Homework 2 (40 pts for undergraduates, 60 pts for graduate students)

Academic Integrity Statement

This work is subject to the Swanson School of Engineering Academic Integrity Policy. For this assignment, you may consult with other students in the course to discuss solution strategies and seek help, but do not present work that is not your own. You may use online references and resources, and course notes and materials provided by the instructors. Don't copy solutions from others. *If you can't explain all the steps required to solve the problem the following day without seeing the solution, you probably need to reconsider your methods.*

Submission Instructions

Please submit the entire assignment as a Jupyterlab notebook via Canvas. Use Markdown to comment on your results and answer the questions in each task.

Problem 1 (20 pts): Analysis of Runge's phenomenon and Chebyshev nodes

Consider the following four functions on the interval $I = [-1, 1]$:

$$f_1(x) = \frac{1}{1 + 10x^2}, \;\; f_2(x) = \frac{1}{1 + 3x^2}, \;\; f_3(x) = \cos(x), \quad f_4(x) = e^x \tag{1}$$

Your tasks:

A (4 pt.) For each of the above functions, calculate an approximation via interpolation with Lagrange polynomials on 21 equidistant nodes within $[-1, 1]$ and plot it along with the exact function. Create a separate plot for each function. Label your plots with different colors and line styles. You may reuse the notebook "Week3_Lagrange" available on Canvas. Using your plots which of the Lagrange approximations to a function is more susceptible to Runge's phenomenon of overshooting?

B (4 pt.) For the function $f_1(x)$, calculate an approximation via interpolation with Lagrange polynomials as well as Chebyshev polynomials using 6 Chebyshev points within $[-1, 1]$, and plot both approximations along with the exact function on the same graph. Note that Lagrange polynomials must use the same points as the Chebyshev polynomials. You may reuse any appropriate weekly notebooks available on Canvas. How do the approximations with Lagrange and Chebyshev polynomials compare to each other? Consider the accuracy of Lagrange interpolation from Part A and the current Lagrange interpolation with using 6 Chebyshev nodes, which one is better? Explain your observations based on function approximation theory.

The following information may be useful for later parts:

The approximation error $e_n(x) = f(x) - p_n(x)$ at a point $x \in [-1, 1]$ for the general polynomial approximation with polynomial $p_n(x)$ via interpolation on $n + 1$ points $x_0, x_1, \ldots, x_n \in [-1, 1]$, not necessarily equidistant, is given by

$$|e_n(x)| = \frac{f^{(n+1)}(\zeta)}{(n+1)!} \prod_{i=0}^{n} (x - x_i) \leq \frac{f^{(n+1)}(\zeta) h^{n+1}}{4(n+1)} \tag{2}$$

where ζ is a point that lies within $[-1, 1]$, $f^{(n)}$ represents the nth the derivative of function f, and $\prod$ is the product symbol. The upper bound formula is based on an equidistant interval of h between data points (see lecture notes).

C (4 pt.) For functions $f_1(x)$ and $f_3(x)$ given in part A, derive the 1st, 2nd, and 3rd derivatives analytically (or use symbolic math software) and plot them in the interval [-1,1]. Now, pick the point $x = -0.9$. What is the trend of derivatives for each function at this point, with increasing order of differentiation? Do the values stay constant, increasing or decreasing trend? Use this information to explain what you observed in part A?

D (8 pt.) Now for $x = 0.9$, calculate the expressions

$$P_L(n) = \prod_{i=0}^{n}(x - x_i^L), \quad P_C(n) = \prod_{i=0}^{n}(x - x_i^C) \tag{3}$$

for $n = 4, 7, 10$, where $x_i^L, i = 0, 1, \ldots, n$ are the $n + 1$ equidistant Lagrange points and $x_i^C, i = 0, 1, \ldots, n$ are the Chebyshev points between $[-1, 1]$. Plot $P_L(n), P_C(n)$ as a function of n running from 4, 7, 10 in the same plot. How do they compare to each other? Based on equation (2), can you explain why the interpolation on Chebyshev nodes is more accurate than interpolation on equidistant points?

