Assignment – Week 2

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07/05/2025

Preliminary remarks

You are expected to complete **two tasks for this week**, ideally of **different types** (e.g., one essay, one simulation, one formal derivation) — though this is not a strict requirement. The deadline is **midnight on the 16th of May**. You are strongly encouraged to explore additional tasks — this is an opportunity to deepen your understanding by tackling what most interests or challenges you.

The rationale behind offering many different tasks is twofold: to give you **maximum choice**, and to help you develop a sense of what it means to **examine models carefully**. Whether through formal derivation, coding, or conceptual writing, the aim of these assignments is to stretch yourselves intellectually. Try things, fail at them, and then try again. **Hitting walls is not only normal — it is essential** when working with formal models and doing research more generally. Much of the value lies in grappling with structure, assumptions, and logic — not just in arriving at the "correct" answer.

Please feel free to choose the level of difficulty that suits you best. Some tasks are more involved than others, and it is entirely fine to focus on those you find most manageable or intriguing. **There is absolutely no reason to feel bad about choosing easier options** — this material is tricky, and any serious, good-faith effort is genuinely appreciated.

This is a **formative course** — your marks do not count, so this is also a chance to take risks and venture out of your comfort zone. If time is tight and you cannot get to some of the more challenging exercises, **that is completely fine**. Do what you can, and feel free to keep the sheet for future reference.

The use of **generative AI is actively encouraged**. It can help illuminate reasoning gaps, accelerate coding, or suggest alternative formulations. That said, a word of caution:

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outsourcing your thinking too early or too often can undermine your learning. The long-run payoff comes from engaging deeply first, and only then consulting AI to refine or test your ideas. I will not police the use of AI — the responsibility for learning is ultimately yours.

Submission instructions: Please email your completed tasks to me by the deadline. Ideally, submit your work as a single **integrated document** — such as an `R Markdown` or `Jupyter Notebook` file - combining code, output, and explanatory text. This format is especially useful for simulations and makes your reasoning easier to follow. You may also submit essays in Word, LaTeX, or plain Markdown. Although zipped folders are acceptable (for example, if you submit multiple files or raw code), they are **not encouraged**.

Sunk costs and the loyalty-competence trade-off

This assignment builds on the model of sunk-cost induced party loyalty discussed in lecture. You will extend the baseline model in two directions:

1. Uncertainty over outside options.
2. Competence-contingent promotion probabilities.

Each part includes both formal derivations and interpretive components. You are strongly encouraged to include graphs or simple simulations to support your analysis.

Part A – Joining Under Uncertain Outside Options

Objective: Analyse how uncertainty in outside options affects the decision to join a party under sunk cost constraints.

Agents face uncertainty regarding their outside options. Each agent has a type $\alpha \in \{\alpha_L, \alpha_H\}$, denoting low or high competence, which is privately known to the agent. The outside option w is drawn from a uniform distribution:

$$w \sim U[\underline{w}(\alpha), \bar{w}(\alpha)]$$

Assume that:

$$\bar{w}(\alpha_H) > \bar{w}(\alpha_L), \quad \underline{w}(\alpha_H) > \underline{w}(\alpha_L)$$

The party offers a fixed reward $R > 0$, known to all agents. Agents are risk-neutral. In period 1, they must decide whether to join the party by incurring a sunk cost $c > 0$. In period 2, they observe their realised outside option w and decide whether to accept the party reward R or take w.

(a) Derivation – Joining Condition Under Uncertainty

Let $\mathbb{E}[\max(R, w)]$ denote the expected period-2 payoff conditional on joining. The agent joins if: $\mathbb{E}[\max(R, w(\alpha))] \geq c$. The expectation is given for the second case, but not for the first and third. Please derive these.

- **Case 1:** If $R \leq \underline{w}$, then $w \geq R$ for all realisations.
- **Case 2:** If $R \geq \bar{w}$, then $R \geq w$ for all realisations. Thus,

$$\mathbb{E}[\max(R, w)] = R$$

- **Case 3:** If $R \in (\underline{w}, \bar{w})$, then we have mixed case (Hint: Use the sum of two integrals).

Based on the above, compute $\mathbb{E}[\max(R, w)]$ in closed form.

