HOMEWORK 01
MATH 8300

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Problem 1.

(1) Describe all the isomorphism classes of representations of $\mathbb{C}[x]$ of dimension 1. How many are there?

Proof. The modules $\mathbb{C}[x]/(x-z)$ for $z \in \mathbb{C}$ are all of dimension one (they are isomorphic to $\mathbb{C}$) and they are not mutually isomorphic for $(x-z) \neq (x-s)$, as the action of x differs between them. Appealing to the structure theorem of modules over a PID, we see that these are indeed the only possible modules of dimension 1.

There are uncountably many such modules of distinct isomorphism classes when considered as $\mathbb{C}[x]$-modules. Base changing to $\mathbb{C}$ collapses these to a single isomorphism class. $\square$

(2) Describe also the isomorphism classes of representations of $\mathbb{C}[X]$ of dimension 2. Can they all be generated by a single element? If not, identify the representations that can be generated by a single element. Are any of these representations of dimension 2 simple?

Proof. Once again, an appeal to the structure theorem says that any such module must be of the form

$$\mathbb{C}[x] \bigoplus_i \frac{\mathbb{C}[x]}{I}.$$

Indeed, such modules of dimension two are necessarily of the form $\mathbb{C}[x]/(x-a) \oplus \mathbb{C}[x]/(x-b)$ for $a, b \in \mathbb{C}$ or $\mathbb{C}[x]/(x-a)^2$ for $a \in \mathbb{C}$. By the Chinese remainder theorem, $\mathbb{C}[x]/(x-a) \oplus \mathbb{C}[x]/(x-b) \cong \mathbb{C}/\left((x-a)(x-b)\right)$ for a, b distinct. Thus, we reduce our classification to the modules which can be generated by a single element and those which cannot. The former class is of the form $\mathbb{C}[x]/(f)$ for f degree 2. The latter is of the form $\mathbb{C}[x]/(x-a) \oplus \mathbb{C}[x]/(x-a)$. To see this cannot be generated by a single element, suppose that there was such a generator z. Then, as the map from $\mathbb{C}[x]$ that multiplies by z is surjective, which gives an isomorphism $\mathbb{C}[x]/(a)$ with the aforementioned module. However, this cannot be true, as the action of $x-a$ on the module $\mathbb{C}[x]/(x-a) \oplus \mathbb{C}[x]/(x-a)$ is trivial, and if that were true for $\mathbb{C}[x]/(z)$, then $z = x-a$ up to units and the $\mathbb{C}[x]/(z)$ has dimension 1 as a module, which does not equal the dimension of $\mathbb{C}[x]/(x-a) \oplus \mathbb{C}[x]/(x-a)$, which is 2. $\square$

Problem 2.

(1) Let $f \in \mathbb{Q}[x]$ be an irreducible polynomial. Show that every finitely generated module for the ring $A = \mathbb{Q}[x]/(f^r)$ is a direct sum of modules isomorphic to $V_s := \mathbb{Q}[x]/(f^s)$, where $1 \leq s \leq r$. Show that A has only one simple module up to isomorphism. When $r = 5$, calculate $\dim \mathrm{Hom}_A(V_2, V_4)$ and $\dim \mathrm{Hom}(V_4, V_2)$.

Proof. Of course, that every finitely generated module over A is a direct sum of V_s comes immediately from the structure theorem, as we have that the module must be of the form

$$\frac{\mathbb{Q}[x]}{f^r} \bigoplus_i \frac{\mathbb{Q}[x]/(f^r)}{(f^s)/(f^r)},$$

which is isomorphic to $\bigoplus_i V_{s_i}$ for $s_i \leq r$.

That A has only one simple module comes from a more general fact that any (commutative, Noetherian) local ring (R, m) has a unique simple module isomorphic to R/m—to demonstrate this fact, consider the map $A \to M$ given by fixing a nonzero element a of the simple module M and

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taking the map to be multiplication by a. Surjectivity is implied by simplicity of M, and thus M is only of length 1 if the kernel of this map is m.

