) \longrightarrow (3)$$

Equating (1) = (3)

$$\rho_0 \frac{\cancel{4}}{3} \cancel{\pi} r^3 = D_r (\cancel{4\pi} r^2)$$

$$D_r = \frac{r}{3}\rho_0$$

$$0 < r \le a \rightarrow (4)$$

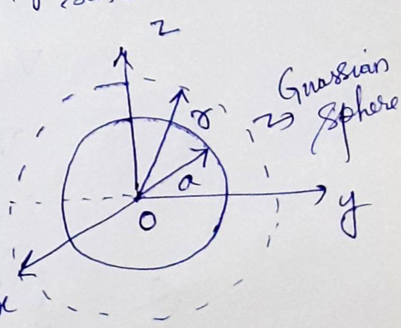
$$\vec{D} = \frac{r}{3}\rho_0 \hat{a_r} \text{ for } 0 < r \le a$$

Case (ii): $r \geq a$

$$Q_{enc} = \int_v \rho_v dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \rho_0 r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$Q_{enc} = \rho_0 \frac{4\pi}{3} a^3 \rightarrow (5)$$

$$\psi = \oint_S \vec{D}\cdot\vec{ds} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} D_r \hat{a_r}\cdot r^2 \sin\theta \, d\theta \, d\phi \, \hat{a_r}$$



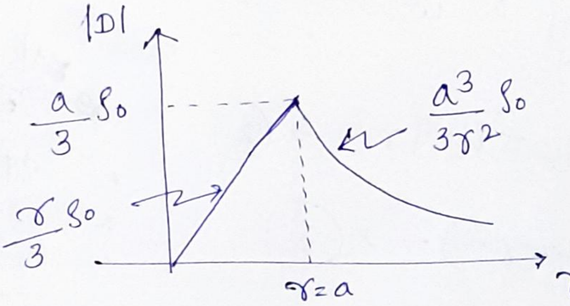
$$\psi = D_r (4\pi r^2) \rightarrow (6)$$

(5) = (6)

$$\rho_0 \frac{\cancel{4\pi}}{3} a^3 = D_r \cancel{4\pi} r^2$$

$$D_r = \frac{\rho_0 a^3}{3 r^2} \text{ for } r \geq a$$

$$\vec{D} = \begin{cases} \dfrac{r}{3}\rho_0 \,\hat{a}_r & 0 < r \le a \\[2mm] \dfrac{a^3}{3r^2}\rho_0 \,\hat{a}_r & r \geq a \end{cases}$$



① Given that $\vec{D} = z\rho\cos^2\phi \,\hat{a}_z$ c/m², calculate the charge density at $(1, \pi/4, 3)$ and the total charge enclosed by the cylinder of radius 1m with $-2 \le z \le 2$m

Solution:

charge density $= \rho_v = \nabla\cdot\vec{D}$

$$\nabla\cdot\vec{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z}$$

$$\vec{D} = 0\,\hat{a}_\rho + 0\,\hat{a}_\phi + z\rho\cos^2\phi\,\hat{a}_z$$

$$\nabla\cdot\vec{D} = \frac{\partial}{\partial z}(z\rho\cos^2\phi)$$

$$\nabla\cdot\vec{D} = \rho\cos^2\phi$$

$$\nabla\cdot\vec{D} = \rho_v \Big|_{(1,\pi/4,3)} = \rho\cos^2\phi = 1\times\cos^2\frac{\pi}{4} = 1\times\frac{1}{2} = 0.5$$

$$\nabla\cdot\vec{D} = \rho_v$$

$$Q = \int_V \rho_v \, dv = \int_{z=-2}^{2}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{1} \rho\cos^2\phi \;\rho \, d\phi \, d\rho \, dz$$

$$Q = \int_{z=-2}^{2} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{1} \rho^2 d\rho \cos^2\phi \, d\phi \, dz$$

$$= \int_{z=-2}^{2} dz \int_{\phi=0}^{2\pi} \cos^2\phi \, d\phi \int_{\rho=0}^{1} \rho^2 d\rho$$

