Lab 1 Problems Written Questions

1.

	$X=0$	$X=1$
$Y=0$	$1/4$	$1/4$
$Y=1$	$1/6$	$1/3$

a) What is the probability that $X=1$?

$$P(X=1) = P(X=1, Y=0) + P(X=1, Y=1)$$
$$= 1/4 + 1/3$$
$$= \boxed{7/12}$$

b) What is the probability that $X=1$ conditioned on $Y=1$?

$$P(X=1 \mid Y=1) = \frac{P(X=1 \mid Y=1)}{P(Y=1)} = \frac{1/3}{1/6+1/3} = \frac{1/3}{1/2} = \boxed{\frac{2}{3}}$$

c) What is the variance of the random variable X?

$$P(X=0) = 1/4 + 1/6 = 5/12$$

$$P(X=x) = \begin{cases} 5/12, & \text{if } x=0 \\ 7/12, & \text{if } x=1 \end{cases}$$

$$E(X) = 0(5/12) + 1(7/12)$$
$$= 7/12$$
$$E(X^2) = 0^2(5/12) + 1^2(7/12)$$
$$= 7/12$$
$$Var(X) = E(X^2) - E^2(X)$$
$$= 7/12 - (7/12)^2$$
$$= 7/12 - 49/144$$
$$= 84/144 - 49/144$$
$$= \boxed{35/144}$$

d) What is the variance of the random variable X conditioned that Y = 1?

$$P(X=0 \mid Y=1) = \frac{P(X=0 \mid Y=1)}{P(Y=1)} = \frac{1/6}{1/6 + 1/3} = \frac{1/6}{3/6} = 1/3$$

$$P(X=x \mid Y=1) = \begin{cases} 1/3, & \text{if } x=0 \\ 2/3, & \text{if } x=1 \end{cases}$$

$$E(X \mid Y=1) = 0(1/3) + 1(2/3) = 2/3$$
$$E(X^2 \mid Y=1) = 0^2(1/3) + 1^2(2/3) = 2/3$$
$$Var(X \mid Y=1) = E(X^2 \mid Y=1) - E^2(X \mid Y=1)$$
$$= \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{3} - \frac{4}{9}$$
$$= 6/9 - 4/9 = \boxed{2/9}$$

e) What is $E[X^3 + X^2 + 3Y^7 [Y=1]]$?

$$E[X^3 \mid Y=1] + E[X^2 \mid Y=1] + 3E[Y^7 \mid Y=1]$$
$$= (0^3)(1/6 + 1/3) + (1^3)(1 - 1/6 + 1/3) + (0^2)(1/6 + 1/3) + (1^2)(1/6 + 1/3)$$
$$+ 3(1^7)$$
$$= 2/3 + 2/3 + 3(1)$$
$$= 4/3 + 3$$
$$= \boxed{13/3}$$

2. $v_1 = [1,1,1]$ and $v_2 = [1,0,0]$
$P_1 = [3,3,3]$, $P_2 = [1,2,3]$, $P_3 = [0,0,1]$

$$P_1|_{\mathbb{R}^3} = \frac{\langle P_1, v_1\rangle}{\langle v_1, v_1\rangle} v_1 + \frac{\langle P_1, v_2\rangle}{\langle v_2, v_2\rangle} v_2$$

Projection $= \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$= 3\langle 1,1,1\rangle + 0\langle 1,0,0\rangle$
$= \langle 3,3,3\rangle + \langle 0,0,0\rangle$
$= \langle 3,3,3\rangle$

$$P_2|_{\mathbb{R}^3} = \frac{\langle P_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 + \frac{\langle P_2, v_2\rangle}{\langle v_2, v_2\rangle} v_2$$

projection $= \begin{bmatrix} 2.5 \\ -1.5 \end{bmatrix}$

$= 2.5\langle 1,1,1\rangle - 1.5\langle 1,0,0\rangle$
$= \langle 2.5, 2.5, 2.5\rangle - \langle 1.5, 0, 0\rangle$
$= \langle 1, 2.5, 2.5)$

$$P_3|_{\mathbb{R}^3} = \frac{\langle P_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 + \frac{\langle P_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2$$

projection $= \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$

$= 0.5(\langle 1,1,1\rangle) - 0.5(\langle 1,0,0\rangle)$

$= \langle 0, 0.5, 0.5)$

coordinates of 3 projected points =
$\boxed{\langle 3,3,3\rangle, \langle 1, 2.5, 2.5\rangle, \langle 0, 0.5, 0.5\rangle}$

3. $P(H) = 2/3$, $X = \#$ of heads

$X \sim \text{Binom}(n = 100, p = 2/3)$

$E[X] = np = 200/3 = \mu$

$Var(x) = np(1-p) = 200/9 = \sigma^2$

n is large, thus by CLT

$$P[X \le 50] = \int_{-\infty}^{50} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$P\left[\frac{X - 200/3}{\sqrt{200/9}} \le \frac{50 - 200/3}{\sqrt{200/9}}\right]$$

$$P\left[\frac{X - 200/3}{\sqrt{200/9}} \le -3.535\right] = \Phi(-3.535)$$

Look up table

$$\Phi(-3.535) \approx \boxed{0.00019}$$