Double-Entry Accounting: a Group-Theoretical Treatment

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1 Background

According to a 2017 article by the BBC [8], the first known use of the double-entry system for bookkeeping was in fourteenth century Venice. It was then known as 'bookkeeping alla Veneziana' (Eng. Venetian bookkeeping). It is possible that the Venetians may have taken the idea from elsewhere in history; however, it may have just been a local invention.

The article goes on to outline how two centuries later, the now most famous accountant to ever live, Luca Pacioli (1447 − 1517), would formalise the double-entry system in his 1494 book *Summa de Arithmetica, Geometria, Proportioni et Proportionalita* (Eng. *'Summary of Arithmetic, Geometry, Proportion and Proportionality'*).

As its title suggests, his work was not an accounting textbook by any means. The book was a summary of Renaissance era mathematics. It was broad in its scope, covering a wide variety of mathematics known at the time. In his 2021 article, Sangster [10] alludes to how Luca Pacioli taught accounting as he did algebra - from an axiomatic perspective[1]. He would lay out five generalisable statements which formed the entire foundation for double-entry accounting. This approach will be familiar to any pure mathematicians. Despite this initial lack of distinction between the two fields, the connection is widely unrecognised in the modern era (DaRin [3]).

In 1982, mathematician and economist David Ellerman (b. 1943) wrote his book, *Economics, Accounting and Property Theory* [4]. The book explores connections between economic theory, accounting and property rights - Ellerman discusses the limitations of traditional economic theory introduced by the latter two concepts. However, included within the appendices is a full and mathematically rigorous definition of double-entry accounting using group theory. Ellerman, it seems, was keenly aware of the connection between mathematics and accounting and sought to formalise this for his readers.

Ellerman would at times continue to expand upon this subject in his works, including in his 2014 article *On Double-Entry Bookkeeping: The Mathematical Treatment* [5] - of which many definitions in this work, such as for our group of differences[2], will be taken.

Another noteworthy contribution in this field comes from Cruz Rambaud, García Pérez, Nehmer and Robinson in their 2010 book *Algebraic Models for Accounting Systems* [9]. In said book, the authors develop algebraic models to describe accounting processes and relationships; including the use of monoids in a similar fashion to Ellerman in his 1982 work [4]. Also of particular relevance is the use of balance vectors in chapter 2 - which we shall use to in order to generalise the approach taken by Ellerman. [5]

[1] Sangster [10] rightfully mentions in his article that this terminology was yet to be invented at the time.
[2] Also called the 'Pacioli group' by Ellerman, after Luca Pacioli. [5]

2 The Group of Differences

It is well known in mathematics that the set of the natural numbers inclusive of 0, which we shall denote $\mathbb{N}_0$, does not form a group under the binary operation of addition.

Indeed, Artin [1] states that a defining property of a group G is the presence of some inverse $b \in G$ such that $ab = 1$ for $a \in G$, where 1 is the identity element $1 \in G$, which satisfies $1a = a1 = a$ for all $a \in G$.

It is trivial to see that such an inverse does not exist within $\mathbb{N}_0$. There can be no positive integer b such that $1 + b = 0$ (noting that 0 **does** satisfy the conditions of the identity element).

2.1 Constructing the Group of Differences

One is likely to be left pondering on why we should even need to mention a result so trivial. The purpose of such an acknowledgment is to further our understanding of the motivation behind what Ellerman [5] describes as the *group of differences*, originally introduced by Bourbaki [2] in 1974. This group shall become the key to the mathematical formulation of accounting.

Definition 2.1. Let $\mathbb{N}_0$ denote the set of natural numbers inclusive of 0, i.e. $\mathbb{N} \cup \{0\}$.

Define a relation $\sim$, such that for $a, b, c, d \in \mathbb{N}_0$,

$$(a, b) \sim (c, d) \text{ if and only if } a + d = b + c.^3$$

Proposition 2.2. *$\sim$ is an equivalence relation.*

Proof. We have **reflexivity** is trivial, as

$$a + b = b + a \text{ and thus } (a, b) \sim (a, b).$$

For **transitivity**, assuming that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, we have

$$c + f = d + e \implies c = d + e - f.$$

If we insert this into $a + d = b + c$ we get

$$\begin{aligned} a + d &= b + d + e - f \\ \implies a &= b + e - f \\ \implies a + f &= b + e. \end{aligned}$$

So we have that $(a, b) \sim (e, f)$ and transitivity holds.