(b) Interpretation

Briefly address the following conceptual questions:

- How does uncertainty (i.e., the width of the interval $\bar{w} - \underline{w}$) affect the party's ability to screen for loyalty?
- What are the implications when the outside option distributions of the two types overlap?
- Under what conditions is the selection effect strong or weak? Explain the logic intuitively.

Optional Extension: Simulating Joining Probabilities under Uncertainty
Explore how the probability of joining varies with the outside option distribution and the party reward R. You can assume:

- Uniform distribution for $w \sim U[\underline{w}, \bar{w}]$
- Agents join if $\mathbb{E}[\max(R, w)] \geq c$
- Vary $\underline{w}$, $\bar{w}$, and R to see how the joining condition changes

Hint: Plot $\mathbb{E}[\max(R, w)]$ as a function of R for different values of $\underline{w}, \bar{w}$. Then compare it to the cost c.

R (using ggplot2):

```
library(ggplot2)

expected_max <- function(R, w_low, w_high) {
  if (R <= w_low) {
    return((w_low + w_high) / 2)
  } else if (R >= w_high) {
    return(R)
  } else {
    return((R * (R - w_low) + 0.5 * (w_high^2 - R^2)) / (w_high - w_low))
  }
}

R_vals <- seq(0, 10, length.out = 1000)
cost <- 5
w_ranges <- list(c(3, 6), c(0.1, 8), c(0.01, 9))

df <- data.frame()
entry_lines <- data.frame()

for (w in w_ranges) {
  E_vals <- sapply(R_vals, expected_max, w_low = w[1], w_high = w[2])
  range_label <- paste0("[", w[1], ", ", w[2], "]")

  df <- rbind(df, data.frame(R = R_vals, E = E_vals, Range = range_label))

  sign_diff <- diff(sign(E_vals - cost))
  cross_idx <- which(sign_diff != 0)
  cross_points <- numeric()

  for (i in cross_idx) {
    R1 <- R_vals[i]
    R2 <- R_vals[i + 1]
```

```
    E1 <- E_vals[i]
    E2 <- E_vals[i + 1]
    if (E2 != E1) {
      R_cross <- R1 + (cost - E1) * (R2 - R1) / (E2 - E1)
      cross_points <- c(cross_points, R_cross)
    }
  }

  if (length(cross_points) >= 1) {
    entry_lines <- rbind(entry_lines,
                         data.frame(R_join = round(max(cross_points), 2),
                                    Range = range_label))
  }
}

# Print joining conditions
cat("Joining conditions:\n")
for (i in 1:nrow(entry_lines)) {
  cat(entry_lines$Range[i], ": Agents join if R \geq", entry_lines$R_join[i], "\n")
}

# Plot
ggplot(df, aes(x = R, y = E, colour = Range)) +
  geom_line(size = 1.2) +
  geom_hline(yintercept = cost, linetype = "dashed", color = "black", size = 0.8) +
  labs(title = "Expected Payoff vs Reward Threshold (R)",
       y = expression(E[max(R, w)]), x = "Reward R") +
  theme_minimal(base_size = 14)
```

Part B – Promotion Probabilities and the Loyalty–Competence Trade-off

Objective: Explore how competence-contingent promotion probabilities affect joining decisions under sunk costs and outside option uncertainty.

Now suppose that receiving the party reward R upon joining is not guaranteed. Instead, it occurs with a probability $p(\alpha)$ that depends on an agent's competence type $\alpha \in \{\alpha_L, \alpha_H\}$, with:

$$p(\alpha_H) > p(\alpha_L)$$

Each agent draws an outside option w from a type-dependent uniform distribution:

$$w \sim U[\underline{w}(\alpha), \bar{w}(\alpha)]$$

An agent who joins pays a sunk cost $c > 0$ and then:

- With probability $p(\alpha)$, receives $\max(R, w)$.
- With probability $1 - p(\alpha)$, receives w only.

