Unit - 5

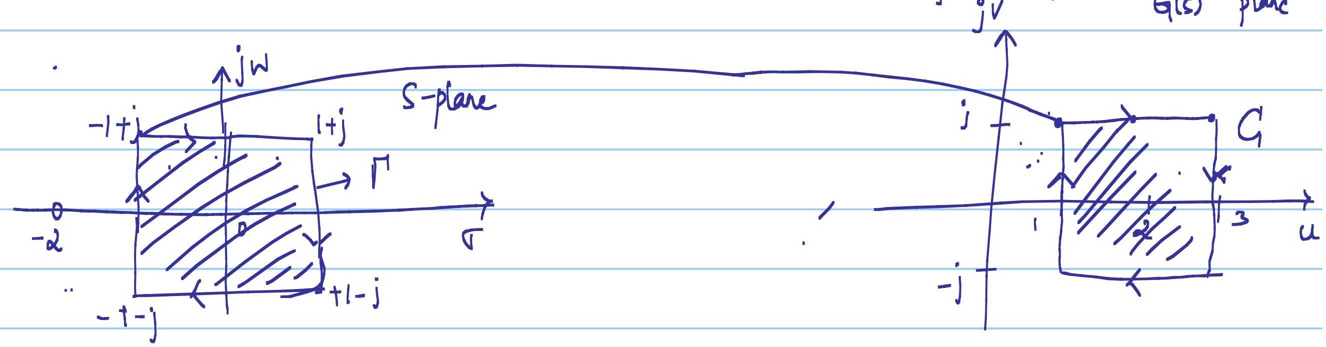
Nyquist Stability Criteria

Contour Mapping / Conformal Mapping

$G(s) = s + 2$

$s = -1+j \quad G(s) = 1+j$

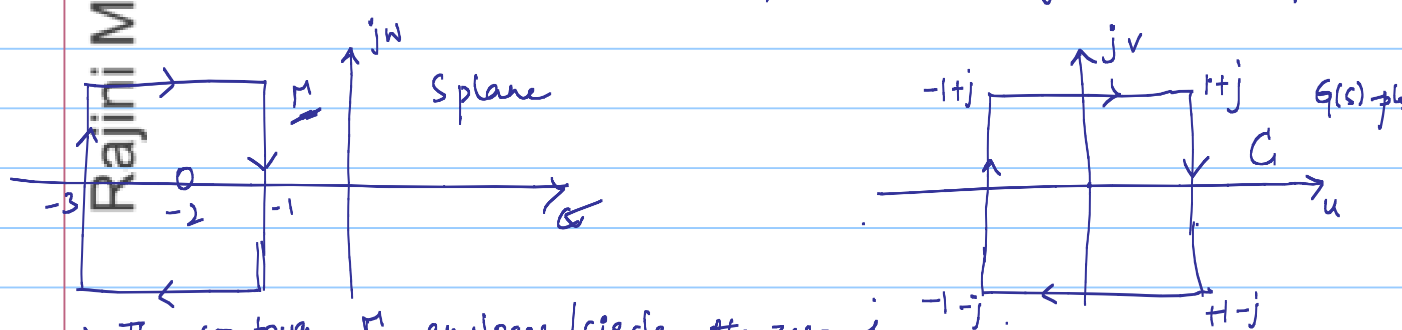
$s = 1+j \quad G(s) = 3+j$



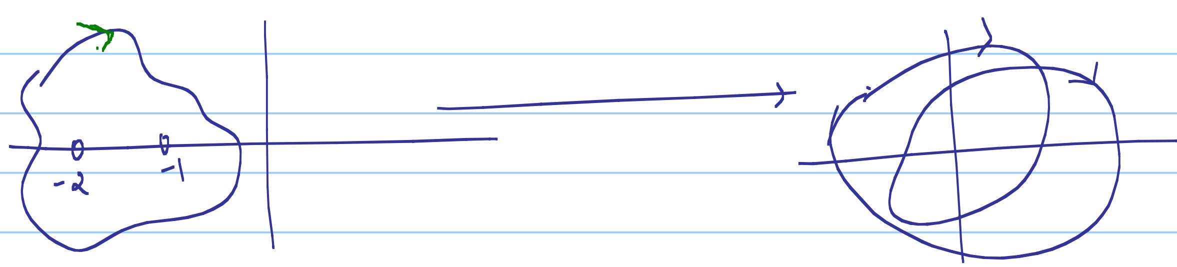
Enclose: Anything to the right of the path is "enclosed" by the contour 'Γ'

We choose contour Γ encircle origin once in clockwise.

→ The contour C_1 doesnot enclose / encircle origin / infinity in G(s) plane



→ The contour Γ encloses / circle the zero of G(s), hence the contour C_1 enclose / encircles origin once in clockwise direction



In general, if the contour Γ in s plane enclose / encircle '3' zeros then contour C_1 in G(s) - plane will enclose / encircle origin 3 times in clockwise direction

If Γ in S-plane enclose / encircle 'p' poles of G(s) then contour C in G(s) - plane will enclose infinity or encircle origin 'p' times in anticlockwise direction

Let N be the no of encirclements, 'z' & 'p' be the no of zeros & poles of G(s) enclosed / encircled by contour Γ.

