CONTROL SYSTEMS

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UNIT 5: STABILITY IN THE FREQUENCY DOMAIN

Introduction

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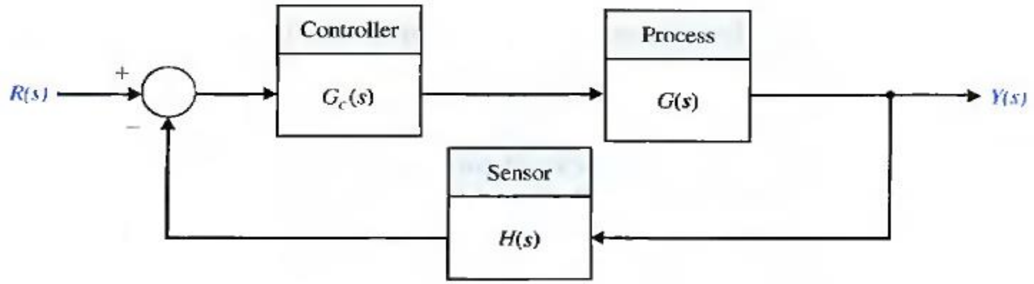
STABILITY IN THE FREQUENCY DOMAIN

Introduction

- The major advantage of Polar Plot lies in stability study of systems
- Harry Nyquist related the stability of a dynamical system to these plots (1932 at Bell Labs)
- His work on Polar plots (applied to stability of systems) are called as Nyquist Plots

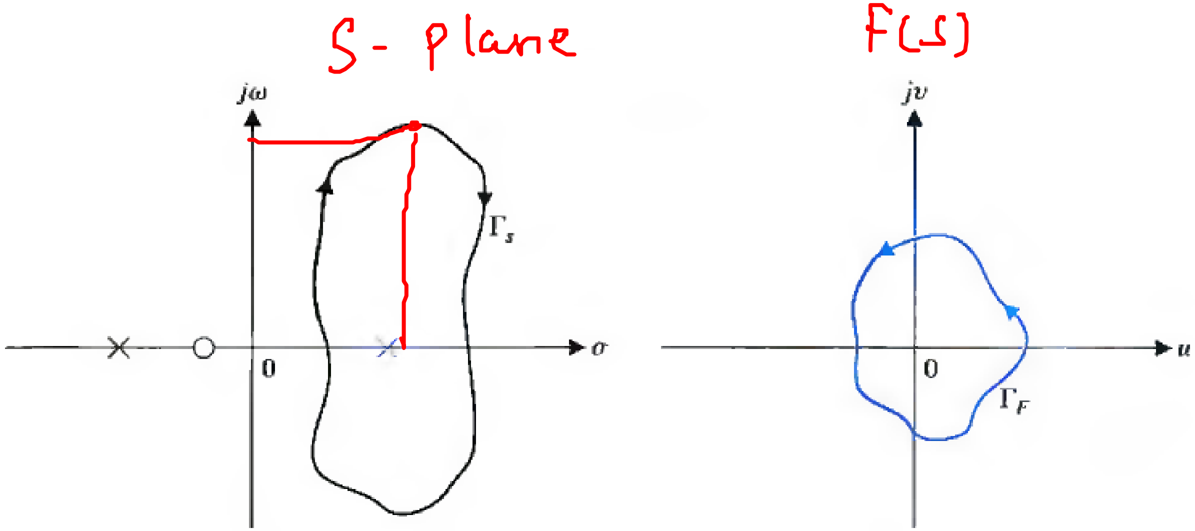


STABILITY IN THE FREQUENCY DOMAIN

Introduction

- Given open loop frequency response , Nyquist Plots determines closed loop system stability

- For a given continuous closed path in the s-plane that does not go through the singular points, there corresponds a closed curve in the F(s) plane.



$F(s) = 1 + G(s)H(s)$

$u + jv$

Introduction

- Number and direction of encirclements plays an important role in the stability of the system.
- For each point in the s-plane , there corresponds a point in the F(s) plane. i.e., for $s = \sigma + j\omega$, there would be F(s) = u + jv in F(s) plane. This is called mapping.

UNIT 5: STABILITY IN THE FREQUENCY DOMAIN

Mapping of Contours in s - plane

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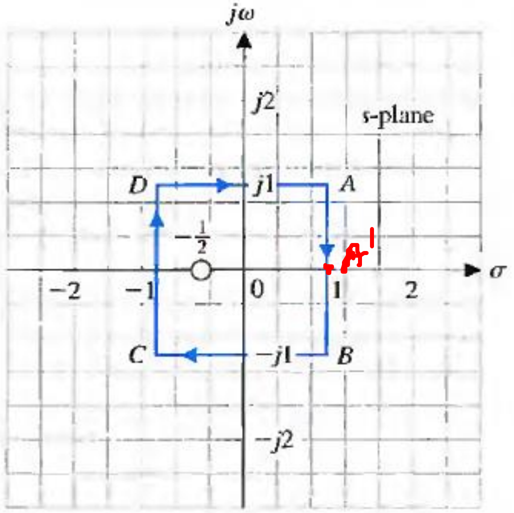
Mapping of Contours in s - plane

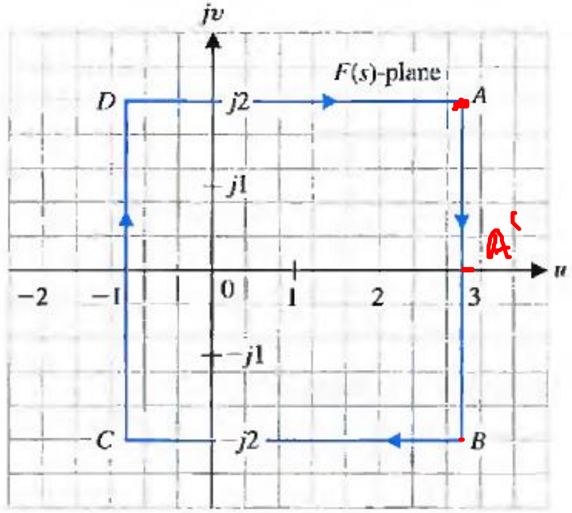
- A contour map is a contour or trajectory in one plane mapped or translated into another plane by a relation F(s).
- For example, $F(s) = 2s + 1$

$s = \sigma + j\omega$, $F(s) = 2(\sigma + j\omega) + 1$

$= \underbrace{2\sigma + 1}_{u} + j\underbrace{2\omega}_{v}$

- This type of mapping, which retains the angles of the s-plane contour on the F(s)-plane is called conformal mapping.





$\sigma = 1, \omega = -1$

$A = 1 + j$, $F(s) = 3 + 2j$; $F(s)_A = 3 + j2$ $(3, 2)$

$A' = 1$, $F(s) = 3$

$B = 1 - j$, $F(s) = 3 - 2j$

$C = -1 - j$, $F(s) = -1 - 2j$

$D = -1 + j$, $F(s) = -1 + 2j$

STABILITY IN THE FREQUENCY DOMAIN

Mapping of Contours in s - plane

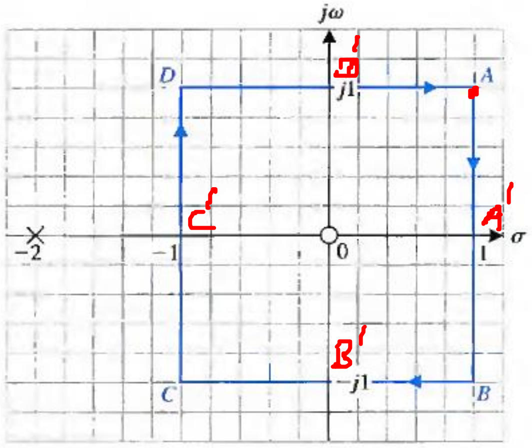
For example, $F(s) = \frac{s}{s+2}$

$$s = \sigma + j\omega, \quad F(s) = \frac{\sigma + j\omega}{\sigma + j\omega + 2}$$

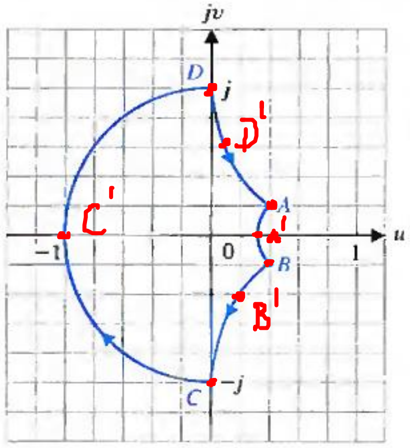
$$F(s) = \frac{(\sigma + j\omega)(\sigma + 2 - j\omega)}{(\sigma+2)^2 + \omega^2}$$

$$= \underbrace{\frac{\sigma^2 + \omega^2 + 2\sigma}{(\sigma+2)^2 + \omega^2}}_{u} + j \underbrace{\frac{2\omega}{(\sigma+2)^2 + \omega^2}}_{v}$$

s - plane



F(s) - plane



	A	A'	B	B'	C	C'	D	D'
s	$1+j$ (σ ω)	1	$1-j$	$-j$	$-1-j$	-1	$-1+j$	j
F(s)	$0.4+0.2j$	$1/3$	$0.4-0.2j$	$\frac{1-2j}{5}$	$-j$	-1	j	$\frac{1+2j}{5}$

Mapping of Contours in s - plane

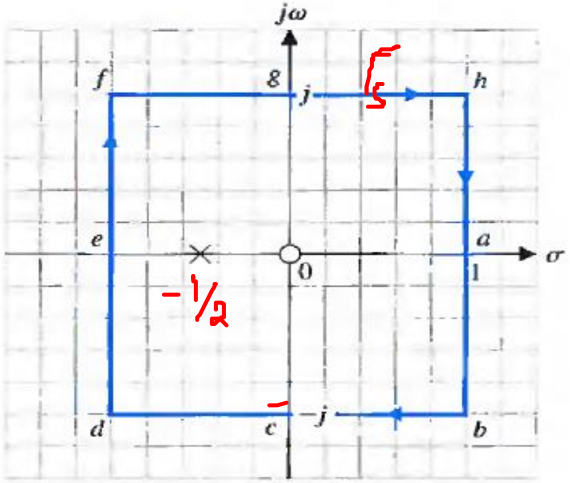
Cauchy's Theorem(Principle of Argument)

