Chapter 1

Real Analysis

1.1 Algebra of Sets

Definitions 1.1. **set** *a well-defined collection of objects*

subset *set A is a subset of set B if each element of set A is also in set B*

equal *Two sets A and B equal if they have the same elements*

Remark. $A = B \iff A \subset B$ *and* $B \subset A$

Null set, ϕ is the set which contains no elements.

Power set, $P(X)$ is the family of all subsets of the set X.

$\mathbb{N}$ set of all natural numbers

$\mathbb{Z}$ set of all integers

$\mathbb{Q}$ set of all rationals

$\mathbb{R}$ set of all real numbers

$\mathbb{C}$ set of all complex numbers

Definitions 1.2. *Set operations,*

Union/Join $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Intersection/Meet $A \cap B = \{x : x \in A \text{ and } x \in B\}$

Complement/Difference $A - B = \{x \in A : x \notin B\}$

Symmetric Difference $A \Delta B = (A - B) \cup (B - A)$

Remark. *Inclusive property,*

1. $A \subset A \cup B$
2. $A \cap B \subset A$
3. $A - B \subset A$

Remark. $A \subset B \iff A \cap B = A \iff A \cup B = B \iff A - B = \phi$

Remark. *Union and Intersection are idempotent, commutative, associative and distributive.*

disjoint Two sets A and B are disjoint if $A \cap B = \phi$

Definitions 1.3. *The (cartesian) product,* $A \times B = \{ (x, y) : x \in A,\ y \in B \}$

Theorem 1.4. *Augustus de Morgan*

1. $X - (A \cup B) = (X - A) \cap (X - B)$
2. $X - (A \cap B) = (X - A) \cup (X - B)$

1.2 Functions

Definitions 1.5. *A binary relation R from a set A to a set B is a subset of their cartesian product. ie,* $R \subset A \times B$.

Remark. *Let* $X = \{ 1,\ 2,\ 3 \}$. *Then* $R = \{ (1, 2),\ (1, 3),\ (2, 3) \}$ *is a relation on X (to X itself). This relation R is the 'less than' ($<$) relation on X.*

Definitions 1.6. *A function $f : A \to B$ is a subset of $A \times B$ such that for every $x \in A$ there exists a unique $y \in B$ where $(x, y) \in f$.*

image of x Let $(x, y) \in f$, then the image of x, $f(x) = y$

Inverse image $f^{-1}(y) = \{x \in A : y = f(x)\}$ is the fibre of f over y[**?**]

domain of f is the set A

co-domain of f is the set B

range of f is the set $\{y \in B : \exists x \in A,\ y = f(x)\}$

Remark. $f(x) = y \implies x \in f^{-1}(y)$

Definitions 1.7. *Let functions $f : A \to B$ and $g : B \to C$, then the composition $g \circ f : A \to C$ is function such that $g \circ f(x) = g(f(x))$.*

Theorem 1.8. *Let $f : A \to B$ and $g : B \to C$ be functions, then*

1. *domain of $g \circ f$ is the domain of f*
2. *co-domain of $g \circ f$ is the co-domain of g*
3. *range of $g \circ f$ is the range of image of A*

Remark. *Function composition is not commutative.*

Definitions 1.9. *Let $f : A \to B$ be a function, then*

injective *f is injective if images of distinct elment of A are distinct.*

surjective *f is surjective if elements of B are images of some elements of A.*

bijective = *injective + surjective = one-one + onto*

Definitions 1.10. *Let* $f : A \to B$ *and* $g : B \to A$ *are both injective functions such that* $g = \{(b, a) \in B \times A : (a, b) \in f\}$*, then* g *is the inverse function of* f*,* f^{-1}*.*

Remark. $f : X \to Y$ *is injective iff*

1. f *has left inverse. ie,* $\exists g : Y \to X,\ gf = id_X$
2. $C \cap D = \phi \iff f(C) \cap f(D) = \phi$

$f : X \to Y$ *is surjective function iff*

1. f *has right inverse. ie,* $\exists g : Y \to X,\ fg = id_Y$

$f : A \to B$ *is a bijection iff*

1. $f^{-1} \circ f = id_A$ *and* $f \circ f^{-1} = id_B$

Theorem 1.11 (Schroder-Bernstein)**.** $\exists$ *injective function* $f : A \to B$*,* $\exists$ *injective function* $g : B \to A \implies \exists$ *bijection* $h : A \to B$*.*

Remark. *functions & sets*

1. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
2. $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
3. $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$
4. $f(A \cup B) = f(A) \cup f(B)$

Remark. *Let* $f : A \to B$

$$\forall y \in B, |f^{-1}(y)| = \begin{cases} \leq 1, & f \text{ is injective} \\ = 1, & f \text{ is bijection} \\ \geq 1, & f \text{ is surjective} \end{cases}$$

Remark. *The following statements are equivalent:*

1. $f : X \to Y$ *is injective*
2. $\forall A, B \subset X,\ f(A \cap B) = f(A) \cap f(B)$
3. $\forall A, B \subset X,\ f(A - B) = f(A) - f(B)$
4. $\forall A \subset X,\ A = f^{-1} \circ f(A)$

Remark. *The following statements are equivalent:*

1. $f : X \to Y$ *is surjective*
2. $\forall A \subset Y,\ A = f(f^{-1}(A))$

Remark. *Some non-equal sets,*

1. $f(A \cap B) \neq f(A) \cap f(B), \qquad \because f(A) \cap f(B) \not\subset f(A \cap B)$

2. $f(A-B) \neq f(A) - f(B), \quad \because f(A-B) \not\subset f(A) - f(B)$

3. $A \neq f \circ f^{-1}(A), \quad \because A \not\subset f \circ f^{-1}(A)$

4. $A \neq f^{-1} \circ f(A), \quad \because A \not\subset f^{-1} \circ f(A)$

Definitions 1.12. *The restriction of a function $f : A \to B$ into a subset $C \subset A$ is the function $f_{|C} : C \to B$, such that $\forall x \in C,\ f_{|C}(x) = f(x)$*

Definitions 1.13. *An extension of a function $f : A \to B$ into a superset $C \supset A$ is the function $F : C \to B$, such that $\forall x \in A,\ F(x) = f(x)$.*

1.3 Partial Order

Definitions 1.14. *A (binary) relation $\leq$ between set A and set B is a subset of the cartesian product, $A \times B$. $(x, y) \in \leq$ may be written as $x \leq y$.*
A relation on S is a relation from S into S.

reflexive $x \in S \implies xRx$

symmetric $\forall xRy,\ x \neq y \implies yRx$

antisymmetric $\forall xRy,\ x \neq y \implies \neg yRx$

transitive $xRy \ \wedge\ yRz \implies xRz$

connex $\forall x, y \in R \implies (xRy) \ \vee\ (yRx)$

Remark. *Every connex relation is reflexive.*
An equivalence relation R is a reflexive, symmetric and transitive relation.

Definitions 1.15. *An order $<$ is a transitive relation such that,*

$$\forall x, y \in S,\ x < y \ OR\ x = y \ OR\ y < x$$

An order is a reflexive, antisymmetric and transitive relation.

ordered set is a set with an order on it.

strict order is a non-reflexive, antisymmetric and trasitive relation.

diagonal on S is the set $\Delta S = \{(x, x) \in S \times S \ :\ x \in S\}$

total/linear/simple order is a antisymmetric, trasitive and connex relation.

Remark. *For any set X, The set inclusion $\subset$ is a partial order on $P(X)$.*

Axiom 1.16 (Choice)**.** *If $\{A_i : i \in I\}$ is a non-empty family of sets such that A_i is non-empty for each $i \in I$, then $\prod A_i$ is non-empty*

Lemma 1.17 (Zorn)**.** *If every chain in a partially ordered set X has an upper bound in X, then X has a maximal element.*

Remark. *Zorn's lemma is equivalent to the axiom of choice.[**?**]*

1.4 Cardinality

Definitions 1.18. *Set A,B are have same cardinality, if there exists a bijection* $f : A \to B$.

Remark. $card(X) \leq card(Y)$ *if there exists an injective function,* $f : X \to Y$
$card(Y) \leq card(X)$ *if there exists an surjective function,* $f : X \to Y$

Remark. *Cardinality is an equivalence relation on the family of all sets.*

Definitions 1.19. *A set S is finite if* $\exists n \in \mathbb{N}$*, such that* $card(S) = n$.
A set S is countably infinite if $card(S) = card(\mathbb{N})$.
A set S is countable if it is finite or countably infinite.
A set S is uncountable if it neither finite nor countably inifinite.

Theorem 1.20. *Every infinite subset of a countably set is countable.*

Remark. *Countability is the smallest infinity.*

Theorem 1.21. *Every infinite set contains a countable subset.*

Theorem 1.22. *For an infinite set A, the following statements are equivalent:[**?**]*

1. *A is countable*
2. *There exists a subset* $B \subset \mathbb{N}$ *and a surjective function* $f : B \to A$
3. *There exists an injective function* $g : A \to \mathbb{N}$

Theorem 1.23. *Every subset of finite(countable) set is finite(countable).*

Theorem 1.24. *Finite union of finite sets is finite. Countable union of countable sets is countable.*

Remark. *The sets* $\mathbb{N}, \mathbb{Q}$ *are countable. The sets* $(0, 1)$, $\mathbb{R}$ *are uncountable.*

Theorem 1.25. *The set of all sequences in* $\{0, 1\}$ *is uncountable.*

Remark. *The set of all di-adic real numbers is uncountable.*
The set of all integerts is not uncountable ?