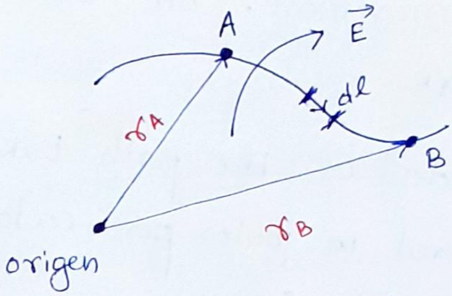
$$= z \Big|_{z=-2}^{2} \left[\int_{\phi=0}^{2\pi} \frac{1+\cos 2\phi}{2} \right] \left[\frac{\rho^3}{3} \Big|_0^1 \right]$$

$$= \frac{4}{3} \times \frac{1}{2} \left[\phi \Big|_0^{2\pi} + \cancel{\frac{\sin 2\phi}{2}}^{\,0} \Big|_0^{2\pi} \right]$$

$$= \frac{4}{3 \times \cancel{2}} \times \cancel{2}\pi = \frac{4\pi}{3} \, C$$

Electric Potential

Suppose we wish to move a point charge from point A to point B in an electric field. $\vec{E}$ as shown in figure



From coulomb's law $\vec{F} = Q\vec{E}$

∴ The workdone in displacing the charge by $\vec{dl}$ is

$$dW = -\vec{F}.\vec{dl} = -Q\vec{E}.\vec{dl}$$

The negative sign indicates that the work is being done by external agent. Thus the total workdone or the potential energy required in moving Q from A to B, is

$$W = -Q\int_A^B \vec{E}.\vec{dl}$$

$$\frac{W}{Q} = -\int_A^B \vec{E}.\vec{dl}$$

$$\text{Potential difference} = V_{AB} = \frac{W}{Q} = -\int_A^B \vec{E}.\vec{dl}$$

*) In determining V_{AB}, A is the initial point while B is the final point.

*) If V_{AB} is negative, there is a loss in potential energy in moving Q from A to B. This implies that the work is being done by the field. However, if V_{AB} is positive, there is a gain in potential energy in the movement. An external agent performs the work.

*) V_{AB} is independent of the path taken

*) V_{AB} is measured in joules per coulomb, commonly referred to as Volts (V).

Consider Electric field due to a point charge "Q" located at the origen, then

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r$$

$$V_{AB} = -\int_{r_A}^{r_B} \vec{E}\cdot\vec{dl}$$

$$= -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r \cdot dr\,\hat{a}_r$$

$$= -\frac{Q}{4\pi\varepsilon_0}\int_{r_A}^{r_B} \frac{1}{r^2}dr = \left[\frac{Q}{4\pi\varepsilon_0}\times\frac{1}{r}\right]_{r_A}^{r_B}$$

$$V_{AB} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right] = V_B - V_A$$

In problems involving point charges, it is customary to choose infinity as reference. That is We assume the potential at infinity is zero.

As $r_A \to \infty$ $V_A = \frac{Q}{4\pi\varepsilon_0}\times\frac{1}{r_A} = 0$

$\therefore$ The potential at any point ($r_B \to r$) due to a point charge 'Q' located at the origen is given by

where $r_B \to r$:

$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

The potential at any point is the potential difference between the point and a chosen point (or reference point) at which the potential is zero.

"V" can also be expressed as $V = -\int_{-\infty}^{r} \vec{E} \cdot d\vec{l}$

If the point charge "Q" is not located at the origen but at a point whose position vector is r', the potential $V(x, y, z)$ or $V(\vec{r})$ is given by

$$V(r) = \frac{Q}{4\pi\varepsilon_0 |\vec{r} - \vec{r}'|}$$

If there multiple charges such that Q_1 is located at point whose position vector is $\vec{r}_1$, Q_2 is located at point whose position vector is $\vec{r}_2$ etc

$$V(r) = \frac{Q_1}{4\pi\varepsilon_0 |\vec{r} - \vec{r}_1|} + \frac{Q_2}{4\pi\varepsilon_0 |\vec{r} - \vec{r}_2|} + \frac{Q_3}{4\pi\varepsilon_0 |\vec{r} - \vec{r}_3|} + \cdots + \frac{Q_N}{4\pi\varepsilon_0 |\vec{r} - \vec{r}_N|}$$

{ wher $\vec{r}_i$ = Position vector of the charge Q_i

where $\vec{r}$ = position vector of the point at which potential is being evaluated.

Similarly for continuous charge distributions.