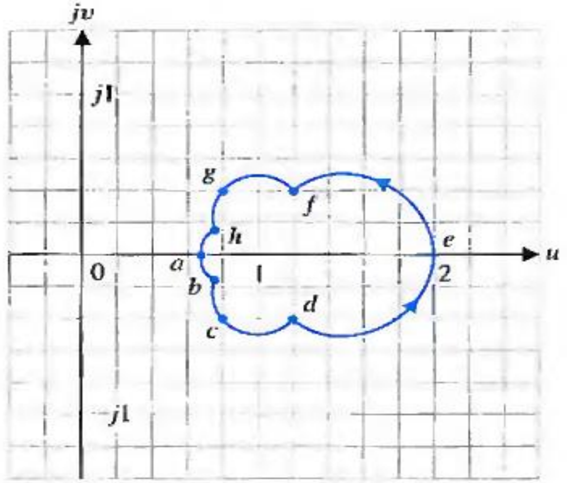
If a contour Γ_s in the s- plane encircles z zeros and p poles of F(s) and does not pass through any pole or zero of F(s) and the traversal is in the clockwise direction along the contour, the corresponding contour Γ_F in the F(s) plane encircles the origin of the F(s) – plane N = Z – P times in the clockwise direction

$$F(s) = \frac{s}{s + 1/2}$$

Z = 1
P = 1

N = 1 − 1 = 0





STABILITY IN THE FREQUENCY DOMAIN

Mapping of Contours in s - plane

s- plane   F(s)- plane

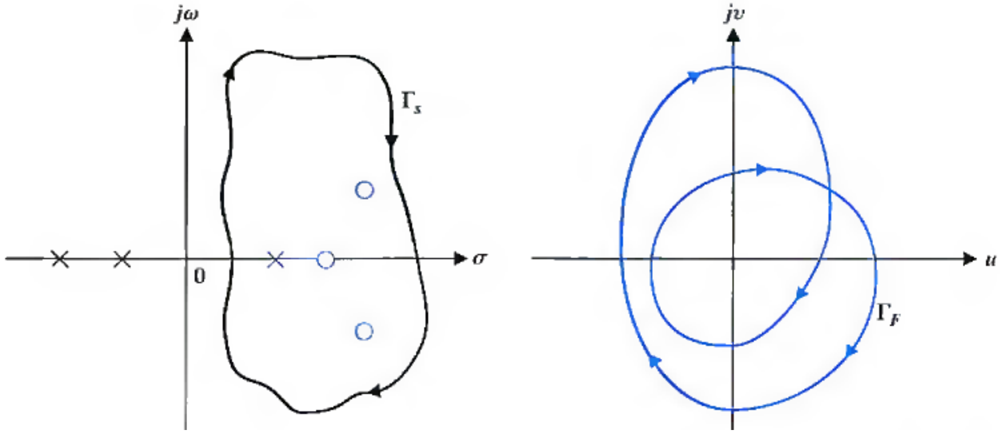
$Z = 3$

$P = 1$

$N = Z - P$

$= 3 - 1$

$= 2$



$Z = 3$

$P = 1$

$N = Z - P = 3 - 1 = 2$

Mapping of Contours in s - plane

Different closed contours in s – plane gives rise to different closed curves in F(s) – plane.

$Z = 0$

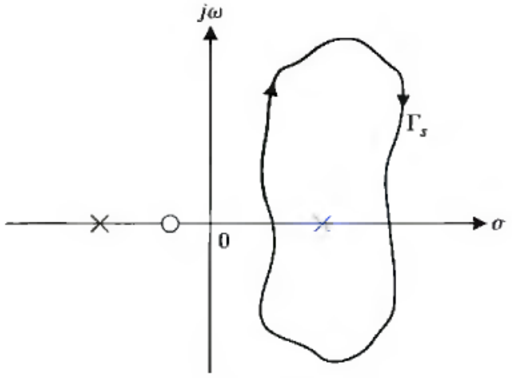
$P = 1$

$N = Z - P$

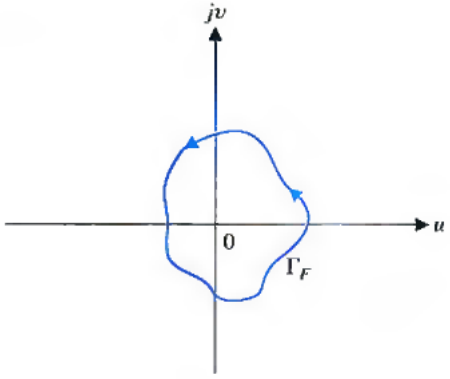
$= 0 - 1$

$= -1$

s- plane



F(s) - plane

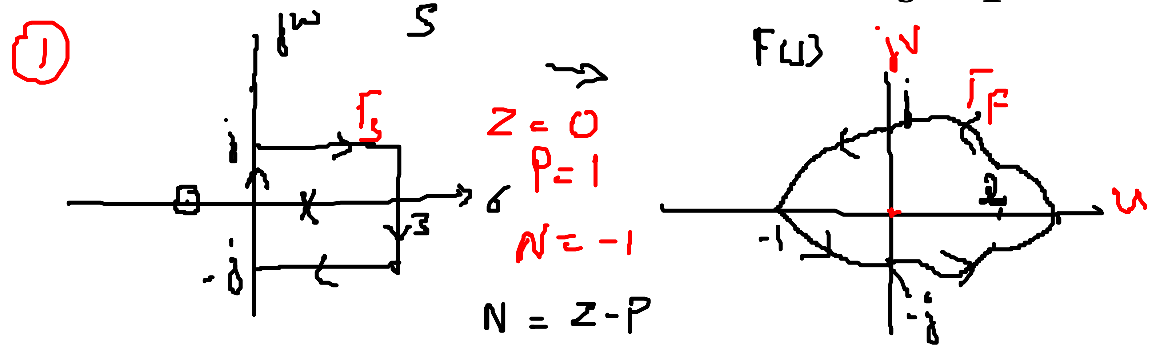


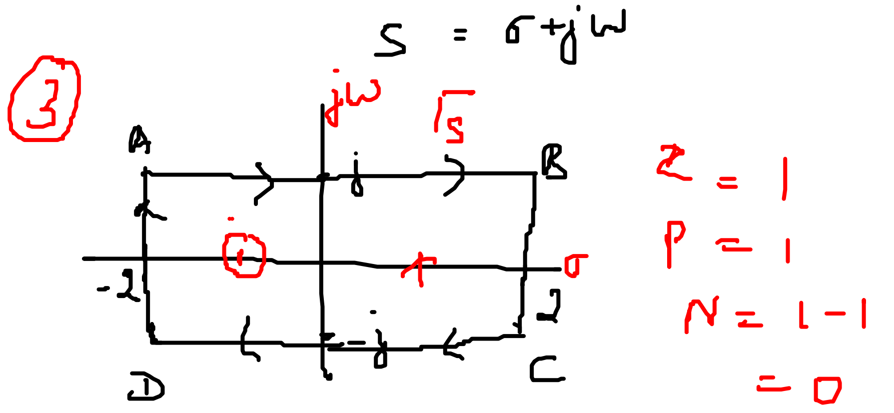
STABILITY IN THE FREQUENCY DOMAIN

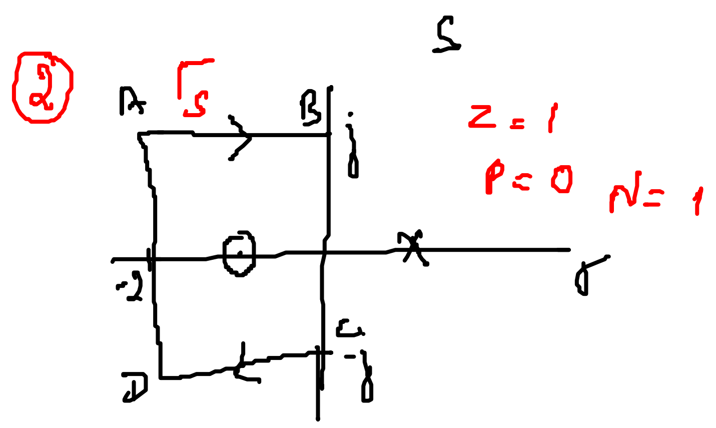
Mapping of Contours in s - plane

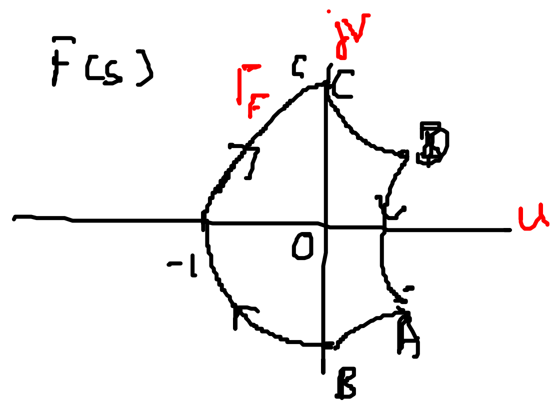
Different closed contours in s – plane gives rise to different closed curves in F(s) – plane.

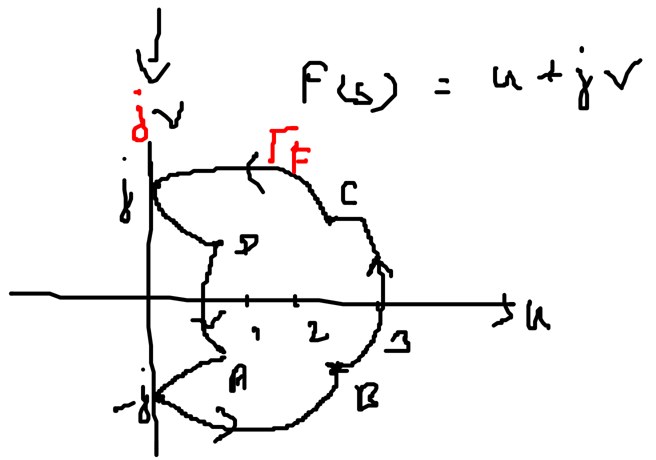
$$F(s) = \frac{s+1}{s-1}$$











UNIT 4: STABILITY IN THE FREQUENCY DOMAIN

Nyquist Stability Criterion

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STABILITY IN THE FREQUENCY DOMAIN

Nyquist Stability Criterion

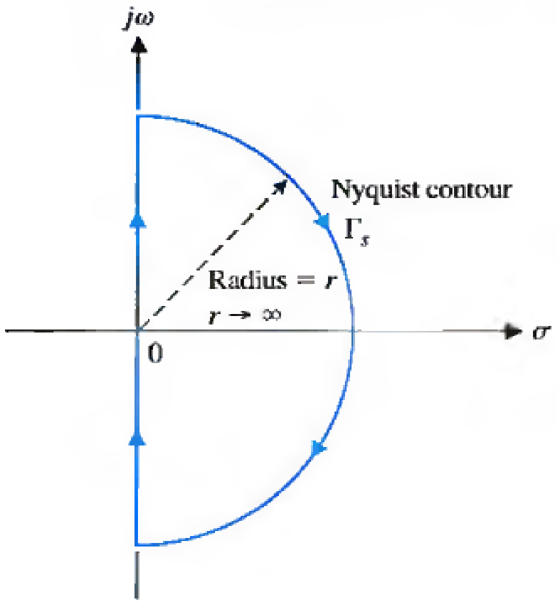
$G(s) = \frac{N(s)}{D(s)}$

$1 + G(s) = 0$

$1 + \frac{N(s)}{D(s)} = 0$

$D(s) + N(s) = 0$

The Nyquist path encloses the entire RHS plane and encloses all the zeros and poles of 1 + G(s)H(s) that have positive real parts.