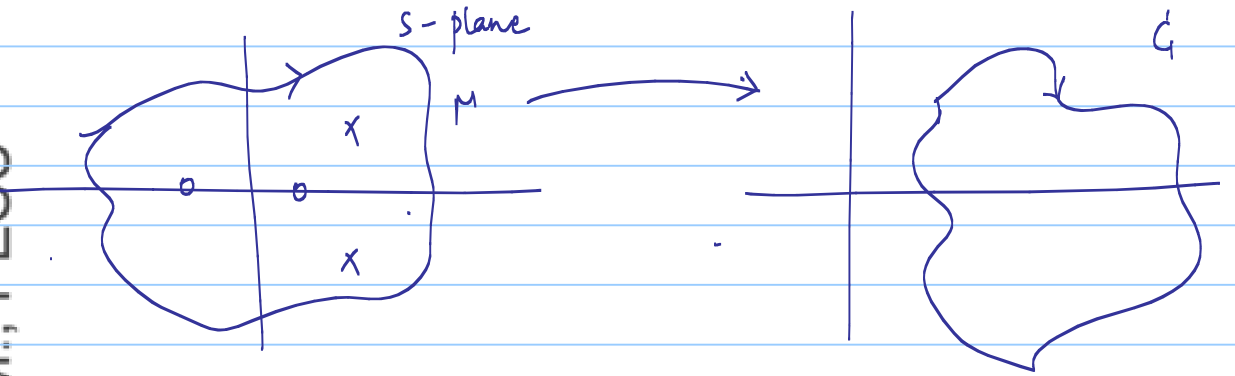
Define

$$N \triangleq Z - P$$

N — +ve clockwise direction

N — -ve anticlockwise direction

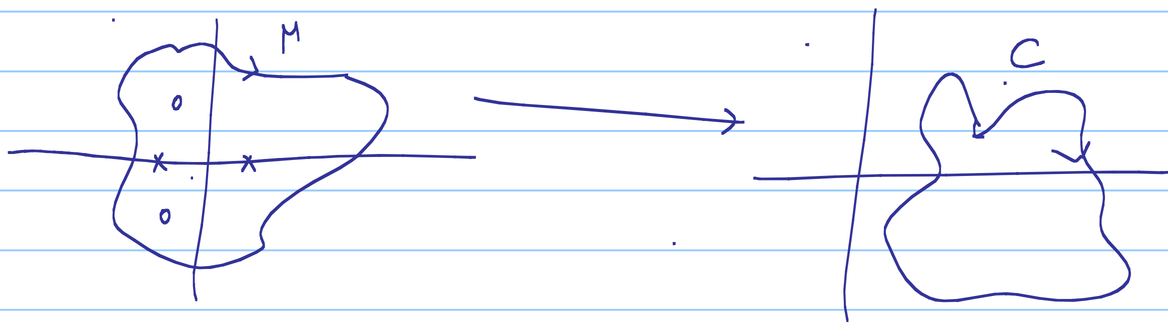
N — 0



1. $Z > P$

N is positive. Hence Origin is encircled by contour C, N times in clockwise direction

2 $Z = P$



N is zero. Hence, origin not encircled by the contour 'C'

3 $Z < P$

N is negative. Hence, the contour C enclose infinity / encircle origin |N| times in anticlockwise direction

Nyquist Stability Criteria:

Let $G(s)H(s)$ be the open loop transfer function. then the closed loop transfer function is

$$T(s) = \frac{G(s)}{1 + \underbrace{G(s)H(s)}_{OLTF}}$$

The charecteristic equation

$$q(s) = 1 + G(s)H(s)$$

Let $$G(s)H(s) = \frac{K(s+z_1)\ldots(s+z_m)}{(s+p_1)\ldots(s+p_n)} \quad \text{--- (1)}$$

then

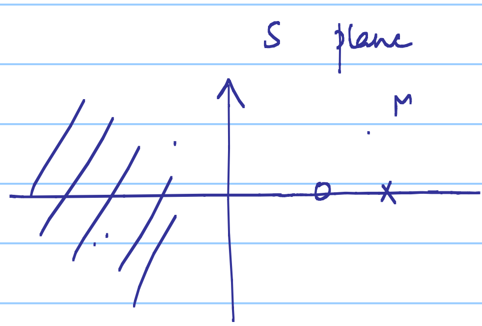
$$q(s) = 1 + \frac{K(s+z_1)\ldots(s+z_m)}{(s+p_1)\ldots(s+p_n)}$$

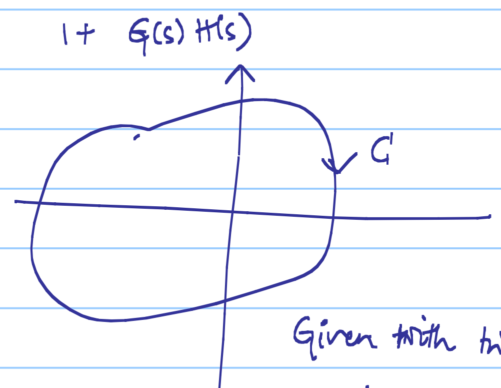
$$q(s) = \frac{(s+p_1)(s+p_2)\ldots(s+p_n) + K(s+z_1)\ldots(s+z_m)}{(s+p_1)\ldots(s+p_n)} \quad \text{--- (2)}$$

Comparing eq^n (1) with (2)

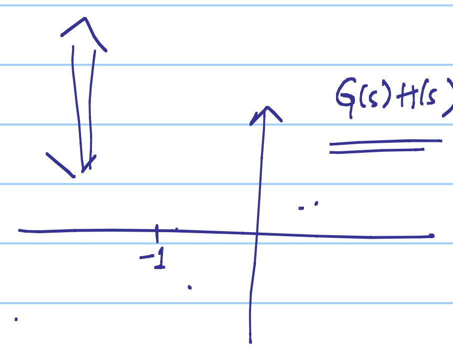
→ The poles of $G(s)H(s)$ is same as poles $q(s)$

→ The zeros of $q(s)$ are poles of closed loop transfer function





Given with this contour C



Statement of Nyquist Stability Criteria.

If the OLTF $G(s)H(s)$ has P poles in right half S-plane then for stability, the contour in $G(s)H(s)$ plane must encircle $-1+j0$ point P times in anti/counter clockwise direction

WKT, $N = Z - P$

$Z = N + P = 0$

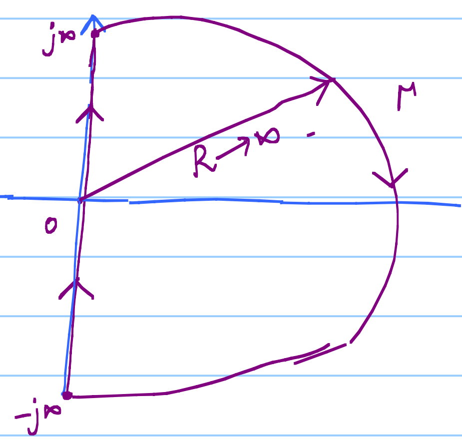
$\Rightarrow N = -P$

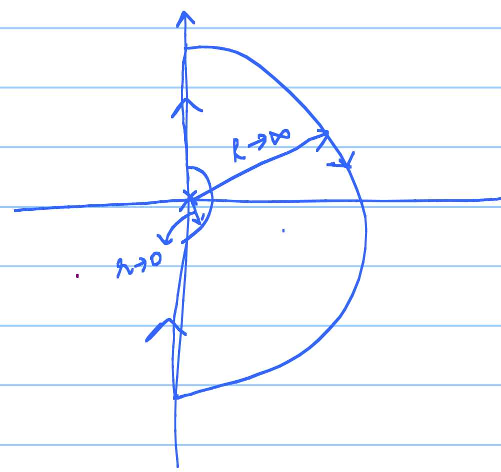
Z — no of closed loop system unstable poles / zero of $q(s)$

