CEE 958, Spring 2023
HW1
Due date: February 6

Part 1

Consider the PDE model:

$$\frac{d^2u}{dx^2} + \frac{du}{dx} + x = 0, \quad 0 \leq x \leq 1 \tag{1}$$

subject to the boundary conditions:

$$u(0) = u(1) = 0. \tag{2}$$

Solve this equation using the residual least square method described by Eq ((**??**)) in the Topic III lecture slides.

Here, $L[u(x)] = \frac{d^2u}{dx^2} + \frac{du}{dx}$, $f(x) = -x$ and $R(x) = L[u(x)] - f(x) = \frac{d^2u}{dx^2} + \frac{du}{dx} + x$.

It is always advisable to find the approximation $\tilde{u}$ that satisfies the boundary conditions so that the second term in the *loss function* in Eq (8) (Topic III lecture slides) drops out, yielding the following loss function

$$\mathcal{L}(\mathbf{a}) = \int_0^1 R(x, \mathbf{a})^2 dx \tag{3}$$

For these BCs, we can write the approximate solution as

$$\tilde{u}(x, \mathbf{a}) = \tilde{u}'(x, \mathbf{a})x(x-1)$$

where $\tilde{u}'(x, \mathbf{a})$ is any approximating function with the parameter vector $\mathbf{a} = (a_1, ..., a_N)$ (could be a polynomial, DNN etc).

In this HW assignment, you are asked to assume the polynomial form of $\tilde{u}'(x, \mathbf{a})$:

$$\tilde{u}'(x) = a_1 + a_2 x$$

The parameter vector $\mathbf{a}$ can be found by minimizing the loss function:

$$\mathbf{a}^* = \min_{\mathbf{a}} \mathcal{L}(\mathbf{a}) \tag{4}$$

Questions:

- Compute and plot the solution $\tilde{u}(x, \mathbf{a})$.
- How to estimate the error in this solution if you do not know the true solution?
- Plot the residual $R(x)$ as a function of x.

Hints:

$$\tilde{u}(x, \mathbf{a}) = \tilde{u}'(x, \mathbf{a})x(x-1) = (a_1 + a_2 x)(x^2 - x) = a_1(x^2 - x) + a_2(x^3 - x^2)$$

To obtain the residual function R, compute:

$$\frac{d\tilde{u}(x, \mathbf{a})}{dx} = ...$$

and

$$\frac{d^2\tilde{u}(x,\mathbf{a})}{dx^2} = ...$$

Then,

$$R(x;a_1,a_2) = \frac{d^2\tilde{u}}{dx^2} + \frac{d\tilde{u}(x)}{dx} + x = ...$$

To find

$$(a_1,a_2)^* = \min_{(a_1,a_2)} \mathcal{L}(a_1,a_2)$$

solve the system of two equations

$$\frac{\partial}{\partial a_1}\mathcal{L} = \int_\Omega R(x,\mathbf{a})\frac{\partial R(x,\mathbf{a})}{\partial a_1}dx = 0 \tag{5}$$

$$\frac{\partial}{\partial a_2}\mathcal{L} = \int_\Omega R(x,\mathbf{a})\frac{\partial R(x,\mathbf{a})}{\partial a_2}dx = 0 \tag{6}$$

Part 2

Solve the same equation as in Part 1, but now assuming that one measurement of u is available: $u^* = 0.06377$ at the location $x^* = 0.5$.

Recall that data can be included in the residual least-square solution by adding the data missmatch term in the loss function:

$$\mathcal{L} = \int_\Omega R(x,\mathbf{a})^2 dx + \lambda(\tilde{u}(x^*,\mathbf{a}) - u^*)^2 \tag{7}$$

The coefficients a_1 and a_2 can be found by solving the equations:

$$\frac{\partial}{\partial a_1}\mathcal{L} = \int_\Omega R(x,\mathbf{a})\frac{\partial R(x,\mathbf{a})}{\partial a_1}dx + (\tilde{u}(x^*,\mathbf{a}) - u^*)\frac{\partial \tilde{u}(x^*,\mathbf{a})}{\partial a_1} = 0 \tag{8}$$

$$\frac{\partial}{\partial a_2}\mathcal{L} = \int_\Omega R(x,\mathbf{a})\frac{\partial R(x,\mathbf{a})}{\partial a_2}dx + (\tilde{u}(x^*,\mathbf{a}) - u^*)\frac{\partial \tilde{u}(x^*,\mathbf{a})}{\partial a_2} = 0 \tag{9}$$

- Compute a_1 and a_2 for $\lambda = 1$, 100, 1000.
- Plot solution $\tilde{u}(x,\mathbf{a})$ as a function of x for $\lambda = 1$, 100, 1000.
- On the same figure, plot the analytical solution $u(x) = -\frac{x^2}{2} + x + \frac{1}{2(1-e^{-1})}(e^{-x} - 1)$. Discuss the results.
- Plot the residual $R(x,a)$ versus x for $\lambda = 0$, 1, 100, 1000. Discuss the results.