Lecture 4
Image Transforms

ECE 1390/2390

Learning Objectives:

DFT for images

DCT

Haramord-Walsh transform

Haar transform

- Why?
  - Isolate patterns of interest
  - Represent data in a more compact form
  - Reversable

DFT

$$f(x) = [1\ 2\ 3\ 2\ 1\ 0]$$

$$N = 6$$

$$F(w) = \int_{-\infty}^{\infty} dt \cdot f(t) \cdot e^{-iwt}$$

$$F(k) = \sum_{x=0}^{5} f(x) \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot x / N}$$

$$F(k) = 1 \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot \frac{0}{5}} + 2 \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot \frac{1}{5}} + 3 \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot \frac{2}{5}} + 2 \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot \frac{3}{5}} + 1 \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot \frac{4}{5}} + 0 \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot \frac{5}{5}}$$

$$F(k) = \sum_{x=0}^{N-1} f(x) \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot x / N}$$

$$\mathrm{f}(x) = \frac{1}{N} \sum_{k=0}^{N-1} F(k) \cdot e^{j \cdot 2 \cdot \pi \cdot k \cdot x / N}$$

K=0 = 9
K=1 = -2.11 – 1.53*j
K=2 = 0.12 + 0.36*j
K=3 = 0.12 - 0.36*j
K=4 = -2.11 + 1.53*j
K=4 = 9

DFT-2D

$$F(u,v) = \sum_{y=0}^{M-1} \sum_{x=0}^{N-1} f(x,y) \cdot e^{-j \cdot 2 \cdot \pi \cdot (\frac{u}{N} * x + \frac{v}{M} * y)}$$

$$f(x,y) = \frac{1}{M * N} \sum_{v=0}^{M-1} \sum_{u=0}^{N-1} F(u,v) \cdot e^{j \cdot 2 \cdot \pi \cdot (\frac{u}{N} * x + \frac{v}{M} * y)}$$

DFT Matrix form

$$F = H * \mathrm{x}$$

$$H_{2x2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_{4x4} = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$H_{8x8} = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1-j}{\sqrt{2}} & -j & \frac{-1-j}{\sqrt{2}} & -1 & \frac{-1+j}{\sqrt{2}} & j & \frac{1+j}{\sqrt{2}} \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & \frac{-1-j}{\sqrt{2}} & j & \frac{1-j}{\sqrt{2}} & -1 & \frac{1+j}{\sqrt{2}} & -j & \frac{-1+j}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{-1+j}{\sqrt{2}} & -j & \frac{1+j}{\sqrt{2}} & -1 & \frac{1-j}{\sqrt{2}} & j & \frac{-1-j}{\sqrt{2}} \\ 1 & j & -1 & -j & 1 & -j & -1 & -j \\ 1 & \frac{1+j}{\sqrt{2}} & j & \frac{-1+j}{\sqrt{2}} & -1 & \frac{-1-j}{\sqrt{2}} & -j & \frac{1-j}{\sqrt{2}} \end{bmatrix}$$

DFT Matrix forr

$$F = H * \mathrm{x}$$

$$W = \frac{1}{\sqrt{8}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\ \omega^0 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\ \omega^0 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\ \omega^0 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\ \omega^0 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49} \end{bmatrix}$$

$$w = e^{-2\pi j/8}$$

DFT-2D Matrix form

$$\mathrm{F} = \mathrm{H} * \mathrm{x} * \mathrm{H}^{\mathrm{T}}$$

DCT

1D transform

$$\mathrm{F} = \mathrm{C} * \mathrm{x}$$

$$C = \begin{bmatrix} C(0,0) & C(1,0) & C(2,0) \\ C(0,1) & C(1,1) & \\ C(0,2) & & \ddots \end{bmatrix}$$

2D transform

$$\mathrm{F} = \mathrm{C} * \mathrm{x} * \mathrm{C}^{\mathrm{T}}$$

$$C(u,v) = \begin{cases} \sqrt{\dfrac{1}{N}} & u = 0 \\ & 0 \le v \le N-1 \\ \sqrt{\dfrac{2}{N}} \cos\left(\dfrac{(2v+1)\cdot\pi\cdot u}{2N}\right) & 0 \le u \le N-1 \\ & 0 \le v \le N-1 \end{cases}$$

DCT

1D transform

$$F = C * x$$

2D transform

$$F = C * x * C^T$$

DCT^{-1}

$$x = C^T * F$$

$$x = C^T * F * C$$

Hadamard Transform

$$F = \frac{1}{N^2} \mathrm{H}_{\mathrm{nxn}} * \mathrm{x} * \mathrm{H}_{\mathrm{nxn}}{}^{\mathrm{T}}$$

$$\mathrm{x} = \mathrm{H} * \mathrm{F} * \mathrm{H}^{\mathrm{T}}$$

$$F = \frac{1}{N*M} \mathrm{H}_{\mathrm{nxn}} * \mathrm{x} * \mathrm{H}_{\mathrm{mxm}}{}^{\mathrm{T}}$$

The 2x2 is identical to the DFT 2x2

$$H_{2x2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_{4x4} = (H_{2x2} \otimes H_{2x2}) \qquad H_{4x4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$H_{8x8} = (H_{2x2} \otimes H_{4x4}) \qquad H_{8x8} = \begin{bmatrix} H_{4x4} & H_{4x4} \\ H_{4x4} & -H_{4x4} \end{bmatrix}$$

Welch Transform

$$\mathrm{F} = \mathrm{W} * \mathrm{x} * \mathrm{W}^{\mathrm{T}}$$

$$\mathrm{x} = \mathrm{W} * \mathrm{F} * \mathrm{W}^{\mathrm{T}}$$

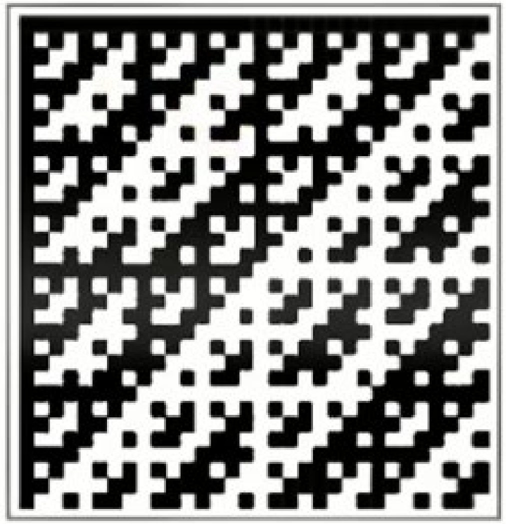
Number of sign changes

$$H_{4x4} = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{array}{l} \rightarrow 0 \\ \rightarrow 3 \\ \rightarrow 1 \\ \rightarrow 2 \end{array}$$

