TOPOLOGIES ON SCHEMES

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ABSTRACT. This short note aims to be a readable exposition of the basics of alternative (Grothendieck) (pre)topologies on the category of schemes. Proofs are provided where they provide intuition. Definitions are provided for all material one would not likely encounter in an introductory commutative algebra and geometry course, which makes the intended audience graduate students who have finished these courses.

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1. Introduction

Often, when dealing with a geometric object X, one studies its properties by endowing it with some topology and computing invariants H^*_{singular}. This is useful, for instance, in distinguishing various affine spaces. Endowed with the standard metric topology, we have that

$$H^i_{\text{singular}}(\mathbb{C}^n \setminus \{p\}, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 2n \\ 0, & \text{else} \end{cases},$$

for any $p \in \mathbb{C}^n$. This implies that there is no homeomorphism from $\mathbb{C}^n$ to $\mathbb{C}^m$ if $m \neq n$. We would like to use similar invariants to study objects in other categories, namely **Sch**. Unfortunately, the singular invariants are not well-behaved in the Zariski topology. Namely, the singular invariants of a scheme X factor through the forgetful functor to **Top**, but this implies $H^*_{\text{singular}}(\mathbb{A}^1) = H^*_{\text{singular}}(\mathbb{P}^1)$ as the Zariski topology of the projective and affine line are each simply the cofinite topology (over any closed field). Indeed, the situation is quite bad for studying any purely topological invariants of a scheme, which is demonstrated by the following proposition.

Proposition 1.1. *Let k be an algebraically closed field of cardinality greater than or equal to $|\mathbb{R}|$ and let X be an irreducible variety over k. Then, X is topologically contractible.*

Thus, it is clear that the elementary tools afforded to us by algebraic topology will not suffice. Instead, we turn to sheaf cohomology: given $\mathcal{F} \in \mathbf{Ab}(X)$, it is a result of Grothendieck that there exists an injective resolution of $\mathcal{F}$. We apply global sections to this resolution, and take its cohomology. These groups, $H^i(X, \mathcal{F})$ are useful invariants of a scheme. For instance, the vanishing of these groups in every dimension for $\mathcal{F}$ quasicoherent is equivalent to $X = \operatorname{Spec}(R)$. However, these groups are not quite sufficient for our needs. For instance, the cohomology of projective space with coefficients in any constant sheaf vanishes, which is not analogous to the singular case: $H^{2i}_{\text{singular}}(\mathbb{P}^n(\mathbb{C}), \mathbb{Z}_p) = \mathbb{Z}_p$ if and only if $i = n$. This expository paper aims to develop a cohomology theory which captures this behavior.

2. Topologies on categories

The end goal of this paper is to develop a cohomology theory that more closely models cohomology in the analytic category. Seeing as (Zariski) sheaf cohomology satisfies many desireable properties, it is natural to consider ways to generalize this that preserve the nice theoretical properties of Zariski cohomology while also yielding the results we want, particularly on projective space.

2.1. **Pretopologies.**

Definition 2.1. A *Grothendieck pretopology* on a category $\mathcal{C}$ equipped with fiber products is a set of distinguished sets of morphisms $\{X_\alpha \to X\}_{\alpha \in \Lambda}$ called *coverings* such that:

(1) Every isomorphism is a covering.
(2) If $\{X_\alpha \to X\}_{\alpha \in \Lambda}$ is a covering and, for each α, $\{Y_{\alpha\beta} \to X_\alpha\}_{\beta \in \Gamma}$ is a covering, then $\{Y_{\alpha\beta} \to X\}_{(\alpha,\beta) \in \Lambda \times \Gamma}$ is a covering.

(3) If $Y \to X$ is a morphism and $\{X_\alpha \to X\}_{\alpha\in\Lambda}$ is a covering, then $\{X_\alpha \times_X Y \to Y\}_{\alpha\in\Lambda}$ is a covering.

If $\{X_\alpha \to X\}_{\alpha\in\Lambda}$ is a covering, we say $\{X_\alpha\}$ *covers* X.

Observe that we require that each collection is indeed a set—for now, note that this is nontrivial. Given a category $\mathcal{C}$ with fibre products and where the collection of objects is a set and the collection $\mathrm{Hom}(X, Y)$ is a set, we automatically have the following:

Proposition 2.2. *Given the assumptions above, the following two desciptions of coverings define Grothendieck pretopologies on $\mathcal{C}$:*

(1) $\{\phi : X \to Y\}$ where ϕ is an isomorphism.
(2) $\{\psi : X \to Y\}$ where $\psi \in \mathrm{Hom}(X, Y)$.

We say the first is the discrete pretopology and the second is the indiscrete pretopology.

Proof. Each automatically satisfies the first axiom of a Grothendieck pretopology. The indiscrete pretopology trivially satisfies the second and third axioms automatircally. Isomorphisms compose to isomorphisms, so the discrete pretopology satisfies the second axiom. For the third axiom, consider that if $X' \to X$ is an isomorphism and $Y \to X$ a morphism, then $Y \times_X X' = Y \times_X X = Y$, so $\{Y \times_X X' \to Y\}$ is a covering. □

2.2. **Sites.**

Definition 2.3. A *site* is a pair $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$ where $\mathcal{C}$ is a category and $\mathrm{Cov}(\mathcal{C})$ is a Grothendieck pretopology on $\mathcal{C}$.

For any topological space X, we define the (little) Zariski site on X to be the category whose objects are open subsets of X and morphisms are inclusions of open sets, equipped with the indiscrete pretopology. It is denoted X_{Zar}. We define the big Zariski site of X to be the comma category $\mathbf{Top}/X$ equipped with the pretopology consisting of families of open immersions sharing a common codomain, whose union of images equal the codomain. Recall an *open immersion* of topological spaces is a morphism which is open and a homeomorphism on its image.

