Part I

Real Analysis

Chapter 1

Real Field

1.1 Algebra of Sets

set a well-defined collection of objects

subset set A is a subset of set B if each element of set A is also in set B

equal Two sets A and B equal if they have the same elements

Remark. $A = B \iff A \subset B$ *and* $B \subset A$

Null set, ϕ is the set which contains no elements.

Power set, $P(X)$ is the family of all subsets of the set X.

$\mathbb{N}$ set of all natural numbers

$\mathbb{Z}$ set of all integers

$\mathbb{Q}$ set of all rationals

$\mathbb{R}$ set of all real numbers

$\mathbb{C}$ set of all complex numbers

Union/Join $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Intersection/Meet $A \cap B = \{x : x \in A \text{ and } x \in B\}$

Complement/Difference $A - B = \{x \in A : x \notin B\}$

Symmetric Difference $A \Delta B = (A - B) \cup (B - A)$

Remark. *Inclusive property,*

1. $A \subset A \cup B$
2. $A \cap B \subset A$
3. $A - B \subset A$

Remark. $A \subset B \iff A \cap B = A \iff A \cup B = B \iff A - B = \phi$

Remark. *Union and Intersection are idempotent, commutative, associative and distributive.*

disjoint Two sets A and B are disjoint if $A \cap B = \phi$

Definitions 1.1. *The (cartesian) product,* $A \times B = \{ (x, y) : x \in A,\ y \in B \}$

Theorem 1.2. *Augustus de Morgan*

1. $X - (A \cup B) = (X - A) \cap (X - B)$
2. $X - (A \cap B) = (X - A) \cup (X - B)$

1.2 Functions

Definitions 1.3. *A binary relation R from a set A to a set B is a subset of their cartesian product. ie, $R \subset A \times B$.*

Remark. *Let $X = \{ 1, 2, 3 \}$. Then $R = \{ (1, 2), (1, 3), (2, 3) \}$ is a relation on X (to X itself).*

Definitions 1.4. *A function $f : A \to B$ is a subset of $A \times B$ such that for every $x \in A$ there exists a unique $y \in B$ where $(x, y) \in f$.*

image of x Let $(x, y) \in f$, then the image of x, $f(x) = y$

Inverse image $f^{-1}(y) = \{x \in A : y = f(x)\}$ is the fibre of f over y[**?**]

domain of f is the set A

co-domain of f is the set B

range of f is the set $\{y \in B : \exists x \in A,\ y = f(x)\}$

f **is injective** if images of distinct element of A are distinct.

f **is surjective** if each element of B is an image of some element of A.

f **is bijective** if f is both injective and surjective.

Remark. $f(x) = y \implies x \in f^{-1}(y) \subset X$.

Definitions 1.5. *Let functions $f : A \to B$ and $g : B \to C$, then the composition $g \circ f : A \to C$ is function such that $g \circ f(x) = g(f(x))$.*

Theorem 1.6. *Let $f : A \to B$ and $g : B \to C$ be functions, then*

1. *domain of $g \circ f$ is the domain of f*
2. *co-domain of $g \circ f$ is the co-domain of g*
3. *range of $g \circ f$ is the range of image of A*

Remark. *Function composition is not commutative. ie, $f \circ g \neq g \circ f$.*
Let $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R},\ g(x) = x + 1$.
Then $f \circ g(2) = f(g(2)) = f(3) = 9$ and $g \circ f(2) = g(f(2)) = g(4) = 5$.

Definitions 1.7. *Let $f : A \to B$ and $g : B \to A$ are both injective functions such that $g = \{(b, a) \in B \times A : (a, b) \in f\}$, then g is the inverse function of f, f^{-1}.*

Remark. *$f : X \to Y$ is injective iff*

1. *f has left inverse. ie, $\exists g : Y \to X,\ g \circ f = id_X$*
2. *$C \cap D = \phi \iff f(C) \cap f(D) = \phi$*

Remark. *$f : X \to Y$ is surjective function iff*

1. *f has right inverse. ie, $\exists g : Y \to X,\ f \circ g = id_Y$*

Remark. *$f : A \to B$ is a bijection iff*

1. *$f^{-1} \circ f = id_A$ and $f \circ f^{-1} = id_B$*

Theorem 1.8 (Schroder-Bernstein)**.** *$\exists$ injective function $f : A \to B$, $\exists$ injective function $g : B \to A \implies \exists$ bijection $h : A \to B$.*

Remark. *functions & sets*

1. *$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$*
2. *$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$*

3. $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$

4. $f(A \cup B) = f(A) \cup f(B)$

Remark. *Let* $f : A \to B$

$$\forall y \in B, |f^{-1}(y)| = \begin{cases} \leq 1, & f \text{ is injective} \\ = 1, & f \text{ is bijection} \\ \geq 1, & f \text{ is surjective} \end{cases}$$

Remark. *The following statements are equivalent:*

1. $f : X \to Y$ *is injective*

2. $\forall A, B \subset X,\ f(A \cap B) = f(A) \cap f(B)$

3. $\forall A, B \subset X,\ f(A - B) = f(A) - f(B)$

4. $\forall A \subset X,\ A = f^{-1} \circ f(A)$

Remark. *The following statements are equivalent:*

1. $f : X \to Y$ *is surjective*

2. $\forall A \subset Y,\ A = f(f^{-1}(A))$

Remark. *Some non-equal sets,*

1. $f(A \cap B) \neq f(A) \cap f(B), \qquad \because f(A) \cap f(B) \not\subset f(A \cap B)$

2. $f(A - B) \neq f(A) - f(B), \qquad \because f(A - B) \not\subset f(A) - f(B)$

3. $A \neq f \circ f^{-1}(A), \qquad \because A \not\subset f \circ f^{-1}(A)$

4. $A \neq f^{-1} \circ f(A), \qquad \because A \not\subset f^{-1} \circ f(A)$

Definitions 1.9. *The restriction of a function* $f : A \to B$ *into a subset* $C \subset A$ *is the function* $f_{|C} : C \to B$*, such that* $\forall x \in C,\ f_{|C}(x) = f(x)$

Definitions 1.10. *An extension of a function* $f : A \to B$ *into a superset* $C \supset A$ *is the function* $F : C \to B$*, such that* $\forall x \in A,\ F(x) = f(x)$*.*

1.3 Partial Order

Definitions 1.11. *A (binary) relation $\leq$ between set A and set B is a subset of the cartesian product, $A \times B$. $(x, y) \in \leq$ may be written as $x \leq y$.*
A relation on S is a relation from S into S.

reflexive $x \in S \implies xRx$

symmetric $\forall xRy,\ x \neq y \implies yRx$

antisymmetric $\forall xRy,\ x \neq y \implies \neg yRx$

transitive $xRy\ \wedge\ yRz \implies xRz$

connex $\forall x, y \in R \implies (xRy)\ \vee\ (yRx)$

Remark. *Every connex relation is reflexive.*
An equivalence relation R is a reflexive, symmetric and transitive relation.

Definitions 1.12. *An order $<$ is a transitive relation such that,*

$$\forall x, y \in S,\ x < y\ OR\ x = y\ OR\ y < x$$

An order is a reflexive, antisymmetric and transitive relation.

ordered set is a set with an order on it.

strict order is a non-reflexive, antisymmetric and trasitive relation.

diagonal on S is the set $\Delta S = \{(x, x) \in S \times S\ :\ x \in S\}$

total/linear/simple order is a antisymmetric, trasitive and connex relation.

Remark. *For any set X, The set inclusion $\subset$ is a partial order on $P(X)$.*

Axiom 1.13 (Choice)**.** *If $\{A_i : i \in I\}$ is a non-empty family of sets such that A_i is non-empty for each $i \in I$, then $\prod A_i$ is non-empty*

Lemma 1.14 (Zorn)**.** *If every chain in a partially ordered set X has an upper bound in X, then X has a maximal element.*

Remark. *Zorn's lemma is equivalent to the axiom of choice.[**?**]*

1.4 Cardinality

Definitions 1.15. *Set A,B are have same cardinality, if there exists a bijection $f : A \to B$.*

Remark. $card(X) \leq card(Y)$ *if there exists an injective function,* $f : X \to Y$
$card(Y) \leq card(X)$ *if there exists an surjective function,* $f : X \to Y$

Remark. *Cardinality is an equivalence relation on the family of all sets.*

Definitions 1.16. *A set S is finite if $\exists n \in \mathbb{N}$, such that $card(S) = n$.*
A set S is countably infinite if $card(S) = card(\mathbb{N})$.
A set S is countable if it is finite or countably infinite.
A set S is uncountable if it neither finite nor countably inifinite.

Theorem 1.17. *Every infinite subset of a countably set is countable.*

Remark. *Countability is the smallest infinity.*

Theorem 1.18. *Every infinite set contains a countable subset.*

Theorem 1.19. *For an infinite set A, the following statements are equivalent:[**?**]*

1. *A is countable*
2. *There exists a subset $B \subset \mathbb{N}$ and a surjective function $f : B \to A$*
3. *There exists an injective function $g : A \to \mathbb{N}$*

Theorem 1.20. *Every subset of finite(countable) set is finite(countable).*

Theorem 1.21. *Finite union of finite sets is finite. Countable union of countable sets is countable.*

Remark. *The sets $\mathbb{N}, \mathbb{Q}$ are countable. The sets $(0, 1)$, $\mathbb{R}$ are uncountable.*

Theorem 1.22. *The set of all sequences in $\{0, 1\}$ is uncountable.*

Remark. *The set of all di-adic real numbers is uncountable.*
The set of all integerts is not uncountable ?

