Chapter 1

Analysis

1.1 Sequence

Definitions 1.1. *Sequence x_n in a set X is a function $x : \mathbb{N} \to X$ where $x_n = x(n)$.*

Definitions 1.2. *Subsequence x_{n_k} of a sequence x_n is a function $x \circ n$ where $n : \mathbb{N} \to \mathbb{N},\ n_k = n(k)$ is a strictly increasing sequence.*

1.1.1 Convergence

Definitions 1.3 (metric)**.** *A sequence x_n converges to x if there exists $N \in \mathbb{N}$ such that $\forall n > N,\ d(x_n, x) < \varepsilon$.*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n > N,\ d(x_n, x) < \varepsilon \tag{1.1}$$

Definitions 1.4 (norm)**.** *A sequence x_n converges to x if there exists $N \in \mathbb{N}$ such that $\forall n > N, \|x_n - x\| < \varepsilon$.*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n > N,\ \|x_n, x\| < \varepsilon \tag{1.2}$$

Definitions 1.5 (neighbourhood)**.** *A sequence x_n converges to x if any neighbourhood N of x contains all except finitely many x_n's.*

$$\forall U \in \mathcal{N}_x,\ \exists N \in \mathbb{N},\ \forall n > N,\ x_n \in U \tag{1.3}$$

Remark (subsequence)**.** *A sequence x_n converges to x if and only if every subsequence has a convergent subsequence.*

1.1.2 Limit Point

Definitions 1.6. *x is a limit point of sequence x_n if x_n converges to x.*

Definitions 1.7. *x is a cluster point of sequence x_n, there exists a subsequence x_{n_k} converging to x.*

1.1.3 Cauchy Criterion

Definitions 1.8 (metric). *A sequence x_n is Cauchy if there exists $N \in \mathbb{N}$ such that $\forall n, m > N,\ d(x_n, x_m) < \varepsilon$.*

Definitions 1.9 (norm). *A sequence x_n is Cauchy if there exists $N \in \mathbb{N}$ such that $\forall n, m > N,\ \|x_n - x_m\| < \varepsilon$.*

1.1.4 Complete Space

Definitions 1.10 (complete). *A space is complete if every Cauchy sequence in it converges.*

1.2 Limit Superior/Inferior

Definitions 1.11.

$$\limsup_{n\to\infty} x_n = \inf_{n\geq 0} \sup_{m\geq n} x_n$$

Definitions 1.12.

$$\liminf_{n\to\infty} x_n = \sup_{n\geq 0} \inf_{m\geq n} x_n$$

Remark. $\liminf x_n = I, \limsup x_n = S$ *are the bounds for cluster points of* x_n. *Thus, there are at most finitely many terms outside* $(I - \varepsilon, S + \varepsilon)$. *However,* $[I, S]$ *may not contain any term of* x_n. *For example,* $x_n = (-1)^n(1 + \frac{1}{n})$.

1.2.1 Properties of limit superior/inferior

$$\inf x_n \leq \liminf x_n \leq \limsup x_n \leq \sup x_n$$
$$\liminf a_n + \liminf b_n \leq \liminf(a_n + b_n) \leq \limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$$
$$\liminf a_n \liminf b_n \leq \liminf(a_n b_n) \leq \limsup(a_n b_n) \leq \limsup a_n \limsup b_n$$

Theorem 1.13 (Stolz-Cesaro).

$$\liminf_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} \leq \liminf_{n\to\infty} \frac{a_n}{b_n} \leq \limsup_{n\to\infty} \frac{a_n}{b_n} \leq \limsup_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

1.3 Limit of a function

Definitions 1.14 (limit). *If $f(x_n) \to L$ as $x_n \to a$, then $\lim_{x\to a} f(x) = L$.*

Definitions 1.15 (continuity). *A function $f : X \to Y$ is continuous at $a \in X$, if $\lim_{x\to a} f(x) = f(\lim_{x\to a} x) = f(a)$.*

Theorem 1.16. *Limit is algebraic.*

Suppose $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ exists, then

$$\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) \tag{1.4}$$

$$\lim_{x\to a} f(x) \pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) \tag{1.5}$$

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \; \lim_{x\to a} g(x) \tag{1.6}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \tag{1.7}$$

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} f(x)^{\lim_{x\to a} g(x)} \tag{1.8}$$

Remark (exceptions).

$$\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, \; 0 \pm \infty, \; \infty - \infty, \; 0^0, \; \infty^0, \; 1^{\pm\infty}$$

Theorem 1.17 (L'Hospital/Bernouli).

