$$f''(x_0) = \frac{1}{12h^2}\Big[-f(x_0-2h) + 16f(x_0-h) - 30f(x_0) + 16f(x_0+h) - f(x_0+2h)\Big] + \frac{h^4}{90} f^{vi}(x_0)$$

4^{th} order accurate.

$$f'_j + \sum_{k=0}^{2} a_k f_{j+k} = (\underbrace{a_0 + a_1 + a_2}_{0}) f_j$$

$$+ (\overbrace{1 + a_1 h + 2 a_2 h}^{0}) f'_j$$

$$+ \left(\overbrace{a_1 \frac{h^2}{2} + a_2 \frac{(2h)^2}{2}}^{0}\right) f''_j$$

$$+ \left(a_1 \frac{h^3}{6} + a_2 \frac{(2h)^3}{6}\right) f'''_j$$

$$\left.\begin{aligned} &\checkmark \quad a_0 + a_1 + a_2 = 0 \\ &\checkmark \quad a_1 h + 2h\, a_2 = -1 \\ &\checkmark \quad a_1 \frac{h^2}{2} + \frac{2a_2}{h^2} = 0 \end{aligned}\right\} \qquad \begin{aligned} a_1 &= -\frac{2}{h} \\ a_2 &= 1/2h \\ a_0 &= 3/2h \end{aligned}$$

$$f'_j = \frac{-3 f_j + 4 f_{j+1} - f_{j+2}}{2h} + O(h^2)$$

Modified Wave Number Approach

how well a FD formula differentiates a sinusoidal function

pure harmonic function.

$$f(x) = e^{ikx}$$

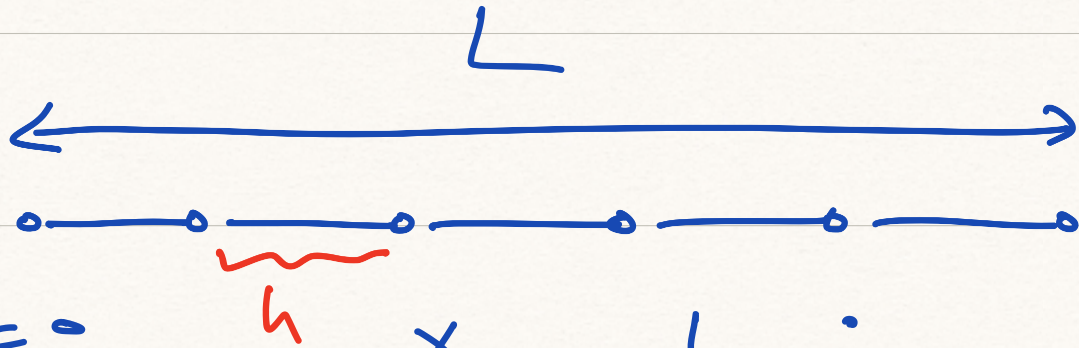
$$= \cos kx + i \sin kx$$

$$f(x) = e^{ikx} \qquad k = \frac{2\pi n}{L}$$

$$n = 0, 1, 2, \ldots N/2$$

exact derivative

$$f'(x) = ik\, e^{ikx} = \boxed{ik f}$$



$j = 0$

$$x_j = \frac{L}{N} j \qquad h = \frac{L}{N}$$

$$j = 0, 1, 2, \ldots N$$

$$\left.\frac{\delta f}{\delta x}\right|_j = \frac{f_{j+1} - f_{j-1}}{2h}$$

2nd order accurate central difference

$$\frac{\delta f}{\delta x} = \frac{e^{i 2\pi n (j+1)/N} - e^{i 2\pi n (j-1)/N}}{2h}$$

$$e^{ikx_j} = e^{i \frac{2\pi n}{\cancel{L}} \left(\frac{\cancel{L}}{N} j\right)} = e^{i \frac{2\pi n}{N} j} \quad f_j$$

$$= \frac{e^{i 2\pi \frac{n}{N} j} e^{i 2\pi \frac{n}{N}} - e^{i 2\pi \frac{n}{N} j} e^{-i 2\pi \frac{n}{N}}}{2h} \quad f_j$$

$$\frac{\delta f}{\delta x} = \frac{e^{i 2\pi n/N} - e^{-i 2\pi n/N}}{2h} f_j$$

$\cos(-x) = \cos x$

$\sin(-x) = -\sin x$

$$\frac{\delta f}{\delta x} = \frac{\cancel{2} i \sin\left(\frac{2\pi n}{N}\right)}{\cancel{2} h} f_j$$

$$\frac{\delta f}{\delta x} = \frac{\cancel{2}\, \underline{i} \sin\left(\frac{2\pi n}{N}\right)}{\cancel{2}h} \underline{\underline{f_j}}$$

$$f_j' = \underline{i}\, \underline{\underline{k}}\, \underline{\underline{f_j}}$$

$$\tilde{k} = \frac{1}{h} \sin\left(\frac{2\pi n}{N}\right)$$ modified wave number