DIGITAL IMAGE PROCESSING-1

Unit 2: Lecture 17-18

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Unit 2: Image Transforms

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Last Session

- Image transforms preliminaries
  - Orthogonal transforms
  - Unitary transforms
  - Separable transforms

Today's Session

- Image basics cont..
- Desirable properties of image transforms
- 1 D and 2D DFT

Orthogonal and Orthonormal Vectors

Let $\mathcal{V}$ be an inner-product space. A set of vectors

$$\{u_1, u_2, \ldots, u_n, \ldots\} \in \mathcal{V}$$

is called **orthonormal** if and only if

$$\forall i, j : \langle u_i, u_j \rangle = \delta_{ij}$$

where δ_{ij} is the Kronecker delta and $\langle \cdot, \cdot \rangle$ is the inner product defined over $\mathcal{V}$.

Orthogonal Vectors

- In linear algebra, two **vectors** in an inner product space are **orthonormal** if they are **orthogonal** (perpendicular) and are **unit vectors**.
- An **orthonormal** set which forms a basis is called an **orthonormal** basis.

- So vectors being **orthogonal** puts a restriction on the angle **between** the vectors whereas vectors being **orthonormal** puts restriction on both the angle **between** them as well as **the length of those vectors.**

Unitary and Hermitian Matrix

- **Unitary Matrix:** A Matrix 'A' is a unitary matrix if $\mathbf{A^{-1} = A^{*T}}$

  where A* is conjugate of A

- **Hermitian Matrix:** $\mathbf{A^{H} = A^{*T}}$
  - **For a real matrix** $\mathbf{A^{H} = A^{T}}$
- **So for a Unitary Matrix:** $\mathbf{A^{-1} = A^{H}}$

- **Unitary transform:** A discrete linear transform is unitary if its transform matrix conforms to the unitary condition $A\,A^{*T} = A \times A^{H} = I$

  where A = transformation matrix and **A^H = Hermitian matrix and $A^H = A^{*T}$**

- **When the transform matrix A is unitary, the defined transform is called unitary transform**
  - **Determinant and eigen value of a unitary matrix have unity magnitude**
- **Image Transforms represent the given image as a series summation of a set of Unitary Matrices (basis matrices)**

2D Separable and Unitary Transform

- The **forward kernel** is said to be separable if

$$a_{k,l}(m,n) = a_k(m).a_l(n) = a(k,m).b(l,n)$$

Where

$a(k,m) = A$ **and** $b(l,n) = B$ are Unitary matrices

$AA^{*T} = I$. and $BB^{*T} = I$

Summary: 1D and 2D Unitary transform

- 1D unitary transform, $V=AU$
- 1D Inverse unitary transform, $U=A^{*T} V$
- 2D unitary transform, $V=AUA^{T}$
- 2D Inverse unitary transform, $U=A^{*T} V A^{*}$

Basis Images

V=AU

- Any image can be represented as a linear combination of **basis images.**
- **Basis images :** Images can be expanded in terms of discrete set of basis arrays called "Basis images"
- The basis images can be generated by unitary matrices.
- Basis image is represented as

$$B^*{}_{k,l} = b^*{}_k \,.\, b^{*T}{}_l$$

Where

$b^*{}_k$ is k^{th} column of A^{*T}

$b^*{}_l$ is l^{th} column of A^{*T}

- Using the transformed image v and these basis images, original image can be generated as follows:

$$u(m,n) = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} v(k,l) B^*_{k,l}$$

Basis Images: Example

2. (a)Find transformed image $\boldsymbol{V}$. Find inverse transformed image $\boldsymbol{U}$ using inverse transform formula.

(b) Calculate basis images, and find inverse transformed image $\boldsymbol{U}$ using basis images.

$$\boldsymbol{A} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{bmatrix} \quad \boldsymbol{U} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \end{bmatrix}$$

Sol: w.k.t $\boldsymbol{V} = \boldsymbol{A}\,\boldsymbol{U}\,\boldsymbol{A}^{\boldsymbol{T}}$

$$\rightarrow A^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\rightarrow AUA^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Basis Images: Example

$$= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1+3 & 2+4 \\ 1-3 & 2-4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & 6 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 4+6 & 4-6 \\ -2-2 & -2+2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 10 & -2 \\ -4 & 0 \end{bmatrix}$$

$$\boxed{V = \begin{bmatrix} 5 & -1 \\ -2 & 0 \end{bmatrix}}$$

Basis Images: Example

Inverse transform u is given by

$$U = A^{*T} V A^{*}$$

$* \; A = A^{*}$ ($\because$ A is real matrix).