Finally, **symmetry**. We have that $a + d = b + c$ if and only if $c + b = d + a$, it follows that

$$(a, b) \sim (c, d) \text{ if and only if } (c, d) \sim (a, b).$$

$\square$

We are now able to define our group of differences, as per Ellerman. [5]

Definition 2.3 (The Group of Differences)**.** Let $\mathbb{N}_0$, $\sim$ be as above. For $a, b \in \mathbb{N}_0$, denote the equivalence class of (a, b) under $\sim$ as $[a, b]$.

Define $G = \{[a, b] : a, b \in \mathbb{N}_0\}$ as the set of equivalence classes of ordered pairs of non-negative integers under $\sim$.

Define some binary operation $+$ to be such that

$$[a, b] + [c, d] = [a + c,\ b + d] \text{ for } [a, b], [c, d] \in G. \text{ [5]}$$

We must prove that the properties of a group hold.

Theorem 2.4. *$(G, +)$ is indeed a group. [5]*

[3] Ellerman [5] refers to this operation as the *cross-sum*.

Proof. Let $[a,b],[c,d] \in G$. We have

$$[a,b]+[c,d]=[a+c,\ b+d].$$

We note that $a+c,\ b+d \in \mathbb{N}_0$, as the sum of any non-negative integers must also be a non-negative integer. Thus, $[a+c,\ b+d] \in G$, i.e. G is closed under $+$.

Consider $[0,0] \in G$. Let us show that this is the identity element of our group. We have

$$[a,b]+[0,0]=[a+0,b+0]=[a,b]=[0+a,0+b]=[0,0]+[a,b].$$

As such, [0,0] is the identity element.

Let also $[e,f] \in G$. To show **associativity**, we note that

$$[a,b]+([c,d]+[e,f])=[a,b]+[c+e,d+f]$$

$$=[a+c+e,b+d+f]$$

$$=[a+c,b+d]+[e+f]=([a,b]+[c,d])+[e,f]$$

Finally, we show the existence of some inverse for each element of G. We claim that the inverse of $[a,b]$ is given by $[b,a]$. Indeed, we have

$$[a,b]+[b,a]=[a+b,b+a],$$

which represents the equivalent class of $(a+b,b+a)\sim(0,0)$, as

$$a+b+0=b+a+0.$$

We thus conclude our group properties hold. $\square$

Going forward, we shall use standard brackets to denote our equivalence classes. I.e. (x,y) instead of $[x,y]$.

3 An Introduction to Double-Entry Accounting

We have shown that the so-called group of differences is indeed a group as per the definition given by Artin [1], which is representative of the standard definition. We infer, naturally, that the common properties of groups hold.

What may not be inferred naturally, however, is the need for such a group. The set of integers, $\mathbb{Z}$, forms a group under addition [1], negating the need to define our group, if the reader would excuse the pun.

The following section introduces the basic premises of double-entry accounting, to give appropriate context to our newly-defined group of differences before we begin to apply this in context. Much of what follows may already be known by the reader, in which case it would be acceptable to skip.

3.1 Assets, Liabilities and Capital

Assets may be defined as resources owned by an entity (such a business) that have the potential to provide future economic benefits (Marriot et al. [7]). Examples of assets include cash, production machinery or credit sales due from customers (known as **trade receivables**).

On the contrary, **liabilities** may be defined as amounts repayable by the entity (Marriot et al. [7]). Examples of liabilities include loans and credit purchases from suppliers (known as **trade payables**).

The amount of money invested in the entity by the owner(s) is known as **capital** (Marriot et al. [7]). Capital can come from many different sources, such as direct cash investments, the issuing of shares, or profits from trade. Capital can also be decreased when an entity makes a loss, or chooses to withdraw a specific amount.

3.2 The Accounting Equation

The above definitions give an informal outline of the main components of a **balance sheet** (also known as a **statement of financial position**). We now establish a relation between these components.

Again from Marriot et al. [7], we have that the **accounting equation** is as follows:

$$A = L + C,$$

where A, L and C denote assets, liabilities and capital respectively.

What this means in precise terms will become clear later. For now, it is enough to acknowledge that the sum of the liabilities and capital of an entity must be equal to their total assets.