Unfortunately, the Zariski sites on a scheme X are not recovered by the forgetful functor $\mathbf{Sch} \to \mathbf{Top}$. However, the sites are defined analogously as the previous case. The little Zariski site of X, denoted $X_{\text{ét}}$ is the category whose objects are schemes Y over X whose structure morphism is an open immersion. The big Zariski site, denoted $(\mathbf{Sch}/X)_{\text{ét}}$ is defined as the category $\mathbf{Sch}/X$ with the Zariski topology. In each case, the coverings are families of open immersions whose image set-theoretically cover X.

In general, given a "nice" class of morphisms τ and a scheme X, we define the *little τ-site* to be the category whose objects are schemes over X whose structure morphism is τ, and coverings are families of τ-morphisms whose image set theoretically cover X. The *big τ-site* is defined as the category $\mathbf{Sch}/X$ with the same notion of covering as in the little site. We denote the little site by X_τ and the big site by $(\mathbf{Sch}/X)_\tau$.

Generally, the big site will reveal more information about the underlying scheme X, but will pose more technical problems. The little site will behave more closely to objects in classical geometry.

Remark 2.4. There is an obvious set-theoretic problem here if we work with the naive definitions of **Top** and **Sch** (the collection of objects is indeed a proper class), however this will be resolved in a later section.

Remark 2.5. In older texts (namely EGA and SGA), the big and small sites are referred to as the *gros* and *petit* site, respectively.

Example 2.6. Let G be a group and denote by $\mathcal{T}_G$ the category of G-sets. Let coverings in this category be families of G-equivariant maps with shared image such that the family set-theoretically covers this image. In other words, a family of G-equivariant maps $\{U_i \to U\}$ is a covering iff $\bigcup U_i = U$.

This is a well-known example of a site, and this particular telling of the story follows that of [1, Tag 03NP].

3. Sheaves

For this section, we assume all categories enjoy all finite fibre products. This is not strictly necessary, but makes the exposition straightfoward. A *presheaf* on a category $\mathcal{C}$ is an object of the functor category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$. The category of presheaves on $\mathcal{C}$ should be seen as the canonical co-completion of $\mathcal{C}$.

Definition 3.1. Let $\mathcal{C}$ be a site and $\mathcal{F}$ a presheaf on $\mathcal{C}$. We say that $\mathcal{F}$ is a *sheaf* if, for every covering $\{U_i \to U\}_{i \in I}$, the following diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j),$$

is an equalizer in **Set**. We refer to the category of sheaves on a site by $\mathbf{Sh}(\mathcal{C}$. Moreover, any category $\mathcal{T}$ equipped with an equivalence to $\mathbf{Sh}(\mathcal{D})$ for some site $\mathcal{D}$ is defined to be a *(Grothendieck) topos*.

We now pivot to focus our attention on the category **Sch**. Let $\mathcal{F}$ be a sheaf on X a scheme. We say that $\mathcal{F}$ is *quasicoherent* if, for each $\mathrm{Spec}(A) \subset X$, $\mathcal{F}|_{\mathrm{Spec}(A)} \cong \widetilde{M}$ for an A-module M. It is *coherent* if furthermore M is finitely presented.

The following proposition requires notions developed later, but it is best placed here for reference.

Proposition 3.2. *For any quasicoherent sheaf $\mathcal{F}$ on S, the functor*

$$\mathcal{F}^a : \mathbf{Sch}/S \to \mathbf{Ab}$$
$$(f : T \to S) \mapsto \Gamma(T, f^*\mathcal{F})$$

is an $\mathcal{O}$-module which satisfies the sheaf condition in the fpqc topology.

Proof. See [1, Tag 03OG]. □

4. Morphisms in Sch

4.1. Finiteness properties.

4.1. **Finiteness properties.** These definitions are standard in the literature. We collect them here for the reader's convenience.

Definition 4.1. Let $f : X \to Y$ be a morphism of schemes.

(i) f is *locally of finite type* if there exists an open cover $\{V_i\}$ of Y such that $V_i = \mathrm{Spec}(B_i)$ and for each i, $f^{-1}(V_i)$ can be covered by open sets $U_j = \mathrm{Spec}(A_j)$ so that A_j is a finitely generated B_i-algebra.

(ii) f is *of finite type* if there exists an open cover $\{V_i\}$ of Y such that $V_i = \mathrm{Spec}(B_i)$ and for each i, $f^{-1}(V_i)$ can be covered by **finitely many** open sets $U_j = \mathrm{Spec}(A_j)$ so that A_j is a finite generated B_i-algebra.

(iii) f is *finite* if there exists an open cover $\{V_i\}$ of Y such that $V_i = \mathrm{Spec}(B_i)$ and $f^{-1}(V_i) = \mathrm{Spec}(A_i)$ for A_i a finitely generated B_i-algebra.