Theorem 1.23 (well-ordering). *Every nonempty subset of $\mathbb{N}$ has a smallest element in it.*

Theorem 1.24 (induction). *If $p(1) \wedge (p(k) \implies p(k+1))$, then $\forall n \in \mathbb{N},\ p(n)$*

Remark. *Well-ordering & induction principles are equivalent.*

Theorem 1.25. *Countable union of countable sets is countable.*

Theorem 1.26. *Finite product of countable sets is countable.*

Remark. *Countable product of countable sets is not necessarily countable.*

Remark. *If $card(A) \leq card(B)$ and $card(B) \leq card(A)$, then there exists a bijection $f : A \to B$. Thus $card(A) = card(B)$.*

Theorem 1.27 (Cantor)**.** *If A is a set, then $card(A) \leq card(P(A))$ and $card(A) \neq card(P(A))$.*

Remark. *Cardinality of the null set is 0.*
$card(\mathbb{N}) = \aleph_0$
$card(P(\mathbb{N})) = card(\mathbb{R}) = \aleph_1$

Remark (Continuum hypothesis)**.** *There is no cardinal number between $\aleph_0$ and $\aleph_1$.*

Remark (Generalised Continuum hypothesis)**.** *For any infinite cardinal $\aleph_k$, there is no cardinal number between $\aleph_k$ and $\aleph_{k+1}$.*

1.5 Real Field

Definitions 1.28. *A binary operation on the set A is a function $\star : A \times A \to A$.*

Remark. *$\star(a, b) = c$ may be written as $a \star b = c$ instead of $(a, b) \star c$*

Axiom 1.29 (Field)**.** *A set F with two binary operations $+, \times$ is a field if it satifies*

1. $\forall x, y \in F,\ x + y \in F$
2. $\forall x, y \in F,\ x + y = y + x$
3. $\forall x, y, z \in F,\ (x + y) + z = x + (y + z)$
4. $\exists$ *a unique* $0 \in F,\ \forall x \in F,\ x + 0 = x$
5. $\forall x \in F,\ \exists (-x) \in F,\ x + (-x) = 0$
6. $\forall x, y \in F,\ x \times y \in F$

7. $\forall x, y \in F,\ x \times y = y \times x$
8. $\forall x, y, z \in F,\ (x \times y) \times z = x \times (y \times z)$
9. $\exists$ *a unique* $1 \in F,\ \forall x \in F,\ x \times 1 = x$
10. $\forall x \in F,\ x \neq 0,\ \exists x^{-1} \in F,\ x \times x^{-1} = 1$
11. $\forall x, y, z \in F,\ x \times (y + z) = (x \times y) + (x \times z)$

Remark. *Let* $0, 1$ *be additive and multiplicative identities, then* $\forall x, y, z \in \mathbb{R}$,

1. $x + y = x + z \iff y = z$
2. $x + y = x \iff y = 0$
3. $x + y = 0 \iff y = -x$
4. $x + y = z \iff x = z + (-y)$
5. $-(-x) = x$
6. *For* $x \neq 0, xy = xz \iff y = z$
7. *For* $x \neq 0, xy = x \iff y = 1$
8. $xy = 1 \iff y = x^{-1}$
9. $xy = z \iff x = zy^{-1}$
10. $(x^{-1})^{-1} = x$
11. $0x = 0$
12. $(-1)x = -x$
13. $(-1)(-1) = 1$
14. $xy = 0 \iff a = 0$ *or* $b = 0$
15. $(-x)(-y) = xy$

Remark. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ *are fields.*

Theorem 1.30. *There doesn't exist a rational number* r *such that* $r^2 = 2$.

Axiom 1.31 (Order)**.** *An ordered field* F *is a field with an order* $<$ *such that,*

1. $\forall a, b \in F$ exactly one of the statements $a < b,\ a = b,\ b < a$ is true.
2. $\forall x, y, z \in F,\ y < z \implies (x + y) < (x + z)$
3. $\forall x, y \in F,\ 0 < x,\ 0 < y \implies 0 < (x \times y)$

Remark. *Let x, y, z in ordered field $\mathbb{R}$,*

1. $x < 0 \iff -x > 0$
2. $x - y > 0 \iff x > y$
3. $x > 0, y < z \implies xy < xz$
4. $x < 0, y < z \implies xy > xz$
5. $x \neq 0 \implies x^2 > 0$
6. $1 > 0$
7. $0 < x < y \iff 0 < y^{-1} < x^{-1}$
8. $x < y \implies x < \frac{x+y}{2} < y$

Definitions 1.32. *Absolute value of a real number r, $|r| = r$ if $r \geq 0$ and $|r| = -r$ if $r < 0$*

Remark. *Properties,*

1. $|-a| = |a|$
2. $|ab| = |a||b|$
3. $|a + b| \leq |a| + |b|$
4. $|a| \leq b \iff -b \leq a \leq b$
5. $-|a| \leq a \leq |a|$
6. $|a + b| \leq |a| + |b|$
7. $(1 + a)^n \leq 1 + na, \forall n \in \mathbb{N}$

Remark. *Two real numbers a, b are equal iff $\forall \epsilon > 0,\ |a - b| < \epsilon$.*

Definitions 1.33. *Given $a_j < b_j, \forall j$, the set of all points $\boldsymbol{x} \in \mathbb{R}^n$ such that $a_j \leq x_j \leq b_j,\ \forall j$ is an n-cell.*

Definitions 1.34. *Let* $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)$. *Then,* $\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$, $c\boldsymbol{x} = (cx_1, cx_2, \cdots, cx_n)$ *and*

$$|\boldsymbol{x}.\boldsymbol{y}| = \sum_{j=1}^{n} x_j y_j \quad \& \quad |\boldsymbol{x}| = \left(\sum_{j=1}^{n} |x_j|^2\right)^{\frac{1}{2}}$$

Remark. *Let* $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \mathbb{R}^n$,

1. *For* $\boldsymbol{x} \neq \boldsymbol{0}$, $|\boldsymbol{x}| > 0$
2. $|c\boldsymbol{x}| = |c||\boldsymbol{x}|$
3. $|\boldsymbol{x}.\boldsymbol{y}| \leq |\boldsymbol{x}||\boldsymbol{y}|$
4. $|\boldsymbol{x}+\boldsymbol{y}| \leq |\boldsymbol{x}| + |\boldsymbol{y}|$

Lagrange's identity,

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 = \sum_{j=1}^{n} a_j^2 \sum_{k=1}^{n} b_k^2 - \frac{1}{2} \sum_{j,k=1}^{n} (a_j b_k - b_k a_j)^2$$

Cauchy's inequality,

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \sum_{j=1}^{n} a_j^2 \sum_{k=1}^{n} b_k^2$$

Triangular inequality,

$$\left(\sum_{j=1}^{n} (a_j + b_j)^2\right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^{n} a_j\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} b_j^2\right)^{\frac{1}{2}}$$

Bernouli's inequality,

$$(1 + x)^n \geq 1 + nx$$

By Mean Value Theorem,

$$a^r b^{(1-r)} \leq ra + (1 - r)b, \quad 0 < r < 1, a > 0, b > 0$$

$$\implies AB \leq \frac{A^p}{p} + \frac{B^q}{q}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1$$

Holder's inequality,

$$\sum_{j=1}^{n} a_j b_j \leq \left(\sum_{j=1}^{n} a_j^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} b_j^q\right)^{\frac{1}{q}}$$

Minkowski's inequality,

$$\left(\sum_{j=1}^{n}(a_j+b_j)^r\right)^{\frac{1}{r}} \leq \left(\sum_{j=1}^{n}a_j^r\right)^{\frac{1}{r}} + \left(\sum_{j=1}^{n}b_j^r\right)^{\frac{1}{r}}$$

Chebyshev's inequality,

$$\left(\frac{1}{n}\sum_{j=1}^{n}a_j^r\right)^{\frac{1}{r}}\left(\frac{1}{n}\sum_{j=1}^{n}b_j^r\right)^{\frac{1}{r}} \leq \left(\frac{1}{n}\sum_{j=1}^{n}(a_jb_j)^r\right)^{\frac{1}{r}}, \quad a_j \leq a_{j+1},\ b_j \leq b_{j+1}$$

Definitions 1.35. *A subset X of $\mathbb{R}^n$ is convex if for any two points $\boldsymbol{x},\boldsymbol{y} \in X$ and real number λ such that $0 < \lambda < 1$, every points $\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in X$.*

maximal An element $x \in A$ is maximal if $\nexists y \in A$ such that $x < y$.

minimal An element $x \in A$ is minimal if $\nexists y \in A$ such that $y < x$.

maximum An element $x \in A$ is maximum if $\forall y \in A,\ y < x$.

minimum An element $x \in A$ is minimum if $\forall y \in A,\ x < y$.