N — No of encirclement around $-1+j0$ point in $G(s)H(s)$ - plane

P — No of poles of $G(s)H(s)$ / poles of $q(s)$ in RH S-plane

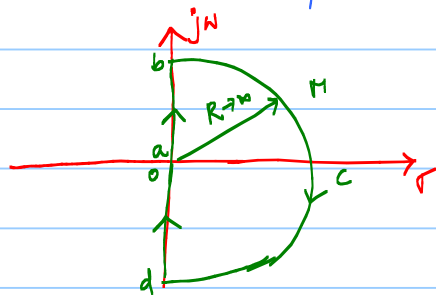
"A feedback system is stable iff the contour in the $G(s)H(s)$ plane doesnot encircle the $(-1,0)$ point when the poles of $G(s)H(s)$ (OLTF) in the right half s-plane is zero"





$$OLTF \cdot G(s)H(s) = \frac{100}{(s+1)(0.1s+1)}$$

1. No of OLTF poles in RHS-plane : $P = 0$
2. The contour in S-plane



Segment Γ_1: Path ab

$s \to 0$ to $j\infty$

Segment Γ_2: Path bcd

$s = Re^{j\theta}$, $R \to \infty$

$\theta \to \pi/2 \to 0 \to -\pi/2$

Segment Γ_3: path da

$s \to -j\infty$ to 0

3. $s = j\omega$

$$G(j\omega)H(j\omega) = \frac{100}{(j\omega+1)(0.1j\omega+1)}$$

$$= \frac{100}{\sqrt{\omega^2+1}\ \sqrt{0.1^2\omega^2+1}} \ \lfloor -\tan^{-1}(\omega) - \tan^{-1}(0.1\omega)$$

$$= \frac{100}{(j\omega+1)(0.1j\omega+1)} \times \frac{(1-j\omega)(1-0.1j\omega)}{(1-j\omega)(1-0.1j\omega)}$$

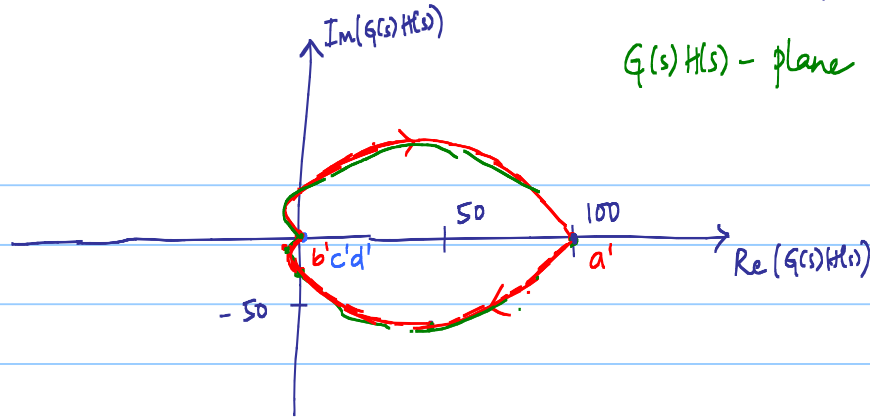
$$= \frac{100(1-1.1j\omega - 0.1\omega^2)}{(1^2-(-j\omega)^2)(1^2-(0.1j\omega)^2)}$$

$$G(j\omega)H(j\omega) = \frac{100(1-0.1\omega^2)}{(1+\omega^2)(1+0.01\omega^2)} - j\frac{100\times 1.1\,\omega}{(1+\omega^2)(1+0.01\omega^2)}$$

Γ_1 mapping: Path ab.

$s = j\omega$ $\qquad \omega \to 0$ to ∞

ω	M	θ	$Re(G(j\omega)H(j\omega))$	$Im(G(j\omega)H(j\omega))$
0	100	0	100	0
1	71	-50.7	44.5	-55
100	0.1	-173	-0.1	-0.01
∞	0	-180	0	-0



put $Re\{G(s)H(s)\} = 0$

$$\frac{100(1-0.1\omega^2)}{(1+\omega^2)(1+0.1^2\omega^2)} = 0$$

$$100(1-0.1\omega^2) = 0$$

$$\Rightarrow \quad \omega^2 = \frac{1}{0.1}$$

$$\omega = 3.16 \text{ rad/sec}$$

$$Im\{G(s)H(s)\}\Big|_{\omega=3.16} = -j\,\frac{100 \times 1.1 \times \sqrt{10}}{(1+10)(1+0.1^2 \times 10)} = -j\,28.76$$

Segment M_2: path bcd

$$\underline{\underline{S = Re^{j\theta}}} \qquad R \to \infty$$
$$\theta \to \pi/2 \text{ to } 0 \text{ to } -\pi/2$$

$$G(s)H(s)\Big|_{s=Re^{j\theta}} = \frac{100}{(Re^{j\theta}+1)(0.1Re^{j\theta}+1)}$$

$$\lim_{R\to\infty} G(Re^{j\theta})H(Re^{j\theta}) = \lim_{R\to\infty} \frac{100}{(Re^{j\theta}+1)(0.1Re^{j\theta}+1)}$$

$$= \lim_{R\to\infty} \frac{100}{Re^{j\theta}\, Re^{j\theta} \times 0.1}$$

$$= 0\, e^{-j2\theta}$$

$$= 0 \;\underline{\big|-180 \to 0 \to 180}$$

Segment M_3: path da

$$S = -j\omega \qquad \omega \to \infty \text{ to } 0$$

The contour of path da is the mirror image of the contour of the path ab

$$Re\{G(j\omega)H(j\omega)\} = Re\{G(j\omega)H(j\omega)\}$$

$$Im\{G(-j\omega)H(-j\omega)\} = -Im\{G(j\omega)H(j\omega)\}$$

$$Z = N + P$$

Since the contour C_1 in $G(s)H(s)$ plane does not encircle $(-1, 0)$ point therefore $N = 0$

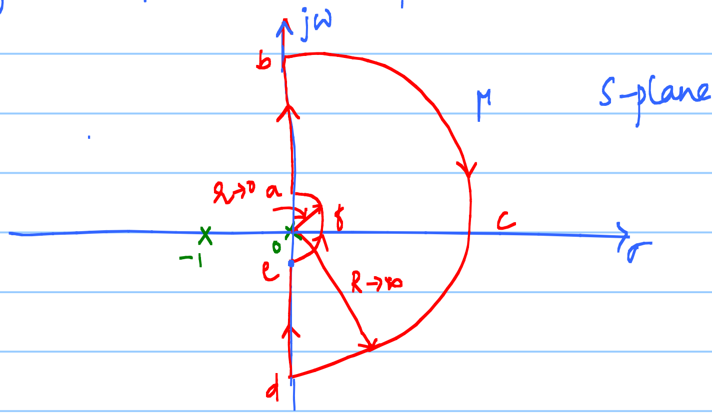
$$Z = 0 + 0$$

$$\boxed{Z = 0}$$

The closed loop system is stable

2. $G(s) = \dfrac{5}{s(s+1)}$

No of OLTF poles in RHP $P = 0$



Segment M_1: Path ab.