4.2. **Smooth morphisms.**

Definition 4.2. A morphism $f : X \to Y$ is *smooth of relative dimension n* if it satifies either of the following equivalent conditions:

(i) **(Local Jacobian condition)** There exist open covers $\{V_i\}$ of Y and $\{U_i\}$ of X such that, for every i, there is a commutative diagram

$$\begin{tikzcd} U_i \arrow[r, "\sim"] \arrow[d, "f|_{U_i}"] & W \arrow[d, "\rho|_W"] \\ V_i \arrow[r, "\sim"] & \mathrm{Spec}(B) \end{tikzcd}$$

where there exists a morphism $\mathrm{Spec}(B[x_1, \ldots, x_{n+r}]/(f_1, \ldots f_r)) \to \mathrm{Spec}(B)$ induced by the inclusion of the coefficient ring, and W is an open subscheme of $\mathrm{Spec}(B[x_1, \ldots, x_{n+r}]/(f_1, \ldots, f_r))$. Moreover, we require the morphism ρ to satisfy the *Jacobian condition*, the determinant of the matrix of partials of f_i with respect to the first r variables of $B[x_1, \ldots, x_{n+r}$ is everywhere invertible, that is,

$$\det\left(\frac{\partial f_i}{\partial x_j}\right)$$

is a nowhere zero function on W.

(ii) **(Algebraic condition)** f is locally of finite presentation, f is flat of relative dimension n, and $\Omega_{X/Y} = 0$.

We say a morphism is *smooth* if it is smooth of some arbitrary dimension n.

Theorem 4.3. *The above definitions are indeed equivalent.*

Proof. Omitted. See [2]. □

Remark 4.4. By either definition, smoothness is local on both the source and target. A consequence of this is that the locus where a morphism is smooth is open! Moreover, a smooth morphism is necessarily locally of finite presentation.

Proposition 4.5. *Let $f : X \to Y$ be a smooth morphism of relative dimension n where X and Y are schemes over S. Then, the base-change $f \times Z : X \times Z \to Y \times Z$ is smooth of relative dimension n.*

Proof. We use the algebraic condition for smoothness. Each of the criteria (locally of finite presentation, flat of dimension n, and vanishing of the sheaf of differentials) is preserved by base change by arbitrary schemes. □

Proposition 4.6. *Let $f : X \to Y$ be smooth of relative dimension n and $f : Y \to Z$ smooth of relative dimension m. Then, $g \circ f$ is smooth of relative dimension $n+m$.*

Proof. Omitted. See [2]. □

4.3. **Flat morphisms.**

> "The concept of flatness is a riddle that comes out of algebra, but which technically is the answer to many prayers" (David Mumford, *The Red Book*)

Definition 4.7. A R-module M is *flat* if any of the following equivalent conditions are satisfied:

(i) For $0 \to A \to B \to C \to 0$ exact, $0 \to A \otimes M \to B \otimes M \to C \otimes M \to 0$ is exact.
(ii) The functor $M \otimes -$ takes monomorphisms to monomorphisms.
(iii) $\mathrm{Tor}_1^R(M, N) = 0$ for all R-modules N.
(iv) $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$ and R-modules N.

These conditions say the functor $M \otimes -$ is *exact*. We say that M is *faithfully flat* if the converse to (i) holds.

Definition 4.8. A morphism of rings $f : A \to B$ is *flat* (resp. *faithfully flat*) if B enjoys the structure of a flat (resp. faithfully flat) A-module under f.

Definition 4.9. Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F}$ quasicoherent on X. We say $\mathcal{F}$ is *flat* over Y at $x \in X$ if $\mathcal{F}_x$ is flat as a $\mathcal{O}_{Y,f(x)}$ module (with the structure morphism being given by $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$). The map f is *flat* at $x \in X$ if $\mathcal{O}_X$ is flat over Y at x. f is *flat* if it is flat at every point of X. f is *faithfully flat* if it is flat and surjective.

We now list helpful properties of flatness taken from [3], without proof.

1. The composition of two flat morphisms is flat.
2. The fibre product of two flat (resp. faithfully flat) morphisms remain flat (resp. faithfully flat).
3. Flatness and faithful flatness are preserved by base change.
4. For f finitely presented, flatness is an *open* condition (the set $\{x \in X \mid f \text{ is flat at } x\}$ is open, although possibly empty).
5. Open immersions are flat (they induced isomorphisms on stalks).
6. The pullback functor $QCoh(Y) \to QCoh(X)$ is exact if induced by a flat morphism.
7. Flatness and locally of finite presentation imply open.
8. **(Generic flatness)** If $f : X \to Y$ is of finite type and Y is integral, then there is a nonempty open subset $U \subset Y$ such that the restrict of f to $f^{-1}(U)$ is flat.
9. **(Dimension of fibers)** If $f : X \to Y$ is flat at x, then
$$\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,f(x)} + \dim f^{-1}(f(x)).$$
10. **(Stalks pushforward)** If $f : X \to Y$ is flat, then $f(\mathcal{O}_{X,x}) = \mathcal{O}_{Y,f(x)}$.
11. **(Topological quotient)** If $f : X \to Y$ is faithfully flat, then the topology on Y is the quotient topology of X induced by f.

Example 4.10. We now give a collection of examples of morphisms which are not flat. The last two are taken from [3].

1. Every blowup is a nonexample because the fibers are not equidimensional.
2. Open immersions are flat because they induce isomorphisms on stalks.
3. Closed immersions are never flat unless they are also open immersions (so of the form $X \to X \amalg Y$, the inclusion of a component).

(4) Coverings of the form $\mathrm{Spec}(R^n) \to \mathrm{Spec}(R)$ because free modules are injective so $\mathrm{Tor}_i^R(R^n, R) = 0$ for $i > 0$.
(5) Either coordinate projection $\mathbb{A}^2 \to \mathbb{A}$ is faithfully flat as $A[x_1, \dots, x_n]$ is flat over A.
(6) The normalization $\mathrm{Spec}(k[t^2, t^3]) \to \mathbb{A}^1$ is not flat, although it is surjective.

4.4. **Unramified morphisms.**

Definition 4.11. A morphism of local rings $f : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ is *unramified* if the induced extension of fields is separable.