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho_L(\vec{r}\,')\,dl'}{|\vec{r}-\vec{r}\,'|} \quad \text{For line charge}$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_s(\vec{r}\,')\,ds'}{|\vec{r}-\vec{r}\,'|} \quad \text{For surface charge}$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_v(\vec{r}\,')\,dv'}{|\vec{r}-\vec{r}\,'|} \quad \text{For Volume charge}$$

r = position vector of point at which potential is evaluated

r' = Position vector of the point at which charge density is evaluated

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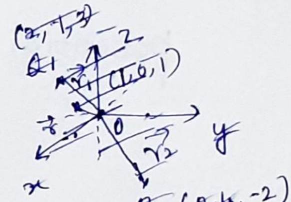
(1) Two point charges $-4\mu C$ and $5\mu C$ are located at $(2,-1,3)$ and $(0,4,-2)$ respectively. Find the potential at $(1,0,1)$ assuming zero potential at infinity.



Solution:-

$Q_1 = -4\mu C$ at $(2,-1,3)$

$Q_2 = 5\mu C$ at $(0,4,-2)$

$$V(r) = \frac{Q_1}{4\pi\varepsilon_0|\vec{r}-\vec{r}_1|} + \frac{Q_2}{4\pi\varepsilon_0|\vec{r}-\vec{r}_2|} + C_0$$

If $V(\infty) = 0$, then $C_0 = 0$

$$|\vec{r}-\vec{r}_1| = |(1,0,1)-(2,-1,3)| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$$

$$|\vec{r}-\vec{r}_2| = |(1,0,1)-(0,4,-2)| = \sqrt{1^2+4^2+3^2} = \sqrt{26}$$

$$V(r) = \frac{-4\times10^{-6}}{\frac{\cancel{4\pi}\times10^{-9}}{36\pi 9}\sqrt{6}} + \frac{5\times10^{-6}}{\frac{4\pi\times10^{-9}}{36\pi 9}\sqrt{26}} = -5.872\ kV$$

$\longrightarrow 8.825\times10^{3}\ V$

$$= -\frac{36\times9\times4}{\sqrt{6}}\times10^{+3} = -14.69\times10^{3}\ V$$

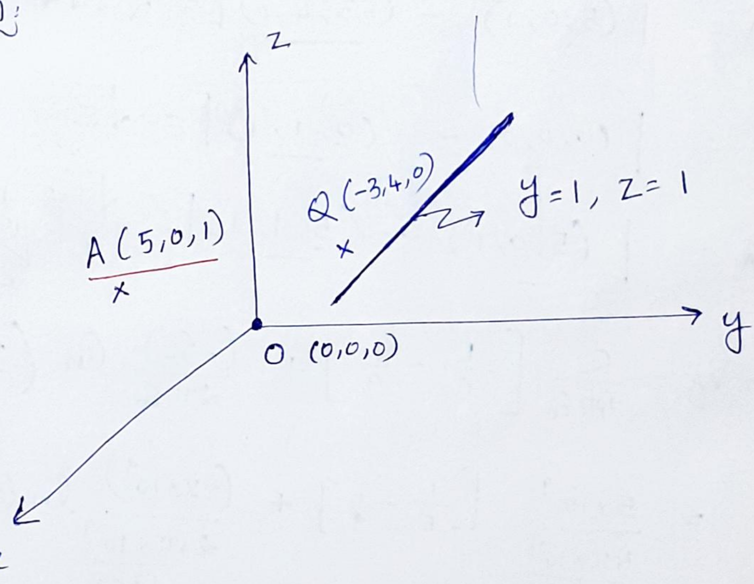
2) A point charge of $5\eta C$ is located at $(-3, 4, 0)$, while line $y=1$, $z=1$ carries uniform charge $2\eta C/m$.

(a) If $V = 0V$ at $O(0,0,0)$, find V at $A(5,0,1)$

(b) If $V = 100V$ at $B(1,2,1)$, find V at $C(-2,5,3)$

(c) If $V = -5V$ at O, find V_{BC}

Solution:



Let the potential at any point

$$V = V_Q + V_L \rightarrow (1)$$

$$V_Q = \frac{Q}{4\pi\varepsilon_0 r} + C_1 \rightarrow (2)$$

$$V_L = -\int \vec{E}\cdot d\vec{l} = -\int \frac{\rho_L \hat{a}_\rho}{2\pi\varepsilon_0 \rho} \cdot d\rho\,\hat{a}_\rho + C_2$$

$$V_L = -\frac{\rho_L}{2\pi\varepsilon_0}\ln(\rho) + C_2 \rightarrow (3)$$

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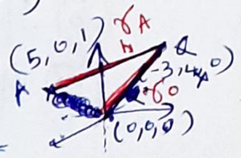
$$V_O - V_A = [V_{Q \text{ at } O} + V_{L \text{ at } O}] - [V_{Q \text{ at } A} + V_{L \text{ at } A}]$$

