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Assignment 2

Part I

a) For a single random variable X, cumularive probability function $F(x)$ represents the probability X takes on a value $\leq x$

$F(x) = P(X \leq x)$

For continuous variables $X_1, \ldots, X_K$ and $g = \max\{X_1, \ldots, X_K\}$, the CDF $G(x)$ represents the probability g takes on a value $\leq x$

$G(x) = P(g \leq x)$

Because "g" finds the max, all variables $X_1, \ldots, X_K$ will be $\leq x$

$G(x) = P(X_1 \leq x) \ldots P(X_2 \leq x)$

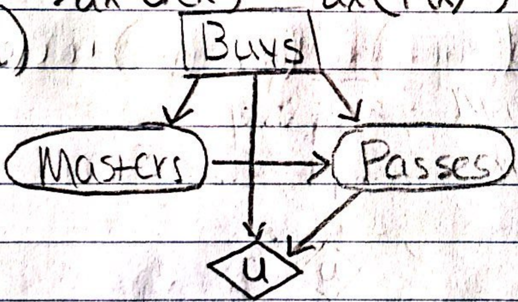
Because $X_1, \ldots, X_K$ have the same PDF and are independent

$G(x) = (F(x))^k$

Because the CDF is the integral of the PDF, we can differentiate

$\frac{d}{dx} G(x) = \frac{d}{dx}(F(x)^k) = k\,F(x)^{k-1} F'(x) = kf(x)(F(x))^{k-1}$

b) i)



ii) $EV = P(b)U(b) + P(\neg b)U(\neg b) + P(p)U(p) + P(\neg p)U(\neg p)$

$EV_b = U(b) + [P(p|b,m)P(m|b) + P(p|b,\neg m)P(\neg m|b)]U(p) + P(\neg p)U(\neg p)$

$EV_b = -100 + [(0.9)(0.9) + (0.5)(0.1)](2000) + \cancel{P(\neg p)(0)}^{0}$

$EV_b = -100 + 1720$

$EV_b = 1620$

$EV_{\neg b} = U(\neg b) + [P(p|\neg b, m)P(m|\neg b) + P(p|\neg b, \neg m)P(\neg m|\neg b)]U(p) + \cancel{P(\neg p)U(\neg p)}^{0}$

$EV_{\neg b} = 0 + [(0.8)(0.7) + (0.3)(0.3)](2000)$

$EV_{\neg b} = 1300$

iii) Sam should by the book because the expected utility for buying the book is larger than not buying the book. This follows the Maximum Expected Utility Principle.

c) $b_{t+1} = \alpha O_{t+1} T^T b_t$

$$b_1 = \alpha \begin{bmatrix} 0.0 \\ 0.5 \\ 0.75 \end{bmatrix} \circ \begin{bmatrix} 0.81 & 0 & 0 \\ 0.18 & 0.9 & 0 \\ 0.01 & 0.1 & 1.0 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$b_1 = \alpha \begin{bmatrix} 0.0 \\ 0.5 \\ 0.75 \end{bmatrix} \circ \begin{bmatrix} 0.27 \\ 0.36 \\ 0.37 \end{bmatrix}$$

$$b_1 = \alpha \begin{bmatrix} 0.0 \\ 0.18 \\ 0.2775 \end{bmatrix}$$

$$b_1 = \langle 0.0, 0.3934, 0.6066 \rangle$$

$$b_2 = \alpha \begin{bmatrix} 1.0 \\ 0.5 \\ 0.25 \end{bmatrix} \circ \begin{bmatrix} 0.81 & 0.0 & 0.0 \\ 0.18 & 0.9 & 0.0 \\ 0.01 & 0.1 & 1.0 \end{bmatrix} \begin{bmatrix} 0.0 \\ 0.3934 \\ 0.6066 \end{bmatrix}$$

$$b_2 = \alpha \begin{bmatrix} 1.0 \\ 0.5 \\ 0.25 \end{bmatrix} \circ \begin{bmatrix} 0.0 \\ 0.35406 \\ 0.64594 \end{bmatrix}$$

$$b_2 = \alpha \begin{bmatrix} 0.0 \\ 0.17703 \\ 0.161485 \end{bmatrix}$$

$$b_2 = \langle 0.0, 0.523, 0.477 \rangle$$

d) i) For $n=3$,

$$EC = p_1(C_1^o + C_1^r) + p_2(C_1^o + C_2^o + C_2^r) + p_3(C_1^o + C_2^o + C_3^o + C_3^r)$$

$$EC = \sum_{i=1}^{3} p_i\left(C_i^r + \sum_{j=1}^{i} C_j^o\right)$$

$$EC = \sum_{i=1}^{n} p_i\left(C_i^r + \sum_{j=1}^{i} C_j^o\right) \leftarrow \text{Replace 3 with } n$$