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Remark (application).

$$\lim_{x\to 0} (2+x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\log(1+x)} = e^{\lim_{x\to 0} \frac{\log(2+x)}{x}} = e^{\lim_{x\to 0} \frac{1}{2+x}} = \sqrt{e}$$

Squeeze theorem Suppose $f(x) \leq g(x) \leq h(x)$ for each x in an open interval containing a (except a). If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then

$$\lim_{x\to a} g(x) = L \tag{1.9}$$

Theorem 1.18 (chain rule). *Suppose $\lim_{x\to a} g(x) = b$ and f is continuous at b, then*

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b) = c \tag{1.10}$$

Remark. *The existence of limit $\lim_{y\to b} f(y) = c$ does not imply $f(b) = c$. If g assumes value b in some neighbourhood of a, then*

$$\lim_{x\to a} g(x) = b, \; \lim_{y\to b} f(x) = c \not\Longrightarrow \lim_{x\to a} f \circ g(x) = c$$

1.4 Limit Inferior/Superior of Functions

Definitions 1.19 (metric).

$$\limsup_{x\to a} f = \lim_{\varepsilon\to 0} \sup_{x\in B(a,\varepsilon)^*} \{f(x)\} = \inf_{\varepsilon>0} \sup_{x\in B(a,\varepsilon)^*} \{f(x)\}$$

$$\liminf_{x\to a} f = \lim_{\varepsilon\to 0} \inf_{x\in B(a,\varepsilon)^*} \{f(x)\} = \sup_{\varepsilon>0} \inf_{x\in B(a,\varepsilon)^*} \{f(x)\}$$

1.5 Sequence of Functions

1.5.1 Notions of Convergence

Definitions 1.20 (pointwise). *Sequence of functions are pointwise convergent if for each $x_0 \in X$, the sequence $f_n(x_0)$ converges to $f(x_0)$.*

$$\textit{(metric)} \quad \forall x \in X,\ \forall \varepsilon > 0,\ \exists N_{x,\varepsilon} \in \mathbb{N},\ \forall n > N_{x,\varepsilon},\ d(f_n(x), f(x)) < \varepsilon \tag{1.11}$$
$$\textit{(norm)} \quad \forall x \in X,\ \forall \varepsilon > 0,\ \exists N_{x,\varepsilon} \in \mathbb{N},\ \forall n > N_{x,\varepsilon},\ \|f_n(x), f(x)\| < \varepsilon \tag{1.12}$$
$$\textit{(nbd)} \quad \forall x \in X,\ \forall U \in \mathcal{N}_{f(x)},\ \exists N_{x,U} \in \mathbb{N},\ \forall n > N_{x,U},\ f_n(x) \in U \tag{1.13}$$

Definitions 1.21 (uniform). *Sequence of functions are uniformly convergent if for each $x \in X$, all the sequences $f_n(x)$ converges to $f(x)$ uniformly.*

$$\textit{(metric)} \quad \forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N},\ \forall x \in X,\ \forall n > N_\varepsilon,\ d(f_n(x), f(x)) < \varepsilon \tag{1.14}$$
$$\textit{(norm)} \quad \forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N},\ \forall x \in X,\ \forall n > N_\varepsilon,\ \|f_n(x), f(x)\| < \varepsilon \tag{1.15}$$
$$\textit{(nbd)} \quad ?? \tag{1.16}$$

1.5.2 Notions of Boundedness

Definitions 1.22 (pointwise). *Sequence of functions is pointwise bounded if for each $x_0 \in X$, the sequence $f_n(x_0)$ is bounded.*

$$\forall x \in X,\ \exists M_x \in \mathbb{R},\ |f_n(x)| < M_x \tag{1.17}$$

Definitions 1.23 (uniform). *Sequence of functions if uniformly bounded if the pointwise bounds are uniform.*

$$\exists M \in \mathbb{R},\ \forall x \in X,\ |f_n(x)| < M \tag{1.18}$$

1.6 Limit of a Set

Definitions 1.24.

$$\liminf X = \inf\{\textit{limit points}\}$$
$$\limsup X = \sup\{\textit{limit points}\}$$

1.7 Sequence of Sets

Definitions 1.25.

$$\liminf X_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} X_n$$
$$\limsup X_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} X_n$$