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On Knot Invariants and the Alexander Polynomial

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Contents

1 Introduction: Knot Theory

Before we consider the Alexander polynomial, we must first introduce knot theory. Knot theory is, as the name would suggest, the study of knots. It is then only natural to ask one's self 'what is a knot?'. With some thought, many of us could define what we know as a 'knot' in a non-mathematical context. It is then interesting perhaps, to work from this everyday knowledge to develop a mathematical definition for a knot, and to see what sort of assumptions and limitations occur.

1.1 Knots in a Piece of String

We begin with an informal definition for knots in the context of a piece of string. Suppose we have a piece of string, with the ends intersecting (attached). We assume that the string is breadthless, so when laid out straight end-to-end it will represent a line segment.

Figure 1: A piece of string. Note that the end points are the same.

We ask ourselves two questions: firstly, does the arrangement shown above constitute a knot? Second, how can we represent this arrangement mathematically? For the latter, we simply note that (since we've before mentioned that the string is breadthless) the string is simply a curve in $\mathbb{R}^3$. Although the string is of course strewn out across the table due to gravity and therefore upon first sight only appears to be in two-dimensions, we can clearly show this is not the case as we can lift the string up.

We also note that the curve represented by the string is simple and closed. We remind ourselves of what it means to be simple and closed.

Definition 1.1. A curve is said to be *simple* if it does not intersect itself, unless at its endpoints.

Figure 2: A simple curve.

Definition 1.2. A curve is said to be *closed* if it encloses an area within the space, i.e. its endpoints are equal.

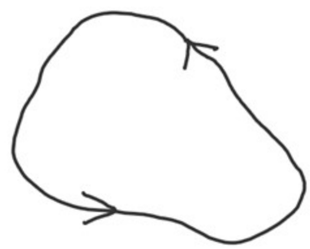


Figure 3: A closed curve. Note that this curve is also simple.

Now we consider the former question. In our everyday understanding of the real world, the string as shown above is not considered a knot. However, with some simple manipulation, we can tie a knot in this piece of string. For now, we shall call this unmodified closed string an *unknot*.

We now twist the string, keeping the ends attached.

Figure 4: The string after being twisted. Observe that one part of the string lies on top of another. They do NOT intersect.

Upon glancing at figure 4, one might mistakenly think that the curve represented by the string is no longer simple. However, recall that the string exists in three-dimensions, so it does not intersect (except at the ends) as at no point does one part of the string pass through another, only over. We also note that the perspective of this photograph gives rise to the idea that this knot has been 'projected' onto two-dimensions such that it is easy for us to view; each point on the curve could be given a two-dimensional coordinate on the image. This is why it appears as though the knot intersects despite it not doing so. Since we have yet to tie a knot, we shall once again refer to this an unknot.

In order to tie a knot in the string, we shall have to separate the ends and later reattach them. We separate the ends of our string, bring one end over the other and then pass this end through the hole that is created in the string. We then reattach the ends. In everyday terms, this is known as the overhand knot. This process is shown in figures 5–7 below.

Figure 5: Step 1 of tying an overhand knot.

Figure 6: Step 2 of tying an overhand knot.

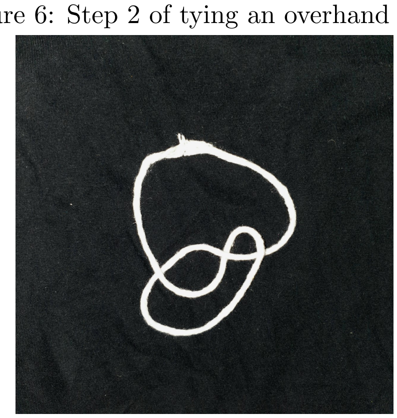


Figure 7: Step 3 of tying an overhand knot.

As before, we note that this string is still simple, and since we reattached the ends, still closed. However, this is now a knot as we traditionally know it. Hence it is no longer an unknot. This raises a number of questions. We ask what distinguishes knots from unknots in a mathematical sense? How can we define knots? What if we were to repeat this process an infinite number of times? This is where we must introduce limitations and abstraction in order to make our definition of knots mathematically viable.

1.2 Modelling Knots as Polygonal Curves

In order to develop an intuitive, useful definition of knots that is as close to real life as possible, we shall model our knots as *polygonal curves* rather than continuous curves. We remind ourselves of the definition of polygonal curves:

Definition 1.3. A *polygonal curve* is a connected series of line segments between consecutive points $P_1, P_2, \ldots, P_n \in \mathbb{R}^3$ for some $n \in \mathbb{N}$, i.e. it is the union of the line segments $[P_1, P_2], [P_2, P_3], \ldots, [P_{n-1}, P_n] \subset \mathbb{R}^3$. A polygonal curve is *closed* if $P_n = P_1$. A polygonal curve is *simple* if the line segments do not intersect, except at end points.

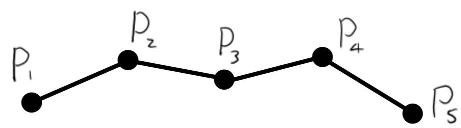


Figure 8: A polygonal curve. This curve is not closed, but it is simple.

The main reason for modelling our knots this way is that it limits our knots to exclude *wild knots*, knots with an infinite number of crossings. In our piece of string it would of course be impossible to have an infinite amount of crossings, so modelling our knots this way brings our mathematical definition closer to real life. It takes little observation to note that our piece of string cannot be drawn accurately as a polygonal curve as it can a continuous curve, however it can be approximated as such, and we shall see that it has little bearing on the real life accuracy of our definition.

So, we have seen that we can model our string as a simple, closed, polygonal curve in $\mathbb{R}^3$. We have shown that we can tie a knot (specifically an overhand knot) into this piece of string. Now we can produce some definitions.

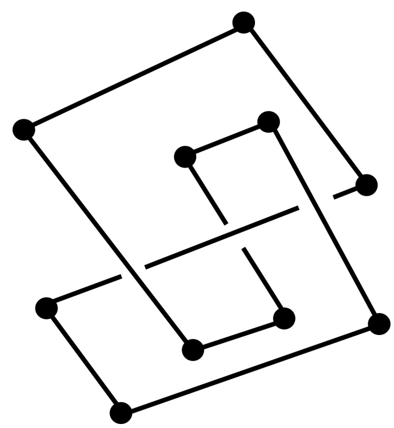


Figure 9: The overhand knot from figure **??** as a polygonal curve. Note here that crossings are represented by breaks in the edges, where another edge that passes over the broken edge represents that part of the knot passing over, this convention will be used throughout.

1.3 Defining Knots

Definition 1.4. A *knot* is a simple, closed, polygonal curve in $\mathbb{R}^3$.

This is our first definition of a knot. It is simple yet incredibly powerful. We now look at the points connecting line segments in detail.

Definition 1.5. Two or more points $P_i \in \mathbb{R}^n$ are said to be *collinear* if they all lie on the same straight line. Otherwise they are *non-collinear*.

Remark 1.6. Any two points in $\mathbb{R}^n$ are obviously collinear as you can draw a line segment connecting them.

If three or more consecutive points in our knot are collinear, then we can remove the middle points without changing the shape of the knot, i.e. we can say the knot before removing the points and the knot after removing the points are equivalent. We note that the ordered set of points of the latter knot is a subset of the ordered set of points of the former knot.

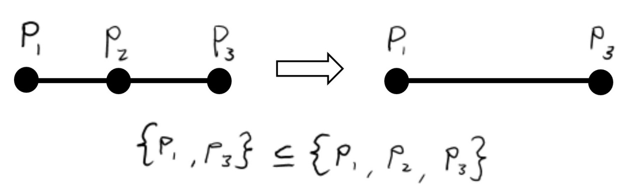
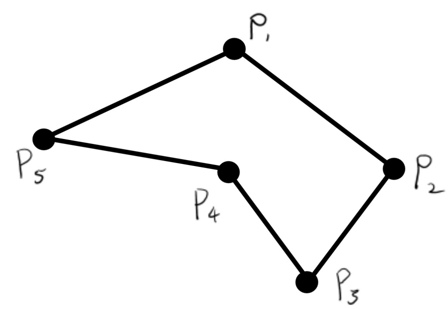


Figure 10: Removing P_2 does not change the shape of this line, and hence does knot change the shape of the knot containing this line. Notice that $\{P_1, P_3\} \subset \{P_1, P_2, P_3\}$.

Using this, we can define *vertices*:

Definition 1.7. The points $(P_1, P_2, \ldots, P_n)$ are called the *vertices* of a knot if there exists no proper ordered subset of $\{P_1, P_2, \ldots, P_n\}$ that defines the same knot.

Example 1.8. Consider the following polygonal curve:



The vertices are labelled P_i, for $i \in \mathbb{N}$. This polygonal curve is a knot because it is simple and closed.

1.3.1 Oriented Knots

We define oriented knots, which become useful later on.

Definition 1.9. An *oriented knot* is a knot along with an ordering of its vertices. The ordering must be such that it determines the original knot. Two orderings are equivalent if they differ by some cyclic permutation. [15]

Example 1.10. Consider K, where K is given by the line segments

$$[P_1, P_2], [P_2, P_3], \ldots, [P_{n-1}, P_1] \subset \mathbb{R}^3$$

. Then an ordering of K is given by $\{P_1, \ldots, P_{n-1}\}$. An equivalent ordering is given by $\{P_3, \ldots, P_{n-1}, P_1, P_2\}$.

The ordering $\{P_{n-1}, \ldots, P_1\}$ is not equivalent to the previous two as it cannot be obtained via a cyclic permutation, and thus gives an alternate orientation.

1.3.2 Links

We also define links. Going back to our string, we have that a 'real life' link is just two knots with one tied inside the other, as shown below.

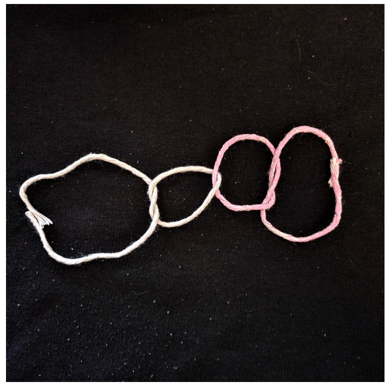


Figure 11: Two knots tied together. Both are overhand knots. The knots are different colours.

We want to express this mathematically.

Definition 1.11. A *link* of n components is a subset of $\mathbb{R}^3$ consisting of n disjoint, simple closed polygonal curves (knots). [14]

Figure 12: A link with two components. Both knots are drawn in different colours.

Remark 1.12. A link with one component is just a knot. [14]

So we can think of links as just a collection of knots that are perhaps tied together.

1.3.3 Defining Knots as an Embedding of the Unit Circle

We also present an alternate definition of knots. This definition does not model knots as polygonal curves. It will not be as useful as our original definition for defining equivalence of knots etc., but it is still insightful nonetheless. We take our definition from ideas presented in the opening chapter of Lickorish's book. [14]

Definition 1.13. A *knot* is a smooth embedding $K : S^1 \to \mathbb{R}^3$, i.e. an embedding of the unit circle into three-dimensional Euclidean space. [14]

1.4 Equivalence of Knots and Links

In the real world, if I have tied a knot in a closed piece of string and then decide to deform the string however I please, we would still say that this is the same knot. Only if I disconnected the ends and tied a different knot in the string would one say that it is a different knot. This is true mathematically too. We can deform our curves in certain ways without changing the knot itself, any deformation of this sort is equivalent to the original knot; they form an equivalence class.