3.3 Transactions

Consider a transaction of any sort. A common example is the sale of inventory (an asset) for cash (an asset). The effect of this transaction on the accounting equation is as follows:

1. A will decrease as the inventory is sold;

2. A will increase as cash is received in return.

However, for the accounting equation to hold under these circumstances the net change of A must be 0, noting that there is no direct change in L or C. As such, the value of the inventory sold must be equal to the cash received for the sale - which is often not the case in the real world.

This is the motivation behind **profit** and **loss**.

3.4 Profit and Loss

Returning to our previous example, let us denote the total assets before the sale as A_0, such that our accounting equation can be written as

$$A_0 = L + C$$

Let A be the total assets after the sale. The following holds:

1. If $A - A_0 = 0$ then the inventory was sold at its value and $A = L + C$.

2. Otherwise, $A \neq L + C$.

Define $p = A - A_0$. Then

$$A = L + (C + p).$$

p is our **profit/loss** on the sale, it is considered capital.

Remark. Throughout the above outline of the accounting equation, the reader should note that we have not defined $+$ rigorously, unlike when we discussed the group of differences. The actual meaning of $+$ shall become clear as we begin to discuss **debits** and **credits**. For now, we can just think of $+$ as some additive function that follows the rules we'd expect.

3.5 Ledgers

For practical reasons, assets, liabilities and capital are split into multiple accounts. For example, an entity may have a cash account and a trade receivables account under its asset accounts. These accounts are known as **ledgers** (Marriot et al. [7]).

Ledgers are the foundation of double-entry accounting. When a transaction takes place, there must be a dual effect on the ledgers - an 'increase' and a 'decrease', in order for the accounting equation to continue to hold.

3.6 Debits and Credits

These dual effects on the ledgers are known as **debits** and **credits**; often abbreviated to **Dr** and **Cr** respectively (Marriot et al. [7]).

For some transaction, the total debits MUST equal the total credits. Debits and credits are inverses of one another, i.e. a credit to a ledger will cancel an equal debit to the same ledger.

3.7 T-Accounts

We can represent ledgers as **t-accounts**. In a t-account, we write debits on the left hand side and credits on the right hand side, as follows: (Marriot et al. [7])

[Ledger Name]

Debits	**Credits**
x_0	y_0
x_1	y_0

To better understand how this system works, we present an example of a transaction and its effect on three ledgers.

We again return to a sale. Consider the following t-accounts:[4]

Inventory

1000	

Cash

2000	

Profit/Loss

Now, suppose a sale is made: 500 worth of inventory for 800 in cash.[5] Profit on the sale is $800 - 500 = 300$. The double-entry is as follows:

- Cr Inventory 500
- Dr Cash 800

[4] In accounting, a debit in an asset ledger represents a positive value. For example, here we have 1000 worth of inventory and 2000 in cash.

[5] Under International Financial Reporting Standards (IFRS), the inventory ledger is not adjusted until the end of the accounting period, regardless of when changes actually occurred. Instead, any inventory purchased is debited to profit, offsetting inventory sold in the calculation of profit. Inventory is then manually adjusted based upon recorded closing inventory at a point in time.[6]

- Cr Profit/Loss 300

Note that the debits and credits both sum to 800, as required. Our t-accounts then become:

Inventory

1000	500

Cash

2000	
800	

Profit/Loss

	300

Remark. Thus concludes our introduction to the most basic of double-entry accounting concepts. The reader should be careful to note that we have intentionally excluded much of the context and application throughout this section. Accounting is a broad field, with many intricacies and conventions. We've instead taken an abstract view - one which will lay the groundwork for the mathematical approach we set out to achieve.[6]

[6]For a detailed, contextual introduction to the field of accounting, refer to Introduction to Accounting (Marriot, Pru and Edwards, J.R. and Mellett, Howard J.) [7], where the definitions from this section originate.

4 Double-Entry Accounting using the Group of Differences

Let $x_i \in \mathbb{N}_0$ for $i \in \mathbb{N}_0$. Consider some ledger represented by the following t-account:

$$\begin{array}{c|c} x_0 & y_0 \\ x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{array}$$

Recall that the left and right hand columns represent debits and credits respectively, then we can sum these columns to obtain

$$x = \sum_{i=1}^{n} x_i$$
$$y = \sum_{i=1}^{n} y_i$$

x and y then represent the total debits and total credits for this ledger, respectively.

4.1 Transactions as Ordered Pairs

Let G be the group of differences. From the above, we have shown that a t-account may be represented as an ordered pair of positive integers $(x, y) \in G$, as per Ellerman [5]. From this observation, the significance of our earlier defined group becomes clear.