$$W_{4x4} = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Hadamard

Natural Ordering



32x32

$$H_{4x4} = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

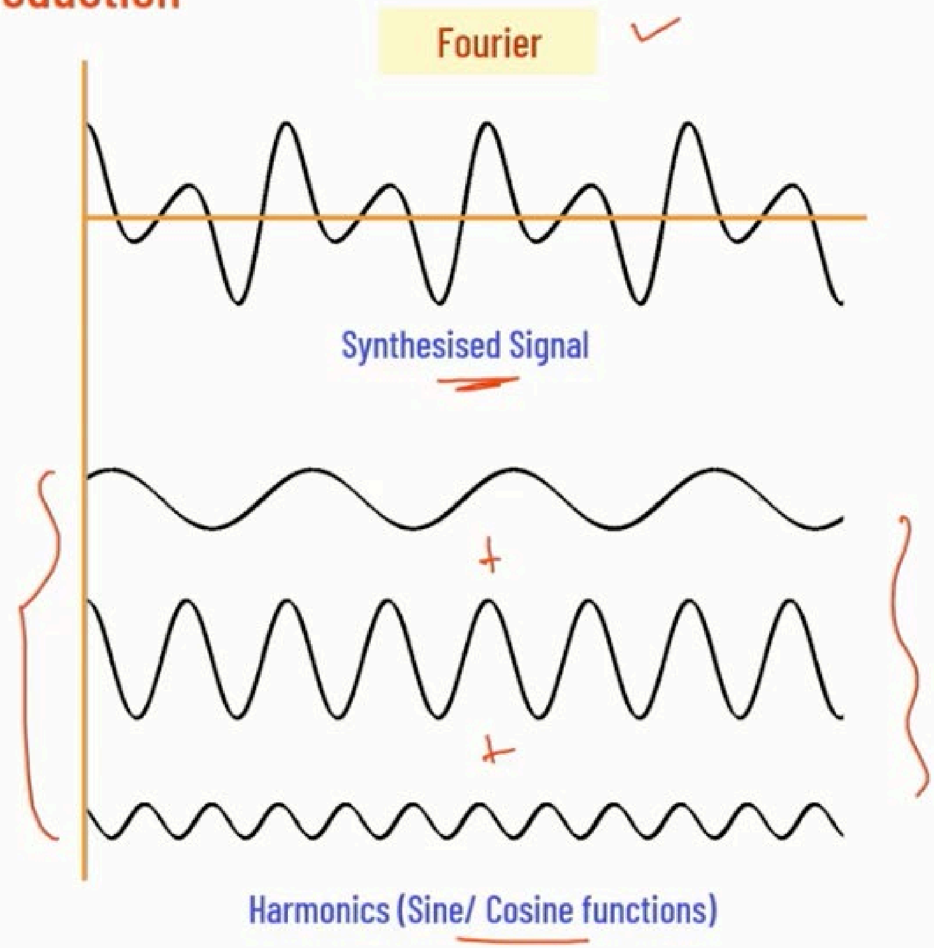
Welch

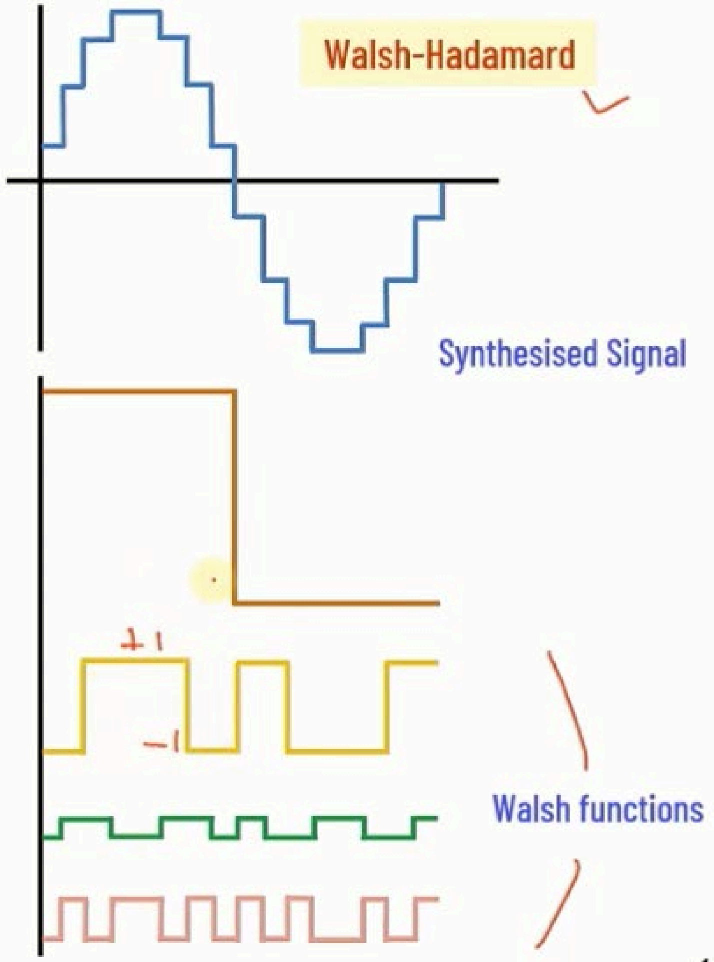
Sequency Ordering

32x32

$$W_{4x4} = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

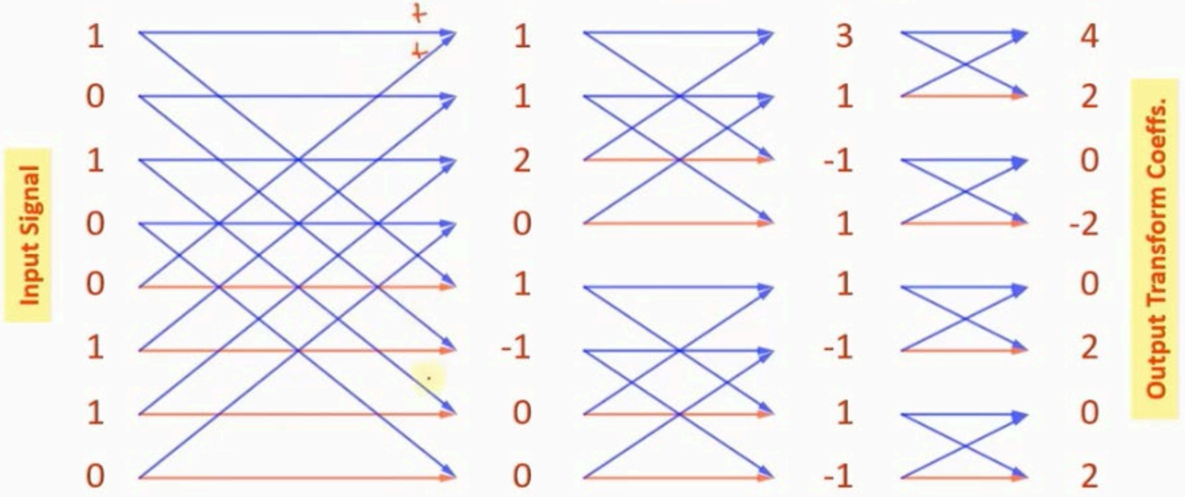
Introduction





Fast WHT

- Similar to the Cooley-Tukey algorithm (FFT), Fast Walsh-Hadamard Transform is developed with complexity $O(N \log N)$. This uses only "**Addition and Subtraction and No Multiplication**".