Theorem 1.26 (well-ordering)**.** *Every nonempty subset of* $\mathbb{N}$ *has a smallest element in it.*

Theorem 1.27 (induction)**.** *If* $p(1) \wedge (p(k) \implies p(k+1))$*, then* $\forall n \in \mathbb{N},\ p(n)$

Remark. *Well-ordering & induction principles are equivalent.*

Theorem 1.28. *Countable union of countable sets is countable.*

Theorem 1.29. *Finite product of countable sets is countable.*

Remark. *Countable product of countable sets is not necessarily countable.*

Remark. *If* $card(A) \leq card(B)$ *and* $card(B) \leq card(A)$*, then there exists a bijection* $f : A \to B$*. Thus* $card(A) = card(B)$.

Theorem 1.30 (Cantor)**.** *If A is a set, then* $card(A) \leq card(P(A))$ *and* $card(A) \neq card(P(A))$.

Remark. *Cardinality of the null set is 0.*
$card(\mathbb{N}) = \aleph_0$
$card(P(\mathbb{N})) = card(\mathbb{R}) = \aleph_1$

Remark (Continuum hypothesis)**.** *There is no cardinal number between* $\aleph_0$ *and* $\aleph_1$*.*

Remark (Generalised Continuum hypothesis)**.** *For any infinite cardinal* $\aleph_k$*, there is no cardinal number between* $\aleph_k$ *and* $\aleph_{k+1}$*.*

1.5 Real Field

Definitions 1.31. *A binary operation on the set* A *is a function* $\star : A \times A \to A$*.*

Remark. $\star(a, b) = c$ *may be written as* $a \star b = c$ *instead of* $(a, b) \star c$

Axiom 1.32 (Field)**.** *A set* F *with two binary operations* $+, \times$ *is a field if it satifies*

1. $\forall x, y \in F,\ x + y \in F$
2. $\forall x, y \in F,\ x + y = y + x$
3. $\forall x, y, z \in F,\ (x + y) + z = x + (y + z)$
4. $\exists$ *a unique* $0 \in F,\ \forall x \in F,\ x + 0 = x$
5. $\forall x \in F,\ \exists (-x) \in F,\ x + (-x) = 0$
6. $\forall x, y \in F,\ x \times y \in F$
7. $\forall x, y \in F,\ x \times y = y \times x$
8. $\forall x, y, z \in F,\ (x \times y) \times z = x \times (y \times z)$
9. $\exists$ *a unique* $1 \in F,\ \forall x \in F,\ x \times 1 = x$
10. $\forall x \in F,\ x \neq 0,\ \exists x^{-1} \in F,\ x \times x^{-1} = 1$
11. $\forall x, y, z \in F,\ x \times (y + z) = (x \times y) + (x \times z)$

Remark. *Let* $0, 1$ *be additive and multiplicative identities, then* $\forall x, y, z \in \mathbb{R}$*,*

1. $x + y = x + z \iff y = z$
2. $x + y = x \iff y = 0$
3. $x + y = 0 \iff y = -x$
4. $x + y = z \iff x = z + (-y)$
5. $-(-x) = x$
6. *For* $x \neq 0, xy = xz \iff y = z$
7. *For* $x \neq 0, xy = x \iff y = 1$
8. $xy = 1 \iff y = x^{-1}$

9. $xy = z \iff x = zy^{-1}$
10. $(x^{-1})^{-1} = x$
11. $0x = 0$
12. $(-1)x = -x$
13. $(-1)(-1) = 1$
14. $xy = 0 \iff a = 0$ *or* $b = 0$
15. $(-x)(-y) = xy$

Remark. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ *are fields.*

Theorem 1.33. *There doesn't exist a rational number r such that $r^2 = 2$.*

Axiom 1.34 (Order)**.** *An ordered field F is a field with an order $<$ such that,*

1. $\forall a, b \in F$ *exactly one of the statements* $a < b,\ a = b,\ b < a$ *is true.*
2. $\forall x, y, z \in F,\ y < z \implies (x + y) < (x + z)$
3. $\forall x, y \in F,\ 0 < x,\ 0 < y \implies 0 < (x \times y)$

Remark. *Let x, y, z in ordered field $\mathbb{R}$,*

1. $x < 0 \iff -x > 0$
2. $x - y > 0 \iff x > y$
3. $x > 0, y < z \implies xy < xz$
4. $x < 0, y < z \implies xy > xz$
5. $x \neq 0 \implies x^2 > 0$
6. $1 > 0$
7. $0 < x < y \iff 0 < y^{-1} < x^{-1}$
8. $x < y \implies x < \frac{x+y}{2} < y$

Definitions 1.35. *Absolute value of a real number r, $|r| = r$ if $r \geq 0$ and $|r| = -r$ if $r < 0$*

Remark. *Properties,*

1. $|-a| = |a|$
2. $|ab| = |a||b|$
3. $|a + b| \leq |a| + |b|$
4. $|a| \leq b \iff -b \leq a \leq b$
5. $-|a| \leq a \leq |a|$
6. $|a + b| \leq |a| + |b|$

7. $(1+a)^n \leq 1+na, \forall n \in \mathbb{N}$

Definitions 1.36. *Given $a_j < b_j, \forall j$, the set of all points $\boldsymbol{x} \in \mathbb{R}^n$ such that $a_j \leq x_j \leq b_j,\ \forall j$ is an n-cell.*

Definitions 1.37. *Let $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)$. Then, $\boldsymbol{x}+\boldsymbol{y} = (x_1+y_1, x_2+y_2, \cdots, x_n+y_n)$, $c\boldsymbol{x} = (cx_1, cx_2, \cdots, cx_n)$ and*

$$|\boldsymbol{x}.\boldsymbol{y}| = \sum_{j=1}^{n} x_j y_j \quad \& \quad |\boldsymbol{x}| = \left(\sum_{j=1}^{n} |x_j|^2\right)^{\frac{1}{2}}$$

Remark. *Let $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \mathbb{R}^n$,*

1. *For $\boldsymbol{x} \neq \boldsymbol{0}$, $|\boldsymbol{x}| > 0$*
2. $|c\boldsymbol{x}| = |c||\boldsymbol{x}|$
3. $|\boldsymbol{x}.\boldsymbol{y}| \leq |\boldsymbol{x}||\boldsymbol{y}|$
4. $|\boldsymbol{x}+\boldsymbol{y}| \leq |\boldsymbol{x}| + |\boldsymbol{y}|$

Remark.

$$\text{Lagrange's identity,} \quad \left(\sum_{j=1}^{n} a_j b_j\right)^2 = \sum_{j=1}^{n} a_j^2 \sum_{k=1}^{n} b_k^2 - \frac{1}{2}\sum_{j,k=1}^{n} (a_j b_k - b_k a_j)^2 \tag{1.1}$$

$$\text{Cauchy's inequality,} \quad \left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \sum_{j=1}^{n} a_j^2 \sum_{k=1}^{n} b_k^2 \tag{1.2}$$

$$\text{Triangular inequality,} \quad \left(\sum_{j=1}^{n} (a_j+b_j)^2\right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^{n} a_j\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} b_j^2\right)^{\frac{1}{2}} \tag{1.3}$$

Remark.

$$\text{Bernouli's inequality,} \ (1+x)^n \geq 1+nx \tag{1.4}$$

$$\text{by Mean Value Theorem,} \ a^r b^{(1-r)} \leq ra + (1-r)b, \quad 0<r<1, a>0, b>0 \tag{1.5}$$

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q}, \quad \text{where } \frac{1}{p}+\frac{1}{q} = 1 \tag{1.6}$$

$$\text{Holder's inequality,} \ \sum_{j=1}^{n} a_j b_j \leq \left(\sum_{j=1}^{n} a_j^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} b_j^q\right)^{\frac{1}{q}} \tag{1.7}$$

$$\text{Minkowski's inequality,} \ \left(\sum_{j=1}^{n} (a_j+b_j)^r\right)^{\frac{1}{r}} \leq \left(\sum_{j=1}^{n} a_j^r\right)^{\frac{1}{r}} + \left(\sum_{j=1}^{n} b_j^r\right)^{\frac{1}{r}} \tag{1.8}$$

$$\text{Chebyshev's inequality,} \ \left(\frac{1}{n}\sum_{j=1}^{n} a_j^r\right)^{\frac{1}{r}} \left(\frac{1}{n}\sum_{j=1}^{n} b_j^r\right)^{\frac{1}{r}} \leq \left(\frac{1}{n}\sum_{j=1}^{n} (a_j b_j)^r\right)^{\frac{1}{r}}, \quad a_j \leq a_{j+1},\ b_j \leq b_{j+1} \tag{1.9}$$

