Homework 1

ME 2060: Numerical Methods

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Abstract

Homework discussion, calculations, and answers are provided herein.

Keywords Numerical Methods, Mechanical Engineering

1 Effects of roundoff and truncation errors on numerical accuracy

The one-sided finite difference scheme to approximate the first derivative of a function f is defined as:

$$f_h(x) = \frac{f(x+h) - f(x)}{h} \approx f'(x), \tag{1}$$

where h is the step size if no roundoff error exists, then the accuracy of the scheme is dermined soley by **truncation error**

$$t_e(h) = |f'(x) - f_h(x)| = O(h), \quad f'(x) = f_h(x) + O(h), \tag{2}$$

where the big O - notation means there exists a constant K such that $O < K \cdot h$ for all h. In the presence of roundoff errors, x cannot be represented exactly; instead, it is represented by the rounded value $\tilde{x}$ with the associated roundoff error $r = |\tilde{x} - x|$.

1.1 Part A

Show that the total error of the finite-difference approximation consists of both truncation error $t_e(h)$ due to the finite-difference scheme and the roundoff error r:

$$\epsilon(h) := |f'(x) - f_h(\tilde{x})| = O(h) + \frac{r}{h} \tag{3}$$

Hint: Start from $f'(x) = f_h(x) + O(h)$, and consider that the computation of the finite difference $f_h(x)$ approximation already involves roundoff errors, e.g., $f(x) = \tilde{f}(x) + r$, $f(x+h) = \tilde{f}(x) + h + r$.

Answer 1A
Answer goes here.

1.2 Part B

From Part A, what can you say about the numerical accuracy of the finite difference scheme as the step size h is continually decreased?

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Answer 1B
Answer goes here.

1.3 Part C

The second order central difference approximation is defined as

$$f_h^c(x) = \frac{f(x+h-f(x-h))}{2h} \approx f'(x) \tag{4}$$

and has a truncation error of $t_e(h) = |f'(x) - f_h^c(x)| = O(h^2)$.

Within the notebook `Week2_FD_students.ipynb` available on Canvas Module 1, add a function to evaluate the finite difference approximation f_h^c for the derivative of $f(x) = \sin(x)$ with step size h at a fixed $x = x_0 = 1$. For the same array h of step sizes as in the notebook, calculate the array consisting of errors between the finite difference approximation f_h^c and the exact dervative f'. Plot the approximation error for the central differences scheme as a function of step size h. Display your plot in log format in addition tot he previous plot for forward differences.

Answer 1C
Answer goes here.

1.4 Part D

Now repeat the same above steps for the function $g(x) = \sin(100x)$ and its derivative $g'(x)$. Add your plots for the forward difference and central difference numerical errors to the same plot. Make sure to use a different linestyle, and specify your legend entries. Looking at your plot, what can you say about the accuracy of each method to approximate the derivative for $\sin(x)$ and $\sin(100x)$ at $x_0 = 1$? Can you explain why?

Answer 1D
Answer goes here.

2 Errors in Scientific Computing

The sine function is given by the infinite series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{5}$$

2.1 Part A

Calculate the forward and backward errors if we approximate the sine function by using only the first term in the series for $x = 0.1,\ 0.5,\ 1.0$.

Answer 2A
Answer goes here.

2.2 Part B

Calculate the forward and backward errors if we approximate the sine function by using only the first two terms in the series for $x = 0.1,\ 0.5,\ 1.0$.

Answer 2B
Answer goes here.

3 Condition Number & Stability

This is Exercise 1 from Section 1.4, page 26 of Driscoll and Braun [2018]. Exercises are also available at the end of each section of a chapter in the online textbook. Refer to your textbook for cross-referenced equations and tables.

Consider the formulas

$$f(x) = \frac{1 - \cos x}{\sin x}, \quad g(x) = \frac{2 \sin^2 x/2}{\sin x}, \tag{6}$$

which are mathematically equivalent, but they suggest evaluation alrogirthms that can behave quite differently in floating point arithmetic.

3.1 Part A

Using (1.2.6), find the relative condition number of f. Note that because f & g are mathematically equivalent, their condition numbers are the same. Show that the condition number approaches to 1 as $x \to 0$, which means it is possible to compute the function accurately near zero.

Answer 3A
Answer goes here.

3.2 Part B

Compute $f(10^{-6})$ using a sequence of four elementary operations. Using Table 1.1 on page 13, make a table like the one shown in Demo 1.4.1 in the book that shows the result of each elementary result and the numerical value of the condition number of that step.

Answer 3B
Answer goes here.

3.3 Part C

Repeat Part B for $g(10^{-6})$, which has six elementary steps.

Answer 3C
Answer goes here.

3.4 Part D

Based on your answers to Part B & C, is $f(10^{-6})$ or $g(10^{-6})$ more accurate (provide your answer as Markdown text in your Jupyter notebook)?

Answer 3D
Answer goes here.

4 Floating Point Arithmetic

Create an example calculation to demonstrate that floating point addition is not associative. Repeat the exercise for floating point multiplication.

Answer 4
Answer goes here.

Original article

This article is available online at the following URL: https:
www.google.com

References

T. A. Driscoll and R. J. Braun. *Fundamentals of Numerical Computation*. 2018.