In general, $G(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$ — (1) → OLTF

C.E, $F(s) = 1 + G(s)H(s) = 0$

$= 1 + \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = 0$

CLTF → $\frac{(s+z_1')(s+z_2')\cdots(s+z_m')}{(s+p_1)(s+p_2)\cdots(s+p_n)} = 0$ — (2)

$(s+z_1')(s+z_2')\cdots(s+z_m') = 0$

$N = Z - P$ → from $G(s)$ on RHS (unstable poles)

↓ after mapping

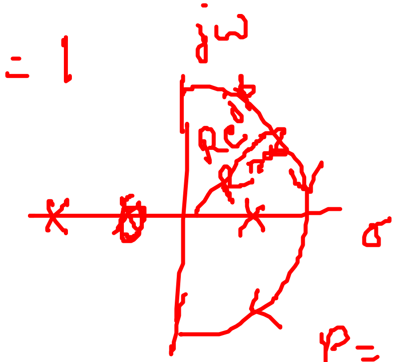
$Z = N + P$

$= 0 \Rightarrow$ stable

Nyquist Stability Criterion

$$G(s)H(s) = \frac{s+1}{(s+2)(s-1)}$$

$P = 1$



$P = 1$

Comparing equations 1 & 2,

- we observe that the poles of F(s) and poles of G(s)H(s) are same.
- The zeros of F(s) are the roots of characteristic equation.
- For the system to be stable, the roots of the characteristic equation must lie in the LHS plane. i.e., zeros of F(s) must lie in the LHS plane.
- The number of RH plane zeros of F(s) is equal to the number of poles 1+G(s)H(s) in the RH plus the number of encirclements of the origin of the F(s) plane.

$N = Z - P$

$Z = N + P$ → No. of poles of G(s)H(s) which lie in RHS of s-plane or unstable poles

STABILITY IN THE FREQUENCY DOMAIN

Nyquist Stability Criterion

$F(s) = 1 + G(s)H(s) \rightarrow$ origin is 0

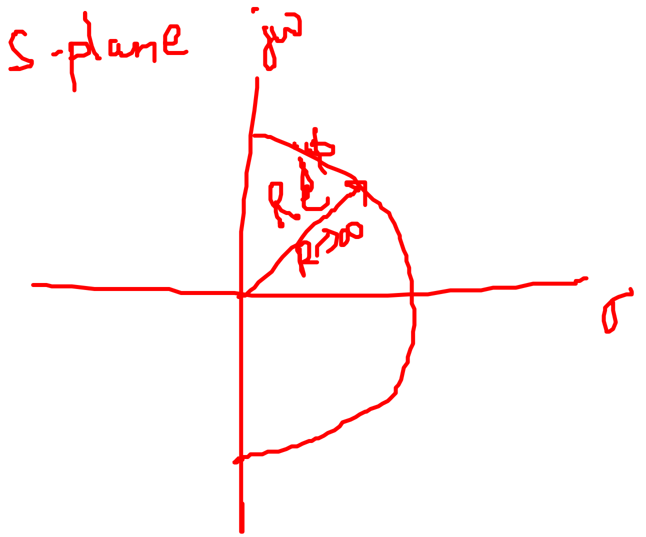
$G(s)H(s) = 1 - F(s)$

$1 + GH = 0$

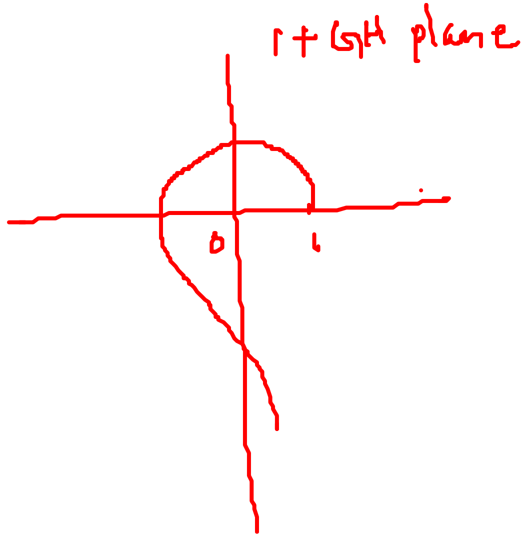
$GH = -1 + j0$

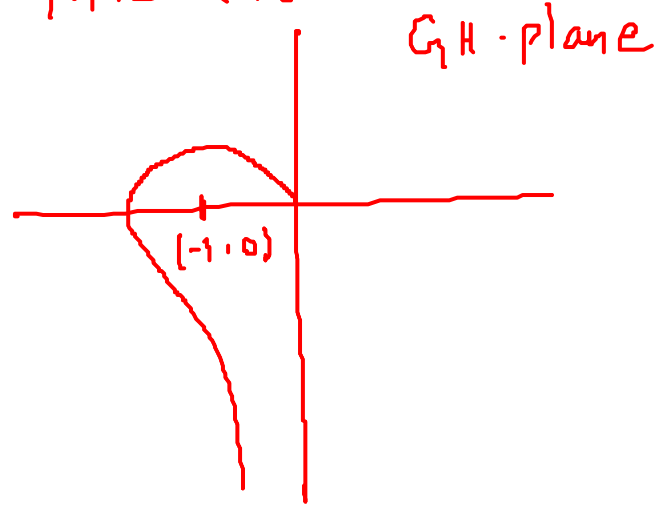
F(s)=1+L(s) => L(s) = F(s) – 1 = $F'(s)$

F(s) plane·



$\Rightarrow$





always mapping is between s-plane to GH plane instead of F = 1 + GH plane

Nyquist Stability Criterion

- A feedback system is stable if and only if the contour Γ_L in the L(s) plane does not encircle the (-1, 0) point when the number of poles of L(s) in the right hand s-plane is zero(P = 0) $Z = 0, N = 0, P = 0$
- A feedback control system is stable if and only if, for the contour Γ_L, the number of encirclement of the (-1,0) point is equal to the number of poles of L(s) with positive real parts. $Z = 0, N = P$
- The basis for the 2 statements lies in the following expression
- $Z = N + P$

STABILITY IN THE FREQUENCY DOMAIN

Nyquist Stability Criterion

G(s)H(s)

Example, $L(s) = \dfrac{K}{(s+1)(0.1s+1)}$

Sol: path ab, $s = j\omega$, $\omega \to 0$ to ∞

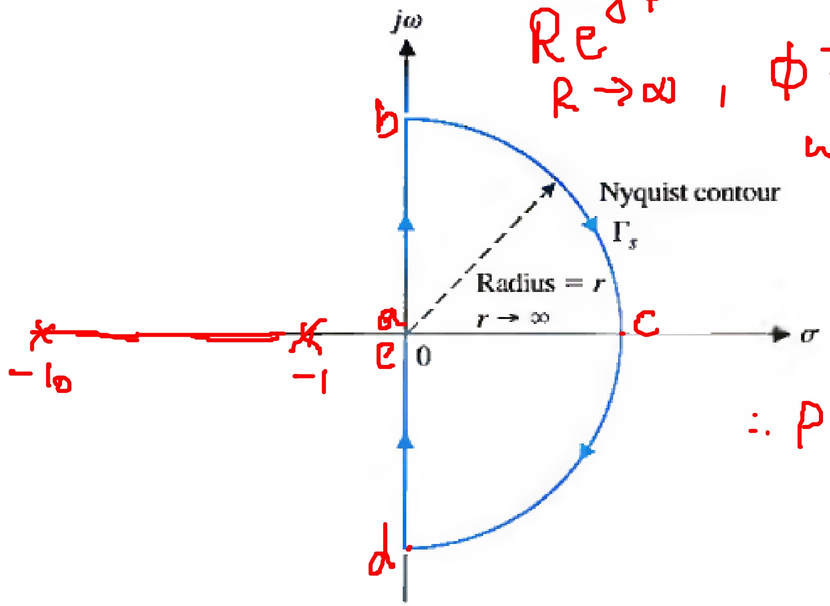
$L(j\omega) = \dfrac{K}{(j\omega+1)(0.1j\omega+1)}$

let $K = 100$

$|L(j\omega)| = \dfrac{100}{\sqrt{\omega^2+1}\,\sqrt{(0.1\omega)^2+1}}$

$\angle L(j\omega) = 0 - \tan^{-1}\omega - \tan^{-1}0.1\omega$

$L(j\omega) = \dfrac{100(1-j\omega)(1-0.1j\omega)}{(\omega^2+1)((0.1\omega)^2+1)} = \dfrac{100(1-0.1\omega^2) - j(100 \times 1.1\omega)}{(\omega^2+1)(0.01\omega^2+1)}$



$Re^{j\phi}$

$R \to \infty$, $\phi \to 90°$ to $-90°$

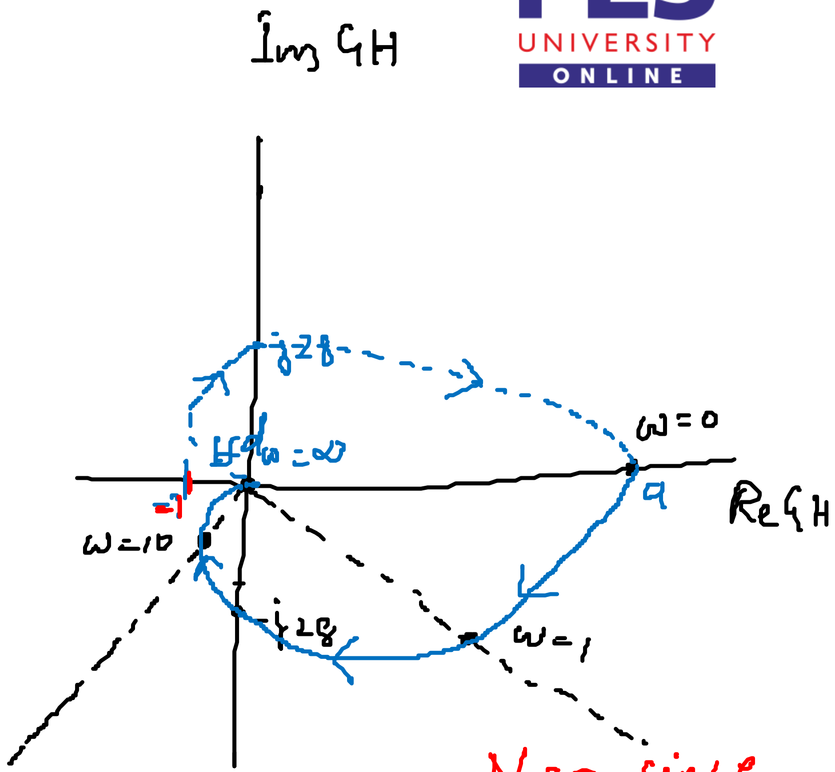
$\omega = 0$ to ∞

$\therefore P = 0$

STABILITY IN THE FREQUENCY DOMAIN

Nyquist Stability Criterion

ω	$\lvert L(j\omega)\rvert$	$\angle L(j\omega)$	$Re\, L(j\omega)$	$Im\, L(j\omega)$
0	100	0	100	0
1	70.36	−50.7		
10	7.036	−129		
∞	0	−180°	0	0



To know where it cuts $j\omega$-axis

$$Re\, L(j\omega) = 0$$

$$\frac{100(1-0.1\omega^2)}{(1+\omega^2)(1+0.01\omega^2)} = 0$$

$$\omega^2 = 10 \Rightarrow \omega = 3.16 \text{ rad/sec}$$

$$Im\, L(j\omega)\Big|_{\omega=3.16} = \frac{-100\times 0.1\times 3.16}{(1+3.16^2)(1+0.01\times 3.16^2)} = -j28.76$$

N = 0 since it is not encircling (−1, 0) point

$\therefore$ system is stable $\Leftarrow$ $Z = N + P = 0$

path bcd , $s = \lim_{R\to\infty} Re^{j\phi}$, $\phi = +90^\circ$ to 0° to -90°

$$L(s) = \lim_{R\to\infty} \frac{100}{(1+Re^{j\phi})(0.1Re^{j\phi}+1)} = \lim_{R\to\infty} \frac{100}{Re^{j\phi} \times 0.1Re^{j\phi}} = \lim_{R\to\infty} \frac{100}{0.1R^2 e^{j2\phi}}$$