Definitions 1.36. *An element x of an ordered set S,R is an upper bound a subset $E \subset S$ if $\forall y \in E,\ \neg xRy$. A subset E of the ordered set S is bounded above if $\exists$ a upper bound of E, x in S.*

Definitions 1.37. *An element x of an ordered set S,R is a lower bound a subset $E \subset S$ if $\forall y \in E,\ \neg yRx$. A subset E of the ordered set S is bounded below if $\exists$ a lower bound of E, x in S.*

Definitions 1.38. *The supremum of a subset E, $\sup E$ of an ordered set S is the lower bound of all upper bounds of the set E in S.*

Definitions 1.39. *The infimum of a subset E, $\inf E$ of an ordered set S is the upper bound of all lower bounds of the set E in S.*

Remark. *ϵ Characterisation,*

1. $x = \sup E \iff \forall\epsilon > 0,\ \exists y \in E$, *such that* $x - \epsilon < y$
2. $x = \inf E \iff \forall\epsilon > 0,\ \exists y \in E$, *such that* $y < x + \epsilon$

Remark. *Properties,*

1. $\sup_{x,y} f(x,y) = \sup_x \sup_y f(x,y) = \sup_y \sup_x f(x,y)$

2. $\sup_y \inf_x f(x,y) \leq \inf_x \sup_y f(x,y)$
3. $\sup(a + f(x)) = a + \sup f(x)$
4. $\inf f + g(x) \leq \inf f(x) + \inf g(x) \leq \sup f(x) + \sup g(x) \leq \sup f + g(x)$

Definitions 1.40. *An ordered set S is complete if every nonempty subset $E \subset S$, which is bounded above, has $\sup E \in S$.*

Theorem 1.41. *There exists a unique complete ordered field $\mathbb{R}$, that contains $\mathbb{Q}$.*

Axiom 1.42 (Completeness)**.** *A set S is complete, if every nonempty subset $E \subset S$, which is bounded above has a least upper bouned in S.*

Remark. *Every cachy sequence in a complete space is convergent.*

Theorem 1.43. *There exists a complete ordered field $\mathbb{R}$. Moreover $\mathbb{Q} \subset \mathbb{R}$.*

Remark. $S \subset T \implies \inf S \leq \inf T \leq \sup T \leq \sup S$
$\sup A \cup B = \max\{\sup A, \sup B\} \quad \inf A \cup B = \min\{\inf A, \inf B\}$

Remark. *It is possible for a set to have no maximum and yet be bounded above. But, if a set is not bounded above, it doesn't have a maximum.*
For example : Open interval, $(0, 1)$ *doesn't have it's extrema.*

Remark. $\inf \phi = \infty, and \sup \phi = -\infty$
Set E is unbounded above, then $\sup E = \infty$
Set E is unbounded below, then $\inf E = -\infty$

Theorem 1.44. $\mathbb{N}$ *is not bounded above.*

Theorem 1.45 (Archimedean)**.** $\forall x, y \in \mathbb{R},\ 0 < x,\ \exists n \in \mathbb{N}$ *such that* $y < nx$

Remark. *The following statements are equivalent,*

1. $\exists n \in \mathbb{N}$ *such that* $y < nx$
2. $\exists n \in \mathbb{N}$ *such that* $0 < \frac{1}{n} < y$
3. $\exists n \in \mathbb{N}$ *such that* $n - 1 \leq y < n$

Theorem 1.46. *A subset A of $\mathbb{R}$ is open iff it is a countable union of open intervals.*

Theorem 1.47. *For every positive real number $x \in \mathbb{R}$, there exists a unique $y \in R$, such that $y^n = x$. We write, $y = x^{\frac{1}{n}}$.*

Corollary 1.47.1. *Let a, b be positive real numbers and n be a positive integer.*

$$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$$

.

Theorem 1.48 (nested interval)**.** *Let $I_1 \supset I_2 \cdots I_n$ be a sequence of closed, bounded, non-empty intervals, then there exists $x \in \mathbb{R}$ such that $x \in J_k, \ \forall k$*

Remark. *The family of closed, bounded intervals have countable intersection property. The nested interval theorem fails for open intervals.*

Theorem 1.49 (nested cell)**.** *Let $\{J_n\}$ be a sequence of non-empty, closed nested cells in $\mathbb{R}^k$, then there exists $\boldsymbol{x} \in \mathbb{R}^k$ such that $\boldsymbol{x} \in J_k, \ \forall k$*

Theorem 1.50. *Every nonempty finite subset of $\mathbb{R}$ has its extrema in it.*

Theorem 1.51. *$\mathbb{Q}$ is dense in $\mathbb{R}$.*
$\forall x, y \in \mathbb{R}, \ \exists q \in \mathbb{Q}$ such that $x < q < y$

1.6 Complex Field

Definitions 1.52. *A complex number $z \in \mathbb{C}$ is an ordered pair of real numbers, $(u, v) \in \mathbb{R} \times \mathbb{R}$.*

Theorem 1.53. *The set of all complex numbers is a field with*
addition, $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, defined by $(a, b) + (c, d) = (a + c, b + d)$ and
multiplication, $\cdot : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, defined by $(a, b)(c, d) = (ac - bd, ad + bc)$

Remark. *$\mathbb{C}$ is a field on $\mathbb{R}^2$. There doesn't exists a field on $\mathbb{R}^n$ for $n > 2$.*[1]

Definitions 1.54. $i = (0, 1)$

Theorem 1.55. $i^2 = (-1, 0)$, $(a, b) = a + ib$

Definitions 1.56. *The conjugate of a complex number $a + ib$ is $a - ib$.*

Theorem 1.57. *If $z, w \in \mathbb{C}$,*

1. $\overline{z + w} = \bar{z} + \bar{w}$
2. $\overline{zw} = \bar{z}\bar{w}$
3. $z + \bar{z} = 2\Re(z)$

[1] proof reference : not found yet

4. $z - \bar{z} = 2\Im(z)$

5. *For* $z \neq 0$, $z\bar{z}$ *is a positive real number*

Definitions 1.58. *The absolute value* $|z|$ *is the non-negative square root of* $z\bar{z}$.

Theorem 1.59. *Let* $z, w \in \mathbb{C}$,

1. $|z| = |\bar{z}|$

2. $|zw| = |z||w|$

3. $|\Re(z)| \leq |z|$

4. $|z + w| < |z| + |w|$

Definitions 1.60. *Let* $\boldsymbol{z},\boldsymbol{w} \in \mathbb{C}^n$, $\boldsymbol{z} = (a_1, a_2, \cdots, a_n)$, $\boldsymbol{w} = (b_1, b_2, \cdots, b_n)$. *Then,*

$$|\boldsymbol{z}.\boldsymbol{w}| = \left|\sum_{j=1}^{n} a_j \bar{b_j}\right| \quad \& \quad \|\boldsymbol{z}\| = \left(\sum_{j=1}^{n} |a_j|^2\right)^{\frac{1}{2}}$$

Theorem 1.61 (Cauchy-Schwarz-Buniakowsky)**.** $|\boldsymbol{z}.\boldsymbol{w}| \leq \|\boldsymbol{z}\|\|\boldsymbol{w}\|$

$$\left|\sum_{j=1}^{n} a_j \bar{b_j}\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \; \sum_{j=1}^{n} |b_j|^2$$

Chapter 2

Sequences and Series

2.1 Sequences

Definitions 2.1 (sequence)**.** *A sequence* $\{x_n\}$ *in* $\mathbb{R}$ *is a function* $x : \mathbb{N} \to \mathbb{R}$ *where the* n^{th} *term of the sequence,* $x_n = x(k)$*.*

Definitions 2.2. *The range of a sequence* $\{x_n\}$ *is the set* $\{x_k : k \in \mathbb{N}\}$*.*

Remark. *The range of any sequence is countable.*

Definitions 2.3 (subsequence)**.** *Given a sequence* $\{x_k\},\ x : \mathbb{N} \to X,\ x_k = x(k)$ *and a monotone function* $N : \mathbb{N} \to \mathbb{N},\ n_k = n(k)$*, then the sequence* $\{y_k\} : \mathbb{N} \to X,\ y_k = x_{n_k} = (x \circ n)(k)$ *is a subsequence of* $\{x_k\}$*.*

Axiom 2.4 (Dependent Choice)**.** *Let* $\leq$ *be a relation on* X *such that every element* $x \in X$ *is related to some element of* X*, then there exists a sequence for each element* $x \in X$ *such that* $x_1 = x$ *and* $x_k \leq x_{k+1}$ *for every integer* $k \in \mathbb{N}$*.[1]*

Theorem 2.5 (Recursive Definition)**.** *Given a function* $f : X \to X$*, for every* $x \in X$ *there exists a unique sequence* $\{x_n\}$ *such that* $x_1 = x,\ x_{k+1} = f(x_k),\ \forall k \in \mathbb{N}$

Corollary 2.5.1 (Generalised Recursive Definition)**.** *Given a sequence of functions* $f_n : X^n \to X$*, for every* $x \in X$ *there exists a unique sequence* $\{x_n\}$ *such that* $x_1 = x,\ x_k + 1 = f_k(x_1, x_2, \cdots, x_k),\ \forall k \in \mathbb{N}$

2.2 Convergence of Sequences

Definitions 2.6 (convergence)**.** *A sequence* $\{x_n\}$ *converges to* $x \in \mathbb{R}$ *if for every* $\epsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that* $|x - x_n| < \epsilon$ *for every* $k \geq N$*. The* *real number x is the limit of the sequence $\{x_n\}$. ie, $\{x_n\} \to x$. A sequence $\{x_n\}$ is divergent if it is not convergent. ie, $\{x_n\} \to \infty$.*

Remark. *Examples,*
Convergent sequence, $\{x_n\},\ x_n = \frac{1}{n},\ \{x_n\} \to 0$
Divergent sequence, $\{x_n\},\ x_n = -1^n,\ \{x_n\} \to \infty$

Remark (ϵ-neighbourhood of x)**.** *A sequence $\{x_n\} \to x \iff \forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N,\ x_n \in (x - \epsilon, x + \epsilon)$.*

Definitions 2.7 (bounded)**.** *A sequence $\{x_n\}$ is bounded if $\exists M > 0$ such that $\forall n \in \mathbb{N},\ |x_n| \leq M$. ie, $\forall n \in \mathbb{N},\ x_n \in [-M, M]$.*

Remark. *Every convergent sequence is bounded.*

Theorem 2.8 (algebraic limit)**.** *Suppose $\{x_n\} \to x,\ \{y_n\} \to y$, then*

1. $\forall n \in \mathbb{N},\ x_n \leq y_n \implies x \leq y$

2. $\{\alpha x_n + \beta y_n\} \to \alpha x + \beta y$

3. $\{x_n y_n\} \to xy$

4. $\{\frac{x_n}{y_n}\} \to \frac{x}{y}$*, provided* $y \neq 0$.