We shall first define these so-called 'elementary deformations' before proving that they form equivalence classes. The results in this section are analogous for both knots and links, where one should recall that a link of one component is a knot.

Definition 1.14. Suppose we have two knots, $K_1 \subseteq L_1$ and $K_2 \subseteq L_2$, where L_1 and L_2 are links, K_1 is defined by the polygonal curve $\{P_1, P_2, \ldots, P_n\}$, and K_2 is defined by the polygonal curve $\{P_0, P_1, P_2, \ldots, P_n\}$. Then L_2 is an *elementary deformation* of L_1 if the following two conditions are satisfied:

1. P_0 is not collinear with P_1 and P_n.
2. The triangle bounded by the edges $[P_0, P_1]$, $[P_1, P_n]$ and $[P_n, P_0]$ intersects L_1 ONLY at the edge $[P_1, P_n]$.

L_1 is similarly an *elementary deformation* of L_2. [15]

If we didn't have condition (1), then two completely identical polygonal curves with a different number of points would be considered different knots, this is obviously not a useful definition of equivalence. If we add a new point to our knot but it doesn't change shape, then it is still the same knot. The purpose of condition (2) is to ensure the link does not cross itself.

Thus, we define equivalence of knots:

Definition 1.15. Two KNOTS (i.e. links with one component) K_0 and K_n are *equivalent* if there exists a sequence of knots $K_0, K_1, \ldots, K_n$ such that for all $i \in \{0, \ldots, n\}$ we have K_{i+1} an elementary deformation of K_i. [15]

And similarly, equivalence of links:

Definition 1.16. Two LINKS L_0 and L_n are *equivalent* if there exists a sequence of links $L_0, L_1, \ldots, L_n$ such that for all $i \in \{0, \ldots, n\}$ we have L_{i+1} an elementary deformation of L_i. [15]

We immediately follow with a proposition:

Proposition 1.17. *The sets of knots (or links) equivalent under elementary deformations form an equivalence class.*

Proof. We prove the proposition for knots. The proof is analagous for links.

Let $\sim$ be the relation of knot equivalence, i.e. if $K_0 \sim K_n$ then there exists a sequence of knots $K_0, K_1, \dots, K_n$ such that for all $i \in \{0, \dots, n\}$ we have K_{i+1} an elementary deformation of K_i.

We have that $K_0 \sim K_0$, the sequence of knots is simply K_0. So $\sim$ is **reflexive**.

Suppose that $K_0 \sim K_n$, then there exists a sequence of knots $K_0, K_1, \dots, K_n$ such that for all $i \in \{0, \dots, n\}$ we have K_{i+1} an elementary deformation of K_i. By the definition of elementary deformations, we also have that for all $i \in \{0, \dots, n\}$, K_i an elementary deformation of K_{i+1}. Hence we have that $K_n \sim K_0$ with sequence $K_n, K_{n-1}, \dots, K_0$ of elementary deformations. So $\sim$ is **symmetric**.

Suppose we have $K_0 \sim K_n$ with sequence

$$K_0, K_1, \dots, K_n,$$

and $K_n \sim K_m$ with sequence

$$K_n, K_{n+1}, \dots, K_m.$$

Then clearly $K_0 \sim K_m$ with sequence

$$K_0, \dots, K_n, K_{n+1}, \dots, K_m.$$

So $\sim$ is **transistive**.

We have shown $\sim$ is reflexive, symmetric and transistive. Hence, it is an equivalence relation. $\square$

We turn our attention back to the previously mentioned unknot. How can we define the unknot?

Definition 1.18. An *unknot* is defined as a knot in the equivalence class of some knot given by three non-collinear points.

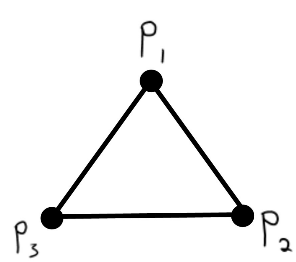


Figure 13: The unknot.

Remark 1.19. When we say 'unknot', we are referring to either the knot given by three non-collinear points shown above or the equivalence class of the unknot. It should be clear from the context which of these we refer to in a given instance.

So now we have determined what it means for knots to be equivalent.

1.5 Knot Diagrams

Remark 1.20. Throughout this section, we give our definitions and theorems in terms of knots. However, these definitions and theorems are analagous for links. We can have link projections and thus link diagrams.

We first define knot projections.

Definition 1.21. A *knot projection* is simply a map $P : \mathbb{R}^3 \to \mathbb{R}^2$ such that $P(x, y, z) = (x, y)$ for any point (x, y, z) on a knot K.

So a projection simply takes our three-dimensional curve and projects it onto a two-dimensional plane. This makes it very easy to draw our knots, we just draw the projection. But what about when two points on our curve share the same x and y coordinates? When projected we will lose any information regarding which of these points lies above the other in the z-axis! Cue knot diagrams. A knot diagram is an easy way to preserve this information whilst still having our knots easy to draw and hence visualise. First, we must discuss a limitation of certain knot projections.

Definition 1.22. A knot projection is called a *regular projection* if no THREE points on the knot project to the same point on the plane, i.e. no THREE points on the knot share the same x and y coordinates. [20]

In order to draw a diagram, we need our projection to be regular. If our knot does not give a regular projection, we can thankfully turn to its equivalence class to find one that does.

Theorem 1.23. *Any knot either gives a regular projection, or is equivalent to a knot that does so.*

Proof. Suppose we have a knot K that does not give a regular projection. Consider a point $\mathbf{x} = (x, y, z)$ on K that shares its x and y coordinates with arbitrarily many other points on K.

Consider the point $\mathbf{x_0} = (x+\epsilon, y, z)$, where the triangle bounded by the edge E on which (x, y, z) lies and the two edges E_1 and E_2 connecting the vertices at either end of E to the point $\mathbf{x_0}$ does not intersect the knot anywhere except E.

Then the knot wherein E is replaced by E_1 and E_2 is an elementary deformation of K and hence equivalent. We no longer have the issue of $\mathbf{x}$ being a point our knot and hence sharing its x and y coordinates with other points.

Repeat this process with all points that share x and y coordinates with more than one other point and you get an equivalent knot that gives a regular projection. □

Definition 1.24. A *knot diagram* is a picture of a regular projection of a knot. Knot diagrams are often drawn as continuous curves, despite the fact we model them as polygonal curves. Points where the projection instersects itself are drawn so as to preserve which point of the original knot crosses above the other in the z-axis, as shown below:

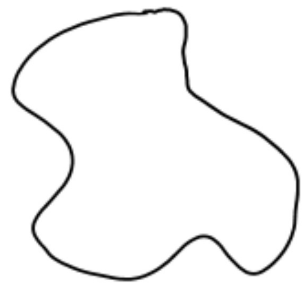


Figure 14: A diagram of the unknot.

Figure 15: A diagram of the *trefoil knot*, the simplest nontrivial knot.

Figure 15 shows the *trefoil knot*, the simplest non-trivial knot. This knot will reappear numerous times throughout.

Remark 1.25. Throughout this project, we shall draw our knot diagrams as smooth curves. One might wonder why we do not stay true to our definition of knots as polygonal curves. We draw our knots this way for a number of reasons, but mostly as they become easier to draw and easier to interpret.

Our definition is not invalidated however, as if we wish, we can interpret our smooth diagrams as polygonal curves with a number of vertices that tends to infinity, such that they appear to be smooth when viewed.

Knot diagrams are a useful tool for determining when knots are equivalent, as we can manipulate the picture in certain ways so as to find equivalent knots. These manipulations are known as *Reidemeister moves* (after German mathematician Kurt Reidemeister (1893 - 1971)). Knot diagrams are also aesthetic and easy to understand.

To summarise, *knot projections* are functions that take knots to the two-dimensional plane and *knot diagrams* are just an intuitive way of representing knot projections whilst preserving the properties of the original knot's crossings. We will draw many knot diagrams throughout; they will be useful tools for us to explore properties of knots and eventually derive the Alexander polynomial.

Remark 1.26. Note that figure 12 from earlier is a diagram. It is in fact a *link diagram.*

We introduce some definitions that will be useful later on.

Definition 1.27. A *strand* of a knot diagram is a piece of the knot that goes from one undercrossing to another with only overcrossings inbetween, as shown below: [1]

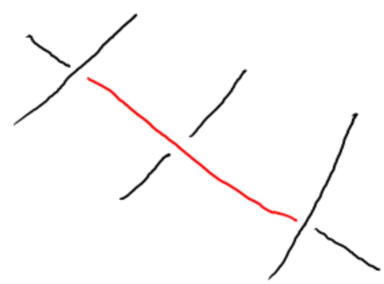


Figure 16: A strand, highlighted in red. Note that this is only a portion of the full knot diagram.

Definition 1.28. The *crossing number* of a knot diagram is the number of crossings on the diagram.

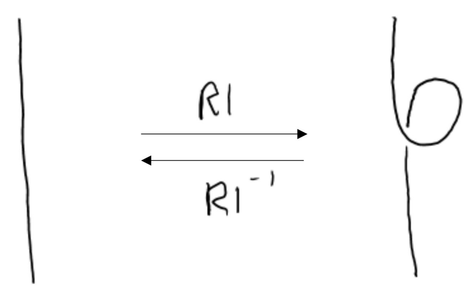
Proposition 1.29. *The number of strands on a knot diagram is equal to the crossing number.*

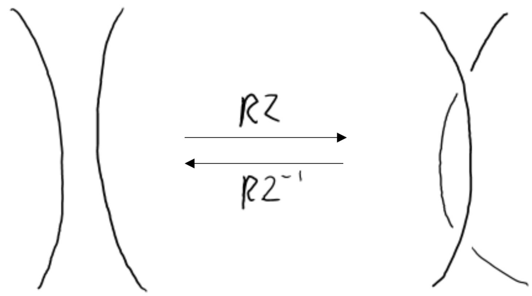
Proof. A new strand begins when we have a crossing, hence this is trivial. $\square$

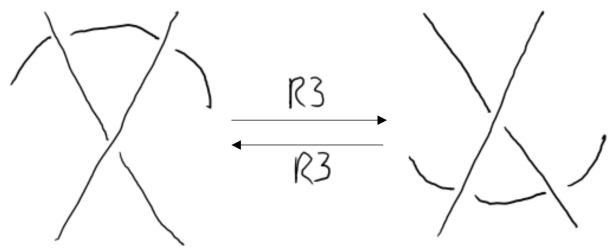
1.6 Reidemeister Moves

Let us introduce the Reidemeister moves.

Definition 1.30. The three *Reidemeister moves* that can be applied to knot diagrams are the *twist/untwist (R*1*)*, *poke/unpoke (R*2*)* and *slide (R*3*)*. These moves are reversible, we shall denote their inverses by *untwist* ($R1^{-1}$), *unpoke* ($R2^{-1}$) and *slide* ($R3^{-1}$) respectively. Note that $R3$ and $R3^{-1}$ are the same move, and neither changes the number of crosses in the diagram, so we shall use them interchangeably. The Reidemeister moves are shown below: [19]







This leads us to *Reidemeister's theorem*, a theorem that relates diagrams to our notion of equivalence.