Let $r = 5$. $\mathrm{Hom}_A(V_2, V_4)$ is the collection of maps $A \to V_4$ which annihilate f^2. This only occurs if 1 is mapped into (f^2). The length of $\mathrm{Hom}_A(V_2, V_4)$ is given by considering that the submodules are classified by which ideal 1 is sent to in the underlying map $A \to V_4$, of which there are three choices, $0 \subset (f^3) \subset (f^2)$. This gives the desired length: 2. The module $\mathrm{Hom}_A(V_4, V_2)$ is similarly classified, and 1 may be mapped into either of the ideals $0 \subset (f) \subset V_2$, which gives a length of 2 as well. ☐

Problem 2. Show that $\mathbb{Q}[x]/((x-1)^5) \cong \mathbb{Q}[x]/((x-2)^5)$ as algebras.

Proof. As $\mathbb{Q}$-algebras, they are each isomorphic to $\mathbb{Q}[\epsilon]/(\epsilon^5)$ under the map sending ϵ to $x-1$ and $x-2$, respectively.

They are not isomorphic as $\mathbb{Q}[x]$-algebras, however, as there is no surjective $\mathbb{Q}[x]$-linear arrow making the following diagram commute:

$$\begin{tikzcd} & \mathbb{Q}[x] \arrow[dl] \arrow[dr] & \\ \frac{\mathbb{Q}[x]}{(x-1)^5} & & \frac{\mathbb{Q}[x]}{(x-2)^5} \end{tikzcd}$$

This can be seen by the fact that $x-2$ is sent to a unit in the left map and sent to a nonunit in the right map. To prove that $x-2$ is a unit in $\mathbb{Q}[x]/((x-1)^5)$, it is sufficient to observe that as $(x-1)^5$ and $(x-2)$ are coprime in $\mathbb{Q}[x]$, there are polynomials $p(x)$, $q(x)$ so that

$$p(x)(x-2) + q(x)(x-1)^5 = 1,$$

and so $p(x)$ is the multiplicative inverse for $(x-2)$ in $\mathbb{Q}[x]/((x-1)^5)$.

Perhaps easier to see is that each of these algebras are local, and their residue fields are not isomorphic as $\mathbb{Q}$-algebras as the action of x are different in either. ☐

Problem 3. Let A be a ring and let V be an A-module.

(1) Show that V is simple if and only if for all nonzero $x \in V$, x generates V.

Proof. Suppose there is a nonzero element x which does not generate V. What it generates is a nonzero submodule strictly contained in V which is absurd as V is simple. In the other direction, there can be so nonzero submodules of V as all of their elements generate the entirity of V. ☐

(2) Show that V is simple if and only if V is isomorphic to A/I for some maximal left ideal I.

Proof. Fix a nonzero $v \in V$ and consider the map $A \to V$ which sends a to am. This is surjective by the prior question. The kernel of this map must be maximal, otherwise the maximal ideal containing the kernel would yield a nonzero strict submodule of V as a quotient of A. Thus $V \cong A/I$ for I maximal. ☐

(3) Show that if A is a finite dimensional algebra over a field then every simple A-module is a composition factor of the free rank 1 module ${}_AA$, and hence that a finite dimensional algebra only has finitely many isomorphism classes of simple modules.

Proof. Fix a simple module $M \cong A/m$ for m a maximal left ideal. We write a composition series for ${}_AA$ which begins with the inclusion $m \to A$ (we will drop the left-module subscript notation, e.g. ${}_AA$, as from now on everything in sight is acted on the left by A), this begins a composition series as $A/m \cong M$ is simple. We may extend this to 0 by observing that ideals are vector subspaces of A, and thus we may inductively choose a maximal subideal of each element of the composition series, which terminates as each step decreases the dimension by 1. By Jordan-Hölder, there are only finitely many such (isomorphism classes of) simple modules, as the composition series is unique up to reordering of the quotient modules. ☐

Problem 4.. Let K be a field, and let $Q_2 = y\bullet \xleftarrow{\beta} \bullet x$ be the quiver in the notes with representations $S_x = 0 \xleftarrow{0} K$, $S_y = K \xleftarrow{0} 0$, and $V = K \xleftarrow{1} K$.

(1) Compute $\dim \mathrm{Hom}_{K(F(Q_2))}(S_x, V)$, $\dim \mathrm{Hom}_{K(F(Q_2))}(V, S_x)$ and $\dim \mathrm{Hom}_{K(F(Q_2))}(V, V)$.

