A SMOOTH CATEGORY PROOF OF THE HOSOKAWA-KAWAUCHI SOLID TORUS UNKNOTTING CRITERION FOR 2-KNOTS

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Abstract. Hosokawa and Kawauchi showed that in the piecewise linear case, a closed connected surface of genus n in $\mathbb{R}^4$ is unknotted if and only if it bounds a (piecewise linear) solid torus of genus n. Restricting to the case of 2-knots (genus 0), we provide an explicit and self-contained proof of this unknotting criterion in the smooth case, i.e. we show that some smooth 2-knot in $\mathbb{R}^4$ is ambient isotopic to a surface in $\mathbb{R}^3 \times \{0\}$ (unknotted) if and only if it bounds some smooth 3-ball in $\mathbb{R}^4$. Although this result is well-known as a formal consequence of previous work, including by Moise and Munkres, we choose to provide a direct argument using standard machinery such as collaring and isotopy extension for expository purposes.

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1. Introduction

In classical knot theory, Rolfson [Rol03] (chapter 2.B, exercise 10) gives that some knot $K_1 \subset \mathbb{R}^3$ is ambient isotopic to the *unknot* if it lies in a plane. In informal terms, an unknot is subset of $\mathbb{R}^3$ that can be 'untangled'. If K_1 were a piece of string, then, intuitively, the string is unknotted if it doesn't have any knots in it. [Rol03] gives an alternate yet equivalent definition for K_1 to be the unknot: that K_1 is equivalent to the unknot if and only if it is bounded by an embedded 2-disk in $\mathbb{R}^3$.

Now consider a 2-knot $K_2 \subset \mathbb{R}^4$. The higher dimensional analogue for the plane-definition of classical unknotting is for K_2 to be ambient isotopic to $\mathbb{R}^3 \times \{0\}$, i.e. it can be deformed ambiently to the 3-dimensional plane (Klingmann [Kli73]). The connection to the classical definition is easy to see. We call such a 2-knot *unknotted*. In practice, this definition extends to any closed connected surface in $\mathbb{R}^4$, i.e. it is possible for any such surface to be considered unknotted. A surface that is not unknotted is called a *knotted surface*.

In 1979, Hosokawa and Kawauchi [HK79] proved that for some general piecewise linear closed connected surface $F_n \subset \mathbb{R}^4$ of genus n (for which a piecewise linear K_2 is an example of), F_n is unknotted if and only if it bounds a piecewise linear solid torus T_n (of genus n). This harkens back to the two equivalent unknot definitions in the classical case.

In this note, we provide an explicit and self-contained proof of the equivalence of the two previously mentioned unknotting criteria, isolated to the case of smooth 2-knots. This particular case is not covered by Hosokawa and Kawauchi's [HK79] piecewise linear proof, but is often cited as a formal consequence in geometric topology due to the uniqueness of piecewise linear and smooth structures on 3-manifolds (Moise [Moi77] and Munkres [Mun60] respectively), and the absence of pseudo-isotopy obstructions in 3-dimensions[1] (Cerf [Cer68]). Here, we shall avoid the complex machinery of Cerf [Cer68] and rely on standard isotopy extension arguments as presented by Hirsch in his 1997 book [Hir97] for our proofs.

[1] Cerf's key result in his 1968 paper [Cer68] is that in 3-dimensions, pseudo-isotopy (a weaker definition than standard isotopy) and isotopy coincide. This results underpins the modern isotopy extension theorems, such as in Hirsch [Hir97].

2. Background and Preliminaries

We define smooth 2-knots as follows, with definitions taken from Kamada [Kam17].

Definition 2.1. A *smooth surface knot* is a smooth submanifold of $\mathbb{R}^4$ that is diffeomorphic to a closed connected surface. Similarly, a *piecewise linear surface knot* is a piecewise linear submanifold of $\mathbb{R}^4$ that is homeomorphic to a closed connected surface.

Definition 2.2. A 2-*knot* is a surface knot that is diffeomorphic to the 2-sphere, S^2.