Definition 4.12. A morphism of schemes $f : X \to Y$ is *unramified* if the induced map on stalks is unramified as a morphism of local rings.

4.5. **Étale morphisms.** The underlying problem with the Zariski topology is that there are simply not enough open sets. For instance, every two paths $p_1, p_2 : [0, 1] \to \mathbb{P}^1_{\mathbb{C}}$ are homotopic, but certainly the same is not true for $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C})$. To resolve this, we require a notion of *étale open sets*, which are essentially étale morphisms.

Étale morphisms are to the algebraic category as submersions are to the differentiable one.

Definition 4.13. Let $f : X \to Y$ be a morphism of schemes. We say f is étale if it satifies one of two equivalent properties:
(i) **(Geometric definition)** f is smooth of relative dimension zero.
(ii) **(Algebraic definition)** f is flat and unramified.

There are many more definitions. For instance, one can define a class of *formally étale* morphisms via deformations and say an étale map is a formally étale map subject to finiteness conditions. For varieties over a closed field, one may say $f : V \to W$ is étale if it induces isomorphisms on the Zariski tangent space of each point.

It is nonobvious that the given definitions are equivalent. The following proposition proves it.

Proposition 4.14. *Let $f : X \to Y$ be a morphism of schemes. Then, the following are equivalent.*
(i) f is smooth of relative dimension zero.
(ii) f is flat and unramified.

The easiest case to classify are étale morphisms to a field.

Theorem 4.15. *Suppose $f : \mathrm{Spec}(S) \to \mathrm{Spec}(k)$ is étale. Then, $R \cong k_1 \times \cdots \times k_n$ where each k_i is a separable finite field extensions of k.*

5. Topologies on **Sch**

We simply give the definitions and provide a few propositions that guide the reader through the idea of the relation of the topology to the others in the section.

Here is the general picture. The arrows correspond to increasing coarseness.

$$\text{canonical} \longrightarrow \text{fpqc} \longrightarrow \text{fppf} \longrightarrow \text{étale} \longrightarrow \text{Zariski}$$

The canoncial topology is the finest topology for which the functor of points for a scheme h_X is a sheaf. A topology is deemed *subcanonical* if it is coarser than the canoncial topology.

5.1. **The Zariski topology.** This is the most well-known, and it is strictly a formalization of the standard point-set topoloy on a scheme. We define it anyways.

Definition 5.1. A *Zariski covering* of a scheme U is a family of morphisms of schemes with fixed target $\{\phi_i : U_i \to U\}$ such that

(1) Each ϕ_i is an open immersion.
(2) The family set-theoretically covers U, i.e. $\bigcup \phi_i(U_i) = U$.

The Zariski topology is the most coarse topology we will define.

Lemma 5.2. *For any Zariski covering $\{\phi_i : U_i \to U\}$ for U affine, there exists a Zariski covering $\{\psi_j : T_j \to U\}$ which is finite and refines the family $\{\phi_i\}$. Furthermore, each T_j is a standard open of U.*

Proof. Affine schemes are quasicompact and the standard opens are a base for the topology. □

Proof. The functor $(\mathbf{Aff}/S)_{\text{Zar}} \to (\mathbf{Sch}/S)_{\text{Zar}}$ for a scheme S induces an equivalence of topoi $\mathbf{Sh}((\mathbf{Aff}/S)_{\text{Zar}})$ and $\mathbf{Sh}((\mathbf{Sch}/S)_{\text{Zar}})$. □

Proof. Omitted. See [1, Tag 020W] □

5.2. **The fpqc topology.**

Definition 5.3. A *fpqc* covering is a family $\{\phi_i : U_i \to U\}_{i\in I}$ of morphisms of schemes of fixed target which satifies:

(1) Each ϕ_i is flat.
(2) For each affine open $T \subset U$, there exists a $n > 1$ and a map $a : [n] \to I$ and affine opens $V_j \subset U_{a(j)}$ so that
$$\bigcup_{j=1}^{n} \phi_{a(j)}(V_j) = U.$$

In particular, this implies the usual covering condition that $\bigcup_{i\in I} \phi_i(U_i) = U$.

5.3. **The étale topology.**

Definition 5.4. An *étale covering* of a scheme U is a family of morphisms of schemes with fixed target $\{\phi_i : U_i \to U\}$ such that

(1) Each ϕ_i is etale.
(2) The family set-theoretically covers U, i.e. $\bigcup \phi_i(U_i) = U$.

Proposition 5.5. *For a scheme X, the collection of families of étale coverings endow the following categories the structure of a site:*

(1) $X_{ét}$.
(2) $(\mathbf{Sch}/X)_{ét}$.

Proof. Recall that $X_{\text{ét}}$ (the *small étale site*) is the category whose objects are étale morphisms to X. It is a fact that any triangle of schemes whose structure morphisms are étale is automatically an étale triangle (see [1] for a proof). We check the three conditions for a Grothendieck pretology:

(1) Isomorphisms are indeed étale.
(2) Compositions of étale morphisms are étale.
(3) The base change of an étale morphism is étale.

The proof for the big étale site over X is exactly the same. ☐

The following lemma implies the étale topology is subcanonical.

Lemma 5.6. *Étale coverings are fpqc.*

Proof. Étale morphisms are flat, and an étale covering is surjective on its target, so the first condition is satisfies.

