Homework 1

Jacob Skinner

April 12, 2018

Abstract

Here we examine Euler's method of numerical approximation as applied to modeling nuclear decay. We explore both its effectiveness and its limitations, for a variety of possible physical situation.

1 Introduction

Consider a sample which consists of three substances, A, B, and C. Substance A radioactively decays into B with a time constant τ_A, which itself decays into substance C with a time constant τ_B. Suppose that we cannot directly measure the amounts of A, B, or C. Luckily, we can easily measure the amount of substance B relative to A: N_B/N_A. It also just so happens that substance B only appears as a decay product of A, so we can use their ratio to measure the age of the sample.

In order to find the age of a given sample, we should be able to simulate the aging process of that sample, assuming we know the values of the time constants τ_A and τ_B. Simply run the simulation forward, and record the time at which the simulated value of N_B/N_A matches the measured value.

2 The Analytical Solution

2.1 N_B/N_A as a Function of Time

The equations describing the concentrations of A and B relative to the initial amount ($N_A(0)$) as functions of time are already known for 2 possible cases:

$$\frac{N_B}{N_A}(T) = \begin{cases} T & \gamma = 1 \\ \frac{1-\mathrm{e}^{-T(\gamma-1)}}{\gamma-1} & \gamma \neq 1 \end{cases}$$

Note that $\gamma = \tau_A/\tau_B$ and $T = t/\tau_A$

This gives rise to distinct behaviors depending on whether $\gamma < 1$, $\gamma = 1$, or $\gamma > 1$. Interestingly, if $\gamma > 1$, the curve is range limited such that N_B/N_A asymptotically approaches the value $\frac{1}{\gamma-1}$ from below. More on that in section 4. Figure 1 shows the different behaviors that $\frac{N_B}{N_A}(T)$ takes on depending on the value of γ.

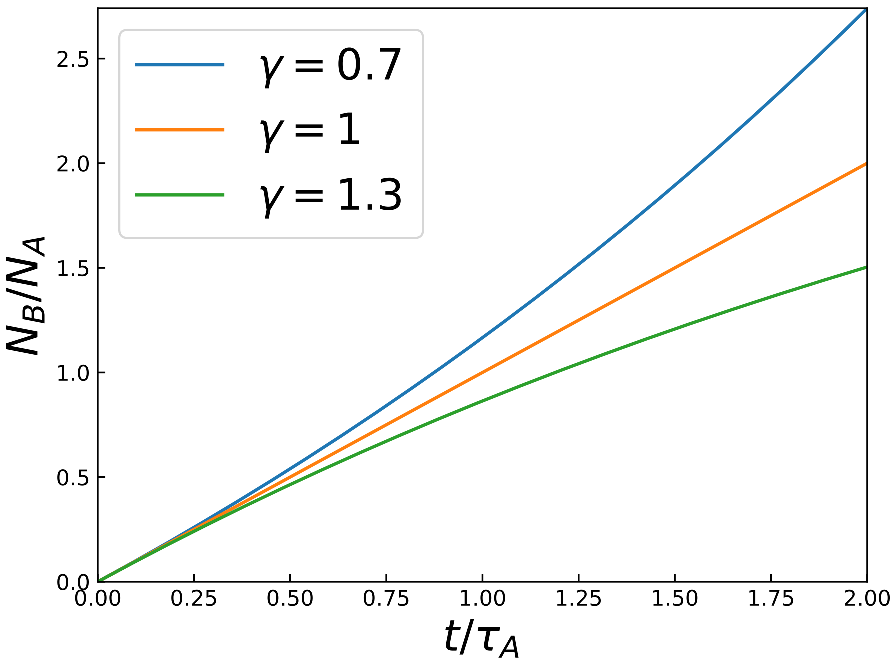

Figure 1: N_B/N_A as a function of t/τ_A. Note that the $\gamma < 1$ case exhibits upwards concavity, while the $\gamma > 1$ case exhibits downwards concavity. If $\gamma = 1$, the result is the unity line, i.e. $N_B/N_A = t/\tau_A$ for all t.

3 The Numerical Approximation

3.1 Physical System

The system is described by a pair of ordinary differential equations:

$$\frac{dN_A}{dt} = \frac{-N_A}{\tau_A}$$

$$\frac{dN_B}{dt} = \frac{N_A}{\tau_A} - \frac{N_B}{\tau_B}$$

Their solution is described in section 2. For each tiny advancement in time, a small amount dN_A decays into B, and a tiny amount dN_B decays into C.

3.2 Algorithmic Approach

The application of Euler's method to this problem was relatively straightforward, consisting of few steps. An important simplifying assumption to note is that $\frac{N_B}{N_A}(0) = 0$, i.e. we let $t = 0$ correspond to the time at which none of substance A has yet decayed. Also, we chose $dt = 0.001(\tau_A)$, estimating that something like 0.05% of substance A would decay in that time frame, giving adequate resolution over several multiples of τ_A.

1. At $T = 0$ set $N_B = 0$ and $N_A = 1$
2. Use the given differential equations to update the values of N_B and N_A.
3. Update the simulation time ($T = T + dt$).
4. Repeat steps 2-3 until N_B/N_A equals the measured value.
5. Record the value of T at which the simulation ended.

