Chapter 1

Analysis

1.1 Sequence

Definitions 1.1. *Sequence x_n in a set X is a function $x : \mathbb{N} \to X$ where $x_n = x(n)$.*

Definitions 1.2. *Subsequence x_{n_k} of a sequence x_n is a function $x \circ n$ where $n : \mathbb{N} \to \mathbb{N},\ n_k = n(k)$ is a strictly increasing sequence.*

1.1.1 Convergence

Definitions 1.3 (metric)**.** *A sequence x_n converges to x if there exists $N \in \mathbb{N}$ such that $\forall n > N,\ d(x_n, x) < \varepsilon$.*

Definitions 1.4 (norm)**.** *A sequence x_n converges to x if there exists $N \in \mathbb{N}$ such that $\forall n > N, \|x_n - x\| < \varepsilon$.*

Definitions 1.5 (neighbourhood)**.** *A sequence x_n converges to x if any neighbourhood N of x contains all except finitely many x_n's.*

Remark (subsequence)**.** *A sequence x_n converges to x if and only if every subsequence has a convergent subsequence.*

1.1.2 Limit Point

Definitions 1.6. *x is a limit point of sequence x_n if x_n converges to x.*

Definitions 1.7. *x is a cluster point of sequence x_n, there exists a subsequence x_{n_k} converging to x.*

1.1.3 Cauchy Criterion

Definitions 1.8 (metric)**.** *A sequence x_n is Cauchy if there exists $N \in \mathbb{N}$ such that $\forall n, m > N,\ d(x_n, x_m) < \varepsilon$.*

Definitions 1.9 (norm)**.** *A sequence x_n is Cauchy if there exists $N \in \mathbb{N}$ such that $\forall n, m > N,\ \|x_n - x_m\| < \varepsilon$.*

1.1.4 Complete Space

Definitions 1.10 (complete)**.** *A space is complete if every Cauchy sequence in it converges.*

Theorem 1.11 (Stolz-Cesaro)**.**

$$\liminf_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} \le \liminf_{n\to\infty} \frac{a_n}{b_n} \le \limsup_{n\to\infty} \frac{a_n}{b_n} \le \limsup_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$$

1.2 Limit of a function

Definitions 1.12 (limit)**.** *If $f(x_n) \to L$ as $x_n \to a$, then $\lim_{x\to a} f(x) = L$.*

Definitions 1.13 (continuity)**.** *A function $f : X \to Y$ is continuous at $a \in X$, if $\lim_{x\to a} f(x) = f(\lim_{x\to a} x) = f(a)$.*

Theorem 1.14. *Limit is algebraic.*

Suppose $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ exists, then

$$\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x) \tag{1.1}$$

$$\lim_{x\to a} f(x) \pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) \tag{1.2}$$

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \ \lim_{x\to a} g(x) \tag{1.3}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \tag{1.4}$$

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} f(x)^{\lim_{x\to a} g(x)} \tag{1.5}$$

Remark (exceptions)**.**

$$\frac{0}{0},\ \frac{\pm\infty}{\pm\infty},\ 0 \pm \infty,\ \infty - \infty,\ 0^0,\ \infty^0,\ 1^{\pm\infty}$$

Theorem 1.15 (L'Hospital/Bernouli)**.**

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Remark (application)**.**

$$\lim_{x\to 0} (2+x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\log(1+x)} = e^{\lim_{x\to 0} \frac{\log(2+x)}{x}} = e^{\lim_{x\to 0} \frac{1}{2+x}} = \sqrt{e}$$

Squeeze theorem Suppose $f(x) \le g(x) \le h(x)$ for each x in an open interval containing a (except a). If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then

$$\lim_{x\to a} g(x) = L \tag{1.6}$$

Theorem 1.16 (chain rule)**.** *Suppose* $\lim_{x\to a} g(x) = b$ *and* f *is continuous at* b*, then*

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b) = c \tag{1.7}$$

Remark. *The existence of limit* $\lim_{y\to b} f(y) = c$ *does not imply* $f(b) = c$*. If* g *assumes value* b *in some neighbourhood of* a*, then*

$$\lim_{x\to a} g(x) = b,\ \lim_{y\to b} f(x) = c \not\Longrightarrow \lim_{x\to a} f\circ g(x) = c$$