(a) Derivation – Expected Payoff Under Uncertain Promotion

Let the expected utility of joining for an agent of type α be denoted $\mathbb{E}_\alpha$, defined by:

$$\mathbb{E}_\alpha = p(\alpha) \cdot \mathbb{E}[\max(R, w(\alpha))] + (1 - p(\alpha)) \cdot \mathbb{E}[w(\alpha)]$$

Here:

$$\mathbb{E}[w(\alpha)] = \frac{\underline{w}(\alpha) + \bar{w}(\alpha)}{2}$$

For the term $\mathbb{E}[\max(R, w(\alpha))]$, consider three cases:

- **Case 1:** $R \leq \underline{w}(\alpha)$
$$\Rightarrow \mathbb{E}[\max(R, w)] = \frac{\underline{w}(\alpha) + \bar{w}(\alpha)}{2}$$
- **Case 2:** $R \geq \bar{w}(\alpha)$
$$\Rightarrow \mathbb{E}[\max(R, w)] = R$$
- **Case 3:** $R \in (\underline{w}(\alpha), \bar{w}(\alpha))$

$$\Rightarrow \mathbb{E}[\max(R, w)] = \frac{R(R - \underline{w}(\alpha)) + \frac{1}{2}(\bar{w}(\alpha)^2 - R^2)}{\bar{w}(\alpha) - \underline{w}(\alpha)}$$

Derive the joining condition:

$$\mathbb{E}_\alpha \geq c$$

Example: Suppose:

$$p(\alpha_H) = 0.9, \quad p(\alpha_L) = 0.5, \quad c = 5, \quad R = 6$$

$$\underline{w}(\alpha_H) = 3, \quad \bar{w}(\alpha_H) = 8; \quad \underline{w}(\alpha_L) = 1, \quad \bar{w}(\alpha_L) = 6$$

Plug in values to compute $\mathbb{E}_\alpha$ and compare to c.

(b) Interpretation

Reflect on the following conceptual questions:

- How does increasing $p(\alpha_H)$ affect the loyalty–competence trade-off?
- When can competence-based promotion *undo* the selection effect created by sunk costs?
- Can strong rewards for competence backfire by crowding out loyal but less competent joiners?

Part C – Simulation and Visualisation

Objective: Use simulation to build intuition about how sunk costs, competence-based promotion probabilities, and outside option uncertainty shape agents' decisions to join.

You will simulate and visualize the *joining condition* for two assumed agent types:

- α_H: high competence
- α_L: low competence

These types are fixed model inputs — not computed — and determine other parameters such as outside option distributions and promotion probabilities. The goal is to examine how the expected utility from joining compares to the sunk cost c, and how this relationship shifts under different assumptions.

1. **Plot expected utility curves:**
    - Choose parameter values for each type:
        - Outside option range: $[\underline{w}(\alpha), \bar{w}(\alpha)]$
        - Promotion probability: $p(\alpha)$
        - Party reward: R

- For each type, compute and plot the expected utility of joining $\mathbb{E}_\alpha$ for a range of sunk costs $c \in [0, 10]$.
- **Plot specification:**
  - **x-axis:** Sunk cost c
  - **y-axis:** Expected utility from joining $\mathbb{E}_\alpha$
  - Add the line $y = c$ to indicate the indifference condition $\mathbb{E}_\alpha = c$
  - Add horizontal lines for each agent's $\mathbb{E}_\alpha$ across c

2. **Identify joining thresholds:**
   - Locate and mark the point where each $\mathbb{E}_\alpha$ curve intersects the $y = c$ line.
   - Interpret this point $c^*(\alpha)$ as the **maximum sunk cost** an agent is willing to pay in order to join.
   - **Plot update:**
     - Add a vertical line at $c^*(\alpha)$ for each type.
     - Label it clearly as the agent's "joining threshold."

3. **Explore parameter changes:**
   - Repeat the plot for different scenarios:
     - Vary the promotion probability $p(\alpha)$
     - Vary the outside option range, i.e., the width $\bar{w}(\alpha) - \underline{w}(\alpha)$
   - **Each plot should use:**
     - **x-axis:** Sunk cost c
     - **y-axis:** Expected utility $\mathbb{E}_\alpha$
   - Compare how the shapes and positions of the utility curves shift.
   - Identify which agent type is more sensitive to each parameter.

4. **Interpretation:**
   - Reflect on your plots to assess:
     - When does the sunk cost serve as an effective screening tool?
     - How do competence incentives (via $p(\alpha)$) and outside option uncertainty jointly influence selection?
     - Can strong rewards for competence backfire by discouraging loyal but less competent joiners?

Analysing our stylised model of logrolling

This exercise explores the logic of cross-issue vote trading (logrolling) between two legislators with asymmetric preferences.