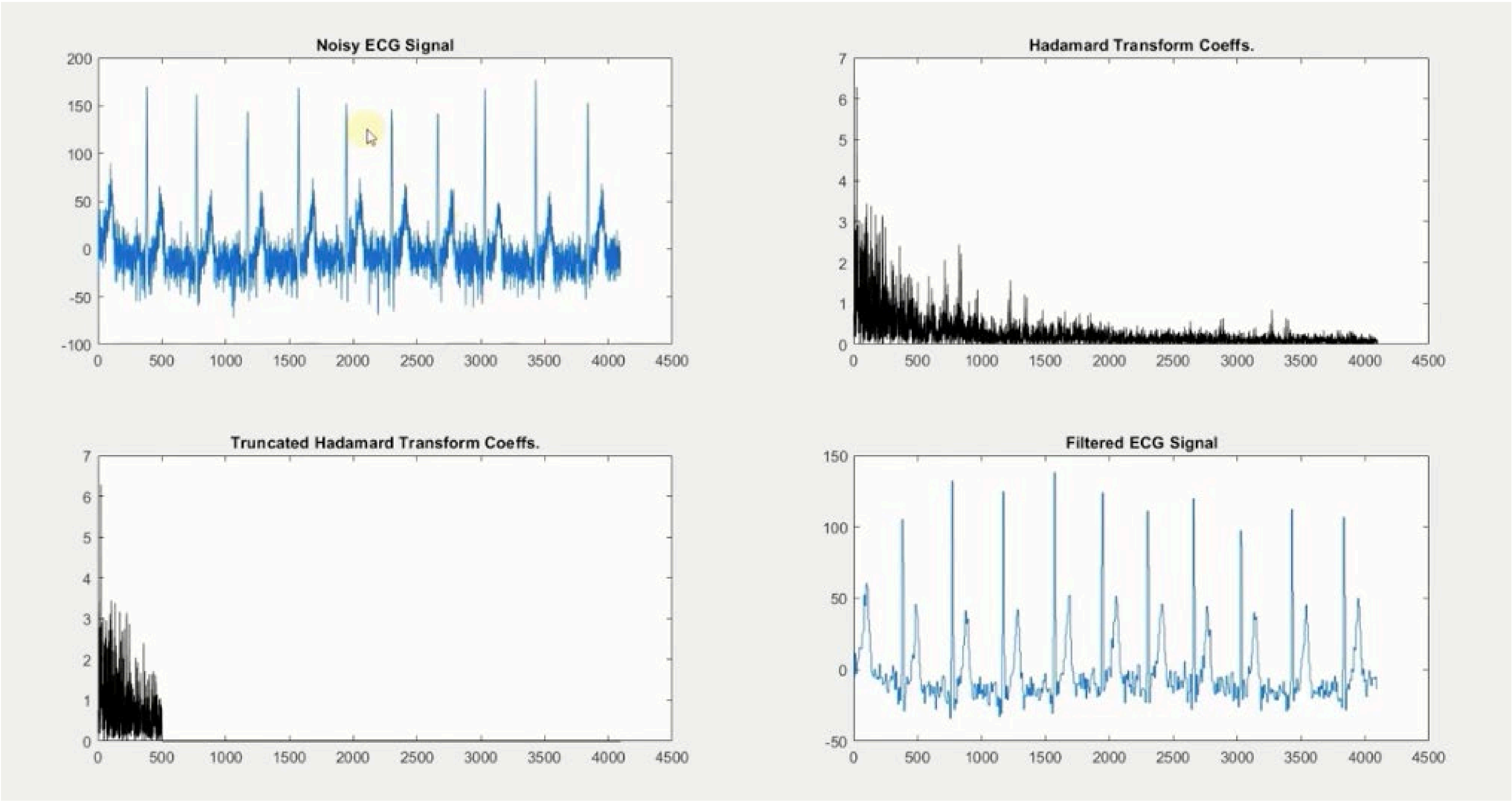
BLUE IS PLUS
RED IS MINUS

MATLAB Code to implement Signal Filtering using FWHT

```
%Program for 1D Hadamard Transform
load ecg; %Loading ECG signal

n=log(length(x))/log(2); %Adjusting length to make it power of 2
n=floor(n);
x=x(1:2^n);

x = x + 0.1.*randn(1,length(x));  % Adding noise
y = fwht(x);  %Walsh-Hadamard Transform (WHT)
yo=y; %Backup
y(500:end) = 0;  %Removing the higher coefficients
xr = ifwht(y);   %Signal reconstruction using inverse WHT
```



Noisy ECG Signal
200
150
100
50
0
-50
-100
0 500 1000 1500 2000 2500 3000 3500 4000 4500
Hadamard Transform Coeffs.
7
6
5
4
3
2
1
0
0 500 1000 1500 2000 2500 3000 3500 4000 4500
Truncated Hadamard Transform Coeffs.
7
6
5
4
3
2
1
0
0 500 1000 1500 2000 2500 3000 3500 4000 4500
Filtered ECG Signal
150
100
50
0
-50
0 500 1000 1500 2000 2500 3000 3500 4000 4500

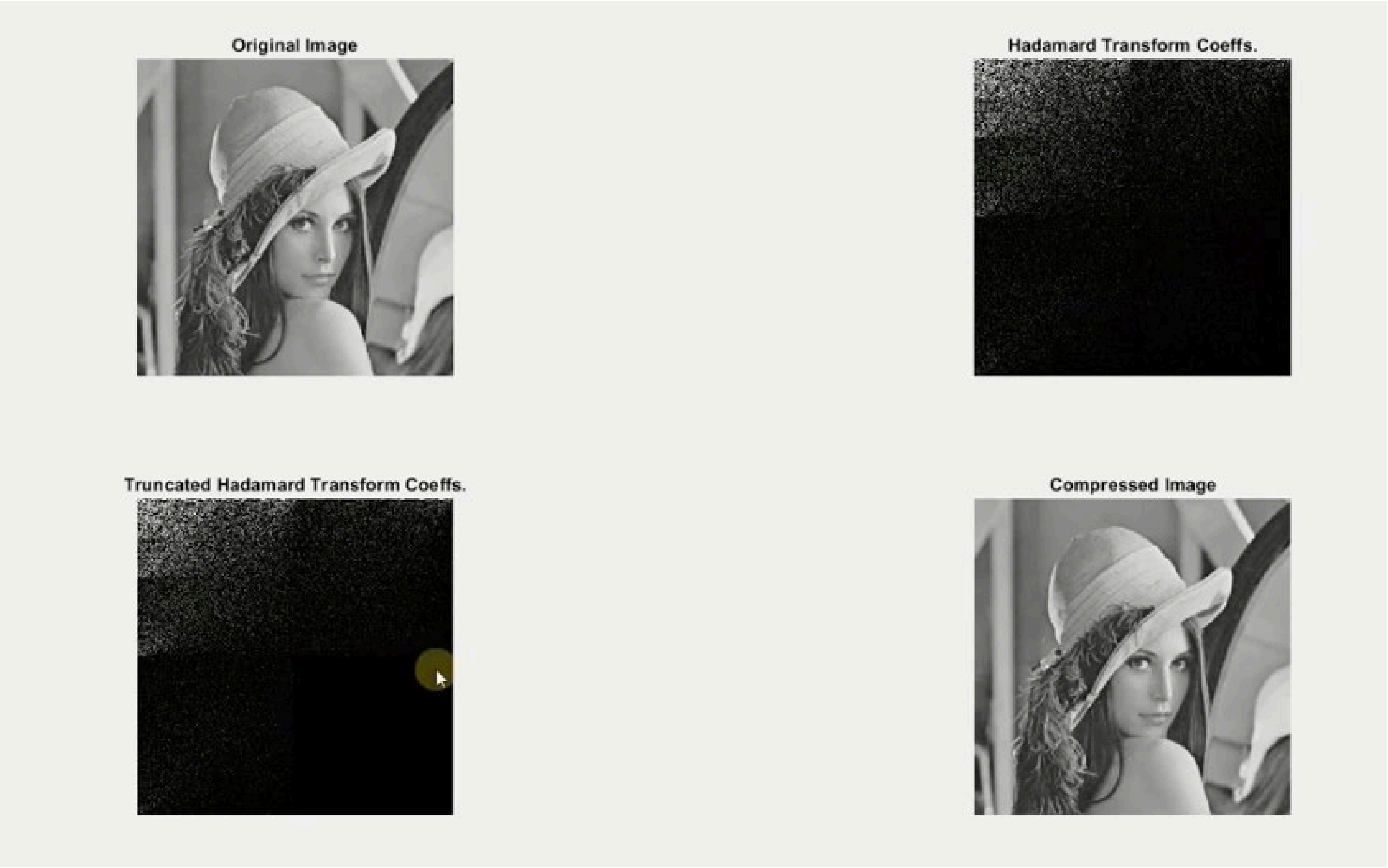
MATLAB Code to implement Image Compression using FWHT

Energy compaction

```
%Program for 2D Hadamard Transform
[filename,pathname] = uigetfile('*.*','Select grayscale Image');
filewithpath=strcat(pathname,filename);
img = imread(filewithpath);
[r,c]=size(img); %Getting image size
imgg=double(img);

%Forward WHT
yc=fwht(imgg); %Column wise operation
yr=fwht(yc'); %Row wise operation
y=yr’; %WHT Coeffs.
yo=y; %Coeff. Backup

y(256:r,256:c)=0; %Truncating WHT Coeffs.
```


Original Image
Hadamard Transform Coeffs.
Truncated Hadamard Transform Coeffs.
Compressed Image