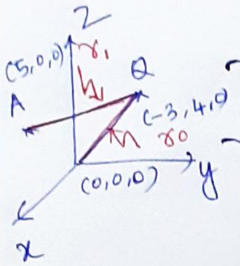
$$V_O - V_A = \left[V_{Q \text{ at } O} - V_{Q \text{ at } A} \right] + \left[V_{L \text{ at } O} - V_{L \text{ at } A} \right]$$

[constants will get cancel by subtraction]

$$= \left[\frac{Q}{4\pi\varepsilon_0 r_0} - \frac{Q}{4\pi\varepsilon_0 r_A} \right] + \left[\frac{-\rho_L}{2\pi\varepsilon_0} \ln(\rho_0) + \frac{\rho_L}{2\pi\varepsilon_0} \ln(\rho_A) \right]$$

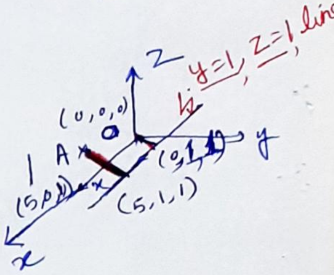
$$= \frac{Q}{4\pi\varepsilon_0} \left[\frac{1}{r_0} - \frac{1}{r_A} \right] + \left(\frac{-\rho_L}{2\pi\varepsilon_0} \ln(\rho_0/\rho_A) \right)$$





$$r_0 = |(0,0,0) - (-3,4,0)| = \sqrt{3^2+4^2} = 5$$

$$r_A = |(5,0,1) - (-3,4,0)| = \sqrt{8^2+4^2+1^2} = 9$$



$$\rho_0 = |(0,0,0) - (0,1,1)| = \sqrt{2}$$

$$\rho_A = |(5,0,1) - (5,1,1)| = \sqrt{1} = 1$$

$$V_O - V_A = \frac{Q}{4\pi\varepsilon_0} \left[\frac{1}{5} - \frac{1}{9} \right] + \frac{(-\rho_L)}{2\pi\varepsilon_0} \ln\left(\frac{\sqrt{2}}{1}\right)$$

$$= \frac{5\times10^{-9}}{\frac{4\pi\times10^{-9}}{36\pi}} \left[\frac{1}{5} - \frac{1}{9} \right] + \frac{(-2\times10^{-9})}{\frac{2\times\pi\times10^{-9}}{36\pi}} \ln(\sqrt{2})$$

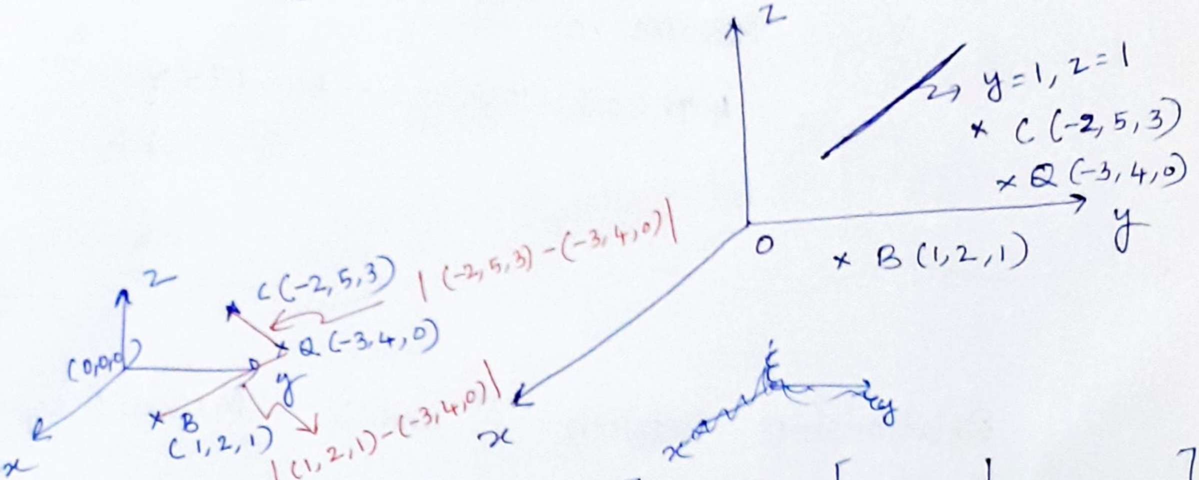
$$V_O - V_A = 4 - 12.4766 = -8.4766 \text{ V}$$