Setup

Two legislators, L and R, bargain over a bundle of policies $(x, y) \in [0, 1]^2$. Their ideal points are:

$$(x_L, y_L) = (0.2, 1.0), \qquad (x_R, y_R) = (1.0, 0.2)$$

Each legislator has a weighted Euclidean utility function:

$$U_i(x, y) = -\left[\alpha_i(x - x_i)^2 + (1 - \alpha_i)(y - y_i)^2\right], \qquad i \in \{L, R\}$$

Assume symmetric but opposing salience weights:

$$\alpha_L = \alpha, \qquad \alpha_R = 1 - \alpha, \qquad \text{where } \alpha \in (0, 1)$$

If no agreement is reached, the policy outcome defaults to the majority-rule median:

$$(x^*, y^*) = (0.5, 0.5)$$

Part A – Utilities and Status Quo Payoffs

1. Derive the utility of each legislator under the default outcome $(0.5, 0.5)$. Express each utility as a function of α.

2. Derive the gains from cooperation when one legislator cares more about a given issue and gets her preferred outcome on that issue, while the other legislator gets her preferred outcome on the other issue.

$$\Delta U_i(x, y) = U_i(x, y) - U_i(0.5, 0.5)$$

Part B – A Stylised Bargaining Solution (Weighted-Average Approach)

Motivation: To focus on the core logic of logrolling, we adopt a stylised and transparent alternative that reflects the trade-offs involved when legislators have different issue priorities.

Suppose that legislators L and R engage in bargaining over two policy dimensions, x and y, each of which matters more to one legislator than the other. We assume that the final policy bundle (x^*, y^*) is a **weighted average of ideal points**, where the weights reflect how much each legislator cares about each issue.

More specifically, let the salience parameter $\alpha \in (0, 1)$ reflect how much legislator L cares about issue x (and, symmetrically, how much R cares about issue y). Then define the negotiated outcome as:

$$x^* = \alpha \cdot x_L + (1 - \alpha) \cdot x_R, \qquad y^* = (1 - \alpha) \cdot y_L + \alpha \cdot y_R$$

This formulation captures the logic of **mutual deference on less important issues**: each legislator concedes more on the dimension they care less about in order to get closer to their ideal point on the issue they prioritise.

Interpretation: The outcome is pulled toward L's ideal point in x when α is large (indicating x is highly salient to L), and toward R's ideal point in y for the same reason.

Tasks

1. Plot $x^*(\alpha)$ and $y^*(\alpha)$ for $\alpha \in \{0.05, 0.10, \ldots, 0.95\}$. Label the axes and indicate each legislator's ideal points for reference.
2. Interpret your results: Who benefits as α increases? When is the scope for logrolling largest?
3. Compare this stylised solution to the default (majority-rule) policy outcome $(0.5, 0.5)$. When does cooperation lead to large gains over the status quo?

Part C – A Dynamic Extension of Logrolling

We now extend the model to a dynamic setting where policies are set in two periods, allowing legislators to trade across time.

- In **Period 1**, policy x is set (e.g., business regulation),
- In **Period 2**, policy y is set (e.g., social spending).

The legislators' ideal points are:

$$(x_L, y_L) = (0.2, 1.0), \qquad (x_R, y_R) = (1.0, 0.2)$$

Each legislator has a weighted Euclidean utility function that captures the salience of each issue. As before, let $\alpha \in (0,1)$ denote how much L cares about x (with R caring more about y in that case):

$$U_L = -\left[\alpha(x - x_L)^2 + \delta(1-\alpha)(y - y_L)^2\right]$$

$$U_R = -\left[(1-\alpha)(x - x_R)^2 + \delta\alpha(y - y_R)^2\right]$$

The discount factor, $\delta \in (0,1)$, reflects how much each legislator values future payoffs. If no agreement is reached, a default (moderate) outcome is imposed in each period:

$$x^{\text{default}} = y^{\text{default}} = 0.5$$

Please answer the following:

1. Suppose legislators agree to trade across time: R gets her preferred outcome x^R in Period 1, and L gets her favoured y^L in Period 2. Write out the total utility with discounting.

2. Let the period-specific outcomes be stylised weighted averages:

   $$x^* = \lambda x_L + (1-\lambda)x_R, \quad y^* = \mu y_L + (1-\mu)y_R$$

   Compute each legislator's total utility as a function of λ, μ, α, and δ.