$$EC = \sum_{i=1}^{n} p_i C_i^r + \sum_{i=1}^{n} p_i \sum_{j=1}^{i} C_j^o$$

ii) Let's say we have two components, x and y

$$EC_{xy} = p_x C_x^r + p_y C_y^r + p_x C_x^o + p_y(C_x^o + C_y^o)$$

$$EC_{yx} = p_y C_y^r + p_x C_x^r + p_y C_y^o + p_x(C_y^o + C_x^o)$$

Let's also say that the Expected cost of choosing x then y is greater than choosing y then x

$$EC_{xy} > EC_{yx}$$

$$p_x C_x^r + p_y C_y^r + p_x C_x^o + p_y(C_x^o + C_y^o) > p_y C_y^r + p_x C_x^r + p_y C_y^o + p_x(C_y^o + C_x^o)$$

$$p_x C_x^o + p_y(C_x^o + C_y^o) > p_y C_y^o + p_x(C_y^o + C_x^o)$$

$$p_x C_x^o + p_y C_x^o + p_y C_y^o > p_y C_y^o + p_x C_y^o + p_x C_x^o$$

$$p_y C_x^o > p_x C_y^o$$

$$\frac{p_y}{C_y^o} > \frac{p_x}{C_x^o}$$

As you can see the smaller p_i/C_i^o ratio leads to a smaller cost. This proves the optimal action sequence is to choose components in order of ascending p_i/C_i^o ratios.

$P(TL_0|e_{1:t})$ and $P(TR_0|e_{1:t})$ is belief

e) $P(TL_1|GL_1, L) = ?$

$P(X_{t+1}|e_{1:t+1}) = \alpha P(e_{t+1}|X_{t+1}) \sum_{x_t} P(X_{t+1}|x_t) P(x_t|e_{1:t})$

$P(TL_1|GL_1, L) = \alpha P(GL_1|TL_1) \sum_{x_t} P(TL_1|x_t) P(x_t|e_{1:t})$

$P(TL_1|GL_1, L) = \alpha P(GL_1|TL_1) [P(TL_1|TL_0, L) P(TL_0|e_{1:t}) + P(TL_1|TR_0, L) P(TR_0|e_{1:t})]$

$P(TL_1|GL_1, L) = \alpha (0.85) [1.0(0.85) + 0.0(0.15)]$

$P(TL_1|GL_1, L) = \alpha\, 0.7225$

$P(TR_1|GR_1, L) = \alpha P(GL_1|TR_1, L) \sum_{x_t} P(TR_1|x_t) P(x_t|e_{1:t})$

$P(TR_1|GR_1, L) = \alpha P(GL_1|TR_1 L) [P(TR_1|TL_0, L) P(TL_0|e_{1:t}) + P(TR_1|TR_0, L) P(TR_0|e_{1:t})]$

$P(TR_1|GR_1, L) = \alpha (0.15) [0.0(0.85) + 1.0(0.15)]$

$P(TR_1|GR_1, L) = \alpha\, 0.0225$

$\alpha = 1/_{0.7225 + 0.0225} = 1.3423$

Updated Belief = $\langle 0.970, 0.030 \rangle$

f) If we apply a Bellman update, B, on value function, V^t, we get the value function at the next step, V^{t+1}

$$BV = V^{t+1}$$

The contraction property of the Bellman update states that

$$\|BV^{t+1} - BV^t\| \leq \gamma \|V^{t+1} - V^t\|$$

but let's replace BV^{t+1} and BV^t with first statement

$$\|V^{t+2} - V^{t+1}\| \leq \gamma \|V^{t+1} - V^t\|$$

The value function for the optimal policy, V^*, cannot be improved, so...

$$BV^* = V^*$$

Let's now start with the following,

$$\|BV^* - BV^t\| \leq \gamma \|V^* - V^t\|$$

$$\|V^* - V^{t+1}\| \leq \gamma \|V^* - V^t\|$$

$$\|V^* - V^{t+1}\| \leq \gamma (\|V^* - V^{t+1}\| + \|V^{t+1} - V^t\|)$$

$$\|V^* - V^{t+1}\| \leq \gamma \|V^* - V^{t+1}\| + \gamma \|V^{t+1} - V^t\|$$

$$(1-\gamma)\|V^* - V^{t+1}\| \leq \gamma \|V^{t+1} - V^t\|$$

$$(1-\gamma/\gamma)\|V^* - V^{t+1}\| \leq \|V^{t+1} - V^t\|$$

Let's now insert the inequality from the problem statement

$$(1-\gamma/\gamma)\|V^* - V^{t+1}\| \leq \|V^{t+1} - V^t\| < \epsilon(1-\gamma)/\gamma$$

$$(1-\gamma/\gamma)\|V^* - V^{t+1}\| < \epsilon(1-\gamma)/\gamma$$

$$\|V^* - V^{t+1}\| < \epsilon$$