Towards a Complete Classification of 2-Knots up to Smooth Isotopy

Jacob Russell MSci

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Contents

1 The Fundamental Group

We begin our work with a discussion of the fundamental group, a key consideration in the field of Algebraic Topology. In doing so, we refer primarily to Hatcher's Algebraic Topology [Hat02], a great introductory text to the field.

1.1 Homotopy of Paths

Definition 1.1. Let $a, b \in \mathbb{R}$. A *path* in some topological space X is a continuous map $f : [a, b] \to X$.

A space is said to be *path connected* if any two points $x_0, x_1 \in X$ can be joined by a path such that $f(a) = x_0$ and $f(b) = x_1$. [Hat02]

This is an intuitive definition. It is easy to understand. We now introduce what is known as a homotopy of paths. This formalises the idea of deforming paths whilst keeping the endpoints as they are. Think of a piece of string fixed at its endpoints, we can deform it but not change its endpoints. The collection of all possible deformations is the homotopy of the path, defined mathematically as such:

Definition 1.2. A *homotopy of paths* in X is a family $f_t : [0, 1] \to X$, with $0 \leq t \leq 1$, such that:

1. The endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t.
2. The map $F : [0, 1] \times [0, 1] \to X$ where $F(s, t) = f_t(s)$ is continuous.

We say that two paths f_0 and f_1 belonging to the same family f_t are *homotopic*, denoted $f_0 \sim f_1$. [Hat02]

Proposition 1.3. *All paths with given endpoints x_0 and x_1 are homotopic to one another in $\mathbb{R}^3$.*

Proof. Let f_0 and f_1 be paths in X such that $f_0(0) = f_1(0) = x_0$ and $f_0(1) = f_1(1) = x_1$ (so f_0 and f_1 have the same endpoints). We claim that f_0 and f_1 are homotopic via the homotopy $f_t(s) = (1 - t)f_0(s) + tf_1(s)$.

We note that the line segment between $f_0(s)$ and $f_1(s)$ is linearly parametrised as $f_0(s) + t(f_1(s) - f_0(s)) = (1 - t)f_0(s) + tf_1(s) = f_t(s)$. When we have $f_0(s) = f_1(s)$ then clearly

$$f_t(s) = f_0(s) + t(f_1(s) - f_0(s)) = f_0(s) + t(f_0(s) - f_0(s)) = f_0(s)$$

for all $0 \leq t \leq 1$. This obviously occurs for $s = 0, 1$ and so each f_t is a path from x_0 to x_1. So the first condition for f_t to be a homotopy is satisfied.

We have that continuity of $F(s, t) = f_t(s)$ simply follows from the fact scalar multiplication and vector addition preserve continuity in $f_t(s) = (1 - t)f_0(s) + tf_1(s)$ as $f_0(s)$ and $f_1(s)$ are continuous.

So $f_t(s)$ is a homotopy in X containing $f_0(s)$ and $f_1(s)$. □

Proposition 1.4. *$\sim$ is an equivalence relation.*

Proof. Let f_0, f_1, g_0 and g_1 be paths.

Suppose $f_0 \sim f_1$. Then clearly there is some homotopy f_t such that this is the case. Hence we also have $f_1 \sim f_0$ via the homotopy f_{1-t}. $F(s, 1 - t)$ (as defined in definition 1.2) is clearly continuous as $F(s, t)$ is continuous. So $\sim$ is **symmetric**.

We have that $f_0 \sim f_0$ via the homotopy $f_t = f_0$. $F(s, t)$ is trivially continuous as it is just a path. So $\sim$ is **reflexive**.

Let $f_1 = g_0$. Suppose $f_0 \sim f_1$ via the homotopy f_t and $f_1 = g_0 \sim g_1$ via the homotopy g_t. Define a new homotopy h_t such that

$$h_t = \begin{cases} f_{2t} & \text{for } 0 \leq t \leq 1/2 \\ g_{2t-1} & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

Then clearly $h_0 = f_0$, $h_{1/2} = f_1 = g_0$ and $h_1 = g_1$. So this homotopy works so long as we can show $H(s,t) = h_t(s)$ is continuous. We have that

$$H(s,t) = \begin{cases} F(s,2t) & \text{for } 0 \leq t \leq 1/2 \\ G(s,2t-1) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

So H, from the continuity of F and G, is continuous on $[0,1] \times [0,1/2]$ and $[0,1] \times [1/2,1]$. From the fact that a function defined on the union of two closed sets is continuous if it is continuous when restricted to both of the closed sets individually (theorem A.31), H is continuous on $[0,1] \times [0,1]$. Hence

$$f_0 \sim f_1 = g_0 \sim g_1$$

and so $\sim$ is **transitive**.

We have shown $\sim$ is reflexive, symmetric and transitive. Hence, it is an equivalence relation. □

Definition 1.5. The equivalence class of a path f under the equivalence relation $\sim$ is called the *homotopy class* of f and is denoted $[f]$.

From this point forward, unless stated otherwise, when we refer to a 'path' in X we are actually referring to its homotopy class.

1.2 Composition and Loops

Definition 1.6. The *composition* or *product path* of two paths $f, g : [0,1] \to X$ with $f(1) = g(0)$ is denoted $f \cdot g$, or simply fg if the context is clear, and is defined by [Hat02]

$$f \cdot g(s) = \begin{cases} f(2s) & \text{for } 0 \leq s \leq 1/2 \\ g(2s-1) & \text{for } 1/2 \leq s \leq 1. \end{cases}$$

This definition may not be immediately clear. One should note what this actually defines, the product path is essentially a way of traversing f and then immediately traversing g. This is possible as the paths have a common endpoint, so it is possible to traverse one immediately after the other. We note that in order for $f \cdot g$ to be traversed in unit time, we have modified the arguments such that f and g are traversed twice as fast.

Definition 1.7. A path f with the same starting and ending point, i.e. $f(0) = f(1) = x_0$, is called a *loop*. x_0 is known as the *basepoint*. We denote the set of all homotopy classes $[f]$ of loops with basepoint x_0 by $\pi_1(X, x_0)$. [Hat02]

We are now ready to define the fundamental group.

1.3 The Fundamental Group $\pi_1(X, x_0)$

The following proof is slightly modified from Hatcher's. [Hat02]

Proposition 1.8. *$(\pi_1(X, x_0), \cdot)$ is a group, with $\cdot$ denoting composition of paths. This is known as the fundamental group of X at the basepoint x_0. [Hat02]*

Proof. Since every loop in $\pi_1(X, x_0)$ has the same basepoint, x_0, we can guarantee that fg is well defined for some loops f, g with basepoint x_0. So clearly $\pi_1(X, x_0)$ is closed under $\cdot$ as $[f][g] = [f \cdot g]$ is a homotopy class of loops with basepoint x_0 and so is contained in $\pi_1(X, x_0)$ (as fg must have the same endpoints as anything contained in its homotopy class).

Define a *reparametrisation* of a path f to be a composition $f \circ \phi$, where $\phi : [0,1] \to [0,1]$ is any continuous map with $\phi(0) = 0$ and $\phi(1) = 1$. We have that reparametrising a path preserves its homotopy class (i.e. $f \sim f\phi$) via the homotopy $f\phi_t$, where

$$\phi_t(s) = (1-t)\phi(s) + ts,$$

noting that $\phi_0(s) = \phi(s)$ and $\phi_1(s) = s$.

We first show **associativity**. Suppose we have loops f, g, h with basepoint x_0. We have

$$([f][g])[h] = [fg][h] = [fgh] = [f][gh] = [f]([g][h]).$$

To see this, we note that $[f]([g][h])$ is a reparametrisation of $([f][g])[h]$ by ϕ given by

$$\phi(s) = \begin{cases} s/2 & \text{for } 0 \le s \le 1/2 \\ s - 1/4 & \text{for } 1/2 \le s \le 3/4 \\ 2(s-1/2) & \text{for } 3/4 \le s \le 1. \end{cases}$$

We now show the existence of some **identity element**. Let e be a loop with basepoint x_0 where $e(s) = x_0$ for all $s \in [0,1]$. We claim $[e]$ is the identity element. To see this, note that fe is a reparametrisation of f by ϕ given by

$$\phi(s) = \begin{cases} 2s & \text{for } 0 \le s \le 1/2 \\ 1 & \text{for } 1/2 \le s \le 1. \end{cases}$$

Similarly, ef is a reparametrisation of f by ϕ given by

$$\phi(s) = \begin{cases} 0 & \text{for } 0 \le s \le 1/2 \\ 2(s-1/2) & \text{for } 1/2 \le s \le 1. \end{cases}$$

So we get that

$$[f][e] = [fe] = [f] = [ef] = [e][f],$$

as desired.

The final group axiom to verify is the existence of some **inverse element** of $[f]$ in $\pi_1(X, x_0)$, that we shall denote $[f^{-1}]$, i.e. the homotopy class of some path f^{-1} that we shall define. Let us define f^{-1} as

$$f^{-1}(s) = f(1-s).$$

Clearly, f^{-1} is the same loop as f but travelling in the opposing direction. Define a homotopy by $h_t(s) = f_t(s) \cdot f_t^{-1}(s)$ where

$$f_t(s) = \begin{cases} f(s) & \text{for } 0 \le s \le 1-t \\ f(1-t) & \text{for } 1-t \le s \le 1. \end{cases}$$

and

$$f_t^{-1}(s) = \begin{cases} f^{-1}(t) & \text{for } 0 \le s \le t \\ f^{-1}(s) & \text{for } t \le s \le 1. \end{cases}$$

is the inverse loop of $f_t(s)$. We see that $f_0(s) = f(s)$ and $f_1(s) = f(1) = x_0 = e$ (as f is a loop with basepoint x_0). As a result, we can deduce that h_t is a homotopy from $f(s) \cdot f^{-1}(s)$ to $e(s) \cdot e^{-1}(s) = e(s)$. So $f \cdot f^{-1} \sim e$. Swapping f_t and f_t^{-1} in h_t we instead get $f^{-1} \cdot f \sim e$. Putting these results together we deduce that

$$[f][f^{-1}] = [ff^{-1}] = [e] = [f^{-1}f] = [f^{-1}][f].$$

So we have our inverse.

The fundamental group is then indeed a group, as it satisfies the group axioms. □

1.3.1 Induced Homomorphisms

The following definition from Munkres' chapter on the fundamental group [Mun00] shall become useful later on. We include it here as it is relevant to the fundamental group.

Definition 1.9. Let X, Y be topological spaces with $x_0 \in X$ and $y_0 \in Y$. Let $h : X \to Y$ be a continuous map with $h(x_0) = y_0$. Define $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by

$$h_*([f]) = [h \circ f].$$

The map h_* is called the *homomorphism induced by* h relative to x_0.

1.4 Simple Connectedness

We end this section by defining what it means for a space X to be *simply connected*.

Definition 1.10. A topological space X is said to be *simply connected* if it is path connected and for every $x_0 \in X$ we have that $\pi_1(X, x_0) = \{[e]\}$, where e is the constant loop at x_0. [Mun00]

Remark 1.11. In other words, a simply connected space is one in which every loop is homotopic to the constant loop.

2 Covering Spaces and Deck Transformations

2.1 Components

We begin with the definition of a *component*, an equivalence class that equates two points in a connected subspace.

Definition 2.1. Let X be a topological space with $x_0, x_1 \in X$. Set $x_0 \sim x_1$ if there exists some connected subspace $U \subseteq X$ such that $x_0, x_1 \in U$. The equivalence classes are called *components*[1] of X. [Mun00]

Proposition 2.2. *$\sim$, as defined in definition 2.1, is an equivalence relation.*

Proof. Clearly $\sim$ is **reflexive** (i.e. $x_0 \sim x_0$) by setting $U = \{x_0\}$, which is trivially connected.

To show **symmetry**, suppose that $x_0 \sim x_1$. Then there exists some connected $U \subseteq X$ such that $x_0, x_1 \in U$. Then $x_1 \sim x_0$ by the same subspace U.

Transitivity requires slightly more thought. Suppose $x_0 \sim x_1$ and $x_1 \sim x_2$. Let $U \subseteq X$ denote the connected subspace containing x_0 and x_1. Similarly, let $V \subseteq X$ denote the connected subspace containing x_1 and x_2. Consider $U \cup V$. As both U and V contain x_1, $U \cup V \subseteq X$ is necessarily a connected subspace (theorem A.15). Thus, as $x_0, x_2 \in U \cup V$, we have $x_0 \sim x_2$.

We have shown $\sim$ is reflexive, symmetric and transitive. Hence, it is an equivalence relation. □

Theorem 2.3. *The components C_α of X are connected, disjoint subspaces of X such that $\bigcup C_\alpha = X$.*

Any non-empty connected subspace $U \subseteq X$ intersects only one component. [Mun00]

Proof. We have already shown that the components C_α of X are equivalence classes. As such, they form a partition of X by definition and it is clear that they are disjoint with $\bigcup C_\alpha = X$.

Now suppose some connected subspace $U \subseteq X$ is such that for some components C_0 and C_1, we have $U \cap C_0 = \{x_0\}$ and $U \cap C_1 = \{x_1\}$. Then $x_0 \sim x_1$ by definition. Clearly, this is a contradiction as $x_1 \notin C_0$ (and similarly $x_0 \notin C_1$) by definition and disjointness of components. So U can only intersect one component.

We show that some C_0 is connected. For every $x \in C_0$, we have $x_0 \sim x$; i.e. there exists a connected subspace $U_x \subseteq X$ such that $x_0, x \in U_x$. We know that $U_x \subseteq C_0$ as U_x can only intersect one component. As such, we have

$$C_0 = \bigcup_{x \in C_0} U_x.$$

Since all U_x are connected and share the point x_0, their union, i.e. C_0, is connected. □

Remark 2.4. From this theorem, a visual image becomes clear to the reader. We can view components as disjoint, connected sections that total to the entire space. The components are then the maximal connected subsets of a topological space.

2.1.1 Path Components

In a similar vain to before, we discuss path components and show that these also partition X into equivalence classes.

Definition 2.5. Let X be a topological space with $x_0, x_1 \in X$. Set $x_0 \sim x_1$ if there exists some path from x_0 to x_1 in X. The equivalence classes are called *path components* of X. [Mun00]

Proposition 2.6. *$\sim$, as defined in definition 2.5, is an equivalence relation. [Mun00]*

[1] Or *'connected components'*.

Proof. **Reflexivity** follows by considering $f : [a, b] \to X$, $f(s) = x_0$. This is trivially a path from x_0 to x_0 and so $x_0 \sim x_0$.

Let $f : [a, b] \to X$ be a path from x_0 to x_1. So $x_0 \sim x_1$. Define $g : [a, b] \to X$ to be such that $g(s) = f(1 - s)$. Then clearly g is just the reverse path of f. So $x_1 \sim x_0$ via g. Hence, we have **symmetry**.

For **transitivity**, let $f : [a, b] \to X$ be a path from x_0 to x_1 and let $g : [b, c] \to X$ be a path from x_1 to x_2. We acknowledge that if there is some path with domain $[a, b]$ then there is also some path between the same points with domain $[a_0, b_0]$ from theorem A.29. Hence, we are able to choose f and g such that they meet at b without loss of generality. Define $h : [a, c] \to X$ with

$$h(s) = \begin{cases} f(s) & \text{for } a \leq s \leq b \\ g(s) & \text{for } b \leq s \leq c. \end{cases}$$

h is clearly continuous via the pasting lemma (theorem A.31) as $f(b) = x_1 = g(b)$ and so we have $x_0 \sim x_2$ via h. □

The following theorem and subsequent proof is analogous to theorem 2.3.

Theorem 2.7. *The path components P_α of X are path connected, disjoint subspaces of X such that $\bigcup P_\alpha = X$.*

Any non-empty path connected subspace $U \subseteq X$ intersects only one path component. [Mun00]

Proof. We have already shown that the components P_α of X are equivalence classes. As such, they form a partition of X by definition and it is clear that they are disjoint with $\bigcup P_\alpha = X$.

Now suppose some path connected subspace $U \subseteq X$ is such that for some path components P_0 and P_1, we have $U \cap P_0 = \{x_0\}$ and $U \cap P_1 = \{x_1\}$. Then $x_0 \sim x_1$ by definition. Clearly, this is a contradiction as $x_1 \notin P_0$ (and similarly $x_0 \notin P_1$) by definition and disjointness of path components. So U can only intersect one path component.

We show that some P_0 is path connected. For every $x \in P_0$, we have $x_0 \sim x$; i.e. there exists a path connected subspace $U_x \subseteq X$ such that $x_0, x \in U_x$. We know that $U_x \subseteq P_0$ as U_x can only intersect one path component. As such, we have

$$P_0 = \bigcup_{x \in P_0} U_x.$$

Since all U_x are path connected and share the point x_0, their union, i.e. P_0, is path connected. □

Remark 2.8. Similarly to as before, we can see the path components as the maximal path connected subsets of a topological space.

2.2 Local Connectedness and Local Path Connectedness

Definition 2.9. Some topological space X is said to be *locally connected at $x \in X$* if for every neighbourhood U of x, there exists some connected neighbourhood $V \subseteq U$ of x.

We describe X as *locally connected* if all $x \in X$ are locally connected. [Mun00]

Recall the definition of *path connectedness* (definition 1.1).

Definition 2.10. Some topological space X is said to be *locally path connected at $x \in X$* if for every neighbourhood U of x, there exists some path connected neighbourhood $V \subseteq U$ of x.

We describe X as *locally path connected* if all $x \in X$ are locally path connected. [Mun00]

We now prove some basic theorems that arise from the above definitions.

Theorem 2.11. *A topological space X is locally connected if and only if for every open set $U \in X$, each component of U is open in X. [Mun00]*

Proof. Let X be locally connected with $U \subseteq X$ an open set in X and C some component of U. If x_0 is a point of C, then we can choose some connected neighbourhood $V \subseteq U$ of x_0. As V is connected and C is a maximal connected subspace of X, V must lie entirely in C. As V is open in X and contained in C, every x_0 in C has an open neighbourhood lying entirely in C. Thus, C is open in X.

Now suppose instead that the components of open sets in X are open. We show that X is locally connected. Given some point $x_0 \in X$ and a neighbourhood U of x_0, let C be the component of U containing x_0. We have assumed that C is open. Since C is open in X, it is an open, connected neighbourhood of x_0 contained in U. As x_0 is arbitrary, we can conclude that X is locally connected. $\square$

We have a similar theorem for local path-connectedness.

Theorem 2.12. *A topological space X is locally path connected if and only if for every open set $U \in X$, each path component of U is open in X. [Mun00]*

Proof. Let X be locally path connected with $U \subseteq X$ an open set in X and P some path component of U. If x_0 is a point of P, then we can choose some path connected neighbourhood $V \subseteq U$ of x_0. As V is path connected and P is a maximal path connected subspace of X, V must lie entirely in P. As V is open in X and contained in P, every x_0 in P has an open neighbourhood lying entirely in P. Thus, P is open in X.

Now suppose instead that the path components of open sets in X are open. We show that X is locally path connected. Given some point $x_0 \in X$ and a neighbourhood U of x_0, let P be the path component of U containing x_0. We have assumed that P is open. Since P is open in X, it is an open, path connected neighbourhood of x_0 contained in U. As x_0 is arbitrary, we can conclude that X is locally path connected. $\square$

RELATIONSHIP BETWEEN COMPONENTS AND PATH COMPONENTS - 25.5 MUNKRES

2.3 Covering Spaces

Munkres introduces the notion of *covering spaces* in the ninth chapter of his book [Mun00].

Definition 2.13. Let E, B be topological spaces. Let $p : E \to B$ be a continuous map that is surjective. Let $V \subseteq B$ be an open set.

We say that V is *evenly covered* by p if the inverse image $p^{-1}(V)$ can be written as $\bigcup U_\alpha$, where $U_\alpha \in E$ are disjoint open sets each with the property that the restriction of p to U_α, denoted $p|_{U_\alpha}$, is a homeomorphism of U_α onto V.

We say that the collection $\{U_\alpha\}$ is a partition of $p^{-1}(V)$ into *sheets*[2]. [Mun00]

Remark 2.14. We can think of the above definition as follows: we take some open set in B, and if p is such that its inverse image is composed of a number of disjoint open sets in E, with each of these sets homeomorphic to the open set in B, then the open set in B is *evenly covered* by p. See figure 1.

[2]Or '*slices*'.

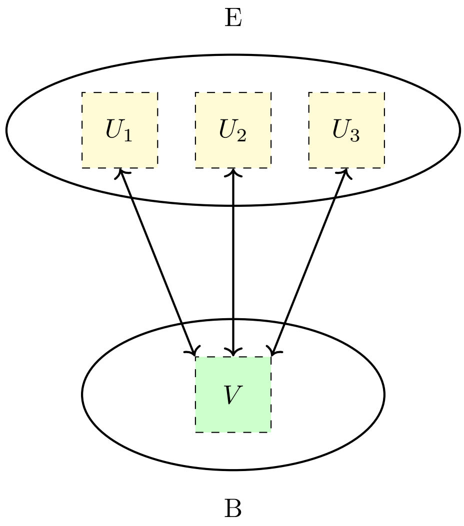


Figure 1: V is evenly covered by some p. Here, the U_α are each homeomorphic to V via $p|_{U_\alpha}$, represented by the arrows.

Definition 2.15. Let $p : E \to B$ be as above. If every $y_0 \in B$ has some open neighbourhood $V \in B$ that is evenly covered by p, then p is called a *covering map*.

We say that E is a *covering space* of B. [Mun00]

We now prove a theorem regarding covering maps.

Theorem 2.16. *Let $p : E \to B$ be a covering map. Let $V \subseteq B$ be a subspace. If $U = p^{-1}(V)$, then the restriction*

$$p|_U : U \to V$$

is a covering map. [Mun00]

Proof. Let $b_0 \in V$, $b_0 \in V_0$ with V_0 an open set in B that is evenly covered by p. Let $\{U_\alpha\}$ be a partition of $p^{-1}(V_0)$ into sheets.

Then we have that $V \cap V_0$ is a neighbourhood of b_0 in V and the sets $U \cap U_\alpha$ are disjoint open sets in U such that $\bigcup U \cap U_\alpha = p^{-1}(V \cap V_0)$, and each $p|_{U \cap U_\alpha}$ is a homeomorphism onto $V \cap V_0$ due to it being a restriction of a known homeomorphism, in this case $p|_{U_\alpha}$. Hence, $p|_U : U \to V$ is a covering map by the definition. $\square$

2.4 Uniqueness of Path Liftings

Unless stated otherwise, from this point onwards we shall assume that E and B are locally path connected and path connected. Assume also that this applies for any covering spaces equivalent to E (see section 2.6).

Definition 2.17. Let $p : E \to B$ be a map. Let $f : X \to B$ be a continuous mapping of X into B. Then a map $\tilde{f} : X \to E$ with the property that $p \circ \tilde{f} = f$ is called a *lifting* of f. [Mun00]

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

We will now show that for some covering map p, paths have unique liftings. We will go on to prove this for homotopies of paths. Our proof argument uses induction, it comes from Munkres. [Mun00]

Lemma 2.18. *Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Let $f : [0,1] \to B$ be a path with $f(0) = b_0$. Then f has a unique lifting to some path $\tilde{f} : [0,1] \to E$ with $\tilde{f}(0) = b_0$. [Mun00]*

Proof. Cover B by open sets $V_\alpha \subseteq B$, each of which is evenly covered by p. Choose some subdivision of $[0,1]$, say $s_1, s_2, \ldots, s_n$, such that for each $i \in 1, \ldots, n$, the set $f([s_i, s_{i+1}])$ lies in such an open set, say V_0. We are able to choose such a subdivision due to the Lebesgue number lemma (lemma A.22), by noting that $[0,1]$ is a compact metric space (see the Heine-Borel theorem in chapter 2 of Knapp [Kna07] for proof that any closed, bounded set in $\mathbb{R}$ is compact).

We shall define $\tilde{f}$ step-by-step. First, set $\tilde{f}(0) = e_0$ as per the definition. Assume that $\tilde{f}(s)$ is defined for $0 \leq s \leq s_i$.

Now we define $\tilde{f}$ on $[s_i, s_{i+1}]$. We have that $\tilde{f}([s_i, s_{i+1}])$ lies in some open set that is evenly covered by p, which we have denoted V_0. Let $\{U_\alpha\}$ be a partition of $p^{-1}(V_0)$ into sheets. Each U_α is homeomorphically mapped onto V_0 by p. $\tilde{f}(s_i)$ must lie in one of these sets, say U_0. Define $\tilde{f}(s)$ for $s_i \leq s \leq s_{i+1}$ by

$$\tilde{f}(s) = (p|_{U_0})^{-1}(f(s)).$$

Here, $p|_{U_0}$ denotes the restriction of p to U_0. As $p|_{U_0} : U_0 \to V_0$ is a homeomorphism, $\tilde{f}$ will be continuous on $[s_i, s_{i+1}]$.

On each interval $[s_i, s_{i+1}]$, we define $\tilde{f}$ using the local inverse $(p|_{U_0})^{-1}$, where U_0 is the component of $p^{-1}(V_0)$ containing $\tilde{f}(s_i)$ as previously defined. Again since $p|_{U_0}$ is a homeomorphism, $(p|_{U_0})^{-1}$ is also continuous (a necessary property for homeomorphisms). Furthermore, since $\tilde{f}$ is already defined at s_i and we always select the same connected component U_0 containing $\tilde{f}(s_i)$, we can guarantee that the values of $\tilde{f}$ match at each subdivision boundary $[s_i, s_{i+1}]$. Hence, we can apply the pasting lemma (theorem A.31) at each boundary and conclude that $\tilde{f}$ is continuous.

We note that

$$p \circ \tilde{f}(s) = p((p|_{U_0})^{-1})(f(s)) = f(s)$$

and thus $p \circ \tilde{f} = f$ as required. We have shown the existence of $\tilde{f}$.

It remains to show the uniqueness of $\tilde{f}$. We again prove this step-by-step. Assume there exists some other lifting of f beginning at e_0, say $\tilde{\tilde{f}}$. We show that $\tilde{f} = \tilde{\tilde{f}}$.

As they begin at e_0, we have $\tilde{f}(0) = \tilde{\tilde{f}}(0) = e_0$. Now suppose that $\tilde{f}(s) = \tilde{\tilde{f}}(s)$ for all $0 \leq s \leq s_i$. With U_0 as before, recall that for $s_i \leq s \leq s_{i+1}$,

$$\tilde{f}(s) = (p|_{U_0})^{-1}(f(s)).$$

We will show that $\tilde{\tilde{f}}(s)$ necessarily equals $(p|_{U_0})^{-1}(f(s))$ also.

As $\tilde{\tilde{f}}$ is a lifting of f, it must carry the interval $[s_i, s_{i+1}]$ into $\bigcup U_\alpha$ by the definition of an even covering. As each U_α is open and disjoint, we must have that the connectedness of $f([s_i, s_{i+1}])$ implies that it is contained entirely in one of the U_α, else it would not be connected.

However, we previously assumed that $\tilde{\tilde{f}}(s_i) = \tilde{f}(s_i)$ for $0 \leq s \leq s_i$. As $\tilde{f}(s_i) \in U_0$, so must $\tilde{\tilde{f}}(s_i)$ be. Therefore, the U_α in which $f([s_i, s_{i+1}])$ must be contained is U_0. As such, for $s_i \leq s \leq s_{i+1}$, $\tilde{\tilde{f}}(s)$ must equal some $p^{-1}(f(s)) \in U_0$, yet this is precisely the point $(p|_{U_0})^{-1}(f(s))$. As such $\tilde{f} = \tilde{\tilde{f}}$ is unique. □

Lemma 2.19. *Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Let $F : [0,1] \times [0,1] \to B$ be a continuous map with $F(0,0) = b_0$. Then F has a unique lifting to a continuous map*

$$\tilde{F} : [0,1] \times [0,1] \to E$$

such that $F(0,0) = e_0$. [Mun00]

Proof. We begin, similar to the previous lemma, by setting $\tilde{F}(0,0) = e_0$ as per the definition.

We are able to use the previous lemma to extend $\tilde{F}$ to $\tilde{F}|_{0\times[0,1]}$ and $\tilde{F}|_{[0,1]\times 0}$ by noting that these are paths beginning at e_0 (so such a unique lifting exists). Now we must extend $\tilde{F}$ to the entire square $[0,1]\times[0,1]$.

We may appeal to the Lebesgue number lemma (lemma A.22) once again, noting that $[0,1]\times[0,1]$ is compact (for proof that the Cartesian product of two compact sets is compact, see chapter 3 of Munkres [Mun00]); in doing so we can choose subdivisions $s_0 < s_1 < \cdots < s_n$, $t_0 < t_1 < \cdots < t_m$ such that for each $i \in \{1,\dots,n\}$, $j \in \{1,\dots,m\}$, we have that each rectangle $[s_{i-1}, s_i]\times[t_{j-1}, t_j]$ is mapped by F into some open set, say V_0, of B that is evenly covered by p.

We shall define $\tilde{F}$ step-by-step, as is all too familiar at this point. We shall define row-by-row, beginning with $[s_0, s_1]\times[t_0, t_1]$, then $[s_0, s_1]\times[t_1, t_2]$ and so on. Then we move to the next row and define $[s_1, s_2]\times[t_0, t_1]$, then $[s_1, s_2]\times[t_1, t_2]$ and so on. We do this until we have defined the entirety of the rectangle $[0,1]\times[0,1]$.

Given i_0 and j_0, we assume that $\tilde{F}$ is defined on some set A, defined as the union between $0\times[0,1]$, $[0,1]\times 0$ and all $[s_{i-1}, s_i]\times[t_{j-1}, t_j]$ with $j < j_0$ or $j = j_0$, $i < i_0$. These are the rows previous to our current subdivision, and the part of the current row already defined. We assume also that $\tilde{F}|_A$ is continuous.