$s = j\omega \qquad \omega \to 0$ to ∞

$$G(j\omega) = \frac{5}{j\omega(j\omega+1)}$$

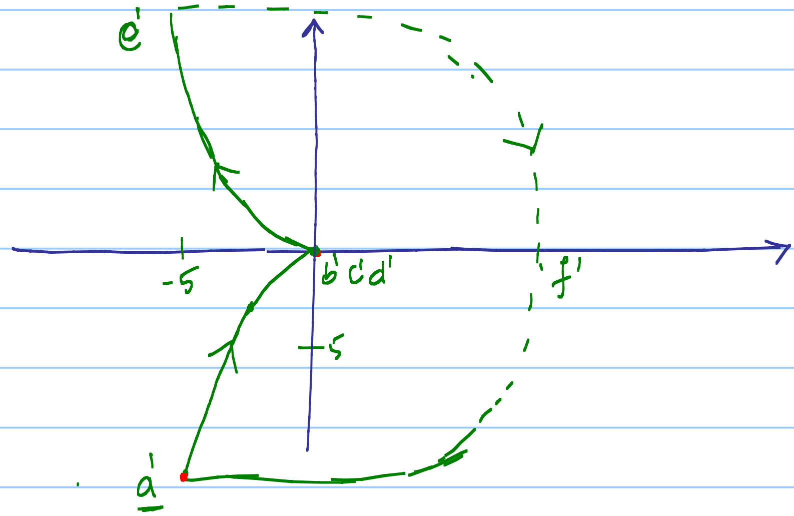
$$= \frac{5}{\omega\sqrt{\omega^2+1}} \angle -90 - \tan^{-1}(\omega)$$

$$= -j\frac{5}{\omega(j\omega+1)} \times \frac{(1-j\omega)}{(1-j\omega)}$$

$$= \frac{-5j(1-j\omega)}{\omega(1+\omega^2)}$$

$$= -\frac{5}{(1+\omega^2)} - j\frac{5}{\omega(1+\omega^2)}$$

ω	M	$\angle G(j\omega)$	Re	Im
0	∞	-90	-5	$-\infty$
1	3.5	-135	-2.5	-2.5
100				
∞	0	-180	-0	-0



Segment M_2: Path bcd

$S = Re^{j\theta}$ $\quad R \to \infty \quad$ $\theta \to \pi/2$ to $0 \to -\pi/2$

$$\lim_{R\to\infty} G(Re^{j\theta}) = \lim_{R\to\infty} \frac{5}{Re^{j\theta}(Re^{j\theta}+1)} = \lim_{R\to\infty} \frac{5}{R^2 e^{j2\theta}} \approx 0\, e^{-j2\theta}$$

M $\quad \phi$

$$= 0 \angle -180 \to 0 \to 180$$

Segment M_3: Path de

Mirror image of path ab

Segment M_4: Path efa

$S = re^{j\phi}$ $\quad r \to 0 \quad$ $\phi \to -\pi/2$ to 0 to $\pi/2$

$$\lim_{r\to 0} G(re^{j\phi}) = \lim_{r\to 0} \frac{5}{re^{j\phi}(\underbrace{re^{j\phi}+1}_{1})}$$

$$\approx \lim_{r\to 0} \frac{5}{re^{j\phi}}$$

$$= \infty \angle -\phi$$

$$= \infty \angle \pi/2 \to 0 \to -\pi/2$$

$N = 0$ & $P = 0$ $\Rightarrow$ $Z = N + P$

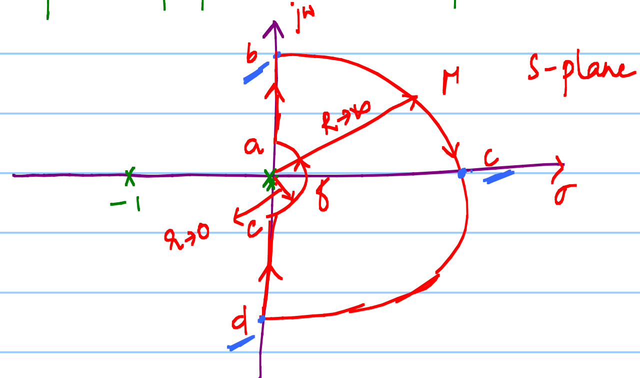
$\boxed{Z = 0}$

The closed loop system is stable

3. $K\,G(s) = \dfrac{K}{s^2(\tau s+1)}$. Let $\tau = 1$

Let $K = 1$

No of open loop poles in RHP $p = 0$



Segmenet Γ_1 : path ab

$S = j\omega$

$$G(j\omega) = \frac{1}{(j\omega)^2(j\omega+1)}$$

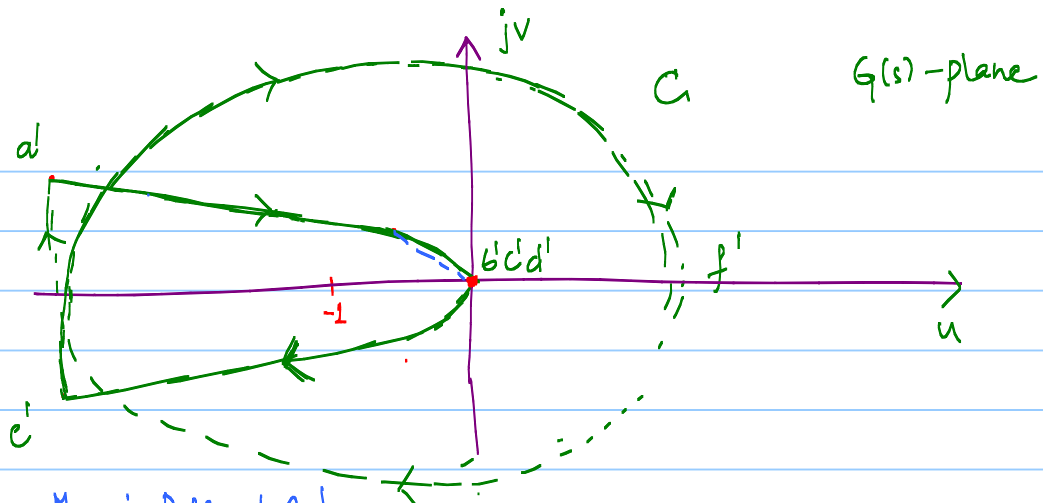
$$G(j\omega) = \frac{-1}{\omega^2(j\omega+1)}$$

$$= \frac{1}{\omega^2\sqrt{\omega^2+1}} \angle -180 - \tan^{-1}(\omega)$$

$$G(j\omega) = \frac{-1}{\omega^2(j\omega+1)} \cdot \frac{(1-j\omega)}{(1-j\omega)}$$

$$= \frac{-1}{\omega^2(\omega^2+1)} + j\frac{\omega}{\omega^2(\omega^2+1)}$$

ω	M	$\angle G(j\omega)$	Re	Im
0	∞	-180	$-\infty$	$+\infty$
1	0.7	-225	-0.5	0.5
100	9.9×10^{-7}	-269.4	-9.9×10^{-9}	$+9.9\times10^{-7}$
∞	0	$\underline{\underline{-270}}$	-0	0



Segment M_2 : Path bcd

$$S = Re^{j\theta} \qquad R \to \infty \qquad \theta \to 90 \to 0 \to -90^\circ$$

$$\lim_{R\to\infty} G(Re^{j\theta}) = \lim_{R\to\infty} \frac{1}{R^2 e^{j2\theta}(\underbrace{Re^{j\theta}+1}_{Re^{j\theta}})}$$

$$\simeq \lim_{R\to\infty} \frac{1}{R^3 e^{j3\theta}}$$

$$= 0 \,\underline{|-3\theta} = 0\,\underline{|-270 \to 0 \to +270^\circ}$$

Segment M_3 : path de

Mirror image of path ab.