For those interested, more detailed explanation for the Runge example can be found in the following reference: *Epperson, James F. "On the Runge example." The American Mathematical Monthly 94.4 (1987): 329-341.*

Problem 2 (20 pts): Analysis of B-Spline convergence with increasing order and knot number (Continuation of in-class exercise)

Let the following two functions be defined on the interval $I = [-3, 3]$:

$$f_1(x) = \sin(x), \quad f_2(x) = \cos(x^2) \tag{4}$$

In the following, knots will be referring to points within the interval I on which the exact function values are known and are used for B-Spline interpolation of the functions given above.

A (4 pt.) Use three equidistant knots within $I = [-3, 3]$ (with distance $d = 3$) to construct the B-Spline interpolated approximation for the function $f_1(x) = \sin(x)$, with increasing B-Spline order $n = 2, 3, 4, 5, 6, 7, 8$. For each B-Spline order $i.e., n = 2, 3, \ldots, 8$, evaluate the exact function and the B-spline approximation to that exact function on an equidistant grid for $[-3, 3]$ with 1001 points ($x_0 = -3, \ldots, x_i, \ldots, x_{1000} = 3$). Calculate the maximal approximation error as $\max(|f_e(x_i) - F_B(x_i)|),\ i = 0, \ldots, 1000$, where f_e is the exact function and F_B is the B-Spline approximation of the exact function. Then plot the error $\epsilon(n)$ as function of order n. In a separate figure, plot the B-Spline approximations with order $n = 2, 3, 4, 5$ along with the exact function $f_1(x)$ to visualize the quality of the interpolated approximations with increasing B-Spline order. Comment on your observations.

B (4 pt.) For a fixed B-Spline order $n = 3$, vary the number of equidistant knots $m = 3, 4, 5, \ldots, 13$ within the interval $I = [-3, 3]$ to construct the B-Spline interpolated approximation for the function $f_1(x) = \sin(x)$. For each of the knots number m used to construct the B-Spline interpolation, evaluate the exact function and the B-spline approximation to that exact function on an equidistant grid for $[-3, 3]$ with 1001 points ($x_0 = -3, \ldots, x_i, \ldots, x_{1000} = 3$). Calculate the maximal approximation error $\epsilon(m)\ \max(|f_e(x_i) - F_B(x_i)|),\ i = 0, \ldots, 1000$, where f_e is the exact function and F_B is the B-Spline approximation of the exact function. Then plot the error $\epsilon(m)$ as function of knots number m. Comment on your observations taking into account your observations from Part A. In terms of decreasing the error, which strategy would you prefer? Which one is more efficient?

C (4 pt.) Repeat part A, but now for the function $f_2(x) = \cos(x^2)$ for B-Spline orders $n = 2, \ldots, 10$ on a fixed set of 3 knots (i.e., $m = 3$) How does the approximation error improve with increasing B-Spline order n? How does the quality of approximation compare to the function $f_1(x)$ used in part A?

D (4 pt.) Repeat part B, but now for the function $f_2(x) = \cos(x^2)$ for equidistant knots number $m = 3, \ldots, 13$. How does the approximation improve with increasing knots number m for the B-Spline with order $n = 3$? How does the quality of approximation compare to function $f_1(x)$?

E (4 pt.) Now, considering your work from part A through D, how does the function to be approximated affect your choice of an interpolating function? Would you always use a high order B-Spline or would you keep the order constant and increase the number of the knots?

Problem 3 (20 pts, graduate students only)

Standard Chebyshev polynomials are defined in the domain $x \in [-1, 1]$. If the problem is defined on a different domain such as $x \in [a, b]$, a transformation for the independent variable is needed.

To use Chebyshev interpolation for a general interval $x \in [a, b]$, we apply the affine transformation that maps $\psi \in [-1, 1]$ onto $[a, b]$ using the following transformation thar shift and scale Chebyshev points

$$x = a\frac{b-a}{2}(t+1), \quad t \in [-1, 1], \tag{5}$$

and we redefine the interpolation abscissae as

$$x_i \leftarrow a + \frac{b-a}{2}(x_i + 1), \quad i = 0, \cdots, n. \tag{6}$$

Let us now consider a different form of the Runge's function

$$f(x) = \frac{1}{1+x^2}. \tag{7}$$

Use the above transformation to interpolate the Runge's function in the interval [-2,5] using Chebyshev polynomials with 6 Chebyshev points. Plot both the exact function and the interpolations on the same graph.