Remark. *Application 1*
Let $\{x_n\},\ x_n = \frac{n+3}{2n} \implies \forall n \in \mathbb{N},\ x_n = \frac{1}{2} + \frac{3}{2n}$
Thus, $x_n = 0.5y_n + 1.5z_n$, where $\forall n \in \mathbb{N},\ y_n = 1,\ z_n = \frac{1}{n}$
We have, $\{y_n\} \to 1$ and $\{z_n\} \to 0$. Thus, $\{x_n\} \to 0.5$

Remark. *Application 2*
Let $\{x_n\}$ is convergent and given recursive definition, $x_1 = c,\ x_{n+1} = f(x_n)$. Then $\{x_n\} \to x$ where $x = f(x)$. In case of multiple solutions, use x_1.
For example, $x_1 = 3,\ x_{n+1} = \frac{1}{4-x_n} \implies x = \frac{1}{4-x} \implies x = 2 \pm \sqrt{3},\ x \neq 4$.
However, $\forall n \geq 3,\ x_n < 1$. Thus $x = 2 - \sqrt{3}$.

Theorem 2.9 (order limit)**.** *Suppose $\{x_n\} \to x,\ \{y_n\} \to y$, then*

1. $\forall n \in \mathbb{N},\ x_n \geq 0 \implies x \geq 0$.

2. $\forall n \in \mathbb{N},\ x_n \geq y_n \implies x \geq y$.

3. $\exists c \in \mathbb{R},\ \forall n \in \mathbb{N},\ x_n \geq c \implies x \geq c$.

4. $\exists c \in \mathbb{R},\ \forall n \in \mathbb{N},\ x_n \leq c \implies x \leq c$.

Theorem 2.10 (Squeeze). *Suppose* $\{x_n\},\ \{y_n\},\ \{z_n\}$ *are sequences. Then* $\forall n \in \mathbb{N},\ x_n \leq z_n \leq y_n,\ \{x_n\} \to k,\ \{y_n\} \to k \implies \{z_n\} \to k.$

Remark (shuffled sequence). *Let* $\{x_n\}, \{y_n\}$ *be sequences and* $\{z_n\}$ *be a shuffled sequence given by* $\{x_1, y_1, x_2, y_2, \cdots\}$*, then*
$\{z_n\} \to k \iff \{x_n\} \to k,\ \{y_n\} \to k.$

Definitions 2.11 (absolute convergence). *A sequence* $\{x_n\}$ *is absolutely convergent if the sequence* $\{|x_n|\}$ *is convergent.*

Remark. *Every convergent sequence is absolutely convergent.*
$\{x_n\} \to x \implies \{|x_n|\} \to |x|$

Remark. *Suppose* $\{x_n\}$ *is bounded and* $\{y_n\} \to 0$*, then* $\{x_n y_n\} \to 0$

Remark. $\{x_n\} \to 0$ *and* $\forall n \in \mathbb{N},\ |y_n - y| \leq x_n$*, then* $\{y_n\} \to y$.

Definitions 2.12 (Cesaro Means). *The sequence* $\{y_n\}$ *where* $y_k = \frac{1}{k}\sum_{n=1}^{k} x_k$ *is the sequence of the averages of a sequence* $\{x_n\}$.

Remark. *Let* $\{y_n\}$ *be the sequence of averages of the sequence* $\{x_n\}$,

1. $\{y_n\} \to \infty \implies \{x_n\} \to \infty$
2. $\{x_n\} \to k \implies \{y_n\} \to k$

Definitions 2.13 (cluster point). *A real number* x *is a cluster point of the sequence* $\{x_n\}$ *if for every* $\epsilon > 0$ *and every integer* $N \in \mathbb{N}$*, there exists an integer* $k > N$ *such that* $|x_n - x| < \epsilon$.

Remark. *A real number* x *is a cluster point of the sequence* $\{x_n\}$ *iff there is a subsequence converging to* x.

Definitions 2.14 (cauchy). *A sequence is cauchy if for every* $\epsilon > 0$*, there exists an integer* $N \in \mathbb{N}$ *such that for every* $n, m > N$*,* $|x_n - x_m| < \epsilon$.

Remark. *A sequence of real numbers converges iff cauchy.*

2.3 Monotone Convergence Theorem

Theorem 2.15. *Every bounded monotone sequence* $\{x_n\}$ *in* $\mathbb{R}$ *converges.*

Remark. *If monotone decreasing and bounded below,* $\{x_n\} \to \inf\{x_n\}$.
If monotone increasing and bounded above, $\{x_n\} \to \sup\{x_n\}$.

Remark. *Every convergent sequence is bounded.*

Remark. *A sequence of real numbers can have atmost one limit.*

Definitions 2.16. .
limit inferior, $\underline{\lim}\{x_n\} = \sup\{\inf\{x_m : m \geq n\}\}$
limit superior, $\overline{\lim}\{x_n\} = \inf\{\sup\{x_m : m \geq n\}\}$

Remark. *Let $\{x_n\}$ be a bounded sequence, then limit superior and limit inferior are the largest and smallest limit points of it.[2]*
$\underline{\lim}\{x_n\} \leq \overline{\lim}\{x_n\}$

Remark. *For $\{x_n\}$, $\{y_n\}$ such that $\forall n \in \mathbb{N}$, $x_n \leq y_n$,*

1. $\overline{\lim}\{x_n\} \leq \overline{\lim}\{y_n\}$
2. $\underline{\lim}\{x_n\} \leq \underline{\lim}\{y_n\}$

2.4 Convergence Test by Sandwitch Lemma

Convergence of sequences can be tested using the Sandwitch Lemma.

Remark.

$$\begin{aligned}\underline{\lim}\{x_n\} + \underline{\lim}\{y_n\} &\leq \underline{\lim}\{x_n + y_n\}\\ &\leq \overline{\lim}\{x_n + y_n\}\\ &\leq \overline{\lim}\{x_n\} + \overline{\lim}\{y_n\}\end{aligned}$$

Remark. *Let $\{x_n\}$ such that $\sqrt{n} = (1 + x_n)^n$.*
By Bernouli's inequality, $\sqrt{n} = (1 + x_n)^n \geq 1 + nx \implies 0 < x < \frac{1}{\sqrt{n}}$.

$$\underline{\lim}\{\frac{x_n + 1}{x_n}\} \leq \underline{\lim}\{\sqrt[n]{n}\} \leq \overline{\lim}\{\sqrt[n]{n}\} \leq \overline{\lim}\{\frac{x_n + 1}{x_n}\}$$

And $\lim \frac{x_n+1}{x_n} = 1$. *Thus* $\lim \sqrt[n]{n} = 1$

Remark. *Let $\{y_n\}$ be the sequence of averages of the sequence $\{x_n\}$, then* $\underline{\lim}\, x_n \leq \underline{\lim}\, y_n \leq \overline{\lim}\, y_n \leq \overline{\lim}\, x_n$

Remark. *Results by Average Sequence*

1. $x_{n+1} - x_n \to x \implies \frac{x_n}{n} \to x$
2. *If $\{x_n\}$ is bounded and*

$$2x_n \leq x_{n+1} + x_{n-1} \implies \{x_n + 1 - x_n\}$$

*is monotone increasing to 0. pending pp.28 exr 12b[**?**]*

3. $0 < x_1 < 1,\ x_{n+1} = 1 - \sqrt{1-x_n}$, *then* x_n *monotone decreasing to 0 and* $\frac{x_{n+1}}{x_n}$ *convergent to* $\frac{1}{2}$.

4. $\{x_n\},\ x_n = \left(1+\frac{1}{n}\right)^n$ *is convergent*

5. $\{x_n\},\ |x_n - x_{n-1}| \leq \alpha|x_{n+1}-x_n|,\ 0<\alpha<1$ *is convergent*

6. $x_1 = 1,\ x_{n+1} = \frac{1}{3+x_n}$ *is convergent to ?*

7. $x_1 = 1,\ x_{n+1} = 1 + \frac{1}{1+x_n}$ *is convergent to* $\sqrt{2}$

8. $x_1 = 1,\ x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$ *is convergent to* $\sqrt{2}$

2.5 Series

Definitions 2.17 (series)**.** *A series is given by* $\sum_{n=1}^{\infty} x_n$.

Definitions 2.18 (convergence)**.** *A series* $\sum_{n=1}^{\infty} x_n$ *is convergent if the sequence of partial sums* $\{y_k\}$ *such that* $y_k = \sum_{n=1}^{k} x_n$ *is convergent. And the limit of the sequence of partial sums is the sum of the series. ie,* $\{y_k\} \to y$, *then* $\sum_{n=1}^{\infty} x_n = y$.