Definitions 1.38. *A subset X of $\mathbb{R}^n$ is convex if for any two points $\boldsymbol{x},\boldsymbol{y} \in X$ and real number λ such that $0 < \lambda < 1$, every points $\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in X$.*

Definitions 1.39. *extrema of A,*

maximal *An element $x \in A$ is maximal if $\nexists y \in A$ such that $x < y$.*

minimal *An element $x \in A$ is minimal if $\nexists y \in A$ such that $y < x$.*

maximum *An element $x \in A$ is maximum if $\forall y \in A,\ y < x$.*

minimum *An element $x \in A$ is minimum if $\forall y \in A,\ x < y$.*

Definitions 1.40. *An element x of an ordered set S,R is an upper bound a subset $E \subset S$ if $\forall y \in E,\ \neg xRy$. A subset E of the ordered set S is bounded above if $\exists$ a upper bound of E, x in S.*

Definitions 1.41. *An element x of an ordered set S,R is a lower bound a subset $E \subset S$ if $\forall y \in E,\ \neg yRx$. A subset E of the ordered set S is bounded below if $\exists$ a lower bound of E, x in S.*

Definitions 1.42. *The supremum of a subset E, $\sup E$ of an ordered set S is the lower bound of all upper bounds of the set E in S.*

Definitions 1.43. *The infimum of a subset E, $\inf E$ of an ordered set S is the upper bound of all lower bounds of the set E in S.*

Remark. *ϵ Characterisation,*

1. $x = \sup E \iff \forall \epsilon > 0,\ \exists y \in E,$ *such that* $x - \epsilon < y$
2. $x = \inf E \iff \forall \epsilon > 0,\ \exists y \in E,$ *such that* $y < x + \epsilon$

Remark. *Properties,*

1. $\sup_{x,y} f(x,y) = \sup_x \sup_y f(x,y) = \sup_y \sup_x f(x,y)$
2. $\sup_y \inf_x f(x,y) \leq \inf_x \sup_y f(x,y)$
3. $\sup(a + f(x)) = a + \sup f(x)$
4. $\inf f + g(x) \leq \inf f(x) + \inf g(x) \leq \sup f(x) + \sup g(x) \leq \sup f + g(x)$

Definitions 1.44. *An ordered set S is complete if every nonempty subset $E \subset S$, which is bounded above, has $\sup E \in S$.*

Theorem 1.45. *There exists a unique complete ordered field $\mathbb{R}$, that contains $\mathbb{Q}$.*

Axiom 1.46 (Completeness)**.** *A set S is complete, if every nonempty subset $E \subset S$, which is bounded above has a least upper bouned in S.*

Remark. *Every cachy sequence in a complete space is convergent.*

Theorem 1.47. *There exists a complete ordered field $\mathbb{R}$. Moreover $\mathbb{Q} \subset \mathbb{R}$.*

Remark. $S \subset T \implies \inf S \leq \inf T \leq \sup T \leq \sup S$
$\sup A \cup B = \max\{\sup A, \sup B\} \quad \inf A \cup B = \min\{\inf A, \inf B\}$

Remark. *It is possible for a set to have no maximum and yet be bounded above. But, if a set is not bounded above, it doesn't have a maximum. For example : Open interval,* $(0, 1)$ *doesn't have it's extrema.*

Remark. $\inf \phi = \infty$*, and* $\sup \phi = -\infty$
Set E *is unbounded above, then* $\sup E = \infty$
Set E *is unbounded below, then* $\inf E = -\infty$

Theorem 1.48. $\mathbb{N}$ *is not bounded above.*

Theorem 1.49 (Archimedean)**.** $\forall x, y \in \mathbb{R},\ 0 < x,\ \exists n \in \mathbb{N}$ *such that* $y < nx$

Remark. *The following statements are equivalent,*

1. $\exists n \in \mathbb{N}$ *such that* $y < nx$
2. $\exists n \in \mathbb{N}$ *such that* $0 < \frac{1}{n} < y$
3. $\exists n \in \mathbb{N}$ *such that* $n - 1 \leq y < n$

Theorem 1.50. *A subset* A *of* $\mathbb{R}$ *is open iff it is a countable union of open intervals.*

Theorem 1.51. *For every positive real number* $x \in \mathbb{R}$*, there exists a unique* $y \in R$*, such that* $y^n = x$*. We write,* $y = x^{\frac{1}{n}}$*.*

Corollary 1.51.1. *Let* a, b *be positive real numbers and* n *be a positive integer.*

$$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$$

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Theorem 1.52 (nested interval)**.** *Let* $I_1 \supset I_2 \cdots I_n$ *be a sequence of closed, bounded, non-empty intervals, then there exists* $x \in \mathbb{R}$ *such that* $x \in J_k,\ \forall k$

Remark. *The family of closed, bounded intervals have countable intersection property. The nested interval theorem fails for open intervals.*

Theorem 1.53 (nested cell)**.** *Let* $\{J_n\}$ *be a sequence of non-empty, closed nested cells in* $\mathbb{R}^k$*, then there exists* $\boldsymbol{x} \in \mathbb{R}^k$ *such that* $\boldsymbol{x} \in J_k,\ \forall k$

Theorem 1.54. *Every nonempty finite subset of* $\mathbb{R}$ *has its extrema in it.*

Theorem 1.55. $\mathbb{Q}$ *is dense in* $\mathbb{R}$*.*
$\forall x, y \in \mathbb{R},\ \exists q \in \mathbb{Q}$ *such that* $x < q < y$

1.6 Complex Field

Definitions 1.56. *A complex number* $z \in \mathbb{C}$ *is an ordered pair of real numbers,* $(u, v) \in \mathbb{R} \times \mathbb{R}$*.*

Theorem 1.57. *The set of all complex numbers is a field with*
addition, $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$*, defined by* $(a, b) + (c, d) = (a + c, b + d)$ *and*
multiplication, $\cdot : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$*, defined by* $(a, b)(c, d) = (ac - bd, ad + bc)$

Remark. *$\mathbb{C}$ is a field on $\mathbb{R}^2$. There doesn't exists a field on $\mathbb{R}^n$ for $n > 2$.*[1]

Definitions 1.58. $i = (0, 1)$

Theorem 1.59. $i^2 = (-1, 0)$, $(a, b) = a + ib$

Definitions 1.60. *The conjugate of a complex number $a + ib$ is $a - ib$.*

Theorem 1.61. *If $z, w \in \mathbb{C}$,*

1. $\overline{z + w} = \bar{z} + \bar{w}$
2. $\overline{zw} = \bar{z}\bar{w}$
3. $z + \bar{z} = 2\Re(z)$
4. $z - \bar{z} = 2\Im(z)$
5. *For $z \neq 0$, $z\bar{z}$ is a positive real number*

Definitions 1.62. *The absolute value $|z|$ is the non-negative square root of $z\bar{z}$.*

Theorem 1.63. *Let $z, w \in \mathbb{C}$,*

1. $|z| = |\bar{z}|$
2. $|zw| = |z||w|$
3. $|\Re(z)| \leq |z|$
4. $|z + w| < |z| + |w|$

Definitions 1.64. *Let $\boldsymbol{z},\boldsymbol{w} \in \mathbb{C}^n$, $\boldsymbol{z} = (a_1, a_2, \cdots, a_n)$, $\boldsymbol{w} = (b_1, b_2, \cdots, b_n)$. Then,*

$$|\boldsymbol{z}.\boldsymbol{w}| = \left|\sum_{j=1}^{n} a_j \bar{b_j}\right| \quad \& \quad \|\boldsymbol{z}\| = \left(\sum_{j=1}^{n} |a_j|^2\right)^{\frac{1}{2}}$$

Theorem 1.65 (Cauchy-Schwarz-Buniakowsky)**.** $|\boldsymbol{z}.\boldsymbol{w}| \leq \|\boldsymbol{z}\|\|\boldsymbol{w}\|$

$$\left|\sum_{j=1}^{n} a_j \bar{b_j}\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

1.7 Sequences

Definitions 1.66. *A sequence $\{x_n\}$ in set $\mathbb{R}$ is a function $x : \mathbb{N} \to \mathbb{R}$ where $x_n = x(k)$.*[2]

Definitions 1.67. *The range of a sequence $\{x_n\}$ is the set $\{x_k : k \in \mathbb{N}\}$. The range of any sequence is countable.*

[1] proof reference : not found yet

[2] A sequence $\{x_n\}$ on the set X is a function $x : \mathbb{N} \to X$. The kth term x_k of the given sequence is the image $x(k) \in X$.