It follows that a *smooth* 2-*knot* is a smooth surface knot that is diffeomorphic to the 2-sphere, S^2. Equivalently, a *smooth* 2-*knot* is a smoothly embedded 2-sphere in $\mathbb{R}^4$.[2]

2.1. Smooth 3-Balls.

We also must define our smooth 3-balls, for which we extend the definition for 3-balls in a manifold given by Munkres' book [Mun00] to the smooth case.

Definition 2.3. A 3-manifold B^3 is called a 3-*ball* if it is homeomorphic as manifolds to the closed unit ball of dimension 3.

Definition 2.4. A 3-manifold B^3 is called a *smooth* 3-*ball* if it is diffeomorphic as manifolds to the closed unit ball of dimension 3.

An immediate consequence of this definition is that the boundary of B^3, denoted ∂B^3, is such that $\partial B^3 = S^2$ i.e. the 2-sphere, which makes ∂B^3 trivially a 2-knot when embedded in $\mathbb{R}^4$.

2.2. Unique Structures on 3-Balls.

We turn to Moise [Moi77], who proved that any 3-manifold (such as a 3-ball) admits a unique piecewise linear structure. Munkres' [Mun60] prior work takes this further, proving that every piecewise linear 3-manifold admits a unique smooth structure.[3] We put this together to attain the following theorem:

Theorem 2.5. *Any* 3-*ball admits a unique smooth structure up to diffeomorphism.*

2.3. Collaring Theorem.

Morris W. Hirsch's book *Differential Topology* [Hir97] will provide many of our preliminary definitions and some of the theorems that we shall use later in our proofs. Others will be taken from Lee [Lee12]. The first of these relate to collars of boundaries of manifolds and tubular neighbourhoods of submanifolds.

Definition 2.6. A *collar* on some manifold M is an embedding

$$f : \partial M \times [0, \delta) \to M$$

such that $f(x, 0) = x$.

The collaring theorem, proven by Hirsch, states that every manifold boundary has a collar.

Theorem 2.7. *Let M be some manifold. Then ∂M has a collar.*

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[2] We see that these statements are equivalent by picking some diffeomorphism from the 2-sphere to the smooth 2-knot and composing it with the inclusion map, which gives a smooth embedding. Conversely, the image of some smoothly embedded 2-sphere gives a closed embedded 2-dimensional submanifold of $\mathbb{R}^4$, precisely our smooth surface knot.

[3] A result that, interestingly, does not hold in some higher dimensions.

2.4. **Isotopy Extension.** We have the isotopy extension theorem, an important result that we shall use to extend isotopies from our 3-balls to the entirety of $\mathbb{R}^4$. This is contained in chapter 8 of [Hir97], as theorem 1.3.[4] We note here the specific case for $\mathbb{R}^4$, as is relevant to us.

Theorem 2.8. *Let $V \subset \mathbb{R}^4$ be a compact submanifold. Let $F : V \times [0,1] \to \mathbb{R}^4$ be an isotopy of V. If $F(V \times [0,1]) \subset \mathbb{R}^4$ is contained in some open set $U \subset \mathbb{R}^4$, then F extends to an ambient isotopy of $\mathbb{R}^4$ with compact support.*[5]

2.5. **Hirsch's Smoothing Theorem.** Also in his book *Differential Topology*, Hirsch [Hir97] proves the following corollary to the isotopy extension theorems regarding n-manifolds, which shall be relevant to us.

Theorem 2.9. *For $i = 0, 1$, let W_i be n-manifolds without boundary that are the union of two closed n-dimensional submanifolds M_i, N_i with the property that*

$$M_i \cap N_i = \partial M_i = \partial N_i,$$

which we denote as V_i.

Let $h : W_0 \to W_1$ be a homeomorphism that diffeomorphically maps M_0 and N_0 to M_1 and N_1, respectively. Then there exists a diffeomorphism $f : W_0 \to W_1$ such that $f(M_0) = M_1$ and $f(N_0) = f(N_1)$ with $f|_{V_0} = h|_{V_0}$. Additionally, f can be chosen to coincide with h outside a given neighbourhood of V_i.