An initial limitation of this approach is that t-accounts and transactions themselves are indistinguishable. Take our t-account (x, y). Suppose some transaction has the effect of debiting this account by x_{n+1}. We have that the following are equal:

$$\begin{aligned} (x, y) + (x_{n+1}, 0) &= (x + x_{n+1}, y) \\ &= (x_0 + \cdots + x_n + x_{n+1}, y_0 + \cdots + y_n) \\ &= (x_0, 0) + (0, y_0) + \cdots + (x_n, 0) + (0, y_n) + (x_{n+1}, 0) \end{aligned}$$

From the above, we see clearly that our t-account is just the sum of the transactions affecting the t-account, also as ordered pairs. Each time a debit or credit is posted to the t-account, we simply add the transaction to the existing t-account as shown above. In the following we refer to t-accounts, however it is worth noting that these definitions are also applicable to any transactions or partial t-accounts.

4.2 T-Accounts as Equivalence Classes

Ellerman [5] defines a 'debit isomorphism' and a corresponding 'credit isomorphism'[7]. These take elements from G to the integers $\mathbb{Z}$ to determine the balance of the t-account. For our purposes, we shall not use these definitions and instead focus on a 'balance' homomorphism that has a codomain in G. We wish to instead define some function to find the unique reduced form of a t-account (i.e. its balance).

Definition 4.1. Suppose we have some t-account $g = (x, y)$ where $g \in G$. Then the **account balance** $\beta : G \mapsto G$ is given by

$$\beta(g) = \begin{cases} (x - y, 0) & \text{if } x \geq y, \\ (0, y - x) & \text{if } x < y. \end{cases}$$

This definition is an alternative version of Ellerman's [5] definition for the unique reduced form of an ordered pair in the group of differences. We choose to refer to this as the 'account balance'.

Example 4.2. Suppose Elinor's cash t-account is given by $(4300, 2000)$. Then her account balance is given by

$$\beta((4300, 2000)) = (2300, 0),$$

where we have noted that $4300 \geq 2000$.

[7] Cruz Rambaud et al. also refer to such isomorphisms in their book.[9]

Example 4.3. Suppose Anna's trade-payables t-account is given by $(640, 640)$. Then her account balance is given by

$$\beta((640, 640)) = (0, 0),$$

where we have noted that $0 \geq 0$.

Our next theorem is again by Ellerman [5], however we have adjusted the phrasing to suit the context.

Theorem 4.4. *Let $g_1, g_2 \in G$ be t-accounts. Then $g_1 = g_2$ if and only if $\beta(g_1) = \beta(g_2)$.*

Proof. First we show that $g_1 = g_2 \implies \beta(g_1) = \beta(g_2)$. Let $g_1 = (x_1, y_1)$ and $g_2 = (x_2, y_2)$. Without loss of generality, suppose $x_1 \geq y_1$ such that $\beta(g_1) = (x_1 - y_1, 0)$.

Suppose $g_1 = g_2$. Appealing to our notion of equivalence, we obtain

$$\begin{aligned} x_1 + y_2 &= y_1 + x_2 \\ \implies x_1 - y_1 &= x_2 - y_2. \end{aligned}$$

As $x_1 \geq y_1$, it follows that $x_1 - y_1 = x_2 - y_2 \geq 0$ and as such $x_2 \geq y_2$.

From this, we have $\beta(g_2) = (x_2 - y_2, 0)$. We must now show that $(x_1 - y_1, 0) = (x_2 - y_2, 0)$, i.e. $x_1 - y_1 + 0 = 0 + x_2 - y_2$. However, we have already shown this previously. Thus, we have shown that

$$\begin{aligned} g_1 &= g_2 \\ \implies x_1 + y_2 &= y_1 + x_2 \\ \implies x_1 - y_1 &= x_2 - y_2 \\ \implies (x_1 - y_1, 0) &= (x_2 - y_2, 0) \\ \implies \beta(g_1) &= \beta(g_2). \end{aligned}$$

We must now prove that $\beta(g_1) = \beta(g_2) \implies g_1 = g_2$. Suppose then that $\beta(g_1) = \beta(g_2)$. From this, we may deduce that $x_1 - y_1 = x_2 - y_2$ or $y_1 - x_1 = y_2 - x_2$ from our definition of β. Noting that these expressions are equivalent, we have that

$$\begin{aligned} x_1 - y_1 &= x_2 - y_2 \\ \implies x_1 + y_2 &= y_1 + x_2, \end{aligned}$$

which is precisely our definition of equivalence. Thus, $g_1 = g_2$. □

The next corollary follows directly.