But $V_O = 0$

$$0 - V_A = -8.4766 \Rightarrow V_A = 8.4766 \text{ V}$$

$$V_B - V_C = \left[V_{Q\,at\,B} + V_{L\,at\,B}\right] - \left[V_{Q\,at\,C} + V_{L\,at\,C}\right]$$
$$= \left[V_{Q\,at\,B} - V_{Q\,at\,C}\right] + \left[V_{L\,at\,B} - V_{L\,at\,C}\right]$$

(b) $$V_B - V_C = \begin{bmatrix} V_Q & - & V_Q \\ \text{at } B & & \text{at } C \end{bmatrix} + \begin{bmatrix} V_L & - & V_L \\ \text{at } B & & \text{at } C \end{bmatrix}$$



$$V_B - V_C = \frac{Q}{4\pi\varepsilon_0} \times \left[\frac{1}{|(1,2,1) - (-3,4,0)|}\right] - \frac{Q}{4\pi\varepsilon_0} \times \left[\frac{1}{(-2,5,3) - (-3,4,0)}\right]$$

$$+ \left[\left(\frac{-\rho_L \ln \rho_B}{2\pi\varepsilon_0}\right) - \left(\frac{-\rho_L \ln \rho_C}{2\pi\varepsilon_0}\right)\right]$$

$$\rho_B = |(1,2,1) - (\underline{1},1,1)| = \sqrt{(1-1)^2 + (2-1)^2 + (1-1)^2} = 1$$

$$\rho_C = |(-2,5,3) - (-\underline{2},1,1)| = \sqrt{0^2 + 4^2 + 2^2} = \sqrt{20}$$

$$V_B - V_C = \frac{5\times 10^{-9}}{\frac{4\pi \times 10^{-9}}{36\pi}} \left[\frac{1}{\sqrt{21}} - \frac{1}{\sqrt{11}}\right]$$

$$+ \frac{(2\times 10^{9})}{\frac{2\pi\times 10^{-9}}{36\pi}} \left[\cancel{\ln\sqrt{1}}^{\,0} - \ln\sqrt{20}\right]$$

$$V_B - V_C = 50.1749 \Rightarrow V_B = 100 \Rightarrow 100 - V_C = 50.1749$$

$$V_C = 49.8251\ V$$

c) If $V = -5\,V$ at O find V_{BC}

$V_{BC} = V_C - V_B$

$V_{BC} = 49.825 - 100 = -50.175\,V$

Relationship between $\vec{E}$ and $\vec{V}$ - Maxwell's equation

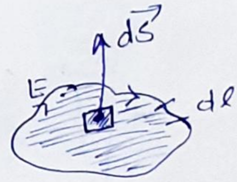
We know $V_{AB} = V_B - V_A$

$V_{BA} = V_A - V_B$

$V_{AB} = -V_{BA}$

$V_{AB} + V_{BA} = 0$

$$\oint_L \vec{E}\cdot d\vec{l} = \int_S (\nabla\times\vec{E})\cdot d\vec{S} = 0$$



$\oint \vec{E}\cdot dl = 0$ Implies physically no network is done in moving a charge along a closed path in an Electrostatic fied [From Stroke's theorem]

$$\boxed{\oint_L \vec{E}\cdot d\vec{l} = 0} \text{ or } \boxed{\nabla\times\vec{E} = 0}$$ is

referred to as Second Maxwell's Equation for static fields.