2.6. **Fixing a Collar of the 3-Ball.** We also present a lemma that will be used twice across our proofs. This lemma uses a result from Smale [Sma59] to show that any orientation preserving diffeomorphism on some boundary of a smooth 3-ball extends to a diffeomorphism of the whole ball that is the identity on some collar of the 3-ball.

Lemma 2.10. *Let $B^3_{C^\infty}$ be a smooth 3-ball. Let $c : \partial B^3_{C^\infty} \times [0,1] \to B^3_{C^\infty}$ be a smooth collar, which exists via the collaring theorem (2.7).*

Then for every orientation preserving diffeomorphism $\varphi : \partial B^3_{C^\infty} \to \partial B^3_{C^\infty}$ there is a diffeomorphism $\Phi : B^3_{C^\infty} \to B^3_{C^\infty}$ such that

$$\Phi|_{B^3_{C^\infty}} = \varphi \text{ and } \Phi|_{c(\partial B^3_{C^\infty} \times [\epsilon, \delta])} = \mathrm{id}$$

for some $0 < \epsilon < \delta$.

Proof. By [Sma59], we have that $\mathrm{Diff}^+(S^2)$, the group of orientation preserving diffeomorphisms of S^2, is path connected. Note that as $\partial B^3_{C^\infty}$ and S^2 are diffeomorphic by definition, we have that $\mathrm{Diff}^+(S^2) \cong \mathrm{Diff}^+(\partial B^3_{C^\infty})$.

Let $\varphi \in \mathrm{Diff}^+(\partial B^3_{C^\infty})$ be arbitrary. Now, we can choose some smooth isotopy $\varphi_t : \partial B^3_{C^\infty} \to \partial B^3_{C^\infty}$ such that $\varphi_0 = \mathrm{id}$ and $\varphi_1 = \varphi$.

Let $0 < \epsilon < \delta$. We define a new isotopy on the collar $c : \partial B^3_{C^\infty} \times [0,1] \to B^3_{C^\infty}$ by

$$\varphi'_t(c(x,s)) = c(\varphi_{\rho(s)t}(x), s)$$

where $\rho(s) = 1$ for $0 \le s \le \epsilon/2$ and $\rho(s) = 0$ for $\epsilon \le s$, varying smoothly between.

By the general isotopy extension theorem (see Hirsch [Hir97] chapter 8) we can extend this isotopy to some ambient isotopy $\Phi_t : B^3_{C^\infty} \to B^3_{C^\infty}$ that is such that

$$\Phi_1|_{B^3_{C^\infty}} = \varphi \text{ and } \Phi_1|_{c(\partial B^3_{C^\infty} \times [\epsilon, \delta])} = \mathrm{id}\,.$$

Hence, Φ_1 satisfies our conditions. □

[4]Hirsch uses the word *diffeotopy* instead of *ambient isotopy*, but the meaning is the same.

[5]*Compact support* here simply guarantees that there are no changes near infinity.

3. The Hosokawa-Kawauchi Piecewise Linear Solid-Torus Unknotting Criterion

Throughout this section until the next, we take all objects to be in the piecewise linear category.

In their 1979 paper *Unknotted Surfaces in Four-Spaces*, Hosokawa and Kawauchi [HK79] define *unknottedness* of some closed, connected surface as the existence of some solid torus of the same genus that bounds the surface.

Definition 3.1. Formally, for some closed, connected surface F_n of genus n in $\mathbb{R}^4$, F_n is *unknotted* if there exists some solid torus[6] of genus n, T_n, with the boundary of T_n, denoted ∂T_n, such that

$$\partial T_n = F_n.$$

Otherwise, F_n is said to be *knotted*.