Corollary 4.5. *Let g_1 be as above. Then $g_1 = \beta(g_1)$.*

Proof. Trivial. □

Remark 4.6. The purpose of the account balance function is to bring a t-account to its unique reduced form. Going forward, we shall appeal to the above theorem and its corollary implicitly. This should be clear from the context.

Remark 4.7. Thus far, we have established a solid foundation. We defined a group, the group of differences G, such that t-accounts of ledgers in a double-entry accounting system may be represented by its elements. We have shown that transactions may also be represented by elements of this group. Further, we defined an account balance, a group homomorphism that returns the reduced form of some t-account. Next, we will define the accounting equation using this framework.

4.3 The Accounting Equation in the Group of Differences

This definition is again adapted from Ellerman.[5]

Definition 4.8. Let $h \in G$ be such that $h = g_0 + g_1 + \cdots + g_n$ for $n \in N_0$. Then h is **balanced** if we have that

$$\beta(h) = (0, 0).$$

Remark 4.9. The above definition is analagous to the accounting equation. Rather, it is the properties required for some sum of t-accounts to be 'balanced' in the sense that its debits and credits sum to 0.

We present a detailed example that brings together all that we have discussed thus far.

Example 4.10. Let G be the group of differences.

Suppose Isaac runs a business that sells socks. At the beginning of the accounting period, his assets include £640 worth of inventory (socks ready to be sold), a sock-creating machine[8] he purchased for £2200 and the £20 cash in his bank. His only liability is £320 in trade payables, owed to his supplier, Quinn.

Isaac has never prepared accounts for his business before and his knowledge of accounting is very limited. He asks us to work out the value of his business' capital.

Let $(x_0, y_0) \in G$ be Isaac's opening capital. We drop the currency signs for ease. Recall that an asset is expressed as a debit and a liability is expressed as a credit (see section 3.7). Then, in order for the sum of our t-accounts to be balanced (i.e. for the accounting equation to be satisfied), we must have that

$$(640, 0) + (2200, 0) + (20, 0) + (0, 320) + (0, y_0) = (0, 0).$$

Simplifying, we have

$$\begin{aligned}
(640, 0) + (2200, 0) + (20, 0) + (0, 320) + (0, y_0) &= (0, 0) \\
\implies (640 + 2200 + 20, 320) + (0, y_0) &= (0, 0) \\
\implies (640 + 2200 + 20 + 0, 320 + y_0) &= (0, 0) \\
\implies (2860, 320 + y_0) &= (0, 0).
\end{aligned}$$

Appealing to the our notion of equivalence, we have

$$\begin{aligned}
(2860, 320 + y_0) = (2540, y_0) &= (0, 0) \\
\implies y_0 &= 2540
\end{aligned}$$

as our opening capital. Here, we have used the fact that $\beta((2540, 2540)) = (0, 0) \implies (2540, 2540) = (0, 0)$.

Now, suppose the following transactions occur:

1. Isaac purchases an additional £100 of raw materials[9] from Quinn, £50 by cash and £50 on credit to be paid at a later date.

2. The sock-creating machine breaks and must be repaired, costing Isaac £120 as an expense, paid in cash.

3. Isaac sells £200 worth of socks for £400. All sales are received as cash.

4. Isaac repays £160 of his debt to Quinn.

We write these transactions as double-entries (see section 3.6), respectively given by:

1. Dr Inventory 100
   Cr Cash 50
   Cr Trade Payables 50

2. Dr Profit/Loss 120
   Cr Cash 120

3. Dr Cash 400
   Cr Profit/Loss 200
   Cr Inventory 200

4. Dr Trade Payables 160
   Cr Cash 160

[8] Not yet invented.

[9] For this example, presume that raw materials are immediately converted to inventory upon purchase.

First, consider the **Cash** ledger. The ordered pairs representing the effects of the above transactions give

$$\begin{aligned}(0,50)+(0,120)+(400,0)+(0,160)\\ =(400,50+120+160)\\ =(400,330)\\ =(70,0).\end{aligned}$$

Similarly, for **Inventory** we have

$$\begin{aligned}(100,0)+(0,200)\\ =(100,200)\\ =(0,100).\end{aligned}$$

Trade Payables gives

$$\begin{aligned}(160,0)+(0,50)\\ =(160,50)\\ =(110,0).\end{aligned}$$

Finally, for **Profit/Loss** we have

$$\begin{aligned}(120,0)+(0,200)\\ =(120,200)\\ =(0,80),\end{aligned}$$

thus giving £80 as our profit.