1.7 Reidemeister's Theorem

Theorem 1.31 (Reidemeister's Theorem)**.** *Suppose two knots K_1 and K_2 are equivalent. Then the diagram of K_2 can be obtained via a finite application of the Reidemeister moves (or their inverses) on the diagram of K_1 and vice versa. [19]*

Proof. A proof was first detailed in Reidemeister's 1927 paper *Elementare Begründung der Knotentheorie (Eng.: Elementary Justification of the Knot Theory).* [**?**] We shall not go into detail here, since said proof is beyond the scope of this project. □

Reidemeister's theorem is incredibly powerful. It allows us to clearly identify equivalent knots using just their diagrams.

Example 1.32. The two knot diagrams shown below represent equivalent knots, as they differ only by the application of $R2$.

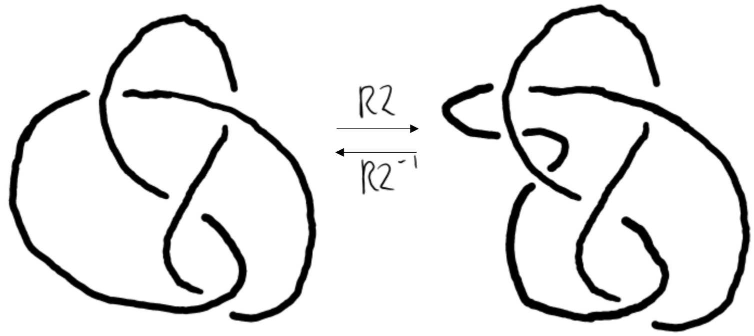


Figure 17: Here we have applied $R2$. These diagrams thus represent knots that are equivalent.

1.8 Knot Invariants

Definition 1.33. A *knot invariant* is a function that assigns a mathematical expression to a knot. Knot invariants are preserved by equivalence, i.e. the knot invariant must take two equivalent knots to the same expression. [23]

Example 1.34. A rather useless example of an invariant is the *constant invariant*. Let our function take all knots to a constant c. Then this is, by definition, a knot invariant, albiet a very unpractical one. This knot invariant tells us nothing about differences between knots.

1.8.1 Tricolourability

We will now give a significantly more useful example of a knot invariant.

Definition 1.35. We say that a knot diagram is *tricolourable* if each of the strands in the diagram can be coloured with one of three colours such that at each crossing, either:

- three DIFFERENT colours come together, **or**
- the SAME colour comes together from all three strands.

We also require that at least two distinct colours be used in the diagram (so a diagram cannot be composed of all one colour). [1]

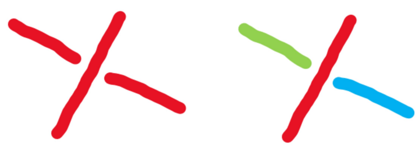
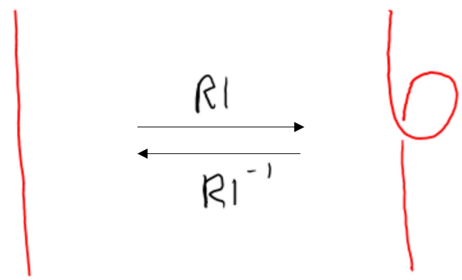


Figure 18: Crossings must either be composed entirely of one colour (left), or be composed of three distinct colours (right).

Theorem 1.36. *If the diagram of a knot is tricolourable, then its tricolourability is preserved by the Reidemeister moves.[1]*

Proof. We sketch a proof to show that tricolourability is preserved for each of the Reidemeister moves.

For $R1$, we separate one strand into two separate strands. If we leave both these new strands the same colour, then tricolourability is still satisfied. This is shown below.

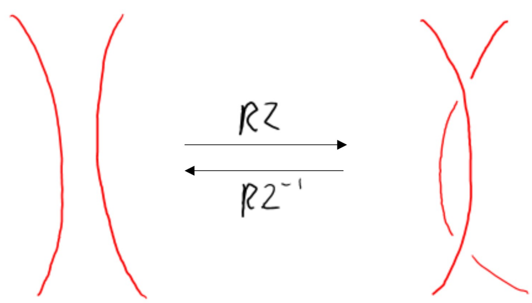


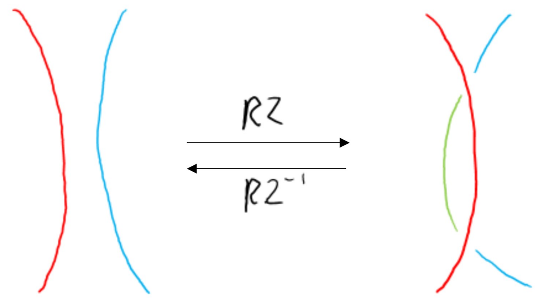
We have that removing a crossing via $R1^{-1}$ also similarly preserves tricolourability, as seen in the figure above. [1]

Now we consider $R2$. $R2$ introduces two new crossings.

- If the original two strands are different colours, then we colour the new strand with the third colour and tricolourability is preserved.
- If the two original strands are the same colour, we colour the new strand with the same colour as the other two and tricolourability is preserved.

This is shown below.

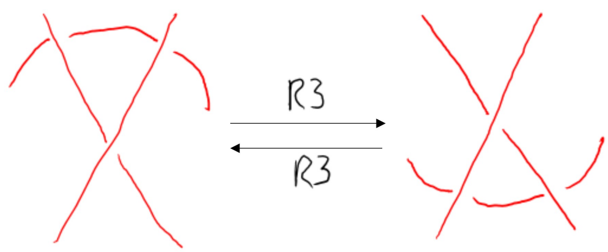


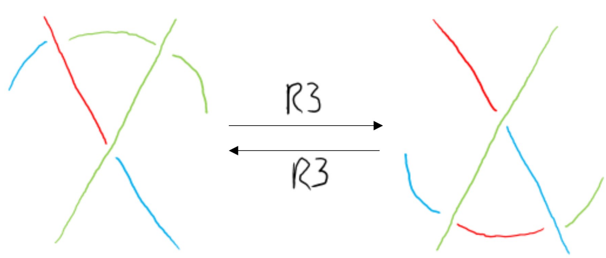


We have that $R2^{-1}$ also preserves tricolourability, as seen in the figures above.

- If all of the strands in the diagram are the same colour, we colour the new strands created by this move with that colour.
- If three distinct colours come together at both the crossings on the diagram, we colour the two new strands as follows. The strand that remains a straight line stays the colour that it is. The strand that is created from the three strands that have moved become the colour that appeared twice in the diagram before $R2^{-1}$ acted on it. [1]

For $R3$ (and similarly it's equivalent inverse, $R3^{-1}$), tricolourability is preserved in the diagram, as shown below.





$\square$

Corollary 1.37. *If a knot is tricolourable, then any equivalent knot is also tricolourable.*

This corollary leads to an obvious result:

Corollary 1.38. *Tricolourability is a knot invariant. [12]*

Example 1.39. The *trefoil* is tricolourable, as shown below.

Figure 19: Tricolouring of the trefoil knot.

Example 1.40. The *unknot* is not tricolourable.

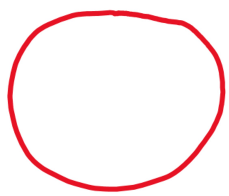


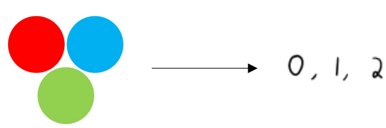
Figure 20: This unknot diagram can only be coloured with one colour.

1.8.2 Integer Labellings

In this section we describe another, equivalent knot invariant to tricolourability.

Recall our condition at a crossing for tricolourability: our three strands involved at a crossing are either all the same colour or three distinct colours.

Suppose that, instead of labelling our strands with colours as before, we were to label them with three integers: $0, 1$ and 2. [15]



Now, how does our condition for tricolourability at crossings translate into our new setting?

Proposition 1.41. *Our condition for tricolourability at crossings is preserved in our new setting if the following holds:*

- *our overcrossing is labelled with an integer x and the other two strands are labelled with integers y and z,*
- $2x - y - z \equiv 0 \mod 3$. *[15]*

Proof. We shall prove that these two conditions are equivalent.

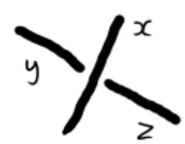
First, suppose that our three strands are all labelled with the same integer x, as shown below.

Then we see that clearly

$$2(x) - x - x \equiv 0 \mod 3,$$

so our condition is satisfied.

Now suppose that all three strands are labelled with different integers, x, y and z, where x is the integer used to label the overcrossing. This is shown below.



Note that swapping y and z has no effect on the value of $2x - y - z$. With this in mind, we try all distinct combinations of x, y and z:

- if $x = 0$, $y = 1$ and $z = 2$, $2(0) - 1 - 2 = -3 \equiv 0 \mod 3$,
- if $x = 1$, $y = 0$ and $z = 2$, $2(1) - 0 - 2 = 0 \equiv 0 \mod 3$,
- if $x = 2$, $y = 0$ and $z = 1$, $2(2) - 0 - 1 = 3 \equiv 0 \mod 3$.

So our condition is satisfied.

Hence our condition for tricolourability at a crossing is equivalent to our condition for labelling the strands with the integers $0, 1$ and 2 given above. □

2 Deriving the Alexander Polynomial: Fox's Algorithm

Now that we have established a good understanding of the fundamentals of knot theory, we can begin to derive the Alexander polynomial itself. But we must first begin with a discussion of the fundamental group; this serves as our motivation for the Alexander polynomial.

2.1 Homotopy of Paths

We define paths and homotopy, these definitions are taken from Allen Hatcher's book on *Algebraic Topology.* [9]

Definition 2.1. A *path* in some space X is a continuous map $f : [0, 1] \to X$.[9]

Remark 2.2. Later we will take X to be a specific space, the subset of $\mathbb{R}^3$ that is all points not occupied by a knot. However during this section we will define things for some general space X.

This is an intuitive definition. It is easy to understand. We now introduce what is known as a homotopy of paths. This formalises the idea of deforming our paths whilst keeping the endpoints as they are. Think of a piece of string fixed at its endpoints, we can deform it but not change it's endpoints. The collection of all possible deformations is the homotopy of the path, defined mathematically as such:

Definition 2.3. A *homotopy of paths* in X is a family $f_t : [0, 1] \to X$, with $0 \leq t \leq 1$, such that:

1. The endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t.
2. The map $F : [0, 1] \times [0, 1] \to X$ where $F(s, t) = f_t(s)$ is continuous.

We say that two paths f_0 and f_1 belonging to the same family f_t are *homotopic*, denoted $f_0 \simeq f_1$. [9]

Proposition 2.4. *All paths with given endpoints x_0 and x_1 are homotopic to one another in $\mathbb{R}^3$.*

Proof. Let f_0 and f_1 be paths in X such that $f_0(0) = f_1(0) = x_0$ and $f_0(1) = f_1(1) = x_1$ (so f_0 and f_1 have the same endpoints). We claim that f_0 and f_1 are homotopic via the homotopy $f_t(s) = (1 - t)f_0(s) + tf_1(s)$.