$$\rightarrow A^{*T} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\rightarrow A^{*T} V A^{*} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Basis Images: Example

$$A^{*T} V A^{*} = \frac{1}{2}\begin{bmatrix} 5-2 & -1+0 \\ 5+2 & -1+0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 3 & -1 \\ 7 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 3-1 & 3+1 \\ 7-1 & 7+1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

$$\boxed{U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}$$ (Same as input image)

Basis Images: Example

Now find the inverse using the basis image:
w.k.t basis image is given by

$$B^*_{k,l} = b^*_k \, . \, b^{*T}_l$$

Where

b^*_k is k^{th} column of A^{*T}

b^*_l is l^{th} column of A^{*T}

Steps to calculate basis image:

1. Find A^{*T}
2. Find. b^*_k Which is kth column of A^{*T}
3. Find all basis images (B^*_{00} , B^*_{01} , B^*_{10} , B^*_{11}) by using the above formula ; for k=0,1 and l=0,1

Basis Images: Example

→ $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $A = A^*$ ($\because$ A is real matrix).

→ $A^{*T} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

→ $b_0^* = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ 0^{th} column of A^{*T}

→ $b_1^* = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ 1^{st} column of A^{*T}

→ $B_{00}^* = b_0^* \cdot b_0^{*T} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Basis Images: Example

$$B^*_{01} = b^*_0 \; b^{*T}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}$$

$$B^*_{10} = b^*_1 \cdot b^{*T}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\-1 & -1\end{bmatrix}$$

$$B^*_{11} = b^*_1 \cdot b^{*T}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

Computation of v using basis images:

w.k.t. u(m,n)= $\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} v(k,l)B^*_{k,l}$

Here $N=2$

$$\therefore \; U = \sum_{k=0}^{1}\sum_{l=0}^{1} v(k,l) \cdot B^*_{kl}$$

$$= V(0,0)\,B^*_{00} + V(0,1)\,B^*_{01} + V(1,0)\,B^*_{10} + V(1,1)\,B^*_{11}$$

Basis Images: Example

$$= 5 \times \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + (-1)\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} + (-2)\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} + (0)\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix} + 0\right\}$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 5-1-2 & 5+1-2 \\ 5-1+2 & 5+1+2 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

Desirable Properties of Image Transforms

- **Energy Conservation** (Lossless)
  - **Entropy (average information) of a vector is preserved under unitary transformation**
- **Energy compaction** (Energy distributed in lesser number of coefficients)
- **Decorrelation** (reduced dependency/similarity/correlation between pixels)

Desirable Properties of Unitary Transform

Property 1: Energy Conservation (Lossless)

- Entropy (average information) of a vector is preserved under unitary transformation
- Input signal energy is equal to output signal energy
- w.k.t 1D unitary transform, V=AU

$$\|V\|^2 = \|U\|^2$$

$$\|V\|^2 = \sum_{k=0}^{N-1} |v(k)|^2 = V^T\,V = (AU)^T A\,U$$

$$= U^T\,A^T\,A\,U = U^T\,U = \|U\|^2$$

Similarly for 2D DFT

$$\sum_{m,n=0}^{N-1}\sum |u(m,n)|^2 = \sum_{k,l=0}^{N-1}\sum |v(k,l)|^2$$

Desirable Properties of Unitary Transform

Ex. Check for Energy Conservation for the following case (1D).

$$A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$V = AU = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\|V\|^2 = \left(\frac{7}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{49}{2} + \frac{1}{2} = 25$$

$$\|U\|^2 = 3^2 + 4^2 = 9 + 16 = 25$$

Desirable Properties of Unitary Transform

Ex. Check for Energy Conservation for the following case (2D).

$$U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$V = A U A^T$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{bmatrix} 1+3 & 2+4 \\ -1+3 & -2+4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 6 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4+6 & -4+6 \\ 2+2 & -2+2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 10 & 2 \\ 4 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix}$$

$$\|U\|^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$$

$$\|V\|^2 = 5^2 + 1^2 + 2^2 + 0^2 = 30$$

$$\therefore \|U\|^2 = \|V\|^2$$

Desirable Properties of Unitary Transform

Property 2: Energy compaction (Energy distributed in lesser number of coefficients)

- The energy in a unitary transformed matrix is unevenly distributed among its coefficients.
- Only few coefficients of transform have significant values and most of the coefficients have zero or nearly zero values.
- In lossy compression, these coefficients are rounded off to zero and the image is reconstructed with only few coefficients having significant values.

