Appendix B

Notation and proofs for Chapter 6

PART 1: THE VOTER'S DECISION PROBLEM

This appendix considers how a voter motivated solely by a desire to affect the outcome of the election decides which candidate to vote for, given that she votes. There are two parameters in the voter's decision (subscript i's are suppressed and the distribution of utility types F is taken as given): First, the voter's preferences over the candidates, given by $u \in U$; second, the voter's expectations about how well each candidate will do at the polls.

I model voter expectations as follows. Each voter i views the candidates' vote totals (exclusive of her own vote) as random variables $V_1, \ldots, V_K$ governed by a joint distribution function, $g_n(v_1, \ldots, v_K)$. I assume that the mean of g_n does not depend on n (the number of voters), although n may affect higher-order moments. It may be, for example, that g_n is the K-nomial distribution with parameters $\pi = (\pi_1, \ldots, \pi_K)$ and $n - 1$. This is the case considered by Palfrey (1989), Cox (1994), and in Chapter 4.

I assume that the joint distribution g_n is common knowledge. This entails common knowledge of the expected vote shares of the candidates, denoted $\pi = (\pi_1, \ldots, \pi_K) = E(V_1/(n-1), \ldots, V_K/(n-1) \mid g_n)$, and of the tie-probabilities relevant in the voter's expected utility calculation.

I also assume that the probability beliefs g_n satisfy a version of Myerson and Weber's (1993:105) *ordering condition*. Supposing without loss of generality that the candidates' numbers refer to their order of expected finish, i.e., that $\pi_1 \geq \pi_2 \geq \ldots \geq \pi_K$, the version of the ordering condition that I shall use says the following: If $\pi_j < \pi_3$ then each voter believes the probability of the event "candidate j is tied for second" is *negligible* in comparison to the probability of the event "candidate 3 is tied for second," for large enough electorates. That is, no voter believes a fourth or lower place candidate really has a non-negligible chance of being tied for second, even conditional on there being a tie for second between some candidates. This condition, which is used below in deriving Proposition 1 but not in deriving Proposition 2, emerges naturally in models in which each voter's decision is statistically independent of every other's and the electorate is large (for then g_n collapses around its mean as n grows without bound; cf. Palfrey 1989; Cox 1994).

Another thing that the ordering condition says is that, if $\pi_1 > \pi_2$ then each voter believes the probability of the event "candidate 1 is tied for second" is *negligible* in comparison to the probability of the event "candidate 2 is tied for second," for large enough electorates. Thus, just as it is unlikely that fourth- or lower-place candidates will end up in a tie for second, relative to the probability that the third-place candidate will, so also it is unlikely that a first-place candidate will end up in a tie for second, relative to the probability that the second-place candidate will. This condition is used only in Proposition 2 and is unnecessary in deriving Proposition 1.

Finally, I also assume that expectations are rational:

Rational expectations condition: The expectation g_n is rational with respect to the distribution F if, for all j:

$$\pi_j = \int_{H_j(g_n)} dF$$

Here, $H_j(g_n)$ is the set of all voters for whom casting a vote for candidate j is optimal, given that g_n describes the distribution of other voters' votes.

I shall denote by $V(u;g_n) \subseteq K = \{1, \ldots ,K\}$ the optimal vote(s) of a voter of type u facing an electorate described by g_n. The purpose of this appendix is to show that the parameters identified are indeed sufficient to yield a well-defined decision problem, and to reveal such of the technical details of solving this problem as are necessary for proving the theorem to come.

By voting for candidate j, the focal voter can affect her utility either by breaking or making ties for second in the first round or by giving a candidate the last vote needed for a majority in the first round. I shall denote the probability (under g_n) that candidate j is one vote shy of a first-round majority, with k in second, by $q^j{}_k$. This gives the probability of putting j over the top rather than facing a jk pairing in the runoff. As far as breaking and making ties, there are two abstract possibilities to consider (if, following Hoffman (1982) and Myerson and Weber (1993), I ignore the possibility of r-way ties, $r > 2$, for the second seat): the voter's vote may put j into a tie with k for the second runoff spot or break a tie for the second spot between j and k. I shall let $q^i{}_{jk}$ equal the probability that candidates j and k end up tied for second, with candidate i in first.

Assuming (again following Hoffman and Myerson and Weber) that the probability of the event "*k* is in second, tied with *j*, behind *i*" equals the probability of the event "*k* is in second, one vote ahead of *j*, behind *i*," then q^i_{jk} equals the probability of *breaking* a tie for second between *j* and *k*, with *i* in first, and it also equals the probability of *making* a tie for second between *j* and *k*, with *i* in first.

In terms of the notation just introduced, the expected utility increment from voting for *j* rather than abstaining can be written:

$$\xi_j = \sum_{k \neq j} q^j_k (u_j - (p_{jk}u_j + p_{kj}u_k)) + \sum_{i \neq j} \sum_{\substack{k \neq j \\ k \neq i}} q^i_{jk}((p_{ji}u_j + p_{ij}u_i) - (p_{ki}u_k + p_{ik}u_i))$$

where p_{jk} equals the probability that candidate *j* will defeat candidate *k* in a runoff pairing of the two. Thus, $V(u;g_n) = \arg\max_{j \in K} \xi_j$.