$\oint \vec{E}\cdot d\vec{l} = 0$, Second Maxwell's equation in integral form

$\nabla\times\vec{E} = 0$, Second Maxwell's equation in differential form

$$V = -\int \vec{E} \cdot \vec{dl}$$

$$dV = -\vec{E} \cdot \vec{dl}$$

$$dV = -\left[E_x \hat{a}_x + E_y \hat{a}_y + E_z \hat{a}_z\right] \cdot \left[dx\, \hat{a}_x + dy\, \hat{a}_y + dz\, \hat{a}_z\right]$$

$$dV = -\left[E_x dx + E_y dy + E_z dz\right] \rightarrow (1)$$

From Calculus

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \rightarrow (2)$$

Equating (1) = (2)

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

$$\therefore \boxed{E = -\nabla V} \qquad E = -\text{grad} V$$

1) Given the potential $V = \frac{10}{r^2} \sin\theta \cos\phi$

(i) Find the electric flux density $\vec{D}$ at $(2, \frac{\pi}{2}, 0)$

(ii) Calculate the work done in moving a $10\mu C$ charge from point A $(1, 30°, 120°)$ to $B(4, 90°, 60°)$

Solution :

(i) $E = -\nabla V$

$$\vec{E} = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi\right]$$

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}\left(\frac{10}{r^2}\sin\theta\cos\phi\right) = -\frac{20}{r^3}\sin\theta\cos\phi$$

$$\frac{\partial V}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{10}{r^2}\sin\theta\cos\phi\right) = \frac{10}{r^2}\cos\theta\cos\phi$$

$$\frac{\partial V}{\partial \phi} = \frac{\partial}{\partial \phi}\left(\frac{10}{r^2}\sin\theta\cos\phi\right) = \frac{10}{r^2}\sin\theta(-\sin\phi)$$

$$\vec{E} = +\frac{20}{r^3}\sin\theta\cos\phi\,\hat{a}_r - \frac{10}{r^3}\cos\theta\cos\phi\,\hat{a}_\theta + \frac{10\cancel{\sin\theta}\sin\phi}{r^3\cancel{\sin\theta}}\hat{a}_\phi$$

$$\vec{E} = \frac{20}{r^3}\sin\theta\cos\phi\,\hat{a}_r - \frac{10}{r^3}\cos\theta\cos\phi\,\hat{a}_\theta + \frac{10}{r^3}\sin\phi\,\hat{a}_\phi$$

$$D = \varepsilon_0\vec{E}$$

$$\vec{D}\Big|_{(2,\frac{\pi}{2},0)} = \varepsilon_0\left[\frac{20}{2^3}\sin 90°\cos 0\,\hat{a}_r - \frac{10}{2^3}\cos 90°\cos 0\,\hat{a}_\theta + \frac{10}{2^3}\sin 0°\,\hat{a}_\phi\right]$$

$$= \varepsilon_0[2.5]\hat{a}_r = 22.1\times10^{-12}\ C/m^2$$

(b) $\frac{W}{Q} = -\int \vec{E} \cdot d\vec{l}$

$$W = -QV = Q[V_B - V_A]$$

$V_{AB} = V_B - V_A$ $\quad A = (1, 30^\circ, 120^\circ)$

$\quad B = (4, 90^\circ, 60^\circ)$

$$|x| = 10 \times 10^{-6} \left[\underbrace{\frac{10}{r^2} \sin\theta \cos\phi \Big|_{(4, 90^\circ, 60^\circ)}}_{V_B} - \underbrace{\frac{10}{r^2} \sin\theta \cos\phi \Big|_{(1, 30^\circ, 120^\circ)}}_{V_A} \right]$$

$$W = 10 \times 10^{-6} \left[\frac{10}{4^2} \times 1 \times \frac{1}{2} - \frac{10}{1^2} \times \frac{1 \times}{2} \left(-\frac{1}{2}\right) \right]$$

$$= 28.125 \, \mu J$$

<u>Note</u>: (1) Gradient in Rectangular Coordinate System

grad V $\quad \nabla V = \frac{\partial V}{\partial x} \hat{a}_x + \frac{\partial V}{\partial y} \hat{a}_y + \frac{\partial V}{\partial z} \hat{a}_z$

(2) Gradient in Cylindrical Coordinate System

$$\nabla V = \frac{\partial V}{\partial \rho} \hat{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{a}_\phi + \frac{\partial V}{\partial z} \hat{a}_z$$

(3) Gradient in Spherical Coordinate System

$$\nabla V = \frac{\partial V}{\partial r} \hat{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{a}_\theta + \frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi} \hat{a}_\phi$$