Desirable Properties of Unitary Transform

Check for Energy compaction for the following case.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 100 \\ 98 \\ 98 \\ 100 \end{bmatrix}$$ **and threshold is 64**

1D unitary Transform $V = A.U$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 100 \\ 98 \\ 98 \\ 100 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 100+98+98+100 \\ 100-98+98-100 \\ 100+98-98-100 \\ 100-98-98+100 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 396 \\ 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 198 \\ 0 \\ 0 \\ \cancel{2}\,0 \end{bmatrix}$$

198 → Significant Value

0, 0, 0 → Insignificant Co-efficients.

If input data has distributed energy and the transformed output has only few significant values(most energy is in first element)

Desirable Properties of Unitary Transform

Property 3: Decorrelation

- When the input elements are highly correlated, the transform coefficients tends to be uncorrelated.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{DCT}} A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If input data is highly correlated the transformed output has only one non zero element

Classification of Image Transforms

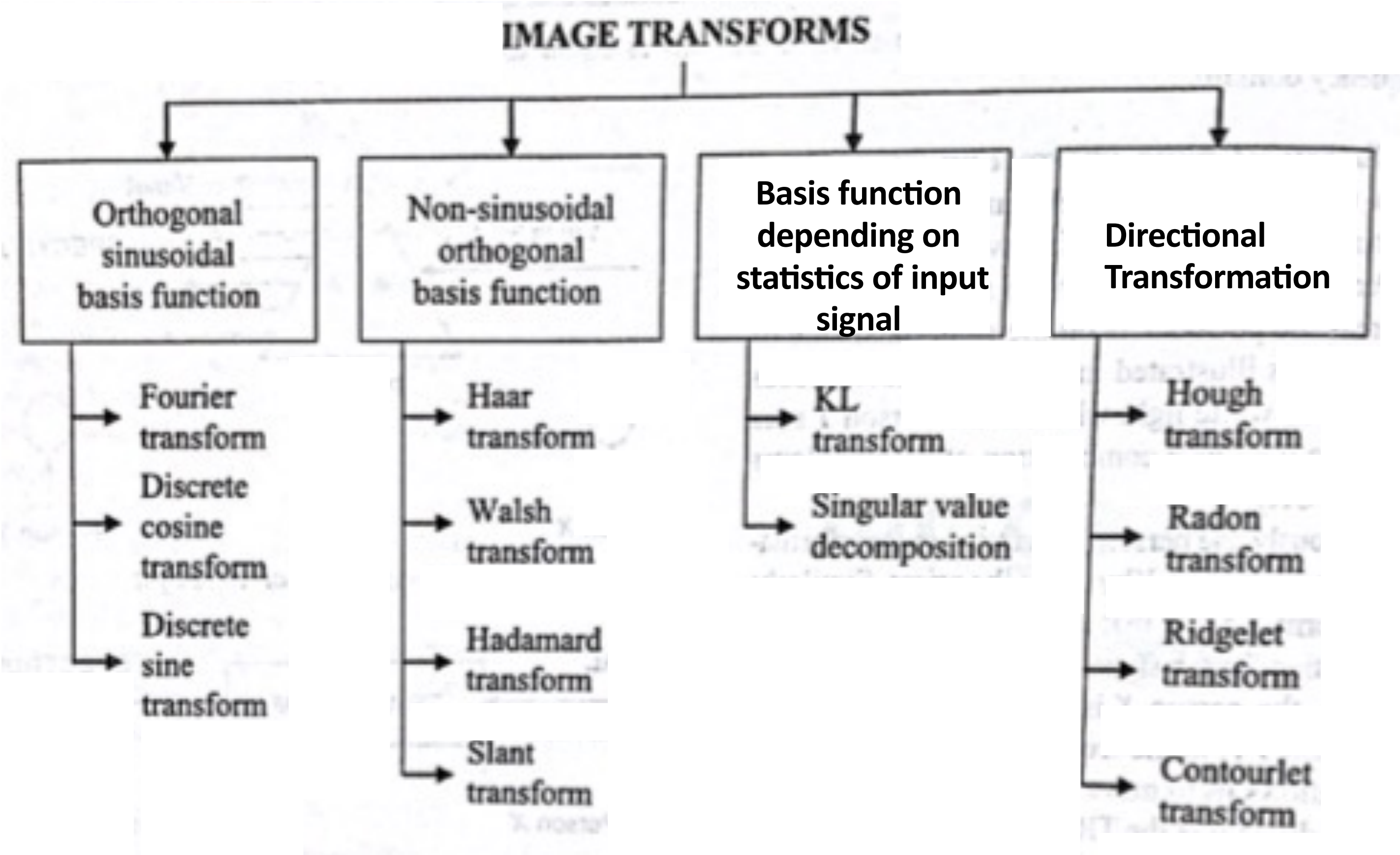

Image Transforms: Types

- DFT, DCT, DST, Hadamard, Slant, Haar, Walsh, KL,SVD etc.

- Basic function’s dependency on statistics of input data: KL, SVD

- Independent of input:
  - Sinusoidal orthogonal basis function: DFT, DCT, DST
  - Non sinusoidal orthogonal basis function – Hadamard , Slant, Haar , Walsh

1D Discrete Fourier Transform

- Consider a sequence $u(n)$ having N elements. DFT of $u(n)$ is defined as

$$v(k) = \sum_{n=0}^{N-1} u(n){W_N}^{kn} \quad k = 0,1, \ldots N-1$$

Where $W_N = exp\left\{-j\frac{2\pi}{N}\right\}$

Inverse DFT is given by