$= 0, e^{-j2\phi}$

$-2\phi \Rightarrow -180^\circ$ to 0° to 180°

path de , $s = -j\omega$, $\omega = -\infty$ to 0

mirror image of path ab

C_2, $s = Re^{j\theta}$, $R \to \infty$, $\theta \to \pi/2$ to $-\pi/2$

$$L(s) = \lim_{R \to \infty} \frac{100}{(\underbrace{1 + Re^{j\theta}}_{Re^{j\theta}})(\underbrace{1 + 0.1\,Re^{j\theta}}_{0.1\,Re^{j\theta}})}$$

$$= \lim_{R \to \infty} \frac{100}{0.1 R^2 e^{j2\theta}}$$

$$= 0\, e^{-j2\theta}$$

$= 0$, $-2(\pi/2)$ to $-2(-\pi/2)$

$-\pi$ to π

C_3, $s = -j\omega$

mirror image of polar plot obtained for segment C_1

STABILITY IN THE FREQUENCY DOMAIN

Nyquist Stability Criterion – Example 2

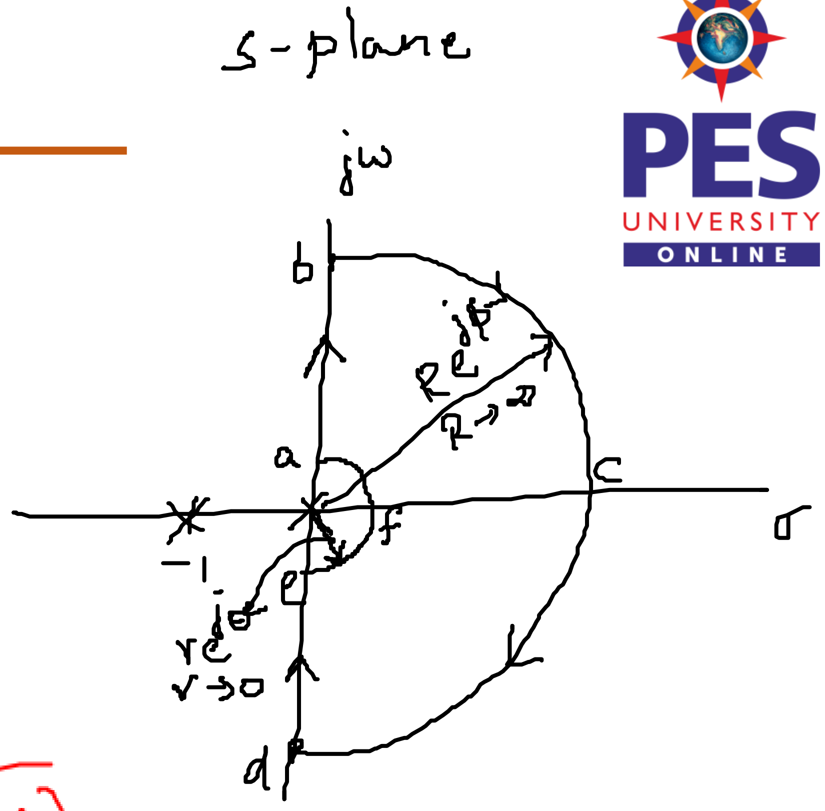
$$L(s) = \frac{5}{s(s+1)}$$

path ab, $s = j\omega$

$$L(j\omega) = \frac{5}{j\omega(j\omega+1)} \times \frac{-j\omega(-j\omega+1)}{-j\omega(-j\omega+1)}$$

$$= \frac{5(-\omega^2 - j\omega)}{\omega^2(\omega^2+1)} = \frac{-5\omega^2}{\omega^2(\omega^2+1)} - j\frac{5\omega}{\omega^2(\omega^2+1)}$$

$$|L(j\omega)| = \frac{5}{\omega\sqrt{\omega^2+1}}, \quad \angle L(j\omega) = -90^\circ - \tan^{-1}\omega$$



STABILITY IN THE FREQUENCY DOMAIN

Nyquist Stability Criterion

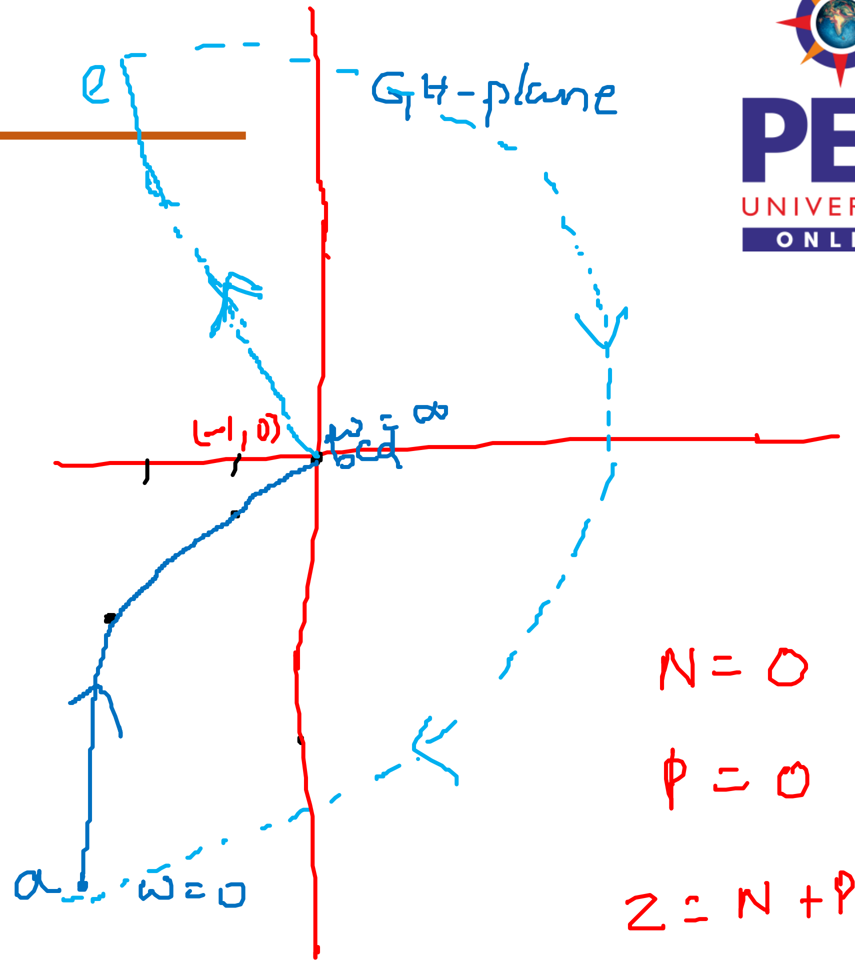
ω	0	1	2	∞
$\lvert L(j\omega)\rvert$	∞	3.53	1.11	0
$\angle L(j\omega)$	-90°	-135°	-153°	-180°
Re	∞	-2.5	-1	
Im		-2.5	-0.5	

path bcd, $s = \underset{R\to\infty}{R e^{j\theta}}$, $\theta \to 90$ to -90°

$$L(s) = \lim_{R\to\infty} \frac{5}{Re^{j\theta}(Re^{j\theta}+1)} = \lim_{R\to\infty} \frac{5}{R^2 e^{j2\theta}}$$

$$= 0\, e^{-j2\theta}$$

$$= 0,\ -2\times 90 \text{ to } -2(-90) \Rightarrow -180^\circ \text{ to } 180^\circ$$



$N = 0$

$P = 0$

$Z = N + P = 0$

$\Rightarrow$ system is stable

STABILITY IN THE FREQUENCY DOMAIN

Nyquist Stability Criterion

$L(s) = \dfrac{5}{s(s+1)}$

Path efa, $s = re^{j\phi}$, $\phi \to -90^\circ \to 0 \to 90^\circ$
$r \to 0$

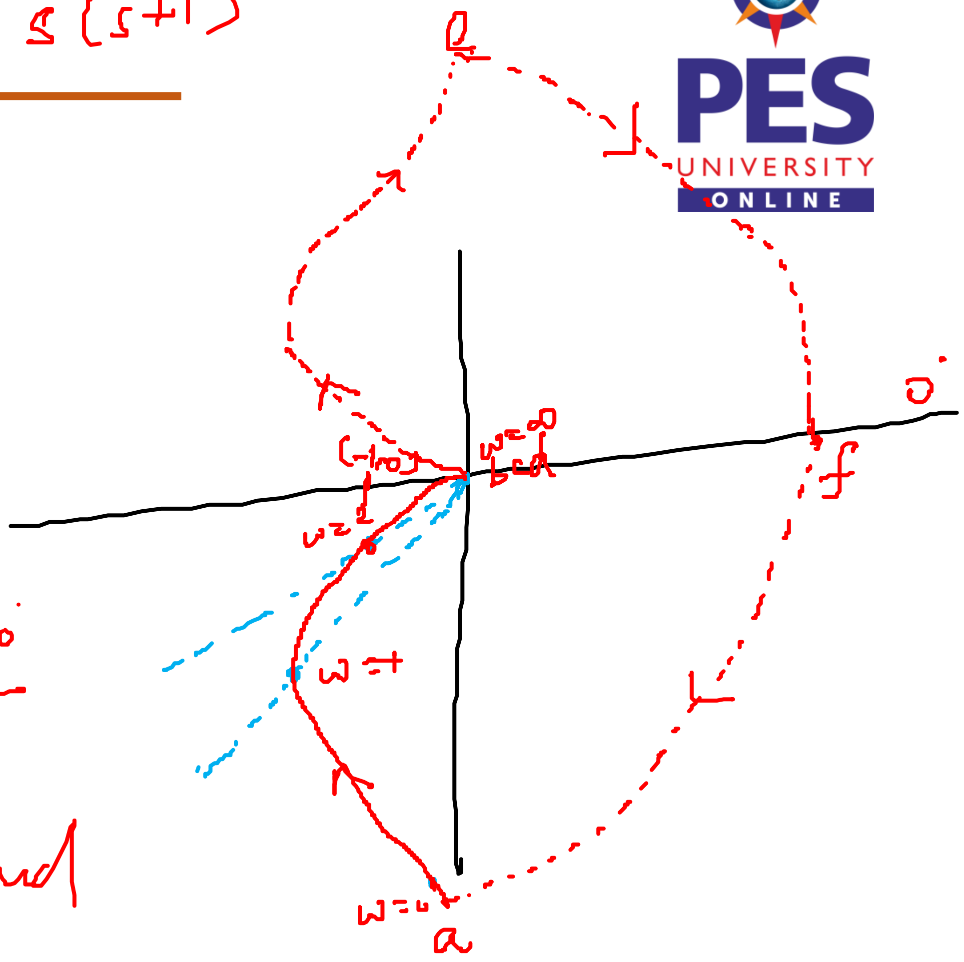
$L(s) = \lim_{r \to 0} \dfrac{5}{re^{j\phi}(\cancel{re^{j\phi}}^{\,0} + 1)}$