To check the quasicompactness property, let $V \subset U$ be an affine open, and let V_{ij} be a covering of $\phi_i^{-1}(V)$ for some affine opens $V_{ij} \subset U_i$. Étale morphisms are open, so

$$\bigcup_{i \in I} \bigcup_{j \in J} \phi(V_{ij})$$

is an open covering of V. By the qc assumption on V, this covering has a finite refinement. ☐

5.4. **The fppf topology.**

Definition 5.7. A *fppf covering* of a scheme U is a family of morphisms of schemes with fixed target $\{\phi_i : U_i \to U\}$ such that

(1) Each ϕ_i is flat.
(2) Each ϕ_i is locally of finite presentation.
(3) The family set theoretically covers U, i.e. $\bigcup \phi_i(U_i) = U$.

Proposition 5.8. *The fppf coverings satisfy the definition of a pretopology.*

Proposition 5.9. *Any Zariski or Étale coveing is an fppf covering.*

Proof. Clear from the definitions (open immersions are flat and locally of finite presentation because they are local isomorphisms, étale morphisms are flat and locally of finite presentation by definition). ☐

6. Cohomology

6.1. **Derived functors.**

Definition 6.1. Let $\mathcal{F} : \mathbf{A} \to \mathbf{B}$ be a functor. Suppose that

(i) $\mathcal{A}$ and $\mathbf{B}$ are abelian.
(ii) For every $X \in \mathbf{A}$, there is an injective object I and a monomorphism $X \to I$.
(iii) $\mathcal{F}$ is left exact.

Then, the *ith derived functor* of $\mathcal{F}$, written $R^i\mathcal{F}$, is the ith cohomology group of any complex of the form

$$0 \to I^1 \to I^2 \to \cdots$$

where $X \to I^\bullet$ is an injective resolution of X.

Proposition 6.2. *The definition of the functors $R^i\mathcal{F}$ is independent of the choice of injective resolution.*

Lemma 6.3. *Given a short exact sequence* $0 \to A \to B \to C \to 0$*, there is a long exact sequence*

$$
\begin{tikzcd}
0 \arrow[r] & F(A) \arrow[r] & F(B) \arrow[r] & F(C) \arrow[dll] & \\
& R^1F(A) \arrow[r] & R^1F(B) \arrow[r] & R^1F(C) \arrow[dll] & \\
& R^2F(A) \arrow[r] & R^2F(B) \arrow[r] & R^2F(C) \arrow[r] & \cdots
\end{tikzcd}
$$

Proof. Take an injective resolution of A, B, C and apply the snake lemma. □

6.2. **Sheaf cohomology.** Fix a site $\mathcal{C}$ over the category **Sch** and consider the category of abelian sheaves on $\mathcal{C}$. For $\mathcal{F} \in \mathbf{Ab}(\mathcal{C})$, we define the *ith sheaf cohomology of* $\mathcal{F}$, $H^i(-, \mathcal{F})$ to be the ith right derived functor of $\Gamma(-, \mathcal{F})$. Recall that $\Gamma(-, \mathcal{F})$ is the global sections functor for $\mathcal{F}$ on a covering U.

For this definition to make any sense, we require the following theorem.

Theorem 6.4. *The category of abelian sheaves on a site is abelian, and has enough injectives.*

Proof. See [1, Tag 03NT]. □

6.3. **Čech cohomology.** Unfortunately, (derived functor) sheaf cohomology is not easy to compute in general. In this section, we introduce an alternative cohomology theory, which is (sometimes) more computable and (sometimes) agrees with sheaf cohomology. All material in this section follows the exposition in [1, Tag 03OK]

Definition 6.5. Let **C** be a category, $\mathcal{U} = \{U_i \to U\}_{i \in I}$ a family of morphisms with fixed target, and $\mathcal{F}$ an abelian presheaf. The *Čech complex* $\check{C}(\mathcal{U}, \mathcal{F})$ by

$$\prod_{i_0 \in I} \mathcal{F}(U_{i_0}) \to \prod_{i_0, i_1 \in I} \mathcal{F}(U_{i_0} \times_U U_{i_1}) \to \prod_{i_0, i_1, i_2 \in I} \mathcal{F}(U_{i_0} \times_U U_{i_1} \times_U U_{i_2}) \to \cdots$$

where the first term is in degree 0. We allow the case $i_0 = i_1$, etc.

The *Čech cohomology groups* are defined by

$$\check{H}^p(\mathcal{U}, \mathcal{F}) := H^p(\check{C}(\mathcal{U}, \mathcal{F})).$$

After some work, it is possible to show the following theorem.

Theorem 6.6. *The functors* $\check{H}^p(\mathcal{U}, -)$ *for* $p > 1$ *are the right derived functors* $\check{H}^0(\mathcal{U}, -)$.

Outline. Demonstrate the Čech cohomology is a δ-functor. Argue that $\check{H}^p(\mathcal{U}, \mathcal{I}) = 0$ for all $p > 0$ and injective $\mathcal{I}$. Invoke Grothendieck's result that effaceable δ-functors are unique up to unique isomorphism, and that the derived functor is effeceable. □

We recall that there is an adjunction between the categories $\mathbf{Ab}(\mathcal{C})$ and $\mathbf{PAb}(\mathcal{C})$ given by sheafification on the left and the forgetful functor on the right. This, purely formally, gives the following lemma.

Lemma 6.7. *The forgetful functor* $\mathbf{Ab}(\mathcal{C}) \to \mathbf{PAb}(\mathcal{C})$ *transforms injectives to injectives.*

Proof. As do all functors which possess a left adjoint. ☐

Theorem 6.8. *Let $\mathcal{C}$ be a site. For any covering $\mathcal{U} = \{U_i \to U\}_{i\in I}$ and any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ there exists a spectral sequence*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(U, \mathcal{F}),$$

where $\underline{H}^q(\mathcal{F})$ is the abelian presheaf $v \mapsto H^q(V, \mathcal{F})$.