Remark. $\sum_{n=1}^{\infty} \frac{1}{n^2} = r < 2$

Remark (harmonic series)**.** $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

Theorem 2.19 (cauchy condensation test)**.** *Suppose* $\{x_n\}$ *is decreasing and* $x_n \geq 0$.
$\sum_{n=1}^{\infty} x_n$ *converges iff* $\sum_{n=1}^{\infty} 2^{n-1} x_{2^{n-1}}$ *converges.*

Corollary 2.19.1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges iff* $p > 1$.

Definitions 2.20 (rearrangement invariant)**.** *A series* $\sum_{n=1}^{\infty} x_n$ *is rearrangement invariant if for every bijection* $\sigma : \mathbb{N} \to \mathbb{N}$, $\sum_{k=1}^{\infty} x_{\sigma_k}$ *converges and is invariant.*

Theorem 2.21. *A series* $\sum_{n=1}^{\infty} x_n$ *is rearrangement invariant if for every integer* $n \in \mathbb{N}$, $x_n \geq 0$.

Remark. *Series that are rearrangement invariant are unconditionally convergent.*

2.6 Double Series

Definitions 2.22 (double series)**.** *A double series*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{n,m} = \lim_{k\to\infty}\sum_{n=1}^{k}\sum_{m=1}^{\infty} x_{n,m}$$

Theorem 2.23 (rearrangement invariance)**.** $\forall n, m \in \mathbb{N},\ a_{n,m} \geq 0,$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{n,m} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{n,m}$$

Theorem 2.24 (rearrangement invariant double series into single series)**.** $\forall n, m \in \mathbb{N},\ x_{n,m} \geq 0,\ \forall$ *bijection* $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N},$

$$\sum_{k=1}^{\infty} x_{\sigma_k} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{n,m}$$

Corollary 2.24.1. $\forall n, m \in \mathbb{N},\ x_{n,m} \geq 0,\ \forall$ *bijection* $\sigma : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N},$

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} x_{\sigma_{j,k}} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{n,m}$$

Chapter 3

Bolzano Weierstrass Theorem

3.1 Bolzano-Weierstrass Theorem

Theorem 3.1. *Subsequence of convergent sequences converges to the same limit.*

Remark (Divergence)**.** *Suppose $\{x_n\}$ has two subsequences converging to distinct real numbers, then $\{x_n\}$ is divergent.*

Theorem 3.2. *Every sequence $\{x_n\}$ in $\mathbb{R}$ has subsequences converging to $\underline{\lim}\{x_n\}$ and $\overline{\lim}\{x_n\}$.*

Remark. *A sequence $\{x_n\}$ is convergent iff $\underline{\lim}\{x_n\} = \overline{\lim}\{x_n\}$*

Theorem 3.3 (Bolzano Weierstrass)**.** *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Remark. *Every bounded sequence in $\mathbb{R}$ has subsequences converging to $\underline{\lim}$ and $\overline{\lim}$. [2]*

Theorem 3.4. *Every bounded sequence $\{x_n\}$ in $\mathbb{R}$ converges iff $\underline{\lim}\{x_n\} = \overline{\lim}\{x_n\}$.*

Definitions 3.5 (cauchy)**.** *A sequence $\{x_k\}$ is Cauchy, if $\forall\epsilon > 0,\ \exists N \in \mathbb{N}$ such that $\forall n, m > N,\ |a_n - a_m| < \epsilon$.*

Lemma 3.6. *Cauchy sequences are bounded.*

Theorem 3.7 (cauchy)**.** *A sequence in $\mathbb{R}$ is convergent iff it is a Cauchy sequence.*

Remark. *For a sequence $\{x_n\}$ in $\mathbb{R}$ following statements are equivalent :*

1. $\{x_n\}$ *converges to* x

2. *Every subsequence of* $\{x_n\}$ *has a subsequence converging to* x

3. $\{n_k\}, \{m_k\}$ *be sequences in* $\mathbb{N}$, *then* $\lim\{x_{n_k}\} = \lim\{x_{m_k}\} = x$

Remark. $\{x_n\} \to x$ *iff* $\lim\{x_{2n}\} = \lim\{x_{2n+1}\} = x$

Theorem 3.8 (Bolzano-Weierstrass)**.** *Every bounded, infinite subset* A *of* $\mathbb{R}^k$ *has a cluster point.*

Remark. *Cluster points of* $A \cup B$ *are either cluster points of* A *or of* B.

Remark. *Completeness axiom, Nested Interval Property, Monotone Convergence Theorem, Bolzano-Weierstrass Theorem and Cauchy Crieterion are equivalent.*

3.2 Series Convergence Tests

Definitions 3.9 (convergence)**.**

$$\sum_{k=1}^{\infty} x_k = s \iff \{s_j\} \to s, \text{ where } s_j = \sum_{k=1}^{j} x_j$$

Theorem 3.10 (algebraic limit)**.**

$$\sum_{k=1}^{\infty} x_k = s, \ \sum_{k=1}^{\infty} y_k = t \implies \sum_{k=1}^{\infty} ax_k + by_k = as + bt$$

Theorem 3.11 (cauchy criterion)**.**

$$\sum_{k=1}^{\infty} x_k = s \iff \forall \epsilon > 0, \ \exists N \in \mathbb{N} \text{ such that } |s_n - s_m| < \epsilon$$

where $|s_n - s_m| = |x_{m+1} + x_{m+2} + \cdots + x_n|$

Theorem 3.12.

$$\sum_{k=1}^{\infty} x_k = s \implies \{x_k\} \to 0$$

Theorem 3.13 (comparison test)**.** *Suppose* $\forall k \in \mathbb{N}, \ 0 \leq x_k \leq y_k$. *If* $\sum_{k=1}^{\infty} y_k$ *converges then* $\sum_{k=1}^{\infty} x_k$ *converges. And if* $\sum_{k=1}^{\infty} x_k$ *diverges, then* $\sum_{k=1}^{\infty} y_k$ *diverges.*

Remark. *Comparison test holds if* $\exists M \in \mathbb{N},\ \forall k > M,\ 0 \leq x_k \leq y_k$.

Definitions 3.14 (geometric series)**.** *A series of the form* $\sum_{n=1}^{\infty} ar^n$ *is a geometric series.*

Remark. $\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}$, *provided* $|r| < 1$. *If* $a \neq 0$, *it diverges for* $|r| \geq 1$

Theorem 3.15 (absolute convergence test)**.** .
If a series $\sum_{k=1}^{\infty} |x_k|$ *converges, then* $\sum_{k=1}^{\infty} x_k$ *converges.*

Definitions 3.16. .
If $\sum_{k=1}^{\infty} |x_k|$ *converges, then* $\sum_{k=1}^{\infty} x_k$ *converges absolutely.*
If $\sum_{k=1}^{\infty} x_k$ *converges but* $\sum_{k=1}^{\infty} |x_k|$ *diverges, then* $\sum_{k=1}^{\infty} x_k$ *converges conditionally.*

Theorem 3.17 (alternating series test)**.** .
If a monotone decreasing sequence $\{x_k\} \to 0$, *then the alternating series* $\sum_{k=1}^{\infty} -1^{k+1} x_k$ *converges.*

Theorem 3.18 (rearrangement)**.** *Suppose* $\sum_{k=1}^{\infty} x_k$ *converges absolutely, then any rearrangement of it converges to the same limit.*

Remark. *Suppose series* $\sum_{k=1}^{\infty} x_k$ *converges absolutely and sequence* $\{y_k\}$ *is bounded, then series* $\sum_{k=1}^{\infty} x_k y_k$ *converges.*

Theorem 3.19 (ratio test)**.** *Suppose sequence of ratios* $\{r_k\}$ *is absolutely convergent and* $\{|r_k|\} \to r < 1$ *where* $r_k = \frac{x_{k+1}}{x_k}$, *then series* $\sum_{k=1}^{\infty} x_k$ *converges absolutely.*

3.3 Heine-Borel Theorem

Theorem 3.20. *A subset of* $\mathbb{R}^p$ *is compact iff closed and bounded.*

Remark. *Applications*

1. *Cantor Intersection Theorem*
2. *Lebesgue Covering Theorem*
3. *Nearest Point Theorem*
4. *Circumscribing Contour Theorem*

Theorem 3.21 (Cantor intersection)**.** *Let* $F_1, F_2, \cdots$ *be non-empty, closed, bounded subsets of* $\mathbb{R}^p$ *such that* $F_1 \supset F_2 \supset \cdots$. *Then there exists a point* y *such that* $y \in F_k,\ \forall k$.