We define $\tilde{F}$ on $[s_{i_0-1}, s_{i_0}]\times[t_{j_0-1}, t_{j_0}]$. Choose some open set $V_0 \subseteq B$ that is evenly covered by p and contains the set $F([s_{i_0-1}, s_{i_0}]\times[t_{j_0-1}, t_{j_0}])$, which we have necessarily exists. Let $\{U_\alpha\}$ be a partition of $p^{-1}(V_0)$ into sheets, so each U_α is mapped homeomorphically onto V_0 by p.

We have, by our assumptions, that $\tilde{F}$ is already defined on the 'boundary' $C = A \cap [s_{i_0-1}, s_{i_0}]\times[t_{j_0-1}, t_{j_0}]$. It is trivial to see that this set, which is the bottom and left hand 'sides' of the subdivision $[s_{i_0-1}, s_{i_0}]\times[t_{j_0-1}, t_{j_0}]$, is connected. Therefore, $\tilde{F}(C)$ is connected by preservation of connectedness in continuous functions (theorem A.30) and must lie entirely within some U_α, say U_0.

As $\tilde{F}$ is a lifting of $F|_A$, we have that for $(s,t)\in C$,

$$p|_{U_0}(\tilde{F}(s,t)) = F(s,t).$$

So we may extend $\tilde{F}$ by defining

$$\tilde{F}(s,t) = (p|_{U_0})^{-1}(F(s,t))$$

for $(s,t) \in [s_{i_0-1}, s_{i_0}]\times[t_{j_0-1}, t_{j_0}]$. Continuity follows from the pasting lemma (lemma A.31). Extending this across the entirety of $[0,1]\times[0,1]$ using our assumptions and trivial case, we prove existence.

It remains to show uniqueness. $\tilde{F}$ is completely determined by the chosen value of $\tilde{F}(0,0)$. To see this, note that $\tilde{F}$ must agree between subdivisions on C, the boundary between subdivisions - which is shared between adjacent subdivisions. Thus if we are able to show uniqueness in each subdivision $[s_{i_0-1}, s_{i_0}]\times[t_{j_0-1}, t_{j_0}]$ then we can conclude that $\tilde{F}$ is unique across all of $[0,1]\times[0,1]$.

To see the uniqueness in the subdivision, note that $\tilde{F}(C)$ lies within $U_0 \subseteq p^{-1}(V_0)$ as previously shown. $p|_{U_0} : U_0 \to V_0$ is a homeomorphism, and as such each element of U_0 has a unique inverse in V_0. From these facts, we havee that the only way to define $\tilde{F}$ in each subdivision $[s_{i_0-1}, s_{i_0}]\times[t_{j_0-1}, t_{j_0}]$ is by

$$\tilde{F}(s,t) = (p|_{U_0})^{-1}(F(s,t)).$$

As such, $\tilde{F}$ is unique. □

We now show that the previous lemma implies that if F is a path homotopy, then so is $\tilde{F}$.

Lemma 2.20. *Let F, $\tilde{F}$ be as above. Then if F is a path homotopy, $\tilde{F}$ is a path homotopy. [Mun00]*

Proof. Suppose F is a path homotopy. Then F carries the entirety of $0\times[0,1]$ into some $b_0 \in B$. To see this, note that for a path homotopy $F(0,t) = f_t(0) = b_0$ where f is some path beginning at b_0.

As $\tilde{F}$ is a lifting of F, it carries $0\times[0,1]$ into the set $p^{-1}(b_0)$. As this is a fibre, it is a discrete subset of E. However, $\tilde{F}(0\times[0,1]) \subseteq p^{-1}(b_0)$ must be connected as the continuous image of a connected set is connected. As such, $p^{-1}(b_0)$ must be a singleton as by discreteness of $p^{-1}(b_0)$, the only connected subsets are singletons. As such, $\tilde{F}(0 \times [0,1])$ is a single point.

The same argument shows $\tilde{F}(1 \times [0,1])$ is a single point. As such, $\tilde{F}$ is a family of paths as per the definition of a path homotopy. □

Theorem 2.21. *Let $p : E \to B$ be a covering map. Let f_0 and f_1 be two paths in B from b_0 to b_1, with $\tilde{f}_0$ and $\tilde{f}_1$ their respective liftings to paths in E beginning at e_0. If f_0 and f_1 are path homotopic, then $\tilde{f}_0$ and $\tilde{f}_1$ are path homotopic and end at the same point of E. [Mun00]*

Proof. Let $F : [0,1] \times [0,1] \to B$ be the path homotopy containing f_0 and f_1 such that $F(s,0) = f_0(s)$ and $F(s,1) = f_1(s)$. Then $F(0,0) = b_0$ as the paths begin at b_0. Let $\tilde{F} : [0,1] \times [0,1] \to E$ be the lifting of F to E such that $\tilde{F}(0,0) = e_0$, which exists and is a path homotopy by the previous two lemmas. We have $\tilde{F}(0 \times [0,1]) = \{e_0\}$ and $\tilde{F}(1 \times [0,1]) = \{e_1\}$ by definition of path homotopies.

We note that the restriction $\tilde{F}|_{[0,1]\times 0}$ is a path on E beginning at e_0 (as $\tilde{F}(0 \times [0,1]) = \{e_0\}$) that is a lifting of $F|_{[0,1]\times 0}$. We have

$$\tilde{F}(s,0) = \tilde{f}_0(s)$$

by the uniqueness of path liftings shown in a previous lemma.

By the same argument of uniqueness of path liftings for the restriction $\tilde{F}|_{[0,1]\times 1}$, we have

$$\tilde{F}(s,1) = \tilde{f}_1(s).$$

Therefore, $\tilde{f}_0(1) = \tilde{f}_1(1) = e_1$, with $\tilde{F}$ the path homotopy containing them both. □

2.4.1 Unique Lifting Property

Following on from the above, we can prove a more general theorem regarding uniqueness of lifts. This theorem and its subsequent proof come from section 1.3 of Hatcher [Hat02] - which coincides nicely with Munkres' [Mun00] work on covering spaces. This theorem is referred to as the *unique lifting property.*

Theorem 2.22. *Let $p : E \to B$ be a covering map. Let $f : X \to B$ be a continuous map. Let $\tilde{f}_0, \tilde{f}_1 : X \to E$ be two liftings of f that agree at some $x_0 \in X$, i.e. $\tilde{f}_0(x_0) = \tilde{f}_1(x_0)$. Then if X is connected, $\tilde{f}_0$ and $\tilde{f}_1$ agree on the entirety of X, i.e.*

$$\tilde{f}_0(x) = \tilde{f}_1(x) \text{ for all } x \in X.$$

Proof. Let V be an evenly covered neighbourhood of $f(X_0)$ in B. We have that $p^{-1}(V)$ is partitioned into disjoint sheets $\{U_\alpha\}$, such that each U_α is mapped homeomorphically onto V by p. Let U_0 and U_1 be the sheets containing $\tilde{f}_0$ and $\tilde{f}_1$ respectively.

By definition of continuity, there exists some neighbourhood $N \subseteq X$ of x_0 mapped into U_0 and U_1 by $\tilde{f}_0$ and $\tilde{f}_1$ respectively.

Suppose that $\tilde{f}_0(x_0) \neq \tilde{f}_1(x_0)$. Then clearly $U_0 \neq U_1$ and they must be disjoint. So $\tilde{f}_0(x) \neq \tilde{f}_1(x)$ for all $x \in N$.

Suppose instead that $\tilde{f}_0(x_0) = \tilde{f}_1(x_0)$. Then necessarily we have $U_0 = U_1$. As such, $\tilde{f}_0(x) = \tilde{f}_1(x)$ for all $x \in N$ by the injectivity of p on $U_0 = U_1$. Therefore, N is open in X.

Since we have that N is also closed in X (theorem ????), then we must have by definition of a topological space that $N = X$, as N is nonempty. So $\tilde{f}_0(x) = \tilde{f}_1(x)$ for all $x \in X$. □

2.5 The Lifting Correspondence

We have already alluded to our work on the fundamental group through theorem 2.21, which showed that path homotopies have a unique lift to a path homotopy in a covering space. The aim of this subsection is to begin draw up this connection more explicitly - to expand upon it in such a way that it will become useful to us. We begin with a definition courtesy of Munkres. [Mun00]

Definition 2.23. Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Given some loop $[f] \in \pi_1(B, b_0)$, let $\tilde{f}$ be the lifting of f to a path in E that begins at e_0. Let $\phi([f])$ denote the end point $\tilde{f}(1)$ of $\tilde{f}$. Then ϕ is a well-defined set map

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0).$$

ϕ is called the *lifting correspondence* derived from the covering map p. ϕ is dependent on the choice of e_0. [Mun00]

Remark 2.24. The intuition here is that ϕ tells us how loops in B 'move around' the fibre over b_0. ϕ is well-defined by the unique lifting of homotopies shown previously.

Theorem 2.25. *Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. By path connectedness of E, the lifting correspondence $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is surjective.*

If E is simply connected then ϕ is bijective.

Proof. By path connectedness of E, given $e_1 \in p^{-1}(b_0)$ there must exist some path $\tilde{f}$ in E such that $\tilde{f}(0) = e_0$ and $\tilde{f}(1) = e_1$. Then we have that $f = p \circ \tilde{f}$ is a loop in B at b_0 (since $e_0, e_1 \in p^{-1}(b_0)$). Thus, $\phi([f]) = e_1$ and by the arbitrary nature of our choices we may conclude that ϕ is surjective.

Now assume E is simply connected. It falls upon us to show bijectivity. Let $[f], [g] \in \pi_1(B, b_0)$ such that $\phi([f]) = \phi([g])$. Let $\tilde{f}$ and $\tilde{g}$ be the liftings of f and g respectively, to paths in E that begin at e_0 (which we know necessarily exist). Then we have $\tilde{f}(1) = \tilde{g}(1)$ by the uniqueness of path liftings.

Since E is simply connected, there must exist a path homotopy $\tilde{f}_t$ in E such that $\tilde{f}_0 = \tilde{f}$ and $f_1 = \tilde{g}$. Then $p \circ \tilde{f}_t$ is a path homotopy in B between f and g by preservation of homotopies. Thus $[f] = [g]$ and we can conclude that ϕ is injective - and hence bijective. $\square$

The following proposition is presented stand-alone by Hatcher [Hat02], we choose to take the same approach here.

Proposition 2.26. *Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Consider the map $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ given by*

$$p_*([f]) = [p \circ f]$$

for some $[f] \in \pi_1(E, e_0)$. This map is injective.

The image subgroup $p_*(\pi_1(E, e_0)) \subseteq \pi_1(B, b_0)$ consists of the homotopy classes of loops in B based at b_0 whose lifts to E starting at e_0 are loops. [Hat02]

Proof. Let $[f] \in \pi_1(E, e_0)$ be such that $[f] \in \ker p_*$, i.e.

$$p_*([f]) = [p \circ f] = e_{b_0} \in \pi_1(B, b_0),$$

where e_{b_0} is the identity element of $\pi_1(B, b_0)$. So $p \circ f$ is homotopic to the constant loop e_{b_0}.

Let $g_t : [0, 1] \to B$ be a homotopy such that $g_0 = p \circ f$ and $g_1 = e_{b_0}$, which necessarily exists. Since p is a covering map and f is a loop based at e_0, the homotopy between $p \circ f$ and e_{b_0} can be lifted to some homotopy $\tilde{g_t}$ with $\tilde{g_0} = f$ because f already lifts and a homotopy lifting is unique as per lemma 2.20.

Since the constant loop at b_0 lifts uniquely to a path starting at e_0 and constant paths lift to constant paths, this path must be the constant loop at e_0. Thus $\tilde{g_t}$ is a homotopy between $g_0 \tilde{=} f$ and the constant loop at e_0, $\tilde{g_1} = e_{e_0}$. Since our choice of $[f] \in \ker p_*$ was arbitrary, we have that $\ker p_* = [e_0]$ and thus by theorem B.22 we must have injectivity of p_*.

We prove the second part of our proposition. Clearly, loops lifted from e_0 to b_0 must be elements of the image subgroup $p_*(\pi_1(E, e_0)) \subseteq \pi_1(B, b_0)$. Now consider some element of $p_*(\pi_1(E, e_0)) \subseteq \pi_1(B, b_0)$.

Clearly such an element is homotopic to some loop lifted from e_0 to b_0, so by lemma 2.20 this element must itself be a loop lifted from e_0 to b_0. ☐

Munkres [Mun00] takes the previous proposition even further, with the following results:

Proposition 2.27. *Let $p : E \to B$ be a covering map with $p(e_0) = b_0$ as above. Let p_* be as above. Let $H = p_*(\pi_1(E, e_0))$. Then the lifting correspondence ϕ induces an injective map*

$$\Phi : \pi_1(B, b_0)/H \to p^{-1}(b_0)$$

of the collection of right cosets of H into the fibre $p^{-1}(b_0)$.

This map is bijective as E is path connected. [Mun00]

$$\begin{tikzcd}
\pi_1(E,e_0) \arrow[r, hook, "p_*"] & \pi_1(B,b_0) \arrow[r, two heads] \arrow[dr, two heads, bend left, "\phi"'] & \pi_1(B,b_0)/p_*(\pi_1(E,e_0)) \arrow[d, leftrightarrow, "\Phi"] \\
& & p^{-1}(b_0)
\end{tikzcd}$$

Proof. Let $f, g \in \pi_1(B, b_0)$ and let $\tilde{f}$, $\tilde{g}$ be their unique liftings in E beginning at e_0. Then by the lifting correspondence definition we have $\phi([f]) = \tilde{f}(1)$ and $\phi([g]) = \tilde{g}(1)$. We will show that $\phi([f]) = \phi([g])$ if and only if $[f] \in H[g]$, i.e. $[f]$ and $[g]$ are contained in the same right coset and thus Φ is injective.

Suppose then that $\phi([f]) = \phi([g])$. Then $\tilde{f}(1) = \tilde{g}(1)$, i.e. $\tilde{f}$ and $\tilde{g}$ end at the same point in E. Consider the product of $\tilde{f}$ and the reverse of $\tilde{g}$, $\tilde{g}^{-1}(s) = \tilde{g}(1-s)$[3], which is clearly a loop from the fact that $\tilde{f}(1) = \tilde{g}(1)$ and both $\tilde{f}$ and $\tilde{g}$ begin at e_0. Call this loop (with basepoint e_0) $\tilde{h}$. Clearly

$$[\tilde{h}\tilde{g}] = [\tilde{f}\tilde{g}^{-1}g] = [\tilde{f}].$$

So let $\tilde{f}_t$ be the path homotopy between $\tilde{h}\tilde{g}$ and $\tilde{f}$. Then $p \circ \tilde{f}_t$ is a path homotopy in B between hg and f, where $h = p \circ \tilde{h}$, by preservation of homotopies. Thus, $[f] \in H[g]$. From the fact E is path connected, we have already shown ϕ is surjective. As Φ is induced by ϕ, we have that it must be surjective also (theorem **??**).

Conversely, assume that $[f] \in H[g]$. Then $[f] = [hg]$ where $h = p \circ \tilde{h}$ for some loop $\tilde{h}$ in E with basepoint e_0. The product $\tilde{h}\tilde{g}$ is defined - it is a lifting of hg. As we have $[f] = [hg]$, we must have that the liftings $\tilde{f}$ and $\tilde{h}\tilde{g}$ (which both begin at e_0) end at the same point in E. So we have shown that $\phi([f]) = \phi([g])$ if $[f] \in H[g]$. ☐

Proposition 2.28. *Let $p : E \to B$ be a covering map with $p(e_0) = b_0$ as above. Let p_*, H be as above. If some $f \in \pi_1(B, b_0)$, then $[f] \in H$ if and only if f lists to a loop in $\tilde{f} \in \pi_1(E, e_0)$.*

Proof. With Φ defined as in the previous proposition, we have already shown its injectivity. So $\phi([f]) = \phi([g])$ if and only if $[f] \in H[g]$. Suppose g is the constant loop. Then we have that $\phi([f]) = \phi([g]) = e_0$ (as g begins and ends at e_0) if and only if $[f] \in H[g]$. However, $\phi([f]) = e_0$ precisely when the lift of f is a loop at e_0. ☐

2.6 Equivalence of Covering Spaces

We define what it means for covering spaces to be equivalent. Recall that we continue to assume path connectedness and local path connectedness of E and B.

[3]Here we have extended our notion of the inverse of a loop from proposition 1.8 to any path, it is simple to see that $\tilde{g}(1-s)$ is simply 'travelling' the path in reverse.

Definition 2.29. Let $p : E \to B$ and $p' : E' \to B$ be covering maps. Suppose there exists some homomorphism $h : E \to E'$ such that

$$p = p' \circ h.$$

Then h is called an *equivalence of covering maps/spaces.*

p and p' are said to be *equivalent.*[Mun00]

The following lemma provides general condition for when a lifting exists and is unique. As such, it is known as the *general lifting lemma.*

Lemma 2.30. *Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Let $f : X \to B$ be a continuous map. Let $x_0 \in X$ be such that $f(x_0) = b_0$ and let X be path connected and locally path connected. Then f can be lifted to some unique map $\tilde{f} : X \to E$ such that $\tilde{f}(x_0) = e_0$ if and only if [Mun00]*

$$f_*(\pi_1(X, x_0)) \subseteq p_*(\pi_1(E, e_0)).$$

Proof. Recall that f_* is defined by

$$f_*([\alpha]) = [f \circ \alpha]$$

for $\alpha \in \pi_1(X, x_0)$ (definition 1.9).

Suppose the lifting $\tilde{f}$ exists, so $\tilde{f}(x_0) = e_0$. Then clearly we have

$$f_*(\pi_1(X, x_0)) = p_*(\tilde{f}_*(\pi_1(X, x_0))$$

by the definition of a lifting. We note that

$$p_*(\tilde{f}_*(\pi_1(X, x_0)) \subseteq p_*(\pi_1(E, e_0))$$

by the definition of $\tilde{f}_*$ and we are done.

It remains to show uniqueness of $\tilde{f}$. Let $x_1 \in X$. Choose some path $\alpha : [0, 1] \to X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. We can take the path $f \circ \alpha$ in B and lift it to some path γ in E such that $\gamma(0) = e_0$. Lemma 2.18 necessitates the existence of such a γ. We have assumed $\tilde{f}$ exists, and we have that path liftings are unique. From this, as $\tilde{f} \circ \alpha$ is a path lifting of $f \circ \alpha$ with $\tilde{f} \circ \alpha(0) = e_0$, it must be true that $\tilde{f} \circ \alpha = \gamma$ and so

$$\tilde{f} \circ \alpha(1) = \tilde{f}(x_1) = \gamma(1).$$

Thus, our $\tilde{f}$ is unique.

Now, conversely, suppose that $f_*(\pi_1(X, x_0)) \subseteq p_*(\pi_1(E, e_0))$ as per the lemma. We set out to prove that $\tilde{f}$ exists, and again that it is unique. Throughout, take $x_1 \in X$. Let $\alpha : [0, 1] \to X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$ again. We once again lift $f \circ \alpha$ to some path γ in E with $\gamma(0) = e_0$, which we know exists and is unique. During our proof of the converse, we showed that this implies $\tilde{f}(x_1) = \gamma(1)$.

We must show that independent of the choice of α, $\tilde{f}$ is well-defined. Suppose we also have some path $\beta : [0, 1] \to X$ such that $\beta(0) = x_0$ and $\beta(1) = x_1$. Let δ, a path in E, be the unique lift of $f \circ \beta$ to E with $\delta(0) = e_0$ - which is the same argument as before but for a new path β.

Consider the loop with basepoint x_0 formed by moving α and then along β^{-1} (defined as $\beta^{-1}(s) = \beta(1 - s)$). We see clearly that this is a loop as it moves to x_1 at $s = 1/2$ and then back to x_0 at $s = 1$. Denote this loop $\alpha\beta^{-1}$. Then clearly $f \circ \alpha\beta^{-1}$ is also a loop with basepoint b_0. To see this, note that

$$f \circ \alpha\beta^{-1}(0) = f \circ \alpha\beta^{-1}(1) = f(x_0) = b_0.$$

By our assumption, we have that $f_*(\pi_1(X, x_0)) \subseteq p_*(\pi_1(E, e_0))$ and thus the homotopy class $[f \circ \alpha\beta^{-1}] \in p_*(\pi_1(E, e_0))$. We wish to lift $f \circ \alpha\beta^{-1}$ to some path in E beginning at e_0. By the uniqueness of path liftings, we must have that $\gamma\delta^{-1}$ is the unique lift of $f \circ \alpha\beta^{-1}$ to E, starting at e_0. However, by our assumption, we have that $f_*(\pi_1(X, x_0)) \subseteq p_*(\pi_1(E, e_0))$ and thus the homotopy class $[f \circ \alpha\beta^{-1}] \in p_*(\pi_1(E, e_0))$. So our $\gamma\delta^{-1}$ must also end at e_0 and is thus a loop.

As δ is the inverse of δ^{-1}, we have

$$\delta(1) = \delta^{-1}(0) = \gamma(1).$$

So we have that regardless of our choice of path in X, the end point of the lift $f \circ \alpha$ is the same. Thus we have shown that $\tilde{f}$ is well-defined.

We can now demonstrate continuity of $\tilde{f}$. Choose a path connected neighbourhood V of $f(x_1)$ that is evenly covered by p. Break $p^{-1}(V)$ up into sheets and let U_0 be the sheet containing $\tilde{f}(x_1)$ (we can replace V by a smaller neighbourhood of $f(x_1)$ if required, as our choice was arbitrary). Given a neighbourhood N of $\tilde{f}(x_1)$, we can assume that $U_0 \subseteq N$. Restrict p to U_0 to obtain $p|_{U_0} : U_0 \to V$. $p|_{U_0}$ is necessarily a homeomorphism by definition of covering spaces. By continuity of f at x_1 and local path connectedness of X, we are able to find some path connected neighbourhood W of x_1 such that $f(W) \subseteq V$.

We show that $\tilde{f}(W) \subseteq U_0$, which will satisfy the conditions for continuity. Given some $x \in W$, choose some path $\beta : [0,1] \to W$ with $\beta(0) = x_1$ and $\beta(1) = x$. As $\tilde{f}$ is well defined, we can obtain $\tilde{f}(x)$ by taking the path $\alpha\beta$ from x_0 to x, then lifting $f \circ \alpha\beta$ to some path in E beginning at e_0 with $\tilde{f}(x)$ as the end point of the lifted path. We have that the previously defined γ is a lifting of α that begins at e_0. Since $f \circ \beta$ is in V, the path $\delta = p|_{U_0}^{-1} \circ f \circ \beta$ is a lifting of $f \circ \beta$ beginning at $\tilde{f}(x_1)$. Then $\gamma\delta$ is a lifting of $f \circ \alpha\beta$ beginning at e_0 and ending at $\delta(1) \in U_0$. So $\tilde{f}(W) \subseteq U_0$ and thus $\tilde{f}$ is continuous.

□

We now use this lemma to prove a theorem that provides a specific instance wherein an equivalence h between covering spaces exists.

Theorem 2.31. *Let $p : E \to B$ and $p' : E' \to B$ be covering maps with*

$$p(e_0) = p'(e'_0) = b_0.$$

Then there exists some equivalence $h : E \to E'$, that is unique, such that $h(e_0) = e'_0$ if and only if [Mun00]

$$p_*(\pi_1(E, e_0)) = p'_*(\pi_1(E', e'_0)).$$

Remark 2.32. In the above, recall that p_* is a group homomorphism. Recall also that the image of a homomorphism is a subgroup of its codomain (theorem B.24). As such, $p_*(\pi_1(E, e_0))$ and $p'_*(\pi_1(E', e'_0))$ are both subgroups of $\pi_1(B, b_0)$.

Proof. Suppose that there exists some equivalence $h : E \to E'$ such that $h(e_0) = e'_0$. Then, given that h is a homeomorphism (by the definition of equivalence) implies that

$$h_*(\pi_1(E, e_0)) = \pi_1(E', e'_0)$$

by bijectivity. Note that in the definition of equivalence we have $p = p' \circ h$. So we have

$$\begin{aligned} p_*(\pi_1(E, e_0)) &= p'_* \circ h_*(\pi_1(E, e_0)) \\ &= p'_*(\pi_1(E', e'_0)). \end{aligned}$$

Now we must show the converse. Assume that $p_*(\pi_1(E, e_0)) = p'_*(\pi_1(E', e'_0))$. We will show that h exists. As p' is a covering map and E is assumed to be path connected and locally path connected, we must have that there exists some map (by the preceding lemma, lemma 2.30) $h : E \to E'$, with

$h(e_0) = e_0'$, that is a lifting of p, i.e. $p' \circ h = p$.

$$\begin{tikzcd} & E' \arrow[d, "p'"] \\ E \arrow[ur, "h"] \arrow[r, "p"] & B \end{tikzcd}$$

By the same argument, we can derive some map $h' : E' \to E$, with $h'(e_0') = e_0$, that is a lifting of p', i.e. $p \circ h' = p'$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ E' \arrow[ur, "h'"] \arrow[r, "p'"] & B \end{tikzcd}$$

Now consider the map $h' \circ h : E \to E$, which is a lifting of p as

$$p \circ h' \circ h = p' \circ h = p.$$

Note also that

$$\begin{aligned} & p(e_0) = p \circ h' \circ h(e_0) \\ \implies & h' \circ h(e_0) = e_0. \end{aligned}$$

So $h' \circ h$ takes e_0 to itself. By uniqueness in lemma 2.30, we have that $h' \circ h$ must be the identity map of E. The same argument shows that $h \circ h'$ must be the identity map of E'.

So we have that h exists with $h(e_0) = e_0'$ and is unique. □

The next lemma will be used in our proof of an important result concerning the relationship between equivalence of covering spaces and conjugacy classes of subgroups of the fundamental group in B.

Lemma 2.33. *Let $p : E \to B$ be a covering map. Let e_0, e_1 in the fibre $p^{-1}(b_0)$. Let*

$$H_i = p_*(\pi_1(E, e_i))$$

.

Let γ be a path in E such that $\gamma(0) = e_0$ and $\gamma(1) = e_1$. Let $\alpha = p \circ \gamma$ be a loop in B. This is necessarily a loop. Then

$$H_0 = [\alpha] H_1 [\alpha]^{-1},$$

i.e. H_0 and H_1 are conjugate.

Conversely, suppose we have e_0. Then given some arbitrary $H \leq \pi_1(B, b_0)$ conjugate to H_0, there exists e_1 (in the fibre $p^{-1}(b_0)$) such that $H_1 = H$. [Mun00]

Proof. Let $h \in H_1 = p_*(\pi_1(E, e_1))$ with $[h] = p_*([\tilde{h}])$ for some loop $\tilde{h} \in \pi_1(E, e_1)$. Let $\tilde{k} = (\gamma\tilde{h})\gamma^{-1}$. Note that $\tilde{k} \in \pi_1(E, e_0)$. To see this, first note that

$$(\gamma\tilde{h})\gamma^{-1} = \begin{cases} \gamma(4s) & \text{for } 0 \leq s \leq 1/4 \\ \tilde{h}(4s-1) & \text{for } 1/4 \leq s \leq 1/2 \\ \gamma^{-1}(2s-1) & \text{for } 1/2 \leq s \leq 1. \end{cases}$$

Substituting $s = 0$ and $s = 1$ shows clearly that this is a loop at e_0.

From this, we apply p_* to obtain

$$p_*([\tilde{k}]) = [(\alpha h)\alpha^{-1}] = [\alpha][h][\alpha^{-1}].$$

As $[\alpha][h][\alpha^{-1}] \in p_*(\pi_1(E, e_0))$ and our choice of $h \in H_1$ was arbitrary, we can conclude that

$$H_0 \subseteq [\alpha]H_1[\alpha^{-1}].$$

Using the exact same argument with γ^{-1} and $\alpha^{-1} = p \circ \gamma^{-1}$, we can easily show that this is true. So $H_0 \subseteq [\alpha]H_1[\alpha^{-1}]$ and $[\alpha]H_1[\alpha^{-1}] \subseteq H_0$, i.e. H_0 and H_1 are conjugate.

Now we prove the converse. Let e_0 be given. Let H be conjugate to H_0. Then we have

$$H_0 = [\alpha]H[\alpha]^{-1}$$

for some $\alpha \in \pi_1(B, b_0)$. Let γ be the lifting of α to a path in E (which necessarily exists) with $\gamma(0) = e_0$. Let $e_1 = \gamma(1)$, the endpoint of γ. Then our previous argument remains true and we have

$$H_0 = [\alpha]H_1[\alpha]^{-1}.$$

So $H = H_1$. □

Now we come to our main result for this section. We show that covering maps are equivalent if their images of the homomorphism induced by p relative to some corresponding basepoints are conjugate.

Theorem 2.34. *Let $p : E \to B$ and $p' : E' \to B$ be covering maps with*

$$p(e_0) = p'(e'_0) = b_0.$$

Then p and p' are equivalent if and only if the subgroups $H_0, H'_0 \subseteq \pi_1(B, b_0)$ are conjugate, where [Mun00]

$$H_0 = p_*(\pi_1(E, e_0))$$
$$\text{and } H'_0 = p'_*(\pi_1(E', e'_0)).$$

Proof. Here, we use theorem 2.31 and the previous lemma, 2.33.

Suppose $h : E \to E'$ is an equivalence with $h(e_0) = e'_1$. Let $H'_1 = p'_*(\pi_1(E', e'_1))$. By 2.31 we have $H_0 = H'_1$ (to see this, note that we have $p(e_0) = p'(e'_0) = b_0$). Then by 2.33 we must have that H'_1 is conjugate to H'_0 (as we have there must exist some path between e'_0 and e'_1). So $H_0 = H'_1$ is conjugate to H'_0.