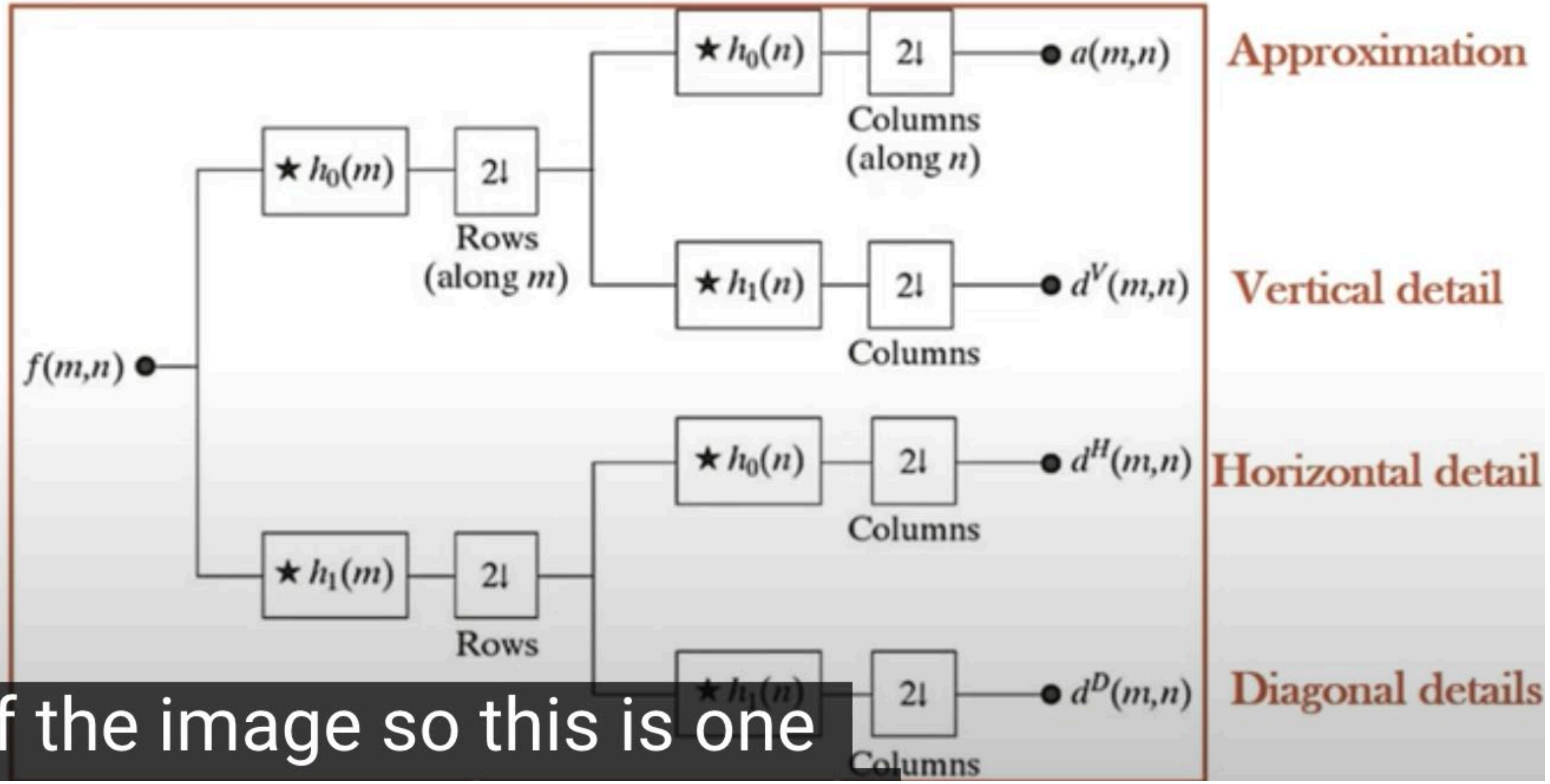
PSNR.

Tau- time/shift
Scale – frequency
Large S = stretched wavelet (slow changes)
Small S = compressed (fine details)

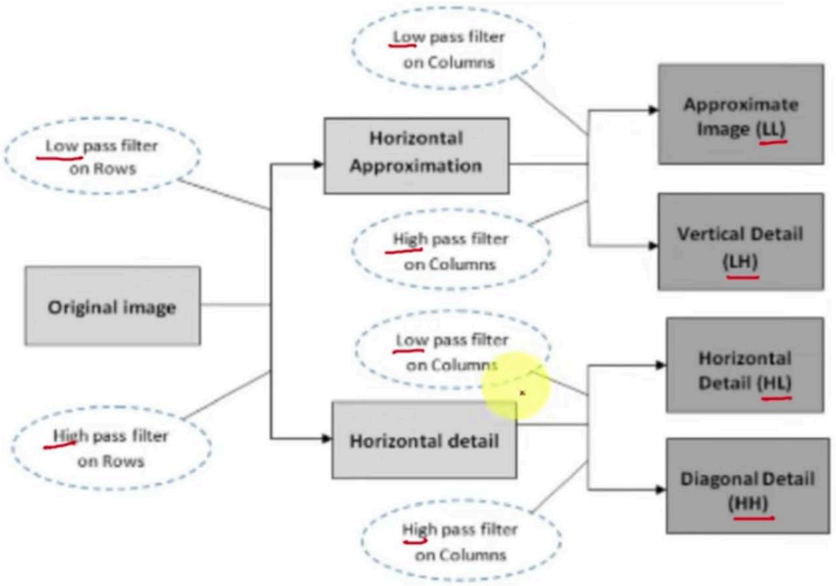
$$F(w) = \int_{-\infty}^{\infty} dt \cdot f(t) \cdot e^{-iwt}$$

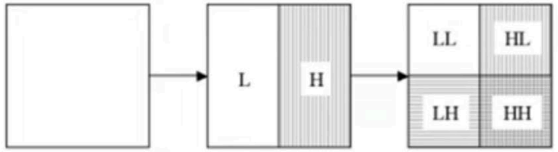
$$CWT(\tau, s) = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} dt \cdot x(t) \cdot \psi\left(\frac{t-\tau}{s}\right)$$

