Academic Integrity Statement

This work is subject to the Swanson School of Engineering Academic Integrity Policy. For this assignment, you may consult with other students in the course to discuss solution strategies and seek help, but do not present or submit work that is not your own. You may use online references and resources, and course notes and materials provided by the instructors. Do not obtain solutions from others. *If you cannot explain all the steps required to solve the assignment the following day without seeing the solution, you need to reconsider your methods.*

Submission

Please submit the written part of the assignment as a PDF document (`hw1_lastName.pdf`) via Canvas. Any Julia code must be uploaded to canvas as a JupyterLab notebook with the following filename: `hw1_code_lastName.ipnyb`. Use the markdown language liberally throughout your notebook to better explain your code.

Clearly mark your answers for each sub-problem. Please put a box around your answer and label it with the sub-problem, e.g "2B".

Homework 1 (50 pts total)

Problems:

Problem 1 (20 pts): Effects of roundoff and truncation errors on numerical accuracy

The one-sided finite difference scheme to approximate the first derivative of a function f is defined as:

$$f_h(x) = \frac{f(x+h) - f(x)}{h} \approx f'(x), \tag{1}$$

where h is the step size if no roundoff error exists, then the accuracy of the scheme is determined solely by the **truncation error**

$$t_e(h) = |f'(x) - f_h(x)| = \mathcal{O}(h), \quad f'(x) = f_h(x) + \mathcal{O}(h), \tag{2}$$

where the big $\mathcal{O}-$ notation means that there exists a constant K such that $\mathcal{O}(h) < K \cdot h$ for all h.

In the presence of roundoff errors, x can not be represented exactly; instead, it is represented by the rounded value $\tilde{x}$ with the associated roundoff error $r = |\tilde{x} - x|$.

Your tasks:

A. (6 pt.) Show that the total error of the finite-difference approximation consists of both truncation error $t(h)$ due to the finite-difference scheme and the roundoff error r:

$$\epsilon(h) := |f'(x) - f_h(\tilde{x})| = O(h) + \frac{r}{h} \tag{3}$$

Hint: Start from $f'(x) = f_h(x) + \mathcal{O}(h)$, and consider that the computation of the finite difference $f_h(x)$ approximation already involves roundoff errors, e.g. $f(x) = \tilde{f}(x) + r$, $f(x+h) = \tilde{f}(x) + h + r$.

B. (4 pt.) From part A, what can you say about the numerical accuracy of the finite difference scheme as the step size h is continually decreased? .

C. (5 pt.) The second order central difference approximation is defined as

$$f_h^c(x) = \frac{f(x+h - f(x-h))}{2h} \approx f'(x) \tag{4}$$

and has a truncation error of $t(h) = |f'(x) - f_h^c(x)| = \mathcal{O}(h^2)$.

Within the notebook "Week2_FD_students" available on Canvas Module 1, add a function to evaluate the finite difference approximation f_h^c for the derivative of $f(x) = \sin(x)$ with step size h at a fixed $x = x_0 = 1$. For the same array h of step sizes as in the notebook, calculate the array consisting of errors between the finite difference approximataion f_h^c and the exact derivative f'. Plot the approximation error for the central differences scheme as a function of the step size h. Display your plot in log format in addition to the previous plot for forward differences.

D. (5 pt.) Now repeat the same above steps for the function $g(x) = \sin(100x)$ and its derivative $g'(x)$. Add your plots for the FD and CD numerical errors to the same plot. Make sure to use a different linestyle, and specify your legend entries. Looking at your plot, what can you say about the accuracy of each method to approximate the derivative for $\sin(x)$ and $\sin(100x)$ at $x_0 = 1$? Can you explain why? .

Problem 2 (20 pts): Errors in Scientific Computing

The sine function is given by the infinite series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

A. (10 pts) Calculate the forward and backward errors if we approximate the sine function by using only the first term in the series for $x = 0.1, 0.5, 1.0$

B. (10 pts) Calculate the forward and backward errors if we approximate the sine function by using only the first two terms in the series for $x = 0.1, 0.5, 1.0$

Problem 3 (20 pts): Condition Number & Stability

This is Exercise 1 from Section 1.4, page 26 of the textbook (Exercises are also available at the end of each section of a chapter in the online textbook) Refer to your textbook for cross-referenced equations and Tables.

Consider the formulas

$$f(x) = \frac{1-\cos x}{\sin x}, \qquad g(x) = \frac{2\sin^2 x/2}{\sin x},$$

which are mathematically equivalent, but they suggest evaluation algorithms that can behave quite differently in floating point arithmetic.

A. (6 pt.) Using (1.2.6), find the relative condition number of f. Note that because f & g are mathematically equivalent, their condition numbers are the same. Show that the condition number approaches to 1 as $x \to 0$, which means it is possible to compute the function accurately near zero.

B. (6 pt.) (Computer exercise) Compute $f(10^{-6})$ using a sequence of four elementary operations. Using Table 1.1 on page 13, make a table like the one shown in Demo 1.4.1 in the book that shows the result of each elementary result and the numerical value of the condition number of that step.

C. (6 pt.) Repeat part B for $g(10^{-6})$, which has six elementary steps

D. (2 pt.) Based on your answers to part B & C, is $f(10^{-6})$ or $g(10^{-6})$ more accurate (provide your answer as Markdown text in your Jupyter notebook)?

Problem 4 (10 pts): Floating Point Arithmetic

(Computer exercise) Create an example calculation to demonstrate that floating point addition is not associative. Repeat the exercise for floating point multiplication.