We note that the line segment between $f_0(s)$ and $f_1(s)$ is linearly parametrised as $f_0(s) + t(f_1(s) - f_0(s)) = (1 - t)f_0(s) + tf_1(s) = f_t(s)$. When we have $f_0(s) = f_1(s)$ then clearly

$$f_t(s) = f_0(s) + t(f_1(s) - f_0(s)) = f_0(s) + t(f_0(s) - f_0(s)) = f_0(s)$$

for all $0 \leq t \leq 1$. This obviously occurs for $s = 0, 1$ and so each f_t is a path from x_0 to x_1. So the first condition for f_t to be a homotopy is satisfied.

We have that continuity of $F(s,t) = f_t(s)$ simply follows from the fact scalar multiplication and vector addition preserve continuity in $f_t(s) = (1-t)f_0(s) + tf_1(s)$ as $f_0(s)$ and $f_1(s)$ are continuous.

So $f_t(s)$ is a homotopy in X containing $f_0(s)$ and $f_1(s)$. □

Proposition 2.5. *$\simeq$ is an equivalence relation.*

Proof. Let f_0, f_1, g_0 and g_1 be paths.

Suppose $f_0 \simeq f_1$. Then clearly there is some homotopy f_t such that this is the case. Hence we also have $f_1 \simeq f_0$ via the homotopy f_{1-t}. $F(s, 1-t)$ (as defined in the definition) is clearly continuous as $F(s,t)$ is continuous. So $\simeq$ is **symmetric**.

We have that $f_0 \simeq f_0$ via the homotopy $f_t = f_0$. $F(t,s)$ is trivially continuous as it is just a path. So $\simeq$ is **reflexive**.

Let $f_1 = g_0$ Suppose $f_0 \simeq f_1$ via the homotopy f_t and $f_1 = g_0 \simeq g_1$ via the homotopy g_t. Define a new homotopy h_t such that

$$h_t = \begin{cases} f_{2t} & \text{for } 0 \leq t \leq 1/2 \\ g_{2t-1} & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

Then clearly $h_0 = f_0$, $h_{1/2} = f_1 = g_0$ and $h_1 = g_1$. So this homotopy works so long as we can show $H(s,t) = h_t(s)$ is continuous. We have that

$$H(s,t) = \begin{cases} F(s, 2t) & \text{for } 0 \leq t \leq 1/2 \\ G(s, 2t-1) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

So H, from the continuity of F and G, is continuous on $[0,1] \times [0,1/2]$ and $[0,1] \times [1/2,1]$. From the fact that a function defined on the union of two closed sets is continuous if it is continuous when restricted to both of the closed sets individually (see Munkres [18]), H is continuous on $[0,1] \times [0,1]$. Hence

$$f_0 \simeq f_1 = g_0 \simeq g_1$$

and so $\simeq$ is **transitive**.

We have shown $\simeq$ is reflexive, symmetric and transistive. Hence, it is an equivalence relation. □

Definition 2.6. The equivalence class of a path f under the equivalence relation $\simeq$ is called the *homotopy class* of f and is denoted $[f]$.

From this point forward, unless stated otherwise, when we refer to a 'path' in X we are actually referring to its homotopy class.

2.2 The Fundamental Group

Definition 2.7. The *composition* or *product path* of two paths $f, g : [0,1] \to X$ with $f(1) = g(0)$ is denoted $f \cdot g$, or simply fg if the context is clear, and is defined by the formula

$$f \cdot g(s) = \begin{cases} f(2s) & \text{for } 0 \leq t \leq 1/2 \\ g(2s-1) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

[9]

This definition may not be immediately clear. One should note what this actually defines, the product path is essentially a way of traversing f and then immediately traversing g. This is possible as the paths have a common endpoint, so it is possible to traverse one immediately after the other. We note that in order for $f \cdot g$ to be traversed in unit time, we have modified the arguments such that f and g are traversed twice as fast.

Definition 2.8. A path f with the same starting and ending point, i.e. $f(0) = f(1) = x_0$, is called a *loop*. x_0 is known as the *basepoint*. We denote the set of all homotopy classes $[f]$ of loops with basepoint x_0 by $\pi_1(X, x_0)$.[9]

We are now ready to define the fundamental group.

Proposition 2.9. $(\pi_1(X, x_0), \cdot)$ *is a group, with* $\cdot$ *denoting composition of paths. This is known as the fundamental group of* X *at the basepoint* x_0*.[9]*

Proof. This proof is slightly modified from Allen Hatcher's *Algebraic Topology.* [9]

Since every path in $\pi_1(X, x_0)$ has the same endpoint, x_0, we can guarantee that fg is well defined for some loops f, g with basepoint x_0. So clearly $\pi_1(X, x_0)$ is closed under $\cdot$ as $[f][g] = [f \cdot g]$ is a homotopy class of loops with basepoint x_0 and so is contained in $\pi_1(X, x_0)$ (as fg must have the same endpoints as anything contained in its homotopy class).

Define a *reparametrisation* of a path f to be a composition $f\phi$, where $\phi : [0,1] \to [0,1]$ is any continuous map with $\phi(0) = 0$ and $\phi(1) = 1$. We have that reparametrising a path preserves its homotopy class (i.e. $f \simeq f\phi$) via the homotopy $f\phi_t$, where

$$\phi_t(s) = (1-t)\phi(s) + ts,$$

noting that $\phi_0(s) = \phi(s)$ and $\phi_1(s) = s$.

We first show **associativity**. Suppose we have loops f, g, h with basepoint x_0. We have

$$([f][g])[h] = [fg][h] = [fgh] = [f][gh] = [f]([g][h]).$$

To see this, we note that $[f]([g][h])$ is a reparametrisation of $([f][g])[h]$ by ϕ given by

$$\phi(s) = \begin{cases} s/2 & \text{for } 0 \le s \le 1/2 \\ s - 1/4 & \text{for } 1/2 \le s \le 3/4 \\ 2(s - 1/2) & \text{for } 3/4 \le s \le 1. \end{cases}$$

We now show the existence of some **identity element**. Let e be a loop with basepoint x_0 where $e(s) = x_0$ for all $s \in [0,1]$. We claim $[e]$ is the identity element. To see this, note that fe is a reparametrisation of f by ϕ given by

$$\phi(s) = \begin{cases} 2s & \text{for } 0 \le s \le 1/2 \\ 1 & \text{for } 1/2 \le s \le 1. \end{cases}$$

Similarly, ef is a reparametrisation of f by ϕ given by

$$\phi(s) = \begin{cases} 0 & \text{for } 0 \le s \le 1/2 \\ 2(s - 1/2) & \text{for } 1/2 \le s \le 1. \end{cases}$$

So we get that

$$[f][e] = [fe] = [f] = [ef] = [e][f],$$

as desired.

The final group axiom to verify is the existence of some **inverse element** of $[f]$ in $\pi_1(X, x_0)$, that we shall denote $[f^{-1}]$, i.e. the homotopy class of some path f^{-1} that we shall define. Let us define f^{-1} as

$$f^{-1}(s) = f(1 - s).$$

Clearly, f^{-1} is the same loop as f but travelling in the opposing direction. Define a homotopy by $h_t(s) = f_t(s) \cdot f_t^{-1}(s)$ where

$$f_t(s) = \begin{cases} f(s) & \text{for } 0 \le s \le 1 - t \\ f(1 - t) & \text{for } 1 - t \le s \le 1. \end{cases}$$

and

$$f_t^{-1}(s) = \begin{cases} f^{-1}(t) & \text{for } 0 \le s \le t \\ f^{-1}(s) & \text{for } t \le s \le 1. \end{cases}$$

is the inverse loop of $f_t(s)$. We see that $f_0(s) = f(s)$ and $f_1(s) = f(1) = x_0 = e$ (as f is a loop with basepoint x_0). As a result, we can deduce that h_t is a homotopy from $f(s) \cdot f^{-1}(s)$ to $e(s) \cdot e^{-1}(s) = e(s)$. So $f \cdot f^{-1} \simeq e$. Swapping f_t and f_t^{-1} in h_t we instead get $f^{-1} \cdot f \simeq e$. Putting these results together we deduce that

$$[f][f^{-1}] = [ff^{-1}] = [e] = [f^{-1}f] = [f^{-1}][f].$$

So we have our inverse.

The fundamental group is then indeed a group, as it satisfies the group axioms. □

2.3 Presentations of Groups

We now take a slight tangent to discuss presentations of groups. This will become relevant later on when we make the link between the fundamental group and our knot diagrams.

Our definition is based upon the description of presentations given in Dummit and Foote's *Abstract Algebra*. **??**

Definition 2.10. A *presentation* of a group G is denoted $\langle S : R \rangle$, where

- $S = \{s_1, s_2, \ldots, s_n\} \subseteq G$ is a set of generators, i.e. a set of elements such that every element $g \in G$ can be written as a product of powers of elements of S.

- $R = \{R_1, R_2, \ldots, R_m\}$ is a set of relations, that is, an equation in the elements from $S \cup \{e\}$ (here e denotes the identity of G) such that any relation among the elements of S can be deduced from R.

To make this definition clear, we give an example.

Example 2.11. A presentation of the *dihedral group* D_n is given by

$$D_n = \langle r, s : r^n = s^2 = e, srs = r^{-1} \rangle.$$

2.3.1 Tietze Transformations

We introduce the Tietze transformations, transformations of a group presentation that give a simpler presentation of the same group.

We define these transformations, with our definitions taken from the fifth chapter of Baumslag's *Topics in Combinatorial Group Theory.* [2]

Definition 2.12. Let $\langle S : R \rangle$ be a presentation of some group G. We define the *Tietze moves* $T1$, $T1^{-1}$, $T2$ and $T2^{-1}$ below. Let $T \subset G$.

- $T1$ *(adding generators)*: $G = \langle S : R \rangle = \langle S \sqcup T : R \cup \{g = w_g : g \in T\} \rangle$[1], where for each g, w_g is some product of elements in S such that in G, $g = w_g$.

- $T1^{-1}$ *(removing generators)*: $G = \langle S : R \rangle = \langle S \backslash T : R \backslash \{g = w_g : g \in T\} \rangle$, where again for each g, w_g is some product of elements in S such that in G, $g = w_g$.

- $T2$ *(adding relations)*: $G = \langle S : R \rangle = \langle S : R \cup R' \rangle$, where R' is some set of relations derived from the relations in R.

- $T2^{-1}$ *(removing relations)*: $G = \langle S : R \rangle = \langle S : R \setminus R' \rangle$, where R' is a set of relations that can be derived completely from R.

Remark 2.13. We present some clearer, albiet informal, definitions for the Tietze moves:

- $T1$ and $T1^{-1}$ allow us to add and remove generators, respectively. Our new generators must be elements of the group that can be expressed via the existing set of generators. Hence, when we introduce new generators, we must also introduce a new relation that expresses the new generator in terms of the existing generators.

- $T2$ and $T2^{-1}$ allow us to add and remove relations, respectively. Any new relations must be derived from the existing set of relations. Any relations that are being removed must be able to be derived from the relations that will remain.

[1] $\sqcup$ denotes the disjoint union of two sets.