Segment M_4 : path efa

$$S = re^{j\phi} \qquad r \to 0 \qquad \phi \to -90 \text{ to } 0 \text{ to } 90$$

$$\lim_{r\to 0} G(re^{j\phi}) = \lim_{r\to 0} \frac{1}{(re^{j\phi})^2(\underbrace{re^{j\phi}+1}_{1})}$$

$$\simeq \lim_{r\to 0} \frac{1}{r^2 e^{j2\phi}}$$

$$= \infty \,\underline{|-2\phi}$$

$$= \infty \,\underline{|180 \to 90} \to 0 \to -90 \to -180$$

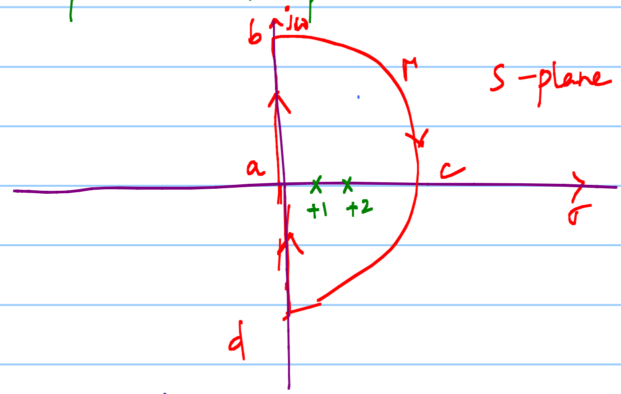
The contour C_1 in G(s)-plane encircles $-1+j0$ point 2 times in clockwise direction $\therefore$ $N = 2$

$$Z = N + P$$
$$= 2 + 0 = 2$$

$\Rightarrow$ Two poles of CLS are in RH s-plane

4. $G(s) = \dfrac{K(s+4)}{(s-1)(s-2)}$ Take $K = 8$

No of OL poles in RHP $p = 2$



Segment r_1 : path ab

$s = j\omega$

$$G(j\omega) = \frac{K(j\omega + 4)}{(j\omega-1)(j\omega-2)}$$

$$= \frac{K\sqrt{\omega^2+4^2}}{\sqrt{\omega^2+1}\ \sqrt{\omega^2+2^2}} \quad \underline{\left| \tan^{-1}(\omega/4) - (180 - \tan^{-1}(\omega)) - (180 - \tan^{-1}(\omega/2)) \right.}$$

WKT $\tan^{-1}\left(\frac{\omega}{-1}\right) = 180 - \tan^{-1}\left(\frac{\omega}{1}\right)$

$$G(j\omega) = \frac{K(j\omega+4)(-j\omega-1)(-j\omega-2)}{(\omega^2+1)(\omega^2+4)}$$

$K(j\omega+4)(j\omega+1)(j\omega+2) = K(j\omega+4)(2+3j\omega-\omega^2)$

$= K(8-7\omega^2) + jK(14\omega-\omega^3)$

$$\checkmark\ G(j\omega) = \frac{K(8-7\omega^2)}{(\omega^2+1)(\omega^2+4)} + j\frac{K(14\omega-\omega^3)}{(\omega^2+1)(\omega^2+4)}$$

ω	M	$\lvert G(j\omega)$	Re	Im
0	2K	$-360°$	2K	0
1	1.3K	$-274°.4$	K/10	13K/10
∞	0	-90	-0	-0

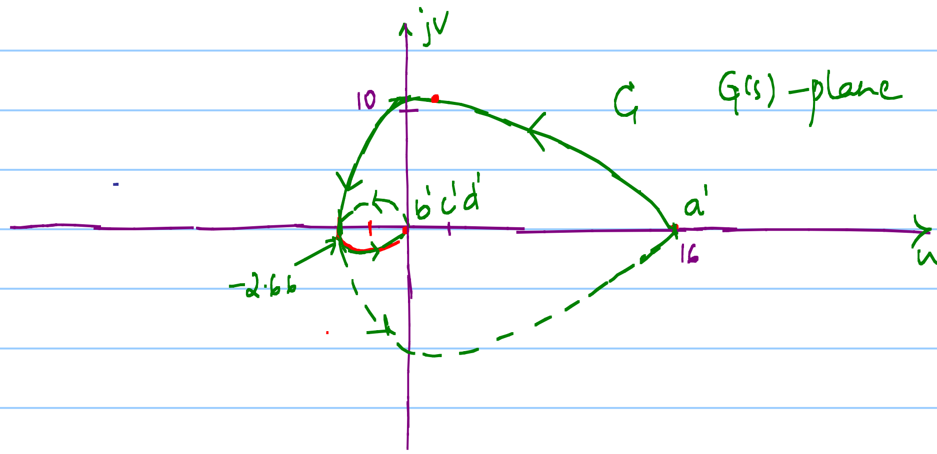
$90 - (180-90) - (180-90)$

$\therefore$ $Im[G(j\omega)] = 0 \Rightarrow (14\omega - \omega^3) = 0$

$\Rightarrow \omega^2 = 14$

$\omega = \sqrt{14}$

$$\mathrm{Re}\left[G(j\omega)\right]\Big|_{\omega=\sqrt{14}} = \frac{K(8-7\times 14)}{15\times 18} = -K/3 = -2.66 \qquad K=8$$



Segment M_2 : Path bcd

$$s = Re^{j\theta} \qquad R\to\infty \qquad \theta \to +90 \to 0 \to -90$$

$$\lim_{R\to\infty} G(Re^{j\theta}) = \lim_{R\to\infty} \frac{K(Re^{j\theta}+4)}{(Re^{j\theta}-1)(Re^{j\theta}-2)} \simeq \lim_{R\to\infty} \frac{K\cancel{Re^{j\theta}}}{\cancel{Re^{j\theta}}\, Re^{j\theta}}$$

$$= 0 \angle -\theta$$

Segment M_3 : Path da

Mirror image of path ab

The contour G encircles -1 point twice in anticlockwise direction $\therefore$ $N = -2$

$$Z = N + P$$
$$= -2 + 2$$
$$Z = 0$$

To find the range of k for system to be stable.

$$\mathrm{Re}\{G(j\omega)\} < -1$$

$$\frac{K(8-7\omega^2)}{(\omega^2+1)(\omega^2+4)} < -1$$

$$\left.\frac{K(8-7\omega^2)}{(\omega^2+1)(\omega^2+4)}\right|_{\omega=\sqrt{14}} < -1$$

$$K\left(\frac{8-7\times 14}{15\times 18}\right) < -1$$

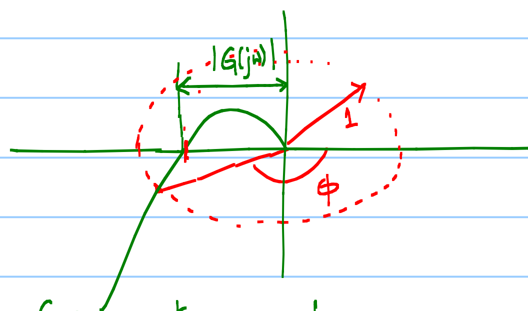
$$-K \times \frac{1}{3} > 1$$

$$\Rightarrow K > 3$$

The CL system is stable for $K > 3$

Gain Margin :

The gain margin is the reciprocal of the magnitude $|G(j\omega)|$ ~~at~~ at the frequency at which the phase angle is -180°.