We now show that our accounting equation continues to hold. Adding the above ledgers, we obtain

$$\begin{aligned}(70,0)+(0,100)+(110,0)+(0,80)\\ =(70+110,100+80)\\ =(180,180)=(0,0).\end{aligned}$$

From this, we conclude that there are no 'gaps' in our double-entries - the accounting equation has held in spite of these transactions. In other words, Isaac's books 'balance'.

4.4 Limitations of a Single-Dimensional Approach

As the reader is likely to have already observed, this approach of using the single-dimensional group of differences to represent financial statements is incredibly limited. We have shown, without issue, that the presentation is functional and mathematically sound; however, it is not particularly helpful for a number of reasons.

Remark 4.11. The inability to distinguish between t-accounts and transactions is perhaps the biggest flaw of this approach. The purpose of accounting is to encode information. It is no good for us that an entire swath of transactions and balances be encoded in one element of an infinite group.

Remark 4.12. Whilst not useful in terms of encoding information, it is possible to verify that financial statements balance using this approach - see example 4.10.

It is for this reason that one may be inclined to develop the ideas thus far discussed into a broad framework that is more practical. We shall attempt to do exactly that.

5 Multi-Dimensional Double-Entry Accounting using the Group of Differences

In his 2014 paper, Ellerman [5] generalises the group of differences to n-dimensions. Ellerman's motivation for doing so is to allow for 'incommensurate physical quantities' to be measured alongside common monetary units.

Remark 5.1. In our work thus far, we have taken an abstract approach and not specified what elements of G actually measure. We will go forward under the assumption that all elements of G are 'comparable', in the sense that they represent the same common units. This will be one of the key distinctions between Ellerman's work and ours.

5.1 The Cartesian Product of G

Let us first consider $G^n = G \times G \times \cdots \times G$, the cartesian product of G n-times. Let $g_1, g_2 \ldots, g_2 \in G$. Then we have that

$$\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} \in G^n.$$

Define $+$ to be component-wise, such that for some $h_1, h_2 \ldots, h_2 \in G$ we have

$$\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} = \begin{bmatrix} g_1 + h_1 \\ g_2 + h_2 \\ \vdots \\ g_n + h_n \end{bmatrix}.$$

Recall that the elements of G are ordered pairs. Addition between some g_i and h_i for $i \in 1, \ldots, n$ behaves as previously defined in definition 2.3.

Theorem 5.2. *G^n forms a group under component-wise addition.*

Proof. For a general proof that the cartesian product of some group also forms a group under a component-wise operation, please see Chapter 2 of Artin's book. [1] □

5.2 Balance Vectors

Remark 5.3. Cruz Rambaud et al. [9] model accounting systems using an ordered integral domain, as opposed to a group. This allows for the introduction of a multiplicative operation in addition to the additive operation we have already defined. The need for a multiplicative operation lies beyond the scope of our work and thus we shall continue to use our well-defined group; however, it is a model that provides valuable insight worthy of further exploration.

We adapt the definition of a balance vector from Cruz Rambaud et al. [9] to suit our group.

Definition 5.4. Let G^n be as above. Let $g_1, g_2, \ldots, g_n \in G$. Then we have that

$$\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} \in G^n$$

is a **balance vector** if

$$g_1 + g_2 + \cdots + g_n = (0, 0)$$

under our notion of equivalence. Denote the set of balance vectors of G^n by $Bal(G^n)$. [9]

Remark 5.5. In simpler terms, the above defines a balance vector to be an element of G^n such that its components are **balanced** (definition 4.8).

Using the notion of a balance vector, we are able to better encode the properties of individual t-accounts that make up the broader financial statements, whilst still ensuring our accounts sum to $(0, 0)$ (i.e. the accounting equation is still satisfied). This is demonstrated in the below example.

Example 5.6. Roman runs a herbal tea store. He has £1050 cash in the bank. The shop premises have a value of £4000. Roman owes £1200 to a supplier, Lucas. Roman's capital is £3850.