Definitions 1.68. *Given a sequence* $\{x_n\},\ f : \mathbb{N} \to X,\ x_n = f(n)$ *and a monotone function* $g : \mathbb{N} \to \mathbb{N},\ n_k = g(k)$*, then the sequence* $\{y_k\},\ y_k = x_{n_k} = (f \circ g)(k)$ *is a subsequence of* $\{x_n\}$*.*

Axiom 1.69 (Dependent Choice)**.** *Let* $\leq$ *be a relation on* X *such that every element* $x \in X$ *is related to some element of* X*, then there exists a sequence for each element* $x \in X$ *such that* $x_1 = x$ *and* $x_k \leq x_{k+1}$ *for every integer* $k \in \mathbb{N}$*.[**?**]*

Theorem 1.70 (Recursive Definition)**.** *Given a function* $f : X \to X$*, for every* $x \in X$ *there exists a unique sequence* $\{x_n\}$ *such that* $x_1 = x,\ x_{k+1} = f(x_k),\ \forall k \in \mathbb{N}$

Corollary 1.70.1 (Generalised Recursive Definition)**.** *Given a sequence of functions* $f_n : X^n \to X$*, for every* $x \in X$ *there exists a unique sequence* $\{x_n\}$ *such that* $x_1 = x,\ x_k + 1 = f_k(x_1, x_2, \cdots, x_k),\ \forall k \in \mathbb{N}$

1.8 Convergence of Sequences

Definitions 1.71. *A sequence* $\{x_n\}$ *converges to* $x \in \mathbb{R}$ *if for every* $\epsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that* $|x - x_n| < \epsilon$ *for every* $k \geq N$*. The real number* x *is the limit of the sequence* $\{x_n\}$*.*

Definitions 1.72. *A sequence is cauchy if for every* $\epsilon > 0$*, there exists an integer* $N \in \mathbb{N}$ *such that for every* $n, m > N$*,* $|x_n - x_m| < \epsilon$

Remark. *A sequence of real numbers converges iff cauchy.*

Theorem 1.73. *Every bounded monotone sequence* $\{x_n\}$ *in* $\mathbb{R}$ *converges. If monotone decreasing, limit is* $\inf\{x_n\}$*. If monotone increasing, limit is* $\sup\{x_n\}$*.*

Definitions 1.74. *A real number* x *is a limit point (cluster point) of the sequence* $\{x_n\}$ *if for every* $\epsilon > 0$ *and every integer* $N \in \mathbb{N}$*, there exists an integer* $k > N$ *such that* $|x_n - x| < \epsilon$*.*

Remark. *A real number* x *is a limit point of the sequence* $\{x_n\}$ *iff there is a subsequence converging to* x*.*

Remark. *Every convergent sequence is bounded.*

Remark. *A sequence of real numbers can have atmost one limit.*

Remark. *Suppose* $\lim x_n = x,\ \lim y_n = y$

1. $x_n \leq y_n \implies x \leq y$
2. $\lim(\alpha x_n + \beta y_n) = \alpha x + \beta y$
3. $\lim x_n y_n = xy$
4. *Suppose* $|y_n| \geq \delta$ *for some* $\delta > 0,$ *then* $\lim \frac{x_n}{y_n} = \frac{x}{y}$

Lemma 1.75 (Sandwitch)**.** *Suppose* $\{x_n\},\ \{y_n\},\ \{z_n\}$ *are sequences such that* $x_n \leq z_n \leq y_n,\ \forall n \in \mathbb{N}$*. If* $\lim x_n = \lim y_n = x$*, then* $\lim z_n = x$*.*

1.9 limits superior, limits inferior

Definitions 1.76. *limit inferior,* $\underline{\lim}\{x_n\} = \sup\{\inf\{x_m : m \geq n\}\}$
limit superior, $\overline{\lim}\{x_n\} = \inf\{\sup\{x_m : m \geq n\}\}$

Remark. *Let $\{x_n\}$ be a bounded sequence, then limit superior and limit inferior are the largest and smallest limit points of it.[**?**]*
$\underline{\lim}\{x_n\} \leq \overline{\lim}\{x_n\}$

Remark. *For $\{x_n\},\ \{y_n\}$ such that $x_n \leq y_n$ for every integer $n \in \mathbb{N}$*

1. $\overline{\lim}\{x_n\} \leq \overline{\lim}\{y_n\}$
2. $\underline{\lim}\{x_n\} \leq \underline{\lim}\{y_n\}$

1.10 Convergence Test by Sandwitch Lemma

Convergence of sequences can be tested using the Sandwitch Lemma.

Remark.

$$\underline{\lim}\{x_n\} + \underline{\lim}\{y_n\} \leq \underline{\lim}\{x_n + y_n\} \leq \overline{\lim}\{x_n + y_n\} \leq \overline{\lim}\{x_n\} + \overline{\lim}\{y_n\}$$

If $\{x_n\}$ or $\{y_n\}$ converges, then $\lim\{x_n + y_n\} = \lim x_n + \lim y_n$

Remark. *Let $\{x_n\}$ such that $\sqrt{n} = (1 + x_n)^n$.*
By Bernouli's inequality, $\sqrt{n} = (1 + x_n)^n \geq 1 + nx \implies 0 < x < \frac{1}{\sqrt{n}}$.

$$\underline{\lim}\,\frac{x_n + 1}{x_n} \leq \underline{\lim}\,\sqrt[n]{n} \leq \overline{\lim}\,\sqrt[n]{n} \leq \overline{\lim}\,\frac{x_n + 1}{x_n}$$

And $\lim \frac{x_n+1}{x_n} = 1$. Thus $\lim \sqrt[n]{n} = 1$

Definitions 1.77. *Sequence $\{a_n\}$ such that $a_k = \frac{1}{k}\sum_{n=1}^{k} x_k$ is the sequence of the averages of a sequence $\{x_n\}$*

Remark. *Let $\{a_n\}$ be the sequence of averages of the sequence $\{x_n\}$,*

1. $\underline{\lim}\, x_n \leq \underline{\lim}\, a_n \leq \overline{\lim}\, a_n \leq \overline{\lim}\, x_n$
2. *If $\{x_n\}$ converges, then $\{a_n\}$ converges.*
3. *If $\{a_n\}$ diverges, then $\{x_n\}$ diverges.*
4. *$\{a_n\}$ converges, then $\{x_n\}$ need not converge.* [3]

Remark (Results by Average Sequence)**.** *1.* $x_{n+1} - x_n \to x \implies \frac{x_n}{n} \to x$

2. *If $\{x_n\}$ is bounded and $2x_n \leq x_{n+1} + x_{n-1} \implies \{x_n + 1 - x_n\}$ is monotone increasing to 0. pending pp.28 exr 12b[**?**]*
3. *$0 < x_1 < 1,\ x_{n+1} = 1 - \sqrt{1 - x_n}$, then x_n monotone decreasing to 0 and $\frac{x_{n+1}}{x_n}$ convergent to $\frac{1}{2}$.*

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[3] $\{x_n\}$ such that $x_n = -1^n$

4. $\{x_n\},\ x_n = \left(1+\frac{1}{n}\right)^n$ *is convergent*

5. $\{x_n\},\ |x_n - x_{n-1}| \le \alpha|x_{n+1} - x_n|,\ 0 < \alpha < 1$ *is convergent*

6. $x_1 = 1,\ x_{n+1} = \frac{1}{3+x_n}$ *is convergent to ?*

7. $x_1 = 1,\ x_{n+1} = 1 + \frac{1}{1+x_n}$ *is convergent to* $\sqrt{2}$

8. $x_1 = 1,\ x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$ *is convergent to* $\sqrt{2}$

1.11 Series

Definitions 1.78. *A series $\sum_{n=1}^{\infty} x_n$ is convergent if the sequence of partial sums $\{y_k\}$ such that $y_k = \sum_{n=1}^{k} x_n$ is convergent.*

Definitions 1.79. *A series $\sum_{n=1}^{\infty} x_n$ is rearrangement invariant if for every bijection $\sigma : \mathbb{N} \to \mathbb{N}$, $\sum_{k=1}^{\infty} x_{\sigma_k}$ converges and is invariant.*

Theorem 1.80. *A series $\sum_{n=1}^{\infty} x_n$ is rearrangement invariant if for every integer $n \in \mathbb{N}$, $x_n \ge 0$.*

Remark. *Series that are rearrangement invariant are unconditionally convergent.*

1.12 Double Series

Definitions 1.81. *A double series $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{n,m}$ is $\lim_{k\to\infty}\sum_{n=1}^{k}\sum_{m=1}^{\infty} x_{n,m}$.*

Theorem 1.82. *If for every $n, m \in \mathbb{N}$, $a_{n,m} \ge 0$, then $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{n,m} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{n,m}$.*

Theorem 1.83 (double series into single series)**.** *If for every $n, m \in \mathbb{N}$, $x_{n,m} \ge 0$ and for every bijection $\sigma : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$, $\sum_{k=1}^{\infty} x_{\sigma_k} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{n,m}$*

Corollary 1.83.1. *If for every $n, m \in \mathbb{N}$, $x_{n,m} \ge 0$ and for every bijection $\sigma : \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N}$, $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} x_{\sigma_{j,k}} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{n,m}$*

1.13 Convergence of series

1.14 Bolzano-Weierstrass Theorem

Theorem 1.84. *Every sequence $\{x_n\}$ in $\mathbb{R}$ has subsequences converging to $\underline{\lim}\{x_n\}$ and $\overline{\lim}\{x_n\}$.*

Remark. *A sequence $\{x_n\}$ is convergent iff $\underline{\lim}\{x_n\} = \overline{\lim}\{x_n\}$*

Theorem 1.85. *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*[4]

Theorem 1.86. *Every bounded sequence $\{x_n\}$ in $\mathbb{R}$ converges iff $\underline{\lim}\{x_n\} = \overline{\lim}\{x_n\}$.*

[4]Every bounded sequence in $\mathbb{R}$ has subsequences converging to $\underline{\lim}$ and $\overline{\lim}$. [**?**]

Definitions 1.87. *A sequence* $< x_k >$ *is Cauchy, if* $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ *such that* $\forall n, m > N,\ |a_n - a_m| < \epsilon$.