Proof. See [1, Tag 03OW]. ☐

Remark 6.9. This spectral sequence is the Grothendieck spectral sequence corresponding to the composition

$$\mathbf{Ab}(\mathcal{C}) \longrightarrow \mathbf{PAb}(\mathcal{C}) \xrightarrow{\check{H}^0} \mathbf{Ab}.$$

6.4. **Cohomology of quasicoherent sheaves.** For better or for worse, the topologies we have introduced thus far cannot be distinguished by the cohomology of $\mathcal{F}$ quasicoherent on X a scheme.

As is usual by now, we follow quite closely the exposition in the Stacks project. The following lemma allows us to reduce the analysis of a covering to a single standard affine cover.

Lemma 6.10. *Let $\tau \in \{fppf, étale, Zariski\}$. Let S be a scheme and $\mathcal{F}$ an abelian sheaf on $(\mathbf{Sch}/S)_\tau$, and let $\mathcal{U} = \{U_i \to U\}$ be a standard τ-covering of this site. Let $V = \coprod_{i\in I} U_i$. Then,*

(1) V is affine.
(2) $\mathcal{V} = \{V \to U\}$ is a τ-covering and an fpqc covering.
(3) The Čech complexes $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ and $\check{C}^\bullet(\mathcal{V}, \mathcal{F})$ agree.

Lemma 6.11. *Let $\mathcal{C}$ a site, $\mathcal{F}$ an abelian sheaf, U an object of $\mathcal{C}$, $p \in \mathbb{Z}_{>0}$. For all $\xi \in H^q(U, \mathcal{F})$, there exists a covering $\mathcal{U} = \{U_i \to U\}$ of U so that $\xi|_{U_i} = 0$ for all i.*

We need one final lemma taken from faithfully flat descent.

Lemma 6.12. *If $A \to B$ is faithfully flat and M is an A-module, then the complex $(B/A)_\bullet \otimes_A M$ is exact in positive degrees, and $H^0((B/A)_\bullet \otimes_A M) = M$.*

Now, the main theorem. We present the proof in a way slightly reversed from the Stacks project, in order to isolate the technical aspects for an inexperienced reader (and, indeed, author). In particular, we have split the proof into a lemma and a theorem.

Lemma 6.13. *Let S be a scheme, $\mathcal{F}$ a quasicoherent $\mathcal{O}$-module. Let $\mathcal{C}$ be the big τ-site over S, $(\mathbf{Sch}/S)_\tau$ for $\tau \in \{fppf, étale, Zariski\}$. For an object $f : U \to S$ in $\mathcal{C}$, U affine, we have that*

$$H^p_\tau(U, \mathcal{F}^a) = 0,$$

for $p > 0$.

Proof. Begin with the case $p = 1$. Choose a class $\xi \in H^1_\tau(U, \mathcal{F}^a)$. By Lemma 6.11, there exists a covering $\mathcal{U} = \{U_i \to U\}$ so that $\xi|_{U_i} = 0$. We may refine the cover so that it is a standard τ-covering.

The spectral sequence from Theorem 6.8 shows that $\xi \in \check{H}^1(\mathcal{U}, \mathcal{F}^a)$. Lemma 6.10 implies that, for the cover $\mathcal{V} = \{\coprod_{i\in I} U_i \to U\}$, $\check{H}^1(\mathcal{U}, \mathcal{F}^a) = \check{H}^1(\mathcal{V}, \mathcal{F}^a)$. But, the cover $\mathcal{V}$ is of the form $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$, so $f^*\mathcal{F} = \widetilde{M}$ for an A-module

M. The Čech complex $\check{C}^\bullet(\mathcal{V}, \mathcal{F})$ is precisely $(B/A)_\bullet \otimes_A M$, however Lemma 6.11 implies this complex has no higher cohomology.

Now, the case $p > 1$. We proceed essentially analogously. Choose $\xi \in H^p_\tau(U, \mathcal{F}^a)$. We may choose a standard τ-covering $\mathcal{U}$ for which ξ locally vanishes. The intersections $U_{i_0} \times_U \cdots \times_U U_{i_p}$ are affine, so by induction, $\check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}^a)) = 0$ for $0 < q < p$. This implies ξ is the image of a $\check{\xi} \in \check{H}^p(\mathcal{U}, \mathcal{F}^a)$. Finally, replace $\mathcal{U}$ with $\mathcal{V}$ as in the $p = 1$ case, reduce to the complex of A-modules, and conclude that it has zero higher cohomology. Thus, $\check{\xi} = 0$ and moreover $\xi = 0$. □

Theorem 6.14. *Let S be a scheme, $\mathcal{F}$ a quasicoherent $\mathcal{O}$-module. Let $\mathcal{C}$ be the big τ-site over S, $(\mathbf{Sch}/S)_\tau$ for $\tau \in \{$fppf, étale, Zariski$\}$. Then,*

$$H^p(S, \mathcal{F}) = H^p_\tau(S, \mathcal{F}^a),$$

for all $p \geq 0$.

Proof. Let $f : U \to S$ be an element of the site $\mathcal{C}$. We need to show that $H^p(U, f^*\mathcal{F}) = H^p_\tau(U, \mathcal{F}^a)$. The case $p = 0$ is a definition. For the case $p > 1$, we invoke the previous lemma.