Electric Fields in Material ~~science~~ Space

Continuity equation and Relaxation time

From the principle of charge conservation, the rate of decrease of charge ~~wh~~ within a given volume must be equal to the net outward current flow through the surface of the volume

$$I_{out} = \oint_S \vec{J} \cdot \vec{ds} = -\frac{dQ_{in}}{dt}$$

$$\oint_S \vec{J} \cdot \vec{ds} = \int_V \nabla \cdot \vec{J}\, dv \qquad \left[\text{From Divergence theorem}\right]$$

$$\therefore \int_V \nabla \cdot \vec{J}\, dv = -\frac{d\, Q_{in}}{dt}$$

$$\int_V \nabla \cdot \vec{J}\, dv = -\frac{d}{dt} \int_V \rho_v\, dv$$

$$\int_V \nabla \cdot \vec{J}\, dv = -\int_V \frac{d\rho_v}{dt}\, dv$$

$$\therefore \nabla \cdot \vec{J} = -\frac{d\rho_v}{dt} \qquad \left[\text{Continuity Equation}\right] \text{ or}$$

Continuity of Current equation

Relaxation time

$$\nabla \cdot \vec{J} = -\frac{d\rho_v}{dt} \rightarrow (1)$$

$$\vec{J} = \sigma \vec{E}$$

$$\nabla \cdot \vec{D} = \rho_v$$

$$\nabla \cdot \varepsilon_0 \vec{E} = \rho_v$$

$$\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon_0}$$

Multiply σ on both sides

$$\nabla \cdot \sigma \vec{E} = \frac{\sigma \rho_v}{\varepsilon}$$

$$\nabla \cdot \vec{J} = \frac{\sigma \rho_v}{\varepsilon} \rightarrow (2)$$

$$(1) = (2) \qquad -\frac{d\rho_v}{dt} = \frac{\sigma \rho_v}{\varepsilon}$$

$$\frac{d\rho_v}{dt} + \frac{\sigma \rho_v}{\varepsilon} = 0$$

This is a homogeneous linear ordinary differential equation. By seperating variables

$$\cancel{\frac{d\rho_v}{\rho_v}} \quad \frac{d\rho_v}{dt} = -\frac{\sigma}{\varepsilon}\rho_v$$

$$\frac{d\rho_v}{\rho_v} = -\frac{\sigma}{\varepsilon} dt$$

Integrating on both sides

$$\ln \rho_v = -\frac{\sigma}{\varepsilon} t + \ln \rho_{v0}$$

where $\ln(\rho_{v0})$ is a constant of integration

$$\ln \rho_v - \ln \rho_{v0} = -\frac{\sigma}{\varepsilon} t$$

$$\ln(\rho_v/\rho_{v0}) = -\frac{\sigma}{\varepsilon} t$$

$$\frac{\rho_v}{\rho_{v0}} = e^{-\frac{\sigma}{\varepsilon} t}$$

$$\frac{\rho_v}{\rho_{v0}} = e^{-\frac{t}{T_r}}$$

$$\rho_v = \rho_{v0}\, e^{-t/T_r} \quad \text{where } T_r = \frac{\varepsilon}{\sigma}$$

$$\boxed{T_r = \frac{\varepsilon}{\sigma} = \varepsilon_0 \varepsilon_r/\sigma}$$

T_r = Relaxation time or rearrangement time.

Relaxation time : Relaxation time is the time it takes a charge placed in the interior of a material to drop to 36.8% (e^{-1}) of its initial value

5.34) Determine the relaxation time for each of the
(5.30) following (a) Hard rubber ($\sigma = 10^{-15}$ s/m, $\varepsilon = 3.1\varepsilon_0$)

(b) Mica ($\sigma = 10^{-15}$ s/m, $\varepsilon = 6\varepsilon_0$)

(c) Distilled water ($\sigma = 10^{-4}$ s/m, $\varepsilon = 80\varepsilon_0$)

Solution:-

(a) For Hard rubber, $T_r = \frac{\varepsilon}{\sigma} = \frac{3.1\varepsilon_0}{\sigma}$

$$= \frac{3.1\times10^{-9}/36\pi}{10^{-15}} = 2.741\times10^{4} \text{ sec}$$

(b) For Mica, $T_r = \frac{\varepsilon}{\sigma} = \frac{6\varepsilon_0}{\sigma} = \frac{6\times10^{-9}/36\pi}{10^{-15}}$

$$= 5.305\times10^{4} \text{ sec}$$

(c) Distilled water, $T_r = \frac{\varepsilon}{\sigma} = \frac{80\times10^{-9}/36\pi}{10^{-4}}$

$$= 7.07\,\mu\text{sec}$$