Theorem 1.88 (Cauchy)**.** *A sequence in* $\mathbb{R}$ *is convergent iff it is a Cauchy sequence.*

Remark. *For a sequence* $\{x_n\}$ *in* $\mathbb{R}$ *following statements are equivalent :*

1. $\{x_n\}$ *converges to* x
2. *Every subsequence of* $\{x_n\}$ *has a subsequence converging to* x
3. $\{n_k\}, \{m_k\}$ *be sequences in* $\mathbb{N}$*, then* $\lim x_{n_k} = \lim x_{m_k} = x$

Remark. $\lim\{x_n\} = x$ *iff* $\lim\{x_{2n}\} = \lim\{x_{2n+1}\} = x$

Theorem 1.89 (Bolzano-Weierstrass)**.** *Every bounded, infinite subset* A *of* $\mathbb{R}^k$ *has a cluster point.*

Remark. *Cluster points of* $A \cup B$ *are either cluster points of* A *or of* B*.*

1.15 Heine-Borel Theorem

Theorem 1.90. *A subset of* $\mathbb{R}^p$ *is compact iff closed and bounded.*

Remark. *Applications*

1. *Cantor Intersection Theorem*
2. *Lebesgue Covering Theorem*
3. *Nearest Point Theorem*
4. *Circumscribing Contour Theorem*

Theorem 1.91 (Cantor intersection)**.** *Let* $F_1, F_2, \cdots$ *be non-empty, closed, bounded subsets of* $\mathbb{R}^p$ *such that* $F_1 \supset F_2 \supset \cdots$*. Then there exists a point* y *such that* $y \in F_k,\ \forall k$*.*

Theorem 1.92 (Lebesgue covering)**.** *Let* K *be a compact subset of* $\mathbb{R}^p$ *and* $\mathcal{U}$ *be a cover of* K*. There exists a real number* $r > 0$ *such that* $\forall x \in K,\ B(x, r) \subset U$ *for some* $U \in \mathcal{U}$*.*

Theorem 1.93 (Nearest Point)**.** *Let* K *be a compact subset of* $\mathbb{R}^p$ *and* $x \notin K$*, then there exists* $y \in K$ *such that* $|x - y| \leq |x - z|,\ \forall z \in K$*.*

Theorem 1.94 (Circumscribing contour)**.** *Let* K *be a compact subset of* $\mathbb{R}^2$ *and* G *be an open set containing* K*. Then there exists a closed curve contained in* G *made up of arcs of finite number of circles in* G *such that* K *is contained in it.*

1.16 Continuity

Definitions 1.95. *$f : X \to Y$ is continuous if $\forall \epsilon > 0,\ \forall x \in X,\ \exists \delta_x > 0$ such that $|x - y| < \delta_x \implies |f(x) - f(y)| < \epsilon$.*

Definitions 1.96 (continuity)**.** *f is continuous at x if*
Cauchy - $\forall \epsilon > 0,\ \exists \delta$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.
Heine - Sequence $< x_k > \to x \implies < f(x_k) > \to f(x)$.
Topology - $\forall V \in \mathcal{N}_{f(x)},\ \exists U \in \mathcal{N}_x$ such that $f(U) \subset V$.

1.17 Uniform Continuity

Definitions 1.97. *$f : X \to Y$ is uniformly continuous if $\forall \epsilon > 0,\ \exists \delta > 0$ such that $\forall x \in X,\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.*

Remark. *f is continuous at y if $\lim_{x \to -y} f(x) = \lim_{x \to +y} f(x) = f(y)$*

Theorem 1.98. *A continuous function from a closed bounded interval into $\mathbb{R}$ is uniformly continuous.*

Theorem 1.99. *A continuous function from a closed bounded interval into $\mathbb{R}$ is bounded and attains its extrema.*

1.18 Differentiability

Definitions 1.100. *f is differentiable at y if $\lim_{x \to y} \frac{f(x) - f(y)}{x - y}$ exists.*

1.19 Mean Value Theorem

Theorem 1.101 (intermediate value)**.** *Let continous function $f : [a, b] \to \mathbb{R}$, then $f([a, b]) = [f(a), f(b)]$.*

1.20 Sequence of functions

1.21 Bounded sequence of functions

Definitions 1.102. *A sequence of functions $\{f_n\}$ is bounded if there exists a real-valued function g such that for every $x \in X$ and for every integer $n \in \mathbb{N}$, $|f_n(x)| < g(x)$.*

Definitions 1.103. *If the sequence $\{f_n\}$ is a bounded sequence of functions, then $\overline{\lim} f_n$ and $\underline{\lim} f_n$ are functions defined by $(\overline{\lim} f_n)(x) = \overline{\lim} f_n(x)$ and $(\underline{\lim} f_n)(x) = \underline{\lim} f_n(x)$*

1.22 Series of functions

1.23 Uniform Convergence

Chapter 2

Linear Algebra

2.1 Vector Space

Axiom 2.1 (Field). *A set F together with two binary operations $+, \cdot$ is a field if it satisfies*

1. *Addition is commutative,* $\forall x, y, z \in F,\ x + y = y + x$
2. *Addition is associative,* $\forall x, y, z \in F,\ x + (y + z) = (x + y) + z$
3. *Existence of additive identity,* $\exists 0 \in F,\ x + 0 = x$
4. *Existence of additive inverses,* $\forall x \in F,\ \exists - x \in F,\ x + (-x) = 0$
5. *Multiplication is commutative,* $\forall x, y \in F,\ xy = yx$
6. *Multiplication is associative,* $\forall x, y, z \in F,\ x(yz) = (xy)z$
7. *Existence of multiplicative identity,* $\exists 1 \in F,\ \forall x \in F,\ 1x = x$
8. *Existence of multiplicative inverses,* $\forall x \in F,\ x \neq 0,\ \exists x^{-1} \in F,\ xx^{-1} = 1$
9. *Multiplication is distributive over addition,* $\forall x, y, z \in F,\ x(y+z) = xy+xz$

Remark. *A few fields,*

$\mathbb{Q}$ *field of all rational numbers*

$\mathbb{R}$ *field of all real numbers*

$\mathbb{C}$ *field of all complex numbers*

$\mathbb{Z}_{p^n}$ *Galois field of prime powers*

$\mathbb{Q}(v)$ *algebraic extensions of* $\mathbb{Q}$

$\mathbb{R}(v)$ *algebraic extensions of* $\mathbb{R}$

Axiom 2.2 (Vector Space). *A set V of vectors and a field F of scalars together with two binary operations, vector addition, $+ : V \times V \to V$ and scalar muliplication $\cdot : F \times V \to V$ is a vector space V over F, if it satifies*

1. *Addition is commutative,* $\forall u, v \in V,\ u + v = v + u$
2. *Addition is associtive,* $\forall u, v, w \in V,\ u + (v + w) = (u + v) + w$
3. *Additive identity,* $\exists 0 \in V$*, such that* $\forall v \in V,\ 0 + v = v$
4. *Additive inverses,* $\forall v \in V$*, there exists* $-v \in V$ *such that* $v + (-v) = 0$
5. *Scalar Multiplication is associtive,* $\forall a, b \in F,\ \forall v \in V,\ a(bv) = (ab)v$
6. *Scalar Multiplication is distributive over vector addition,* $\forall a \in F,\ \forall u, v \in V,\ a(u + v) = au + av$
7. *Scalar Multiplication is distributive over scalar addition,* $\forall a, b \in F,\ \forall v \in V,\ (a + b)v = av + bv$
8. *Scalar Multiplication identity,* $\forall v \in V,\ 1v = v$

Remark. *Scalar multiplication is trivially commutative. ie,* $av = va$

Remark. *A few vector spaces,*

F^n *n-tuple space*

$F^{m \times n}$ *space of all* $m \times n$ *matrices*

F^S *space of all functions* $f : S \to F$

$F(x)$ *space of all polynomial functions on* F

Definitions 2.3. *A vector* b *is a linear combination of the set of vectors* $\{a_1, a_2, \cdots, a_n\}$ *if there exists scalars* $c_i \in F$ *such that* $b = \sum c_i a_i$.

