HOMEWORK 01
MATH 8300

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Problem 1.

(1) Describe all the isomorphism classes of representations of $\mathbb{C}[x]$ of dimension 1. How many are there?

Proof. The modules $\mathbb{C}[x]/(x-z)$ for $z \in \mathbb{C}$ are all of dimension one (they are isomorphic to $\mathbb{C}$) and they are not mutually isomorphic for $(x-z) \neq (x-s)$, as the action of x differs between them. Appealing to the structure theorem of modules over a PID, we see that these are indeed the only possible modules of dimension 1.

There are uncountably many such modules of distinct isomorphism classes when considered as $\mathbb{C}[x]$-modules. Base changing to $\mathbb{C}$ collapses these to a single isomorphism class. $\square$

(2) Describe also the isomorphism classes of representations of $\mathbb{C}[X]$ of dimension 2. Can they all be generated by a single element? If not, identify the representations that can be generated by a single element. Are any of these representations of dimension 2 simple?

Proof. Once again, an appeal to the structure theorem says that any such module must be of the form

$$\mathbb{C}[x] \bigoplus_i \frac{\mathbb{C}[x]}{I}.$$

Indeed, such modules of dimension two are necessarily of the form $\mathbb{C}[x]/(x-a) \oplus \mathbb{C}[x]/(x-b)$ for $a, b \in \mathbb{C}$ or $\mathbb{C}[x]/(x-a)^2$ for $a \in \mathbb{C}$. By the Chinese remainder theorem, $\mathbb{C}[x]/(x-a) \oplus \mathbb{C}[x]/(x-b) \cong \mathbb{C}/\left((x-a)(x-b)\right)$ for a, b distinct. Thus, we reduce our classification to the modules which can be generated by a single element and those which cannot. The former class is of the form $\mathbb{C}[x]/(f)$ for f degree 2. The latter is of the form $\mathbb{C}[x]/(x-a) \oplus \mathbb{C}[x]/(x-a)$. To see this cannot be generated by a single element, suppose that there was such a generator a. Then, $\square$

Problem 2.

(1) Let $f \in \mathbb{Q}[x]$ be an irreducible polynomial. Show that every finitely generated module for the ring $A = \mathbb{Q}[x]/(f^r)$ is a direct sum of modules isomorphic to $V_s := \mathbb{Q}[x]/(f^s)$, where $1 \leq s \leq r$. Show that A has only one simple module up to isomorphism. When $r = 5$, calculate $\dim \operatorname{Hom}_A(V_2, V_4)$ and $\dim \operatorname{Hom}(V_4, V_2)$.

Proof. Of course, that every finitely generated module over A is a direct sum of V_s comes immediately from the structure theorem, as we have that the module must be of the form

$$\frac{\mathbb{Q}[x]}{f^r} \bigoplus_i \frac{\mathbb{Q}[x]/(f^r)}{(f^s)/(f^r)},$$

which is isomorphic to $\bigoplus_i V_{s_i}$ for $s_i \leq r$.

That A has only one simple module comes from a more general fact that any (commutative, Noetherian) local ring (R, m) has a unique simple module isomorphic to R/m—to demonstrate this fact, consider the map $A \to M$ given by fixing a nonzero element a of the simple module M and taking the map to be multiplication by a. Surjectivity is implied by simplicity of M, and thus M is only of length 1 if the kernel of this map is m.

Let $r = 5$. $\operatorname{Hom}_A(V_2, V_4)$ is the collection of maps $A \to V_4$ which annihilate f^2. This only occurs if 1 is mapped into (f^2). The length of $\operatorname{Hom}_A(V_2, V_4)$ is given by considering that the submodules

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are classified by which ideal 1 is sent to in the underlying map $A \to V_4$, of which there are three choices, $0 \subset (f^3) \subset (f^2)$. This gives the desired length: 2. The module $\mathrm{Hom}_A(V_4, V_2)$ is similarly classified, and 1 may be mapped into either of the ideals $0 \subset (f) \subset V_2$, which gives a length of 2 as well. ☐

Problem 2. Show that $\mathbb{Q}[x]/((x-1)^5) \cong \mathbb{Q}[x]/((x-2)^5)$ as algebras.