Gain Margin $\quad K = \dfrac{1}{|G(j\omega)|}$

$$1 + KG(j\omega) = 0$$

$$\Rightarrow KG(j\omega) = -1$$

$$|KG(j\omega)| = 1$$

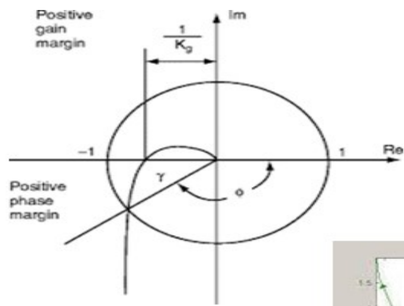
$$\boxed{K = 1/|G(j\omega)|}$$

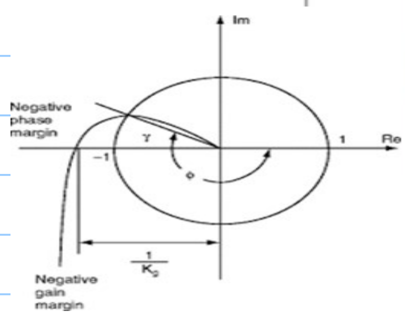
Phase Margin :

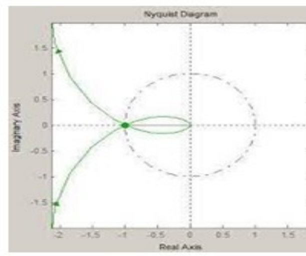
The crossover frequency at which $|G(j\omega)|$ is unity referred to as Gain crossover frequency ω_{GCF}.

The Mark the angle from 0° to the magnitude plot / polar plot $\big|_{s=j\omega}$ referred as 'ϕ'.

Phase margin $= 180^\circ + \phi$

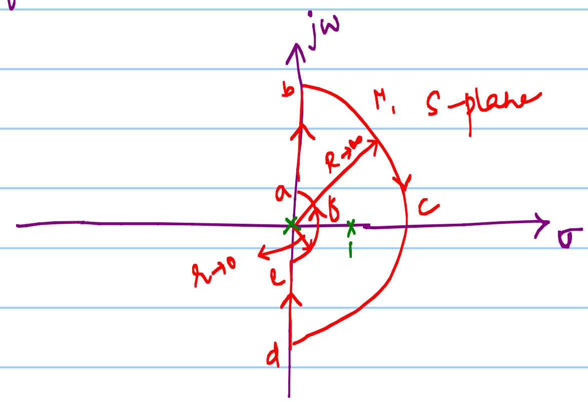

Positive gain margin
$\frac{1}{K_g}$
Im
Re
−1
1
Positive phase margin
γ
φ


Im
Negative phase margin
γ
−1
φ
1
Re
$\frac{1}{K_g}$
Negative gain margin


Nyquist Diagram
Imaginary Axis
Real Axis

5. $G(s) = \dfrac{K_1}{s(s-1)}$

No of OL poles $P = 1$



Segment M_1: Path ab

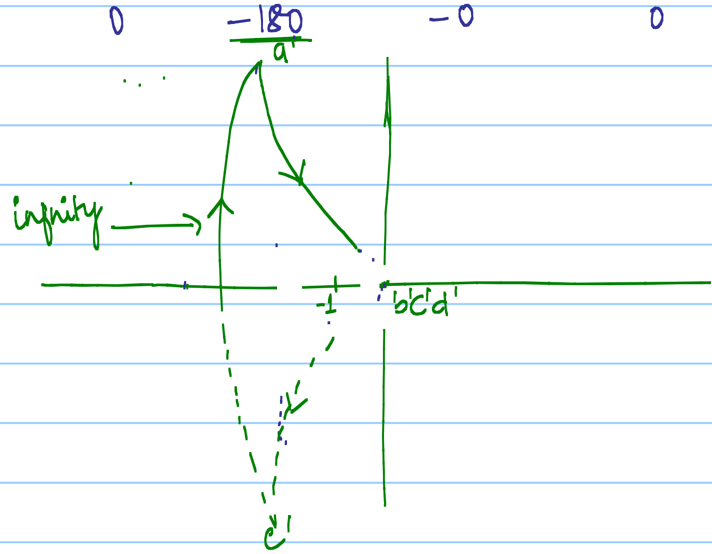
$$s = j\omega$$

$$G(j\omega) = \frac{K_1}{(j\omega)(j\omega-1)}$$

$$= \frac{K_1}{\omega\sqrt{\omega^2+1}} \angle -90 - (180 - \tan^{-1}(\omega))$$

$$= \frac{-jK_1}{\omega(j\omega-1)} \times \frac{(-1-j\omega)}{(-1-j\omega)} = -\frac{K_1}{\omega^2+1} + j\frac{K_1}{\omega(\omega^2+1)}$$

ω	M	∠G(jω)	Re	Im
0	∞	−270	$-K_1$	∞
∞	0	−180	−0	0



Segment M_2: Path bcd

$$s = Re^{j\theta}$$

$$\lim_{R\to\infty} G(Re^{j\theta}) = \lim_{R\to\infty} \frac{k_1}{Re^{j\theta}(Re^{j\theta}-1)}$$

$$= 0 \underline{|-2\theta}$$

Segment M_3 : path de

Mirror image of path ab

Segment M_4 : Path efa

$s = re^{j\phi}$ $\quad r\to 0 \quad \phi \to \underline{-90 \to 0 \to 90}$

$$G(re^{j\phi}) = \lim_{r\to 0} \frac{k_1}{re^{j\phi}(\underbrace{re^{j\phi}-1}_{-1})}$$

$$\Rightarrow \lim_{r\to 0} \frac{k_1}{re^{j\phi}\, e^{-j\pi}} \qquad -1 = e^{-i\pi}$$

$$= \infty \underline{|+180-\phi}$$

$$= \infty \underline{|270 \to 180 \to 90}$$

360

$N = +1 \quad , \; P = 1$

$Z = N + P$

$= 2$

$\Rightarrow$ The closed loop system is unstable

$$G(s) = \frac{k_1(1+k_2 s)}{s(s-1)} \qquad 1+k_2 s = 0, \quad s = -1/k_2$$

$p = 1$

Segment M_1 : path ab

$s = j\omega$

$$G(j\omega) = \frac{k_1(1+k_2 j\omega)}{j\omega(j\omega-1)}$$

$90 \quad -90 - 180 + 90$

$$= \frac{k_1\sqrt{1+k_2^2\omega^2}}{\omega\sqrt{\omega^2+1}} \underline{|\underbrace{\tan^{-1}(k_2\omega) - 90 - (180 - \tan^{-1}(\omega))}}$$

$$= -j\frac{k_1(1+k_2\omega)(-j\omega-1)}{\omega(+j\omega-1)(-j\omega-1)} = -j\,k_1\frac{(-j\omega - 1 - jk_2\omega^2 - k_2\omega)}{\omega(1+\omega^2)}$$