Axiom 2.4. *Let* V *be a vector space, then inner product of* V *is a function,* $\cdot : V \times V \to \mathbb{R}$ *satisfying,*

1. $\forall x \in V,\ x \cdot x \geq 0$
2. $x \cdot x = 0 \iff x = 0$
3. *Inner product is commutative,* $\forall x, y \in V,\ x \cdot y = y \cdot x$
4. $\forall x, y, z \in V,\ x \cdot (y + z) = x \cdot y + x \cdot z$ *and* $(x + y) \cdot z = x \cdot z + y \cdot z$
5. $\forall x, y \in V,\ \forall a \in F,\ (ax) \cdot y = a(x \cdot y) = x \cdot (ay)$

Definitions 2.5. *A vector space* V *with inner product is an inner product space.*

Axiom 2.6. *Let* V *be a vector space, then a norm on* V *is a function* $\|\| : V \to \mathbb{R}$ *satisfying,*

1. $\forall x \in V,\ \|x\| \geq 0$
2. $\|x\| = 0 \iff x = 0$
3. $\forall x \in V,\ \forall a \in F,\ \|ax\| = |a|\|x\|$
4. *triangular inequality,* $\forall x, y \in V,\ \|x + y\| \leq \|x\| + \|y\|$

Definitions 2.7. *A vector space with a norm is a normed space.*

Remark. *Let V be a vector space with inner product $\cdot$. Then inner product induced norm is given by $\forall x \in V,\ \|x\| = \sqrt{x \cdot x}$*

Theorem 2.8. *Let V be a vector space with inner product induced norm, then $x \cdot y \leq \|x\|\|y\|$. And equality holds iff $x = cy$.*

Corollary 2.8.1 (Cauchy-Schwarz-Buniakowsky)**.** $|x \cdot y| \leq \|x\|\|y\|$

Remark. $x = (a, b),\ y = (b, a) \implies$ *Geometric Mean $\leq$ Arithmetic Mean*

Remark (parallelogram identity)**.** $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$

Remark (orthogonal)**.** *$x, y \in V$ are orthogonal if $x \cdot y = 0$.*

Theorem 2.9. *Let $\boldsymbol{x} \in \mathbb{R}^n$, then $|x_j| \leq \|\boldsymbol{x}\| \leq \sqrt{n} \sup\{|x_1|, |x_2|, \cdots, |x_n|\}$.*

2.2 Subspace

Definitions 2.10. *Let V be a vector space over F with vector addition, $+$ and scalar multiplication, $\cdot$, then a subspace W of V is a subset W of V if it is a vector space over F with the same operations restricted to W.*

Theorem 2.11. *A non-empty subset W of V is a subspace of V over the same field F iff for every pair of vectors $u, v \in W$ and every scalar $c \in F$, $cu + v \in W$*

Theorem 2.12. *Let V be a vector space over the field F, then the intersection of any collection of subspace of V is a subspace of V*

Theorem 2.13 (Subspace spanned by W)**.** *Let V be a vector space over the field F and $W \subset V$, then the intersection of all subspace of V that contains W is the subspace spanned/generated by W.*

Theorem 2.14. *The subspace spanned by W is the set of all linear combinations of vectors in W.*

Definitions 2.15. *The sum of subsets $W_1, W_2, \cdots W_n$ of V is the set of all vectors, $w_1 + w_2 + \cdots + w_n$ where $w_k \in W_k$.*

Theorem 2.16. *Let $W_1, W_2, \cdots, W_n$ be subspace of the vector space V, then $W_1 + W_2 + \cdots + W_n$ is the subspace of V containing each of the subspaces W_k.*

Definitions 2.17. *Let A be an $m \times n$ matrix over the field F, then the row space of A is the subspace of F^n spanned by the row vectors of A. And the column space of A is the subspace of F^m spanned by the column vectors of A.*

Remark. *Every linear combination of zero-sum vectors gives zero-sum. Thus there is unique zero-sum $(n-1)$-dimensional subspace for every n-dimensional vector space.*[1]

Remark. *Let a Cube has corners $(x, y, z),\ x, y, z \in \{0, 1\}$, then $(0, 1, 0) - (1, 1, 0)$ is an edge and $(0, 1, 0) - (1, 1, 0) - 1, 1, 1) - (1, 1, 0)$ is a face.*
There are $\binom{n}{k-1} 2^{n-k+1}$ k-dimensional subspaces for such an n-cube. ie, n-cube has 2^n corners, $n2^{n-1}$ edges, $n(n-1)2^{n-3}$ faces ...

[1] zero-sum vectors[**?**] : vectors whose components add to zero. eg. $(1, -2, 1)$.

2.2.1 Basis & Dimension

Definitions 2.18. *A set of vectors* $\{v_1, v_2, \cdots, v_n\}$ *is linearly dependent if there exists scalars* $c_1, c_2, \cdots, c_n \in F$*, not all of them are zero such that* $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$*.*

Remark.

1. *Any set containing the zero vector is linearly dependent.*
2. *Any set containing linearly dependent set is linearly dependent.*
3. *Any subset of a linear independent set is linearly independent.*
4. *S is linearly independent iff every finite subset of S is linearly independent.*$\star$

Definitions 2.19. *A basis for* V *is a linearly independent set of vectors that spans* V*.* V *is finite dimensional if it has a finite basis.*

Remark. *Let* A *be an invertible* $n \times n$ *matrix, then the column vectors of* A *is a basis for* F^n*.*

Theorem 2.20. *Let* V *be a vector space is spanned by a finite set of* n *vectors, then any independent set of vectors in* V *is finite and contains no more than* n *elements.*

Corollary 2.20.1. *If* V *is a finite dimensional vector space, then any two bases of* V *contains the same number of elements.*

Corollary 2.20.2. *If* V *is* n*-dimensional, then any subset of more than* n *vectors is dependent and no subset with fewer than* n *vectors can span* V*.*

Lemma 2.21. *Let* S *be an independent subset of* V *and* $v \in V$ *is not in the subspace spanned by* S*, then* $S \cup \{v\}$ *is an independent subset of* V*.*

Theorem 2.22. *If* W *is a subspace of a finite dimensional vector space* V*, then every independent subset of* W *is finite and is part of a basis for* W*.*

Corollary 2.22.1. *If* W *is a proper subspace of a finite dimensional vector space* V*, then* W *is finite dimensional and* $\dim W < \dim V$*.*

Corollary 2.22.2. *Let* A *be an* $n \times n$ *matrix. If row vectors of* A *are linearly independent in* F^n*, then* A *is invertible.*

Corollary 2.22.3. *If* W_1, W_2 *are finite dimensional subspaces of* V*, then* $W_1 + W_2$ *is finite dimensional and* $\dim W_1 + W_2 = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$

2.2.2 Change of Basis

Definitions 2.23. *The co-ordinates of a vector* $v \in V$ *with respect to an ordered basis* B*,* $[v]_B$ *is the column vector of scalars* $c_1, c_2, \cdots, c_n$ *such that* $v = c_1b_1 + c_2b_2 + \cdots + c_nb_n$ *where* $b_k \in B$*.*

Theorem 2.24 (Change of Basis)**.** *Let* V *be an* n*-dimensional vector space over the field* F *and* B, B' *be two ordered bases for* V*, then there exists an* $n \times n$ *invertible matrix* P *such that* $[v]_B = P[v]_{B'}$*.*

Theorem 2.25. *For every invertible $n \times n$ matrix, P and basis B of the vector V over the field F, there exists another basis B' such that $[v]_B = P[v]_{B'}$ for every vector $v \in V$.*

Definitions 2.26. *Row rank and Column rank,*

row rank *dimension of row space of A*

column rank *dimension of column space of A*

Theorem 2.27. *Row-equivalent matrices have the same row space.*

Theorem 2.28. *Non-zero row vectors of a row-reduced echelon matrix, R forms a basis for the row space of R.*

Theorem 2.29. *For every subspace W of F^n with $\dim W \leq m$, there exists a unique $m \times n$ row-reduced echelon matrix, R such that its row space is W.*

Theorem 2.30. *Every $m \times n$ matrix over the field F is row-equivalent to a unique row-reduced echelon matrix.*

Theorem 2.31. *The following statements are equivalent,*

1. *A, B are row-equivalent.*
2. *A, B have same row space.*
3. *There exists an invertible matrix, P such that $A = PB$*
4. *$AX = 0, BX = 0$ has same solution space.*

2.3 Linear Transformations

Definitions 2.32. *Let V, W be vector spaces over the same field F. A linear transformation T is a function, $T : V \to W$ such that $T(cu + v) = cTu + Tv$ where $u, v \in V$ and $c \in F$.*

Remark. *A few linear transformations,*

1. *The set of all polynomials over the field $\mathbb{C}$ with differentiation.*
2. *$F^{m\times n}$ with matrix multiplication.*
3. *The set of all continuous real functions with integration.*

Remark. *Properties of linear transformations,*

1. *$T(0) = 0$*
2. *T preserves linear combinations*

Theorem 2.33. *Let $B = \{b_1, b_2, \cdots, b_n\}$ be an ordered basis for an n-dimensional vector space V over the field F and W be any vector space over the field F and $b'_1, b'_2, \cdots, b'_n \in W$, then there exists a unique linear transformation $T : V \to W$ such that $T(b_k) = b'_k,\ \forall k$.*