Theorem 3.22 (Lebesgue covering)**.** *Let K be a compact subset of $\mathbb{R}^p$ and $\mathcal{U}$ be a cover of K. There exists a real number $r > 0$ such that $\forall x \in K,\ B(x,r) \subset U$ for some $U \in \mathcal{U}$.*

Theorem 3.23 (Nearest Point)**.** *Let K be a compact subset of $\mathbb{R}^p$ and $x \notin K$, then there exists $y \in K$ such that $|x-y| \leq |x-z|,\ \forall z \in K$.*

Theorem 3.24 (Circumscribing contour)**.** *Let K be a compact subset of $\mathbb{R}^2$ and G be an open set containing K. Then there exists a closed curve contained in G made up of arcs of finite number of circles in G such that K is contained in it.*

Chapter 4

Continuity

4.1 Continuity

Definitions 4.1. *$f : X \to Y$ is continuous if $\forall \epsilon > 0,\ \forall x \in X,\ \exists \delta_x > 0$ such that $|x - y| < \delta_x \implies |f(x) - f(y)| < \epsilon$.*

Definitions 4.2 (continuity)**.** *f is continuous at x if*
Cauchy - $\forall \epsilon > 0,\ \exists \delta$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.
Heine - Sequence $< x_k > \to x \implies < f(x_k) > \to f(x)$.
Topology - $\forall V \in \mathcal{N}_{f(x)},\ \exists U \in \mathcal{N}_x$ such that $f(U) \subset V$.

4.2 Uniform Continuity

Definitions 4.3. *$f : X \to Y$ is uniformly continuous if $\forall \epsilon > 0,\ \exists \delta > 0$ such that $\forall x \in X,\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.*

Remark. *f is continuous at y if $\lim_{x \to -y} f(x) = \lim_{x \to +y} f(x) = f(y)$*

Theorem 4.4. *A continuous function from a closed bounded interval into $\mathbb{R}$ is uniformly continuous.*

Theorem 4.5. *A continuous function from a closed bounded interval into $\mathbb{R}$ is bounded and attains its extrema.*

4.3 Differentiability

Definitions 4.6. *f is differentiable at y if $\lim_{x \to y} \frac{f(x) - f(y)}{x - y}$ exists.*

4.4 Mean Value Theorem

Theorem 4.7 (intermediate value)**.** *Let continous function* $f : [a, b] \to \mathbb{R}$*, then* $f([a, b]) = [f(a), f(b)]$.

Chapter 5

Sequence of Functions

5.1 Sequence of functions

5.2 Bounded sequence of functions

Definitions 5.1. *A sequence of functions* $\{f_n\}$ *is bounded if there exists a real-valued function* g *such that for every* $x \in X$ *and for every integer* $n \in \mathbb{N}$, $|f_n(x)| < g(x)$.

Definitions 5.2. *If the sequence* $\{f_n\}$ *is a bounded sequence of functions, then* $\overline{\lim} f_n$ *and* $\underline{\lim} f_n$ *are functions defined by* $(\overline{\lim} f_n)(x) = \overline{\lim} f_n(x)$ *and* $(\underline{\lim} f_n)(x) = \underline{\lim} f_n(x)$

5.3 Series of functions

5.4 Uniform Convergence

Chapter 6

Riemann Integral

Chapter 7

Lebesgue Integral

Chapter 8

Multivariate Calculus

Chapter 9

Metric Spaces

Part II

Linear Algebra

Chapter 10

Vector Spaces

10.1 Vector Space

Axiom 10.1 (Field)**.** *A set F together with two binary operations $+, \cdot$ is a field if it satisfies*

1. *Addition is commutative,* $\forall x, y, z \in F,\ x + y = y + x$
2. *Addition is associative,* $\forall x, y, z \in F,\ x + (y + z) = (x + y) + z$
3. *Existence of additive identity,* $\exists 0 \in F,\ x + 0 = x$
4. *Existence of additive inverses,* $\forall x \in F,\ \exists -x \in F,\ x + (-x) = 0$
5. *Multiplication is commutative,* $\forall x, y \in F,\ xy = yx$
6. *Multiplication is associative,* $\forall x, y, z \in F,\ x(yz) = (xy)z$
7. *Existence of multiplicative identity,* $\exists 1 \in F,\ \forall x \in F,\ 1x = x$
8. *Existence of multiplicative inverses,* $\forall x \in F,\ x \neq 0,\ \exists x^{-1} \in F,\ xx^{-1} = 1$
9. *Multiplication is distributive over addition,* $\forall x, y, z \in F,\ x(y + z) = xy + xz$

Remark. *A few fields,*

$\mathbb{Q}$ *field of all rational numbers*

$\mathbb{R}$ *field of all real numbers*

$\mathbb{C}$ *field of all complex numbers*

$\mathbb{Z}_{p^n}$ *Galois field of prime powers*

$\mathbb{Q}(v)$ *algebraic extensions of* $\mathbb{Q}$

$\mathbb{R}(v)$ *algebraic extensions of* $\mathbb{R}$

Axiom 10.2 (Vector Space). *A set V of vectors and a field F of scalars together with two binary operations, vector addition, $+ : V \times V \to V$ and scalar muliplication $\cdot : F \times V \to V$ is a vector space V over F, if it satifies*

1. *Addition is commutative,* $\forall u, v \in V,\ u + v = v + u$
2. *Addition is associtive,* $\forall u, v, w \in V,\ u + (v + w) = (u + v) + w$
3. *Additive identity,* $\exists 0 \in V$*, such that* $\forall v \in V,\ 0 + v = v$
4. *Additive inverses,* $\forall v \in V$*, there exists* $-v \in V$ *such that* $v + (-v) = 0$
5. *Scalar Multiplication is associtive,* $\forall a, b \in F,\ \forall v \in V,\ a(bv) = (ab)v$
6. *Scalar Multiplication is distributive over vector addition,* $\forall a \in F,\ \forall u, v \in V,\ a(u + v) = au + av$
7. *Scalar Multiplication is distributive over scalar addition,* $\forall a, b \in F,\ \forall v \in V,\ (a + b)v = av + bv$
8. *Scalar Multiplication identity,* $\forall v \in V,\ 1v = v$

Remark. *Scalar multiplication is trivially commutative. ie,* $av = va$

Remark. *A few vector spaces,*

F^n *n-tuple space*

$F^{m\times n}$ *space of all $m \times n$ matrices*

F^S *space of all functions $f : S \to F$*

$F(x)$ *space of all polynomial functions on F*

Definitions 10.3. *A vector b is a linear combination of the set of vectors $\{a_1, a_2, \cdots, a_n\}$ if there exists scalars $c_i \in F$ such that $b = \sum c_i a_i$.*

Axiom 10.4. *Let V be a vector space, then inner product of V is a function, $\cdot : V \times V \to \mathbb{R}$ satisfying,*

1. $\forall x \in V,\ x \cdot x \geq 0$

2. $x \cdot x = 0 \iff x = 0$

3. *Inner product is commutative,* $\forall x, y \in V,\ x \cdot y = y \cdot x$

4. $\forall x, y, z \in V,\ x \cdot (y + z) = x \cdot y + x \cdot z$ *and* $(x + y) \cdot z = x \cdot z + y \cdot z$

5. $\forall x, y \in V,\ \forall a \in F,\ (ax) \cdot y = a(x \cdot y) = x \cdot (ay)$

Definitions 10.5. *A vector space V with inner product is an inner product space.*

Axiom 10.6. *Let V be a vector space, then a norm on V is a function $\|\| : V \to \mathbb{R}$ satisfying,*

1. $\forall x \in V,\ \|x\| \geq 0$

2. $\|x\| = 0 \iff x = 0$

3. $\forall x \in V,\ \forall a \in F,\ \|ax\| = |a|\|x\|$

4. *triangular inequality,* $\forall x, y \in V,\ \|x + y\| \leq \|x\| + \|y\|$

Definitions 10.7. *A vector space with a norm is a normed space.*

Remark. *Let V be a vector space with inner product $\cdot$. Then inner product induced norm is given by* $\forall x \in V,\ \|x\| = \sqrt{x \cdot x}$

Theorem 10.8. *Let V be a vector space with inner product induced norm, then $x \cdot y \leq \|x\|\|y\|$. And equality holds iff $x = cy$.*

Corollary 10.8.1 (Cauchy-Schwarz-Buniakowsky)**.** $|x \cdot y| \leq \|x\|\|y\|$

Remark. $x = (a, b),\ y = (b, a) \implies$ *Geometric Mean* $\leq$ *Arithmetic Mean*

Remark (parallelogram identity)**.** $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$

Remark (orthogonal)**.** *$x, y \in V$ are orthogonal if $x \cdot y = 0$.*

Theorem 10.9. *Let $\boldsymbol{x} \in \mathbb{R}^n$, then* $|x_j| \leq \|\boldsymbol{x}\| \leq \sqrt{n} \sup\{|x_1|, |x_2|, \cdots, |x_n|\}$.

10.2 Subspace

Definitions 10.10. *Let V be a vector space over F with vector addition, $+$ and scalar multiplication, $\cdot$, then a subspace W of V is a subset W of V if it is a vector space over F with the same operations restricted to W.*

Theorem 10.11. *A non-empty subset W of V is a subspace of V over the same field F iff for every pair of vectors $u, v \in W$ and every scalar $c \in F$, $cu + v \in W$*

Theorem 10.12. *Let V be a vector space over the field F, then the intersection of any collection of subspace of V is a subspace of V*

Theorem 10.13 (Subspace spanned by W)**.** *Let V be a vector space over the field F and $W \subset V$, then the intersection of all subspace of V that contains W is the subspace spanned/generated by W.*

Theorem 10.14. *The subspace spanned by W is the set of all linear combinations of vectors in W.*

Definitions 10.15. *The sum of subsets $W_1, W_2, \cdots W_n$ of V is the set of all vectors, $w_1 + w_2 + \cdots + w_n$ where $w_k \in W_k$.*

Theorem 10.16. *Let $W_1, W_2, \cdots, W_n$ be subspace of the vector space V, then $W_1 + W_2 + \cdots + W_n$ is the subspace of V containing each of the subspaces W_k.*

Definitions 10.17. *Let A be an $m \times n$ matrix over the field F, then the row space of A is the subspace of F^n spanned by the row vectors of A. And the column space of A is the subspace of F^m spanned by the column vectors of A.*

Remark. *Every linear combination of zero-sum vectors gives zero-sum. Thus there is unique zero-sum $(n-1)$-dimensional subspace for every n-dimensional vector space.*[1]

Remark. *Let a Cube has corners (x, y, z), $x, y, z \in \{0, 1\}$, then $(0, 1, 0) - (1, 1, 0)$ is an edge and $(0, 1, 0) - (1, 1, 0) - 1, 1, 1) - (1, 1, 0)$ is a face.*
There are $\binom{n}{k-1} 2^{n-k+1}$ k-dimensional subspaces for such an n-cube. ie, n-cube has 2^n corners, $n2^{n-1}$ edges, $n(n-1)2^{n-3}$ faces ...