We can present this information as an element of G^n, given by

$$\begin{bmatrix} (1050, 0) \\ (4000, 0) \\ (0, 1200) \\ (0, 3850) \end{bmatrix}.$$

Summing the components of this vector, we obtain

$$\begin{aligned} (1050, 0) + (4000, 0) + (0, 1200) + (0, 3850) \\ = (1050 + 4000, 1200 + 3850) \\ = (5050, 5050) \\ = (0, 0). \end{aligned}$$

Our vector is thus a balance vector and represents a balanced financial statement.

Lemma 5.7. *$Bal(G^n) < G^n$.*

Proof. Let $\mathbf{g}, \mathbf{h} \in Bal(G^n)$. Denote

$$\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} \text{ and } \mathbf{h} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}.$$

We need to show that

$$\mathbf{g} + \mathbf{h} = \begin{bmatrix} g_1 + h_1 \\ g_2 + h_2 \\ \vdots \\ g_n + h_n \end{bmatrix} \in Bal(G^n),$$

i.e. that $Bal(G^n)$ is closed under $+$. We have, from the properties of $Bal(G^n)$, that

$$\begin{aligned} g_1 + g_2 + \cdots + g_n &= (0, 0) \\ &= h_1 + h_2 + \cdots + h_n. \end{aligned}$$

From this, we deduce that

$$\begin{aligned} g_1 + g_2 + \cdots + g_n + h_1 + h_2 + \cdots + h_n \\ = g_1 + h_1 + g_2 + h_2 + \cdots + g_n + h_n \\ = (0, 0), \end{aligned}$$

which is the necessary condition for $\mathbf{g} + \mathbf{h} \in Bal(G^n)$. We have shown closure.

We must now show that $\mathbf{g}^{-1} \in Bal(G^n)$. For $i \in \{1, \ldots, n\}$, we have $g_i + g_i^{-1} = (0, 0)$ where $g_i^{-1} \in G$ is the inverse of g_i. From this, we obtain

$$\begin{aligned} g_1 + g_1^{-1} + g_2 + g_2^{-1} + \cdots + g_n + g_n^{-1} = (0, 0) \\ = (0, 0) + g_1^{-1} + g_2^{-1} + \cdots + g_n^{-1} = (0, 0) \\ \implies g_1^{-1} + g_2^{-1} + \cdots + g_n^{-1} = (0, 0) \end{aligned}$$

owing to the fact that $g_1 + g_2 + \cdots + g_n = (0, 0)$, as $\mathbf{g} \in Bal(G^n)$. As such

$$\mathbf{g}^{-1} = \begin{bmatrix} g_1^{-1} \\ g_2^{-1} \\ \vdots \\ g_n^{-1} \end{bmatrix} \in Bal(G^n).$$

We conclude that $Bal(G^n) < G^n$, as all subgroup conditions are satisfied. $\square$

5.3 Transactions

In the above example, we did not consider the effect of transactions on balance vectors. Rather, we were able to show how balance vectors can be used to model a financial statement at a point in time. In reality of course, we must consider the effects of a constant stream of transactions on the financial statements, i.e. our balance vector.

We appeal to Cruz Rambaud et al. [9] to formally define transactions - something we were unable to do in our single-dimensional model.

Definition 5.8. Let G^n be as above. Let $\mathbf{g}, \mathbf{t} \in Bal(G^n)$ be some financial statement. Define a homomorphism

$$\tau_{\mathbf{t}} : Bal(G^n) \to Bal(G^n)$$

such that

$$\tau_{\mathbf{t}}(\mathbf{g}) = \mathbf{g} + \mathbf{t},$$

then $\tau_{\mathbf{t}}$ is called the **transaction** corresponding to the balance vector $\mathbf{t}$.

The set of all such transactions is given by

$$Trans(G^n) = \{\tau_{\mathbf{t}} : \mathbf{t} \in Bal(G^n)\}. \text{ [9]}$$

Remark 5.9. It is important to justify our choice of definition. Recall that the total debits of any transaction must equal the total credits. As such, any transaction must be also a balance vector, otherwise it is incomplete.

Lemma 5.10. *$\tau_{\mathbf{t}}$ is a group automorphism.*

Proof. Trivial. □

Using the above, one can model the effects of multiple transactions over a period of time as a composition of homomorphisms in $Trans(G^n)$ acting on the opening financial statement $\mathbf{g}$.

5.4 An Example Using Balance Vectors

We finish with an example that utilises the simple yet rigorous model we have devised for accounting systems.

Example 5.11. In this example, assume that the financial year runs from 1st January 20XX to 31st December 20XX.