Suppose U is separated. We may choose an affine covering of U. This induces a fpqc covering $\mathcal{U}$. The intersections $U_{i_0} \times_U \cdots \times_U U_{i_p}$ are all affine as U is separated. Thus, the rows of the spectral sequence in Theorem 6.8 are all identically zero by the previous lemma, except the zeroeth row. So, the theorem implies

$$H^p_\tau(U, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}),$$

and the last group is famously equal to $H^p(U, \mathcal{F})$. There is some more work to extend to the nonseparated case, which we will not cover. □

6.5. Analytic comparison theorem.

We simply state this amazing result.

Theorem 6.15. *Let X be a nonsingular variety over $\mathbb{C}$. For any finite abelian group Λ and $r \geq 0$, we have*

$$H^r(X_{\acute{e}t}, \Lambda) \cong H^r(X(\mathbb{C}), \Lambda),$$

where we write Λ for the constant sheaf with value Λ.

7. Set theory

This section is just a collection of interesting facts about the material previously presented.

7.1. A fpqc presheaf which may not be sheafified.

Unfortunately, with all the power that the general theory of sites and topoi allow, some (allegedly) simple properties that one would expect a presheaf to enjoy are not possible. We give an example of an fpqc sheaf for which there is no sheafification. This example is taken from [4].

Example 7.1. Fix a commutative ring R. We work in the relative category $\mathbf{Ring}/R$. For a prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$ for an R-algebra A, let $\kappa(\mathfrak{p})$ denote the fraction field of $A/\mathfrak{p}$. We define the following functor:

$$\mathcal{F}(A) = \left\{ f : \operatorname{Spec}(A) \to \nu \mid f \text{ is locally constant and } \nu < |\kappa(\mathfrak{p})| \text{ for all } \mathfrak{p} \in \operatorname{Spec}(A). \right\}.$$

Actually, we only know it is a map of underlying categorys $\mathbf{Ring}/R \to \mathbf{Set}$. However, if $f : A \to B$ is morphism of R-algebras, then $f^{-1}(\mathfrak{q})$ is a prime of A which satisfies $\left|\kappa(f^{-1}(\mathfrak{q}))\right| \leq |\kappa(\mathfrak{q})|$, so the mapping is functorial.

Let η be any cardinal and let

$$\mathcal{L}_\eta(A) = \left\{ f : \mathrm{Spec}(A) \to \eta \mid f \text{ is locally constant.} \right\}.$$

Indeed, we immediately verify this is the sheaf $h_X = \mathrm{Hom}(-, X)$ where $X = \coprod_{i\in\eta} \mathrm{Spec}(R)$. Consider a map which sends elements of a cardinal μ less than η to themselves, otherwise the elements are mapped to zero. This gives a natural transformation $\mathcal{F} \to \mathcal{L}_\eta$. Thus, $\mathcal{F}$ could be sheafified to $\mathcal{G}$, then we would have a diagram of functors

$$\begin{tikzcd} & \mathcal{G} \arrow[dr] & \\ \mathcal{F} \arrow[ur] \arrow[rr] & & \mathcal{L}_\eta \end{tikzcd}$$

We now note that given *fixed* $A \in \mathbf{Ring}/R$, there exists a cardinal λ for which $|\kappa(\mathfrak{p})| < \lambda$ for all $\mathfrak{p} \in A$. Thus, the induced morphism $\mathcal{F}(A) \to \mathcal{L}_\lambda(A)$ is injective. But, and this is the crucial point, this implies $\mathcal{F}(A) \to \mathcal{G}(A)$ must be injective *for every* $A \in \mathbf{Ring}/R$*!* From this we will derive a contradiction.

Let K/k be an extension of fields, so $R = k$. This is an fpqc covering of k. By the sheaf property, we have that the equalizer of

$$\mathcal{F}(K) \rightrightarrows \mathcal{F}(K \otimes_k K)$$

must inject into $\mathcal{G}(k)$. But, $\mathrm{Spec}(K)$ is a point, so the two arrows in the equalizer are the same, and $\mathcal{F}(K)$ is the equalizer. We have an injection

$$\mathcal{F}(K) \hookrightarrow \mathcal{G}(k),$$

but each object has the same cardinality as its argument field. We may choose an extension K/k of arbitrary size, which is absurd.

Remark 7.2. This example fails if we even move to the fppf topology, as K/k is not an fppf covering for arbitrarily large extensions K.

7.2. **Grothendieck universes.**

Definition 7.3. A *Grothendieck universe* is a set V so that

(1) If $x \in V$ and $y \in x$, then $y \in x$.
(2) If $x, y \in V$, then $\{x, y\} \in V$.
(3) If $x \in U$, then $2^x \in U$.
(4) If $I \in V$ and $\{x_i\}_{i\in I}$ is a collection of elements of V, then $\bigcup_{i\in I} x_i \in V$.

Definition 7.4. A cardinal κ is *inaccessible* if there is no sequence of cardinal operations on smaller cardinals which reach κ. It is *strongly inaccessible* if $\alpha < \kappa$ implies $2^\alpha < \kappa$ and any sum of less than κ cardinals below κ is less than κ.

We have the following (amazing) fact.

Theorem 7.5. *The following statements are equivalent.*

(i) For all x a set, there exists a universe V so that $x \in V$.
(ii) For each cardinal κ, there is a strongly inaccessible cardinal greater than κ.

Proposition 7.6. *Let κ be a strongly inaccessible cardinal. Then, $V_\kappa = \{x \text{ a set} \mid |x| < \kappa\}$ is a Grothendieck universe.*

Proposition 7.7. *Let κ be a strongly inaccessible cardinal. Fix the universe V_κ. In the definition of the functor $\mathcal{F}$, restrict the codomain of locally constant functions to be cardinals less than κ. Then, $\mathcal{F}$ may be sheafified, and its sheaf is $\mathcal{L}_\kappa$.*

Proof. This is straightfoward from the definitions. $\square$

7.3. **Lawvere-Tierney topologies.** This section is a stub—it will later be developed to address Lawvere-Tierney topologies on a Grothendieck topos.