3. What conditions on λ and μ make both legislators strictly better off than under the default (status quo in both periods)? Provide intuition.

4. Suppose the game ends after Period 1 with some probability (e.g., $\pi = 0.7$). How does this affect the scope for cooperation? Note: This introduces a risk that the second period may never occur — for instance, due to a snap election, leadership change, or institutional gridlock.

5. What mechanisms (e.g., party leadership, side payments, reputation) might help enforce intertemporal trades in a setting like this?

One- and Two-Dimensional Party Competition

This task focuses on the strategic logic of party positioning under electoral competition. In Part A, you will derive the classical Downsian result of median voter convergence. In Part

B, you will extend this reasoning to a two-dimensional setting and reflect on how real-world dynamics depart from the canonical model.

Answer each sub-question separately. Suggested word counts exclude equations, tables, and diagrams.

Part A – The One-Dimensional Model

Nash equilibrium in the Downsian model (Max. 350 words) Formally derive why convergence to the median voter's ideal point is a pure-strategy Nash equilibrium:

- Explain the set-up of the model: two parties, unidimensional policy space $x \in [0, 1]$, voters vote for the closest party, etc.
- Define the notion of a **pure-strategy Nash equilibrium**, i.e. both parties playing mutual best responses.
- Use a **proof by contradiction** to show that the pure-strategy Nash equilibrium is for parties to locate at the median: assume one party locates at the median and the other deviates. Show that the deviating party could improve its vote share by moving to the median.
- Conclude by discussing why no party benefits from deviating when both locate at the median.

Contextualising the model's assumptions (Max. 450 words) Use a two-column table to list assumptions of the Downsian model and assess how well they apply to a real-world party system of your choice (e.g., UK, Germany, Sweden, Australia, Uruguay, Mexico).

- Column 1: Model assumptions (e.g., parties maximise vote share, unidimensional space, full turnout, perfect information).
- Column 2: For each assumption, explain whether and how it is approximated in your chosen context. Avoid saying assumptions are simply "unrealistic", instead reflecting on the extent to which they are (or are not) approximated.
- Reflect on whether some parties or party families behave in a more Downsian manner than others.

Part B – Valence in Party Competition

Background: In spatial models of voting, voters choose candidates or parties based on the distance between their ideal point and the party's position. But in reality, voters often care about *valence* characteristics — competence, honesty, leadership, or descriptive attributes. Candidates or parties may differ in valence even when offering identical policies.

Common examples of valence include:

- Competence or perceived effectiveness
- Cognitive capacity or leadership skills
- Descriptive traits (e.g., gender, race, class background)
- Morality or integrity

Valence is especially important when parties converge on policies. A party may gain votes due to perceived competence even if it is ideologically similar to its competitor.

Vote-maximising parties with valence (Max. 450 words + equations) Assume two parties A and B choose locations x_A, x_B on the closed unit interval $[0,1]$. Voters are uniformly distributed and vote for the party providing higher utility:

$$U_i^P = -|x_i - x_P| + v_P$$

where $v_A > v_B$ are fixed valence parameters. Assume parties are **purely vote-seeking**.

Answer the following:

1. Derive the location of the **indifferent voter** between A and B as a function of x_A, x_B, v_A, and v_B.
2. Derive each party's **vote share**.
3. Assume A locates at the median ($x = 0.5$). Does B have an incentive to deviate?
4. How does the **importance of valence** (the role of $v_A - v_B$) change as x_A and x_B move closer together? Explain intuitively.
5. Optional: Apply this logic to the convergence (see here for more detail) of mainstream left and right parties on the economic dimension during the era of **Third Way social democracy** (e.g., Blair, Schröder). Why was valence competition (e.g., competence) particularly intense in this period?

Vote- and policy-seeking parties with valence (Max. 450 words + equations) Now assume parties care about both votes and policy. Let party P's utility be:

$$U_P = \lambda V_P - (1-\lambda)(x_P - x_P^{\text{ideal}})^2$$

where:

- V_P is the party's vote share (from above)
- x_P^{ideal} is the party's policy ideal point
- $\lambda \in [0,1]$ measures how vote-seeking the party is

Assume voters' utility is still given by $U_i^P = -|x_i - x_P| + v_P$.
Answer the following:

1. For fixed $v_A > v_B$, how does increasing λ affect party A's optimal location? Why?
2. Suppose $\lambda = 0.5$ for both parties. Which party has greater flexibility in positioning? Explain using the concept of **valence advantage**.
3. Can a party use valence to compensate for ideological divergence? Under what conditions?