Example 2.14. Returning to our presentation of the *dihedral group* D_n, we apply $T2$ to attain the following:

$$D_n = \langle r, s : r^n = s^2 = e, srs = r^{-1} \rangle$$
$$= \langle r, s : r^n = s^2 = e, srs = r^{-1}, r = s^{-1}r^{-1}s^{-1} \rangle.$$

To see that this works, we multiply $srs = r^{-1}$ by s^{-1} on both the left and the right to attain

$$s^{-1}srss^{-1} = r = s^{-1}r^{-1}s^{-1}$$

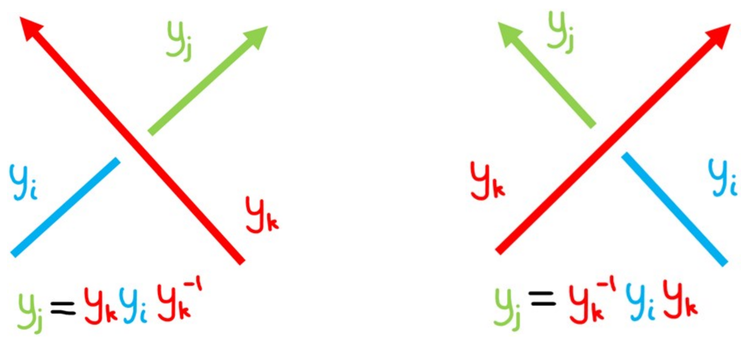
2.4 Knot Groups

We now introduce the knot group of a knot K. From this, we shall find a special presentation of the knot complement of K, known as the *Wirtinger presentation* (after Austrian mathematician Wilhelm Wirtinger (1865 - 1945)). This shall bring us one step closer to deriving the Alexander polynomial.

Definition 2.15. The *knot group* of some knot K is the fundamental group of the knot complement of K, denoted $\pi_1(\mathbb{R}^3 \setminus K, x_0)$, where x_0 is a basepoint.

We now find the Wirtinger presentation. For this we use the method given in Lickorish's book, beginning on page 110. [14]

Let the strands of some ORIENTED knot K be denoted by $y_0, \ldots, y_n$. We consider some crossing, two possible diagrams of which are shown below:



(a) A crossing with a positive sign.

(b) A crossing with a negative sign.

Figure 21: The crossing types.

The diagram on the left shows what is known as a *positive* crossing. Similarly, the diagram on the right is a *negative* crossing.[1] The sign of the crossing is independent of the orientation.

We define a relation $R_k \in R$ by

[1]Figure **??** uses colour to distinguish the strands, it is NOT used to show tricolourability in this instance.

- $y_j = y_k y_i y_k^{-1}$ if the sign of the crossing is **positive**.
- $y_j = y_k^{-1} y_i y_k$ if the sign of the crossing is **negative**.

From this, we get a relation R_k for each of the n crossings. We are almost at the knot group.

For each y_k, let g_k denote a loop that, starting from the basepoint $x_0 \in \mathbb{R}^3 \setminus K$, goes straight to the strand y_k, encircles it in the positive direction, and returns to the basepoint. Then, replacing each y_k with g_k in R, we get the general form of the *Wirtinger presentation*:

$$\pi_1(\mathbb{R}^3 \setminus K, x_0) = \langle g_0, \dots, g_n : R_1, \dots, R_n \rangle.$$

Remark 2.16. It is natural for one to ask what is the motivation here? To put it in informal terms, we are in a sense claiming equality between the homotopy class of the loop around y_j (denoted g_j) and the homotopy class given by looping around y_k, then around y_i, then around y_k in the opposite direction (denoted $g_k g_j g_k^{-1}$ for a positive crossing).

Example 2.17. We give an example of the Wirtinger presentation for the *oriented trefoil knot* K, shown below.

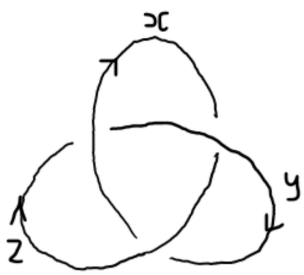


Figure 22: An oriented diagram of the trefoil knot. We have labelled the strands x, y and z.

We note that all three crossings have a positive sign. Hence, we have that the Wirtinger presentation is given by:

$$\pi_1(\mathbb{R}^3 \setminus K, x_0) = \langle g_x, g_y, g_z : g_y = g_x g_z g_x^{-1}, g_x = g_z g_y g_z^{-1}, g_z = g_y g_x g_y^{-1} \rangle,$$

where each g_i corresponds to a loop that, starting from the basepoint $x_0 \in \mathbb{R}^3 \setminus K$, goes straight to the strand i, encircles it in the positive direction, and returns to the basepoint.

We are not done, however, as this presentation can be greatly simplified using the Tietze moves. In fact, we can completely disregard one of the generators $(T1)$. Without loss of generality, let us select g_z to be disregarded. We write

$$g_y = g_x g_y g_x g_y^{-1} g_x^{-1},$$

where we have substituted $g_z = g_y g_x g_y^{-1}$ into $g_y = g_x g_z g_x^{-1}$. We now multiply by $g_x g_y$ on the right to get

$$g_y g_x g_y = g_x g_y g_x g_y^{-1} g_x^{-1} g_x g_y = g_x g_y g_x.$$

So we can rewrite our Wirtinger presentation for the trefoil knot as

$$\pi_1(\mathbb{R}^3 \setminus K, x_0) = \langle g_x, g_y : g_y g_x g_y = g_x g_y g_x \rangle.$$

2.5 Fox Calculus

2.5.1 Background

In the 1950s, American mathematician Ralph H. Fox (1913-1973) detailed what he coined as 'free differential calculus' across five papers:

- *Free Differential Calculus, I: Derivation in the Free Group Ring (1953)*, [4]
- *Free Differential Calculus, II: The Isomorphism Problem of Groups (1954)*, [5]
- *Free Differential Calculus, III: Subgroups (1956)*, [6]
- *Free Differential Calculus, IV: The Quotient Groups of the Lower Central Series (1958)* (with Kuo-Tsai Chen and Roger C. Lyndon), [8]
- *Free Differential Calculus, V: The Alexander Matrices Re-Examined (1960)*. [7]

We shall use these papers to establish the rules of free differential calculus, and use these tools to derive the Alexander polynomial. We shall refer to free differential calculus as 'Fox calculus', as this is how it is now more commonly known.

2.5.2 The Rules of Fox Calculus

From page 550 of Fox's first paper we get the rules for Fox Calculus. [4]

Definition 2.18. Let $g_1, \ldots, g_n$ be noncommuting variables (i.e. the order matters). A *word* in these variables is a product of the powers of the variables.

The *Fox derivative* $\frac{\partial}{\partial g_i}$ on some word satisfies the following two rules:

1. $\frac{\partial}{\partial g_i}(g_j) = \delta_{j\,i}$ and $\frac{\partial}{\partial g_i}(1) = 0$,
2. $\frac{\partial}{\partial g_i}(g_j g_k) = \frac{\partial}{\partial g_i}(g_j) + g_j \frac{\partial}{\partial g_i}(g_k)$.

Proposition 2.19. $\frac{\partial}{\partial g_i}(g_i^{-1}) = -g_i^{-1}$.

Proof. By **1** we have

$$\frac{\partial}{\partial g_i}(g_i g_i^{-1}) = \frac{\partial}{\partial g_i}(1) = 0.$$

Then, by **2** we have

$$\frac{\partial}{\partial g_i}(g_i g_i^{-1}) = \frac{\partial}{\partial g_i}(g_i) + g_i \frac{\partial}{\partial g_i}(g_i^{-1}) = 1 + g_i \frac{\partial}{\partial g_i}(g_i^{-1}).$$

Equating and multiplying by g_i^{-1}, we get

$$\frac{\partial}{\partial g_i}(g_i^{-1}) = -g_i^{-1}$$

as desired. □

2.6 Derivation of the Alexander Polynomial

We shall now use Fox calculus to derive the Alexander polynomial. This algorithm comes from Fox and Crowell's book *Introduction to Knot Theory*[1] [**?**] Some of the simplified notation and explanation is inspired by Edward Long's project on the Alexander polynomial. [16]

First, consider some Wirtinger presentation of some knot group for some knot K, given by $\pi_1(\mathbb{R}^3 \setminus K, x_0) = \langle g_0, \ldots, g_n : R_1, \ldots, R_n \rangle$.

Note that each g_j corresponds to some relation of the form $g_j = g_k g_i g_k^{-1}$ or $g_j = g_k^{-1} g_i g_k$, as given by our earlier algorithm. We can then do a right multiplication by g_j^{-1} to get a relation of the form $1 = w_j$, where w_j is some word in our variables.

Definition 2.20. The $n \times n$ matrix J given by

$$J = \begin{pmatrix} \frac{\partial}{\partial g_1}(w_1) & \frac{\partial}{\partial g_2}(w_1) & \cdots & \frac{\partial}{\partial g_n}(w_1) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial g_1}(w_n) & \frac{\partial}{\partial g_2}(w_n) & \cdots & \frac{\partial}{\partial g_n}(w_n) \end{pmatrix}$$

(where $\frac{\partial}{\partial g_i}$ is the Fox derivative with respect to g_i) is called the *Jacobian* of our presentation.

Definition 2.21. Suppose we have some knot K. Let the Jacobian of the Wirtinger presentation of the knot group of K be denoted J.

We do the following:

1. Choose some row and its corresponding column such that said row's relation contains three DISTINCT generators. Remove this row and column from J to get a new matrix, J'.

2. Replace all instances of variables g_i in J' with a new variable t, and any inverses g_i^{-1} with t^{-1}.

[1] Not to be confused with Lickorish's book of a similar name.

If such a row does not exist, then the Alexander polynomial is 0. Otherwise,

$$\Delta_K(t) := \det J'$$

is *the Alexander polynomial* of K.

Remark 2.22. It is natural for one to wonder why we remove a row and a column of J. As it turns out, this is done to disregard a redundant relation and generator. When considering the Wirtinger presentation in its standard form (prior to the application of Tietze moves), we note that every relation can be derived from the others. Hence we needn't consider n relations and generators but rather $n-1$.

We must be careful however, as this can only be done on a row and column pair such that the relation in said row contains three distinct generators. We shall later see that if we were to remove a relation that is composed of only two distinct generators, then we do not get the Alexander polynomial, in fact our determinant of J' remains unchanged from the determinant of J (up to a factor of -1), which is not very helpful.

We dive straight in with an example.

Example 2.23. We yet again consider the *trefoil knot*. Recall that (prior to our simplification) the Wirtinger presentation of the trefoil knot was given by

$$\langle g_x, g_y, g_z : g_x = g_z g_y g_z^{-1}, g_y = g_x g_z g_x^{-1}, g_z = g_y g_x g_y^{-1} \rangle.$$

We can then rewrite this as

$$\langle g_x, g_y, g_z : 1 = g_z g_y g_z^{-1} g_x^{-1}, 1 = g_x g_z g_x^{-1} g_y^{-1}, 1 = g_y g_x g_y^{-1} g_z^{-1} \rangle.$$

Then the Jacobian J is given by

$$J = \begin{pmatrix} \frac{\partial}{\partial g_x}(g_z g_y g_z^{-1} g_x^{-1}) & \frac{\partial}{\partial g_y}(g_z g_y g_z^{-1} g_x^{-1}) & \frac{\partial}{\partial g_z}(g_z g_y g_z^{-1} g_x^{-1}) \\ \frac{\partial}{\partial g_x}(g_x g_z g_x^{-1} g_y^{-1}) & \frac{\partial}{\partial g_y}(g_x g_z g_x^{-1} g_y^{-1}) & \frac{\partial}{\partial g_z}(g_x g_z g_x^{-1} g_y^{-1}) \\ \frac{\partial}{\partial g_x}(g_y g_x g_y^{-1} g_z^{-1}) & \frac{\partial}{\partial g_y}(g_y g_x g_y^{-1} g_z^{-1}) & \frac{\partial}{\partial g_z}(g_y g_x g_y^{-1} g_z^{-1}) \end{pmatrix}$$

$$= \begin{pmatrix} -g_z g_y g_z^{-1} g_x^{-1} & g_z & 1 - g_z g_y g_z^{-1} \\ 1 - g_x g_z g_x^{-1} & -g_x g_z g_x^{-1} g_y^{-1} & g_x \\ g_y & 1 - g_y g_x g_y^{-1} & -g_y g_x g_y^{-1} g_z^{-1} \end{pmatrix}.$$

Removing the first row and column, we get

$$J = \begin{pmatrix} -g_x g_z g_x^{-1} g_y^{-1} & g_x \\ 1 - g_y g_x g_y^{-1} & -g_y g_x g_y^{-1} g_z^{-1} \end{pmatrix}.$$

Replacing all variables with t, we get

$$J' = \begin{pmatrix} -1 & t \\ 1-t & -1 \end{pmatrix}.$$

Taking the determinant, we get

$$\det J' = \begin{vmatrix} -1 & t \\ 1-t & -1 \end{vmatrix} = (-1)^2 - t(1-t) = \mathbf{1 - t + t^2},$$

our Alexander polynomial for the trefoil knot.