Now suppose the converse is true, i.e. that H'_1 is conjugate to H'_0. Then 2.33 implies that there exists some $e'_1 \in E'$ such that $H'_1 = H_0$. We appeal to 2.31, and the fact that $H'_1 = H_0$, to get the necessary existence of our unique equivalence $h : E \to E'$ with $h(e_0) = e'_1$. □

2.7 Semilocal Simple Connectedness

We introduce the notion of *semilocal simple connectedness* from Munkres [Mun00]. We then show how one may construct a universal covering for any space B that is path connected, locally path connected and semilocally simply connected.

Definition 2.35. Let B be a topological space. B is said to be *semilocally simply connected* if for every $b_0 \in B$ there is some neighbourhood V of b_0 such that the homomorphism

$$\iota_* : \pi_1(V, b_0) \to \pi_1(B, b_0),$$

induced by the inclusion map[4] $\iota : V \to B$, is trivial. [Mun00]

Remark 2.36. Informally, for a space to be semilocally simply connected we must be able to shrink down our view at any point in the space to a small enough neighbourhood such that there are 'no holes' so to speak.

The following lemma will later highlight precisely why the condition of semilocal simple connectedness is necessary for the existence of our universal cover. Its relevance may not be clear immediately, but we shall reference it later.

Lemma 2.37. *Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Suppose E is simply connected. Then b_0 has some neighbourhood V such that the homomorphism*

$$\iota_* : \pi_1(V, b_0) \to \pi_1(B, b_0),$$

induced by the inclusion map $\iota : V \to B$, is trivial.

Proof. Let $V \subseteq B$ be a neighbourhood of b_0 that is evenly covered by p. Let $U_\alpha \subseteq E$ be the sheet that contains e_0. Let γ be some loop in V such that $\gamma(0) = \gamma(1) = b_0$. We have that γ lifts to some loop $\tilde{\gamma}$ by the fact that V and U_α are homeomorphic via $p|_{U_\alpha}$ (proposition 2.26). By simple connectedness of E, $\tilde{f}$ is homotopic to the constant loop at e_0, e_{e_0}, say via $\tilde{\Gamma} : [0, 1] \times [0, 1] \to E$. Thus $p \circ \tilde{\Gamma}$ is a path homotopy in B between γ and the constant loop at b_0, e_{b_0} (proposition 2.26). Thus, for every $[\gamma] \in \pi_1(V, b_0)$ we have

$$\iota_*([\gamma]) = [p \circ \tilde{\gamma}] = [e_{b_0}].$$

So the inclusion-induced map

$$\iota_* : \pi_1(V, b_0) \to \pi_1(B, b_0)$$

is indeed trivial. □

Going forward, if not stated otherwise we shall assume B is also semilocally simply connected.[5]

2.8 The Universal Cover

We are now equipped to construct the universal covering space. We show that the universal covering space exists if and only if B is path connected, locally path connected and semilocally simply connected.

Theorem 2.38. *Let B be a topological space. Let $b_0 \in B$. Let $H \leq \pi_1(B, b_0)$ be a subgroup. Then there exists some covering map $p : E \to B$ and some $e_0 \in p^{-1}(b_0)$ such that [Mun00]*

$$p_*(\pi_1(E, e_0)) = H.$$

if and only if B is as previously assumed, i.e. path connected, locally path connected and semilocally simply connected.

Remark 2.39. The following proof is quite involved, and as such we roughly follow the structure of the proof in Munkres' book. [Mun00] Relative to the statement of the theorem above, we shall actually prove the converse first. The structure of the proof is as follows:

1. Construction of the covering space E and definition of p.

2. Topologising E.

[4] The inclusion map $\iota : V \to B$ is defined by $\iota(v) = v$ for all $v \in V$.

[5] Recall that we have previously assumed that B is locally path connected and path connected unless otherwise stated; these properties still hold.

3. Proof that p is open and continuous.
4. Proof that every point in B has an open neighbourhood such that p is an even cover of the open neighbourhood.
5. Lifting of paths in B beginning at b_0 to E.
6. Proof that E is path connected.
7. Proof that $p_*(\pi(E, e_0)) = H$.
8. Proof that if $p_*(\pi_1(E, e_0)) = H$, then B is path connected, locally path connected and semilocally simply connected.

Proof. We begin our proof using the structure detailed above.

1. Allow $\mathcal{P}$ denote the set of all paths in B that begin at b_0. Let $\alpha, \beta \in \mathcal{P}$. We define an equivalence relation on $\mathcal{P}$ by setting $\alpha \sim \beta$ if $\alpha(1) = \beta(1)$ and $[\alpha\beta^{-1}] \in H$.

   Symmetry is evident from the fact that $(\alpha\beta^{-1})^{-1} = \beta\alpha^{-1}$ (from basic group theory). This necessitates that $[\beta\alpha^{-1}] \in H$ as required (as H is a subgroup). **Reflexivity** follows from the fact that $[\alpha\alpha^{-1}]$ is the trivial loop, which is necessarily in H by the properties of subgroups.

   For **transitivity**, assume for some $\gamma \in \mathcal{P}$ we have $\alpha \sim \beta$ and $\beta \sim \gamma$. Then $[\alpha\beta^{-1}] \in H$ and $[\beta\gamma^{-1}] \in H$. Again by elementary group theory, we have that $[\alpha\beta^{-1}\beta\gamma] = [\alpha\gamma] \in H$. So $\alpha \sim \gamma$.

   We denote the equivalence class of some $\alpha \in \mathcal{P}$ by $\langle\alpha\rangle$[6]. Let E denote the collection of equivalence classes in $\mathcal{P}$, using the equivalence relation defined above. Further, define $p : E \to B$ by

$$p(\langle\alpha\rangle) = \alpha(1).$$

2. We will now topologise our collection E such that p is a covering map.

   Suppose $[\alpha] = [\beta]$. Then $[\alpha\beta^{-1}]$ is the trivial loop and is necessarily in H. So $\langle\alpha\rangle = \langle\beta\rangle$ and we may conclude that

$$[\alpha] = [\beta] \implies \langle\alpha\rangle = \langle\beta\rangle\,. \tag{a}$$

   Suppose that $\langle\alpha\rangle = \langle\beta\rangle$. Let $\delta \in \mathcal{P}$ be such that $\delta(0) = \alpha(1)$. Then clearly

$$\alpha\delta(1) = \begin{cases} \alpha(2s) & \text{for } 0 \leq s \leq 1/2 \\ \delta(2s-1) & \text{for } 1/2 \leq s \leq 1 \end{cases}$$

   ends at the same point, notably $\delta(1)$, as

$$\beta\delta(1) = \begin{cases} \beta(2s) & \text{for } 0 \leq s \leq 1/2 \\ \delta(2s-1) & \text{for } 1/2 \leq s \leq 1. \end{cases}$$

   We also have

$$\begin{aligned} &[(\alpha\delta)(\beta\delta)^{-1}] \\ =&[(\alpha\delta)(\delta^{-1}\beta^{-1})] \\ =&[\alpha\delta\delta^{-1}\beta^{-1}] \\ =&[\alpha\beta^{-1}]. \end{aligned}$$

[6]The reader should be cautious not to confuse the two distinct equivalence classes $\langle\alpha\rangle$, as defined in this proof, and $[\alpha]$, as defined in definition 1.5.

$[\alpha\beta^{-1}] \in H$ by the assumption that $\alpha \sim \beta$. So we have shown that

$$\langle\alpha\rangle = \langle\beta\rangle \implies \langle\alpha\delta\rangle = \langle\beta\delta\rangle \tag{b}$$

under our equivalence relation, for $\delta \in \mathcal{P}$ be such that $\delta(0) = \alpha(1)$.

Let V be some arbitrary path connected neighbourhood of $\alpha(1)$. We define

$$B(V, \alpha) = \{\langle\alpha\delta\rangle : \delta([0,1]) \subseteq V,\ \delta(0) = \alpha(1)\},$$

i.e. δ is some path entirely contained in V that begins at $\alpha(1)$.

Note that $\langle\alpha\rangle \in B(V, \alpha)$. To see this, note that

$$\langle\alpha\rangle = \langle\alpha e_{\alpha(1)}\rangle\,,$$

where $e_{\alpha(1)}$ is the trivial loop at $\alpha(1)$. Clearly $\langle\alpha e_{\alpha(1)}\rangle \in B(V, \alpha)$ by the definition of $B(V, \alpha)$ and so $\langle\alpha\rangle \in B(V, \alpha)$.

We shall show that the collection of sets of the form $B(V, \alpha)$ forms a basis[7] for a topology on E. We must first establish some facts.

Suppose $\langle\beta\rangle \in B(V, \alpha)$. Then $\langle\beta\rangle = \langle\alpha\delta\rangle$ for some path δ in V by definition. We have already shown it to be the case that

$$\langle\beta\delta^{-1}\rangle = \langle(\alpha\delta)\delta^{-1}\rangle$$

by (b). This reduces to just

$$\langle\beta\delta^{-1}\rangle = \langle\alpha\rangle\,.$$

From this, we conclude that $\langle\alpha\rangle \in B(V, \beta)$ via the path δ^{-1} (as $\delta^{-1}(0) = \delta(1) = \beta(1)$).

We shall show that $B(V, \beta) \subseteq B(V, \alpha)$. Write some element of $B(V, \beta)$ as $\langle\beta\gamma\rangle$ where γ is a path in V with $\gamma(0) = \beta(1)$. Then we have

$$\begin{aligned}\langle\beta\gamma\rangle &= \langle(\alpha\delta)\gamma\rangle\\ &= \langle\alpha(\delta\gamma)\rangle\end{aligned}$$

by associativity of the composition of paths[8]. Clearly $\langle\alpha(\delta\gamma)\rangle \in B(V, \beta)$ by noting that $\delta\gamma(0) = \delta(0) = \alpha(1)$ and of course that δ and γ are contained entirely in V by definition. So the arbitrary $\langle\beta\gamma\rangle \in B(V, \alpha)$ and we may conclude that $B(V, \beta) \subseteq B(V, \alpha)$. The same argument shows that we also have $B(V, \alpha) \subseteq B(V, \beta)$ and so

$$B(V, \alpha) = B(V, \beta).$$

Now, we present our argument to show that the collection of sets of the form $B(V, \alpha)$ forms a basis for a topology on E as desired. Clearly the existence property for some $\langle\alpha\rangle$ is satisfied by $B(V, \alpha)$. Suppose that

$$\langle\beta\rangle \in B(V_0, \alpha_0) \cap B(V_1, \alpha_1)$$

for some open path-connected neighbourhoods V_0, V_1 of $\alpha_0(1), \alpha_1(1) \in B$ respectively. Then we can choose some path-connected neighbourhood $V \subseteq V_0 \cap V_1$ of $\beta(1)$. From the definition of the set $B(V, \beta)$ it is clear that we have

$$B(V, \beta) \subseteq B(V_0, \beta) \cap B(V_1, \beta).$$

[7] See definition A.7 for an overview of bases of topological spaces.

[8] Although we proved such an argument for the particular case of loops (proposition 1.8), the same argument can be extended to show that composition of paths is an associative operation on all paths, not just loops.

We have previously shown that it must be that $B(V_0, \beta) = B(V_0, \alpha_0)$ and similarly $B(V_1, \beta) = B(V_1, \alpha_1)$. So we have that

$$\begin{aligned} & B(V, \beta) \subseteq B(V_0, \beta) \cap B(V_1, \beta) \\ \Longrightarrow\ & B(V, \beta) \subseteq B(V_0, \alpha_0) \cap B(V_1, \alpha_1) \end{aligned}$$

as per the definition of a basis, so we are done.

3. Recall that we defined p to be such that $p(\langle\alpha\rangle) = \alpha(1)$. Let $v_0 \in V$. Choose some path δ such that $\delta(0) = \alpha(1)$ and $\delta(1) = v_0$ (via path connectedness). By definition of $B(V, \alpha)$, we must have that $\langle\alpha\delta\rangle \in B(V, \alpha)$. We also have
$$p(\langle\alpha\delta\rangle) = \alpha\delta(1) = \delta(1) = v_0.$$
As our choice of v_0 was arbitrary and we must have that any δ must be contained entirely in V (and thus any $p(\langle\alpha\delta\rangle) = \delta(1)$, we must have
$$p(B(V, \alpha)) = V.$$
So the image of the basis element $B(V, \alpha)$ under p is the open subset V of B and so is open.

   Now we show continuity. Let V_0 be a neighbourhood of $p(\langle\alpha\rangle) = \alpha(1)$. We can choose a path connected neighbourhood of $p(\langle\alpha\rangle) = \alpha(1)$ that lies in V_0, say $V \subseteq V_0$. Then we have that $B(V, \alpha)$ is a neighbourhood of $\langle\alpha\rangle$ such that
$$p(B(V, \alpha)) = V$$
to satisfy our definition of continuity. So p is continuous at $\langle\alpha\rangle$.

4. Let $b_1 \in B$. Choose $V \subseteq B$ such that V is a path connected neighbourhood of b_1 such that the homomorphism $\pi_1(V, b_1) \to \pi_1(B, b_1)$ induced by the inclusion map is trivial (which exists by semilocal simple connectedness). We will show that V is evenly covered by our p.

   Recall the definition of $B(V, \alpha)$. We have shown previously that p maps an arbitrary $B(V, \alpha)$ onto V - so it is clear that
$$\bigcup_\alpha B(V, \alpha) \subseteq p^{-1}(V).$$
We shall show the opposite is true, to conclude that $\bigcup_\alpha B(V, \alpha) = p^{-1}(V)$. Suppose that $\langle\beta\rangle \in p^{-1}(V)$. By our definition of p we have $p(\langle\beta\rangle) = \beta(1)$. So $\beta(1) \in V$. Now choose a path δ in V such that $\delta(0) = b_1$ and $\delta(1) = \beta(1)$. Now choose α to be the path $\beta\delta^{-1}$ which is such that $\alpha(0) = \beta(0) = b_0$ (recall that all paths in $\mathcal{P}$ begin at b_0) and $\alpha(1) = \delta^{-1}(1) = b_1$. Clearly with some minor rearrangement we get $[\beta] = [\alpha\delta]$. Then we have
$$\begin{aligned} & [\beta] = [\alpha\delta] \\ \Longrightarrow\ & \langle\beta\rangle = \langle\alpha\delta\rangle \end{aligned}$$
from (a). We have that $\langle\beta\rangle \in B(V, \alpha)$ via our δ. So any $\langle\beta\rangle \in p^{-1}(V)$ is such that $\langle\beta\rangle \in B(V, \alpha)$ for some α and so we may conclude that
$$p^{-1}(V) \subseteq \bigcup_\alpha B(V, \alpha).$$
Putting our two conclusions together gives
$$\bigcup_\alpha B(V, \alpha) = p^{-1}(V)$$

as desired.

We have previously shown that distinct $B(V,\alpha)$ are disjoint. To see this, recall that if $\langle\beta\rangle \in B(V,\alpha_1)\cap B(V,\alpha_2)$ then $\langle\beta\rangle \in B(V,\alpha_1)$ and $\langle\beta\rangle \in B(V,\alpha_2)$ - this means that

$$B(V,\alpha_1) = B(V,\beta) = B(V,\alpha_2).$$

Recall that we have already shown that p maps $B(V,\alpha)$ onto V. We now show that $p|_{B(V,\beta)} : B(V,\beta) \to V$ is injective. Suppose that

$$p(\langle\alpha\delta_1\rangle) = p(\langle\alpha\delta_2\rangle)$$

where δ_1 and δ_2 are paths in V and clearly $\langle\alpha\delta_1\rangle, \langle\alpha\delta_2\rangle \in B(V,\alpha)$. Then

$$p(\langle\alpha\delta_1\rangle) = \delta_1(1) = \delta_2(1) = p(\langle\alpha\delta_2\rangle)$$

by our definition of p. From our trivial induced inclusion map condition on V, we have that $\delta_1\delta_2^{-1}$ is path homotopic to the constant loop in B. So $[\alpha\delta_1] = [\alpha\delta_2]$. From (a) we may then conclude that

$$\langle\alpha\delta_1\rangle = \langle\alpha\delta_2\rangle,$$

which is precisely our condition for injectivity of $p|_{B(V,\beta)}$ and thus bijectivity. We previously showed that p is continuous and open. So $p|_{B(V,\beta)}$ is a homeomorphism.

This is precisely the definition of what it means for V to be evenly covered by p. As V is for some arbitrary b_1, we may conclude that every $b_1 \in B$ has a neighbourhood that is evenly covered by p.

5. Let $[e_{b_0}]$ denote the equivalence class of the constant loop at b_0. Let $e_0 = \langle e_{b_0}\rangle$. Then we have $p(e_0) = b_0$ by definition. For a path $\alpha \in B$ that begins at b_0 we calculate its lift to some path in E beginning at $e_0 = \langle e_{b_0}\rangle$. We will then show that this lift ends at $\langle\alpha\rangle$.

   Let $c \in [0,1]$. Let $\alpha_c : [0,1] \to B$ be defined by

$$\alpha_c(s) = \alpha(sc).$$

So α_c gives a cut of α that runs from $\alpha(0)$ to $\alpha(c)$. Note that

$$\alpha_0(0) = \alpha_0(1) = \alpha(0)$$

and so α_0 is just the constant loop at $\alpha(0) = b_0$, i.e. e_{b_0}. Clearly, we also have that $\alpha_1 = \alpha$.

Now define $\tilde{\alpha} : [0,1] \to E$ by

$$\tilde{\alpha}(c) = \langle\alpha_c\rangle.$$

We will show that $\tilde{\alpha}$ is continuous.

To do this, we define $\delta_{c,d} : [0,1] \to B$ as follows: for some $d \in [0,1]$ with $c \le d$ we have

$$\delta_{c,d}(s) = \alpha(c + (d-c)s).$$

So $\delta_{c,d}(t)$ runs from $\alpha(c)$ to $\alpha(d)$. Note that α_d (as defined the same as α_c) is path homotopic to $\alpha_c\delta_{c,d}$. To see this, consider the reparametrisation (see proposition 1.8) given by

$$\phi(s) = \begin{cases} 2cs & \text{for } 0 \le s \le 1/2 \\ c + 2(d-c)(s-1/2) & \text{for } 1/2 \le s \le 1 \end{cases}$$

Then we have

$$\alpha(\phi(s)) = \begin{cases} \alpha(2cs) & \text{for } 0 \le s \le 1/2 \\ \alpha(c + 2(d-c)(s-1/2)) & \text{for } 1/2 \le s \le 1 \end{cases}$$

which is precisely the same as

$$\alpha_c\delta_{c,d} = \begin{cases} \alpha_c(2s) = \alpha(2cs) & \text{for } 0 \le s \le 1/2 \\ \delta_{c,d}(2s-1) = \alpha(c + 2(d-c)(s-1/2)) & \text{for } 1/2 \le s \le 1. \end{cases}$$

Let W be some basis element in E about the point $\tilde{\alpha}(c)$. Then we have that W is $B(U, \alpha_c)$ for some path-connected neighbourhood U of $\alpha(c)$. Choose some $\epsilon > 0$ so that for $|c - s| < \epsilon$, the point $\alpha(s)$ lies in U. We show that if d is some point of $[0, 1]$ with $|c - d| < \epsilon$, then $\tilde{\alpha}(d) \in W$ - this suffices to prove continuity of $\tilde{\alpha}$ at c. So then suppose $|c - d| < \epsilon$. Without loss of generality assume $c \le d$. Then since $[\alpha_d] = [\alpha_c\delta_{c,d}]$ we have

$$\tilde{\alpha}(d) = \langle \alpha_d \rangle = \langle \alpha_c\delta_{c,d} \rangle .$$

Since $\delta_{c,d}$ lies in U, we have $\tilde{\alpha}(d) \in W$ and we are done.

$\tilde{\alpha}$ satisfies the definition of a lift of α, as $p(\tilde{\alpha}(c)) = \alpha_c(1) = \alpha(c)$, $\tilde{\alpha}(0) = \langle \alpha_0 \rangle = e_0$ and $\langle \alpha_1 \rangle = \langle \alpha \rangle$ in addition to the continuity of $\tilde{\alpha}$ that we have previously shown.

6. We show that E is path-connected. For some $\langle \alpha \rangle \in E$, then we have already shown that the lift $\tilde{\alpha}$ of the path α is a path in E such that $\tilde{\alpha}(0) = e_0$ and $\tilde{\alpha}(1) = \langle \alpha \rangle$. To connect any two points $\langle \alpha \rangle, \langle \beta \rangle \in E$, we may do the following: let $\tilde{\beta}$ be defined in the same way as $\tilde{\alpha}$. Then $\langle \alpha \rangle$ and $\langle \beta \rangle$ are connected via the path

$$\tilde{\alpha}^{-1}\tilde{\beta} = \begin{cases} \tilde{\alpha}^{-1}(2s) & \text{for } 0 \le s \le 1/2 \\ \tilde{\beta}(2s-1) & \text{for } 1/2 \le s \le 1. \end{cases}$$

One can quickly verify that $\tilde{\alpha}^{-1}\tilde{\beta}(0) = \tilde{\alpha}^{-1}(0) = \langle \alpha \rangle$ and $\tilde{\alpha}^{-1}\tilde{\beta}(1) = \tilde{\beta}(1) = \langle \beta \rangle$. So our condition for path connectedness is met for E.

7. We first appeal to proposition 2.28. We have that $[\alpha] \in p_*(\pi_1(E, e_0))$ if and only if $\tilde{\alpha}$ is a loop in E. So in order for this condition to be satisfied we must have

$$\begin{aligned} \tilde{\alpha}(1) &= \langle \alpha \rangle \\ &= e_0 = \tilde{\alpha}(0). \end{aligned}$$

It is precisely the case that $\langle \alpha \rangle = e_0$ when $[\alpha e_{b_0}^{-1}] = [\alpha] \in H$ by our hypothesis. So any arbitrary $[\alpha] \in H$ if and only if $[\alpha] \in p_*(\pi_1(E, e_0))$ and we may conclude that $[\alpha] \in p_*(\pi_1(E, e_0))$.

8.

□

Definition 2.40. Let B, E be as in theorem 2.38 above. E is called the *universal covering space* of B if H is the trivial subgroup of $\pi_1(B, b_0)$.

2.9 Deck Transformations

Lee [Lee10] provides a definition for *deck transformations*[9], which shall later be relevant to our key theorem of this section.

Definition 2.41. Let $p : E \to B$ be a covering map. A homeomorphism $\varphi : E \to E$ is called a *deck transformation* if $p \circ \varphi = p$.

[9] Or *'covering transformations'*.

Denote the set of deck transformations of E by $\mathrm{Deck}(E)$. [Lee10]

A very simple definition. We now show that the set of deck transformations forms a group under composition of functions.

Proposition 2.42. *Let $p : E \to B$ be a covering map. Then $(\mathrm{Deck}(E), \circ)$ is a group, with $\circ$ denoting the composition of functions. [Lee10]*

Proof. Let $\varphi_0, \varphi_1, \varphi_2 \in \mathrm{Deck}(E)$.

We have that $p \circ \varphi_0 = p$ and $p \circ \varphi_1 = p$ by definition. We consider $\varphi_0 \circ \varphi_1$. To see that $\varphi_0 \circ \varphi_1 \in \mathrm{Deck}(E)$, note that the composition of two homeomorphisms is a homeomorphism; note also that

$$p \circ (\varphi_0 \circ \varphi_1) = (p \circ \varphi_0) \circ \varphi_1 = p \circ \varphi_1 = p.$$

Hence, we have that our group is closed under $\circ$.

Associativity is trivial from the associativity of function composition, i.e.

$$(\varphi_0 \circ \varphi_1) \circ \varphi_2 = \varphi_0 \circ (\varphi_1 \circ \varphi_2).$$

Let us consider some **identity element**, say $\varphi_{id} \in \mathrm{Deck}(E)$, such that $\varphi_{id}(e) = e$ for all $e \in E$. This is trivially a homeomorphism. We also clearly have

$$p \circ \varphi_{id} = p$$

since φ_{id} simply takes every point to itself. Therefore, $\varphi_{id} \in \mathrm{Deck}(E)$. We also have that

$$\varphi_{id} \circ \varphi_0 = \varphi_0 = \varphi_0 \circ \varphi_{id}$$

by again noting that φ_{id} takes every point to itself. As such, φ_{id} is the identity of $\mathrm{Deck}(E)$.

It remains to show the existence of some **inverse**. As φ_0 is a homeomorphism, it necessarily has an inverse, which we can denote φ_0^{-1}. φ_0^{-1} is by definition a homeomorphism (being the inverse of a homeomorphism). We also have that

$$\varphi_0 \circ \varphi_0^{-1} = \varphi_{id} = \varphi_0^{-1} \circ \varphi_0$$

from the fact that φ_0^{-1} is an inverse function of φ_0. It is slightly less trivial to see that

$$\begin{aligned} &p \circ \varphi_0 = p \\ \implies &p \circ \varphi_0 \circ \varphi_0^{-1} = p \circ \varphi_0^{-1} \\ \implies &p = p \circ \varphi_0^{-1}, \end{aligned}$$

which is necessary for $\varphi_0^{-1} \in \mathrm{Deck}\, k(E)$.

$(\mathrm{Deck}(E), \circ)$ is then indeed a group, as it has satisfied the group axioms. □

We prove some important properties of the group of deck transformations. For this we turn to Lee [Lee10] again.

Proposition 2.43. *Let $p : E \to B$ be a covering map. Let $\varphi_0, \varphi_1 \in \mathrm{Deck}(E)$. The following hold:*

1. *If $\varphi_0(e) = \varphi_1(e)$ for some $e \in E$, then $\varphi_0 = \varphi_1$, i.e. they agree at every point.*
2. *$\mathrm{Deck}(E)$ acts continuously on E.*
3. *$\mathrm{Deck}(E)$ acts freely on E.*

4. *For any $b_0 \in B$, each element of* $\mathrm{Deck}(E)$ *permutes the points of the fibre $p^{-1}(b_0)$.*

5. *Let $V \subseteq B$ be an evenly covered open set. Then each element of* $\mathrm{Deck}(E)$ *permutes the components of $p^{-1}(V)$. [Lee10]*

Proof. We prove each point in-turn.

1. Note that some $\varphi \in \mathrm{Deck}(E)$ is a lift of p.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ E \arrow[ur, "\varphi"] \arrow[r, "p"] & B \end{tikzcd}$$

As such, if $\varphi_0(e) = \varphi_1(e)$ then by uniqueness of liftings (theorem 2.22), we have that $\varphi_0 = \varphi_1$.

2. Each $\varphi \in \mathrm{Deck}(E)$ is continuous by definition. So $\mathrm{Deck}(E)$ acts continuously on E.

3. To see that $\mathrm{Deck}(E)$ acts freely on E, we note, from (1), that φ_{id} cannot agree at some $e_0 \in E$ with any other $\varphi \in \mathrm{Deck}(E)$, else they would agree completely. Thus, φ_{id} is the only element of $\mathrm{Deck}(E)$ that fixes points and $\mathrm{Deck}(E)$ acts freely.

4. For all $e \in p^{-1}(b_0)$, we have $\varphi(e) \in p^{-1}(b_0)$ by the nature of deck transformations. Thus, we have

$$p(\varphi_0(e)) = p(e) = b_0.$$

As such, we have shown $\varphi(e)$ remains within the fibre. From the bijectivity of φ_0, we can conclude that φ_0 permutes the elements of the fibre.

5. Let $U_0 \subseteq p^{-1}(V)$ be a component of $p^{-1}(V)$. Since $\varphi_0(U_0)$ is a restriction to a component, it is a connected subset of $p^{-1}(V)$. Thus, it must be contained within a single component, say $U_1 \subseteq p^{-1}(V)$. Similarly, $\varphi_0^{-1}(U_1)$ is contained within a single component, namely $U_0 \subseteq p^{-1}(V)$. From bijectivity, we must have that

$$\varphi_0(U_0) = U_1 \text{ and } \varphi_0^{-1}(U_1) = U_0.$$

So φ_0 permutes the components of $p^{-1}(V)$.

□

We define a normal covering space. We then introduce a proposition to determine when the deck transformation group acts transitively.

Definition 2.44. Let $p : E \to B$ be a covering map. A covering space E is called a *normal covering space* if for each $b \in B$ and each pair of lifts e, e' of b, there exists a deck transformation taking e to e'. [Hat02]

Proposition 2.45. *Let $p : E \to B$ be a covering map. Then the following hold:*

1. *Let $p^{-1}(b_0) = e_0$ and $p^{-1}(b_0) = e_1$, i.e. e_0 and e_1 are two points in the same fibre. Then there exists a deck transformation taking e_0 to e_1 if and only if the induced subgroups $p_*\pi_1(E, e_0)$ and $p_*\pi_1(E, e_1)$ are such that*

$$p_*\pi_1(E, e_0) = p_*\pi_1(E, e_1).$$

2. $\mathrm{Deck}(E)$ *acts transitively on each fibre if and only if p is a normal covering. [Lee10]*

Proof. 1.

□

2.10 Correspondence of Deck Transformations and the Fundamental Group

We conclude this section by proving a result that brings together all we have covered (excuse the pun) thus far. We shall show that there exists an isomorphism between the fundamental group of some space B and the deck transformation group of said space's universal cover. We first prove a more general result.

Theorem 2.46. *Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Suppose E is path-connected. Let $H = p_*(\pi_1(E, e_0))$.*

3 Homotopy Groups

Previously, we derived the fundamental group for some space X. The fundamental group is an example of what is known as a *homotopy group*; in particular, the fundamental group captures information about loops in X that begin at some x_0 (the basepoint). We now wish to consider higher homotopy groups of the form $\pi_n(X, x_0)$ for $n \geq 2$. In the higher-dimensional treatment, subspaces take the place of basepoints. It is $\pi_2(X, x_0)$ that we will be later interested in when we begin to discuss 2-knots.