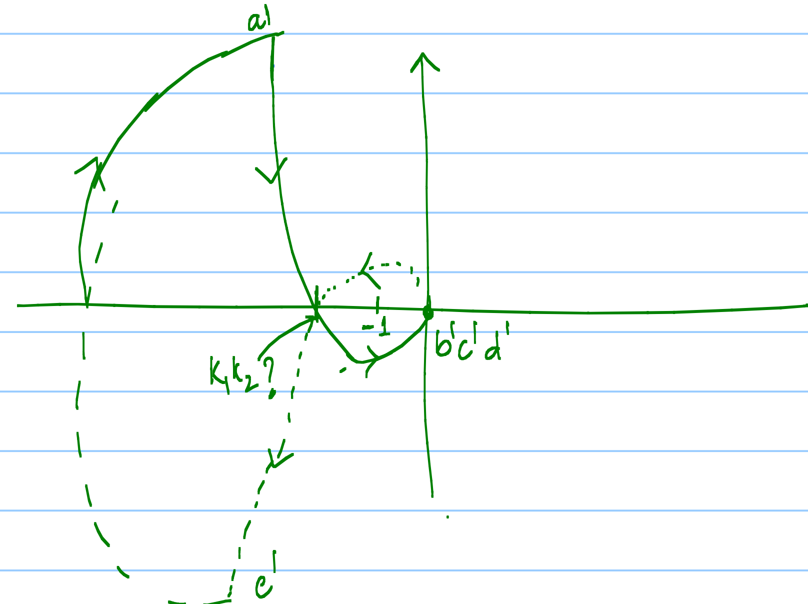
$$G(j\omega) = \frac{+j^2 k_1\omega + jk_1 + j^2 k_1 k_2 \omega^2 + jk_1k_2\omega}{\omega(1+\omega^2)}$$

$$= \frac{-k_1\omega + jk_1 - k_1k_2\omega^2 + jk_1k_2\omega}{\omega(1+\omega^2)}$$

$$= \frac{-k_1\cancel{\omega}(1+k_2\omega)}{\cancel{\omega}(1+\omega^2)} + j\,\frac{k_1(1+k_2\omega)}{\omega(1+\omega^2)}$$

$$= \frac{-k(1+k_2\omega)}{(1+\omega^2)} + j\,\frac{k_1(1+k_2\omega)}{\omega(1+\omega^2)}$$

ω	M	$\angle G(j\omega)$	Re	Im
0	∞	-270	$-k_1$	∞
∞	0	-90		



Segment M_4: Path efa

$s = re^{j\phi}$ $r \to 0$ $\phi \to -90 \to 0 \to 90$

$$\lim_{r\to 0} G(re^{j\phi}) = \lim_{r \to 0} \frac{k_1\overbrace{(1+k_2\, re^{j\phi})}^{1}}{re^{j\phi}\underbrace{(re^{j\phi}-1)}_{-1}}$$

$$= \infty \angle 180 - \phi$$

$$= \infty \angle 270 \to 180 \to 90$$

Condition: $-k_1k_2 < -1$

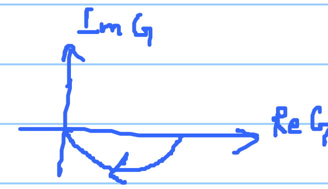
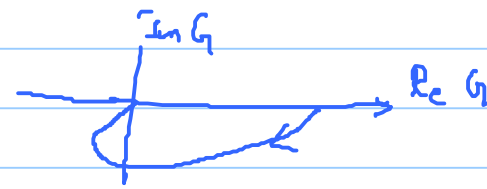
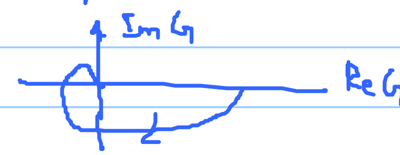
$$\boxed{k_1k_2 > 1}$$

We get $N = -1$

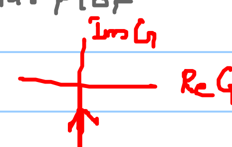
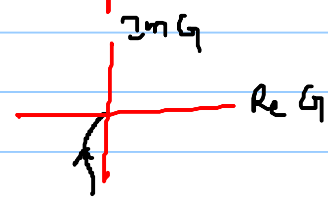
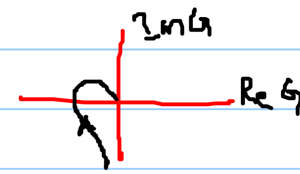
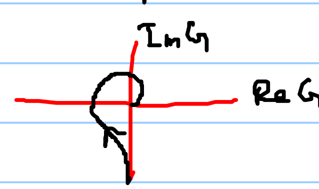
$$Z = N + P$$
$$= -1 + 1$$
$$\boxed{Z = 0}$$

With PD $[\underbrace{1}_{P} + \underbrace{k_2 s}_{D}]$ the closed loop system is stable

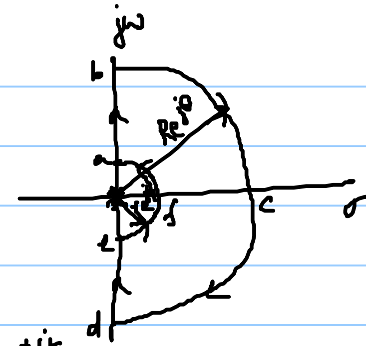
Polar Plot for Type 0 system

	$\omega = 0$	$\omega = \infty$	
$G(s) = \dfrac{k}{\tau s + 1}$	$M = k$, $\phi = 0°$	$M = 0$, $\phi = -90°$	
$G(s) = \dfrac{k}{(\tau_1 s + 1)(\tau_2 s + 1)}$	$M = k$, $\phi = 0°$	$M = 0$, $\phi = -180°$	
$G(s) = \dfrac{k}{(\tau_1 s + 1)(\tau_2 s + 1)(\tau_3 s + 1)}$	$M = k$, $\phi = 0°$	$M = 0$, $\phi = -270°$	