Jaden runs a lab that produces chemicals to be sold to businesses. Due to the complex nature of the chemicals he produces[10], there are not many transactions over the course of the financial year.

The following table summarises his position at the beginning of the financial year:

n	Ledger	Type	Opening Balance
1	Inventory	Asset	£13860
2	Cash	Asset	£2200
3	Trade Receivables	Asset	£1400
4	Non-Current Assets	Asset	£93000
5	Capital	Capital	£4000
6	Loan	Liability	£42000
7	Trade Payables	Liability	£500
8	Tax Payable	Liability	£0

Table 1: Jaden's accounts at the beginning of the financial year.

[10]Despite this, assume that raw materials are immediately converted to inventory.

We can encode this information in a balance vector, as follows:

$$\mathbf{g} = \begin{bmatrix} (13860, 0) \\ (2200, 0) \\ (1400, 0) \\ (63000, 0) \\ (0, 14000) \\ (0, 64600) \\ (0, 1860) \\ (0, 0) \end{bmatrix} \in Bal(G^8).$$

One can easily verify that this is a balance vector.

Transactions that have taken place during the year are summarised in the following table:

Transaction	**Details**	**Date**
$\mathbf{t_1}$	Jaden pays off £1000 of his loan.	02/02/20XX
$\mathbf{t_2}$	Jaden purchased £10000 worth of raw materials from Krissah on credit.	18/03/20XX
$\mathbf{t_3}$	Jaden sold inventory to Hektor for £7200, half of which is paid in cash and the other half paid on credit.	23/11/20XX
$\mathbf{t_4}$	Jaden's non-current assets depreciated by £3000 during the year. Depreciation is an expense.	31/12/20XX
$\mathbf{t_5}$	Jaden owes £560 in tax for the year. Tax is credited to the Tax Payable account and debited as an expense.	31/12/20XX

We let $\mathbf{t_1}, \ldots, \mathbf{t_5}$ be the balance vectors representing their respective transactions. Then the effect of our transactions on the opening financial statements is given by

$$\tau_{\mathbf{t_5}} \circ \tau_{\mathbf{t_4}} \circ \tau_{\mathbf{t_3}} \circ \tau_{\mathbf{t_2}} \circ \tau_{\mathbf{t_1}}(\mathbf{g}).$$

When evaluated, this gives

$$\begin{bmatrix} (13860, 0) \\ (2200, 0) \\ (1400, 0) \\ (63000, 0) \\ (0, 14000) \\ (0, 64600) \\ (0, 1860) \\ (0, 0) \end{bmatrix} + \begin{bmatrix} (0, 0) \\ (0, 1000) \\ (0, 0) \\ (0, 0) \\ (0, 0) \\ (1000, 0) \\ (0, 0) \\ (0, 0) \end{bmatrix} + \begin{bmatrix} (10000, 0) \\ (0, 0) \\ (0, 0) \\ (0, 0) \\ (0, 0) \\ (0, 0) \\ (0, 10000) \\ (0, 0) \end{bmatrix} + \begin{bmatrix} (0, 7200) \\ (3600, 0) \\ (3600, 0) \\ (0, 0) \\ (0, 0) \\ (0, 0) \\ (0, 0) \\ (0, 0) \end{bmatrix} + \begin{bmatrix} (0, 0) \\ (0, 0) \\ (0, 0) \\ (0, 3000) \\ (3000, 0) \\ (0, 0) \\ (0, 0) \\ (0, 0) \end{bmatrix} + \begin{bmatrix} (0, 0) \\ (0, 0) \\ (0, 0) \\ (0, 0) \\ (560, 0) \\ (0, 0) \\ (0, 0) \\ (0, 560) \end{bmatrix}$$

$$= \begin{bmatrix} (23860, 7200) \\ (5800, 1000) \\ (5000, 0) \\ (63000, 3000) \\ (3560, 14000) \\ (1000, 64600) \\ (0, 11860) \\ (0, 560) \end{bmatrix},$$

which becomes

$$\begin{bmatrix} (16660, 0) \\ (4800, 0) \\ (5000, 0) \\ (60000, 0) \\ (0, 10440) \\ (0, 63600) \\ (0, 11860) \\ (0, 560) \end{bmatrix}$$

using theorem 4.4. Again, one can easily verify this is a balance vector.

Thus, Jaden's books 'balance'. The new balance vector becomes our opening balance in the following year.

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