For now, we begin with definitions by Hatcher. [Hat02]

3.1 General Homotopy Classes

The *homotopy of paths* (definition 1.2) served as a specific example of a homotopy class. We now give Hatcher's [Hat02] definition for what it means for maps to be homotopic in general.

Definition 3.1. Let X, Y be topological spaces. Let $f_t : X \to Y$ be a family of continuous maps, with $0 \leq t \leq 1$, such that the map $F : X \times [0, 1] \to Y$ where $F(\mathbf{s}, t) = f_t(\mathbf{s})$ is continuous. We say that two maps f_0 and f_1 belonging to the same family f_t are *homotopic*, denoted $f_0 \sim f_1$. [Hat02]

Remark 3.2. Unlike in definition 1.2, where s was a one-dimensional variable, we here let our $\mathbf{s}$ be multi-variable.

Proposition 3.3. *$\sim$ is an equivalence relation.*

The following proof echoes our proof of proposition 1.4. We include it here for completeness.

Proof. Let f_0, f_1, g_0 and g_1 be continuous maps from X to Y.

Suppose $f_0 \sim f_1$. Then clearly there is some homotopy f_t such that this is the case. Hence we also have $f_1 \sim f_0$ via the homotopy f_{1-t}. $F(\mathbf{s}, 1-t)$ (as defined in definition 1.2) is clearly continuous as $F(\mathbf{s}, t)$ is continuous. So $\sim$ is **symmetric**.

We have that $f_0 \sim f_0$ via the homotopy $f_t = f_0$. $F(\mathbf{s}, t)$ is trivially continuous as it is just f_0. So $\sim$ is **reflexive**.

Let $f_1 = g_0$. Suppose $f_0 \sim f_1$ via the homotopy f_t and $f_1 = g_0 \sim g_1$ via the homotopy g_t. Define a new homotopy h_t such that

$$h_t = \begin{cases} f_{2t} & \text{for } 0 \leq t \leq 1/2 \\ g_{2t-1} & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

Then clearly $h_0 = f_0$, $h_{1/2} = f_1 = g_0$ and $h_1 = g_1$. So this homotopy works so long as we can show $H(\mathbf{s}, t) = h_t(s)$ is continuous. We have that

$$H(\mathbf{s}, t) = \begin{cases} F(\mathbf{s}, 2t) & \text{for } 0 \leq t \leq 1/2 \\ G(\mathbf{s}, 2t-1) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

So H, from the continuity of F and G, is continuous on $X \times [0, 1/2]$ and $X \times [1/2, 1]$. From the fact that a function defined on the union of two closed sets (noting that X is necessarily closed) is continuous if it is continuous when restricted to both of the closed sets individually (theorem A.31), H is continuous on $X \times [0, 1]$. Hence

$$f_0 \sim f_1 = g_0 \sim g_1$$

and so $\sim$ is **transitive**.

We have shown $\sim$ is reflexive, symmetric and transitive. Hence, it is an equivalence relation. $\square$

3.2 The Homotopy Group $\pi_n(X, x_0)$

The following definition simply defines the boundary of some n-dimensional unit cube. For $n = 2$ or $n = 3$ this is very easy to visualise. It is simply the 'edges' of what we know as the square or the 'faces' of a cube, respectively. Going forward, let $I = [0, 1]$ refer to the unit interval and let $I^n = [0, 1]^n$ refer to the n-dimensional unit cube.

Definition 3.4. Let ∂I^n denote the subspace of I^n consisting of all points with at least one coordinate equal to 0 or 1. We call ∂I^n the *boundary* of I^n. [Hat02]

Definition 3.5. Let X be a space and let $x_0 \in X$ be a basepoint. Define $\pi_n(X, x_0)$ to be the set of homotopy classes of maps

$$f : (I^n, \partial I^n) \to (X, x_0)$$

where homotopies f_t satisfy $f_t(\partial I^n) = x_0$ for all t. Denote the homotopy class of some f by $[f]$. [Hat02]

Remark 3.6. As before, it should be clear from the context as to whether f represents a map, or if it represents the equivalence class of that map.

Definition 3.7. For the case $n = 0$, the definition extends by taking I^0 to be a point and ∂I^n to be empty. So $\pi_0(X, x_0)$ is simply the set of path components of X. [Hat02]

We now generalise the composition operation for $n \geq 2$. Recall that f, g in the following definition refers to the equivalence classes of these maps under homotopy. The additive operation that follows is clearly well-defined on homotopy classes.

Definition 3.8. Let $n \geq 2$. Let $f, g : (I^n, \partial I^n) \to (X, x_0)$ be continuous maps. Define $+$ to be such that

$$f + g(s_1, s_2, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n) & \text{for } 0 \leq s_1 \leq 1/2 \\ g(2s_1 - 1, s_2, \ldots, s_n) & \text{for } 1/2 \leq s_1 \leq 1. \end{cases}$$

Proposition 3.9. *$(\pi_n(X, x_0), +)$ is a group.*

Proof. We extend our proof of proposition 1.8.

We first show that $f + g$ is continuous. Note that f and g are continuous. Let $s_1 = 1/2$. Then

$$f + g(1/2, s_2, \ldots, s_n) = f(2(1/2), s_2, \ldots, s_n) = g(2(1/2) - 1, s_2, \ldots, s_n) = x_0$$

owing to the definition of $\pi_n(X, x_0)$. As such, it is clear that $f + g$ is continuous on $0 \leq s_1 \leq 1$ by the pasting lemma (theorem A.31).

Consider $f_0, f_1, g_0, g_1 \in \pi_n(X, x_0)$ with $f_0 \sim f_1$ and $g_0 \sim g_1$. Let $F : I^n \times [0, 1] \to X$ denote a homotopy such that $F(s_1, s_2, \ldots, s_n, t) = f_t(s_1, s_2, \ldots, s_n)$. Note that $F(s_1, s_2, \ldots, s_n, 0) = f_0$, $F(s_1, s_2, \ldots, s_n, 1) = f_1$ and $F(\partial I^n, t) = x_0$. Let $G : I^n \times [0, 1] \to X$ denote a similar homotopy for g_t.

We define

$$H(s_1, s_2 \ldots, s_n, t) = \begin{cases} F(2s_1, s_2, \ldots, s_n, t) & \text{for } 0 \leq s_1 \leq 1/2 \\ G(2s_1 - 1, s_2, \ldots, s_n, t) & \text{for } 1/2 \leq s_1 \leq 1. \end{cases}$$

We again appeal to the pasting lemma (theorem A.31) to determine that H is continuous by noting that F and G are necessarily continuous and evaluating $s_1 = 1/2$. Thus, $H : I^n \times [0, 1] \to X$ is a homotopy. Setting $t = 0$ we have that

$$H(s_1, s_2 \ldots, s_n, 0) = f_0 + g_0(s_1, s_2, \ldots, s_n)$$

and setting $t = 1$ similarly gives

$$H(s_1, s_2 \ldots, s_n, 1) = f_1 + g_1(s_1, s_2, \ldots, s_n)$$

. As such, we have proven that $f_0 + g_0 \sim f_1 + g_1$, i.e. $+$ is well-defined.

We show that $+$ is closed. Recall that f, g must have the necessary condition that $f(\partial I^n)$, $g(\partial I^n) = x_0$. First, consider the case when $s_1 = 0$. Then we have

$$f + g(0, s_2, \dots, s_n) = f(0, s_2, \dots s_n) = x_0$$

as $0 \leq s_1 = 0 \leq 1/2$. Now consider the case $s_1 = 1$. Then we have

$$f + g(1, s_2, \dots, s_n) = g(2(1) - 1, s_2, \dots s_n) = g(1, s_2, \dots s_n) = x_0$$

as $1/2 \leq s_1 = 1 \leq 1$. As such, $f + g \in \pi_n(X, x_0)$ in these cases.

Now consider the case wherein some s_i with $i \in \{2, \dots, n\}$ is such that $s_i = 0$ or $s_i = 1$, i.e. s_i is on the boundary. Assume, without loss of generality, that $0 \leq s_1 \leq 1/2$. Then

$$f + g(s_1, \dots, s_i, \dots, s_n) = f(s_1, \dots, s_i, \dots, s_n) = x_0.$$

So we have that $f + g \in \pi_n(X, x_0)$ in this case. As we have exhausted all possible cases, we conclude that $f + g \in \pi_n(X, x_0)$ and as such, $\pi_n(X, x_0)$ is closed under $+$.

Define a *reparametrisation* of f to be a composition $f + \phi$, where $\phi : I^n \to I^n$ is any continuous map with $\phi(0, s_2, \dots, s_n) = 0$ and $\phi(1, s_2, \dots, s_n) = 1$. We have that reparametrising f preserves its homotopy class (i.e. $f \sim f + \phi$) via the homotopy $f + \phi_t$, where

$$\phi_t(s_1, s_2, ..., s_n) = (1 - t)\phi(s_1, s_2, ..., s_n) + t(s_1, s_2, ..., s_n),$$

noting that $\phi_0(s_1, s_2, ..., s_n) = \phi(s_1, s_2, ..., s_n)$ and $\phi_1(s_1, s_2, ..., s_n) = (s_1, s_2, ..., s_n)$.

We show **associativity**. Suppose we have continuous maps f, g, h with basepoint x_0. We have

$$([f] + [g]) + [h] = [f + g] + [h] = [f + g + h] = [f][g + h] = [f] + ([g] + [h]).$$

To see this, we note that $[f] + ([g] + [h])$ is a reparametrisation of $([f] + [g]) + [h]$ by ϕ given by

$$\phi(s_1, \dots, s_n) = \begin{cases} (s_1/2, \dots, s_n) & \text{for } 0 \leq s_1 \leq 1/2 \\ (s_1 - 1/4, \dots, s_n) & \text{for } 1/2 \leq s_1 \leq 3/4 \\ (2(s_1 - 1/2), \dots, s_n) & \text{for } 3/4 \leq s_1 \leq 1. \end{cases}$$

We now show the existence of some **identity element**. Let e be a continuous map with basepoint x_0 where $e(s_1, s_2, \dots, s_n) = x_0$ for all $(s_1, s_2, \dots, s_n) \in I^n$. We claim $[e]$ is the identity element. To see this, note that $f + e$ is a reparametrisation of f by ϕ given by

$$\phi(s_1, s_2, \dots, s_n) = \begin{cases} (2s_1, s_2, \dots, s_n) & \text{for } 0 \leq s_1 \leq 1/2 \\ (1, s_2, \dots, s_n) & \text{for } 1/2 \leq s_1 \leq 1. \end{cases}$$

Similarly, $e + f$ is a reparametrisation of f by ϕ given by

$$\phi(s) = \begin{cases} (0, s_2, \dots, s_n) & \text{for } 0 \leq s_1 \leq 1/2 \\ (2(s_1 - 1/2), s_2, \dots, s_n) & \text{for } 1/2 \leq s_1 \leq 1. \end{cases}$$

So we get that

$$[f] + [e] = [f + e] = [f] = [e + f] = [e] + [f],$$

as desired.

The final group axiom to verify is the existence of some **inverse element** of $[f]$ in $\pi_n(X, x_0)$, that we shall denote $[f^{-1}]$, i.e. the homotopy class of f^{-1} that we shall define. Let us define f^{-1} as

$$f^{-1}(s_1, s_2, \ldots, s_n) = f(1 - s_1, s_2, \ldots, s_n).$$

Clearly, f^{-1} is the same as f but travelling in the opposing direction in the s_1 coordinate. Define a homotopy by $h_t(s_1, s_2, \ldots, s_n) = f_t(s_1, s_2, \ldots, s_n) + f_t^{-1}(s_1, s_2, \ldots, s_n)$ where

$$f_t(s_1, s_2, \ldots, s_n) = \begin{cases} f(s_1, s_2, \ldots, s_n) & \text{for } 0 \leq s_1 \leq 1 - t \\ f(1 - t, s_2, \ldots, s_n) & \text{for } 1 - t \leq s_1 \leq 1. \end{cases}$$

and

$$f_t^{-1}(s) = \begin{cases} f^{-1}(t, s_2, \ldots, s_n) & \text{for } 0 \leq s_1 \leq t \\ f^{-1}(s_1, s_2, \ldots, s_n) & \text{for } t \leq s_1 \leq 1. \end{cases}$$

is the inverse loop of $f_t(s_1, s_2, \ldots, s_n)$. We see that $f_0(s_1, s_2, \ldots, s_n) = f(s_1, s_2, \ldots, s_n)$ and $f_1(s_1, s_2, \ldots, s_n) = f(1, s_2, \ldots, s_n) = x_0 = e$ (from the definition of $\pi_n(X, x_0)$). As a result, we can deduce that h_t is a homotopy from $f(s_1, s_2, \ldots, s_n) + f^{-1}(s_1, s_2, \ldots, s_n)$ to $e(s_1, s_2, \ldots, s_n) \cdot e^{-1}(s_1, s_2, \ldots, s_n) = e(s_1, s_2, \ldots, s_n)$. So $f + f^{-1} \sim e$. Swapping f_t and f_t^{-1} in h_t we instead get $f^{-1} + f \sim e$. Putting these results together we deduce that

$$[f] + [f^{-1}] = [f + f^{-1}] = [e] = [f^{-1} + f] = [f^{-1}] + [f].$$

So we have our inverse. $\square$

Remark 3.10. Here we have used additive notation instead of multiplicative notation. The reason for this is that $\pi_n(X, x_0)$ is an abelian group for $n \geq 2$, something which we shall prove. [Hat02]

Before we can prove such a property for our homotopy groups, we first prove the *Eckmann-Hilton argument*. This theorem comes from the 1962 paper *Group-Like Structures in General Categories I Multiplications and Comultiplications* by Beno Eckmann and Peter J. Hilton. [EH62] Whilst not explicitly laid out in the original paper, it follows from the arguments. This result from Category Theory will be used to provide a basis for our algebraic proof of the abelian property. A graphical proof of the abelian property can be found in Hatcher [Hat02], should the reader prefer.

Fushida-Hardy [FH] lays out an excellent proof of the abelian property in their paper *A non-visual proof that higher homotopy groups are abelian*, invoking the Eckmann-Hilton argument. We take inspiration from their work in our proof.

Theorem 3.11. Eckmann-Hilton Argument.

Let $\cdot$ and $*$ denote unital (i.e. there exists an identity) binary operations on some set X. Let $x_0, x_1, y_0, y_1 \in X$. Suppose we have

$$(x_0 \cdot x_1) * (y_0 \cdot y_1) = (x_0 \cdot y_0) * (x_1 \cdot y_1).$$

Then $\cdot$ and $*$ are the same operation. Additionally, they are commutative and associative. [FH]

Proof. Let $e.$ and e_* denote the identity elements of $\cdot$ and $*$ respectively. Then we have, by the definition of identity elements and the stated condition in our theorem,

$$e. = e. \cdot e. = (e. * e_*) \cdot (e_* * e.) = (e. \cdot e_*) * (e_* \cdot e.) = e_* * e_* = e_*.$$

So $e. = e_*$. We shall henceforth denote the identity element as e.

For $x_0, y_0 \in X$, we have

$$x_0 \cdot y_0 = (x_0 * e) \cdot (e * y_0) = (x_0 \cdot e) * (e * y_0) = x_0 * y_0.$$

We have again appealed to the condition from the theorem. This result shows that $\cdot$ and $*$ are the same operation. It remains to show that they are commutative and associative.

For commutativity, note that

$$x_0 * x_1 = (e * x_0) \cdot (x_1 * 1) = (e \cdot x_1) * (x_0 \cdot 1) = x_1 * x_0 = x_1 \cdot x_0$$

from the fact that $\cdot$ and $*$ are the same.

Let $z_0 \in X$. For associativity, we have that

$$(x_0 \cdot y_0) \cdot z_0 = (x_0 \cdot y_0) \cdot (e \cdot z_0) = (x_0 \cdot e) \cdot (y_0 \cdot z_0) = x_0 \cdot (y_0 \cdot z_0).$$

Remark 3.12. We included the proof for associativity for the sake of completeness. We have already shown that associativity holds for our homotopy groups.

□

Lemma 3.13. *$(\pi_n(X, x_0), +)$, with $n \geq 2$, is abelian. [Hat02]*

Proof. The structure of this proof comes from Fushida-Hardy's paper. [FH]

Let $n \geq 2$ as per the definition. Let $f_0, g_0, f_1, g_1 \in \pi_n(X, x_0)$. In addition to $+$ as previously defined, define an additional binary operation $+_*$ as follows:

$$f_1 +_* g_1(s_1, s_2, \ldots, s_n) = \begin{cases} f(s_1, 2s_2, s_3, \ldots, s_n) & \text{for } 0 \leq s_2 \leq 1/2 \\ g(s_1, 2s_2 - 1, s_3, \ldots, s_n) & \text{for } 1/2 \leq s_2 \leq 1. \end{cases}$$

It is trivial to show that this is well-defined on equivalence classes using the exact same argument as with $+$.

It remains to show that

$$(f_0 + g_0) +_* (f_1 + g_1) = (f_0 +_* f_1) + (g_0 +_* g_1)$$

so that we may apply the Eckmann-Hilton argument to confirm associativity.

We individually evaluate the left hand side and the right hand side of the above. We have, for the left hand side,

$$\begin{aligned} &(f_0 + g_0) +_* (f_1 + g_1)(s_1, s_2, \ldots, s_n) \\ &= \begin{cases} f_0 + g_0(s_1, 2s_2, s_3, \ldots, s_n) & \text{for } 0 \leq s_2 \leq 1/2 \\ f_1 + g_1(s_1, 2s_2 - 1, s_3, \ldots, s_n) & \text{for } 1/2 \leq s_2 \leq 1. \end{cases} \\ &= \begin{cases} f_0(2s_1, 2s_2, s_3, \ldots, s_n) & \text{for } 0 \leq s_1, s_2 \leq 1/2 \\ g_0(2s_1, 2s_2 - 1, s_3, \ldots, s_n) & \text{for } 0 \leq s_1 \leq 1/2 \leq s_2 \leq 1 \\ f_1(2s_1 - 1, 2s_2, s_3, \ldots, s_n) & \text{for } 0 \leq s_2 \leq 1/2 \leq s_1 \leq 1 \\ g_1(2s_1 - 1, 2s_2 - 1, s_3, \ldots, s_n) & \text{for } 1/2 \leq s_1, s_2 \leq 1. \end{cases} \end{aligned}$$

Similarly, for the right hand side we have

$$
\begin{aligned}
&(f_0 +_* f_1) + (g_0 +_* g_1)(s_1, s_2, \dots, s_n) \\
&= \begin{cases} f_0 +_* f_1(2s_1, s_2, s_3, \dots, s_n) & \text{for } 0 \leq s_1 \leq 1/2 \\ g_0 +_* g_1(2s_1 - 1, s_2, s_3, \dots, s_n) & \text{for } 1/2 \leq s_1 \leq 1. \end{cases} \\
&= \begin{cases} f_0(2s_1, 2s_2, s_3, \dots, s_n) & \text{for } 0 \leq s_1, s_2 \leq 1/2 \\ f_1(2s_1 - 1, 2s_2, s_3, \dots, s_n) & \text{for } 0 \leq s_2 \leq 1/2 \leq s_1 \leq 1 \\ g_0(2s_1, 2s_2 - 1, s_3, \dots, s_n) & \text{for } 0 \leq s_1 \leq 1/2 \leq s_2 \leq 1 \\ g_1(2s_1 - 1, 2s_2 - 1, s_3, \dots, s_n) & \text{for } 1/2 \leq s_1, s_2 \leq 1. \end{cases}
\end{aligned}
$$

These are precisely the same function, and so we have shown equality and can conclude that $+$ is associative by Eckmann-Hilton. Therefore, $(\pi_n(X, x_0), +)$ is abelian. □

Remark 3.14. One will immediately notice that this proof does not hold for $n \leq 2$ due to the fact that $+_*$ requires at least two dimensions to be defined; else the above argument cannot be used.

4 Homology

We begin by introducing *CW (closure-finite weak) complexes*, also known as *cell complexes*. Our definitions will again mostly be drawn from Hatcher. [Hat02]

4.1 CW Complexes

The reader should refer to section A.6.1 should they wish to remind themselves of the common subspaces in Euclidean space.

Definition 4.1. Let X be a Hausdorff topological space. X is a *CW complex* if there exists a filtration

$$\emptyset \subset X^0 \subset X^1 \subset \cdots \subset X^n \subset X$$

such that X may be constructed as follows:

1. Begin with some set X^0 of discrete points. These are the 0-*cells of* X. X^0 is called the 0-*skeleton*.
2. Take a list of closed intervals $D^1_\alpha = [-1, 1]$. For each D^1_α pick some continuous map

   $$\varphi_\alpha : \partial D^1_\alpha \to X^0$$

   . As $\partial D^1_\alpha = \{-1, 1\}$, this map takes each endpoint of D^1_α to a point in X^0. Both endpoints may be taken to the same point.
3. Form the quotient space[10]

   $$X^1 = (X^0 \sqcup D^1_\alpha)/\sim$$

   (this is precisely the set of all equivalence classes under $\sim$, the standard equivalence relation for quotient space (proposition A.36)), identifying each endpoint with its image under φ_α. The image of $\operatorname{Int} D^1_\alpha \to X^1$ is called a 1-*cell*.
4. We proceed inductively. Suppose X^{n-1} has been constructed. For each α choose a closed disk D^n_α. We will form the *n-skeleton* X^n via the continuous *attaching maps*

   $$\varphi_\alpha : \partial D^n_\alpha \to X^{n-1}.$$

   For all boundary points $x \in \partial D^n_\alpha$, we have that $x \sim \varphi_\alpha(x)$ via the same equivalence relation as in the prior step. Then X^n simply becomes the space

   $$X^n = (X^{n-1} \bigsqcup_\alpha D^n_\alpha)/\sim .$$

5. We require the restriction of the quotient map to $\operatorname{Int} D^n_\alpha$ to be a homeomorphism onto its image. We call this image an *n-cell*, denoted e^n_α. We also require that $\bar{e^n_\alpha}$, the closure of e^n_α, be such that it intersects only finitely many other cells of X.
6. Now we endow the *CW complex* X as

   $$X = \bigcup_n X^n$$

   with the weak topology, i.e. a set $U \subset X$ is open if and only if $U \cap X^n$ is open in X^n for each n. [Hat02]

[10]Here, we use $\sqcup$ to denote the disjoint union, see example A.38 for a formal definition.

This is a very heavy definition. It is advised the reader works through the definition slowly, or even tries to construct a 2-dimensional CW complex as they do. However, the below analogy will also help to make things clear.

Remark 4.2. We visualise the low-dimension construction of CW complexes using the construction of a house.

1. We begin by placing wooden posts into the ground to represent the corners of rooms. We mark the tips of these posts with chalk, they become our 0-*cells*. Note that a 0-cell is just a marked point, the remainder of the post is irrelevant to our analogy. The technology does not yet exists to make chalk markings in the air.

2. Our 1-*cells* are pieces of string, connected by us from one chalk point to the next. We also connect the tips of each post with a piece of string. We have chosen our 1-cells this way. This will form the house's foundation, the 1-*skeleton*.

3. We install walls and floors/ceilings in the frames formed by our string. These are the 2-cells.

4. Finally, we add furniture and decorations to the rooms, giving it 'volume' in a sense, as it is no longer hollow. We can call this our 3-*cell*.

This is a crude example and is obviously non-rigorous. Most notable perhaps is the use of furniture and decorations in the place of filling each room with solid concrete (a true 3-cell in our analogy). This abstraction was made to satisfy the author's desire to not live inside a house that is completely filled with solid concrete. Still, the motivation remains.

Definition 4.3. Let X be a CW complex as in the previous definition. For each cell e^n_α, we define a *characteristic map*

$$\Phi_\alpha : D^n_\alpha \to X$$

which extends the attaching map φ_α and is a homeomorphism from the interior of D^n_α onto e^n_α as is required in the definition.

In particular, we take Φ_α to be the composition [Hat02]

$$D^n_\alpha \hookrightarrow X^{n-1} \bigsqcup_\alpha D^n_\alpha \to X^n \hookrightarrow X.$$

Definition 4.4. Let X be a CW complex. A *subcomplex* of X is a closed subspace $A \subset X$ that is a union of cells of X.

4.2 n-Simplexes

Many textbooks, including Hatcher [Hat02], choose to begin the study of homology by introducing *simplicial homology.* This is a more primitive version of the concept, which in our case will not be particularly relevant to our later theorems. As such, we dive straight into the more important *singular homology.* Should the reader wish, they may refer to Hatcher [Hat02] for a more in-depth introduction to homology, including simplicial homology.

We will introduce *n-simplexes*, from which we shall define *singular n-simplexes.*

Definition 4.5. An *n-simplex* is the smallest convex set in m-dimensional Euclidean space $\{v_0, \dots, v_n\} \subset \mathbb{R}^m$ containing $n+1$ points $v_0, \dots, v_n$ such that the difference vectors $v_1 - v_0, \dots, v_n - v_0$ are linearly independent, i.e. $v_0, \dots, v_n$ are affinely independent.

The points $v_0, \dots, v_n$ are the *vertices* of the simplex. We denote the simplex by $[v_0, \dots, v_n]$. [Hat02]

Definition 4.6. The *standard n-simplex*, denoted Δ^n, is given by

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_i t_i = 1\}$$

with the subspace topology inherited from $\mathbb{R}^{n+1}$.

The *vertices* of the simplex are simply the unit vectors along the coordinate axes. [Hat02]

Remark 4.7. The ordering of the vertices is important here. When we say 'n-simplex' we are actually referring to an *n-simplex with an ordering of its vertices.* [Hat02]

Remark 4.8. The n-simplex is the n dimensional analogue of a triangle. [Hat02]

Proposition 4.9. *Let $v_0, \dots, v_n \in \mathbb{R}^m$ be affinely independent. Let Δ^n be the standard n-simplex. Let*

$$\varphi : \Delta^n \to [v_0, \dots, v_n]$$

be defined by

$$\varphi(t_0, \dots, t_n) = \sum_{i=0}^{n} t_i v_i.$$

Then φ is a homeomorphism from Δ^n to the n-simplex $[v_0, \dots, v_n]$. [Hat02]

Proof. We first show that φ is well-defined. The linear combination $\sum_{i=0}^n t_i v_i$ is such that the coefficients t_i are non-negative and sum to 1, by definition of the standard n-simplex. Then we have that $\sum_{i=0}^n t_i v_i$ is a vector contained in the convex hull of $v_0, \dots, v_n$, so each point in the domain Δ^n is taken to some point in $[v_0, \dots, v_n]$. So φ is well-defined.

Now, write each vertex $v_i \in \mathbb{R}^m$ in coordinates, i.e.

$$v_i = (v_i^{(1)}, \dots, v_i^{(m)}).$$

For some $(t_0, \dots, t_n) \in \Delta^n$, the j-th coordinate of $\varphi(t_0, \dots, t_n)$ is

$$(\varphi(t_0, \dots, t_n))^{(j)} = \sum_{i=0}^{n} t_i v_i^{(j)}.$$

This is a polynomial of degree 1 in the variables $t_0, \dots, t_n$ and thus this system can be expressed as a matrix equation. Then continuity of φ follows from the fact that any linear map of the form $L(x) = A(x)$ (where A is a matrix) is continuous (theorem D.5).

We now show injectivity. Assume that

$$\varphi(t_0, \dots t_n) = \varphi(s_0, \dots, s_n)$$

for $(t_0, \dots t_n), (s_0, \dots s_n) \in \Delta^n$. Then we have, by affine independence,

$$\sum_{i=0}^{n} t_i v_i - \sum_{i=0}^{n} s_i v_i = \sum_{i=0}^{n} (t_i - s_i) v_i = 0.$$

Note that since the sum of the t_i (and similarly the sum of the s_i) must be 1, we have that

$$\sum_{i=0}^{n} (t_i - s_i) = 0.$$

As such, the only way for this affine relation to occur whilst $\sum_{i=0}^n (t_i - s_i) = 0.$ is for each $t_i - s_i$ to vanish. Therefore we have

$$(t_0, \dots t_n) = (s_0, \dots, s_n)$$

and we have shown injectivity.

We have already shown that the image of φ, $\varphi(\Delta^n)$, is contained in the convex hull of $\{v_0, \dots, v_n\}$ when we showed that φ was well-defined. Conversely, let x be some arbitrary point in the convex hull of $\{v_0, \dots, v_n\}$. Then we can write

$$x = \sum_{i=0}^{n} t_i v_i$$

where $t_i \geq 0$ and $\sum t_i = 1$, by definition of a convex hull for a finite set. Thus we have

$$x = \varphi(t_0, \dots, t_n) = \sum_{i=0}^{n} t_i v_i$$

and so any such x can be attained this way. Therefore, we have surjectivity.

We have shown that φ is a continuous bijection. As Δ^n is closed and bounded in $\mathbb{R}^{n+1}$, it is compact. As $\mathbb{R}^m$ is Hausdorff and $[v_0, \dots v_n] \subset \mathbb{R}^m$, we have that $[v_0, \dots v_n]$ is Hausdorff. Thus, since φ is a continuous bijection from a compact space to a Hausdorff space, it is a homeomorphism (theorem A.32) and we are done. $\square$

The homeomorphism proven above is key in that it is dependent on the ordering of the vertices. We showed that the homeomorphism between the standard n-simplex and any other n-simplex $[v_0, \dots, v_n]$ exists and that it preserves the order of the vertices. Specifically, we have

$$\varphi(t_0, \dots, t_n) = \sum_{i=0}^{n} t_i v_i.$$

We call the coefficients t_i the *barycentric coordinates* of the point $\sum_{i=0}^{n} t_i v_i$ in $[v_0, \dots, v_n]$.