This approach is agnostic of the analytic solutions and the choice of γ.

4 Precision and Error

Here we examine the long term behavior of the analytic system, and its implications for the numeric solver. See Figure 2 for a comparison of the error between the numeric solver and the analytic solution (as a function of the measured substance ratio) for varying values of γ.

4.1 $\gamma = 1$

We find that for the linear case, a 0.05% error in the measured value of N_B/N_A corresponds to a similar error in the numerically calculated solution. The precision of the input and output are linked in this case. In addition, the linear nature of the analytic solution for this case means that the accuracy of the numeric solution is essentially limited only by machine error. This result is unaffected by the age of the sample, thanks to the linear nature of N_B/N_A.

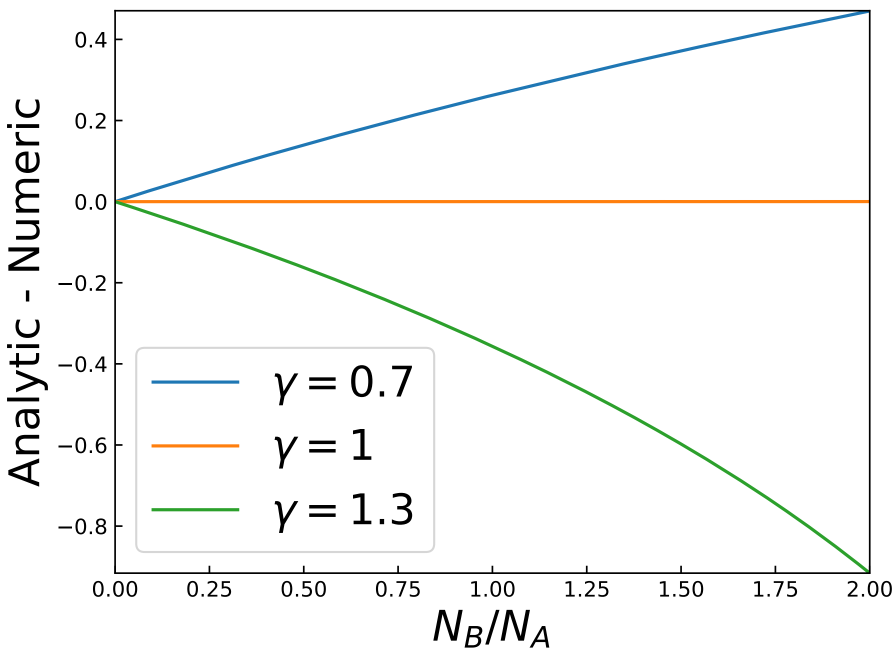

Figure 2: Difference between analytic and numerically calculated sample ages, as a function of N_B/N_A, for 3 values of γ. The difference for $\gamma = 1.3$ eventually reaches a minimum, and returns slowly to 0 (the time scale here is too short for this behavior to be visible). Note the how the error in the linear case is constant.

4.2 $\gamma < 1$

In this case, we find that the analytic solution is concave up, and that the lower γ is, the more stark this concavity. Because Euler's method has no built-in error correction, the numerical solution slowly drifts away from the analytic result. As a result, the solver is less accurate for older samples (error has more time to accumulate), consistently overestimating their age.

It is interesting to note that because N_B/N_A becomes more steep in time, if one were measuring the age analytically, older samples would be known *more* precisely than newer ones. i.e. the effect of measurement precision becomes more and more diminished as the sample has more time to decay. This seems to smell a bit like quantum uncertainty...

4.3 $\gamma > 1$

This case has the most interesting behavior. The analytically calculated value of N_B/N_A asymptotically approaches $1/(\gamma - 1)$ as $t \to \infty$. It becomes flatter and flatter over time. This means that for a sufficiently old sample, even a measurement uncertainty in N_B/N_A as small as 0.005% would correspond to an *enormous* uncertainty in the age of the object. This uncertainty is inherent to the system, not a result of numeric imprecision. It arises because N_B/N_A becomes increasingly "constant" as the sample ages.

An almost opposite behavior is observed when considering the accuracy of the numerically calculated result. The flattening of N_B/N_A means that on long time scales, the numeric solver "catches up" to the analytic result; the inaccuracy shrinks back down from a maximum, rather than gradually increasing like in the $\gamma < 1$ case.

5 Conclusions

Not surprisingly, the accuracy of Euler's method depends on the time step used. Smaller time steps mean more accurate solutions, though more computational time. With respect to the physical system in question, the most important factor is the value of γ, which gives rise to 3 qualitatively different outcomes.

If $\gamma < 1$, the accuracy of the numeric solution and the effect of the precision in the measured value of N_B/N_A both decrease with the age of the object.

if $\gamma = 1$, the numeric solution nearly perfect and independent of age, while the precision of the age calculation is directly coupled to the measurement precision for N_B/N_A.

if $\gamma > 1$, the numeric solution is fairly accurate and does not decrease with the age of the object. Unfortunately, the possible precision decreases with the age of the object, as N_B/N_A changes more and more slowly with time, approaching a constant value of 1/
$gamma - 1$.