Strategic Trade-offs Between Policy and Votes **Background:** A party cares about both vote share and proximity to its ideological ideal. Valence gives it an advantage over its rival. The party's utility is given by:

$$U_P = \lambda V_P - (1-\lambda)(x_P - x_P^{\text{ideal}})^2$$

Voters are uniformly distributed on the unit interval $[0,1]$. The party's vote share is thus:

$$V_P = \frac{x_P + x_R + (v_P - v_R)}{2}$$

where x_R is the rival's position and $v_P > v_R$.

Your Task:

1. Write a function to calculate the utility U_P for a range of positions $x_P \in [0,1]$ given the following parameters:
   - $x_P^{\text{ideal}} = 0.3$

- $x_R = 0.7$
- $v_P = 0.2$, $v_R = 0$
- Try values of $\lambda = 0, 0.25, 0.5, 0.75, 1$

2. Plot U_P against x_P for each value of λ.

3. Answer the following questions:

   (a) How does the optimal x_P (that is, the location that maximises utility) change as λ increases?

   (b) How does valence affect the party's willingness to shift away from its ideal point?

   (c) At what point does vote-seeking completely override policy goals?

   (d) Suppose voter preferences are not uniformly distributed but instead clustered near the centre. How might this affect your results?

Part C – Explaining British Party Positioning (Max. 600 words)

You are given a spatial representation of UK voter and party positions in 2019 and 2022 across economic (left–right) and social (liberal–authoritarian) dimensions.

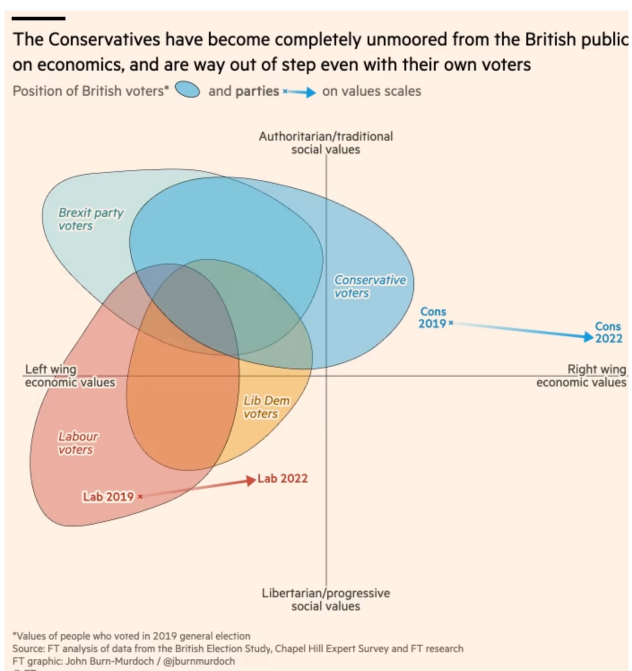

Figure 1: Two-dimensional policy space in Britain (John Burn-Murdoch)

Important note: In two-dimensional spaces, equilibrium outcomes under majority rule are often *not well-defined* (McKelvey's Chaos Theorem) or only exist under very strong assumptions (Plott's theorem). Thus, we should not expect party convergence in the same way as in the one-dimensional model.

Instead, your task is to explain observed positioning patterns in Figure 1 by identifying which **minimal set of assumptions** from the Downsian model must be relaxed – in addition to the assumption of the policy space being two dimensional. Please refer to the slides for the assumptions.

Please answer the following in no more than 500 words:

1. Describe the observed positioning of Labour and Conservative parties between 2019 and 2022.
2. Which assumptions from classical theory do we have to relax to explain their strategies? Try to relax the **fewest necessary**.
3. In relaxing the assumptions, reflect on the relative importance of different types of assumptions: the demand-side, supply-side, and contextual ones.

Optional: Briefly apply the same logic to explain the positioning of smaller parties if you wish.