Definition 4.10. Suppose we have an n-simplex $[v_0, \dots, v_n]$. Now suppose we remove one of the $n+1$ vertices, namely v_i. Then we have an $(n-1)$-simplex $[v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n]$ spanned by the remaining n vertices (as subsets of affinely independent sets preserve this property). This is called a *face* of the n-simplex $[v_0, \dots, v_n]$. [Hat02]

Going forward, we shall assume that the vertices of a face (or any subsimplex) spanned by a subset of the vertices of an n-simplex will always be ordered according to their order in the parent n-simplex.

Definition 4.11. The union of all the faces of Δ^n is denoted $\partial\Delta^n$ and is called the *boundary* of Δ^n.

The *open simplex* $\mathring{\Delta}^n$ is simply the interior of Δ^n, given by $\Delta^n \setminus \partial\Delta^n$.

4.2.1 Singular n-Chain Groups

Definition 4.12. Let X be a topological space. Then a *singular n-simplex* in X is a continuous map [Hat02]

$$\sigma : \Delta^n \to X.$$

Definition 4.13. Let X be a topological space and let $n \geq 0$. The *singular n-chain group* of X is the free abelian group

$$C_n(X) = \bigoplus_{\sigma \in S_n(X)} \mathbb{Z}\sigma$$

whose basis elements are all singular n-simplexes $\sigma : \Delta^n \to X$.

Formal finite sums $\sum_i a_i \sigma_i$ with coefficients $a_i \in \mathbb{Z}$ are called *singular n-chains.* [Hat02]

Definition 4.14. Let $\sigma : \Delta^n \to X$ be a singular n-simplex. Let $d_i : \Delta^{n-1} \to \Delta^n$ be the *face inclusion* that sets the i-th barycentric coordinate to 0, i.e.

$$d_i(u_0, \dots, u_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n),$$

where

$$t_k = \begin{cases} u_k & \text{if } k < i, \\ 0 & \text{if } k = i, \\ u_{k-1} & \text{if } k > i. \end{cases}$$

Define the *boundary homomorphism* $\partial_n : C_n(X) \to C_{n-1}(X)$ by[11]

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ d_i.$$

Here the ordered vertex list $(v_0, \dots, v_n)$ on Δ^n fixes an orientation; the face–inclusion $d_i : \Delta^{n-1} \to \Delta^n$ deletes the i-th vertex, and the sign $(-1)^i$ records the induced orientation of that $(n-1)$-face with respect to the ambient orientation on Δ^n.

We extend ∂_n linearly to all singular n-chains such that we have [Hat02]

$$\partial_n(\sum_i a_i \sigma_i) = \sum a_i \partial_n(\sigma_i).$$

We now prove an important fact relating to boundary homomorphisms. However, we first prove a short lemma related to the face inclusion maps.

Lemma 4.15. *Let $d_j : \Delta^{n-2} \to \Delta^{n-1}, d_i : \Delta^{n-1} \to \Delta^n$ face inclusion maps that set the j-th and i-th barycentric coordinates to 0, respectively. Then we have*

$$d_i \circ d_j = \begin{cases} d_j \circ d_{i-1}, & \text{if } j < i, \\ d_{j+1} \circ d_i, & \text{if } j \geq i. \end{cases}$$

Proof. We prove each case explicitly. First, suppose that $j < i$. Let $(u_0, \dots, u_{n-2}) \in \Delta^{n-2}$. We have

$$d_j(u_0, \dots, u_{n-2}) = (t_0, \dots, t_{j-1}, 0, t_{j+1}, \dots, t_{n-1})$$

where

$$t_k = \begin{cases} u_k & \text{if } k < j, \\ 0 & \text{if } k = j, \\ u_{k-1} & \text{if } k > j. \end{cases}$$

Then we have

$$\begin{aligned} d_i \circ d_j &= d_i(t_0, \dots, t_{j-1}, 0, t_{j+1}, \dots, t_{n-1}) \\ &= (s_0, \dots, s_{j-1}, 0, s_{j+1}, \dots, s_{i-1}, 0, s_{i+1} \dots, s_n) \end{aligned}$$

where

$$s_k = \begin{cases} t_k & \text{if } k < i, \\ 0 & \text{if } k = i, \\ t_{k-1} & \text{if } k > i. \end{cases}$$

[11] Although the ideas here are the same as in Hatcher [Hat02], we differ greatly to in our notation. Hatcher does not use the face inclusion map as we do.

Now we evaluate $d_j \circ d_{i-1}$. We have

$$d_{i-1}(u_0, \dots, u_{n-2}) = (t_0, \dots, t_{i-2}, 0, t_i, \dots, t_{n-1})$$

where

$$t_k = \begin{cases} u_k & \text{if } k < i-1, \\ 0 & \text{if } k = i-1, \\ u_{k-1} & \text{if } k > i-1. \end{cases}$$

Then we have

$$\begin{aligned} d_j \circ d_{i-1} &= d_j(t_0, \dots, t_{i-2}, 0, t_i, \dots, t_{n-1}) \\ &= (s_0, \dots, s_{j-1}, 0, s_{j+1}, \dots, s_{i-1}, 0, s_{i+1} \dots, s_n) \end{aligned}$$

where

$$s_k = \begin{cases} t_k & \text{if } k < j, \\ 0 & \text{if } k = j, \\ t_{k-1} & \text{if } k > j. \end{cases}$$

Clearly, we can conclude that $d_i \circ d_j = d_j \circ d_{i-1}$ when $j < i$.

Now, suppose that $j \geq i$. We assume that $j \neq i$ without loss of generality. We have

$$d_j(u_0, \dots, u_{n-2}) = (t_0, \dots, t_{j-1}, 0, t_{j+1}, \dots, t_{n-1})$$

where

$$t_k = \begin{cases} u_k & \text{if } k < j, \\ 0 & \text{if } k = j, \\ u_{k-1} & \text{if } k > j. \end{cases}$$

Then we have

$$\begin{aligned} d_i \circ d_j &= d_i(t_0, \dots, t_{j-1}, 0, t_{j+1}, \dots, t_{n-1}) \\ &= (s_0, \dots, s_{i-1}, 0, s_{i+1}, \dots, s_j, 0, s_{j+2} \dots, s_n) \end{aligned}$$

where

$$s_k = \begin{cases} t_k & \text{if } k < i, \\ 0 & \text{if } k = i, \\ t_{k-1} & \text{if } k > i. \end{cases}$$

Now we evaluate $d_{j+1} \circ d_i$. We have

$$d_i(u_0, \dots, u_{n-2}) = (t_0, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_{n-1})$$

where

$$t_k = \begin{cases} u_k & \text{if } k < i-1, \\ 0 & \text{if } k = i-1, \\ u_{k-1} & \text{if } k > i-1. \end{cases}$$

Then we have

$$\begin{aligned} d_{j+1} \circ d_i &= d_{j+1}(t_0, \ldots, t_{i-2}, 0, t_i, \ldots, t_{n-1}) \\ &= (s_0, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_j, 0, s_{j+2} \ldots, s_n) \end{aligned}$$

where

$$s_k = \begin{cases} t_k & \text{if } k < j, \\ 0 & \text{if } k = j, \\ t_{k-1} & \text{if } k > j. \end{cases}$$

Clearly, we can conclude that $d_i \circ d_j = d_{j+1} \circ d_i$ when $j \geq i$.

□

Lemma 4.16. *Let X be a space. Then we have [Hat02]*

$$\partial_{n-1}\partial_n(\sigma) = 0.$$

Proof. We have

$$\begin{aligned} \partial_{n-1}\partial_n(\sigma) &= \sum_{i=0}^{n}(-1)^i \partial_{n-1}(\sigma \circ d_i) \\ &= \sum_{i=0}^{n}(-1)^i \sum_{j=0}^{n-1}(-1)^j(\sigma \circ d_i) \circ d_j \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n-1}(-1)^{i+j}\sigma \circ d_i \circ d_j \end{aligned}$$

where we have leveraged the properties of even and odd numbers to simplify in the final line.

Without loss of generality, we can split the sum into two separate parts: those for which $j < i$ and for which $j \geq i$. We obtain

$$\partial_{n-1}\partial_n(\sigma) = \sum_{j<i}(-1)^{i+j}\sigma \circ d_i d_j + \sum_{j \geq i}(-1)^{i+j}\sigma \circ d_i d_j.$$

We appeal to our lemma (4.15). We substitute for each instance of $d_i d_j$ the corresponding equivalent expression (which is dependent on whether $j < i$ or otherwise). We obtain

$$\partial_{n-1}\partial_n(\sigma) = \sum_{j<i}(-1)^{i+j}\sigma \circ d_j d_{i-1} + \sum_{j \geq i}(-1)^{i+j}\sigma \circ d_{j+1} d_i.$$

Now, without loss of generality take some arbitrary $a, b \in \{0, \ldots, n\}$ such that $0 \leq a < b \leq n$. Then the map $\sigma \circ d_a \circ d_b : \Delta^{n-2} \to X$ appears twice in the total sum. To see this, first take $(i, j) = (b+1, a)$ in the former part of the sum. This gives the term

$$(-1)^{(b+1)+a}\sigma \circ d_a \circ d_b.$$

Now take $(i, j) = (a, b)$ in the latter part of the sum. By the $j \geq i$ case in the previous lemma, we have

$$d_{j+1} \circ d_i = d_{b+1} \circ d_a = d_a \circ d_b.$$

So our term is

$$(-1)^{a+b}\sigma \circ d_a \circ d_b.$$

The sum of these two terms is

$$\begin{aligned}&(-1)^{(b+1)+a}\sigma \circ d_a \circ d_b + (-1)^{a+b}\sigma \circ d_a \circ d_b\\ =&-(-1)^{a+b}\sigma \circ d_a \circ d_b + (-1)^{a+b}\sigma \circ d_a \circ d_b\\ =&0\end{aligned}$$

As such, we have shown that every arbitrary $\sigma \circ d_a \circ d_b$ is cancelled out by a corresponding term. So we have $\partial_{n-1}\partial_n(\sigma) = 0$ as desired. □

4.3 Chain Complexes

Definition 4.17. The sequence of homomorphisms of abelian groups

$$\dots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

is called a *chain complex*.

Note here that we have extended the definition of boundary homomorphisms such that $\partial_0(\sigma) = 0$ for $\sigma \in C_0$. [Hat02]

Remark 4.18. We defined C_n to specifically be the **singular** n-chain group $C_n(X)$. However, the previous definition is analogous for any type of n-chain group.

Lemma 4.19. *The equation $\partial_n\partial_{n+1} = 0$ is equivalent to the inclusion $\operatorname{Im}\partial_{n+1} \subset \ker\partial_n$. [Hat02]*

Proof. Let $\partial_n\partial_{n+1}(\sigma) = 0$ for all singular n-simplexes σ. Then clearly

$$\partial_{n+1}(\sigma) \in \ker\partial_n.$$

Since $\partial_{n+1}(\sigma)$ are precisely the elements of $\operatorname{Im}\partial_{n+1}$, we have $\operatorname{Im}\partial_{n+1} \subset \ker\partial_n$.

The converse is also easily shown. Let $\operatorname{Im}\partial_{n+1} \subset \ker\partial_n$. Then any $\partial_{n+1}(\sigma) \in \operatorname{Im}\partial_{n+1}$ must be such that $\partial_{n+1}(\sigma) \in \ker\partial_n$ and so $\partial_n\partial_{n+1}(\sigma) = 0$ and we are done. □

4.3.1 Singular Homology Groups

Proposition 4.20. *Suppose we have a chain complex as in definition 4.17. Then*

$$H_n(X) = \ker\partial_n / \operatorname{Im}\partial_{n+1}$$

is a group, called the singular n-th homology group*.*

The elements of H_n are cosets of $\operatorname{Im}\partial_{n+1}$ *called* homology classes*. Elements of* $\ker\partial_n$ *are called* cycles *and elements of* $\operatorname{Im}\partial_{n+1}$ *are called* boundaries*. Two cycles representing the same homology class are said to be* homologous*. [Hat02]*

Proof. We have that $C_n(X)$ is the free abelian group generated by all singular n-simplices in some space X. As such, $C_n(X)$ is abelian by definition. Recall that by the definition of the kernel and image and the previous lemmas, we have

$$\operatorname{Im}\partial_{n+1} \leq \ker\partial_n \leq C_n(X).$$

So $\operatorname{Im}\partial_{n+1}$ and $\ker\partial_n$ are subgroups of abelian groups and therefore are themselves abelian. As $\operatorname{Im}\partial_{n+1} \subseteq \ker\partial_n$ we may take the quotient group $\ker\partial_n / \operatorname{Im}\partial_{n+1}$. The quotient of an abelian group by a subgroup is again an abelian group (lemma B.17) and so we are done. □

4.4 Homotopy Invariance

Now that we have built our singular homology basics, we may begin to derive some results. The first result will take us back to our notion of homotopy in section 1.

In our upcoming work, we will produce a *commutative diagram*, a concept native to category theory. A *commutative diagram* is a diagram of maps such that any two compositions of maps starting at one point in the diagram and ending at another are equal. [Hat02] This shall become clear as it appears.

Definition 4.21. Let $f : X \to Y$ be a continuous map between spaces X and Y. Define an *induced homomorphism* $f_\# : C_n(X) \to C_n(Y)$ by composing each singular n-simplex $\sigma : \Delta^n \to X$ with f to obtain

$$f_\#(\sigma) = f \circ \sigma : \Delta^n \to Y,$$

then extending $f_\#$ linearly via

$$f_\#(\sum_i n_i\sigma_i) = \sum_i n_i f(\sigma_i).$$

Lemma 4.22. *Let $f : X \to Y$ be a continuous map between spaces X and Y. Let $\partial_n^X : C_n(X) \to C_{n-1}(X)$ and $\partial_n^Y : C_n(Y) \to C_{n-1}(Y)$ be boundary homomorphisms. Then the induced homomorphism $f_\#$ satisfies*

$$f_\#\partial_n^X = \partial_n^Y f_\#.$$

We say that the induced homomorphisms $f_\#$ induce a chain map *from the singular chain complex of X to that of Y. [Hat02]*

Proof. Let σ be some singular n-simplex. We have

$$\begin{aligned} f_\#\partial_n^X(\sigma) &= f_\#(\sum_{i=0}^n (-1)^i \sigma \circ d_i) \\ &= \sum_{i=0}^n (-1)^i (f \circ \sigma) \circ d_i \\ &= \partial_n^Y(f \circ \sigma) \\ &= \partial_n^Y f_\#(\sigma) \end{aligned}$$

by definition and so we are done. □

The previous lemma (4.22) demonstrated that $f_\#\partial_n^X = \partial_n^Y f_\#$ for any choice of n. So we may draw a commutative diagram to better demonstrate this.

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial_{n+1}^X"] \arrow[d, "f_\#"'] & C_n(X) \arrow[r, "\partial_n^X"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\ \cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial_{n+1}^Y"] & C_n(Y) \arrow[r, "\partial_n^Y"] & C_{n-1}(Y) \arrow[r] & \cdots \end{tikzcd}$$

Lemma 4.23. *Let $f : X \to Y$ be a continuous map between spaces X and Y. Then the induced chain map induces homomorphisms $f_* : H_n(X) \to H_n(Y)$ between the homology groups of the two chain complexes. [Hat02]*

Proof. Let $\alpha \in \ker \partial_n^X$ be a cycle. Then by definition we have $\partial_n^X(\alpha) = 0$. Then

$$\partial_n^Y f_\#(\alpha) = f_\#\partial_n^X(\alpha) = f_\#(0) = 0$$

by lemma 4.22 and the definition of $f_\#$. So $f_\#(\alpha) \in \ker \partial_n^Y$.

Let $\partial^X_{n+1}(\beta) \in \mathrm{Im}\partial^{\mathrm{X}}_{\mathrm{n+1}}$ be a boundary for some $\beta \in C_{n+1}(X)$. Then by lemma 4.22 we have

$$f_\#\partial^X_{n+1}(\beta) = \partial^Y_{n+1} f_\#(\beta) \in \mathrm{Im}\,\partial^Y_{n+1}.$$

So we have shown that $f_\#$ maps cycles to cycles and maps boundaries to boundaries. As such, $f_\#$ induces a well-defined homomorphism $f_* : H_n(X) \to H_n(Y)$ such that for equivalence classes we have

$$f_*([\alpha]) = [f_\#(\alpha)].$$

□

We will demonstrate that homotopic maps induce the same homomorphism between homology groups.

Theorem 4.24. *Let $f, g : X \to Y$ be continuous maps between spaces X and Y that are homotopic to one another. Then both induce the same homomorphism between their homology groups. [Hat02]*

$$f_* = g_* : H_n(X) \to H_n(Y).$$

Proof. Fix $n \geq 0$. Let $f_\#, g_\# : C_n(X) \to C_n(Y)$ be chain maps, so $f_\#\partial^X_n = \partial^Y_n f_\#$ and similarly $g_\#\partial^X_n = \partial^Y_n g_\#$. Suppose then that there exists homomorphisms $P_n : C_n(X) \to C_{n+1}(Y)$ such that for every n we have

$$g_\# - f_\# = \partial^Y_{n+1} P_n + P_{n-1}\partial^X_n.$$

Let $\alpha \in \ker \partial^X_n$ be a cycle. Inserting this into our previous formula and noting that $\partial^X_n(\alpha) = 0$ (as α is a cycle) gives

$$g_\# - f_\#(\alpha) = \partial^Y_{n+1} P_n(\alpha) + P_{n-1}\partial^X_n(\alpha) = \partial^Y_{n+1} P_n(\alpha).$$

We then have that $g_\# - f_\#(\alpha) = \partial^Y_{n+1} P_n(\alpha) \in \mathrm{Im}\,\partial^Y_n$, i.e. the difference between $g_\#$ and $f_\#$, is a boundary, so we have that $[f_\#(\alpha)] = [g_\#(\alpha)]$ in $H_n(Y)$. Since all homology classes have a cycle representative, α was arbitrary and this holds for all classes. So it suffices to show that such a P_n exists.

Now, define maps $i_0 : X \to X \times [0,1]$, $i_1 : X \to X \times [0,1]$ by

$$i_0(x) = (x, 0),$$
$$i_1(x) = (x, 1).$$

Then $(i_0)_\#, (i_1)_\# : C_n(X) \to C_n(X \times [0,1])$ are the induced chain maps of i_0 and i_1 respectively. Let $F : X \times [0,1] \to Y$ be the homotopy from f to g. Then we have that

$$F \circ i_0 = f,$$
$$F \circ i_1 = g.$$

Define the *prism operator* $K_n : C_n(X) \to C_{n+1}(X \times [0,1])$ on a singular n-simplex $\sigma : \Delta^n \to X$ by

$$K_n(\sigma) = \sum_{i=0}^{n} (-1)^i (\sigma \times \mathrm{id}_{[0,1]}) \circ k_i$$

where $\sigma \times \mathrm{id}_{[0,1]} : \Delta^n \times [0,1] \to X \times [0,1]$ is the map defined by

$$(\sigma \times \mathrm{id}_{[0,1]})(t_0, \ldots, t_n, s) = (\sigma(t_0, \ldots, t_n), s)$$

and $k_i : \Delta^{n+1} \to \Delta^n \times [0,1]$ is defined by

$$k_i(u_j) = \begin{cases} (t_j, 0) & \text{if } 0 \le j \le i, \\ (t_{j-1}, 1) & \text{if } i+1 \le j \le n+1 \end{cases}$$

extended affinely to all of Δ^{n+1}. The ordered vertices of the image simplex $k_i(\Delta^{n+1})$ are then

$$(t_0, 0), \dots, (t_i, 0), (t_i, 1), \dots, (t_n, 1).$$

This is precisely the ordering that gives our $(-1)^i$ in the definition of the prism operator.

We prove some claims regarding k_i. Let id be the identity map. Recall that two affine maps are equal if and only if they agree on all vertices. So it suffices to show that the following claims agree on vertices. Let d_r be the face inclusion map which acts on each vertex[12] via

$$d_r(t_j) = \begin{cases} u_j & \text{if } j < r, \\ u_{j+1} & \text{if } j \ge r. \end{cases}$$

Going forward, we make clear which part of our chain the maps are acting on with a superscript. For example, $k_i^{(n)} : \Delta^{n+1} \to \Delta^n \times [0,1]$ is the prism map at level n.

1. We claim that
$$k_i^{(n)} \circ d_r^{(n)} = (d_r^{(n-1)} \times \text{id}) \circ k_{i-1}^{(n-1)} \text{ for } 0 \le r \le i-1.$$

   Take some t_j, a vertex in Δ^n. Consider the left hand side, $k_i^{(n)} \circ d_r^{(n)}(t_j)$. We consider all cases.

   (a) Suppose $j < r$. Then
   $$t_j \xrightarrow{d_r^{(n)}} u_j \xrightarrow{k_i^{(n)}} (t_j, 0)$$
   as $j < r$ and so $j \le i-1 < i$.

   (b) Suppose $r \le j \le i-1$. Then
   $$t_j \xrightarrow{d_r^{(n)}} u_{j+1} \xrightarrow{k_i^{(n)}} (t_{j+1}, 0)$$
   as $j \ge r$ and so $j+1 < i$.

   (c) Suppose $j = i$. Then
   $$t_j = t_i \xrightarrow{d_r^{(n)}} u_{i+1} \xrightarrow{k_i^{(n)}} (t_i, 1)$$
   as $j = i \ge r$ and so $j+1 = i+1 \ge i+1$.

   (d) Suppose $j \ge i+1$. Then
   $$t_j \xrightarrow{d_r^{(n)}} u_{j+1} \xrightarrow{k_i^{(n)}} (t_j, 1)$$
   by the assumption.

   Consider the right hand side, $(d_r^{(n-1)} \times \text{id}) \circ k_{i-1}^{(n-1)}$. We consider all cases again. The reader can then verify equality with the left hand side for each case and the claim is proven.

   (a) Suppose $j < r$. Then
   $$t_j \xrightarrow{k_{i-1}^{(n-1)}} (t_j, 0) \xrightarrow{d_r^{(n-1)} \times \text{id}} (t_j, 0),$$
   as $j < r$ and so $j \le i-1 < i$.

[12] Not to be confused with the earlier definition of the face inclusion map, which looked at the entire simplex. Here we are only interested in what happens to specific vertices that existed before, after they have been 'shifted'.

(b) Suppose $r \leq j \leq i-1$. Then

$$t_j \xrightarrow{k_{i-1}^{(n-1)}} (t_j, 0) \xrightarrow{d_r^{(n-1)}\times \mathrm{id}} (t_{j+1}, 0),$$

as $j \geq r$ and so $j+1 < i$.

(c) Suppose $j = i$.

$$t_i \xrightarrow{k_{i-1}^{(n-1)}} (t_{i-1}, 1) \xrightarrow{d_r^{(n-1)}\times \mathrm{id}} (t_i, 1),$$

as $j = i \geq r$ and so $j+1 = i+1 \geq r$.

(d) Suppose $j \geq i+1$. Then

$$t_j \xrightarrow{k_{i-1}^{(n-1)}} (t_{j-1}, 1) \xrightarrow{d_r^{(n-1)}\times \mathrm{id}} (t_j, 1),$$

as $j \geq i+1$ and so $j \geq r$.

2. We also claim that

$$k_i^{(n)} \circ d_r^{(n)} = (d_{r-1}^{(n-1)} \times \mathrm{id}) \circ k_i^{(n-1)} \text{ for } i+2 \leq r \leq n+1.$$

Again take some t_j, a vertex in Δ^n. Consider the left hand side, $k_i^{(n)} \circ d_r^{(n)}(t_j)$. We consider all cases.

(a) Suppose $j \leq i$. Then

$$t_j \xrightarrow{d_r^{(n)}} u_j \xrightarrow{k_i^{(n)}} (t_j, 0)$$

as $j \leq i < i+2 < r$.

(b) Suppose $j = i+1$. Then

$$t_j = t_{i+1} \xrightarrow{d_r^{(n)}} u_{i+1} \xrightarrow{k_i^{(n)}} (t_i, 1)$$

as $j = i+1 < i+2 \leq r$ and of course $j = i+1 \geq i+1$.

(c) Suppose $i+2 \leq j \leq r-1$. Then

$$t_j \xrightarrow{d_r^{(n)}} u_j \xrightarrow{k_i^{(n)}} (t_{j-1}, 1)$$

as $j \leq r-1 < r$ and $j \geq i+2 > i+1$.

(d) Suppose $j \geq r$. Then

$$t_j \xrightarrow{d_r^{(n)}} u_{j+1} \xrightarrow{k_i^{(n)}} (t_j, 1)$$

as $j \geq r \geq i+2 > i+1$.

Consider the right hand side, $(d_{r-1}^{(n-1)} \times \mathrm{id}) \circ k_i^{(n-1)}$. We consider all cases again. The reader can then verify equality with the left hand side for each case and the claim is proven.

(a) Suppose $j \leq i$. Then

$$t_j \xrightarrow{k_i^{(n-1)}} (t_j, 0) \xrightarrow{d_{r-1}^{(n-1)}\times \mathrm{id}} (t_j, 0),$$

as $j \leq i < i+1 < i+2 \leq r$, noting that this gives $j < i+1 \leq r-1$.

(b) Suppose $j = i+1$. Then

$$t_j \xrightarrow{k_i^{(n-1)}} (t_i, 1) \xrightarrow{d_{r-1}^{(n-1)}\times \mathrm{id}} (t_i, 1),$$

as $j = i+1 \geq i+1$ and then by noting that $i+2 \leq r$ gives $i < r-1$.

(c) Suppose $i+2 \leq j \leq r-1$.

$$t_i \xrightarrow{k_i^{(n-1)}} (t_{j-1}, 1) \xrightarrow{d_{r-1}^{(n-1)} \times \mathrm{id}} (t_{j-1}, 1),$$

as $j > i+1$ and $j-1 < r-1$ (from $j \leq r-1$).

(d) Suppose $j \geq r$. Then

$$t_j \xrightarrow{k_i^{(n-1)}} (t_{j-1}, 1) \xrightarrow{d_{r-1}^{(n-1)} \times \mathrm{id}} (t_j, 1),$$

as $j \geq r > i+1$ and $j > r-1$.

3. We claim that $k_i^{(n)} \circ d_i^{(n)}$ acts on some t_j such that

$$t_j \mapsto \begin{cases} (t_j, 0) & \text{if } j < i, \\ (t_j, 1) & \text{if } j \geq i. \end{cases}$$

We prove this for both cases.

(a) Suppose $j < i$. Then we have

$$t_j \xrightarrow{d_i^{(n)}} u_j \xrightarrow{k_i^{(n)}} (t_j, 0),$$

by definition of $d_i^{(n)}$ and $k_i^{(n)}$.

(b) Suppose $j \geq i$. Then we have

$$t_j \xrightarrow{d_i^{(n)}} u_{j+1} \xrightarrow{k_i^{(n)}} (t_j, 1),$$

again by definition of $d_i^{(n)}$ and $k_i^{(n)}$.

$k_i^{(n)} \circ d_i^{(n)}$ is the bottom vertical face of the prism. The image simplex then has ordered vertices

$$(t_0, 0), \ldots, (t_{i-1}, 0), (t_i, 1), \ldots, (t_n, 1).$$

4. We claim that $k_i^{(n)} \circ d_{i+1}^{(n)}$ acts on some t_j such that

$$t_j \mapsto \begin{cases} (t_j, 0) & \text{if } j \leq i, \\ (t_j, 1) & \text{if } j > i. \end{cases}$$

We prove this for both cases.

(a) Suppose $j \leq i$. Then we have

$$t_j \xrightarrow{d_{i+1}^{(n)}} u_j \xrightarrow{k_i^{(n)}} (t_j, 0),$$

by definition of $d_i^{(n)}$ and $k_i^{(n)}$.

(b) Suppose $j > i$. Then we have

$$t_j \xrightarrow{d_{i+1}^{(n)}} u_{j+1} \xrightarrow{k_i^{(n)}} (t_j, 1),$$

again by definition of $d_i^{(n)}$ and $k_i^{(n)}$.

$k_i^{(n)} \circ d_{i+1}^{(n)}$ is the top vertical face of the prism. The image simplex then has ordered vertices

$$(t_0, 0), \ldots, (t_i, 0), (t_{i+1}, 1), \ldots, (t_n, 1).$$

We collect together our rules (1) to (4) proven previously to evaluate the alternating boundary sum for each generator $\sigma : \Delta^n \to X$.

We evaluate

$$\begin{aligned}\partial_{n+1}^{X\times I} K_n(\sigma) &= \sum_{i=0}^{n}(-1)^i \sum_{r=0}^{n+1}(-1)^r(\sigma\times\mathrm{id})\circ k_i^{(n)}\circ d_r^{(n)} \\ &= \sum_{i=0}^{n}(-1)^i\left[(\sigma\times\mathrm{id})\circ k_i^{(n)}\circ d_i^{(n)} - (\sigma\times\mathrm{id})\circ k_i^{(n)}\circ d_{i+1}^{(n)}\right] \\ &+ \sum_{i=0}^{n}(-1)^i \sum_{r=0}^{i-1}(-1)^r(\sigma\times\mathrm{id})\circ k_i^{(n)}\circ d_r^{(n)} \\ &+ \sum_{i=0}^{n}(-1)^i \sum_{r=i+2}^{n+1}(-1)^r(\sigma\times\mathrm{id})\circ k_i^{(n)}\circ d_r^{(n)},\end{aligned}$$

splitting the original sum into four separate sums: one for $0 \le r \le i-1$, one for $i+2 \le r \le n+1$ and then a combined sum for $r = i$ and $r = i+1$. We can now apply our rules.