Type 1 system

	$G(s)$	$\omega = 0$	$\omega = \infty$	Order	Polar Plot
1	k/s	$M = \infty$, $\phi = -90°$	$M = 0$, $\phi = -90°$	1	
2	$\dfrac{k}{s(\tau s + 1)}$	$M = \infty$, $\phi = -90°$	$M = 0$, $\phi = -180°$	2	
3	$\dfrac{k}{s(\tau_1 s + 1)(\tau_2 s + 1)}$	$M = \infty$, $\phi = -90°$	$M = 0$, $\phi = -270°$	3	
4	$\dfrac{K}{s(\tau_1 s + 1)(\tau_2 s + 1)(\tau_3 s + 1)}$	$M = \infty$, $\phi = -90°$	$M = 0$, $\phi = -360°$		

Ex, $GH(s) = \dfrac{k(s\tau_a+1)}{s^3}$



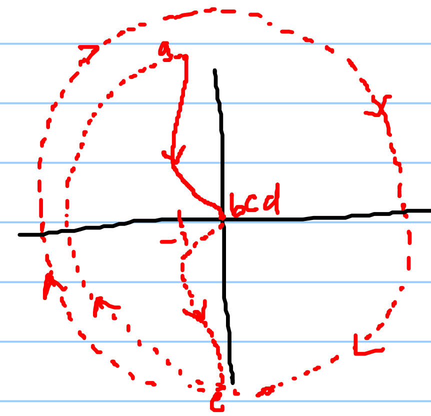
① path ab, $s = j\omega$

$$GH(j\omega) = \frac{k(j\omega\tau_a+1)}{(j\omega)^3} = \frac{k(j\omega\tau_a+1)}{-j\omega^3} \times \frac{j\omega^3}{j\omega^3}$$

$$= \frac{-k\omega^4\tau_a + jk\omega^3}{\omega^6} = \frac{-k\omega\tau_a + jk}{\omega^3}$$

$$|GH(j\omega)| = \frac{k\sqrt{\omega^2\tau_a^2+1}}{\omega^3}, \quad \angle GH(j\omega) = \tan^{-1}\omega\tau_a - 270^\circ$$

ω	$\lvert GH(j\omega)\rvert$	$\angle G(j\omega)$	Re	Im
0	∞	-270°	∞	∞
∞	0	-180°	0	0



path bcd, $s = Re^{j\theta}$, $\theta = 90^\circ \to 0^\circ \to -90^\circ$

$$G(s) = \frac{k(Re^{j\theta}\tau_a+1)}{R^3e^{j3\theta}} = \frac{k}{R^2}\cdot e^{-j2\theta} =$$

$$= 0 \ \angle -2(90^\circ) \to 0^\circ \to -2(-90^\circ)$$

$$= 0 \ \angle -180^\circ \to 0^\circ \to 180^\circ$$

$N = 2$, $P = 1$

$Z = 2+1 = 3$

path de

path efa, $s = re^{j\phi}$, $\phi \Rightarrow -90^\circ \to 0^\circ \to 90^\circ$

$$G(s) = \frac{k(re^{j\phi}\tau_a+1)}{r^3e^{j3\phi}} = \frac{k}{r^3}\cdot e^{-j3\phi}$$

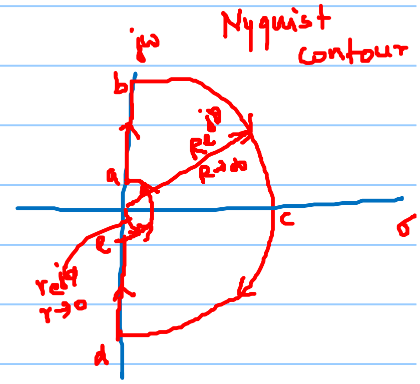
$$= \infty, \ \angle -3(-90^\circ) \to 0^\circ \to -3(90^\circ)$$

$$\angle 270^\circ \to 180^\circ \to 90^\circ \to 0^\circ \to -90^\circ \to -180^\circ \to -270^\circ$$

Example, $G(s) = \dfrac{K(1+2s)}{s(1+s)(1+s+s^2)}$ Find the range of K for which the system is stable

$s = 0, -1, -1 \pm \frac{j\sqrt{3}}{2}$

Sol: path ab, $s = j\omega$



$$G(j\omega) = \frac{K(1+2j\omega)}{j\omega(1+j\omega)(1+j\omega+(j\omega)^2)}$$

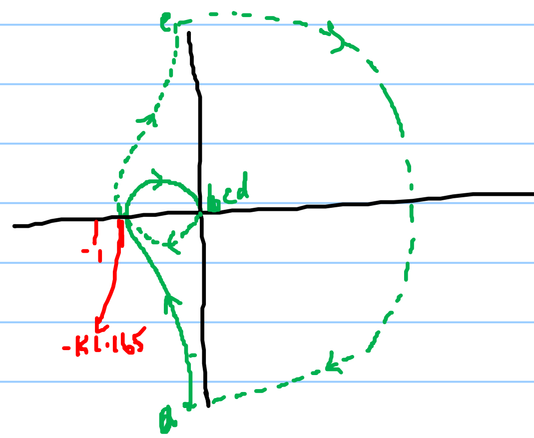
$$= \frac{K(1+2j\omega)}{j\omega(1+j\omega)(1-\omega^2+j\omega)}$$

$$= \frac{K(1+2j\omega)(-j\omega-\omega^2)(1-\omega^2-j\omega)}{\omega^2(1+\omega^2)((1-\omega^2)^2+\omega^2)}$$

$$|G(j\omega)| = \frac{K\sqrt{1+4\omega^2}}{\omega\sqrt{1+\omega^2}\sqrt{1+\omega^4-\omega^2}}$$

$$\angle G(j\omega) = \tan^{-1}(2\omega) - 90^\circ - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{1-\omega^2}$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$	Re	Im
0	∞	-90°	∞	∞
∞	0	-270°	0	0



N=0 , P = 0 ⇒ Z = 0

⇒ system stable for $K < 0.86$

Where it cuts the real axis

$\mathrm{Im}(G(j\omega)) = 0$

$1 + 2\omega^2 - 2\omega^4 = 0$

$2\omega^4 - 2\omega^2 - 1 = 0 \Rightarrow \omega^2 = \dfrac{2 \pm \sqrt{4+8}}{4}$

$\omega = 1.168$

$$\mathrm{Re}(G(j\omega))\Big|_{\omega=1.168} \Rightarrow \frac{-K3\omega^3}{\omega(1+\omega^2)((1-\omega^2)^2+\omega^2)}\Bigg|_{\omega=1.168} = -K1.165$$

For the system to be stable, N = 0 ∵ P = 0, ∴ $-K1.165 > -1$

$K < \dfrac{1}{1.165}$

$0 < K < 0.86$