[1] zero-sum vectors[**?**] : vectors whose components add to zero. eg. $(1, -2, 1)$.

10.2.1 Basis & Dimension

Definitions 10.18. *A set of vectors* $\{v_1, v_2, \cdots, v_n\}$ *is linearly dependent if there exists scalars* $c_1, c_2, \cdots, c_n \in F$*, not all of them are zero such that* $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$*.*

Remark.

1. *Any set containing the zero vector is linearly dependent.*
2. *Any set containing linearly dependent set is linearly dependent.*
3. *Any subset of a linear independent set is linearly independent.*
4. *S is linearly independent iff every finite subset of S is linearly independent.*⋆

Definitions 10.19. *A basis for V is a linearly independent set of vectors that spans V. V is finite dimensional if it has a finite basis.*

Remark. *Let A be an invertible $n \times n$ matrix, then the column vectors of A is a basis for F^n.*

Theorem 10.20. *Let V be a vector space is spanned by a finite set of n vectors, then any independent set of vectors in V is finite and contains no more than n elements.*

Corollary 10.20.1. *If V is a finite dimensional vector space, then any two bases of V contains the same number of elements.*

Corollary 10.20.2. *If V is n-dimensional, then any subset of more than n vectors is dependent and no subset with fewer than n vectors can span V.*

Lemma 10.21. *Let S be an independent subset of V and $v \in V$ is not in the subspace spanned by S, then $S \cup \{v\}$ is an independent subset of V.*

Theorem 10.22. *If W is a subspace of a finite dimensional vector space V, then every independent subset of W is finite and is part of a basis for W.*

Corollary 10.22.1. *If W is a proper subspace of a finite dimensional vector space V, then W is finite dimensional and* $\dim W < \dim V$*.*

Corollary 10.22.2. *Let A be an $n \times n$ matrix. If row vectors of A are linearly independent in F^n, then A is invertible.*

Corollary 10.22.3. *If W_1, W_2 are finite dimensional subspaces of V, then W_1+W_2 is finite dimensional and* $\dim W_1+W_2 = \dim W_1+\dim W_2-\dim W_1\cap W_2$

10.2.2 Change of Basis

Definitions 10.23. *The co-ordinates of a vector $v \in V$ with respect to an ordered basis B, $[v]_B$ is the column vector of scalars $c_1, c_2, \cdots, c_n$ such that $v = c_1b_1 + c_2b_2 + \cdots + c_nb_n$ where $b_k \in B$.*

Theorem 10.24 (Change of Basis)**.** *Let V be an n-dimensional vector space over the field F and B, B' be two ordered bases for V, then there exists an $n \times n$ invertible matrix P such that $[v]_B = P[v]_{B'}$.*

Theorem 10.25. *For every invertible $n \times n$ matrix, P and basis B of the vector V over the field F, there exists another basis B' such that $[v]_B = P[v]_{B'}$ for every vector $v \in V$.*

Definitions 10.26. *Row rank and Column rank,*

row rank *dimension of row space of A*

column rank *dimension of column space of A*

Theorem 10.27. *Row-equivalent matrices have the same row space.*

Theorem 10.28. *Non-zero row vectors of a row-reduced echelon matrix, R forms a basis for the row space of R.*

Theorem 10.29. *For every subspace W of F^n with $\dim W \leq m$, there exists a unique $m \times n$ row-reduced echelon matrix, R such that its row space is W.*

Theorem 10.30. *Every $m \times n$ matrix over the field F is row-equivalent to a unique row-reduced echelon matrix.*

Theorem 10.31. *The following statements are equivalent,*

1. *A, B are row-equivalent.*

2. *A, B have same row space.*

3. *There exists an invertible matrix, P such that $A = PB$*

4. *$AX = 0, BX = 0$ has same solution space.*

10.3 Linear Transformations

Definitions 10.32. *Let V, W be vector spaces over the same field F. A linear transformation T is a function, $T : V \to W$ such that $T(cu + v) = cTu + Tv$ where $u, v \in V$ and $c \in F$.*

Remark. *A few linear transformations,*

1. *The set of all polynomials over the field $\mathbb{C}$ with differentiation.*
2. *$F^{m\times n}$ with matrix multiplication.*
3. *The set of all continuous real functions with integration.*

Remark. *Properties of linear transformations,*

1. *$T(0) = 0$*
2. *T preserves linear combinations*

Theorem 10.33. *Let $B = \{b_1, b_2, \cdots, b_n\}$ be an ordered basis for an n-dimensional vector space V over the field F and W be any vector space over the field F and $b'_1, b'_2, \cdots, b'_n \in W$, then there exists a unique linear transformation $T : V \to W$ such that $T(b_k) = b'_k,\ \forall k$.*

Remark. *A few subspaces from linear transformations,*

1. *$T(V)$ is a subspace of W*
2. *$\{v \in V : Tv = 0\}$ is a subspace of V*

Definitions 10.34. *Subspaces from transformations and their dimensions,*

Null space/Kernel of T $N(T) = \{v \in V \ : \ Tv = 0\}$

Nullity of T $nullity(T) = \dim N(T)$

Range space of T $R(T) = \{w \in W \ : \ Tv = w,\ v \in V\}$

Rank of T $rank(T) = \dim R(T)$

Theorem 10.35 (rank-nullity)**.** *Let T be a linear transformation from a finite dimensional space V into W, then $rank(T) + nullity(T) = \dim V$*

Theorem 10.36. *Let $A \in F^{m\times n}$, then row $rank(A) =$ column $rank(A)$*

Theorem 10.37. *Let V, W be vector spaces over the field F, then the set of all linear transformations $T : V \to W$ with addition, $(T + U)v = Tv + Uv$ and multiplication, $(cT)v = c(Tv)$ is a vector space, $L(V, W)$ over the field F.*

Theorem 10.38. *Let* $\dim V = n,\ \dim W = m$, *then* $\dim L(V, W) = mn$

Theorem 10.39. *Let $T \in L(V, W),\ U \in L(W, Z)$, then $UT =\in L(V, Z)$*

Remark. *A linear operator on V is a linear tranformation, $T : V \to V$.*

Lemma 10.40 (linear algebra with identity)**.** $\star$ *Let V be a vector space over the field F, then the set of all linear operators on V, $L(V, V)$ is a linear algebra with identity. Let $U, T_1, T_2 \in L(V, V)$, then*

1. $\exists I \in L(V, V)$ *such that* $UI = U = IU$

2. $U(T_1 + T_2) = UT_1 + UT_2$ *and* $(T_1 + T_2)U = T_1U + T_2U$

3. $c(UT) = (cU)T$

Theorem 10.41. *Linear tranformation $T : V \to W$ is invertible iff T is bijective, then $T^{-1} : W \to V$ is also bijective.*

non-singular T is non-singular if $Tv = 0 \implies v = 0$ ie, $N(T) = \{0\}$

Theorem 10.42. *A linear transformation, T is injective iff T is non-singular.*

Theorem 10.43. *Linear transformations preserve indepenedence iff non-singular.*$\star$

Remark. *Let V be an n-dimensional vector space over the field F and $T : V \to V$ be a linear operator. For any basis B of V, $B' = \{Tb_1, Tb_2, \cdots, Tb_n\},\ b_j \in B$ is also a basis for V iff the linear opertor $T : V \to V$ is invertible.*

Theorem 10.44. *Let $T : V \to W$ and $U : W \to Z$ be invertible linear transformations, then $UT : V \to Z$ is invertible and $(UT)^{-1} = T^{-1}U^{-1}$*

Theorem 10.45. *Let V, W be finite dimensional vector spaces over the field F such that* $\dim V = \dim W$. *And $T : V \to W$ be a linear transformation, then the following statements are equivalent,*

1. *T is invertible.*

2. *T is non-singular.*

3. T is surjective.

Theorem 10.46. *The set of all invertible linear operators with on V with composition is a non-abelian group.*

Theorem 10.47. *Every n-dimensional vector space over the field F is isomorphic to F^n.*

Theorem 10.48. *Let U, V be vector spaces over the field F, $U : V \to W$ be an isomorphism, then $\phi : L(V, V) \to L(W, W)$, $\phi(T) = UTU^{-1}$ is an isomorphism.*

Chapter 11

Matrices

11.1 Matrix

Definitions 11.1. *An $m \times n$ matrix over the field F is a function $A : \{1, 2, \cdots, m\} \times \{1, 2, \cdots, n\} \to F$.*

upper triangular $a_{ij} = 0,\ i > j$

lower triangular $a_{ij} = 0,\ i < j$

symmetric $a_{ij} = a_{ji}$

skew-symmetric $a_{ij} = -a_{ji}$

hermitian $a_{ij} = \overline{a_{ji}}$

Remark. *There are three elementary row operations on an $m \times n$ matrix*

1. *Multiplication of one row by non-zero scalar*
2. *Replacing row r by row r plus c times row r*
3. *Interchanging two rows*

Remark. *Elementary row operations on A does not affect the set of solutions of the homogenous system of linear equations, $AX = 0$.*
The inverse of any elementary row operation is of the same kind.