Example 2.24. The *unknot* has no crossings. So its Jacobian is a 0×0 matrix. It is hence impossible to remove a row/column.

The determinant of a 0×0 matrix is defined to be 1. So we take the Alexander polyomial of the unknot to be 1.

2.7 Fox's Algorithm

For completeness, here we conclude all that we have discussed thus far and provide a step-by-step algorithm to derive the Alexander polynomial for some knot K. We then simplify this algorithm greatly, to make it more efficient for use.

1. Choose an orientation for a diagram of our knot K.
2. Label the strands of the diagram $y_1, \dots, y_n$.
3. For each of the n crossings, define some relation R_k by
   - $1 = y_k y_i {y_k}^{-1} y_j^{-1}$ if the sign of the crossing is **positive**.
   - $1 = y_k^{-1} y_i y_k {y_j}^{-1}$ if the sign of the crossing is **negative**.
4. In each relation R_k, replace any instance of some y_a $(a \in \{1, \dots n\})$ with g_a, where g_a represents a loop encompassing the strand y_a, with basepoint x_0.
5. Compute the **Jacobian** J of the resulting presentation $\langle g_0, \dots, g_n : R_1, \dots, R_n \rangle$.
6. Choose some row and its corresponding column such that said row's relation contains three distinct generators. Remove this row and column from J to get a new matrix, J'. If no such row exists, then the Alexander polynomial is 0 and we can stop here.
7. Replace all variables g_i in the Jacobian with t and any inverse variables g_i^{-1} with t^{-1}.
8. The determinant of J', $\det J'$, is the Alexander polynomial of K, denoted $\Delta_K(t)$.

2.7.1 Fox's Algorithm Simplified

The algorithm above can be greatly simplified. To do this, we note that after swapping the variables in our Jacobian, we have:

- $\frac{\partial}{\partial g_k}(g_k g_i g_k^{-1} g_j^{-1})$ reduces to $1-t$
- $\frac{\partial}{\partial g_i}(g_k g_i g_k^{-1} g_j^{-1})$ reduces to t
- $\frac{\partial}{\partial g_j}(g_k g_i g_k^{-1} g_j^{-1})$ reduces to -1
- $\frac{\partial}{\partial g_k}(g_k^{-1} g_i g_k g_j^{-1})$ reduces to $1+t$
- $\frac{\partial}{\partial g_i}(g_k^{-1} g_i g_k g_j^{-1})$ reduces to t^{-1}
- $\frac{\partial}{\partial g_j}(g_k^{-1} g_i g_k g_j^{-1})$ reduces to -1
- Every other entry in the corresponding rows reduces to 0.

Using this, our algorithm becomes:

1. Choose an orientation for a diagram of our knot K.
2. Label the strands of the diagram $y_1, \ldots, y_n$.
3. For each of the n crossings, define some relation R_k by
   - $1 = y_k y_i {y_k}^{-1} y_j^{-1}$ if the sign of the crossing is **positive**.
   - $1 = y_k^{-1} y_i y_k {y_j}^{-1}$ if the sign of the crossing is **negative**.
4. We define the k-th row of an $n-1 \times n-1$ matrix J' as follows:
   - if R_k is a positive crossing $y_k y_i y_k^{-1} y_j^{-1}$, then set the **k-th** column entry to be $1-t$, the **i-th** column entry to be t and the **j-th** column entry to be -1,
   - if R_k is a negative crossing $y_k^{-1} y_i y_k y_j^{-1}$, then set the **k-th** column entry to be $1+t$, the **i-th** column entry to be t^{-1} and the **j-th** column entry to be -1.
5. The determinant of J', $\det J'$, is the Alexander polynomial of K, denoted $\Delta_K(t)$.

This version of Fox's algorithm simply provides a faster way of reaching the same answer. It is taken from Livingston's book. [15]

3 Properties of the Alexander Polynomial

3.1 Basic Properties

Proposition 3.1. *The Alexander polynomial has the following properties: [11]*

1. $\Delta_K(1) = \pm 1$.
2. $\Delta_K(t) = t^{\lambda}\Delta_K(t^{-1})$ *for some* $\lambda \in \mathbb{Z}$.

Proof. (1) was proven by Alexander in his 1928 paper *Topological Invariants of Knots and Links*, beginning on page 299. [10]

Similarly, (2) was proven by Seifert in his 1934 paper *Ueber das Geschlecht von Knote*. [21] □

Remark 3.2. The second property comes from orientation. If we reverse the orientation of some knot K, then recompute its Alexander polynomial, we'll find that it differs from the original by a factor of t^{λ} for some $\lambda \in \mathbb{Z}$.

Remark 3.3. Interestingly, these two conditions are sufficient for a polynomial to be an Alexander polynomial for some knot K. For a proof, see Seifert's paper. We shall not go into detail here. [21]

3.2 Invariance of the Alexander Polynomial

We now have a lemma, one that we shall use in our proof that the Alexander polynomial is indeed a knot invariant.

Lemma 3.4. *Let* $\langle g_0, \ldots, g_n : R_1, \ldots, R_n\rangle$ *be the Wirtinger presentation for some knot* K*. Suppose* $\Delta_K(t)$ *is its Alexander polynomial.*

Suppose R_n *is the only instance in which* g_n *appears in a relation. Then suppose we disregard* g_n *and* R_n *to obtain a new presentation* $\langle g_0, \ldots, g_{n-1} : R_1, \ldots, R_{n-1}\rangle$.

Then the Alexander polynomial derived from this new presentation is equal to $\Delta_K(t)$.

Proof. Without loss of generality, suppose K is oriented such that R_n is of the form $g_n = g_1 g_2 g_1^{-1}$. Suppose also that g_n does not appear in any other relation.

Consider the Jacobian for our original presentation, given by

$$J = \begin{pmatrix} \frac{\partial}{\partial g_1}(w_1) & \frac{\partial}{\partial g_2}(w_1) & \frac{\partial}{\partial g_3}(w_1) & \frac{\partial}{\partial g_4}(w_1) & \cdots & \frac{\partial}{\partial g_n}(w_1) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial g_1}(w_n) & \frac{\partial}{\partial g_2}(w_n) & \frac{\partial}{\partial g_3}(w_n) & \frac{\partial}{\partial g_4}(w_n) & \cdots & \frac{\partial}{\partial g_n}(w_n) \end{pmatrix}.$$

We particularly focus our attention on the n-th column. Since g_n only appears in $w_n = g_1 g_2 g_1^{-1} g_n^{-1}$, we have that

$$\frac{\partial}{\partial g_n}(w_i) = 0$$

for $i \in \{1, \ldots, n-1\}$ and

$$\frac{\partial}{\partial g_n}(w_n) = -g_1 g_2 g_1^{-1} g_n^{-1}.$$

After swapping our variables, we get

$$\frac{\partial}{\partial g_n}(w_n) = -ttt^{-1}t^{-1} = -1.$$

So the only non-zero entry of the n-th column is -1.

We remove a row and corresponding column to find our J'. We choose to remove the third row and column. So we get

$$J' = \begin{pmatrix} \frac{\partial}{\partial g_1}(w_1) & \frac{\partial}{\partial g_2}(w_1) & \frac{\partial}{\partial g_4}(w_1) & \cdots & \frac{\partial}{\partial g_n}(w_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial g_1}(w_n) & \frac{\partial}{\partial g_2}(w_n) & \frac{\partial}{\partial g_4}(w_n) & \cdots & \frac{\partial}{\partial g_n}(w_n) \end{pmatrix}.$$

Now suppose we want to compute the determinant of J' using Laplace expansion on the n-th column. Then

$$\det J' = (-1)^{n+1} \begin{vmatrix} \frac{\partial}{\partial g_1}(w_1) & \frac{\partial}{\partial g_2}(w_1) & \frac{\partial}{\partial g_4}(w_1) & \cdots & \frac{\partial}{\partial g_{n-1}}(w_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial g_1}(w_{n-1}) & \frac{\partial}{\partial g_2}(w_{n-1}) & \frac{\partial}{\partial g_4}(w_n) & \cdots & \frac{\partial}{\partial g_{n-1}}(w_{n-1}) \end{vmatrix}.$$

This comes from the fact that the only non-zero entry of the n-th column is -1.

Consider the same presentation, but with g_n and R_n completely removed. Our Jacobian for this presentation is then

$$J_0 = \begin{pmatrix} \frac{\partial}{\partial g_1}(w_1) & \frac{\partial}{\partial g_2}(w_1) & \frac{\partial}{\partial g_3}(w_1) & \frac{\partial}{\partial g_4}(w_1) & \cdots & \frac{\partial}{\partial g_{n-1}}(w_1) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial g_1}(w_{n-1}) & \frac{\partial}{\partial g_2}(w_{n-1}) & \frac{\partial}{\partial g_3}(w_{n-1}) & \frac{\partial}{\partial g_4}(w_{n-1}) & \cdots & \frac{\partial}{\partial g_{n-1}}(w_{n-1}) \end{pmatrix}.$$

We remove the third row and column yet again to get

$$J_0' = \begin{pmatrix} \frac{\partial}{\partial g_1}(w_1) & \frac{\partial}{\partial g_2}(w_1) & \frac{\partial}{\partial g_4}(w_1) & \cdots & \frac{\partial}{\partial g_{n-1}}(w_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial g_1}(w_{n-1}) & \frac{\partial}{\partial g_2}(w_{n-1}) & \frac{\partial}{\partial g_4}(w_{n-1}) & \cdots & \frac{\partial}{\partial g_{n-1}}(w_{n-1}) \end{pmatrix}.$$

Taking the determinant, we have

$$\det J_0' = \begin{vmatrix} \frac{\partial}{\partial g_1}(w_1) & \frac{\partial}{\partial g_2}(w_1) & \frac{\partial}{\partial g_4}(w_1) & \cdots & \frac{\partial}{\partial g_{n-1}}(w_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial g_1}(w_{n-1}) & \frac{\partial}{\partial g_2}(w_{n-1}) & \frac{\partial}{\partial g_4}(w_{n-1}) & \cdots & \frac{\partial}{\partial g_{n-1}}(w_{n-1}) \end{vmatrix} = \det J'.$$

So our Alexander polynomial is unchanged (up to a factor of -1) by the removal of g_n and R_n. $\square$

We now come to the main proof of this project. We prove that the Alexander polynomial is indeed a knot invariant. To do so, we use all of the tools we developed in the first chapter to prove that equivalent knots have the same Alexander polynomial.