In our case, we restrict ourselves to 2-knots as our surfaces. A 2-knot is a surface of genus 0, which follows from the fact it is diffeomorphic to S^2, which has genus 0 by definition. Similarly, we have that a solid torus of genus 0, i.e. a 0 genus handlebody, is a 3-ball.

Let us give this theorem for 2-knots, as will be relevant to us.

Theorem 3.2. *Some 2-knot K in $\mathbb{R}^4$ is unknotted if and only if K bounds a piecewise linear 3-ball in $\mathbb{R}^4$.*

Hosokawa and Kawauchi [HK79] then go on to show that their definition for unknottedness is equivalent to that given by Manfred Klingmann in *Kurven auf orientierbaren Flächen* [Kli73], which is stated explicitly in the former as follows:

Theorem 3.3. *F_n is unknotted in $\mathbb{R}^4$ if and only if F_n is piecewise linear ambient isotopic to a surface in $\mathbb{R}^3 \times \{0\}$.*

We shall be using the smooth version of Klingmann's unknottedness definition to prove our if and only if relationship between 2-knots bounding 3-balls and unknottedness. Hence we can rewrite our theorem 3.2 by replacing the unknottedness property with that given in 3.3, as follows:

Theorem 3.4. *Some 2-knot K bounds a piecewise linear 3-ball in $\mathbb{R}^4$ if and only if K is piecewise linear ambient isotopic to a surface in $\mathbb{R}^3 \times \{0\}$, i.e. K is unknotted.*

So we have established that the main theorem (theorem 3.4) holds in the piecewise linear category. It will now be our task to prove such a statement holds in the smooth category.

[6] A *solid torus of genus n* is a piecewise linear handlebody of genus n.

4. The Solid-Torus Unknotting Criterion for 2-Knots in the Smooth Category

Our proof will be composed of three lemmas.

Lemma 4.1. *Any 2-knot K bounding a piecewise linear 3-ball B^3 embedded in $\mathbb{R}^4$ is such that K also bounds a smooth 3-ball $B^3_{C^\infty}$ in $\mathbb{R}^4$.*

Proof. We have that K bounds B^3. As such, B^3 must agree with the smooth structure of K in some neighbourhood of K. We uniquely smooth B^3 relative to K[7] by theorem 2.5 to obtain some smooth 3-ball $B^3_{C^\infty}$ also bounded by K. We will show that this is smoothly embedded in $\mathbb{R}^4$.

Now, glue a second copy of B^3 along K to obtain a 3-manifold without boundary[8]

$$W = B^3 \cup_K B^3.$$

We denote these piecewise linear 3-balls M and N. Note that $M \cap N = K$. Both M and N admit unique smooth structures by theorem 2.5. We can smooth one copy relative to K, say M, as previously shown, such that M carries the smooth structure $B^3_{C^\infty}$.

Now, take

$$h : W \to W$$

to be the identity homeomorphism. h obviously restricts to diffeomorphisms on M and N (as it is takes them onto themselves). By theorem 2.9, we can choose some global diffeomorphism $\varphi : W \to W$ such that

$$\varphi(M) = h(M) \text{ and } \varphi|_K = h|_K.$$

Restricting this φ to just M we obtain $\varphi(M) = h(M) = B^3_{C^\infty}$. Noting that φ is a diffeomorphism and fixes M at the boundary, we have that $\varphi(M) = B^3_{C^\infty}$ is a smooth 3-ball bounded by K.

We must extend φ to a diffeomorphism in $\mathbb{R}^4$ to show that $B^3_{C^\infty} \subset \mathbb{R}^4$. As W is a smooth submanifold of $\mathbb{R}^4$ and we have that $\varphi : W \to W$ is a diffeomorphism that is the identity on K, the boundary, we can define an isotopy Φ_t such that $\Phi_0 = \text{id}$ and $\Phi_1 = \varphi$. We can then extend this isotopy to all of $\mathbb{R}^4$ (via the isotopy extension theorem 2.8). So Φ_1 is a diffeomorphism on $\mathbb{R}^4$ that takes B^3 to $B^3_{C^\infty}$ whilst keeping K fixed. So K bounds a smooth 3-ball $B^3_{C^\infty}$. □