$$u(n) = \frac{1}{\mathrm{N}} \sum_{k=0}^{N-1} v(k)\, {W_N}^{-kn} \quad n = 0,1, \ldots N-1$$

- The DFT transform pairs are not scaled properly to be unitary transformations.
- In image processing it is more convenient to consider the unitary DFT

1D Discrete Fourier Transform (Unitary)

- Unitary DFT of $u(n)$ having N elements is defined as

$$v(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} u(n) {W_N}^{kn} \quad k = 0,1, \ldots N-1$$

Where $W_N = exp\left\{-j\,\frac{2\pi}{N}\right\}$

Inverse DFT is given by

$$u(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} v(k)\, {W_N}^{-kn} \quad n = 0,1, \ldots N-1$$

- N x N unitary DFT matrix F is given by

$$F = \left\{\frac{1}{\sqrt{N}} W_N^{kn}\right\}, \qquad 0 \le k, \quad n \le N-1$$

1D Discrete Fourier Transform (Unitary) matrix

- N x N unitary DFT matrix F is given by

$$F = \left\{\frac{1}{\sqrt{N}} W_N^{kn}\right\}, \qquad 0 \leq k, \qquad n \leq N-1$$

For N = 4

$$F = \frac{1}{2}\begin{bmatrix} W_4{}^{00} & W_4{}^{01} & W_4{}^{02} & W_4{}^{03} \\ W_4{}^{10} & W_4{}^{11} & W_4{}^{12} & W_4{}^{13} \\ W_4{}^{20} & W_4{}^{21} & W_4{}^{22} & W_4{}^{23} \\ W_4{}^{30} & W_4{}^{31} & W_4{}^{32} & W_4{}^{33} \end{bmatrix}$$

$$W_4{}^{00} = \exp\left\{\frac{-j2\pi \times 0 \times 0}{4}\right\}$$

$$W_4{}^{01} = \exp\left\{\frac{-j2\pi \times 0 \times 1}{4}\right\}$$

$$W_4{}^{11} = \exp\left\{\frac{-j2\pi \times 1 \times 1}{4}\right\}$$

Similarly compute other values

1D Discrete Fourier Transform (Unitary) matrix

$$F = \frac{1}{2}\begin{bmatrix} {W_4}^{00} & {W_4}^{01} & {W_4}^{02} & {W_4}^{03} \\ {W_4}^{10} & {W_4}^{11} & {W_4}^{12} & {W_4}^{13} \\ {W_4}^{20} & {W_4}^{21} & {W_4}^{22} & {W_4}^{23} \\ {W_4}^{30} & {W_4}^{31} & {W_4}^{32} & {W_4}^{33} \end{bmatrix}$$

$$\boldsymbol{F = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}}$$

1D Discrete Fourier Transform (Unitary)