The terms $\{p_{jk}\}$ can be computed as follows. Let $sgn(x - y) = 1$ if $x - y$ is positive, 0 if $x - y$ is zero, and -1 if $x - y$ is negative. Let $U^i = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots u_n)$ be a profile of voter types for voters other than *i*. Let $C^+ = \{U^i \in U^{n-1}: \sum_h sgn(u_{hj} - u_{hk}) > 1\}$ be the set of profiles in which a plurality larger than 1 of the other voters prefer *j* to *k*; $C(a) = \{U^i \in U^{n-1}: \sum_h sgn(u_{hj} - u_{hk}) = a\}$ be the set of profiles in which *j* is *a* votes ahead of *k*, for $a \in \{-1,0,1\}$; and $C^- = \{U^i \in U^{n-1}: \sum_h sgn(u_{hj} - u_{hk}) < -1\}$ be the set of profiles in which a plurality larger than 1 of the other voters prefer *k* to *j*. Finally, for voter *i* let *G* be the distribution over the $n - 1$ other voters' utility types induced by *F*:

$$G = \prod_{h=1}^{n-1} F$$

Then, if voter *i* prefers *j* to *k*,

$$p_{jk} = \int_{C^+ \cup C(1) \cup C(0)} dG + \frac{1}{2} \int_{C(-1)} dG$$

If voter *i* prefers *k* to *j*, then

$$p_{jk} = \int_{C^+} dG + \frac{1}{2} \int_{C(1)} dG$$

PART 2: THE ORDERING CONDITION AND PROOF OF PROPOSITION 1

Given a distribution *F* defined over *U*, I shall say that the expectation *g* is a limit of rational expectations if and only if there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of expectations, each rational with respect to *F*, that converges to *g* in the Whitney-C^∞ topology. In other words, *g* is a limit of rational expectations if and only if arbitrarily large electorates can have rational expectations that are arbitrarily close to g. Given this definition, Proposition 1 can be stated as follows:

> *Proposition 1:* Let g be an expectation (i.e., a joint distribution for $V_1, \dots, V_K$) and let $E(V_1/\Sigma V_k, \dots, V_K/\Sigma V_k \mid g) = \underline{\pi}$. Assume without loss of generality that $\underline{\pi}_1 \geq \underline{\pi}_2 \geq \dots \geq \underline{\pi}_K$. Then if $0 < \underline{\pi}_j < \underline{\pi}_3$ for some $j > 3$, g is not a limit of rational expectations.

In order to prove this proposition, I shall need the following definition.

> *The ordering condition:* A sequence of expectations $\{g_n\}_{n=1}^{\infty}$ satisfies the *ordering condition* if and only if

(1) $\pi_j < \pi_3 \rightarrow \lim_{n\to\infty} \dfrac{q_{jh}^{(n)}}{q_{\bullet\bullet}^{(n)}} = 0$ for all $h \neq j$; and

(2) $\pi_1 > \pi_2 \rightarrow \lim_{n\to\infty} \dfrac{q_{1h}^{(n)}}{q_{\bullet\bullet}^{(n)}} = 0$ for all $h \neq 1$. Here,

$$q_{jh}^{(n)} = \Pr(V_j = V_h \,\&\, V_h < V_k \text{ for exactly one } k | g_n), \text{ and } q_{\bullet\bullet}^{(n)} = \sum_{j\in K} \sum_{\substack{h\in K \\ h>j}} q_{jh}^{(n)}.$$

Proof of Proposition 1

Let g be a limit of rational expectations. Then by definition there exists a sequence of expectations $\{g_n\}_{n=1}^{\infty}$ such that each g_n is rational with respect to F, and such that $\{g_n\}_{n=1}^{\infty} \rightarrow g$ in the Whitney–C^{∞} topology. Suppose that $0 < \underline{\pi}_j < \underline{\pi}_3$. Then since $\{g_n\}_{n=1}^{\infty} \rightarrow g$ it follows that $0 < \pi_j < \pi_3$. Thus, from the ordering condition, we know that

$$\lim_{n\to\infty} \frac{q_{jh}^{(n)}}{q_{\bullet\bullet}^{(n)}} = 0 \text{ for all } h.$$

Thus, in the limit, voting for candidate j is no different from abstaining; the probability of a vote for j affecting the outcome, even given that there is a tie or near-tie of some sort, is virtually nil.

Now consider a voter with arbitrary preferences. In the limit, the only candidates with non-negligible probabilities of being tied for second are some subset of $\{1, 2, \dots, j-1\}$. Will the voter do better to vote for one of these candidates than abstain? I shall deal with only one of many possible cases here, that in which $K > 4$ and $\pi_1 > \pi_2$ and $\pi_3 = \pi_4 > \pi_5$. Letting q_{hk}^{i} equal the probability (under g) that candidates h and k end up tied for second, with candidate i in first, this case is one in which q_{hk}^{i} is non-negligible only if $i = 1$, $h = 2$, and $k \in \{3,4\}$. Thus, the utility of voting for a candidate $k \in \{3, 4\}$, rather than abstaining, is

$$\xi_k = q^1_{2k}((p_{1k}u_1 + p_{k1}u_k) - (p_{12}u_1 + p_{21}u_2))$$

This will be positive for all voters for whom the utility differential in parentheses is positive. If j is such that $0<\underline{\pi}_j<\pi_3$, then voters who prefer a runoff pairing of 1&3 to 1&2 will not vote for j, preferring to vote for 3. Those who prefer a runoff pairing of 1&4 to 1&2 will not vote for j, preferring to vote for 4. Finally, those who prefer a runoff pairing of 1&2 to both 1&3 and 1&4 will not vote for j, preferring to vote for 2. Thus, any voter with a strict ranking of the three probable runoff pairings – 1&2, 1&3, and 1&4 – will not vote for j. Only voters who are indifferent among all three pairings, a measure-zero set, will vote for j. This, however, contradicts the assumption that $0 < \underline{\pi}_j$. Other cases can be dealt with similarly. *QED.*