Lemma 4.2. *Let K be a 2-knot bounding some smooth 3-ball $B^3_{C^\infty} \subset \mathbb{R}^4$. Then K is ambient isotopic to S^2 in $\mathbb{R}^4$.*

Proof. By definition of smooth 3-balls (definition 2.4), we have that $B^3_{C^\infty}$ is diffeomorphic to D^3, the closed unit 3-ball, say via $\varphi : B^3_{C^\infty} \to D^3$. We have that

$$\varphi(\partial B^3_{C^\infty}) = S^2$$

as φ is a diffeomorphism. Choose some collar $c : \partial B^3_{C^\infty} \times [0, \delta) \to B^3_{C^\infty}$ ($\delta \in \mathbb{R}$). Take our φ to be such that $\varphi|_{\partial B^3_{C^\infty}} = \text{id}$ and $\varphi|_{c(\partial B^3_{C^\infty} \times [0,\delta/2])} = \text{id}$ also. This can be done via lemma 2.10, noting that φ is the identity across the collar and the boundary.

Choose some smooth isotopy φ_t such that $\varphi_0 = \text{id}$ the identity map on $B^3_{C^\infty}$, $\varphi_1 = \varphi$ the original diffeomorphism and then for all t, $\varphi_t|_{c(\partial B^3_{C^\infty} \times [0,\delta/2])} = \text{id}$.

[7]When we say 'relative to K' we are choosing the diffeomorphism that keeps a collar of K (which exists by the collaring theorem 2.7) fixed (via 2.10) and extending to the interior via the general isotopy extension theorem (see Hirsch [Hir97] chapter 8) on the isotopy taking id to the diffeomorphism.

[8]The boundary for both copies of the 3-ball cancel and the boundary 'disappears'. This is required to invoke theorem 2.9 later.

We invoke the isotopy extension theorem (2.8) relative to the collar c (i.e. keeping the collar fixed) to extend φ_t to the entirety of $\mathbb{R}^4$. We thus have that there exists some ambient isotopy $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$\Phi_t|_{B^3_{C\infty}} = \varphi_t \text{ and } \Phi_t|_{\partial B^3_{C\infty}} = \mathrm{id}$$

for all $t \in [0,1]$. So we have $\Phi_1(B^3_{C\infty}) = D^3$ and that $\Phi_1(K) = \partial D^3 = S^2$.

We have shown that $B^3_{C\infty}$ is ambient isotopic to $D^3 \subset \mathbb{R}^3 \times \{0\}$ in $\mathbb{R}^4$ via Φ_t. The boundary is preserved under Φ_t such that we may conclude K is ambient isotopic to $S^2 \subset \mathbb{R}^3 \times \{0\}$ in $\mathbb{R}^4$ as required. □

Lemma 4.3. *Let K be a 2-knot that is ambient isotopic to S^2 in $\mathbb{R}^4$. Then K bounds some smooth 3-ball $B^3_{C\infty} \subset \mathbb{R}^4$.*

Proof. By definition, S^2 is ambient isotopic to K, say via some ambient isotopy H_t in $\mathbb{R}^4$. Then H_1 is a diffeomorphism such that

$$H_1(S^2) = K.$$

Taking H_1, a diffeomorphism, on the standard closed unit ball bounded by S^2, $D^3 \subset \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$, gives some

$$H_1(D^3) = B^3_{C\infty}$$

which is a smooth 3-ball by definition (2.4) as it is diffeomorphic to D^3. H_1 preserves the boundary as it is a diffeomorphism, i.e.

$$H_1(S^2) = H_1(\partial D^3) = \partial B^3_{C\infty} = K.$$

and so K bounds a smooth 3-ball $B^3_{C\infty} \subset \mathbb{R}^4$ and we are done. □

Now, we are able to prove our main theorem.

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