In matrix format:

1D forward DFT: $V = FU$

1D Inverse DFT : $U = F^{*T}V$

2D Discrete Fourier Transform (Unitary)

- Unitary DFT of u(m,n) having NxN elements is defined as

$$v(k,l) = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} u(m,n)\, {W_N}^{km} {W_N}^{ln}; 0 \leq k,l \leq N-1$$

Where $W_N \ = exp\left\{-j\,\frac{2\pi}{N}\right\}$

Inverse DFT is given by

$$\mathrm{u}(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} v(k,l) {W_N}^{-km} {W_N}^{-ln}; 0 \leq m,n \leq N-1$$

In matrix format:

2D forward DFT

$$V \ = FUF^T$$

2D Inverse DFT :

$$U \ = {F^*}^T V F^*$$

- Consider 1-D DFT where f(x) ↔ $T(u)$

  Then $f(x) = \frac{1}{\sqrt{N}} \sum T(u) e^{j\frac{2\pi}{N}ux} = \sum T(u) h(x, u)$

  where h(x,u) = $\frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}ux}$

- For 2D DFT:

  Any given transform in 2D can be written as $f(x, y) =$

$$\frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} T(u, v) e^{j\frac{2\pi}{N}[ux+vy]} = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} T(u, v)\, h(x, y, u, v)$$

Inverse transform kernel

$$T(u, v) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y) g(x, y, u, v)$$

Forward transform kernel

2D DFT as a Separable Transform

Now given $T(u, v) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y) g(x, y, u, v)$.

- For 2D DFT g(x,y,u,v) = $\frac{1}{N} e^{j\frac{2\pi}{N}[ux+vy]}$

- Also, g(x,y,u,v) = $\frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}[ux]} \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}[vy]}$

  = $g_1(x,u)\ g_2(y,v)$ → Separable

Here = $g_1(x,u)\ g_1(y,v)$ → Functionally same

- Hence 2D DFT is separable and symmetric

Example on 1D DFT

1. Find Fourier transform of the given matrix U.

$$U = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$F = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

w.k.t $V = FU$

$$V = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & j & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+1+1+1 \\ \cancel{1}\cancel{-j}\cancel{-1}\cancel{+j} \\ \cancel{1}\cancel{-1}\cancel{+j}\cancel{-1} \\ \cancel{1}\cancel{+j}\cancel{-1}\cancel{-j} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$2 \rightarrow$ Significant

$0, 0, 0 \rightarrow$ Insignificant.

Example on 1D DFT

1. Find the Inverse Fourier transform of the given matrix V.

$$V = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$F = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

w.k.t. IDFT is given by

$$U = F^{*T} V$$

$$U = F^{*T} . V = F^{*} V = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}^{*} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Example on 2D DFT

2. Find Fourier transform and Inverse Fourier transform of given matrix.

$$V = FUF^T$$

$$U = F^{*T} V F^*$$

$$\boldsymbol{U} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}$$

→ Size of image = $4 \times 4 \Rightarrow N = 4$

→ 4×4 DFT Kernel is $F = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & 1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$

→ Forward DFT is given by $V = FUF^T$

Example on 2D DFT

$$V = FUF^T$$

$$V = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1+1+1+1 & 1+1+1+1 & 1+1+1+1 & 1+1+1+1 \\ 1-j-1+j & 1-j-1+j & 1-j-1+j & 1-j-1+j \\ 1-1+1-1 & 1-1+1-1 & 1-1+1-1 & 1-1+1-1 \\ 1+j-1-j & 1+j-1-j & 1+j-1-j & 1+j-1-j \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4+4+4+4 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example on 2D DFT

→ Inverse DFT is given by $U = F^{*T} V F^*$ or $U = F^* V F^*$

$$U = F^{*T} V F^*$$

$$U = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Next Session

- Properties of DFT
- Discrete Cosine Transform (DCT)

THANK YOU

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