For when $0 \le r \le i-1$, we use rule (1) to obtain

$$\begin{aligned}&\sum_{i=0}^{n}(-1)^i \sum_{r=0}^{i-1}(-1)^r(\sigma\times\mathrm{id})\circ k_i^{(n)}\circ d_r^{(n)} \\ =&\sum_{i=0}^{n}(-1)^i \sum_{r=0}^{i-1}(-1)^r(\sigma\times\mathrm{id})\circ (d_r^{(n-1)}\times\mathrm{id})\circ k_{i-1}^{(n-1)}.\end{aligned}$$

We then take $j = i-1$ and we have

$$\sum_{j=-1}^{n-1}(-1)^{j+1} \sum_{r=0}^{j}(-1)^r(\sigma\times\mathrm{id})\circ (d_r^{(n-1)}\times\mathrm{id})\circ k_{j}^{(n-1)}$$

as our relevant term. We note that the the terms for $j = -1$ vanishes because $r \le i-1 = -1$ in the inner sum. Considering this, and simplifying $(-1)^{j+1+r}$, gives

$$-\sum_{j=0}^{n-1}\sum_{r=0}^{j}(-1)^{j+r}(\sigma\times\mathrm{id})\circ (d_r^{(n-1)}\times\mathrm{id})\circ k_{j}^{(n-1)}$$

which is precisely equal to

$$-K_{n-1}(\partial_n^X\sigma)$$

by definition of the prism operator.

For when $i+2 \le r \le n+1$, we use rule (2) to obtain

$$\begin{aligned}&\sum_{i=0}^{n}(-1)^i \sum_{r=i+2}^{n+1}(-1)^r(\sigma\times\mathrm{id})\circ k_i^{(n)}\circ d_r^{(n)} \\ =&\sum_{i=0}^{n}(-1)^i \sum_{r=i+2}^{n+1}(-1)^r(\sigma\times\mathrm{id})\circ (d_{r-1}^{(n-1)}\times\mathrm{id})\circ k_{i}^{(n-1)}.\end{aligned}$$

We then take $s = r-1$ and we have

$$\sum_{i=0}^{n}(-1)^i \sum_{s=i+1}^{n}(-1)^{s+1}(\sigma\times\mathrm{id})\circ (d_{s}^{(n-1)}\times\mathrm{id})\circ k_{i}^{(n-1)}$$

as our relevant term. We note that the the terms for $i = n$ vanish because the inner range $s = i + 1$ to $s = n = i$ is of course null. Considering this, and simplifying $(-1)^{i+s+1}$, gives

$$-\sum_{i=0}^{n-1}\sum_{s=i+1}^{n}(-1)^{i+s}(\sigma\times\mathrm{id})\circ(d_s^{(n-1)}\times\mathrm{id})\circ k_i^{(n-1)}.$$

Consider for a moment the term $\sum_{s=i+1}^{n}(-1)^s d_s^{(n-1)}$. We have that

$$\begin{aligned}&\sum_{s=0}^{n}(-1)^s d_s^{(n-1)} = 0\\ =&\sum_{s=0}^{i}(-1)^s d_s^{(n-1)} + \sum_{s=i+1}^{n}(-1)^s d_s^{(n-1)}\end{aligned}$$

from the application of to the identity simplex, which the reader may verify. So we have

$$\begin{aligned}&\sum_{s=0}^{i}(-1)^s d_s^{(n-1)} + \sum_{s=i+1}^{n}(-1)^s d_s^{(n-1)} = 0\\ \implies &\sum_{s=0}^{i}(-1)^s d_s^{(n-1)} = -\sum_{s=i+1}^{n}(-1)^s d_s^{(n-1)}\end{aligned}$$

Substituting this result into our original equation gives

$$\begin{aligned}&-\sum_{i=0}^{n-1}\sum_{s=i+1}^{n}(-1)^{i+s}(\sigma\times\mathrm{id})\circ(d_s^{(n-1)}\times\mathrm{id})\circ k_i^{(n-1)}\\ =&\sum_{i=0}^{n-1}\sum_{s=0}^{i}(-1)^{i+s}(\sigma\times\mathrm{id})\circ(d_s^{(n-1)}\times\mathrm{id})\circ k_i^{(n-1)}\end{aligned}$$

which is precisely equal to

$$K_{n-1}(\partial_n^X\sigma)$$

by definition of the prism operator.

We are left to deal with

$$\sum_{i=0}^{n}(-1)^i\left[(\sigma\times\mathrm{id})\circ k_i^{(n)}\circ d_i^{(n)} - (\sigma\times\mathrm{id})\circ k_i^{(n)}\circ d_{i+1}^{(n)}\right].$$

Consider rules (3) and (4). We showed that $k_i^{(n)}\circ d_{i+1}^{(n)}$ and $k_{i+1}^{(n)}\circ d_{i+1}^{(n)}$ have the same image simplex with the exception of the height of t_{i+1}. However, the composition with $(\sigma\times\mathrm{id})$ gives the same $n+1$ simplex for both, which the reader can easily verify. One can think of the 'top' vertical face of k_i as being the same as the 'bottom' vertical face of k_{i+1}. Recall that a singular simplex is an affine map out of Δ^n; thus, if two such maps agree on all vertices, they agree on the whole simplex. So we have

$$k_i^{(n)}\circ d_{i+1}^{(n)} = k_{i+1}^{(n)}\circ d_{i+1}^{(n)}.$$

As such, due to the alternating signs, the terms of the sum each appear twice except for the first and final terms. We set $j = i + 1$ to obtain

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^i\big[(\sigma \times \mathrm{id}) \circ k_i^{(n)} \circ d_i^{(n)} - (\sigma \times \mathrm{id}) \circ k_i^{(n)} \circ d_{i+1}^{(n)}\big]. \\
&= (\sigma \times \mathrm{id}) \circ k_0^{(n)} \circ d_0^{(n)} - (-1)^n(\sigma \times \mathrm{id}) \circ k_n^{(n)} \circ d_{n+1}^{(n)} + \sum_{j=1}^{n}((-1)^j - (-1)^{j-1})(\sigma \times \mathrm{id}) \circ k_{j-1}^{(n)} \circ d_j^{(n)} \\
&= (-1)^0(\sigma \times \{1\}) - (-1)^n(-1)^{n+1}(\sigma \times \{0\}) + 2\sum_{j=1}^{n}(-1)^j(\sigma \times \mathrm{id}) \circ k_{j-1}^{(n)} \circ d_j^{(n)} \\
&= (\sigma \times \{1\}) - (\sigma \times \{0\}) + 2\sum_{j=1}^{n}(-1)^j(\sigma \times \mathrm{id}) \circ k_{j-1}^{(n)} \circ d_j^{(n)}.
\end{aligned}$$

The first and second terms come from (3) and (4), specifically the fact that the first top face (the former expression) is constant at height 1 and the fact that the last bottom face (the latter expression) is constant at height 0. The -1 factors cancel due to the -1 contained in $d_{n+1}^{(n)}$. This can be further written as

$$\begin{aligned}
&(\sigma \times \{1\}) - (\sigma \times \{0\}) \\
=&(i_1)_\#(\sigma) - (i_0)_\#(\sigma)
\end{aligned}$$

We turn our attention to the remaining term

$$2\sum_{j=1}^{n}(-1)^j(\sigma \times \mathrm{id}) \circ k_{j-1}^{(n)} \circ d_j^{(n)}.$$

Using the fact that $k_{j-1}^{(n)} \circ d_j^{(n)} = k_j^{(n)} \circ d_j^{(n)}$ we may rewrite this as

$$2\sum_{j=1}^{n}(-1)^j(\sigma \times \mathrm{id}) \circ k_j^{(n)} \circ d_j^{(n)}.$$

Using (1), we may write, for $1 \leq j \leq n$,

$$(\sigma \times \mathrm{id}) \circ k_j^{(n)} \circ d_j^{(n)} = \sum_{s=0}^{j-1}(\sigma \times \mathrm{id}) \circ (d_s^{(n-1)} \times \mathrm{id}) \circ k_{j-1}^{(n-1)}$$

Now, we can rewrite

$$\begin{aligned}
\partial_{n+1}^{X \times I} K_n(\sigma) &= \sum_{i=0}^{n}(-1)^i\sum_{r=0}^{n+1}(-1)^r(\sigma \times \mathrm{id}) \circ k_i^{(n)} \circ d_r^{(n)} \\
&= \sum_{i=0}^{n}(-1)^i\big[(\sigma \times \mathrm{id}) \circ k_i^{(n)} \circ d_i^{(n)} - (\sigma \times \mathrm{id}) \circ k_i^{(n)} \circ d_{i+1}^{(n)}\big] \\
&+ \sum_{i=0}^{n}(-1)^i\sum_{r=0}^{i-1}(-1)^r(\sigma \times \mathrm{id}) \circ k_i^{(n)} \circ d_r^{(n)} \\
&+ \sum_{i=0}^{n}(-1)^i\sum_{r=i+2}^{n+1}(-1)^r(\sigma \times \mathrm{id}) \circ k_i^{(n)} \circ d_r^{(n)}
\end{aligned}$$

using all that we have deduced above to obtain

$$\begin{aligned}\partial_{n+1}^{X\times I} K_n(\sigma) &= K_{n-1}(\partial_n^X \sigma) - K_{n-1}(\partial_n^X \sigma) + (i_1)_\#(\sigma) - (i_0)_\#(\sigma) \\ &= (i_1)_\#(\sigma) - (i_0)_\#(\sigma)\end{aligned}$$

FINISH PROOF

Let $P_n : C_n(X) \to C_{n+1}(Y)$ be given by $P_n = F_\# \circ K_n$. Now we compute

$$\partial_{n+1}^Y P_n(\alpha) + P_{n-1}\partial_n^X(\alpha) = \partial_{n+1}^Y F_\#(K_n(\alpha)) + F_\#(K_{n-1}\partial_n^X(\alpha))$$

to verify that we obtain $g_\# - f_\# = \partial_{n+1}^Y P_n + P_{n-1}\partial_n^X$ as desired. In doing so we use the chain map property (lemma 4.22). We have

$$\begin{aligned}&\partial_{n+1}^Y F_\#(K_n(\alpha)) + F_\#(K_{n-1}\partial_n^X(\alpha)) \\ =&F_\#\partial_{n+1}^{X\times[0,1]}(K_n(\alpha)) + F_\#(K_{n-1}\partial_n^X(\alpha))\end{aligned}$$

which we can then simplify (by definition of $F_\#$) to

$$F_\#(\partial_{n+1}^{X\times[0,1]} K_n + K_{n-1}\partial_n^X)(\alpha).$$

□

Remark 4.25. For $f_\#$ and $g_\#$ as in the above proof (4.24), we say that P_n is a *chain homotopy* between the chain maps $f_\#$ and $g_\#$.

4.5 Exact Sequences

We introduce *exact sequences*, an algebraic definition that has many uses across algebraic topology.

Definition 4.26. Let $f_i : X_{i+1} \to X_i$ be homomorphisms for each i. Then the sequence of homomorphisms

$$\dots \xrightarrow{f_{n+2}} X_{n+1} \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \dots$$

is *exact* if for each f_i we have [Hat02]

$$\ker f_i = \operatorname{Im} f_{i+1}.$$

Remark 4.27. Note that $\ker f_i = \operatorname{Im} f_{i+1} \implies \ker f_i \subset \operatorname{Im} f_{i+1}$, so the sequence is a chain complex if the X_i are abelian groups (lemma 4.19). The converse is also true of course, so $\operatorname{Im} f_{i+1} \subset \ker f_i$, i.e. in the case that the X_i are abelian groups, the homology groups of the chain complex are trivial. [Hat02]

We often refer to exact sequences as *long exact* sequences to distinguish them from the special case of the *short exact* sequence. We adopt that convention going forward. Let us define *short exact sequences.*

Definition 4.28. Let $f_i : X_{i+1} \to X_i$ be homomorphisms for each i. Then an exact sequence of the form

$$0 \xrightarrow{0} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{0} 0$$

is called a *short exact* sequence. Here, 0 (contextually as a homomorphism) is the zero homomorphism. [Hat02]

Some results immediately follow.

Lemma 4.29. *Let $f_i : X_{i+1} \to X_i$ be homomorphisms for each i. Then*

$$0 \xrightarrow{0} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{0} 0$$

is a short exact sequence if and only if all of the following statements are true:

1. f_2 *is injective.*
2. f_1 *is surjective.*
3. $\ker f_1 = \operatorname{Im} f_2$.

Proof. Let

$$0 \xrightarrow{0} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{0} 0$$

be a short exact sequence. Then by definition of exact sequences we have $\ker f_1 = \operatorname{Im} f_2$. So clearly for any $x \in X_0$. Also, by exactness we have[13] $\operatorname{Im}(0 \to X_2) = 0 = \ker f_2$, so f_2 is injective by theorem B.22. Finally, by exactness again we have $\operatorname{Im} f_1 = \ker(X_0 \to 0) = X_0$, so clearly f_1 is injective.

Conversely, suppose that f_2 is injective, f_1 is surjective and $\ker f_1 = \operatorname{Im} f_2$. Then clearly $\operatorname{Im}(0 \to X_2) = 0$. By injectivity of f_2 and B.22 we then have $\ker f_2 = 0 = \operatorname{Im}(0 \to X_2)$. Now, the surjectivity of f_1 gives $\operatorname{Im}(f_1) = X_0$ by definition. Clearly $\ker(X_0 \to 0) = X_0$ by nature of the zero homomorphism. Putting these together we obtain $\operatorname{Im}(f_1) = X_0 = \ker(X_0 \to 0)$ and so we have shown exactness across the entire sequence. □

4.5.1 Relative Homology Groups

Our aim in this section is to define *relative homology groups* and to demonstrate that they fit nicely into a long exact sequence we shall construct.

Definition 4.30. Let X be a space. Let $U \subseteq X$ be a subspace. Let $C_n(X, U)$ denote the quotient group $C_n(X)/C_n(U)$. So chains in U are trivial in $C_n(X, U)$ by definition of quotient groups.

The boundary map $\partial_n^X : C_n(X) \to C_{n-1}(X)$ takes $C_n(U)$ to $C_{n-1}(U)$ by the nature of subspaces, it induces a quotient boundary map $\partial_n^{(X,U)} : C_n(X, U) \to C_{n-1}(X, U)$. Then we have a sequence of boundary maps

$$\dots \xrightarrow{\partial_{n+2}^{(X,U)}} C_{n+1}(X, U) \xrightarrow{\partial_{n+1}^{(X,U)}} C_n(X, U) \xrightarrow{\partial_n^{(X,U)}} C_{n-1}(X, U) \xrightarrow{\partial_{n-1}^{(X,U)}} \dots$$

for which the relation $\partial_{n-1}^{(X,U)} \partial_n^{(X,U)} = 0$ holds since it holds in the original group.

Then the homology groups $H_n(X, U)$ of this chain complex are called the *relative homology groups*. Elements of $H_n(X, U)$ are represented by n-chains $\alpha \in C_n(X)$ such that $\partial_n^X(\alpha) \in C_{n-1}(U)$, called *relative cycles*. We call $\alpha = \partial_{n+1}^X(\beta) + \gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(U)$ a *relative boundary*. [Hat02]

Lemma 4.31. *Let X be a space. Let $U \subseteq X$ be a subspace. Let $\alpha \in C_n(X, U)$ be a relative cycle. Then $[\alpha]$ is trivial in $H_n(X, U)$ if and only if $\alpha = \partial_{n+1}^X(\beta) + \gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(U)$, i.e. α is a relative boundary. [Hat02]*

Proof. Assume that $\alpha = \partial_{n+1}^X(\beta) + \gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(U)$. Then we have, by definition of quotient groups,

$$[\alpha] = \partial_{n+1}^X[(\beta)] + [0] = [\partial_{n+1}^X(\beta)] \in \operatorname{Im} \partial_{n+1}^X.$$

So the class of $[\alpha]$ is 0 in $H_n(X, U)$, i.e. $[\alpha]$ is trivial in $H_n(X, U)$.

Now we prove the converse. Suppose that α is trivial in $H_n(X, U)$. Then $[\alpha] = [0]$. As such, there exists $[\beta] \in C_{n+1}(X, U)$ such that

$$[\alpha] = \partial_{n+1}^X[(\beta)] = [\partial_{n+1}^X(\beta)].$$

[13] We abuse notation slightly here. We write $\operatorname{Im}(0 \to X_2)$ to make clear which instance of the 0 homomorphism we refer to.

We then have that $\alpha - \partial^X_{n+1}(\beta) \in C_n(U)$ and we call this chain γ. So we have

$$\alpha = \partial^X_{n+1}(\beta) + \gamma$$

and we are done. $\square$

4.5.2 Long Exact Sequence of Homology Groups

Definition 4.32. Let X be a space. Let $U \subseteq X$ be a subspace. Let $\alpha \in C_n(X)$ be a relative cycle. Let $\delta_n : H_n(X,U) \to H_{n-1}(U)$ be defined by

$$\delta_n([\alpha]) = [\partial^X_n \alpha].$$

We call δ_n a *connecting map.* [Hat02]

We have defined the connecting map. However, we should demonstrate that this map is well-defined and is indeed a homomorphism.

Lemma 4.33. *Let X be a space. Let $U \subseteq X$ be a subspace. Let δ_n be a connecting map as defined in definition 4.32. Then δ_n is indeed a homomorphism. [Hat02]*

Proof. For a proof, please see pg. 116 - 117 of Hatcher. [Hat02] $\square$

Now we move onto one of our main results: the long exact sequence of homology groups.

Theorem 4.34. *Let X be a space. Let $U \subseteq X$ be a subspace. Let $\iota : C_n(U) \to C_n(X)$ be the inclusion map. Let $j : C_n(X) \to C_n(X,U)$ denote the quotient map. Then let $\iota_* : H_n(U) \to H_n(X), j_* : H_n(X) \to H_n(X,U)$ denote the maps induced by ι and j, respectively.*

Then the sequence of homology groups[14]

$$\cdots \xrightarrow{\delta^{(X,U)}_{n+1}} H_n(U) \xrightarrow{\iota^U_*} H_n(X) \xrightarrow{j^{(X,U)}_*} H_n(X,U) \xrightarrow{\delta^{(X,U)}_n} H_{n-1}(U) \xrightarrow{\iota^U_*} H_{n-1}(X) \xrightarrow{j^{(X,U)}_*} \cdots$$

is exact. [Hat02]

Proof. For a proof, please see pg. 115 - 117 of Hatcher. [Hat02] $\square$

[14] Here, the dimension of the ι and j in each instance should be clear from context.

5 Elementary Knot Theory

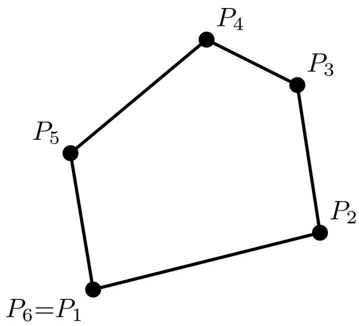
We now introduce some basic elementary knot theory; that is, knots in 3-dimensional space. The reader is no doubt familiar with the concept of a 'knot' as we know it in everyday life. The reader may even be able to tie one themselves. Instead, the question that arises for us is how to define this in mathematical terms. We begin with a recall of some basic definitions.

Definition 5.1. A *polygonal curve* is a connected series of line segments between consecutive points $P_1, P_2, \dots, P_n \in \mathbb{R}^n$ for some $n \in \mathbb{N}$, i.e. it is the union of the line segments $[P_1, P_2], [P_2, P_3], \dots, [P_{n-1}, P_n] \subset \mathbb{R}^n$.

A polygonal curve is *closed* if $P_n = P_1$.

A polygonal curve is *simple* if the line segments do not intersect, except at end points. [Liv93]

Example 5.2. We have the following polygonal curve in $\mathbb{R}^2$, which is both simple and closed.



Definition 5.3. Two or more points $P_i \in \mathbb{R}^n$ are said to be *collinear* if they all lie on the same straight line. Otherwise they are *non-collinear*. [Rus22]

Remark 5.4. Any two points in $\mathbb{R}^n$ are obviously collinear as there trivially exists a line segment connecting them. [Rus22]

5.1 Defining Knots Using Polygonal Curves

Definition 5.5. A *knot* is a simple, closed, polygonal curve in $\mathbb{R}^3$. We may think of this curve as a piece-wise linear embedding $S^1 \hookrightarrow \mathbb{R}^3$. [Liv93]

Remark 5.6. We can make some immediate observations about this definition. Firstly, defining a knot this way limits our knots to exclude *wild knots*, knots with an infinite number of crossings.[15] We are not interested in wild knots.

Additionally, one might notice that we have restricted this definition to $\mathbb{R}^3$. Whilst we shall later consider knots in higher dimensions, for this section we choose to focus solely on knots in 3-dimensional space - for what is known as elementary knot theory.

Remark 5.7. In practice, when we later draw diagrams of knots and links, it shall be assumed that there are so many straight line segments that the curve appears well-rounded. We shall make little mention of the fact that our knots are actually piecewise linear.

5.1.1 Links

Definition 5.8. A *link* of n components is a subset $L \subset \mathbb{R}^3$ consisting of n disjoint knots. [Lic97]

Remark 5.9. A link with one component is just a knot. [Lic97]

[15]the exact meaning of *crossing* shall become clear later on. For now, the reader can just think about what we might consider a 'crossing' in a real knot.

5.2 Equivalence of Knots and Links

In the real world, if I have tied a knot in a closed piece of string and then decide to deform the string however I please, we would still say that this is the same knot. Only if I disconnected the ends and tied a different knot in the string would one say that it is a different knot. This is true mathematically too. We can deform our curves in certain ways without changing the knot itself, any deformation of this sort is equivalent to the original knot; they form an equivalence class.

5.2.1 Ambient Isotopy

When we think about equivalence of our knots, we are motivated by the notion of ambient isotopy. Ambient isotopy for general topological manifolds is defined below.

Definition 5.10. Let M be a topological manifold. An ambient isotopy of M is a continuous map $H : M \times [0,1] \to M$ defined by

$$H(x,t) = H_t(x),$$

such that the following hold: [Kam17]

1. H_0 is the identity map in M.
2. For every $t \in [0,1]$, $H_t : M \to M$ is a homeomorphism of M.
3. The map $(x,t) \mapsto H_t(x)$ is continuous.

Remark 5.11. As we are working in $\mathbb{R}^3$, we assume M to be $\mathbb{R}^3$ for the duration of this section.

5.2.2 Equivalence Classes of Knots

The following definition of equivalence for knots is slighty adapted from Dale Rolfsen's book Knots and Links. [Rol03]

Definition 5.12. Let K_0, K_1 be knots in $\mathbb{R}^3$. Then define a relation $\sim$ such that $K_0 \sim K_1$ if and only if there exists an ambient isotopy $H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that

$$H_1(K_0) = K_1.$$

Then we say that K_0 and K_1 are *ambient isotopic.*

Proposition 5.13. *The relation $\sim$ is an equivalence relation. [Rol03]*

Proof. Let K_0, K_1, K_2 be knots in $\mathbb{R}^3$.

Clearly $K_0 \sim K_1$ via some ambient isotopy H where H_t is the trivial map for all $t \in [0,1]$. So $\sim$ is **reflexive**.

Let $K_0 \sim K_1$ via some ambient isotopy H. Let $\tilde{H} : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ be defined by

$$\tilde{H}(x,t) = H_t^{-1}(x).$$

Clearly each H_t^{-1} is a homeomorphism, being the inverse of a homeomorphism. Also, we have that $H_0^{-1}(x)$ is the trivial map (the inverse of the trivial map is the trivial map).

Then $\tilde{H}(K_1, 1) = H_1^{-1}(K_1) = K_0$ for an ambient isotopy $\tilde{H}$ and so $K_1 \sim K_0$ and we have that $\sim$ is **symmetric**.

Now for **transitivity**. Assume $K_0 \sim K_1$ and $K_1 \sim K_2$ via ambient isotopies F and G respectively. We construct a new ambient isotopy H as follows:

$$H_t = \begin{cases} F_{2t} & \text{for } 0 \leq t \leq 1/2 \\ G_{2t-1} \circ F_1 & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

We shall show that this is indeed an ambient isotopy. We have $H_0 = F_0$ which is the trivial map by definition. We also have that

$$F_{2(\frac{1}{2})} = G_{2(\frac{1}{2})-1} \circ F_1 = G_0 \circ F_1 = F_1$$

and so H is continuous at $t = \frac{1}{2}$ and trivially continuous elsewhere (as F and G are continous). Clearly each H_t is a composition of homeomorphisms and so is indeed a homeomorphism. So we have that H is an ambient isotopy.

Note that $H_1(K_0) = G_1 \circ F_1(K_0) = G_1(K_1) = K_2$ and so $K_0 \sim K_2$ and we are done. □

Definition 5.14. The equivalence class of a knot under $\sim$ is called its *knot type*. [Rol03]

5.2.3 The Unknot

Definition 5.15. An *unknot* is defined as a knot in the equivalence class of some knot given by three non-collinear points.

Remark 5.16. We can think of the unknot as a piece of string that has not been tied in any way.

5.3 Knot Diagrams

Throughout this section, we give our definitions and theorems in terms of knots. However, these definitions and theorems are again analogous for links. We can have link projections and thus link diagrams. We first define knot projections.

Definition 5.17. A *knot projection* is simply a map $P : \mathbb{R}^3 \to \mathbb{R}^2$ such that $P(x, y, z) = (x, y)$ for any point (x, y, z) on a knot K (or a link L). [Rus22]

Remark 5.18. Note here that the choice of the x, y coordinates was completely arbitrary. A knot projection of the form $P(x, y, z) = (x, z)$ is valid.

So a projection simply takes our three-dimensional curve and projects it onto a two-dimensional plane. This makes it very easy to draw our knots, we just draw the projection. But what about when two points on our curve share the same x and y coordinates? When projected we will lose any information regarding which of these points lies above the other in the z-axis. Enter knot diagrams. A knot diagram is an easy way to preserve this information whilst still having our knots easy to draw and hence visualise. First, we must discuss a limitation of certain knot projections.

Definition 5.19. A knot projection is called a *regular projection* if each line segment of some knot K (or link L) projects to a line segment in $\mathbb{R}^2$ such that the projections of two such segments intersect at at most one point for which disjoint segments is not an endpoint. [Lic97]

In order to draw a diagram, we need our projection to be regular. If our knot does not give a regular projection, we can thankfully turn to its equivalence class to find one that does.

Theorem 5.20. *Any knot either gives a regular projection, or is equivalent to a knot that does so. [Rus22]*

Proof. Suppose we have a knot K that does not give a regular projection. Consider a point $\mathbf{x} = (x, y, z)$ on K that shares its x and y coordinates with arbitrarily many other points on K.

Consider the point $\mathbf{x_0} = (x + \epsilon, y, z)$, where the triangle bounded by the edge E on which (x, y, z) lies and the two edges E_1 and E_2 connecting the vertices at either end of E to the point $\mathbf{x_0}$ does not intersect the knot anywhere except E.

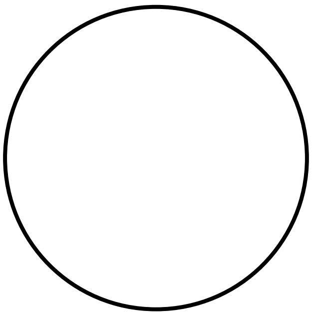
Then the knot wherein E is replaced by E_1 and E_2 is an elementary deformation of K and hence equivalent. We no longer have the issue of $\mathbf{x}$ being a point our knot and hence sharing its x and y coordinates with other points.

Repeat this process with all points that share x and y coordinates with more than one other point and you get an equivalent knot that gives a regular projection. □

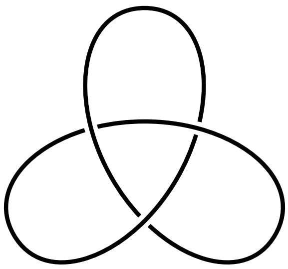
Definition 5.21. A *knot diagram* is a picture of a regular projection of a knot. Knot diagrams are often drawn as continuous curves, despite the fact we model them as polygonal curves.

Points where the projection intersects itself are drawn so as to preserve which point of the original knot crosses above the other in the z-axis, as shown in the latter example below. The upper part of the crossing (higher z value) is called the *over-crossing* and the lower part (smaller z value) is called the *under-crossing*. [Rus22]

Example 5.22. We have the following diagram of the unknot. Note that the unknot diagram is just a circle, as there are no crossings.



Example 5.23. We also have a diagram for the *trefoil knot*, the simplest non-trivial knot.



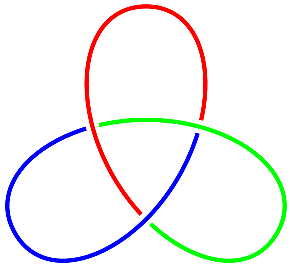
Here we have three crossings, represented by the gaps in the diagram. At each crossing, the part of the line that 'breaks' represents the knot going 'under' itself.

Remark 5.24. To summarise, *knot projections* are functions that take knots to the two-dimensional plane and *knot diagrams* are just an intuitive way of representing knot projections whilst preserving the properties of the original knot's crossings.

5.3.1 Strands and Crossings

Definition 5.25. A *strand* of a knot diagram is a piece of the knot that goes from one under crossing to another with only over crossings in-between, i.e. it is an unbroken part of the knot diagram. [Ada04]

Example 5.26. We return to the diagram for the *trefoil knot*, highlighting each strand in a different colour.