$= \lim_{r \to 0} \dfrac{5}{re^{j\phi}} = \infty e^{-j\phi}$

∞, 90° to 0 to -90°

path de, $s = -j\omega$

mirror image of polar plot obtained for path ab



$N = 0$, $P = 0$

∴ system is stable $\Leftarrow$ $Z = N + P = 0$

STABILITY IN THE FREQUENCY DOMAIN

Nyquist Stability Criterion – Example 3

Ex 3, $G(s) = \dfrac{K}{s^2(\tau s+1)}$

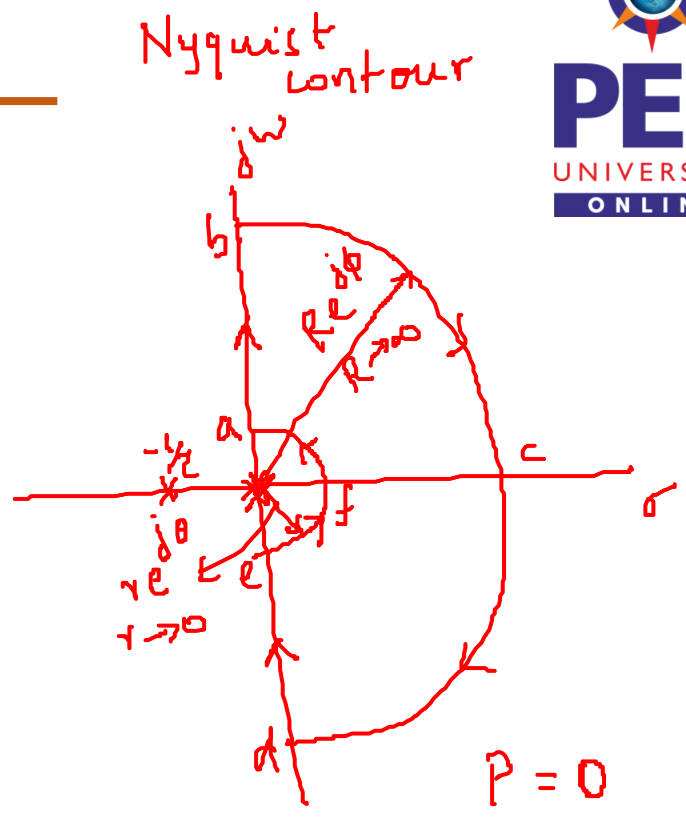
sol: path ab, $s = j\omega$, $K=1$, $\tau=1$

$$|G(j\omega)| = \frac{1}{\omega^2\sqrt{\omega^2+1}}$$

$$\angle G(j\omega) = -180^\circ - \tan^{-1}\omega$$

$$G(j\omega) = \frac{1}{-\omega^2(j\omega+1)} \times \frac{1-j\omega}{(1-j\omega)}$$

$$= \frac{-1+j\omega}{\omega^2(\omega^2+1)}$$

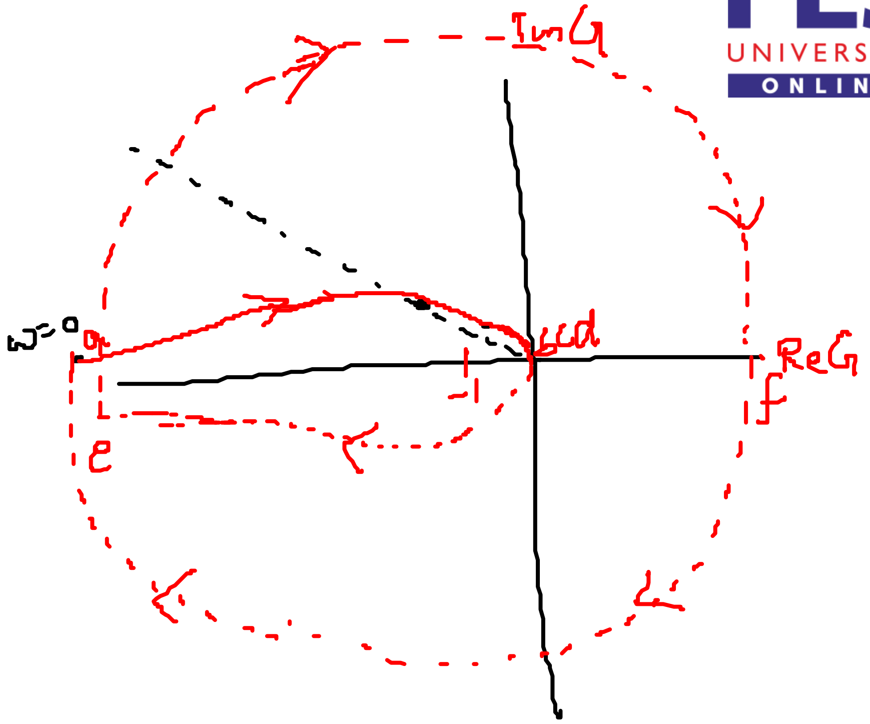


ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$	$Re\, G(j\omega)$	$Im\, G(j\omega)$
0	∞	-180°		
1	0.707	-225°		
2	0.11	-243°		
∞	0	-270°		

Path bcd, $s = \lim_{R\to\infty} Re^{j\phi}$ $\quad \phi \Rightarrow 90^\circ - 0^\circ - -90^\circ$

$$G(s) = \lim_{R\to\infty} \frac{1}{(Re^{j\phi})^2 (Re^{j\phi}+1)}$$

$$= \lim_{R\to\infty} \frac{1}{R^3 e^{j3\phi}} = 0\, e^{-j3\phi}, \; -270^\circ - 0^\circ - 270^\circ$$



$N = 2$

$P = 0$

$Z = 2 + 0 = 2$

$\Rightarrow \therefore$ system is unstable

STABILITY IN THE FREQUENCY DOMAIN

Nyquist Stability Criterion

path de , $s = -j\omega$

mirror image of path ab

path efa , $s = \lim_{r \to 0} re^{j\theta}$, $\theta \Rightarrow -90^\circ - 0^\circ - +90^\circ$

$$G(s) = \lim_{r \to 0} \frac{1}{(re^{j\theta})^2 (re^{j\theta}+1)}$$

$$= \lim_{r \to 0} \frac{1}{r^2 e^{j2\theta}}$$

$$= \infty \cdot e^{-j2\theta}$$

$-2(-90^\circ) - 0^\circ - -2(90^\circ)$

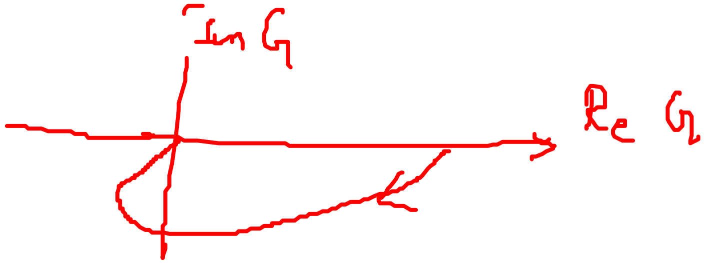
$180^\circ - 0^\circ - -180^\circ$

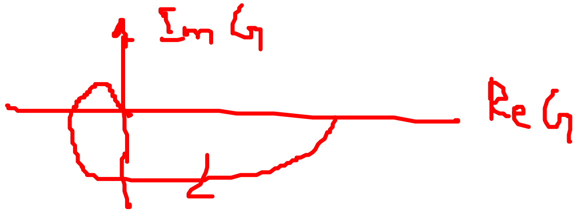
STABILITY IN THE FREQUENCY DOMAIN

Polar plot for different type of systems

Polar Plot for Type 0 System

	$\omega = 0$	$\omega = \infty$
$G(s) = \dfrac{K}{\tau s + 1}$	$M = K$, $\phi = 0^\circ$	$M = 0$, $\phi = -90^\circ$
$G(s) = \dfrac{K}{(\tau_1 s + 1)(\tau_2 s + 1)}$	$M = K$, $\phi = 0^\circ$	$M = 0$, $\phi = -180^\circ$
$G(s) = \dfrac{K}{(\tau_1 s + 1)(\tau_2 s + 1)(\tau_3 s + 1)}$	$M = K$, $\phi = 0^\circ$	$M = 0$, $\phi = -270^\circ$





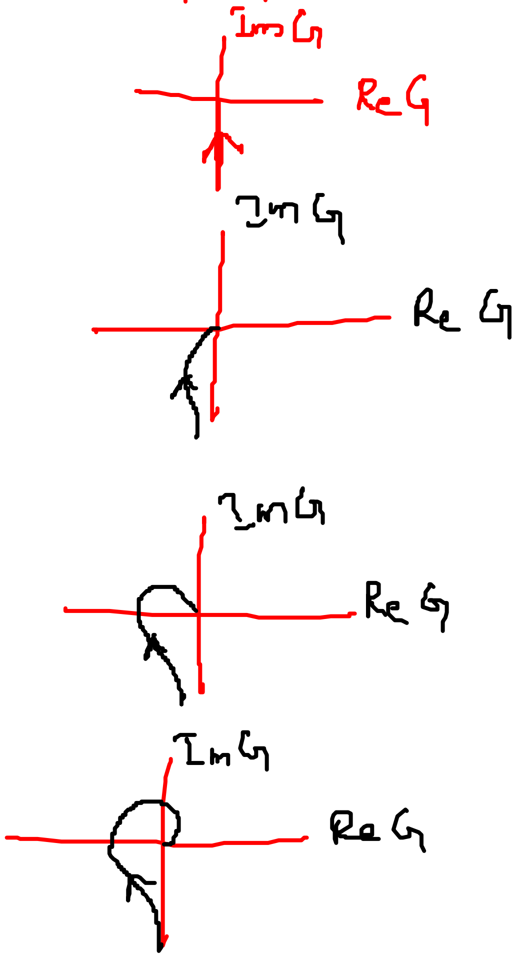
STABILITY IN THE FREQUENCY DOMAIN

Polar plot for different type of systems

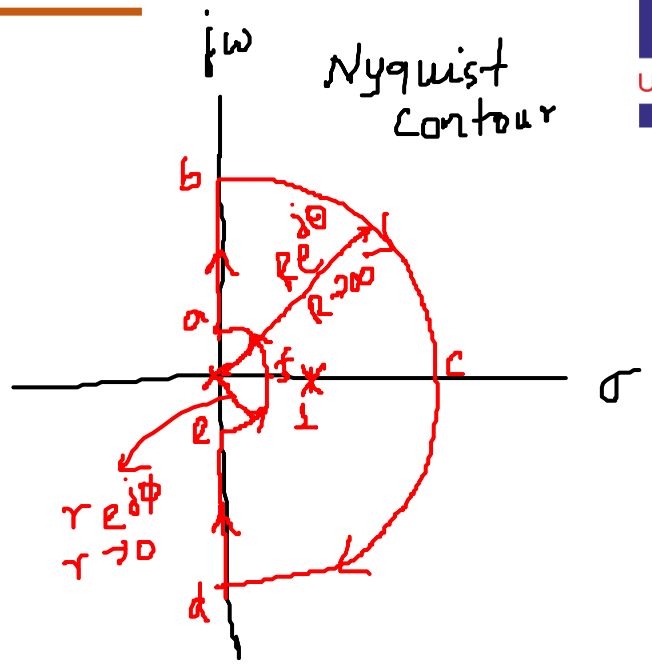
Type 1 system

	$G(s)$	$\omega = 0$	$\omega = \infty$	Order
1	K/s	$M = \infty$ $\phi = -90^\circ$	$M = 0$ $\phi = -90^\circ$	1
2	$\dfrac{K}{s(\tau s+1)}$	$M = \infty$ $\phi = -90^\circ$	$M = 0$ $\phi = -180^\circ$	2
3	$\dfrac{K}{s(\tau_1 s+1)(\tau_2 s+1)}$	$M = \infty$ $\phi = -90^\circ$	$M = 0$ $\phi = -270^\circ$	3
4	$\dfrac{K}{s(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)}$	$M = \infty$ $\phi = -90^\circ$	$M = 0$ $\phi = -360^\circ$	