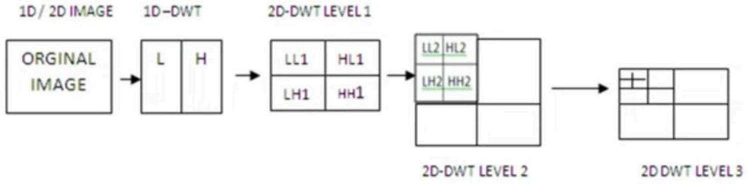
is

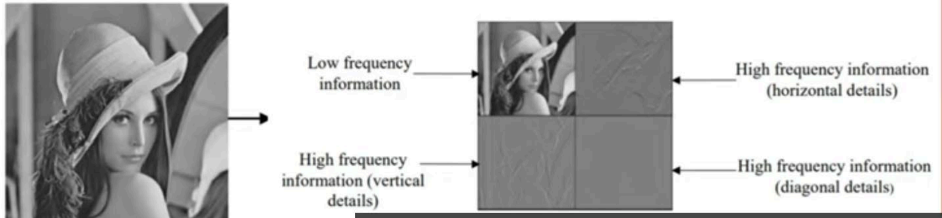


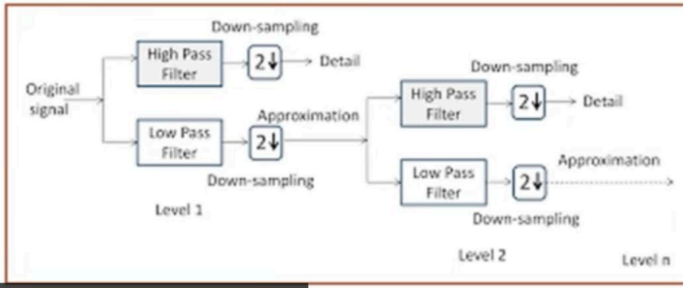
of the image so this is one



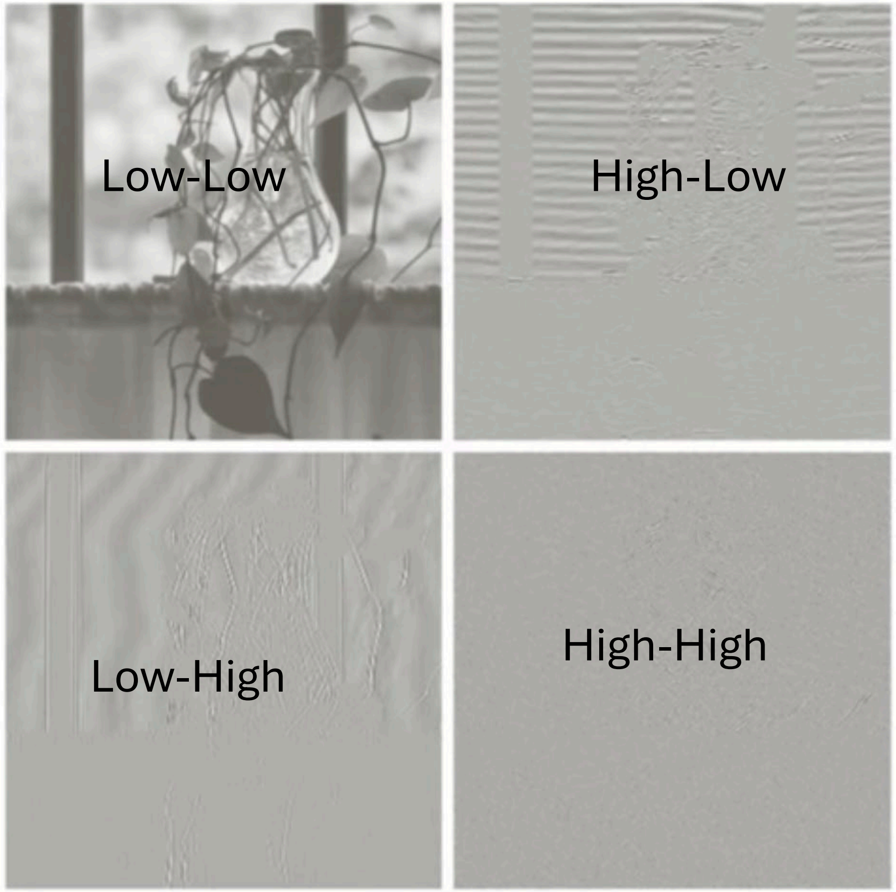








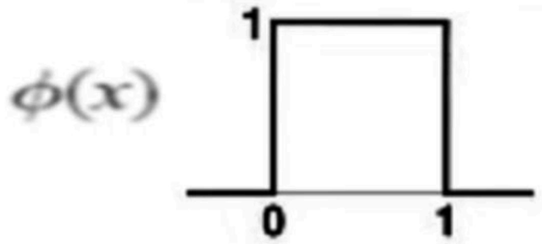
to obtain LL and LH and to the horizontal detail



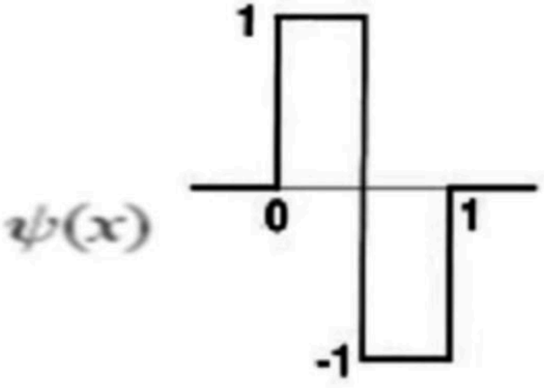
Low-Low
High-Low
Low-High
High-High

HWT- Haar wavelet transform
FHT- Fast Haar transform

- Haar basis functions are the simplest known orthonormal wavelets with matrices having elements either 1, -1 or 0 multiplied by powers of $\sqrt{2}$. Haar transform is computationally efficient, as transform of N size matrix requires only $2(N-1)$ additions and N multiplications. In mathematics, the Haar wavelet is a sequence of rescaled "square-shaped" functions which together form a wavelet family or basis.
- Haar wavelet transform divides the information of an image into approximation and detail sub signals. The approximation sub signal shows the general trend of pixel values and other three detail sub signals show the vertical, horizontal and diagonal details or changes in the images.
- If these details are very small (threshold) then they can be set to zero and greater the number of zeros the greater is the compression ratio.
- Haar transform relies on averaging and differentiating values in an image matrix to produce a matrix which is sparse or nearly sparse. A sparse matrix is a matrix in which a large portion of its entries are 0. A sparse matrix can be stored in an efficient manner, leading to smaller file sizes.



scaling function



wavelet

$$\psi(x) := \begin{cases} 1 & \text{for } 0 \le x < 1/2 \\ -1 & \text{for } 1/2 \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\phi(x) := \begin{cases} 1 & \text{for } 0 \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$H' = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

T=H*F*H’

$$\begin{bmatrix} \phi_x & \psi(x) \\ \phi'_x & \psi'(x) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

❖ Steps to generate Haar Basis matrix for N = 2

Example N=2

k	p	q
0	0	0
1	0	1

Answer

$$H_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{array}{c|cc} & z=0 & z=\frac{1}{2} \\ \hline k=0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ k=1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{array}$$

1. Determine the order of N of the Haar basis (N must be some power of 2). Here order $N=2$
2. Find total bits required n where $n = \log_2 N$. $\therefore n = \log_2 2 = 1$
3. Find p and q : (i) $0 \le p \le n-1$ $\therefore 0 \le p \le 1-1 \iff p=0$
   ✓(ii) If p = 0 then q =0 or q=1
   ✗(iii) If p ≠ 0, $1 \le q \le 2^p$
4. Find k, which is total rows $k = 2^P + q - 1$
   for $p=0, q=0, k=2^0+0-1 = 1-1 = 0$
   for $p=0, q=1, k=2^0+1-1 = 2-1 = 1$
   $\therefore k=0, k=1$
5. Find z, $z = \frac{0}{N}, \frac{1}{N}, \ldots, \frac{N-1}{N} \iff z = 0$ to $\frac{N-1}{N} \Rightarrow z=0$ to $\frac{2-1}{2}$
   $\therefore z \in [0, \frac{1}{2}]$
6. Haar basis functions $h_k(z)$ are the rows of H (k is the k^{th} row)

If $k=0$, $h_0(z) = \frac{1}{\sqrt{N}}$ for all z

(i) $h_0(z) = h_{00}(z) = \frac{1}{\sqrt{N}}$ $\therefore h_0(z) = \frac{1}{\sqrt{2}}$ for $z \in [0, \frac{1}{2}]$ ✓

(ii) If $k=1$,

$$h_k(z) = h_{pq}(z) = \frac{1}{\sqrt{N}}\begin{cases} 2^{p/2} & (q-1)/2^p \le z < (q-0.5)/2^p \\ -2^{p/2} & (q-0.5)/2^p \le z < q/2^p \\ 0 & \text{otherwise } z \in [0,1] \end{cases}$$

when $k=1$, it is for $p=0, q=1$

then, $\frac{(q-1)}{2^p} \le z < \frac{(q-0.5)}{2^p} \iff \frac{1-1}{2^0} \le z < \frac{1-0.5}{2^0}$

$\Rightarrow \frac{0}{1} \le z < \frac{0.5}{1}$

$\Rightarrow 0 \le z < 0.5$

Also, $\frac{q-0.5}{2^p} \le z < \frac{q}{2^p} \iff \frac{1-0.5}{2^0} \le z < \frac{1}{2^0}$

$\Rightarrow \frac{0.5}{1} \le z < \frac{1}{1}$

$\Rightarrow 0.5 \le z < 1$

Hence, $h_k(z) = h_{pq}(z) = h_{01}(z) = \frac{1}{\sqrt{N}}\begin{cases} 2^{0/2}, & 0 \le z < 0.5 \\ -2^{0/2}, & 0.5 \le z < 1 \\ 0, & \text{otherwise} \end{cases}$

