Problem Setup: The objective of this project is to build a parametric reduced order model (ROM) for a transient heat conduction problem in a plate with a cylindrical hole. Please download the MATLAB project associated with this project. Some parts of this MATLAB files are intentionally missing. In Parts 1-5, you need to complete these missing parts. The geometry of the problem is shown in the figure below. The cylinder is centered at $(x_0, y_0) = (0.5, -0.25)$ and its radius is $r_0 = 0.25$. The governing equation is

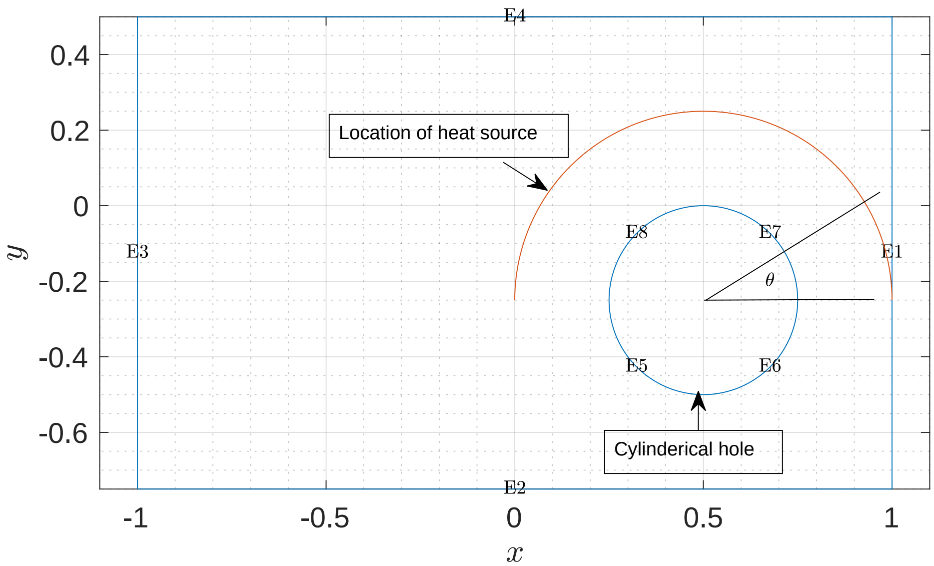

Figure 1: Schematic of the transient heat conduction problem.

given by:

$$\frac{\partial T}{\partial t} = k(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}) + f(x, y) \tag{1}$$

where (x, y) denote the spatial coordinates, T is temperature, t is time and $f(x, y)$ is the heat source, $k = 1$ is the thermal conductivity. The above equation is assumed to be non-dimensionalized. The cylinder is insulated and as a result the boundary condition is $\partial T/\partial n = 0$ at the surface of the cylinder, where n is direction normal to the cylinder surface. The boundary condition at all other edges are $T = 0$. The initial condition is $T = 0$. The heat source is localized and its maximum location is parameterized as in the following:

$$f(x, y) = 10^4 \exp\big(-\frac{(x - x_c)^2 + (y - y_c)^2}{0.05}\big) \tag{2}$$

where (x_c, y_c) are parameters that specify the center of the heat source. In this project (x_c, y_c) can be any where on a circle with the radius $R = 0.5$ with the same center as the hole. Therefore, the angle θ (shown in the figure) parametrizes the location of the heat source as in the following:

$$x_c = R\cos(\theta) + x_0 \tag{3}$$

$$y_c = R\sin(\theta) + y_0 \tag{4}$$

The possible location of heat source is shown by the red line in Figure 1.

Finite Element Solution: For the purpose of this project, the above equations were solved using finite element method (FEM) using MATLAB PDE solver toolbox. The FEM mesh is generated in MATLAB and it is stored in variable `msh`. You can control the size of the mesh by changing the variable `Hmax`. You need to report all of your results with `Hmax=0.05`. However, for debugging purposes you may use a larger value for `Hmax`. The function `SolveFOM` solves the finite element problem and returns the solution snapshots in the interval of $0 \leq t \leq T_f$ at $N_t = 201$ uniformly distributed time snapshots, where $T_f = 0.05$.

1. **Data:** (10 points) Generate a data matrix by solving full order model (FOM) for $Ns = 11$ samples for the following choices of heat sources: $\theta =$`linspace(180,0,Ns)`. Store the resulting data in a matrix named `T`. The size of this matrix must be $N \times (N_t N_s)$, where N is the number of FEM nodes. Plot the last snapshot of temperature for each of the cases.

2. **POD modes:** (25 points) Compute the proper orthogonal decomposition (POD) modes for the data matrix `T`. Plot the first 4 POD modes. Plot the first 50 singular values in a semilogy figure. How many modes are required to capture 99% of the energy? Let that number be denoted by r and use r POD modes in the rest of this project.

3. **ROM:** (25 points) Build the the reduced order model (ROM) low-rank matrix and vector `Kr` and `fr`, where `Kr` $\in \mathbb{R}^{r\times r}$ is the reduced stiffness matrix and `fr` $\in \mathbb{R}^{r\times 1}$ for any given parameter θ. To solve the ROM you must first complete the function `ROM_rhs` and then use `SolveROM`. Check the validity of the ROM by solving ROM for the last sample $\theta = 0$. Compare the FOM and ROM temperature contours at the last snapshot. Also, compare the FOM and ROM temperature profile on the top half of the cylinder at times $t = 0.01, 0.03, 0.05$. Use the function `ExtractCylinderT` to extract temperature on top half of the cylinder surface ($0 \leq \theta \leq 180$). An example of using this function is shown in `main.m`.

4. **Performance of ROM for unseen data:** (25 points) Use the ROM to solve for a new heat source parameter at $\theta = 170$. Note that a heat source at this angle is not included in the training samples. You need to update `fr` according to the new heat source distribution. To this end, you can call the function `[model,FEM_M,FEM_K,FEM_F] = GetFEMMatmodel(xc,yc,model)` to update `FEM_F` by passing the new `xc` and `yc`. Compare the FOM and ROM temperature contours for the last snapshot. Also, compare the FOM and ROM temperature profiles on the top half of the cylinder at times $t = 0.01, 0.03, 0.05$.

5. **Field reconstruction for unseen data:** (15 points) In this case, we assume that we have temperature sensor at the top surface of cylinder ($0 \leq \theta \leq 180$) for a new (unknown) heat source. The objective is to use the measurements of these sensors to reconstruct the temperature field everywhere in the domain. You need to first generate the true temperature field for generating synthetic data for sensors as well for verification purposes. To this end, run the FOM for $\theta = 15$. Use `ExtractCylinderT` function to extract temperature at the top surface of the cylinder. Assume that these are the sensor measurements. Use these measurements along with the POD modes to reconstruct the temperature at all points. Plot the reconstructed temperature field and the true temperature field. Also, plot the contours of error by subtracting the reconstrcunted field from the true field.

Report: Please prepare a report that describes the method and results. There is no requirement on the length of the report. However, the report should be a self-contained document.