Polar Plot



Example, $G(s) = \dfrac{K_1}{s(s-1)}$



Sol:

① path ab, $s = j\omega$

$$G(j\omega) = \frac{K_1}{j\omega(j\omega - 1)} \times \frac{-\omega^2 + j\omega}{-\omega^2 + j\omega}$$

$$= \frac{-K_1\omega^2 + jK_1\omega}{\omega^4 + \omega^2} = \frac{-K_1\omega^2}{\omega^4 + \omega^2} + j\frac{K_1\omega}{\omega^4 + \omega^2}$$

$$|G(j\omega)| = \frac{K_1}{\omega\sqrt{\omega^2 + 1}}, \quad \angle G(j\omega) = -90^\circ - \tan^{-1}\frac{\omega}{-1} = -90^\circ - (\pi - \tan^{-1}\omega)$$

$$= -270^\circ + \tan^{-1}\omega$$

STABILITY IN THE FREQUENCY DOMAIN

Nyquist Stability Criterion

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$	Re	Im
0	∞	-270°	$-K_1$	∞
∞	0	-180°	0	0

② path bcd, $\quad s = \lim_{R\to\infty} Re^{j\theta} \quad \theta \to 90^\circ$ to 0° to -90°

$$G(s) = \lim_{R\to\infty} \frac{K_1}{Re^{j\theta}(Re^{j\theta}-1)} = \lim_{R\to\infty} \frac{K_1}{R^2 e^{j2\theta}}$$

$$= 0\, e^{-j2\theta} = 0, \quad -2(90^\circ) \text{ to } 0^\circ \text{ to } -2(-90^\circ)$$

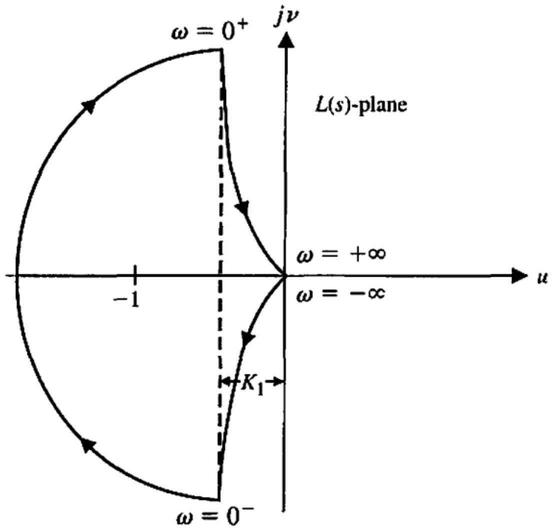
$$= 0, \quad -180^\circ \text{ to } 0^\circ \text{ to } 180^\circ$$

STABILITY IN THE FREQUENCY DOMAIN

Nyquist Stability Criterion

③ path de, $s = -j\omega$

mirror image of polar plot obtained for path ab



N=1, P=1,
Z=1+1=2,
2 CL poles on RHS implies that system is unstable

④ path efa, $s = \lim_{r \to 0} re^{j\phi}$, $\phi \Rightarrow -90^\circ$ to 0° to 90°

$$G(s) = \lim_{r \to 0} \frac{K_1}{re^{j\phi}(\underbrace{re^{j\phi} - 1}_{-1})} = \lim_{r \to 0} \frac{K_1}{re^{j\phi} \cdot e^{-j\pi}} = \infty, e^{j(-\phi+\pi)}$$

$= \infty, -(90^\circ) + 180^\circ)$ to π to $-90^\circ + 180^\circ$

$= \infty, 270^\circ$ to π to 90°

Include derivative controller as $(1 + K_2 s)$, check whether system is stable or not

$$G(s) = \frac{K_1(1 + K_2 s)}{s(s-1)}$$

UNIT 5: STABILITY IN THE FREQUENCY DOMAIN

Relative stability and Nyquist Criterion

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Gain Margin: GM is the reciprocal of the magnitude $|G(j\omega)|$ at the frequency at which phase angle is -180.

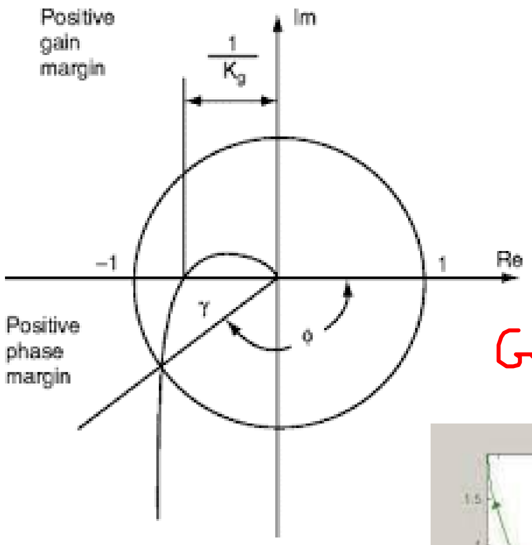
GM in dB = $20 \log \dfrac{1}{|G(j\omega)|} = -20\log_{10}|G(j\omega)|$

Where K_g = $|G(j\omega)|$

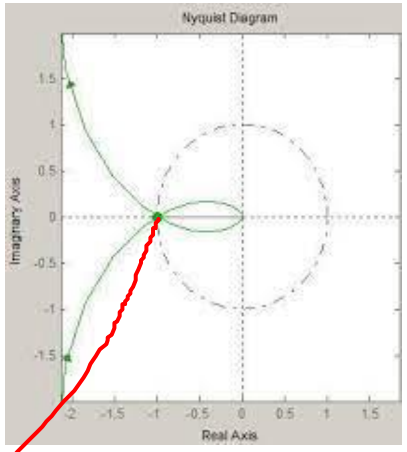
The GM in dB is +ve if K_g > 1

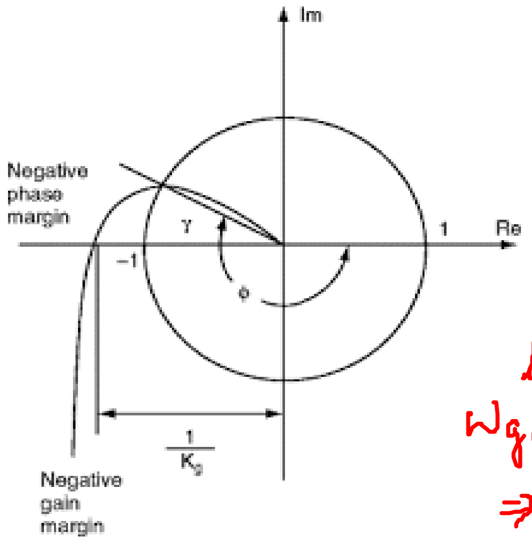
The GM in dB is -ve if K_g < 1

Phase cross over frequency(PCF), ω_{pc} **:** frequency at which phase angle of open loop transfer function equals -180.



GM = PM = 0





$\omega_{gc} = \omega_{pc}$

⇒ marginally stable

STABILITY IN THE FREQUENCY DOMAIN

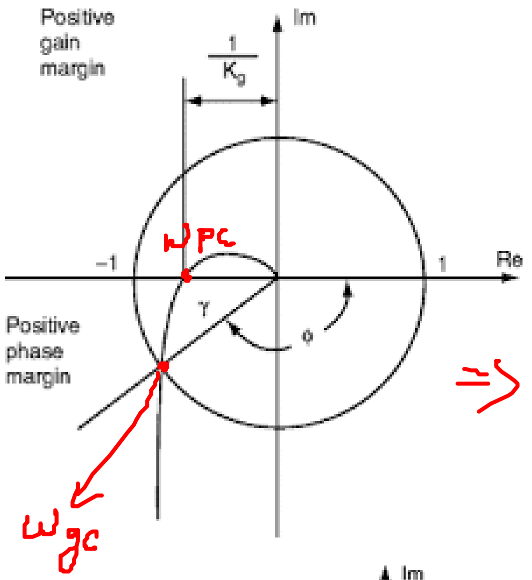
Relative stability & Nyquist Stability Criterion

Phase Margin: PM is the amount of additional phase lag at the gain cross over frequency required to bring the system to the verge of instability.

PM = 180 + ϕ

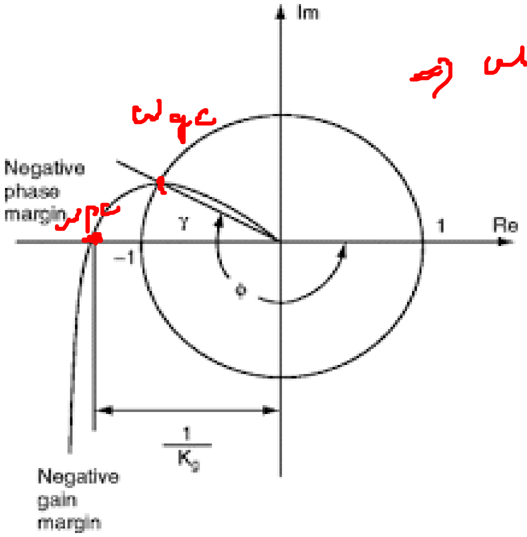
Where $\phi = angle\{G(j\omega)\}|_{\omega=\omega_{gc}}$

Gain cross over frequency(GCF), ω_{gc} : frequency at which magnitude of open loop transfer function equals unity.