$\therefore$ for $z \in [0, \frac{1}{2}]$

$h_1(0) = \frac{1}{\sqrt{N}}[2^0] = \frac{1}{\sqrt{2}}[1] = \frac{1}{\sqrt{2}}$ ✓

$h_1(\frac{1}{2}) = \frac{1}{\sqrt{N}}[-2^0] = \frac{1}{\sqrt{2}}[-1] = \frac{-1}{\sqrt{2}}$ ✓

If $k=0$, $h_0(z) = \frac{1}{\sqrt{N}}$

(i) $h_0(z) = h_{00}(z) = \frac{1}{}$

❖ Steps to generate Haar Basis matrix for N = 4

1. Determine the order of N of the Haar basis (N must be some power of 2). $N = 4 \therefore n = \log_2 4 = \log_2 2^2 = 2$
2. Find total bits required n where $n = \log_2 N$.
3. Find p and q : (i) $0 \le p \le n-1$ $\therefore 0 \le p \le 2-1$ $\therefore p = 0, 1$

   (ii) If $p = 0$ then $q = 0$ or $q = 1$

   If $p = 1$, $1 \le q \le 2^1$ $\Rightarrow q = 1, 2$ (iii) If $p \ne 0$, $1 \le q \le 2^p$
4. Find k, which is total rows $k = 2^P + q - 1$

   If $p=0, q=0, k = 2^0 + 0 - 1 = 0$

   If $p=0, q=1, k = 2^0 + 1 - 1 = 1$

   If $p=1, q=1, k = 2^1 + 1 - 1 = 2$

   If $p=1, q=2, k = 2^1 + 2 - 1 = 3$

5. Find z, $z = \frac{0}{N}, \frac{1}{N}, \ldots, \frac{N-1}{N}$

   $\therefore z \in \left[0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}\right]$

6. Haar basis functions $h_k(z)$ are the rows of H (k is the k^{th} row)

(i) If $k=0$, $h_0(z) = h_{00}(z) = \frac{1}{\sqrt{N}}$ i.e. $h(z) = \frac{1}{\sqrt{4}} = \frac{1}{2}$ for $k=0$

$$h_k(z) = h_{pq}(z) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2} & (q-1)/2^p \le z < (q-0.5)/2^p \\ -2^{p/2} & (q-0.5)/2^p \le z < q/2^p \\ 0 & \text{otherwise } z \in [0,1] \end{cases}$$

	$z = 0$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$
$k = 0$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
2	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0	0
3	0	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$

Example N=4

k	p	q
0	0	0
1	0	1
2	1	1
3	1	2

✓ Answer

$$H_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

(ii) If $k=1$ and $p=0, q=1$

$$h_{01}(z) = \frac{1}{\sqrt{4}} \begin{bmatrix} 2^{0/2}; & \frac{1-1}{2^0} \le z < \frac{1-0.5}{2^0} \\ -2^{0/2}; & \frac{1-0.5}{2^0} \le z < \frac{1}{2^0} \\ 0, & \text{else} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1; & 0 \le z < 0.5 \\ -1; & 0.5 \le z < 1 \\ 0; & \text{else} \end{bmatrix}$$

$\therefore h_1(0) = \frac{1}{2}, h_1\left(\frac{1}{4}\right) = \frac{1}{2}, h_1\left(\frac{2}{4}\right) = -\frac{1}{2}, h_1\left(\frac{3}{4}\right) = -\frac{1}{2}$

(iii) If $k=2, p=1, q=1$

$$h_2(z) = h_{11}(z) = \frac{1}{\sqrt{4}} \begin{bmatrix} 2^{\frac{1}{2}}; & \frac{1-1}{2^1} \le z < \frac{1-0.5}{2^1} \\ -2^{\frac{1}{2}}; & \frac{1-0.5}{2^1} \le z < \frac{1}{2^1} \\ 0; & \text{else} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sqrt{2}; & 0 \le z < \frac{1}{4} \\ -\sqrt{2}; & \frac{1}{4} \le z < \frac{1}{2} \\ 0; & \text{else} \end{bmatrix}$$

$\therefore h_2(0) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}, h_2\left(\frac{1}{4}\right) = -\frac{\sqrt{2}}{2} = \frac{-1}{\sqrt{2}}, h_2\left(\frac{2}{4}\right) = 0, h_2\left(\frac{3}{4}\right) = 0$

(iv) If $k=3, p=1, q=2$

$$h_3(z) = \frac{1}{2} \begin{bmatrix} 2^{\frac{1}{2}}; & \frac{2-1}{2^1} \le z < \frac{2-0.5}{2^1} \\ -2^{\frac{1}{2}}; & \frac{2-0.5}{2^1} \le z < 2/2^1 \\ 0; & \text{else} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sqrt{2}; & \frac{1}{2} \le z < \frac{3}{4} \\ -\sqrt{2}; & \frac{3}{4} \le z < 1 \\ 0, & \text{else} \end{bmatrix}$$

$\therefore h_3(0) = 0, h_3\left(\frac{1}{4}\right) = 0, h_3\left(\frac{2}{4}\right) = \frac{\sqrt{2}}{2}, h_3\left(\frac{3}{4}\right) = -\frac{1}{2}$

Haar transform

Advantages:

- Low computing requirements
- wavelet-like structure (predates formal wavelet models)

Uses:

- image processing
- patten recognition

Real and orthogonal basis
Separable and symmetric
Basis are sequenctially orders

Poor energy compaction (bad for compression)