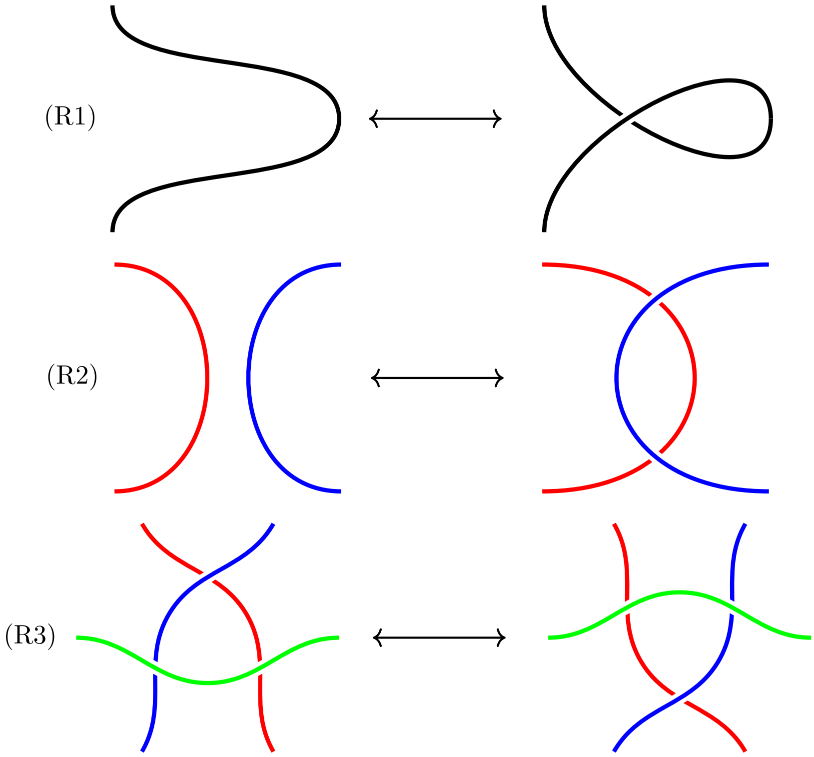


Definition 5.27. The *crossing number* of a knot is the minimum number of crossings over all possible regular knot diagrams. [Rus22]

5.4 Reidemeister Moves

Knot diagrams are a useful tool for determining when knots are equivalent, as we can manipulate the diagram in certain ways so as to find equivalent knots. These manipulations are known as *Reidemeister moves* (after German mathematician Kurt Reidemeister (1893 - 1971)).

Definition 5.28. The three *Reidemeister moves* that can be applied to knot diagrams are the *twist/untwist ($R1$)*, *poke/unpoke ($R2$)* and *slide ($R3$)*. These moves are reversible, we shall denote their inverses by *untwist ($R1^{-1}$)*, *unpoke ($R2^{-1}$)* and *slide ($R3^{-1}$)* respectively. Note that $R3$ and $R3^{-1}$ are the same move, and neither changes the number of crosses in the diagram, so we shall use them interchangeably. The Reidemeister moves are shown below: [Rei83]



This leads us to *Reidemeister's theorem*, a theorem that relates diagrams to our notion of equivalence (definition 5.12).

5.5 Reidemeister's Theorem

Theorem 5.29 (Reidemeister's Theorem)**.** *Suppose two knots K_1 and K_2 are equivalent. Then the diagram of K_2 can be obtained via a finite application of the Reidemeister moves (or their inverses) on the diagram of K_1 and vice versa. [Rei83]*

Proof. A proof was first detailed in Reidemeister's 1927 paper *Elementare Begründung der Knotentheorie (Eng.: Elementary Justification of the Knot Theory).* [Rei27] We shall not go into detail here, since said proof is beyond the scope of this work. □

Reidemeister's theorem is incredibly powerful. It allows us to clearly identify equivalent knots using just their diagrams.

6 Knotted Surfaces and 2-knots

We finally turn our attention to the primary focus of this work. We move away from 3-dimensional Euclidean space and into the unknown.

Many areas of this section will be analogous to section 5. The purpose of that section was to motivate some of the upcoming definitions and provide a 3-dimensional image of what knot theory looks like in the reader's head.

6.1 Surface Knots

Definition 6.1. A *surface knot* is a submanifold $K \subset \mathbb{R}^4$ that is homeomorphic to a closed connected surface.

A *smooth surface knot* is a smooth submanifold of $\mathbb{R}^4$ that is homeomorphic to a closed connected surface. Similarly, a *piecewise linear surface knot* is a piecewise linear submanifold of $\mathbb{R}^4$ that is homeomorphic to a closed connected surface. [Kam17]

Definition 6.2. A surface knot is called *orientable* is it is orientable as a surface. [Kam17]

Definition 6.3. A 2-*knot* is a surface knot that is homeomorphic to the 2-sphere, S^2. [Kam17]

Lemma 6.4. *2-knots are orientable.*

Proof. It follows from the surface classification theorem (theorem XXX) that S^2 is orientable. As a 2-knot is an embedding of S^2, we have that 2-knots are orientable (recall that embeddings are homeomorphic to their image). □

6.2 Equivalence of Surface Knots

Definition 6.5. Let $K_0, K_1 \subset \mathbb{R}^4$ be surface knots. Define a relation $\sim$ such that $K_0 \sim K_1$ if and only if K_0 and K_1 are ambient isotopic. [Kam17]

The following proof is identical

Proposition 6.6. *The relation $\sim$ is an equivalence relation. [Kam17]*

The following proof is identical to the proof of equivalence for standard knots (proposition 5.13).

Proof. Let K_0, K_1, K_2 be surface knots in $\mathbb{R}^4$.

Clearly $K_0 \sim K_1$ via some ambient isotopy H where H_t is the trivial map for all $t \in [0, 1]$. So $\sim$ is **reflexive**.

Let $K_0 \sim K_1$ via some ambient isotopy H. Let $\tilde{H} : \mathbb{R}^4 \times [0, 1] \to \mathbb{R}^4$ be defined by

$$\tilde{H}(x, t) = H_t^{-1}(x).$$

Clearly each H_t^{-1} is a homeomorphism, being the inverse of a homeomorphism. Also, we have that $H_0^{-1}(x)$ is the trivial map (the inverse of the trivial map is the trivial map).

Then $\tilde{H}(K_1, 1) = H_1^{-1}(K_1) = K_0$ for an ambient isotopy $\tilde{H}$ and so $K_1 \sim K_0$ and we have that $\sim$ is **symmetric**.

Now for **transitivity**. Assume $K_0 \sim K_1$ and $K_1 \sim K_2$ via ambient isotopies F and G respectively. We construct a new ambient isotopy H as follows:

$$H_t = \begin{cases} F_{2t} & \text{for } 0 \leq t \leq 1/2 \\ G_{2t-1} \circ F_1 & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

We shall show that this is indeed an ambient isotopy. We have $H_0 = F_0$ which is the trivial map by definition. We also have that

$$F_{2(\frac{1}{2})} = G_{2(\frac{1}{2})-1} \circ F_1 = G_0 \circ F_1 = F_1$$

and so H is continuous at $t = \frac{1}{2}$ and trivially continuous elsewhere (as F and G are continous). Clearly each H_t is a composition of homeomorphisms and so is indeed a homeomorphism. So we have that H is an ambient isotopy.

Note that $H_1(K_0) = G_1 \circ F_1(K_0) = G_1(K_1) = K_2$ and so $K_0 \sim K_2$ and we are done. $\square$

Definition 6.7. The equivalence class of a surface knot under $\sim$ is called its *surface knot type.* [Kam17]

Definition 6.8. Let $K_0, K_1 \subset \mathbb{R}^4$ be oriented surface knots. Then K_0, K_1 are *oriented equivalent* if they are ambient isotopic with respect to their orientations. [Kam17]

6.3 Unknotted 2-knots

We discuss what it means for a 2-knot to be unknotted. This is our 2-knot equivalent of the unknot.

Definition 6.9. Let K be a 2-knot. Then we say that K is *unknotted* if it is ambient isotopic to some 2-knot $K_0 \subset \mathbb{R}^3 \times \{0\}$, i.e. it can be deformed continuously to some 2-knot in the hyperplane $\mathbb{R}^3 \times \{0\}$. [Kam17]

This will serve as our base definition. However, we have an equivalent statement as to what it means to be unknotted, with our proof adapted from a 1979 paper by Fujitsugu Hosokawa and Akio Kawauchi. [HK79]

Hosokawa and Kawauchi define unknotted-ness of some closed, connected and oriented surface F with genus $n \geq 0$ in $\mathbb{R}^4$ by the existence of a solid torus also of genus n whose boundary is F. Here, In the case of 2-knots, which have genus 0, our solid torus is just a 3-ball. We show that our definition for unknotted-ness of a 2-knot is equivalent to the aforementioned specific case.

Theorem 6.10. *Let K be a 2-knot. Then K is unknotted if and only if K bounds a 3-ball embedded in $\mathbb{R}^4$. [Kam17]*

Proof. see PROPOSALS FOR UNKNOTTED SURFACES IN FOUR-SPACES

Suppose that K bounds a piecewise-linear 3-ball B^3 embedded in $\mathbb{R}^4$. $\square$

A Topology

Here we recall some standard definitions and theorems from Topology that are used throughout, for the sake of completeness. These are mostly taken from Munkres' book, although other sources have been used where indicated. [Mun00]

A.1 Open and Closed Sets

Definition A.1. *Topological Space.*

A topological space is a set X together with a *topology* $\mathcal{T}$ such that the following hold:

- $\emptyset \in \mathcal{T}$,
- $X \in \mathcal{T}$,
- for $A_1, \ldots, A_n \in \mathcal{T}$, we have $A_1 \cap \ldots \cap A_n \in \mathcal{T}$,
- for $A_1, A_2, \ldots \in \mathcal{T}$, we have $A_1 \cup A_2 \cup \ldots \in \mathcal{T}$.

The elements of $\mathcal{T}$ are the *open subsets* of X. [Mun00]

Let X be a topological space throughout. Recall that a *collection* is just a set with elements that are sets.

Definition A.2. *Interior.*

Let X be a topological space with $Y \subseteq X$. Then the *interior* of Y, denoted $\operatorname{Int} Y$, is defined as [Mun00]

$$\{\bigcup U_\alpha \mid U_\alpha \subseteq Y, U_\alpha \text{ is open}\}.$$

Definition A.3. *Closed Set.*

A subset $A \subseteq X$ is closed if $X \setminus A$ is open. [Mun00]

Definition A.4. *Closure.*

Let X be a topological space with $Y \subseteq X$. Then the *closure* of Y, denoted $\bar{Y}$, is defined as [Mun00]

$$\{\bigcap U_\alpha \mid Y \subseteq U_\alpha, U_\alpha \text{ is closed}\}.$$

Definition A.5. *Boundary.*

Let X be a topological space with $Y \subseteq X$, the *boundary* of Y in X is the set [Mun00]

$$\partial Y = \bar{Y} \setminus \operatorname{Int}(Y).$$

Proposition A.6. *Let X be some topological space. Then the following holds:*

1. *$\emptyset$ and X are closed.*
2. *Intersections of closed sets are closed.*
3. *Finite unions of closed sets are closed. [Mun00]*

Proof. We prove each individually.

1. $\emptyset$ is the complement of X and is thus closed as X is open. Similarly, X is the complement of $\emptyset$.
2. Let $\{A_\alpha\}$ denote an arbitrary collection of closed sets. We apply DeMorgan's law (refer to chapter 2 of Ashlock and Lee's text [AL20]) to obtain

$$X \setminus \bigcap A_\alpha = \bigcup (X \setminus A_\alpha).$$

By the definition of a closed set, each $X \setminus A_i$ is open. Thus, $\bigcup(X \setminus A_i)$ is an arbitrary union of open sets, which by definition is open. As such, $\bigcap A_i$ is closed.

3. Let A_i be closed for $i = 1, \dots, n$. We have that

$$X \setminus \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X \setminus A_i).$$

To see that this is true, note that the left hand side represents all of x not contained in some A_i, while the right hand side represents the intersection of all of X that is not in some A_i. These are clearly equivalent. As the right hand side is a finite intersection of open sets, it is by definition open. Therefore, $\bigcup_{i=1}^{n} A_i$ is closed as its complement $X \setminus \bigcup_{i=1}^{n} A_i$ is open.

□

Definition A.7. *Basis.*

A *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X such that the following hold:

- for $x_0 \in X$, there exists some $B \in \mathcal{B}$ such that $x_0 \in B$,
- for $B_1, B_2 \in \mathcal{B}$, we have that if $x_0 \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$.

Elements of $\mathcal{B}$ are called *basis elements*. [Mun00]

Definition A.8. *Subspace.*

Let X be a topological space with topology $\mathcal{T}$. Let $Y \subseteq X$. Then the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the *subspace topology.*

We say that Y is a subspace of X. [Mun00]

Definition A.9. *Product Topology.*

Let X and Y be topological spaces with topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$ respectively. The *product topology* on $X \times Y$ is the topology with basis [Mun00]

$$\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}.$$

Definition A.10. *Hausdorff Space.*

X is said to be *Hausdorff* if for each pair x_1, x_2 of distinct points in X, there exists neighbourhoods U_1, U_2 of x_1 and x_2 respectively such that [Mun00]

$$U_1 \cap U_2 = \emptyset.$$

A.2 Compact and Connected Spaces

Definition A.11. *Cover.*

A collection $\mathcal{A}$ of subsets of X is said to *cover* X if $\bigcup \mathcal{A} = X$. We say that $\mathcal{A}$ is a *covering* of X.

$\mathcal{A}$ is an *open covering* of X if its elements are all open subsets of X. [Mun00]

Definition A.12. *Compact Space.*

X is called *compact* if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X. [Mun00]

Definition A.13. *Separation.*

A *separation* of X is a pair of disjoint non-empty open subsets of X, say U and V, such that $U \cup V = X$. [Mun00]

Definition A.14. *Connected Space.*

X is said to be *connected* if there does NOT exist some separation of X. [Mun00]

Theorem A.15. *Suppose $U, V \subseteq X$ are connected subspaces. Let some $x_0 \in X$ be such that $x_0 \in U \cap V$. Then $U \cup V$ is connected. [Mun00]*

Proof. Let $U, V \subseteq X$ be connected subspaces. Let $x_0 \in U \cap V$. Suppose $A \cup B \subseteq X$ is a separation of $U \cup V$, i.e. $U \cup V$ is not connected. Then, without loss of generality, suppose $x_0 \in A$. Since U and V are connected, they must lie entirely within A or B (as $A \cup B$ is a separation of $U \cup V$). As $U \cap V$ contains x_0, it must be the case that $U, V \subseteq A$. This is a contradiction, as $A \cup B$ is a separation of $U \cup V$ and thus B cannot be non-empty. Thus, $U \cup V$ is connected. □

A.3 Metric Spaces

Definition A.16. *Metric Space.*

Let M be a set with $m_0, m_1, m_2 \in M$. Let $d : M \times M \to \mathbb{R}$ be such that

1. $d(m_0, m_1) > 0$ if $m_0 \neq m_1$;
2. $d(m_0, m_0) = 0$;
3. $d(m_0, m_1) = d(m_1, m_0)$;
4. $d(m_0, m_2) \leq d(m_0, m_1) + d(m_1, m_2)$ (the *triangle inequality*).

Then d is called a *metric*. If such a d exists, we say M is *metrisable*. [Mun00]

M, together with d is called a *metric space*, denoted (M, d). We refer to the elements of M as *points*. [Rud76]

Example A.17. *Euclidean Space.*

Let $n \in \mathbb{N}$. The *Euclidean space* of dimension n is the set

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}.$$

Define the *Euclidean distance* $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ by

$$d(x, y) \;=\; \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}, \quad \text{for } x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in \mathbb{R}^n.$$

The pair $(\mathbb{R}^n, d)$ is a metric space because d satisfies, for all $x, y, z \in \mathbb{R}^n$:

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$,
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Definition A.18. *ϵ-Ball.*

Let d be a metric on M. Let $\epsilon > 0$. The *ϵ-Ball* centred at m_0 is given by [Mun00]

$$B_d(m_0, \epsilon) = \{m : d(m, m_0) < \epsilon\}.$$

Definition A.19. *Bounded Subset.*

Let d be a metric on M. Let $A \subseteq M$. A is said to be *bounded* if there exists some $m \in \mathbb{R}$ such that

$$d(a_0, a_1) \leq m$$

for any $a_0, a_1 \in A$.

Definition A.20. *Diameter.*

Let d be a metric on M. Let $A \subseteq M$. Suppose A is bounded and non-empty. The *diameter* of A is given by [Mun00]

$$\text{diam } A = \sup\{d(a_0, a_1) : a_0, a_1 \in A\}.$$

Definition A.21. *Metric Topology.*

Let d be a metric on M. Then the collection of all ϵ-balls $B_d(m_0, \epsilon)$ for $m_0 \in M$ and $\epsilon > 0$ is a basis for a topology on M. This is known as the *metric topology* induced by d. [Mun00]

Lemma A.22. The Lebesgue Number Lemma.

Let $\mathcal{A}$ be an open cover of M with respect to the metric topology induced by d. If M is compact, then there exists $\delta > 0$ such that for each $A \subseteq M$ with diam $A < \delta$, there exists an element of $\mathcal{A}$ containing it.

δ is called a Lebesgue number for the cover $\mathcal{A}$. [Mun00]

Proof. For the trivial case, note that if $M \in \mathcal{A}$, then any positive number is a Lebesgue number for $\mathcal{A}$.

Now suppose that $M \notin \mathcal{A}$. Choose a finite subcollection $\{A_0, \ldots, A_n\} \subseteq \mathcal{A}$ that covers M. For $i \in \{0, \ldots, n\}$, set $A'_i = M \setminus A_i$. Define $d(m, A'_i) = \inf\{d(m, a') : a \in A'_i\}$, i.e. the minimum distance between m and the set A'_i.

Let $f : M \to \mathbb{R}$ be such that

$$f(m) = \frac{1}{n} \sum_{i=1}^{n} d(m, A'_i).$$

This is the average of the shortest distances from m to each A'_i.

Given $m \in M$, choose i such that $m \in A_i$ (which is possible as $\mathcal{A}$ covers M). Then choose $\epsilon > 0$ such that the ϵ-neighbourhood of m lies in A_i (which is possible as $\mathcal{A}$ is an open cover). Then $d(m, A'_i) \geq \epsilon$, so we can deduce that

$$f(m) = \frac{1}{n} \sum_{i=1}^{n} d(m, A'_i) \geq \frac{1}{n} \cdot \epsilon = \frac{\epsilon}{n}.$$

Hence, $f(m) \geq 0$.

We have that f is continuous as it is a linear sum of continuous functions (for proof of the continuity of the distance function, see chapter 2 of Knapp [Kna07]) divided by an integer. As f is continuous it must have a minimum value, say δ, by the extreme value theorem (see chapter 4 of Rudin [Rud76]).

We propose that δ is our required Lebesgue number. Let $B \subseteq M$ be such that diam $B < \delta$. Choose $m_0 \in B$. Then B lies in the δ-neighbourhood of m_0. Let $d(m_0, A'_{max}) = \max_{0 \leq i \leq n} d(m_0, A'_i)$ So we have

$$\delta \leq f(m_0) \leq d(m_0, A'_{max}).$$

Then the δ-neighbourhood of m_0 is contained within the element $A_{max} = X \setminus A'_{max}$ of $\mathcal{A}$. Hence there exists some element of $\mathcal{A}$ containing B and our condition is satisfied. □

A.4 Maps and Homeomorphisms

Definition A.23. *Continuous Function.*

Let X, Y be topological spaces. A function $f : X \to Y$ is said to be *continuous* if for each open subset $V \subseteq Y$, $f^{-1}(V)$ is an open subset of X.[16] [Mun00]

Definition A.24. *Open Function.*

Let X, Y be topological spaces. A function $f : X \to Y$ is said to be *open* if for each open subset $U \subseteq X$, $f(U)$ is an open subset of Y. [Mun00]

Definition A.25. *Embedding.* Let X, Y be topological spaces. Let $f : X \to Y$ be injective and continuous. Let its inverse $f^{-1} : Y \to X$ also be continuous. Then f is called a *embedding* of X into Y.

X is said to be *embedded* in Y by f. [Wil04]

Definition A.26. *Homeomorphism.* Let X, Y be topological spaces. Let $f : X \to Y$ be bijective and continuous. Let its inverse $f^{-1} : Y \to X$ also be continuous. Then f is called a *homeomorphism* from X to Y.

X and Y are said to be *homeomorphic.* [Wil04]

Remark A.27. An embedding can be thought of as a homeomorphism between X and some subspace $V \subseteq Y$, owing to the lack of the surjective condition over the entirety of Y.

Definition A.28. *Fibre.*[17]

Let $f : X \to Y$ be a map. Let $y \in Y$. Then the preimage of a singleton y, $f^{-1}(y) \subseteq X$ where

$$f^{-1}(y) = \{x \in X : f(x) = y\},$$

is called a *fibre* over y. [Lee10]

Theorem A.29. *Let $[a, b], [c, d]$ be closed intervals in $\mathbb{R}$. Then $[a, b], [c, d]$ are homeomorphic. [Wil04]*

Proof. We claim that $[a, d]$ is homeomorphic to $[c, d]$ via the homeomorphism $f : [a, b] \to [c, d]$,

$$f(x) = c + \frac{(x-a)(d-c)}{b-a}.$$

We first note that f is continuous as it is a linear function. All linear functions are continuous everywhere.

Assume $f(x_0) = f(x_1)$ for some $x_0, x_1 \in [a, b]$. Then

$$\begin{aligned} c + \frac{(x_0-a)(d-c)}{b-a} &= c + \frac{(x_1-a)(d-c)}{b-a} \\ \implies x_0 - a &= x_1 - a \\ \implies x_0 &= x_1. \end{aligned}$$

So f is **injective**.

Let $y_0 \in [c, d]$. We need to show that there exists some $x_0 \in [a, b]$ such that $f(x_0) = y_0$. We have

$$\begin{aligned} f(x_0) = y_0 &= c + \frac{(x_0-a)(d-c)}{b-a} \\ \implies x_0 &= a + \frac{(y_0-c)(b-a)}{d-c} \end{aligned}$$

via rearranging. Since $y_0 \in [c, d]$, it follows that $0 \leq \frac{y-c}{d-c} \leq 1$ and so

$$a \leq x_0 = a + \frac{(y_0-c)(b-a)}{d-c} \leq a + b - a = b.$$

[16] We use the terms 'function' and 'map' interchangeably.

[17] Or '*Fiber*' in American English.

So $x_0 \in [a, b]$ and f is **surjective**.

It remains to show that $f^{-1} : [c, d] \to [a, b]$ is continuous. We solve $f(x)$ for x to find the inverse. We have

$$\begin{aligned} f(x) &= c + \frac{(x-a)(d-c)}{b-a} \\ \implies f(x) - c &= \frac{(x-a)(d-c)}{b-a} \\ \implies (f(x) - c)(b-a) &= (x-a)(d-c) \\ \implies \frac{(f(x)-c)(b-a)}{d-c} &= x - a \\ \implies a + \frac{(f(x)-c)(b-a)}{d-c} &= x. \end{aligned}$$

So $f^{-1}(y) = a + \frac{(y-c)(b-a)}{d-c}$ is our inverse function, which is a linear function and is thus continuous.

We have then that f satisfies the conditions of a homomorphism from $[a, b]$ to $[c, d]$. □

Theorem A.30. *Let X be a connected topological space. Let $f : X \to Y$ be a continuous map, with Y some topological space. Then $f(X)$ is connected. [Mun00]*

Proof. Let $V = f(X)$. Define a function $g : X \to V$ to be such that $g(x) = f(x)$ for $x \in X$; i.e. g is f with the range restricted to $V = f(X)$. Then g is necessarily surjective by its definition.

Suppose that $V = V_0 \cup V_1$ is a separation of V. Then $g^{-1}(V_0)$ and $g^{-1}(V_1)$ are disjoint sets with $g^{-1}(V_0) \cup g^{-1}(V_1) = X$ owing to the surjectivity of g. By the definition of continuity of g, we have that $g^{-1}(V_0)$ and $g^{-1}(V_1)$ are open in X. We also have that $g^{-1}(V_0)$ and $g^{-1}(V_1)$ are non-empty by surjectivity of g again. Thus, $g^{-1}(V_0)$ and $g^{-1}(V_1)$ are a separation of X, which contradicts the assumption that X is connected. □

Theorem A.31. The Pasting Lemma.

Let $X = A \cup B$ and Y be topological spaces, where $A \cup B$ is closed in X. Let $f : A \to X$ and $g : B \to Y$ be continuous.

If $f(x_0) = g(x_0)$ for all $x_0 \in A \cap B$, then the function $h : X \to Y$ with

$$h(x_0) = \begin{cases} f(x_0) & \text{for } x_0 \in A \\ g(x_0) & \text{for } x_0 \in B. \end{cases}$$

is continuous. [Mun00]

Proof. Let $C \subseteq Y$ be closed. Clearly, $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ by the definition of h. Since f is continuous, $f^{-1}(C)$ must be closed in A by the definition of continuity and thus must also be closed in X. Similarly, since g is continuous, $g^{-1}(C)$ must be closed in B and thus must also be closed in X. Hence, $h^{-1}(C)$ is closed in X (theorem A.6). So h is continuous. □

Theorem A.32. *TODO(continuous bijection from compact to Hausdorff is a homeomorphism)*

A.5 Quotient Spaces

Definition A.33. *Quotient Map.*

Let X, Y be topological spaces. A surjective function $f : X \to Y$ is said to be a *quotient map* if a subset $V \subseteq Y$ is open in Y if and only if $f^{-1}(V)$ is open in of X. [Mun00]

Definition A.34. *Quotient Topology.*

Let X be a topological space. Let A be a set. Let also $f : X \to A$ be a surjective map. Then the topology $\mathcal{T}$ on A relative to which f is a quotient map is called the *quotient topology* induced by f. [Mun00]

Definition A.35. *Quotient Space.*

Let X be a topological space. Let $X^* = \{X_\alpha\}$ be a collection of disjoint subsets of X such that $\bigcup X_\alpha = X$. Let $f : X \to X^*$ be the surjective map that carries each $x \in X$ to the $X_\alpha \in X^*$ such that $x \in X_\alpha$. In the quotient topology induced by f, the space X^* is called a *quotient space* of X.

We can define an equivalence relation on the elements of X. Munkres [Mun00] touches on this but does not go into detail. We explicitly prove equivalence here, despite its triviality.

Proposition A.36. *Let X be a topological space and let $X^* = \{X_\alpha\}$ be the quotient space of X. Let $f : X \to X^*$ be the surjective map that carries each $x \in X$ to the $X_\alpha \in X^*$ such that $x \in X_\alpha$. Let $x_0, x_1 \in X$.*

Define a relation $\sim$ such that

$$x_0 \sim x_1 \text{ if and only if } f(x_0) = f(x_1).$$

Then $\sim$ is an equivalence relation.

Proof. Suppose $x_0, x_1, x_2 \in X$. Then clearly $x_0 \sim x_0$ as $f(x_0) = f(x_0)$. So we have **reflexivity**.

Suppose $x_0 \sim x_1$, i.e. $f(x_0) = f(x_1)$. Then clearly $f(x_1) = f(x_0)$ by the very nature of equality. So $x_1 \sim x_0$ and we have **symmetry**.

Finally, assume that $x_0 \sim x_1$ and $x_1 \sim x_2$. Then $f(x_0) = f(x_1) = f(x_2)$ and so $x_0 \sim x_2$ and we have shown **transitivity**. □

A.6 Basic Spaces and Constructions

We present the following examples of topological spaces. We do not prove that these are topological spaces, instead we simply define them for later use. Should the reader wish to be convinced that these spaces satisfy our conditions for topological spaces, they can refer to the cited texts.

Example A.37. Discrete Topology.

Let X be a set. Let $\mathcal{T}$ be the collection of all subsets of X. Then $\mathcal{T}$ is a topology on X called the *discrete topology.* [Mun00]

Example A.38. *Disjoint Union Space*

Let $\{X_\alpha\}$ be a collection of topological spaces. Let $\coprod X_\alpha$ denote the disjoint union of the elements of $\{X_\alpha\}$.

Let $U \in \coprod X_\alpha$ be open if and only if each $U \cap X_\alpha$ is open in $\coprod X_\alpha$. Then $\coprod X_\alpha$ is a topological space, known as the *disjoint union space.* [Lee10]

A.6.1 Standard Subspaces of Euclidean Space $\mathbb{R}^n$

Example A.39. *Open Unit Ball.*

The *open unit ball of dimension* n is the subspace $B^n \subset \mathbb{R}^n$ given by [Mun00]

$$B^n = \{(x_0, \dots x_n) \mid \sqrt{x_0^2 + \cdots + x_n^2} < 1\}.$$

Example A.40. *Closed Unit Ball.*

The *closed unit ball of dimension* n is the subspace $D^n \subset \mathbb{R}^n$ given by [Mun00]

$$D^n = \{(x_0, \dots x_n) \mid \sqrt{x_0^2 + \cdots + x_n^2} \leq 1\}.$$

Remark A.41. D^2 is called a *disc*.

Example A.42. *Unit Sphere.*

The *unit sphere of dimension* n is the subspace $S^n \subset \mathbb{R}^{n+1}$ given by [Mun00]

$$S^n = \{(x_0, \dots x_n) \mid \sqrt{x_0^2 + \cdots + x_n^2} = 1\}.$$

B Group Theory

We recall definitions from group theory that shall be relevant in our work. The following definitions are primarily adapted from Dummit and Foote. [DF03]

Definition B.1. *Binary Operation.*
A *binary operation* $\cdot$ on a set G is a function $\cdot : G \times G \to G$. We write $g_0 \cdot g_1$ for $\cdot(g_0, g_1)$. [DF03]

Definition B.2. *Commutativity.*
A binary operation is *commutative* if for all $g_0, g_1 \in G$, we have $g_0 \cdot g_1 = g_1 \cdot g_0$. [DF03]

Definition B.3. *Associativity.*
A binary operation is *associative* if for all $g_0, g_1, g_2 \in G$, we have $g_0 \cdot (g_1 \cdot g_2) = (g_0 \cdot g_1) \cdot g_2$. [DF03]

We introduce the group axioms.