=> stable

=> unstable



when GM & PM are +ve => s/m is stable

when GM & PM are -ve => system is unstable

For example, $G(s) = \frac{1}{s(s+1)(s+0.5)}$

Find a) ω_{gc} , b) ω_{pc}
c) GM , d) PM

Sol:

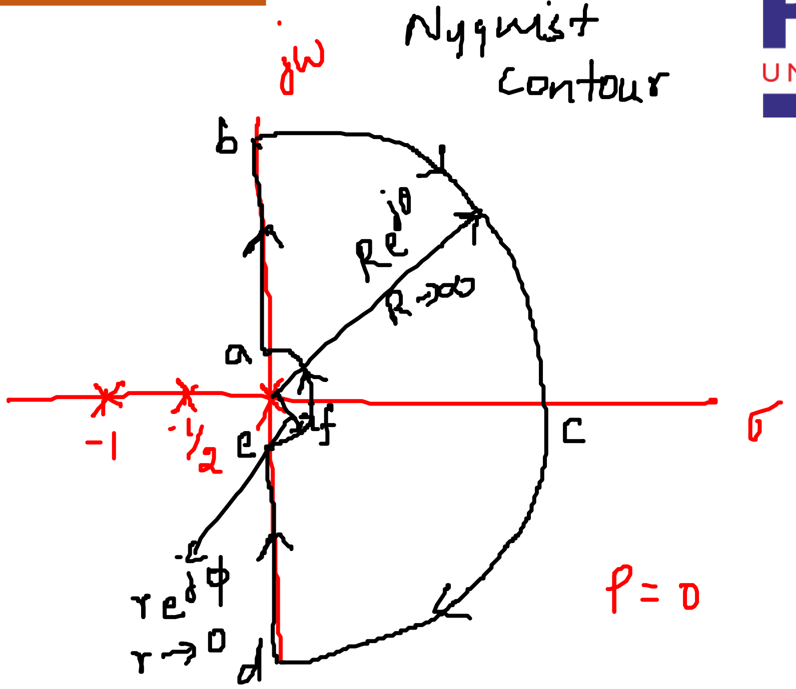
① path ab , $s = j\omega$

$$G(j\omega) = \frac{1}{j\omega(j\omega+1)(j\omega+1/2)}$$

$$G(j\omega) = \frac{-\frac{\omega^2}{2} - \omega^2 + j\left(\omega^3 - \omega/2\right)}{(\omega^4+\omega^2)\left(\omega^2+\frac{1}{4}\right)}$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{\omega^2+1}\sqrt{\omega^2+1/4}} ,$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$



$P = 0$

For example, $G(s) = \frac{1}{s(s+1)(s+0.5)}$

path dc, $s = -j\omega$

mirror image of path ab

path efa, $s = \lim_{r \to 0} re^{j\theta}$, $\theta \Rightarrow -90^\circ - 0^\circ - 90^\circ$

$$G(s) = \lim_{r \to 0} \frac{1}{re^{j\theta}(re^{j\theta}+1)(re^{j\theta}+0.5)}$$

$$= \lim_{r \to 0} \frac{1}{0.5r\, e^{j\theta}}$$

$$= \infty e^{-j\theta} \Rightarrow -\theta \Rightarrow 90^\circ - 0^\circ - -90^\circ$$

STABILITY IN THE FREQUENCY DOMAIN

Relative Stability and Nyquist Stability Criterion

For example, $G(s) = \frac{1}{s(s+1)(s+0.5)}$

To find real axis crossing

$\text{Im}\, G(j\omega) = 0 \Rightarrow \omega^3 - \omega/2 = 0 \Rightarrow \boxed{\omega_{pc} = 1/\sqrt{2} \text{ rad/sec}}$

$\cancel{K_g} = \text{Re}\, G(j\omega)\Big|_{\omega=\omega_{pc}} = \left.\frac{-\frac{\omega^2}{2} - \omega^2}{(\omega^4+\omega^2)(\omega^2+1/4)}\right|_{\omega=\omega_{pc}} = -1.33$

$K_g = |-1.33| = 1.33$

$GM = \frac{1}{K_g} = 0.752$

$GM_{dB} = 20 \log_{10} 0.752 = -2.5\,dB$

$\boxed{GM = -2.5\,dB}$

STABILITY IN THE FREQUENCY DOMAIN

Relative Stability and Nyquist Stability Criterion

For example, $G(s) = \frac{1}{s(s+1)(s+0.5)}$

To find ω_{gc}

$$|G(j\omega)|_{\omega=\omega_{gc}} = 1$$

$$\frac{1}{\omega_{gc}\sqrt{\omega_{gc}^2+1}\sqrt{\omega_{gc}^2+1/4}} = 1$$

$$\omega_{gc}\sqrt{\omega_{gc}^2+1}\sqrt{\omega_{gc}^2+1/4} = 1$$

$$\omega_{gc}^2(\omega_{gc}^2+1)(\omega_{gc}^2+1/4) = 1$$

$$\omega_{gc}^6 + \frac{5}{4}\omega_{gc}^4 + \frac{1}{4}\omega_{gc}^2 = 1$$

$$\omega_{gc}^6 + \frac{5}{4}\omega_{gc}^4 + \frac{1}{4}\omega_{gc}^2 - 1 = 0$$

let $t = \omega_{gc}^2$

$$t^3 + \frac{5}{4}t^2 + \frac{1}{4}t - 1 = 0$$

$$\omega_{gc}^2 = 0.6 \Rightarrow \boxed{\omega_{gc} = 0.813 \text{ rad/sec}}$$

$$\phi = \underline{/G(j\omega)}\Big|_{\omega=\omega_{gc}} = -90^\circ - \tan^{-1}0.813 - \tan^{-1}(2\times0.813)$$

$$= -187.5^\circ$$

$$PM = 180^\circ + \phi$$

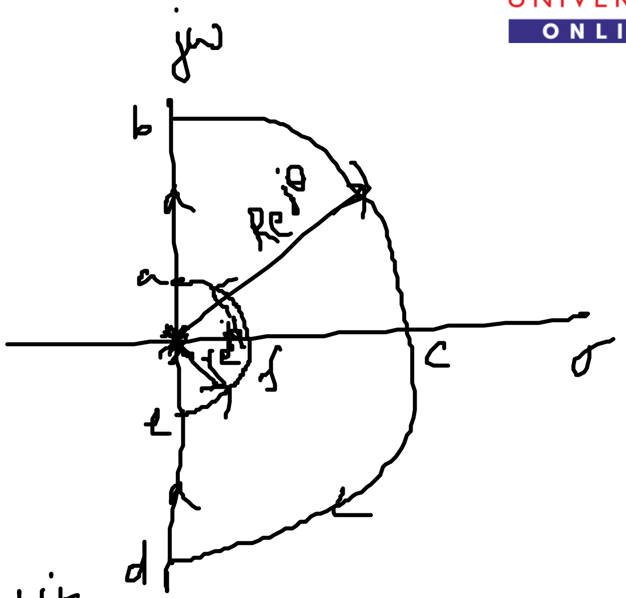
$$= -7.5^\circ$$

$\because$ GM & PM are $-$ve $\Rightarrow$ system is unstable

STABILITY IN THE FREQUENCY DOMAIN

Relative Stability and Nyquist Stability Criterion

Ex, $GH(s) = \dfrac{K(s\tau_a + 1)}{s^3}$



① path ab, $s = j\omega$

$$GH(j\omega) = \frac{K(j\omega\tau_a + 1)}{(j\omega)^3} = \frac{K(j\omega\tau_a + 1)}{-j\omega^3} \times \frac{j\omega^3}{j\omega^3}$$

$$= \frac{-K\omega^4\tau_a + jK\omega^3}{\omega^6} = \frac{-K\omega\tau_a + jK}{\omega^3}$$

$$|GH(j\omega)| = \frac{K\sqrt{\omega^2\tau_a^2 + 1}}{\omega^3}, \quad \underline{|GH(j\omega)} = \tan^{-1}\omega\tau_a - 270^\circ$$

STABILITY IN THE FREQUENCY DOMAIN

Relative Stability and Nyquist Stability Criterion

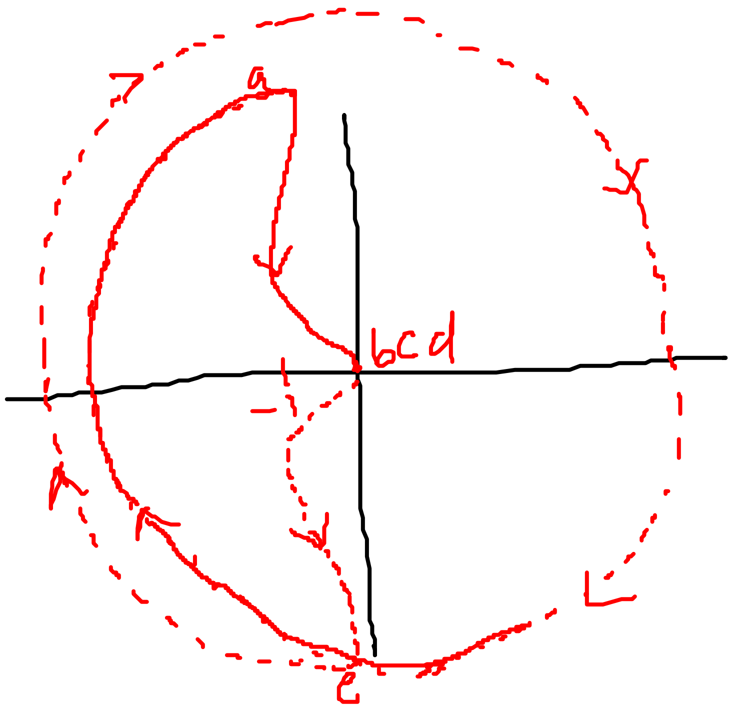
ω	$\lvert GH(j\omega)\rvert$	$\angle G(j\omega)$	Re	Im
0	∞	-270°	∞	∞
∞	0	-180°	0	0

path bcd, $s = Re^{j\theta}$, $\theta = 90^\circ \to 0^\circ \to -90^\circ$

$$G(s) = \frac{K(Re^{j\theta}\tau_a + 1)}{R^3 e^{j3\theta}} = \frac{K}{R^2} \cdot e^{-j2\theta} =$$

$$= 0 \angle -2(+90^\circ) \to 0^\circ \to -2(-90^\circ)$$

$$= 0 \angle -180^\circ \to 0^\circ \to 180^\circ$$



$N = 2$, $P = 0$

$Z = 2 + 0 = 2$

path de
mirror image of polar plot obtained for path ab

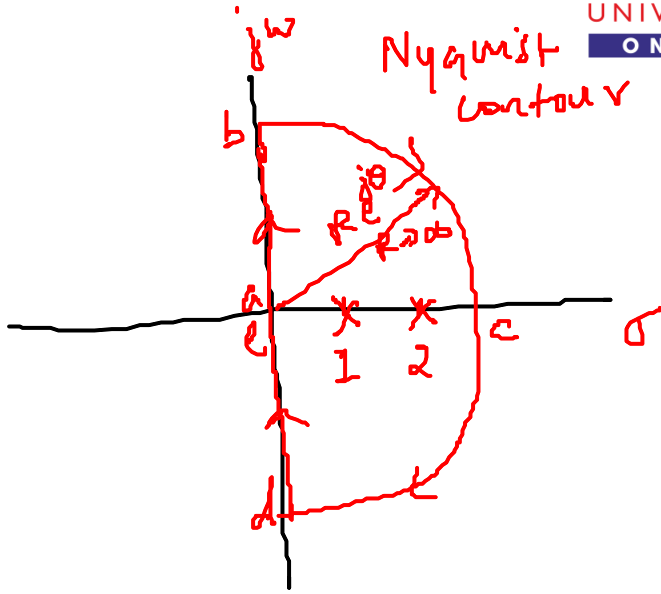
path efa , $s = re^{j\phi}$, $\phi \Rightarrow -90^\circ \to 0^\circ \to 90^\circ$

$$G(s) = \frac{K(re^{j\phi}T_a + 1)}{r^3 e^{j3\phi}} = \frac{K}{r^3} \cdot e^{-j3\phi}$$

$$= \infty , \underline{-3(-90^\circ)} \to 0^\circ \to -3(90^\circ)$$

$$\underline{270^\circ} \to 180^\circ \to 90^\circ \to 0^\circ \to -90^\circ \to -180^\circ \to -270^\circ$$

Example, $G(s) = \dfrac{K(s+4)}{(s-1)(s-2)}$ Find the range of K for which the system is stable.