Remark. *A few subspaces from linear transformations,*

1. *$T(V)$ is a subspace of W*
2. *$\{v \in V : Tv = 0\}$ is a subspace of V*

Definitions 2.34. *Subspaces from transformations and their dimensions,*

Null space/Kernel of T $N(T) = \{v \in V \ : \ Tv = 0\}$

Nullity of T $nullity(T) = \dim N(T)$

Range space of T $R(T) = \{w \in W \ : \ Tv = w, \ v \in V\}$

Rank of T $rank(T) = \dim R(T)$

Theorem 2.35 (rank-nullity)**.** *Let T be a linear transformation from a finite dimensional space V into W, then $rank(T) + nullity(T) = \dim V$*

Theorem 2.36. *Let $A \in F^{m\times n}$, then row $rank(A)$ = column $rank(A)$*

Theorem 2.37. *Let V, W be vector spaces over the field F, then the set of all linear transformations $T : V \to W$ with addition, $(T + U)v = Tv + Uv$ and multiplication, $(cT)v = c(Tv)$ is a vector space, $L(V, W)$ over the field F.*

Theorem 2.38. *Let $\dim V = n, \ \dim W = m$, then $\dim L(V, W) = mn$*

Theorem 2.39. *Let $T \in L(V, W), \ U \in L(W, Z)$, then $UT =\in L(V, Z)$*

Remark. *A linear operator on V is a linear tranformation, $T : V \to V$.*

Lemma 2.40 (linear algebra with identity)**.** ⋆ *Let V be a vector space over the field F, then the set of all linear operators on V, $L(V, V)$ is a linear algebra with identity. Let $U, T_1, T_2 \in L(V, V)$, then*

1. *$\exists I \in L(V, V)$ such that $UI = U = IU$*
2. *$U(T_1 + T_2) = UT_1 + UT_2$ and $(T_1 + T_2)U = T_1U + T_2U$*
3. *$c(UT) = (cU)T$*

Theorem 2.41. *Linear tranformation $T : V \to W$ is invertible iff T is bijective, then $T^{-1} : W \to V$ is also bijective.*

non-singular T is non-singular if $Tv = 0 \implies v = 0$ ie, $N(T) = \{0\}$

Theorem 2.42. *A linear transformation, T is injective iff T is non-singular.*

Theorem 2.43. *Linear transformations preserve indepenedence iff non-singular.*⋆

Remark. *Let V be an n-dimensional vector space over the field F and $T : V \to V$ be a linear operator. For any basis B of V, $B' = \{Tb_1, Tb_2, \cdots, Tb_n\}, \ b_j \in B$ is also a basis for V iff the linear opertor $T : V \to V$ is invertible.*

Theorem 2.44. *Let $T : V \to W$ and $U : W \to Z$ be invertible linear transformations, then $UT : V \to Z$ is invertible and $(UT)^{-1} = T^{-1}U^{-1}$*

Theorem 2.45. *Let V, W be finite dimensional vector spaces over the field F such that $\dim V = \dim W$. And $T : V \to W$ be a linear transformation, then the following statements are equivalent,*

1. *T is invertible.*

2. *T is non-singular.*

3. *T is surjective.*

Theorem 2.46. *The set of all invertible linear operators with on V with composition is a non-abelian group.*

Theorem 2.47. *Every n-dimensional vector space over the field F is isomorphic to F^n.*

Theorem 2.48. *Let U, V be vector spaces over the field F, $U : V \to W$ be an isomorphism, then $\phi : L(V, V) \to L(W, W)$, $\phi(T) = UTU^{-1}$ is an isomorphism.*

2.4 Matrix

Definitions 2.49. *An $m\times n$ matrix over the field F is a function $A : \{1, 2, \cdots, m\}\times\{1, 2, \cdots, n\} \to F$.*

upper triangular $a_{ij} = 0,\ i > j$

lower triangular $a_{ij} = 0,\ i < j$

symmetric $a_{ij} = a_{ji}$

skew-symmetric $a_{ij} = -a_{ji}$

hermitian $a_{ij} = \overline{a_{ji}}$

Remark. *There are three elementary row operations on an $m \times n$ matrix*

1. *Multiplication of one row by non-zero scalar*

2. *Replacing row r by row r plus c times row r*

3. *Interchanging two rows*

Remark. *Elementary row operations on A does not affect the set of solutions of the homogenous system of linear equations, $AX = 0$.*
The inverse of any elementary row operation is of the same kind.

Definitions 2.50. *A matrix B is row-equivalent to matrix A if B can be obtained from A by a finite sequence of row operations.*
If A and B are row-equivalent, then the homogeneous systems of linear equations, $AX = 0$ and $BX = 0$ have exactly the same solutions.

Definitions 2.51. *An $m \times n$ matrix is row-reduced if*

1. *first non-zero entry, pivot of each non-zero row is 1*

2. *each pivot column has all other entries zero*

Remark. *Every $m \times n$ matrix is row-equivalent to a row-reduced $m \times n$ matrix.*

Definitions 2.52. *An $m \times n$ matrix is an echelon matrix if*

1. *non-zero rows are above every zero-rows*
2. *pivot column of any row is less than pivot column of any row below it*

Theorem 2.53. *Every $m \times n$ matrix is row-equivalent to a unique $m \times n$ row-reduced echelon matrix.*

Theorem 2.54. *If A is an $m \times n$ matrix and $m < n$, then $AX = 0$ has a non-trivial solution.*

Theorem 2.55. *A is row-equivalent to $n \times n$ identity matrix iff $AX = 0$ has only the trivial solution.*

Definitions 2.56. $A_{m\times n} \times B_{n\times p} = C_{m\times p},\ c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$

Remark. *Matrix multiplication is not commutative.*
Matrix multiplication is associative.

Definitions 2.57. *An $n \times n$ matrix is an elementary matrix if it can be obtained from $n \times n$ identity matrix by a single elementary row operation.*

Remark. *An elementary operation is equivalent to the left-multiplication by corresponding elementary matrix.*
Two $m \times n$ matrices are row-equivalent if one can be obtained from the other by left-multiplying a finite number of $m \times m$ elementary matrices. (or by right-multiplying a finite number of $n \times n$ elementary matrices)

Definitions 2.58. *Let A be an $n \times n$ matrix over the field F. If $BA = I$, then B is the left-inverse of A. If $AB = I$, then B is the right-inverse of A. If $AB = BA = I$, then B is the inverse of A and A is an invertible matrix.*

Theorem 2.59. *If A, B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$. The product of invertible matrices is invertible.*
Elementary matrices are invertible.

Theorem 2.60. *The following statements are equivalent:*

1. *A is invertible.*
2. *A is row-equivalent to $I_{n\times n}$.*
3. *A is a product of elementary matrices.*
4. *$AX = 0$ has only trivial solution.*
5. *$AX = Y$ has a unique solution X for each Y.*

Corollary 2.60.1. *If A is invertible, a sequence of elementary row operations would reduce A to the identity. The same sequence of elementary row operations would convert I to A^{-1}.*

Corollary 2.60.2. *If A, B are row-equivalent $m \times n$ matrices, then $B = PA$ where P is an invertible $m \times m$ matrix and $A = P^{-1}B$.*
An $n \times n$ matrix with either left or right inverse is invertible.
Let $A = \prod A_k$. A is invertible iff each A_k is invertible.

2.5 Rank

2.6 Determinant

2.7 Linear Equations

2.8 Eigenvalues & Eigenvectors

2.9 Cayley-Hamilton Theorem

2.10 Trasformation Matrix

Theorem 2.61. *Let V, W be vector spaces over field F of dimensions n, m. Let B, B' be ordered bases for V, W, then for each linear transformation $T : V \to W$ there exists an $m \times n$ matrix such that $[Tv]_{B'} = [T]_{BB'}[v]_B$ where the columns of $[T]_{BB'}$ are co-ordinates, $[Tb_j]_{B'}$ for each vector in the ordered basis B of V.*

Theorem 2.62. *Let V, W be vector spaces over the field F of dimensions n, m and B, B' be ordered bases for V, W. For each such pair of ordered bases, the function $\phi : L(V, W) \to F^{m\times n}$, $\phi(T) = [T]_{BB'}$ is an isomorphism.*

Theorem 2.63. *Let V, W, Z be finite dimensional vector spaces over the field F with ordered bases B, B', B'' and T, U be linear transformations, $T : V \to W$, $U : W \to Z$, then $[UT]_{BB''} = [U]_{B'B''}[T]_{BB'}$.*

Corollary 2.63.1. *Let $T : V \to V$ be an invertible linear operator, then $[T^{-1}]_{BB} = [T]^{-1}_{BB}$.*

Corollary 2.63.2. *Let $T : V \to V$ be a linear operator and B, B' be two ordered bases for V, then $[T]_{B'B'} = [I]_{BB'}[T]_{BB}[I]_{B'B}$.*

Theorem 2.64. *Let $T : V \to V$ be a linear operator and B, B' be two ordered bases for V, then there exists an invertible linear operator $U : V \to V$, $Ub_j = b'_j$ such that $[T]_{B'B'} = [U]^{-1}_{BB}[T]_{BB}[U]_{BB}$.$\star$*