Haar Kernel

1) Find the order N;
2) n=log2(N)
2) Determine the value of p and q

    p ranges 0 to n-1

    if p ==0, then q =0 or 1

    else, 1<=q<=2^p

3) Determine value of k

    k=2^p+q-1

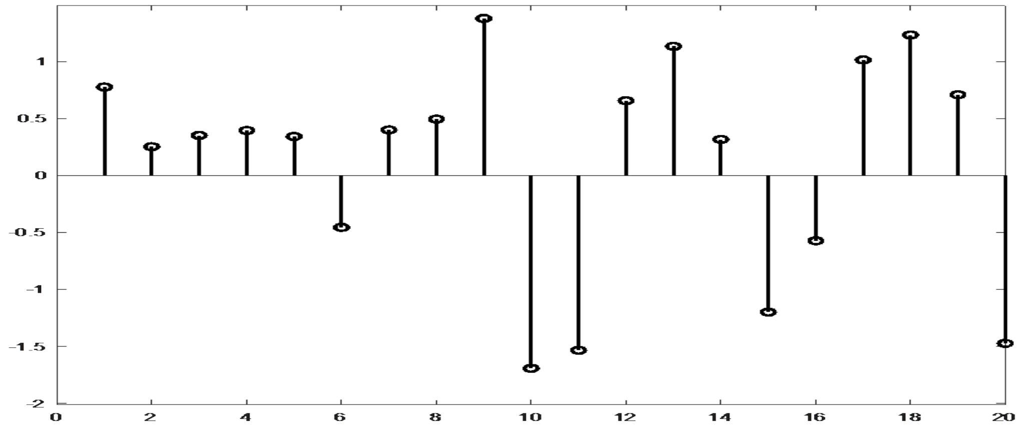
    nz = 0/z,1/z,2/z, ... (N-1)/z

4) If k==0

Decimator

Take every M^{th} sample

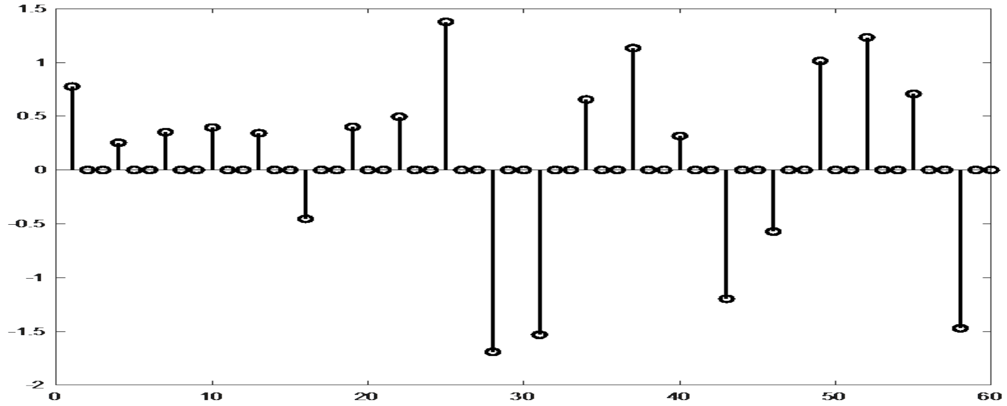
$M\downarrow$

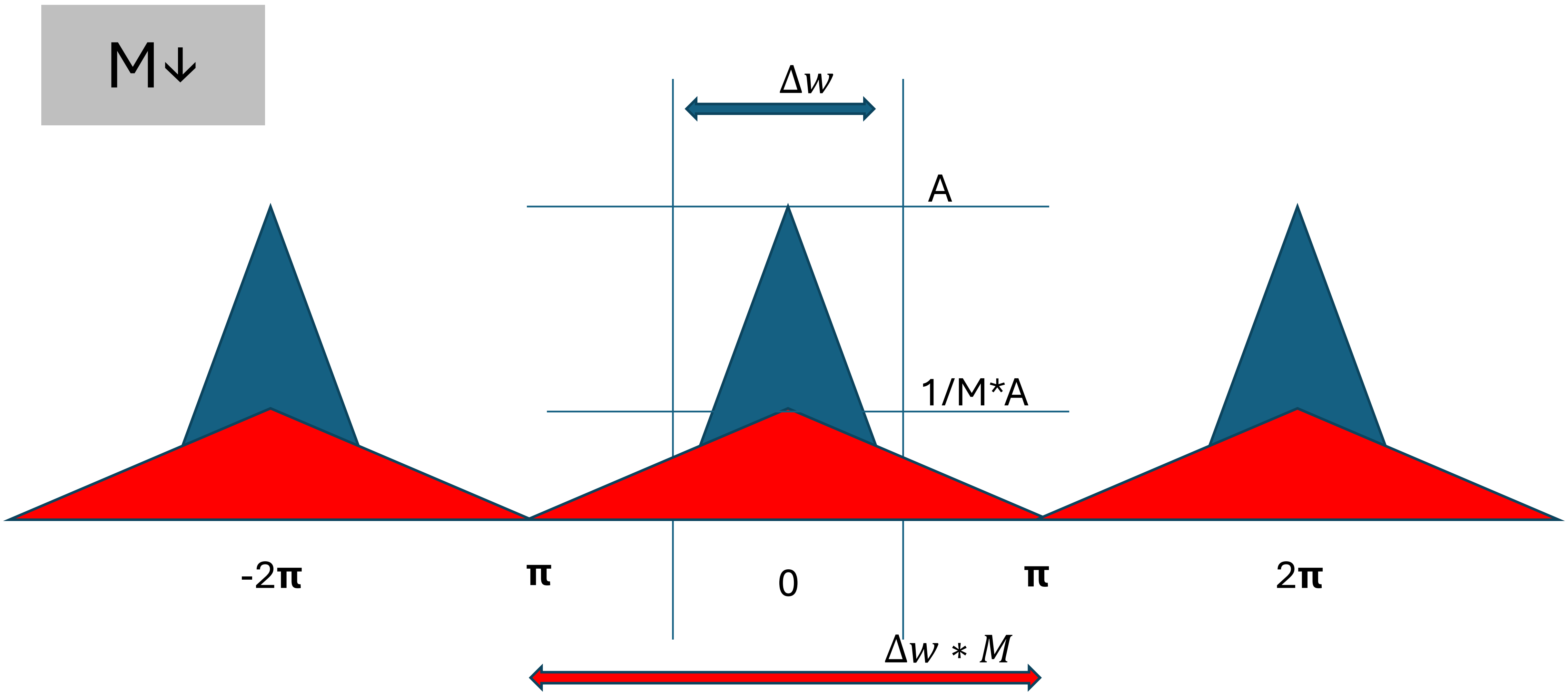


Interpolater

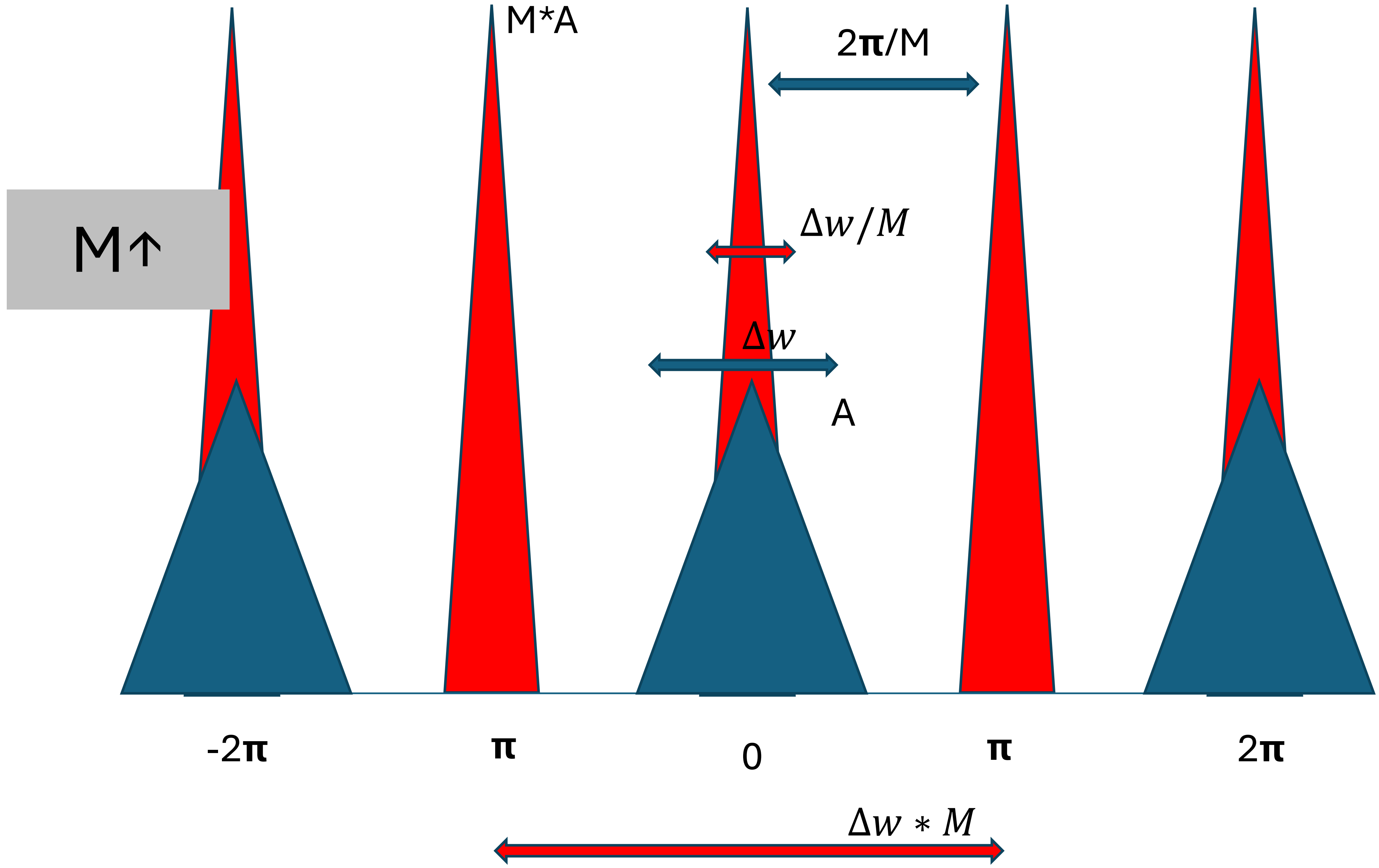
Expand by M samples

$M\uparrow$





M↓
Δw
A
1/M*A
-2π
π
0
π
2π
Δw * M

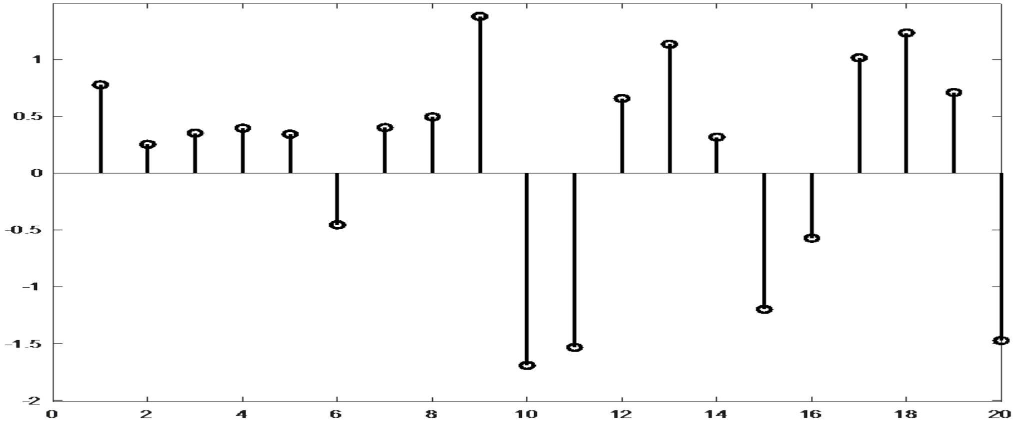


M*A
2π/M
M↑
Δw/M
Δw
A
-2π
π
0
π
2π
Δw * M

Decimator

Take even Mth sample

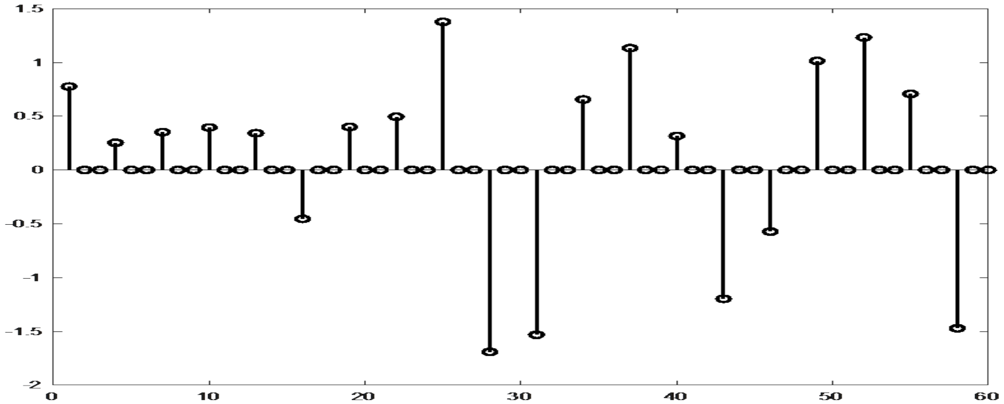
M↓



Interpolator

Expand by M samples

M↑

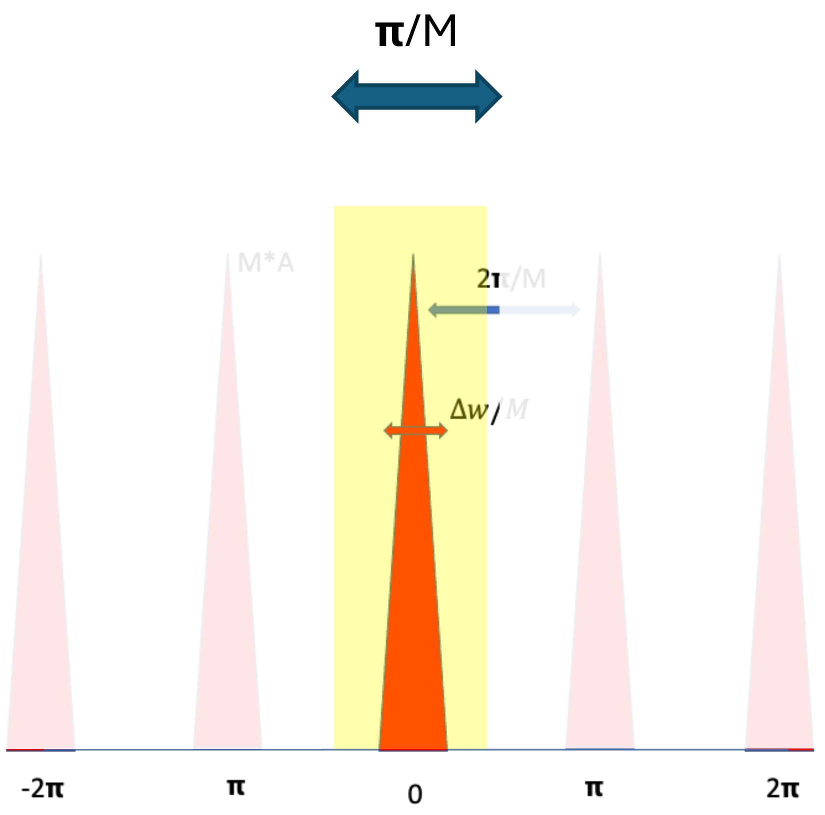


- LPF with cutoff at **π**/M

LPF → M↓

- LPF with cutoff at **π**/M

M↑ → LPF



Practical limit if M and L need to large to get the ratio

- What about non-integers? $\tau = \frac{M}{L}$

$$L\uparrow \rightarrow \overbrace{\text{LPF} \rightarrow \text{LPF}}^{\substack{\text{Combine to single LPF}\\ \text{At minimum } \boldsymbol{\pi}/L \text{ or } \boldsymbol{\pi}/M\\ \text{And gain of } M/L}} \rightarrow M\downarrow$$

- Multirate processing
  - More efficient

Lemmas

1)

$$x[n] \to [M \downarrow] \to x_a[n] \to H(z) \to y[n]$$