Proof. We consider the following diagram

$$\begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d, "0"] & K \arrow[d, "1"] \\ 0 \arrow[r, "\beta"] & K \end{tikzcd}$$

The only choice for β is 0. To make things commute, we require $\alpha = 0$ as well, which implies the dimension of the Hom-module is 0. Similarly, we may consider

$$\begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d, "1"] & K \arrow[d, "0"] \\ K \arrow[r, "\beta"] & 0 \end{tikzcd}$$

β must be 0, and α can be anything. Thus, the dimension of $\mathrm{Hom}(V, S_x)$ is 1. Finally, we may consider

$$\begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d, "1"] & K \arrow[d, "1"] \\ K \arrow[r, "\beta"] & K \end{tikzcd}$$

For which we require $\alpha = \beta$ for commutativity, which yields a dimension of 1 for $\mathrm{Hom}(V, V)$. □

(2) Determine whether or not the path algebra $K(F(Q_2))$ is isomorphic to either $K[x]/(x^2)$ or $K[x]/(x^3)$.

Proof. □

Problem 5. Show that the path algebras of the two quivers $Q_1 = \bullet \to \bullet \leftarrow \bullet$ and $Q_2 = \bullet \leftarrow \bullet \to \bullet$ over R are isomorphic to the algebras of 3×3 matrices over R of the form

$$R_1 = \begin{bmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & d & e \end{bmatrix} \quad \text{and} \quad R_2 = \begin{bmatrix} a & 0 & 0 \\ b & c & d \\ 0 & 0 & e \end{bmatrix}$$

Proof. I claim that $K(F(Q_1)) = R_2$ and $K(F(Q_2)) = R_1$. To see this, we write out the multiplication tables. Write $Q_1 = x \xrightarrow{\alpha} y \xleftarrow{\beta} z$. The multiplication tables are:

$K(F(Q_1))$	1_x	1_y	1_z	α	β
1_x	1_x	0	0	0	0
1_y	0	1_y	0	α	β
1_z	0	0	1_z	0	0
α	α	0	0	0	0
β	0	0	β	0	0

and

R_2	a	c	e	b	d
a	a	0	0	0	0
c	0	c	0	b	d
e	0	0	e	0	0
b	b	0	0	0	0
d	0	0	d	0	0

,

where we write $i \in \{a, b, c, d, e\}$ for the matrix with 1 in the ith place, where i refers to the coordinate of R_1 indicated in the statement of the question. These algebras are necessarily now the same, as mapping obvious elements together, it is immediate that addition is preserved at $1 = 1_x + 1_y + 1_z \mapsto 1 = a + c + e$.

It is not too hard to see that transposing this multiplication table corresponds to transposing R_1 to R_2 and similarly reversing the direction of α, β (as transposing the multiplication table simply conjugates the endpoints of the path). However, for clarity, we write the multiplication tables below:

$K(F(Q_2))$	1_x	1_y	1_z	α	β
1_x	1_x	0	0	α	0
1_y	0	1_y	0	0	0
1_z	0	0	1_z	0	β
α	0	α	0	0	0
β	0	β	0	0	0

and

R_1	a	c	e	b	d
a	a	0	0	b	0
c	0	c	0	0	0
e	0	0	e	0	d
b	0	b	0	0	0
d	0	d	0	0	0

.

□

Problem 6. Let k be a field. Show that the space of column vectors k^n is a simple module for $M_n(k)$. Show that, as a left module, $M_n(k)$ is the direct sum of n modules each isomorphic to k^n. Show that, up to isomorphism, $M_n(k)$ has only one simple module.

Proof. Let $v \in k^n$ be a nonzero column vector. We consider k^n a vector space with standard basis. As v is nonzero, it has a nonzero coordinate i. Indeed, as k is a field, we may choose a matrix which maps v to the ith basis vector. There exists a linear transformation which sends this basis vector to any other basis vector in k^n of our liking, which is expressed as an element of $M_n(k)$. This implies that v generates all of k^n and thus k^n is simple.

Let $N_i \subset M_n(k)$ refer to the submodule of matrices which are zero outside of the ith column. This is a submodule as it is a subset which inherits the addition and action of $M_n(k)$, and it is closed under this action by the definition of matrix multiplication. There is an obvious isomorphism of $N_i \cong k^n$ by identifying k^n with the ith column of any given matrix. Of course, we have a surjection $M_n(k) \to \bigoplus_i^n N_i$ which projects the ith column to N_i, and its kernel is zero by definition.