Proof. As Q-algebras, they are isomorphic by the coordinate change map sending

They are not isomorphic as $\mathbb{Q}[x]$-algebras, however, as there is no $\mathbb{Q}$-linear arrow making the following diagram commute:

$$\begin{tikzcd} & \mathbb{Q}[x] \arrow[dl] \arrow[dr] & \\ \frac{\mathbb{Q}[x]}{(x-1)^5} & & \frac{\mathbb{Q}[x]}{(x-2)^5} \end{tikzcd}$$

This can be seen by ☐

Problem 3. Let A be a ring and let V be an A-module.

(1) Show that V is simple if and only if for all nonzero $x \in V$, x generates V.

Proof. Suppose there is a nonzero element x which does not generate V. What it generates is a nonzero submodule strictly contained in V which is absurd as V is simple. In the other direction, there can be so nonzero submodules of V as all of their elements generate the entirity of V. ☐

(2) Show that V is simple if and only if V is isomorphic to A/I for some maximal left ideal I.

Proof. Fix a nonzero $v \in V$ and consider the map $A \to V$ which sends a to am. This is surjective by the prior question. The kernel of this map must be maximal, otherwise the maximal ideal containing the kernel would yield a nonzero strict submodule of V as a quotient of A. Thus $V \cong A/I$ for I maximal. ☐

(3) Show that if A is a finite dimensional algebra over a field then every simple A-module is a composition factor of the free rank 1 module ${}_AA$, and hence that a finite dimensional algebra only has finitely many isomorphism classes of simple modules.

Proof. Fix a simple module $M \cong A/m$ for m a maximal left ideal. We write a composition series for ${}_AA$ which begins with the inclusion $m \to A$ (we will drop the left-module subscript notation, e.g. ${}_AA$, as from now on everything in sight is acted on the left by A), this begins a composition series as $A/m \cong M$ is simple. We may extend this to 0 by observing that ideals are vector subspaces of A, and thus we may inductively choose a maximal subideal of each element of the composition series, which terminates as each step decreases the dimension by 1. By Jordan-Hölder, there are only finitely many such (isomorphism classes of) simple modules, as the composition series is unique up to reordering of the quotient modules. ☐

Problem 4.. Let K be a field, and let $Q_2 = y\bullet \xleftarrow{\beta} \bullet x$ be the quiver in the notes with representations $S_x = 0 \xleftarrow{0} K$, $S_y = K \xleftarrow{0} 0$, and $V = K \xleftarrow{1} K$.

(1) Compute $\dim \mathrm{Hom}_{K(F(Q_2))}(S_x, V)$, $\dim \mathrm{Hom}_{K(F(Q_2))}(V, S_x)$ and $\dim \mathrm{Hom}_{K(F(Q_2))}(V, V)$.

Proof. We consider the following diagram

$$\begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d, "0"] & K \arrow[d, "1"] \\ 0 \arrow[r, "\beta"] & K \end{tikzcd}$$

The only choice for β is 0. To make things commute, we require $\alpha = 0$ as well, which implies the dimension of the Hom-module is 0. Similarly, we may consider

$$\begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d, "1"] & K \arrow[d, "0"] \\ K \arrow[r, "\beta"] & 0 \end{tikzcd}$$

β must be 0, and α can be anything. Thus, the dimension of $\mathrm{Hom}(V, S_x)$ is 1. Finally, we may consider

$$\begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d, "1"] & K \arrow[d, "1"] \\ K \arrow[r, "\beta"] & K \end{tikzcd}$$

For which we require $\alpha = \beta$ for commutativity, which yields a dimension of 1 for $\mathrm{Hom}(V, V)$. $\square$