Equivalent to

$$x[n] \to H(z^M) \to x_b[n] \to [M \downarrow] \to y[n]$$

$$H(z^M) = \sum_{n=0}^{\infty} h[n] z^{-Mn}$$

Same as running that filter on an expanded signal

- Multirate processing
    - More efficient

Lemmas

1)

$$x[n] \rightarrow [M \downarrow] \rightarrow x_a[n] \rightarrow H(z) \rightarrow y[n]$$

Equivalent to

$$x[n] \rightarrow H(z^M) \rightarrow x_b[n] \rightarrow [M \downarrow] \rightarrow y[n]$$

$$H(z^M) = \sum_{n=0}^{\infty} h[n] z^{-Mn}$$

Proof:

$$X(w) \rightarrow \frac{1}{M} \cdot X\left(\frac{w}{M}\right) \rightarrow \frac{1}{M} \cdot X\left(\frac{w}{M}\right) \cdot H(w) \rightarrow Y(w)$$

$$X(w) \cdot H(wM) \rightarrow \frac{1}{M} \cdot X\left(\frac{w}{M}\right) \cdot H(w) \rightarrow Y(w)$$

- Multirate processing
  - More efficient

Lemmas

2) $$x[n] \rightarrow H(z) \rightarrow x_a[n] \rightarrow [M \uparrow] \rightarrow y[n]$$

This one is more computationally efficient

Equivalent to

$$x[n] \rightarrow [M \uparrow] \rightarrow x_b[n] \rightarrow H(z^M) \rightarrow y[n]$$

Proof:

$$X(w)H(w) \rightarrow L \cdot X(wL) \cdot H(wL) \rightarrow Y(w)$$

$$X(w) \rightarrow L \cdot X(wL) \rightarrow L \cdot X(wL) \cdot H(wL) \rightarrow Y(w)$$

Wavelet transform

Shifted and scaled copies of the mother wavelet
→ multiresolution