Proposition 3.5. *The Alexander polynomial is a knot invariant.*

Proof. Of course the Alexander polynomial assigns a mathematical expression to some knot K, in the form of a polynomial. It suffices to show that equivalent knots have the same Alexander polynomial.

In order to prove such, we recall that two equivalent knots K_1 and K_2 are such that the diagram of K_2 can be obtained via a finite application of Reidemeister moves on the diagram of K_1 (*Reidemeister's theorem*). Hence, if we show that the Reidemeister moves preserve the Alexander polynomial then we are done.

Consider first $R1$. $R1$ increases the number of strands in the diagram by 1, as shown below.

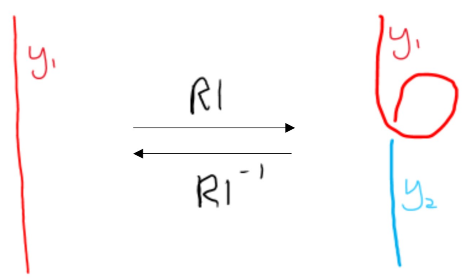


Figure 23: $R1$. Strands are drawn with different colours to highlight how the number of strands changes. The strands have been labelled y_1 and y_2.

Upon application of $R1$, we attain one additional crossing and hence one additional relation in our Wirtinger presentation. Depending on the orientation, this crossing may be positive or negative. Suppose, without loss of generality, that it is positive. Then our relation is

$$\begin{aligned} y_1 &= y_1 y_2 y_1^{-1} \\ &\implies y_1 y_1 = y_1 y_2 \\ &\implies y_1 = y_2. \end{aligned}$$

From this we conclude that we can disregard y_2 (and hence g_2, the loop in the fundamental group that encircles y_2) as a generator and our Wirtinger presentation remains unchanged. So $R1$ does not change the Wirtinger presentation and therefore does not change the Alexander polynomial.

Now for $R2$. $R2$ increases the number of strands in the diagram by 2, as shown below.

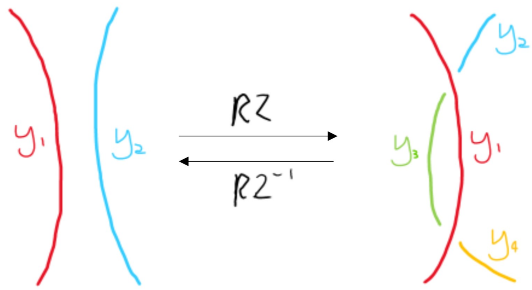


Figure 24: $R2$. Strands are drawn with different colours to highlight how the number of strands changes. The strands have been labelled y_i, with $i \in \{1, \ldots, 4\}$.

Application of $R2$ introduces two new crossings. So there are two extra relations in our Wirtinger presentation to consider. Supposing that the diagram is oriented such that the crossing between y_1, y_2 and y_3 is positive, these are as follows:

- $y_2 = y_1 y_3 y_1^{-1}$,
- $y_3 = y_1^{-1} y_4 y_1$.

From this, we can deduce that $y_2 = y_4$. To see this, note that

$$\begin{aligned} & y_2 = y_1 y_3 y_1^{-1} \\ & \implies y_1^{-1} y_2 y_1 = y_3 \\ & \implies y_1^{-1} y_2 y_1 = y_1^{-1} y_4 y_1 \\ & \implies y_2 = y_4. \end{aligned}$$

We can now disregard y_4 (and hence g_4) as a generator in our presentation.

Since y_3 does not cross with any strands in the complete diagram (not shown), we can also deduce that no other relations are changed as a result of $R2$. We only need to deal with the new relation $y_3 = y_1 y_2 y_1^{-1}$.

We note that by lemma 3.4, as y_3 only appears in this one relation, the Alexander polynomial is unchanged by its introduction. Hence $R2$ does not change the Alexander polynomial.

$R3$ is a little bit different. $R3$ introduces only one new strand but also removes one. It doesn't change the number of crossings. It simply changes the relations.

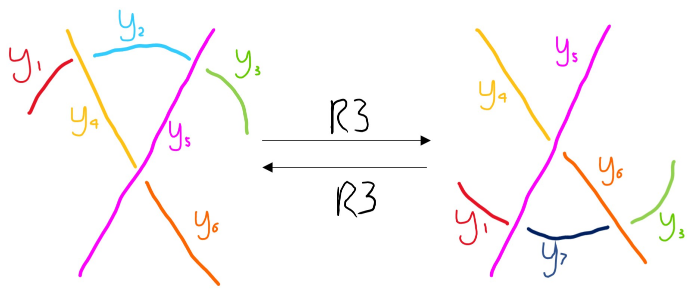


Figure 25: $R3$. Strands are drawn with different colours to highlight how the number of strands changes. The strands have been labelled y_i, with $i \in \{1, \dots, 7\}$. Note that y_6 and y_7 are different strands.

We consider the presentations of the knot before and after $R3$. The crossing in the centre is unchanged by $R3$ and thus we need not consider it.

We have the following relations for pre-$R3$:

- $1 = g_4 g_1 g_4^{-1} g_2^{-1}$,
- $1 = g_5 g_2 g_5^{-1} g_3^{-1}$.

Similarly, for post-$R3$ we have:

- $1 = g_5 g_1 g_5^{-1} g_7^{-1}$,
- $1 = g_6 g_7 g_6^{-1} g_3^{-1}$.

For these sets of relations, we can remove the generators g_2 and g_7 in our presentation using lemma 3.4. We are left with the following relations for pre-$R3$ and post-$R3$ respectively:

- $1 = g_5 g_4 g_1 g_4^{-1} g_5^{-1} g_3^{-1} := w_1$,
- $1 = g_6 g_5 g_1 g_5^{-1} g_6^{-1} g_3^{-1} := m_1$.

We note that

$$(\frac{\partial}{\partial g_1}(w_1), \frac{\partial}{\partial g_3}(w_1), \frac{\partial}{\partial g_4}(w_1), \frac{\partial}{\partial g_5}(w_1), \frac{\partial}{\partial g_6}(w_1))$$
$$= (t^2, 0, -1, 0, 0)$$

after replacing the variables, and similarly

$$(\frac{\partial}{\partial g_1}(m_1), \frac{\partial}{\partial g_3}(m_1), \frac{\partial}{\partial g_4}(m_1), \frac{\partial}{\partial g_5}(m_1), \frac{\partial}{\partial g_6}(m_1))$$
$$= (t^2, 0, -1, 0, 0).$$

We note that these are the rows of the Jacobian pre-$R3$ and post-$R3$ respectively, and thus we deduce our Jacobian remains unchanged and we obtain the same Alexander polynomial pre and post-$R3$.

So we have shown that the Reidemeister moves preserve the Alexander polynomial. So, by Reidemeister's theorem, the Alexander polynomial is a knot invariant. $\square$

4 Deriving the Alexander Polynomial: the Seifert Matrix

We now present an alternate method for deriving the Alexander polynomial for some knot K. This method involves the computation of what is known as the 'Seifert matrix' of K (named after German mathematician Herbert Seifert (1897–1996)). This section is based on the description of the Seifert matrix found in Livingston's book. [15]

4.1 Seifert Surfaces

We begin by defining what is meant by a 'Seifert surface' for some knot K. This definition requires some prior definitions from Topology. These are included in **appendix A** as to keep things concise.

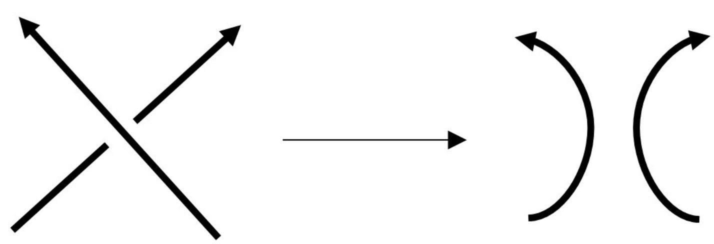
In the following, we denote the boundary of some surface S by ∂S.

From Livingston's book, we have the following theorem.

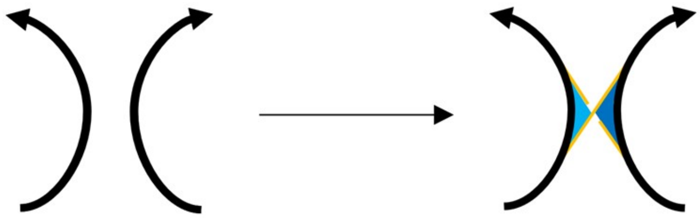
Theorem 4.1. *Every knot K is the boundary of an orientable surface. [15]*

Proof. The following sketch of a proof is known as *Seifert's algorithm*.

Take some oriented knot diagram for K. Every crossing has two incoming strands and two outgoing strands. One of the incoming strands is also one of the outgoing strands. Eliminate each crossing by swapping which incoming strand is connected to which outgoing strand.



We are now left with a set of non-intersecting oriented topological circles, known as *Seifert circles*. We fill in these circles to give discs. If any circles are nested (inside one another), we lift the inner disks above the outer disks perpendicular to the plane. We then connect these discs by attaching twisted bands where the crossings used to be. The twist direction corresponds to the direction of the crossing in the knot. The twist is shown in the diagram below. The yellow line that 'breaks' represents the side of the twist facing away from the reader. The light and dark blue represent the two faces of the surface bound by the twist.



This procedure forms a surface that has the knot as its boundary, as desired. It is easy to see that this surface is orientable by colouring the circles according to their orientation, colour the upward face some colour for clockwise and the downward face another colour for anticlockwise. This will colour the whole surface and show it is orientable. [3] [15] □

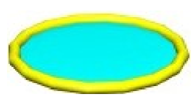
Remark 4.2. Seifert's algorithm allows us to find such a surface for any knot K.

We can now formally define the notion of a *Seifert surface*.

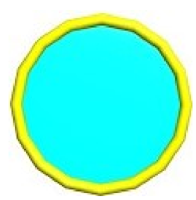
Definition 4.3. Let K be some knot. A *Seifert surface* for K is a compact, orientable, connected surface S with $\partial S = K$. [13]

In the forthcoming examples, we have used software to generate the images. The software in question is *SeifertView*, written and released in 2005 by Jack van Wijk. It can be downloaded for free from the internet. [22]

Example 4.4. We first show the Seifert surface of an *unknot*. This is fairly trivial. It is just the circle with the unknot as its boundary.



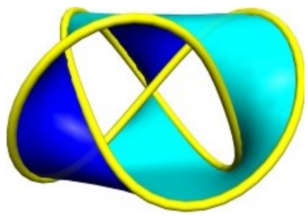
(a) Side view.



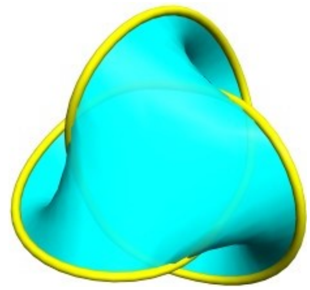
(b) Top-down view.