Definition B.4. *Group.*
An ordered pair consisting of a set G and a binary operation $\cdot$, denoted $(G, \cdot)$[18] is called a *group* if the following hold:

1. $\cdot$ is associative.
2. There exists some $e \in G$, the *identity element*, such that for all $g \in G$ we have
$$e \cdot g = g = g \cdot e.$$
3. For each $g \in G$, there exists some *inverse element* g^{-1} such that [DF03]
$$g \cdot g^{-1} = e = g^{-1} \cdot g.$$

Remark B.5. We often just write $g_0 g_1$ instead of $g_0 \cdot g_1$ when the context is clear.

Remark B.6. Note that the property of $\cdot$ being a function with a range of G necessitates that G is closed under the binary operation $\cdot$, as is often needed to be shown explicitly when proving the group axioms hold.

Definition B.7. *Abelian Group.*
Let $(G, \cdot)$ be a group as above. We say $(G, \cdot)$ is *abelian* if $\cdot$ is commutative. [DF03]

B.1 Subgroups and Cosets

Definition B.8. *Subgroup.*
Let $(G, \cdot)$ be a group. Then some nonempty $H \subseteq G$ is called a *subgroup* of G if $\cdot$ is closed in H. In other words, for all $h_0, h_1 \in H$, $h_0 \cdot h_1 \in H$.

We write $H \leq G$ to denote a subgroup H of G. [DF03]

Definition B.9. *Left Coset.*
Let G be a group and let $H \leq G$ be a subgroup of G. Let $g_0 \in G$ Then
$$g_0 H = \{g_0 h : h \in H\}$$
is called a *left coset* of H in G.

Any element of a coset is called a *representative* for the coset.

The set of left cosets of H in G is denoted G/H. [DF03]

[18]Often shortened to simply G if the context is clear.

Remark B.10. The definition for *right cosets* follows naturally. Whether G/H is the set of left or right cosets is either explicitely stated or clear from context.

Definition B.11. *Index.*

Let G be a group and let $H \leq G$ be a subgroup of G. The number of left cosets of H in G is called the *index* of H in G and is denoted by $|G : H|$. [DF03]

Definition B.12. *Conjugate.*

Let G be a group and let $H \leq G$ be a subgroup of G. Let $g \in G$ and $h \in H$. Then ghg^{-1} is called the *conjugate* of h by g.

The set

$$gHg^{-1} = \{ghg^{-1} : h \in H\}$$

is called the *conjugate* of H by g. [DF03]

Definition B.13. *Conjugate Subgroups.*

Let G be a group and let $H_0, H_1 \leq G$ be subgroups of G. Let $g \in G$. Suppose

$$H_0 = gH_1g^{-1}.$$

Then H_0 and H_1 are said to be *conjugate*. [DF03]

Theorem B.14. *Let G be a group and let $H_0, H_1 \leq G$ be subgroups of G. Let $H_0 \sim H_1$ denote conjugacy of H_0 and H_1.*

Then $\sim$ is an equivalence relation. [DF03]

Proof. We have clearly that $H_0 \sim H_0$ by noting that

$$H_0 = eH_0e^{-1},$$

where $e \in G$ denotes the identity element. So $\sim$ is **reflexive**.

To see **symmetry**, we let $H_0 = g_0H_1g_0^{-1}$ for some $g_0 \in G$, i.e. $H_0 \sim H_1$. Then clearly

$$g_0^{-1}H_0g_0 = H_1$$

and so $H_1 \sim H_0$ also.

Finally, we show **transitivity**. Let $H_2 \leq G$. Suppose $H_0 \sim H_1$ via g_0 and $H_1 \sim H_2$ via g_1. Then we have

$$H_0 = g_0H_1g_0^{-1} = g_0g_1H_2g_1^{-1}g_0^{-1}.$$

Clearly $g_0g_1 \in G$ by the definition of a group. Note also that

$$g_0g_1g_1^{-1}g_0^{-1} = g_0eg_0^{-1} = g_0g_0^{-1} = e.$$

So $g_1^{-1}g_0^{-1}$ is the inverse of g_0g_1 and we have $H_0 \sim H_2$ via g_0g_1. □

Definition B.15. *Normal Subgroup.*

Let G be a group and let $H \leq G$ be a subgroup of G. Some element $g \in G$ is said to *normalise* H if $gHg^{-1} = H$.

H is called a normal subgroup of G, denoted $H \trianglelefteq G$, if every $g \in G$ normalises H. [DF03]

The following definition comes from Artin. [Art10]

Proposition B.16. *Let G be a group and let $H \trianglelefteq G$ be a normal subgroup. Then the set of all left cosets of H in G,*

$$G/H = \{gH : g \in G\},$$

forms a group under the binary operation defined by

$$(g_0H)(g_1H) = g_0g_1H$$

for $g_0, g_1 \in G$.

This group G/H *is known as the* quotient group. *[Art10]*

Proof. We must prove the group axioms hold.

We have clearly that $g_0g_1H \in G/H$, since $g_0g_1 \in G$ by the closure of G and as such g_0g_1H is a left coset of H.

Let $g_2 \in G$. We have

$$\begin{aligned}(g_0H)((g_1H)(g_2H)) &= (g_0H)(g_1g_2H) = g_0g_1g_2H \\ =&(g_0g_1H)(g_2H) = ((g_0H)(g_1H))(g_2H)\end{aligned}$$

and so **associativity** holds.

Consider eH where $e \in G$ is the identity element of G. Then we have

$$(eH)(g_0H) = eg_0H = g_0H = g_0eH = (g_0H)(eH).$$

So eH is the **identity element** of G/H.

It remains to show the existence of some **inverse element** for an arbitrary $g_0H \in G/H$. Let g_0^{-1} denote the inverse element of g_0 in G. We propose that the inverse of g_0H is $g_0^{-1}H$. Note that

$$(g_0^{-1}H)(g_0H) = g_0^{-1}g_0H = eH = g_0g_0^{-1}H = (g_0H)(g_0^{-1}H)$$

and we are done. □

Lemma B.17. *Let* G *be a group that is abelian and let* $H \trianglelefteq G$ *be a normal subgroup. Then* G/H *is abelian.*

Proof. Let $g_0H, g_1H \in G/H$ for elements $g_0, g_1 \in G$. Then we have that $g_0g_1 = g_1g_0$. Then clearly

$$(g_0H)(g_1H) = g_0g_1H = g_1g_0H = (g_1H)(g_0H)$$

and so we have that G/H is abelian. □

B.2 Homomorphisms

Definition B.18. *Homomorphism.*

Let $(G, \cdot)$ and $(H, *)$ be groups. A map $\varphi : G \to H$ such that

$$\varphi(g_0 \cdot g_1) = \varphi(g_0) * \varphi(g_1)$$

for all $g_0, g_1 \in G$ is called a *homomorphism.* [DF03]

Definition B.19. *Isomorphism.*

Let $(G, \cdot)$ and $(H, *)$ be groups. A map $\varphi : G \to H$ such that

$$\varphi(g_0 \cdot g_1) = \varphi(g_0) * \varphi(g_1)$$

for all $g_0, g_1 \in G$ is called an *isomorphism* if it is bijective.

We say G and H are *isomorphic*, written $G \simeq H$. [DF03]

Remark B.20. An isomorphism is a bijective homomorphism.

Definition B.21. *Kernel.*

Let $(G, \cdot)$ and $(H, *)$ be groups. Let $\varphi : G \to H$ be a homomorphism. The *kernel* of φ is defined to be

$$\ker \varphi = \{g \in G : \varphi(g) = e_H\},$$

where e_H denotes the identity element of H. [DF03]

Theorem B.22. *Let $(G, \cdot)$ and $(H, *)$ be groups. Let $\varphi : G \to H$ be a homomorphism. Then φ is injective if and only if $\ker \varphi = \{e_G\} \leq G$, i.e. $\ker \varphi$ is the identity subgroup of G. [DF03]*

Proof. Suppose that $g_0, g_1 \in \ker \varphi$. Note that by definition of a homomorphism and the kernel, we have

$$\varphi(g_0 \cdot g_1) = \varphi(g_0) * \varphi(g_1) = e_H * e_H = e_H,$$

i.e. $g_0 \cdot g_1 \in \ker \varphi$ also. So clearly $\ker \varphi \leq G$.

Now we suppose that φ is injective. Then clearly $\ker \varphi$ may only contain one element (as only one element maps to e_H by injectivity). It must be the case that $\ker \varphi$ contains the identity element of G, otherwise $\ker \varphi$ would not be closed under $\cdot$, which is necessary as $\ker \varphi$ is a subgroup. So we must have $\ker \varphi = \{e_G\}$ is the sole element.

On the contrary, assume instead that $\ker \varphi = \{e_G\}$. Now suppose that for some $g_0, g_1 \in G$ we have that $g_0 = g_1$. This can be written as

$$e_G \cdot g_0 = e_G \cdot g_1$$

by the definition of the identity element e_G. Applying our homomorphism gives

$$\varphi(e_G \cdot g_0) = \varphi(e_G \cdot g_1)$$

Then using the definition of φ and our assumption of the kernel we get

$$\begin{aligned} &\varphi(e_G) * \varphi(g_0) = \varphi(e_G) * \varphi(g_1) \\ \Longrightarrow\ &e_H * \varphi(g_0) = e_H * \varphi(g_1) \\ \Longrightarrow\ &\varphi(g_0) = \varphi(g_1) \end{aligned}$$

so φ is injective. $\square$

Definition B.23. *Image.*

Let $(G, \cdot)$ and $(H, *)$ be groups. Let $\varphi : G \to H$ be a homomorphism. The *image* of G under φ, denoted $\operatorname{Im} \varphi$, is defined as [DF03]

$$\operatorname{Im} \varphi = \{\varphi(g) : g \in G\}.$$

Theorem B.24. *Let $(G, \cdot)$ and $(H, *)$ be groups. Let $\varphi : G \to H$ be a homomorphism. Then we have $\operatorname{Im} \varphi \leq H$. [DF03]*

Proof. Let $\varphi(g_0), \varphi(g_1) \in \operatorname{Im} \varphi$. We have

$$\varphi(g_0)\varphi(g_1) = \varphi(g_0 g_1)$$

by the definition of group homomorphisms. As $g_0 g_1 \in G$ by closure of G, we have that $\operatorname{Im} \varphi$ is closed under the binary operation of H.

By definition of homomorphisms, we also have $\varphi(e_G) = e_H$, where e_G and e_H are the identity elements of G and H respectively. So $e_H \in \operatorname{Im} \varphi$. The necessary presence of the identity element also confirms that $\operatorname{Im} \varphi$ is always nonempty.

It remains to show that each element of $\operatorname{Im}\varphi$ has an inverse that is also in $\operatorname{Im}\varphi$. Let $\varphi(g_0)^{-1} = \varphi(g_0^{-1})$. We show that this is the inverse. Clearly $\varphi(g_0)^{-1} \in \operatorname{Im}\varphi$ as $g_0^{-1} \in G$. Note that

$$\varphi(g_0)\varphi(g_0)^{-1} = \varphi(g_0)\varphi(g_0^{-1}) = \varphi(g_0 g_0^{-1}) = \varphi(e_G) = e_H$$

and so $\varphi(g_0)^{-1}$ is our inverse. □

B.3 The Isomorphism Theorems

Theorem B.25. The First Isomorphism Theorem.

Let $(G, \cdot)$ and $(H, *)$ be groups. Let $\varphi : G \to H$ be a homomorphism. Then $\ker\varphi \trianglelefteq G$ and [Art10]

$$G/\ker\varphi \cong \operatorname{Im}\varphi.$$

Proof. Let e_G, e_H denote the identity elements of G and H respectively. Let $g \in G$, $g' \in \ker\varphi$ be arbitrary. Then we have

$$g(\ker\varphi)g^{-1} = \{gg'g^{-1} \in G : \varphi(gg'g^{-1}) = e_H\}.$$

We have

$$\begin{aligned}
\varphi(gg'g^{-1}) &= \varphi(g)\varphi(g')\varphi(g^{-1}) \\
&= \varphi(g)e_H\varphi(g^{-1}) \\
&= \varphi(g)\varphi(g^{-1}) \\
&= \varphi(gg^{-1}) \\
&= \varphi(e_G) \\
&= e_H.
\end{aligned}$$

Here we have used the properties of homomorphisms and the fact that $g' \in \ker\varphi$. Clearly we have shown that for all $g \in G$, g normalises $\ker\varphi$ and thus it is a normal subgroup.

Now consider the quotient group

$$G/\ker\varphi = \{g\ker\varphi : g \in G\}.$$

We wish to show that there exists a bijection between $G/\ker\varphi$ and $\operatorname{Im}\varphi$.

Let $\tilde{\varphi} : G/\ker\varphi \to \operatorname{Im}\varphi$ be defined by

$$\tilde{\varphi}(g\ker\varphi) = \varphi(g).$$

We will show that this is an isomorphism. It is clear that $\tilde{\varphi}$ is well defined.

Let $g_0, g_1 \in G$. Then

$$\begin{aligned}
&\tilde{\varphi}((g_0\ker\varphi)(g_1\ker\varphi)) \\
&= \tilde{\varphi}(g_0g_1\ker\varphi) \\
&= \varphi(g_0g_1) \\
&= \varphi(g_0)\varphi(g_1).
\end{aligned}$$

We also have that

$$\begin{aligned}&\tilde{\varphi}(g_0 \ker \varphi)\tilde{\varphi}(g_1 \ker \varphi)\\&= \varphi(g_0)\varphi(g_1).\end{aligned}$$

Putting this together gives

$$\tilde{\varphi}((g_0 \ker \varphi)(g_1 \ker \varphi)) = \tilde{\varphi}(g_0 \ker \varphi)\tilde{\varphi}(g_1 \ker \varphi)$$

which is the necessary condition for $\tilde{\varphi}$ to be a homomorphism.

Suppose $g_0, g_1 \in G$ are such that $\tilde{\varphi}(g_0 \ker \varphi) = \tilde{\varphi}(g_1 \ker \varphi)$. Then $\varphi(g_0) = \varphi(g_1)$ by our definition of $\tilde{\varphi}$. Note that

$$\begin{aligned}&\varphi(g_0) = \varphi(g_1)\\ \implies &\varphi(g_0)\varphi(g_1)^{-1} = e_H\\ \implies &\varphi(g_0 g_1^{-1}) = e_H\\ \implies &g_0 g_1^{-1} \in \ker \varphi.\end{aligned}$$

Then by the definition of quotient groups, we must have that $g_0 \ker \varphi = g_1 \ker \varphi$ and so $\tilde{\varphi}$ is injective.

Consider $g_0 \in G$ such that $\varphi(g_0) \in \operatorname{Im} \varphi$. Then

$$\tilde{\varphi}(g_0 \ker(\varphi)) = \varphi(g_0)$$

and so every arbitrary $\varphi(g_0) \in \operatorname{Im} \varphi$ has some corresponding $g_0 \ker \varphi \in G/\ker \varphi$. So $\tilde{\varphi}$ is surjective.

We have shown that $\tilde{\varphi}$ is bijective and thus $G/\ker \varphi \cong \operatorname{Im} \varphi$. $\square$

B.4 Group Actions

Definition B.26. *Group Action.*

A *Group Action* of a group G on some set X is a map $G \times X \to X$, written $g \cdot x$ for some $g \in G, x \in X$, that satisfies the following:

1. $g_0 \cdot (g_1 \cdot x) = (g_0 g_1) \cdot x$ for $g_0, g_1 \in G$.
2. $e \cdot x = x$, where $e \in G$ is the identity of G.

We say the group G acts on the set X. [DF03]

Remark B.27. We must be cautious with abuse of notation here. Note that $\cdot$ in this instance is not a binary operation, as it does not satisfy the definition. In the above, we have used the common shortening of $g_0 g_1$ to represent the product of g_0 and g_1 under the binary operation of G.

Definition B.28. *Kernel of an Action.*

The *kernel* of the action of G on X is defined by

$$\{g : g \cdot x = x \text{ for all } x \in X\}.$$

Definition B.29. *Stabiliser.*

The *stabiliser* of $x_0 \in X$ under the action of G on X, denoted G_{x_0}, is defined by

$$G_{x_0} = \{g : g \cdot x_0 = x_0\}.$$

Definition B.30. *Faithful Action.*

The action of G on X is called *faithful* if its kernel is the identity $e \in G$.

Proposition B.31. *Let G be a group acting on a nonempty set X. Then the relation on X defined by*

$$x_0 \sim x_1 \text{ if and only if } x_0 = g \cdot x_1 \text{ for some } g \in G$$

is an equivalence relation. [DF03]

Proof. We have, by definition of a group action, that $x = e \cdot x$ for all $x \in X$, i.e. $x \sim x$. Thus, we have shown **reflexivity**.

Suppose for some arbitrary $x_0, x_1 \in X$ that $x_0 \sim x_1$. So we have $x_0 = g \cdot x_1$ for some $g \in G$. We see then that

$$g^{-1} \cdot x_0 = g^{-1} \cdot (g \cdot x_1) = (g^{-1} \cdot g) \cdot x_1 = e \cdot x_1 = x_1,$$

so $x_1 \sim x_0$ via g^{-1}. Thus, by the necessary existence of g^{-1} we have shown **symmetry**.

Let $x_2 \in X$. Now suppose $x_0 \sim x_1$ and $x_1 \sim x_2$. Then we have $x_0 = g \cdot x_1$ and $x_1 = g' \cdot x_2$ for some $g, g' \in G$. Note that

$$x_0 = g \cdot x_1 = g \cdot (g' \cdot x_2) = gg' \cdot x_2$$

by the definition of a group axiom. Here we have shown that $x_0 \sim x_2$ via gg', which necessarily exists in G by definition of a group. Thus, **transitivity** holds. □

Definition B.32. *Orbit.*

The equivalence class $[x_0]$ as defined by the equivalence relation in the above proposition is called the *orbit* of G containing x_0. [DF03]

Definition B.33. *Transitive Action.*

The action of G on X is called *transitive* if there is only one orbit, i.e. given $x_0, x_1 \in X$, there exists $g \in G$ such that $x_0 = g \cdot x_1$.

C Manifold Theory

C.1 Topological Manifolds and Surfaces

Definition C.1. *Topological Manifold.*

X is an *topological n-manifold* if it is Hausdorff, has a countable basis and each $x \in X$ has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^n$. [Mun00]

Remark C.2. We refer to topological n-manifolds simply as 'n-manifolds' throughout.

For our purposes, we need only define surfaces as 2-manifolds.

Definition C.3. *Surface.*

A *surface* is a 2-manifold. [Mun00]

Definition C.4. *Orientable Surface.*

A surface is said to be *orientable* if it does not contain a homeomorphic copy of the Möbius strip (an object with only one side). [Mun00]

C.2 Coordinates

Definition C.5. *Coordinate Chart.*

Let M be an n-manifold. A *coordinate chart*[19] on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to \varphi(U)$ is a homeomorphism from U to some $\varphi(U) \subseteq \mathbb{R}^n$.

U is called a *coordinate domain.* φ is called a *(local) coordinate map* and the component functions $(x^1, \ldots, x^n)$ of φ, defined by

$$\varphi(p) = (x^1(p), \ldots, x^n(p)),$$

are called *local coordinates* on U. [Lee12]

Definition C.6. *Atlas.*

Let M be an n-manifold. Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ be a collection of charts of M. Suppose that

$$M = \bigcup_\alpha U_\alpha,$$

then we call $\mathcal{A}$ an *atlas* of M. [Lee12]

Definition C.7. *Transition Map.*

Let M be an n-manifold. Let $(U_0, \varphi_0), (U_1, \varphi_1)$ be two charts of M such that $U_0 \cap U_1 \neq \emptyset$. Then

$$\varphi_1 \circ \varphi_0^{-1} : \varphi_0(U_0 \cap U_1) \to \varphi_1(U_0 \cap U_1)$$

is called the *transition map* from φ_0 to φ_1. [Lee12]

Definition C.8. *Compatible Charts.*

Let M be an n-manifold. Let $(U_0, \varphi_0), (U_1, \varphi_1)$ be two charts of M. If the transition map $\varphi_1 \circ \varphi_0^{-1}$ is a homeomorphism, (U_0, φ_0) and (U_1, φ_1) are said to be *compatible.* [Lee10]

C.3 Euclidean Balls

Definition C.9. Euclidean Ball.

Let M be an n-manifold. Let $B \subset M$ be an open subset. Suppose B is homeomorphic to some ball in $\mathbb{R}^n$. Then we call B a *Euclidean ball*[20] in M. [Lee10]

[19] Or simply *chart.*

[20] Sometimes called a *Euclidean disk* when $n = 2$.

Definition C.10. Regular Euclidean Ball. Let M be an n-manifold. Then some Euclidean ball $B \subset M$ is called a *regular ball* if the following properties hold: [Lee10]

1. There exists some Euclidean ball $B' \subset M$ containing $\bar{B}$, the closure of B.
2. For some $r > 0$, there exists some chart $\varphi : B' \to B_{2r}(0) \subset \mathbb{R}^n$ that sends $\bar{B}$ onto $\bar{B}_r(0)$.

C.4 Differentiability

Definition C.11. *Partial Derivative.*

Suppose $U \subseteq \mathbb{R}^n$ is open and $F : U \to \mathbb{R}$ is a real-valued function. For any $a = (a^1, \ldots, a^n) \in U$ and some $j \in \{1, \ldots, n\}$, the *jth partial derivative* of f at a is given by [Lee12]

$$\frac{\partial f}{\partial x^j}(a) = lim_{h \to 0} \frac{f(a^1, \ldots, a^j + h, \ldots, a^n) - f(a^1, \ldots, a^j, \ldots, a^n)}{h}.$$

Definition C.12. *Jacobian.*

Suppose $U \subseteq \mathbb{R}^n$ is open and $F : U \to \mathbb{R}^m$ is a vector-valued function. Then we can write the coordinates of $F(x)$ as

$$F(x) = (F^1(x), \ldots, F^m(x)).$$

These m functions $F^1, \ldots, F^m : U \to \mathbb{R}$ are called the *component functions* of F.

The matrix

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1} & \cdots & \frac{\partial F^m}{\partial x^n} \end{pmatrix}$$

is called the *Jacobian* of F. [Lee12]

Definition C.13. *Continuously Differentiable.*

Let $j \in \{1, \ldots, n\}$, $i \in \{1, \ldots, m\}$. Suppose $U \subseteq \mathbb{R}^n$ is open and $F : U \to \mathbb{R}^m$ is a vector-valued function. Write the component functions of $F(x)$ as $F^1, \ldots, F^m : U \to \mathbb{R}$. If functions $\frac{\partial F^i}{\partial x^j}$ are continuous for all i, j, then F is said to be *continuously differentiable*, denoted as C^1. [Lee12]

Definition C.14. *Partial Derivatives of Order k.*

Let $j_0, j_1 \in \{1, \ldots, n\}$, $i \in \{1, \ldots, m\}$. Suppose $U \subseteq \mathbb{R}^n$ is open and $F : U \to \mathbb{R}^m$ is a vector-valued function that is C^1. Write the component functions of $F(x)$ as $F^1, \ldots, F^m : U \to \mathbb{R}$. The *second-order partial derivative* of some partial derivative $\frac{\partial F^i}{\partial x^{j_0}}$ is given by

$$\frac{\partial^2 F^i}{\partial x^{j_1} \partial x^{j_0}} = \frac{\partial}{\partial x^{j_1}} \left(\frac{\partial F^i}{\partial x^{j_0}} \right).$$

Proceeding inductively, we can define the *k-th order partial derivatives* to be the partial derivatives of those partial derivatives of order $k - 1$.

If $k \geq 0$ and all the partial derivates of F of order less than or equal to k exist and are continuous on U, then we say that F is *k-times continuously differentiable*, written as C^k. [Lee12]

Remark C.15. A consequence of this definition is that C^0 functions are just continuous functions.

Definition C.16. *Smooth Function.*

A function of class C^k for all $k \geq 0$, i.e. of class C^∞ (infinitely differentiable), is said to be *smooth.* [Lee12]

Definition C.17. *Diffeomorphism.*

Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open subsets. Then some homeomorphism $F : U \to V$ is called a *diffeomorphism* if it is smooth and its inverse F^{-1} is also smooth. [Lee12]

C.5 Smooth Manifolds

Definition C.18. *Smoothly Compatible Charts.*

Let M be an n-manifold. Let $(U_0, \varphi_0), (U_1, \varphi_1)$ be two charts of M. Then $(U_0, \varphi_0), (U_1, \varphi_1)$ are said to be *smoothly compatible* if either the transition map $\varphi_1 \circ \varphi_0^{-1}$ is a diffeomorphism, or $U \cap V = \emptyset$. [Lee12]

Definition C.19. *Smooth Atlas.*

Let M be an n-manifold. Let $\mathcal{A}$ be an atlas of M. $\mathcal{A}$ is called a*smoothl atlas* if any two charts in $\mathcal{A}$ are smoothly compatible with each other. [Lee12]

Definition C.20. *Maximal Smooth Atlas.*

Let M be an n-manifold. Let $\mathcal{A}$ be a smooth atlas of M. Then $\mathcal{A}$ is said to be *maximal* if it is such that $\mathcal{A} \not\subset A_0$ for any larger smooth atlases A_0 of M. [Lee12]

Definition C.21. *Smooth Manifold.*

Let M be an n-manifold. Let $\mathcal{A}$ be a maximal smooth atlas of M. Then we call $\mathcal{A}$ a *smooth structure* on M.

The pair $(M, \mathcal{A})$ is called a *smooth manifold.* [Lee12]

Remark C.22. It is standard to just refer to the pair $(M, \mathcal{A})$ by the manifold itself, M, if the context is clear.

perhaps some additional stuff on smooth structures to make it absolutely clear? see [Lee12] prop. 1.17

D Linear Algebra

D.1 Fields and Vector Spaces

Definition D.1. *Field.*

A *field* is a set $\mathbb{F}$ together with binary operations $+$ (the *additive operation*) and $\cdot$ (the *multiplicative operation*), such that the following properties hold:

1. $\mathbb{F}$ is closed under $+$ and $\cdot$, i.e. for $a, b \in \mathbb{F}$ we have $a + b, a \cdot b \in \mathbb{F}$.
2. $+$ and $\cdot$ are associative, i.e. $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
3. Let also $c \in \mathbb{F}$. Then $a \cdot (b + c) = a \cdot b + a \cdot c$.
4. There exists some element $0 \in \mathbb{F}$ such that $a + 0 = a$ for all $a \in \mathbb{F}$.
5. For all $a \in \mathbb{F}$ there exists some $-a \in \mathbb{F}$ such that $a + -a = 0$.
6. There exists some element $1 \in \mathbb{F}$ such that $a \cdot 1 = a$ for all $a \in \mathbb{F}$.
7. For all $a \in \mathbb{F}$ such that $a \neq 0$ there exists some $a^{-1} \in \mathbb{F}$ such that $a \cdot a^{-1} = 1$. [Hef17]

Definition D.2. *Vector Space.*

Let $\mathbb{F}$ be a field with additive operation $+_{\mathbb{F}}$ and multiplicative operation $\cdot_{\mathbb{F}}$. Let V be a non-empty set. Call elements $a, b \in \mathbb{F}$ *scalars* and elements $\vec{u}, \vec{v}, \vec{w} \in V$ *vectors*.

V together with binary operations $+$ (*vector addition*) and *scalar multiplication* (denoted by $a\vec{u}$ for the scalar product of a with $\vec{u}$) is called a *vector space* over $\mathbb{F}$ if the following properties hold:

1. Vector addition $+$ is associative, i.e. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.
2. Vector addition is commutative, i.e. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
3. There exists an element $\vec{0} \in V$, called the *zero vector*, such that for all $\vec{u} \in V$ we have $\vec{u} + \vec{0} = \vec{u}$.
4. For every $\vec{u} \in V$, there exists some $-\vec{u} \in V$ such that $\vec{u} + -\vec{u} = 0$.
5. $a \cdot_{\mathbb{F}} (b\vec{u}) = (a \cdot_{\mathbb{F}} b)\vec{u}$.
6. $1 \cdot_{\mathbb{F}} \vec{u} = \vec{u}$, where $1 \in \mathbb{F}$ is the multiplicative identity.
7. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$.
8. $(a +_{\mathbb{F}} b)\vec{u} = a\vec{u} + b\vec{v}$. [Hef17]

Remark D.3. It is conventional to drop the field multiplicative notation $\cdot_{\mathbb{F}}$ when the context is clear.

D.2 Linear and Affine Independence

Definition D.4. *Linear Independence.*

Let V be a vector space over $\mathbb{F}$. Let $\{\vec{v_0}, \dots, \vec{v_n}\} \subseteq V$. We say that the vectors $\vec{v_0}, \dots, \vec{v_n}$ are *linearly independent* if for each $i \in \{0, \dots, n\}$ there does NOT exist $\{c_0, \dots, c_{i-1}, c_{i+1}, \dots, c_n\} \subseteq \mathbb{F}$ such that

$$\vec{v_i} = c_0\vec{v_0} + \cdots + c_{i-1}\vec{v_{i-1}} + c_{i+i}\vec{v_{i+1}} + \cdots + c_n\vec{v_n}.$$

In other words, no member of the subset of vectors can be written as a linear combination of the others.

If $\vec{v_0}, \dots, \vec{v_n}$ are not linearly independent then they are said to be *linearly dependent*. [Hef17]

affine independence
convexity
convex hull + for finite set
canonical
affine independence iff linear independence of difference vectors
affine independence is preserved by subsets
any m x n matrix with rank n has a matrix W$^{t}suchthatW^{t}W = I_n$
linear system Ws=x-v$_{0}hasexactlyonesolutionwhenWhasfullcolumnrank$
over R, ever linear (hence affine) map is continuous

Theorem D.5. *vector valued map is continuous when all its coordinate functions are continuous*

affine maps are equal iff they agree on all vertices

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