$M_n(k)$ has only one simple module (up to isomorphism) as the composition series

$$0 \subset N_0 \subset N_0 \oplus N_1 \subset \cdots \subset \bigoplus_{i=0}^{n-1} N_i = M_n(k)$$

has composition factors N_i and the ith place, which is simple as they agree with k^n. By Jordan-Hölder, we conclude. □

Problem 7. Let C_n denote the category with n objects labeled $a_1, \dots, a_n$ and where there is a unique homomorphism $a_i \to a_j$ for every ordered pair of numbers (i, j). Show that the category algebra RC_n is isomorphic to the algebra of $n \times n$ matrices $M_n(R)$.

Proof. This is simply unpacking the definitions. The algebra RC_n has, as a generating set, the arrows $a_i \to a_j =: \beta_{ij}$. The multiplication is defined to be:

$$X = \left(\sum_{i \le j} b_{ij} \beta_{ij} \right) \cdot \left(\sum_{i \le j} c_{ij} \beta_{ij} \right),$$

where the coefficient of β_{ij} in X is exactly the sum $\sum_{k=1}^n b_{ik} c_{kj}$, because the product is zero if the morphisms are not composible and the R-inherited product otherwise. If we consider the map of sets $RC_n \to M_n(R)$ sending β_{ij} to the unit in the ijth coordinate of $M_n(R)$ then extending R-linearly, we find this is precisely the definition of matrix multiplication. □

Problem 8. Let x be an object of a finite category $\mathcal{C}$.

(1) Show that the subset $R\mathcal{C} \cdot 1_x$ of the category algebra $R\mathcal{C}$ is the span of the morphisms whose domain is x, and that $1_x \cdot R\mathcal{C}$ is the span of the morphisms whose codomain is x.

Proof. Of course, this is just the definition of the multiplication in the category algebra. For any $r\beta \in R\mathcal{C}$, we have that $(r\beta) \cdot 1_x$ is 0 if $o(\beta) \neq x$ and $r\beta$ otherwise. This is the definition of the desired span. For the other direction, the argument is the same, but we consider $t(\beta)$ as the multiplication is on the other side. □

(2) Show that $R\mathcal{C} = \bigoplus_{x \in \operatorname{Ob} \mathcal{C}} R\mathcal{C} \cdot 1_x$ as left $R\mathcal{C}$-modules.

Proof. Define a map $\phi : R\mathcal{C} \to \bigoplus R\mathcal{C} \cdot 1_x$ by projecting a formal linear sum of morphisms to the direct summand of its domain. By part (1), this is surjective. The only element mapped to the identity is $\sum_{x \in \operatorname{Ob} \mathcal{C}} 1_x$, so it is a bijection. The projection, of course, preserves the addition. So, we only have to check that the map is stable under the action of the ring. Left multiplication by $R\mathcal{C}$ preserves the domain of a morphism, so the map is a bijective $R\mathcal{C}$-module homomorphism. □

(3) Let $R\operatorname{Hom}_{\mathcal{C}}(x, -)$ denote the functor $C \to R-\mathrm{mod}$ that sends an object y to the free R-module with the set of homomorphisms $\operatorname{Hom}_{\mathcal{C}}(x, y)$ as a basis. Under the correspondence between representations of $\mathcal{C}$ over R and $R\mathcal{C}$-modules, show that the functor $R\operatorname{Hom}(x, -)$ corresponds to the left $R\mathcal{C}$-module $R\mathcal{C} \cdot 1_x$ and that $R\operatorname{Hom}(-, x)$ corresponds to the right $R\mathcal{C}$-module $1_x \cdot R\mathcal{C}$.

Proof. Under the Mitchell-correspondence, the module formed from the representation (functor) given by $\operatorname{Hom}(x, -)$ is precisely the direct sum over the objects x in $\mathcal{C}$, e.g. $\bigoplus_{y \in \operatorname{Ob} \mathcal{C}} R\operatorname{Hom}(x, y)$. This is, by definition, the R-linear span of all morphisms whose domain is x. By (1), this is $R\mathcal{C} \cdot 1_x$. The only thing to check is that the action of $R\mathcal{C}$ agrees on each. Of course, by definition, $R\operatorname{Hom}(x, y)$ is acted on by the basis elements of $R\mathcal{C}$ by composition, which is the same action of $R\mathcal{C} \cdot 1_x$. The same procedure shows that $R\operatorname{Hom}(-, x)$ corresponds to $1_x \cdot R\mathcal{C}$. □