Figure 26: The Seifert surface of the unknot. [22]

Example 4.5. Our next example is all too familiar. We show the Seifert surface of the *trefoil*.



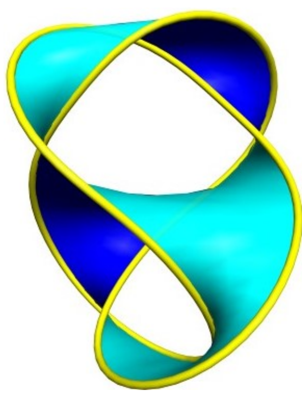
(a) Side view.



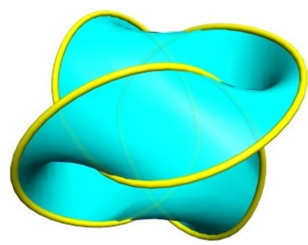
(b) Top-down view.

Figure 27: The Seifert surface of the trefoil. [22]

Example 4.6. For our third example, we have a new knot. This is the *figure-8 knot.*



(a) Side view.



(b) Top-down view.

Figure 28: The Seifert surface of the figure-8 knot. [22]

Definition 4.7. The *genus of an* ***orientable*** *surface* S is the largest number of non-intersecting simple, closed curves that can be drawn on the surface without separating it. We denote the genus of S as $g(S)$. [17]

Remark 4.8. The genus can be thought of informally as the number of 'holes' in a surface.

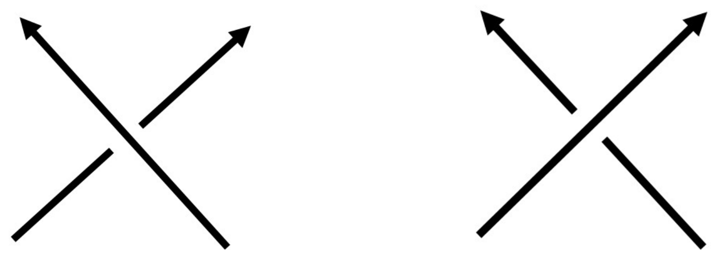
Definition 4.9. The *genus of a knot* K is the MINIMUM possible genus of a Seifert surface for K. We denote the genus of K as $g(K)$ [15]

Remark 4.10. The genus of a knot is quite difficult to compute by hand. Seifert's algorithm, although useful, does not necessarily give the minimum genus surface. [15]

4.2 Linking Numbers

Before we can introduce the Seifert matrix, we require some definitions.

Recall that crossings can be either **positive** or negative, as shown below.



(a) A crossing with a positive sign. (b) A crossing with a negative sign.

Figure 29: The crossing types. The strands are not labelled.

Definition 4.11. Suppose we have a 2-component link $L = \{K_1, K_2\}$. Suppose we have a link diagram of L. The *linking number* of K_1 and K_2 is defined as the SUM of the signs of the crossings in the diagram at which K_1 CROSSES OVER K_2. [15]

The linking number is denoted $lk(K_1, K_2)$.

Proposition 4.12. *The linking number is symmetric, i.e. $lk(K_1, K_2) = lk(K_2, K_1)$ [15]*

Proof. Trivial. □

4.3 The Positive Push Off

We define the 'positive push off' for some curve x on a Seifert surface.

Definition 4.13. Let K be a knot. Fix a Seifert surface S of K.

By definition, a Seifert surface is orientable. Choose a non-vanishing vector perpendicular to the surface. This is the top of the surface. The choice of direction does not matter.

Now let x be some simple, oriented curve on S. One defines the *positive push-off* of x, denoted x^*, as the curve running parallel to x lying just above S.

4.4 The Seifert Matrix

We are now ready to define the Seifert matrix. The definition may seem confusing at first, but with the help of some drawings, it shall become easier to understand. As before, this definition is adapted from Livington's book. [15]

Let K be a knot with Seifert surface S. Choose a non-vanishing vector perpendicular to the surface as before. This is the top of the surface.

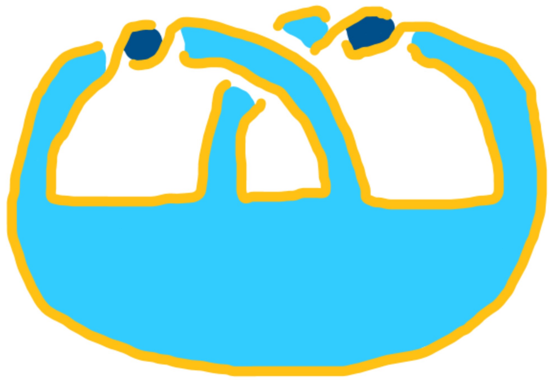


Figure 30: The Seifert surface of the trefoil as a disk with two connected bands.

We require the following theorem, taken from Livingston's book [15], where a sketch of the proof can be found. We shall not include said sketch proof here.

Theorem 4.14. *Classification I.*

*Every **connected** surface with a boundary is homeomorphic to a surface constructed by adding **bands** to a* SINGLE ***disk**.*

A consequence of this theorem is that S can be formed from a single disk with added bands. With these in mind, we have that there arises a family of curves on this surface. The number of such curves is given by $2g(s)$. We denote these curves $x_1, x_2, \ldots, x_{2g}$. [15]

Definition 4.15. The *Seifert matrix* for S is the $2g \times 2g$ matrix V that has (i, j)-component given by

$$v_{i,j} = lk(x_i, x_j^*),$$

where x_j^* denotes the positive push-off of x_j. [15]

Let us give an example. For this example we shall use the following theorem, again taken from Livingston's book [15] without proof.

Theorem 4.16. *If a **connected**, **orientable** surface is formed by attaching bands to a collection of disks, then the genus of the resulting surface is given by [15]*

$$\frac{2 - \text{number of disks} + \text{number of bands} - \text{number of boundary components}}{2}.$$

Example 4.17. As the reader has likely already guessed, we are going to find the Seifert matrix for the *trefoil*.

The Seifert surface for the trefoil in its disk/band form is shown below.

From this, and theorem 4.16, we can deduce that the genus of this surface is $\frac{2-1+2-1}{2} = 1$.

We see that our family of curves, shown in red, are given as below.

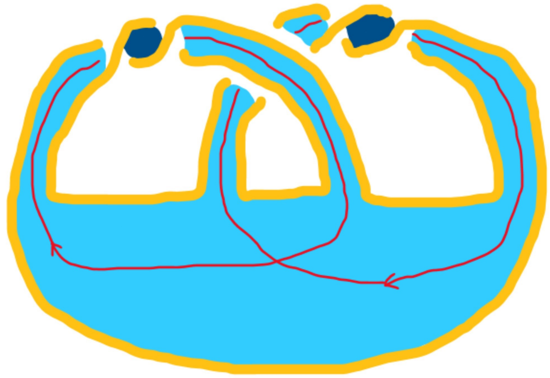


Figure 31: The $2g = 2$ curves that arise on our surface, shown in red.

For both of these curves, say x_1 and x_2 for the left and right curves respectively, we have the positive push-off which lies just above the surface. Using the diagram, we can deduce the following:

- $lk(x_1, x_1^*) = -1$,
- $lk(x_1, x_2^*) = 0$,
- $lk(x_2, x_1^*) = 1$,
- $lk(x_2, x_2^*) = -1$.

From this, we determine that the Seifert matrix for the trefoil is given by

$$V = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}.$$

4.5 Obtaining the Alexander Polynomial from the Seifert Matrix

Theorem 4.18. *Let V be the Seifert matrix for some knot K. Let V^T denote the* ***transpose*** *of V.*

Then the Alexander polynomial $\Delta_K(t)$ is given by [15]

$$\Delta_K(t) = \det(V - tV^T).$$

It is natural for one to wonder how we arrive at the same Alexander polynomial as in the previous section. Unfortunately, it is a very involved proof, and we shall not include it here. However, we shall show that using the Seifert matrix to find the Alexander polynomial of the trefoil gives the same as before (up to a factor of t^λ).

Example 4.19. We have shown that for the *trefoil*:

$$V = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Thus, we have that

$$\Delta_K(t) = \det(V - tV^T) = \det\left(\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} - t\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} -1+t & -t \\ 1 & -1+t \end{pmatrix} = (t-1)^2 + t = t^2 - 2t + 1 + t$$

$$= \mathbf{t^2 - t + 1}.$$

This is exactly the Alexander polynomial we derived in example 2.23.

So we have seen that this method for deriving the Alexander polynomial gives the same result as before for the trefoil. As mentioned, it can be proven that this is the case for any knot.

A Topology

Here we recall some definitions from Topology that are used throughout the project, for the sake of completeness. These definitions are taken from Munkres' book. [18]

Definition A.1. *Topological Space.*
A topological space is a set X together with a *topology* τ such that the following hold:

- $\emptyset \in \tau$,
- $X \in \tau$,
- for $A_1, \ldots, A_n \in \tau$, we have $A_1 \cap \ldots \cap A_n \in \tau$,
- for $A_1, A_2, \ldots \in \tau$, we have $A_1 \cup A_2 \cup \ldots \in \tau$.

The elements of τ are the *open subsets* of X. [18]

Let X be a topological space throughout. Recall that a *collection* is just a set with elements that are sets.

Definition A.2. *Basis.*
A *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X such that the following hold:

- for $x \in X$, there exists some $B \in \mathcal{B}$ such that $x \in B$,
- for $B_1, B_2 \in \mathcal{B}$, we have that if $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$.

Elements of $\mathcal{B}$ are called *basis elements*. [18]

Definition A.3. *Hausdorff Space.*
X is said to be *Hausdorff* if for each pair x_1,x_2 of distinct points of X, there exists neighbourhoods U_1, U_2 of x_1 and x_2 respectively such that [18]

$$U_1 \cap U_2 = \emptyset.$$

A.0.1 Surfaces

Definition A.4. *Manifold.*
X is an *m-manifold* if it is Hausdorff, has a countable basis and each $x \in X$ has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^m$. [18]

For our purposes, we need only define surfaces as 2-manifolds.

Definition A.5. *Surface.*
A *surface* is a 2-manifold. [18]

Definition A.6. *Orientable Surface.*
A surface is said to be *orientable* if it does not contain a homeomorphic copy of the Möbius strip (an object with only one side).

A.0.2 Compact and Connected Spaces

Definition A.7. *Cover.*

A collection $\mathcal{A}$ of subsets of X is said to *cover* X if $\bigcup \mathcal{A} = X$. We say that $\mathcal{A}$ is a *covering* of X.

$\mathcal{A}$ is an *open covering* of X if its elements are all open subsets of X. [18]

Definition A.8. *Compact Space.*

X is called *compact* if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X. [18]

Definition A.9. *Separation.*

A *separation* of X is a pair of disjoint non-empty open subsets of X, say U and V, such that $U \cup V = X$. [18]

Definition A.10. *Connected Space.*

X is said to be *connected* if there does NOT exist some separation of X. [18]

A.0.3 Disks and Bands

Definition A.11. *Disk.*

A *disk* is a surface that is homeomorphic to the circle in $\mathbb{R}^2$. [18]

Definition A.12. *Band.*

A *band* is a surface that is homeomorphic to the square in $\mathbb{R}^2$. [18]

Remark A.13. When we say that a band is 'attached' to a disk, we mean that two opposing sides of the band are attached to this disk.

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