Classically, we may identify the set of subobjects of an object of **Set** as the collection of monomorphisms $\{\phi : Z \to X\}$ modulo isomorphism. But, this is in some sense, a second-order classification of subobjects. We would like to be able to classify them in the first-order language of the category itself (without having to invoke the notion of an equivalence relation). In **Set**, for each subobject $Z \subset X$, there is a map $\chi_Z : X \to \{0,1\}$ which is zero outside of Z and one inside of it. Thus, we may identify the subobjects of X with the set $\mathrm{Hom}(X, \{0,1\})$. We recover the classical notion of a subobject in **Set** by taking the following fibre product:

$$\begin{tikzcd} Z \arrow[r, dashed] \arrow[d, dashed] & \{1\} \arrow[d, "1"] \\ X \arrow[r, "\chi_Z"] & \{0,1\} \end{tikzcd}$$

We generalize this in the following way.

Definition 7.8. Given a category $\mathcal{C}$ with finite limits, a *subobject classifier* is an object Ω with a distinguished monomorphism $\mathrm{true} : \mathbf{1} \to \Omega$, such that for every monomorphism $f : Z \to X$, there exists a unique $\chi_Z : X \to \Omega$ which allows f to arise as the following fibre product:

$$\begin{tikzcd} Z \arrow[r, dashed] \arrow[d, dashed, "f"] & \mathbf{1} \arrow[d, "\mathrm{true}"] \\ X \arrow[r, "\chi_Z"] & \Omega \end{tikzcd}$$

This definition is justified in **Set** by the previous discussion. For more generate $\mathcal{C}$, we have the following proposition.

Proposition 7.9. *Suppose that $\mathcal{C}$ has finite limits and is locally small. Then, $\mathcal{C}$ has a subobject classifier if and only if the functor*

$$\mathrm{Sub} : \mathcal{C} \to \mathbf{Set}$$
$$X \mapsto \{Z \hookrightarrow Z\},$$

is representable. In this instance, $\mathrm{Sub}(-)$ *is represented by the subobject classifier.*

Proof. By definition, the existence of a subobject classifier gives a bijection

$$\theta_X : \mathrm{Sub}(X) \cong \mathrm{Hom}(X, \Omega)$$

defined by sending a subobject Z to the characteristic function χ_Z induced by pullback. To show that this is indeed natural, consider a morphism $f : X \to Y$ and subobjects $Z \subset X$, $W \subset Y$. The functor $\mathrm{Sub}(-)$ is contravariant in pullbacks, so by considering the following diagram,

$$\begin{tikzcd} Z \arrow[r] \arrow[d, hook] & W \arrow[r] \arrow[d, hook] & \mathbf{1} \arrow[d, "\text{true}"] \\ X \arrow[r, "f"] & Y \arrow[r] & \Omega \end{tikzcd}$$

we can observe (by the pasting law for pullbacks) that θ_X is indeed natural.

Now, suppose we are given such a θ_X for all $X \in \mathcal{C}$, natural in X with representing object Θ. In particular, we have that $\mathrm{Sub}(\Omega) \cong \mathrm{Hom}(\Omega, \Omega)$, so there is a unique subobject Ω_0 corresponding to the identity on Ω. Now, consider any map $f : X \to \Omega$. By naturality of θ_X, we have a diagram

$$\begin{tikzcd} \mathrm{Sub}(\Omega) \arrow[r, "\theta_\Omega"] \arrow[d, "\mathrm{Sub}(f)"] & \mathrm{Hom}(\Omega, \Omega) \arrow[d, "h_\Omega(f)"] \\ \mathrm{Sub}(X) \arrow[r, "\theta_X"] & \mathrm{Hom}(X, \Omega) \end{tikzcd} \qquad \begin{tikzcd} \Omega_0 \arrow[r, mapsto] \arrow[d, mapsto] & 1 \arrow[d, mapsto] \\ S \arrow[r, mapsto] & \phi \end{tikzcd}$$

for some ϕ. In particular, for any subobject $S \subset X$, we can write $S = \mathrm{Sub}(f)(\Omega_0)$, and indeed, recover it as a pullback of Ω_0 along some uniquely specified ϕ. This produces the diagram:

$$\begin{tikzcd} S \arrow[r, dashed] \arrow[d, dashed] & \Omega_0 \arrow[d] \\ X \arrow[r, "\phi"] & \Omega \end{tikzcd}$$

Thus, if we can demonstrate that Ω_0 is terminal in $\mathcal{C}$, then we are finished. In the above diagram, let $S = X$. Then, we achieve a map $f_1 : X \to \Omega_0$. Suppose there were another map $f_2 : X \to \Omega_0$. Then, the square equipped with that map in its top horizontal row would still be (trivially) a pullback diagram. If we label the monic $\Omega_0 \to \Omega$ by t, then both $t \circ f_1$ and $t \circ f_2$ must equal ϕ by the uniqueness of ϕ. But, t is monic, so $f_1 = f_2$. This construction gives a unique morphism $X \to \Omega_0$ for all $X \in \mathcal{C}$, so it is terminal. The morphism t is the so-called "true" morphism. $\square$

Corollary 7.10. *Suppose $\mathcal{C}$ has finite limits, is locally small, and enjoys a subobject classifier. Then, $\mathcal{C}$ is well-powered.*

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