Sol: path ab, $s = j\omega$

$$G(j\omega) = \frac{K(j\omega+4)}{(j\omega-1)(j\omega-2)}$$

$$= \frac{K(j\omega+4)(-j\omega-1)(-j\omega-2)}{(\omega^2+1)(\omega^2+4)}$$

$$= \frac{K(4+j\omega)(-\omega^2+3j\omega+2)}{(\omega^2+1)(\omega^2+4)} = \frac{K(8+j12\omega-4\omega^2+2j\omega-3\omega^2-j\omega^3)}{(\omega^2+1)(\omega^3+4)}$$

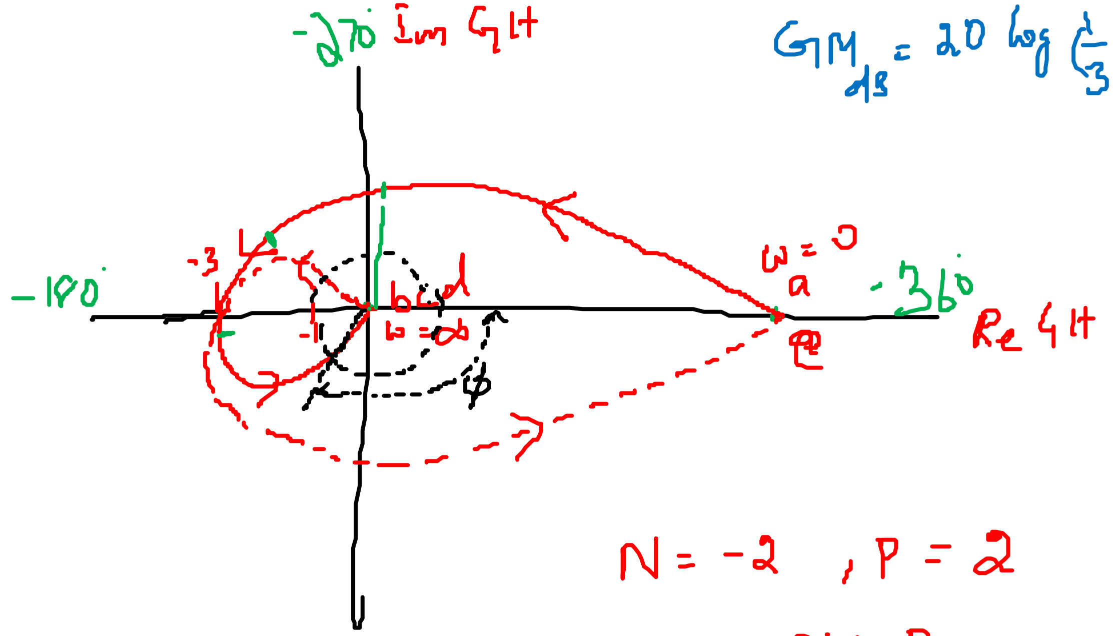
$$= \frac{K(8-7\omega^2)}{(\omega^2+1)(\omega^2+4)} + j\frac{K(14\omega-\omega^3)}{(\omega^2+1)(\omega^2+4)}$$

STABILITY IN THE FREQUENCY DOMAIN

Relative Stability and Nyquist Stability Criterion

$$|G(j\omega)| = \frac{K\sqrt{\omega^2+16}}{\sqrt{\omega^2+1}\,\sqrt{\omega^2+4}}, \quad \angle G(j\omega) = \tan^{-1}\frac{\omega}{4} - (180 - \tan^{-1}\omega) - (180 - \tan^{-1}\frac{\omega}{2})$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	$\cancel{16}$ 2K	$-360°$
1	10.43 1.30 K	$-274°$
2	5.36	$-224°$
4	0.306 K	$-170°$
∞	0	$-90°$



$$GM_{dB} = 20\log\left(\frac{1}{3}\right)$$

$N = -2, \; P = 2$

$Z = N + P$

$= -2 + 2$

$= 0$

$\Rightarrow$ system is stable

(2) Path bcd $\quad s = Re^{j\theta}, \quad R \to \infty, \quad \theta \to 90° \to 0° \to -90°$

$$G(s) = \frac{K(Re^{j\theta}+4)}{(Re^{j\theta}-1)(Re^{j\theta}-2)} = \frac{K}{R}e^{-j\theta} = 0, \; \angle -90° \to 0° \to 90°$$

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(3) path de , $s = -j\omega$

mirror image of polar plot obtained for path ab

To find the real axis crossing , Im = 0

$$\frac{K(14\omega - \omega^3)}{(\omega^2+1)(\omega^2+4)} = 0 \Rightarrow 14\omega - \omega^3 = 0$$

$$\omega^2 = 14 \Rightarrow \omega = \sqrt{14} = \omega_{pc}$$

$$|G(j\omega)|_{\omega=\omega_{pc}} = \left.\frac{K(8-7\omega^2)}{(\omega^2+1)(\omega^2+4)}\right|_{\omega=\sqrt{14}} = \frac{K(8-7\times14)}{(14+1)(14+4)} = -0.33K$$

For the system to be stable, N = -2 , which implies $-0.33K < -1$

$$\Rightarrow K > \frac{1}{0.33}$$

$$K > 3.003$$

STABILITY IN THE FREQUENCY DOMAIN

Relative Stability and Nyquist Stability Criterion

To find ω_{gc} , $|G(j\omega)| = 1$

$$\frac{K\sqrt{\omega_{gc}^2 + 16}}{\sqrt{\omega_{gc}^2 + 1}\ \sqrt{\omega_{gc}^2 + 4}} = 1$$

let $K = 8$, $\omega_{gc} =$

Sub. ω_{gc} in $\angle G(j\omega) = \phi$

$PM = 180^\circ + \phi$

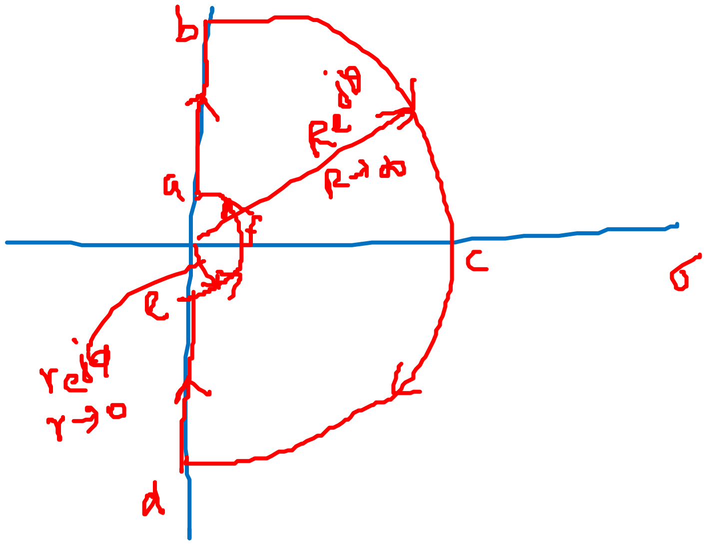
STABILITY IN THE FREQUENCY DOMAIN

Relative Stability and Nyquist Stability Criterion

Example, $G(s) = \dfrac{K(1+2s)}{s(1+s)(1+s+s^2)}$ Find the range of K for which the system is stable

$s = 0, -1, -1 \pm j\frac{\sqrt{3}}{2}$

Sol: path ab, $s = j\omega$



$$G(j\omega) = \frac{K(1+2j\omega)}{j\omega(1+j\omega)(1+j\omega+(j\omega)^2)}$$

$$= \frac{K(1+2j\omega)}{j\omega(1+j\omega)(1-\omega^2+j\omega)}$$

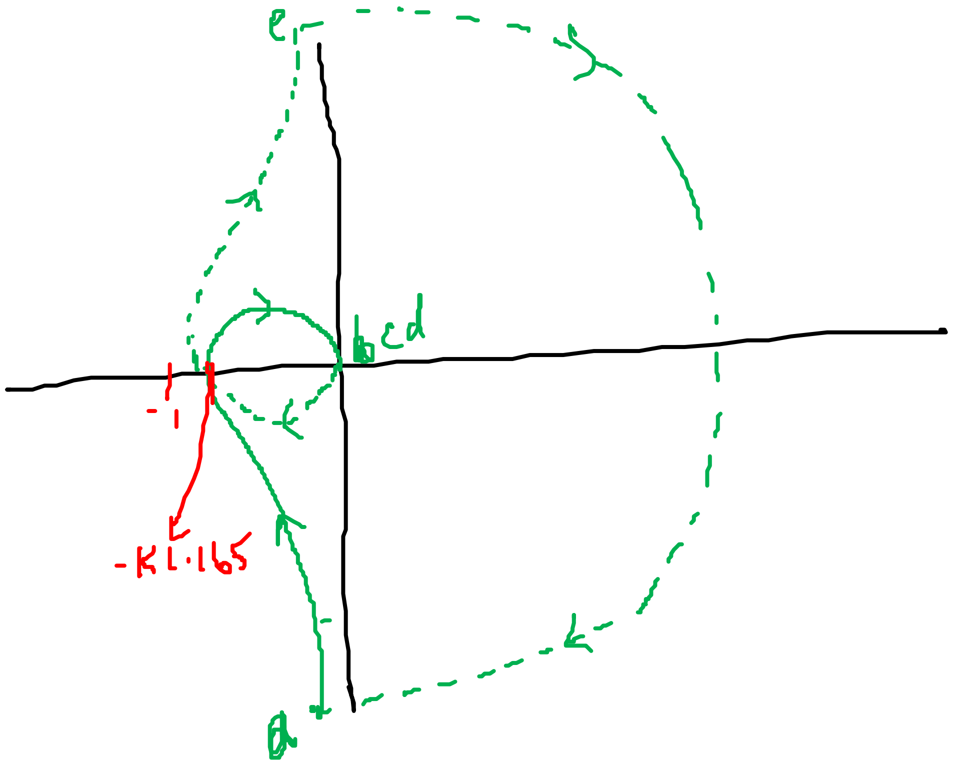
$$= \frac{K(1+2j\omega)(-j\omega-\omega^2)(1-\omega^2-j\omega)}{\omega^2(1+\omega^2)((1-\omega^2)^2+\omega^2)}$$

$$= \frac{K(-3\omega^3 - j(1+2\omega^2-2\omega^4))}{\omega^2(1+\omega^2)((1-\omega^2)^2+\omega^2)}$$

$$|G(j\omega)| = \frac{K\sqrt{1+4\omega^2}}{\omega\sqrt{1+\omega^2}\,\sqrt{1+\omega^4-\omega^2}}$$

$$\angle G(j\omega) = \tan^{-1}(2\omega) - 90^\circ - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{1-\omega^2}$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$	Re	Im
0	∞	-90°	∞	∞
∞	0	-270°	0	0
1				



$N = 0$, $P = 0 \Rightarrow Z = 0$

$\Rightarrow$ system stable for $K < 0.86$

STABILITY IN THE FREQUENCY DOMAIN

Relative Stability and Nyquist Stability Criterion

where it cuts the real axis

$$Im(G(j\omega)) = 0$$

$$1 + 2\omega^2 - 2\omega^4 = 0$$

$$2\omega^4 - 2\omega^2 - 1 = 0 \Rightarrow \omega^2 = \frac{2 \pm \sqrt{4+8}}{4}$$

$$\omega = 1.168$$

$$Re(G(j\omega))\Big|_{\omega=1.168} \Rightarrow \frac{-K3\omega^3}{\omega(1+\omega^2)((1-\omega^2)^2+\omega^2)}\Bigg|_{\omega=1.168} = -K1.165$$

For the system to be stable, $N = 0 \because P = 0$, $\therefore -K1.165 > -1$

$$K < \frac{1}{1.165}$$

$$0 < K < 0.86$$

THANK YOU

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