Remark. *Let V be a vector space over field F with ordered bases B, B', then there exists a linear operator $U : V \to V$, $Ub_j = b'_j$ such that $[I]_{B'B} = [U]_{BB}$.*

Definitions 2.65. *Two $n \times n$ matrices A, B are similar if there exists an invertible $n \times n$ matrix P such that $B = P^{-1}AP$.*

Remark. *Let B, B' be ordered bases for n-dimensional vector space V over field F and $T : V \to V$ be a linear operator, then $[T]_{BB}, [T]_{B'B'}$ are similar.*

Remark. *If linear operator $T : V \to V$ is invertible, then $[T]_{BB}$ is invertible and is similar to $I_{n\times n}$.*

Remark. *Let V be an n-dimensional vector space over field F and A be an invertible $n \times n$ matrix over field F, then there exists a pair of ordered bases B, B' for V such that $[I]_{BB'} = A$. And there exists an invertible linear operator $U : V \to V$, $Ub_j = b'_j$ such that $[U]_{BB} = A$.*

2.11 Linear Functionals

Definitions 2.66. *Let V be a vector space over the field F, then linear transformation $f : V \to F$ is a linear functional on V.*

Definitions 2.67. *Let V be a vector space over the field F, then the set of all linear functionals on V is a dual space $V^* = L(V, F)$ of V.*

Remark. $\dim V^* = \dim V$

Theorem 2.68. *Let $B = \{b_1, b_2, \cdots, b_n\}$ be an ordered basis for the vector space V over the field F, then $B^* = \{f_1, f_2, \cdots, f_n\}$ such that $f_i(b_j) = \delta_{ij}$ is a dual basis for V^*. And for each $v \in V$, the co-ordinates of v are $f_j(v)$.*

Remark. *Let f be a non-zero linear functional on the vector space V over the field F, then* $\dim N(f) = \dim V - 1$. *If V is finite-dimensional, then the null space of any non-zero functional on V is a hyperspace of V.*

Definitions 2.69. *Let S be a subset of the vector space V over the field F, then the annihilator of S, S^0 is the set of all linear functionals on V such that $f(v) = 0,\ \forall v \in S$.*

Remark. *S^0 is the set of all linear functionals on V such that S is contained in the nullspace of all those functionals. The set of all linear functional that map vectors in S into 0.*

Theorem 2.70. *Let W be a subspace of a finite dimensional vector space V over the field F, then* $\dim W + \dim W^0 = \dim V$.

Corollary 2.70.1. *Let W be a k-dimensional subspace of an n-dimensional vector space V over the field F, then W is the intersection of $(n-k)$ hyperspaces in V.*

Corollary 2.70.2. *Let W_1, W_2 be subspaces of a finite dimensional vector space, then $W_1 = W_2$ iff $W_1^0 = W_2^0$.*

Definitions 2.71. *Let V be a vector space over the field F, then double dual V^{**} of V is the set of all linear functionals on the dual space V^*.*

Theorem 2.72. *Let V be a finite-dimensional vector space over the field F, then $\phi : V \to V^{**},\ \phi(v) = L_v$ such that $\forall f \in V^*,\ L_v(f) = f(v)$ is an isomorphism.*

Corollary 2.72.1. *Let V be a finite dimensional vector space over the field F, then for every linear functional $L \in V^{**}$, there exists a unique vector $v \in V$ such that for every linear function $f \in V^*$, $L(f) = f(v)$.*

Corollary 2.72.2. *Let V be a finite dimensional vector space over F, then each basis for V^* is the dual of some basis for V.*

Theorem 2.73. *Let S be a subset of a finite dimensional vector space V over the field F, then S^{0^0} is the subspace spanned by S.*$\star$

Definitions 2.74. *A hyperspace of a vector space is a maximal, proper subspace of it.*

Theorem 2.75. *Let V be a vector space over the field F, then the null space of a non-zero functional f on V is a hyperspace in V. And every hyperspace in V is the null space of some non-zero linear functional on V.*

Lemma 2.76. *Let f, g be linear functionals on a vector space V over the field F, then g is a scalar multiple of f iff null space of g contains null space of f.*

Theorem 2.77. *Let $g, f_1, f_2, \cdots, f_n$ be linear functionals on a vector space V over the field F and $N, N_1, N_2, \cdots, N_n$ be the respective null spaces, then g is a linear combination of f_js iff N contains $\cap_{j=1}^{n} N_j$.*

Theorem 2.78. *Let V, W be vector spaces over the field F, then for each linear transformation, $T : V \to W$ there exists a unique linear transformation, $T^t : W^* \to V^*$ such that for each $v \in V$ and $g \in W^*$, $T^t g(v) = g(Tv)$.*

Definitions 2.79. *Let V, W be vector spaces over the field F and $T : V \to W$ be a linear transformation, then the linear transformation, $T^t : W^* \to V^*$ such that $T_t g(v) = g(Tv),\ \forall v \in V,\ \forall g \in W^*$ is the transpose/adjoint of T.*

Theorem 2.80. *Let V, W be vector spaces over the field F and $T : V \to W$ be a linear transformation, then the null space of T^t is the annihilator of the range of T.*

Corollary 2.80.1. *Let V, W be finite dimensional vector spaces over the field F, then $rank(T^t) = rank(T)$ and $R(T^t) = N(T)^0$*

Theorem 2.81. *Let V, W be finite dimensional vector spaces over the field F with ordered bases B, B' and B^*, B'^* are respective dual bases for V^*, W^* and $T : V \to W$ be a linear transformation with transpose of T, $T^t : W^* \to V^*$, then $[T]_{BB'}$ is the transpose of the matrix $[T^t]_{B'^*B^*}$.*

2.12 Canonical Forms

2.12.1 Diagonal Forms

2.12.2 Triangular Forms

2.12.3 Jordan Forms

2.13 Inner Product Spaces

2.14 Orthonormal Basis

2.15 Quadratic Forms

2.16 Reduction & Classification of Quadratic Forms

Chapter 3

Topology

3.1 Metric Space

Definitions 3.1. *A function $d : X \times X \to \mathbb{R}$ is a metric on X if it satisfies*

1. $d(x, y) \geq 0,\ d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) = d(x, z) + d(z, y)$

Definitions 3.2. *A set X together with a metric on X is a metric space.*

neighbourhood A subset N is a neighbourhood of a point x if there exists a positive real number r such that $y \in N$ for every y satisfying $d(x, y) < r$.

limit point A point x is a limit point of a subset A if every neighbourhood of x has some another point from the subset A.

isolated point A point which is not a limit point of the subset A.

closed A subset A of X is closed if it has every limit point of it. $\overline{A} = A$

interior point A point x is an interior point the subset A if there exists some neighbourhood of x, which is contained in the subset A. $x \in A^0$

open A subset A is open if each point of it is an interior point in it.

complement The subset of all points which are not in the subset A is the complement of A.

perfect A subset A is perfect if it is closed and has each point of it as its limit point.

bounded A subset A is bounded if there is a point x and a positive real number r such that for each point $y \in A$, $d(x, y) < r$.

dense A subset A is dense if every point of X is either in A or is a limit point of A.

boundary A point x is a boundary point of subset A of X if every neighbourhood of x has some point from A as well as $X - A$. $x \in \partial(A)$

Remark. *1.* $\overline{A} \cap \overline{B} \neq \overline{A \cap B}$ *eg.* $\overline{(1,2)} \cap \overline{(2,3)} = \{2\},\ \overline{(1,2) \cap (2,3)} = \{\}$

2. There exists countable, dense subsets of $\mathbb{R}$ *with empty interior. eg.* $\mathbb{Q}$

3. There exists uncountable, dense subsets of $\mathbb{R}$ *with empty interior.*

4. $A \times B$ *is open(or closed) iff both* A, B *are open(or closed).*

5. For Cantor set $\mathcal{C}$*,* $\mathcal{C}^0 = \phi,\ \overline{\mathcal{C}} = \mathcal{C},\ \partial(\mathcal{C}) = \mathcal{C}$

6. $\mathcal{C}$ *can't be expresses as countable union of closed intervals.*

7. $\mathbb{R} - \mathcal{C}$ *can be expresses as countable union of open intervals.*

Remark. *1. For any bounded subset* A*, there exists a point* x *such that* $d(x, y) < r$ *for every* $y \in A$*. Then for any point* $z \in A$*,* $d(z, y) < 2r$ *for every* $y \in A$*.*

Definitions 3.3. radius *The radius of a bounded subset* A *is the smallest real number* r *such that for a particular* $x \in X$*,* $d(x, y) < r$ *for every point* $y \in A$*.*

3.2 Topological Space

Definitions 3.4. *A set* X *together with a family* $\mathcal{T}$ *of subsets of* X *is a topological space if it satisfies*

1. $\mathcal{T}$ *is closed under arbitrary unions*

2. $\mathcal{T}$ *is closed under finite intersections*

usual $d(x, y) = |x - y|$

discrete $d(x, y) = \delta_{xy}$

taxicab $d(x, y) = |x| + |y|$

Definitions 3.5. *A set is separable if it has a countable dense subset.*