Definitions 11.2. *A matrix B is row-equivalent to matrix A if B can be obtained from A by a finite sequence of row operations.*
If A and B are row-equivalent, then the homogeneous systems of linear equations, $AX = 0$ and $BX = 0$ have exactly the same solutions.

Definitions 11.3. *An $m \times n$ matrix is row-reduced if*

1. *first non-zero entry, pivot of each non-zero row is 1*
2. *each pivot column has all other entries zero*

Remark. *Every $m \times n$ matrix is row-equivalent to a row-reduced $m \times n$ matrix.*

Definitions 11.4. *An $m \times n$ matrix is an echelon matrix if*

1. *non-zero rows are above every zero-rows*
2. *pivot column of any row is less than pivot column of any row below it*

Theorem 11.5. *Every $m \times n$ matrix is row-equivalent to a unique $m \times n$ row-reduced echelon matrix.*

Theorem 11.6. *If A is an $m \times n$ matrix and $m < n$, then $AX = 0$ has a non-trivial solution.*

Theorem 11.7. *A is row-equivalent to $n \times n$ identity matrix iff $AX = 0$ has only the trivial solution.*

Definitions 11.8. $A_{m\times n} \times B_{n\times p} = C_{m\times p},\ c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$

Remark. *Matrix multiplication is not commutative.*
Matrix multiplication is associative.

Definitions 11.9. *An $n \times n$ matrix is an elementary matrix if it can be obtained from $n \times n$ identity matrix by a single elementary row operation.*

Remark. *An elementary operation is equivalent to the left-multiplication by corresponding elementary matrix.*
Two $m \times n$ matrices are row-equivalent if one can be obtained from the other by left-multiplying a finite number of $m \times m$ elementary matrices. (or by right-multiplying a finite number of $n \times n$ elementary matrices)

Definitions 11.10. *Let A be an $n \times n$ matrix over the field F. If $BA = I$, then B is the left-inverse of A. If $AB = I$, then B is the right-inverse of A. If $AB = BA = I$, then B is the inverse of A and A is an invertible matrix.*

Theorem 11.11. *If A, B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$. The product of invertible matrices is invertible.*
Elementary matrices are invertible.

Theorem 11.12. *The following statements are equivalent:*

1. *A is invertible.*
2. *A is row-equivalent to $I_{n\times n}$.*
3. *A is a product of elementary matrices.*
4. *$AX = 0$ has only trivial solution.*
5. *$AX = Y$ has a unique solution X for each Y.*

Corollary 11.12.1. *If A is invertible, a sequence of elementary row operations would reduce A to the identity. The same sequence of elementary row operations would convert I to A^{-1}.*

Corollary 11.12.2. *If A, B are row-equivalent $m\times n$ matrices, then $B = PA$ where P is an invertible $m \times m$ matrix and $A = P^{-1}B$.*
An $n \times n$ matrix with either left or right inverse is invertible.
Let $A = \prod A_k$. A is invertible iff each A_k is invertible.

11.2 Rank

11.3 Determinant

11.4 Linear Equations

Chapter 12

Cayley Hamilton Theorem

12.1 Eigenvalues & Eigenvectors

12.2 Cayley-Hamilton Theorem

Chapter 13

Canonical Forms

13.1 Trasformation Matrix

Theorem 13.1. *Let V, W be vector spaces over field F of dimensions n, m. Let B, B' be ordered bases for V, W, then for each linear transformation $T : V \to W$ there exists an $m \times n$ matrix such that $[Tv]_{B'} = [T]_{BB'}[v]_B$ where the columns of $[T]_{BB'}$ are co-ordinates, $[Tb_j]_{B'}$ for each vector in the ordered basis B of V.*

Theorem 13.2. *Let V, W be vector spaces over the field F of dimensions n, m and B, B' be ordered bases for V, W. For each such pair of ordered bases, the function $\phi : L(V, W) \to F^{m\times n}$, $\phi(T) = [T]_{BB'}$ is an isomorphism.*

Theorem 13.3. *Let V, W, Z be finite dimensional vector spaces over the field F with ordered bases B, B', B'' and T, U be linear transformations, $T : V \to W$, $U : W \to Z$, then $[UT]_{BB''} = [U]_{B'B''}[T]_{BB'}$.*

Corollary 13.3.1. *Let $T : V \to V$ be an invertible linear operator, then $[T^{-1}]_{BB} = [T]^{-1}_{BB}$.*

Corollary 13.3.2. *Let $T : V \to V$ be a linear operator and B, B' be two ordered bases for V, then $[T]_{B'B'} = [I]_{BB'}[T]_{BB}[I]_{B'B}$.*

Theorem 13.4. *Let $T : V \to V$ be a linear operator and B, B' be two ordered bases for V, then there exists an invertible linear operator $U : V \to V$, $Ub_j = b'_j$ such that $[T]_{B'B'} = [U]^{-1}_{BB}[T]_{BB}[U]_{BB}$.$\star$*

Remark. *Let V be a vector space over field F with ordered bases B, B', then there exists a linear operator $U : V \to V$, $Ub_j = b'_j$ such that $[I]_{B'B} = [U]_{BB}$.*

Definitions 13.5. *Two $n \times n$ matrices A, B are similar if there exists an invertible $n \times n$ matrix P such that $B = P^{-1}AP$.*

Remark. *Let B, B' be ordered bases for n-dimensional vector space V over field F and $T : V \to V$ be a linear operator, then $[T]_{BB}, [T]_{B'B'}$ are similar.*

Remark. *If linear operator $T : V \to V$ is invertible, then $[T]_{BB}$ is invertible and is similar to $I_{n\times n}$.*

Remark. *Let V be an n-dimensional vector space over field F and A be an invertible $n \times n$ matrix over field F, then there exists a pair of ordered bases B, B' for V such that $[I]_{BB'} = A$. And there exists an invertible linear operator $U : V \to V$, $Ub_j = b'_j$ such that $[U]_{BB} = A$.*

13.2 Linear Functionals

Definitions 13.6. *Let V be a vector space over the field F, then linear transformation $f : V \to F$ is a linear functional on V.*

Definitions 13.7. *Let V be a vector space over the field F, then the set of all linear functionals on V is a dual space $V^* = L(V, F)$ of V.*

Remark. $\dim V^* = \dim V$

Theorem 13.8. *Let $B = \{b_1, b_2, \cdots, b_n\}$ be an ordered basis for the vector space V over the field F, then $B^* = \{f_1, f_2, \cdots, f_n\}$ such that $f_i(b_j) = \delta_{ij}$ is a dual basis for V^*. And for each $v \in V$, the co-ordinates of v are $f_j(v)$.*

Remark. *Let f be a non-zero linear functional on the vector space V over the field F, then* $\dim N(f) = \dim V - 1$. *If V is finite-dimensional, then the null space of any non-zero functional on V is a hyperspace of V.*

Definitions 13.9. *Let S be a subset of the vector space V over the field F, then the annihilator of S, S^0 is the set of all linear functionals on V such that $f(v) = 0, \ \forall v \in S$.*

Remark. *S^0 is the set of all linear functionals on V such that S is contained in the nullspace of all those functionals. The set of all linear functional that map vectors in S into 0.*

Theorem 13.10. *Let W be a subspace of a finite dimensional vector space V over the field F, then* $\dim W + \dim W^0 = \dim V$.

Corollary 13.10.1. *Let W be a k-dimensional subspace of an n-dimensional vector space V over the field F, then W is the intersection of $(n - k)$ hyperspaces in V.*

Corollary 13.10.2. *Let W_1, W_2 be subspaces of a finite dimensional vector space, then $W_1 = W_2$ iff $W_1^0 = W_2^0$.*

Definitions 13.11. *Let V be a vector space over the field F, then double dual V^{**} of V is the set of all linear functionals on the dual space V^*.*

Theorem 13.12. *Let V be a finite-dimensional vector space over the field F, then $\phi : V \to V^{**},\ \phi(v) = L_v$ such that $\forall f \in V^*,\ L_v(f) = f(v)$ is an isomorphism.*

Corollary 13.12.1. *Let V be a finite dimensional vector space over the field F, then for every linear functional $L \in V^{**}$, there exists a unique vector $v \in V$ such that for every linear function $f \in V^*$, $L(f) = f(v)$.*

Corollary 13.12.2. *Let V be a finite dimensional vector space over F, then each basis for V^* is the dual of some basis for V.*

Theorem 13.13. *Let S be a subset of a finite dimensional vector space V over the field F, then S^{0^0} is the subspace spanned by S.$\star$*

Definitions 13.14. *A hyperspace of a vector space is a maximal, proper subspace of it.*

Theorem 13.15. *Let V be a vector space over the field F, then the null space of a non-zero functional f on V is a hyperspace in V. And every hyperspace in V is the null space of some non-zero linear functional on V.*

Lemma 13.16. *Let f, g be linear functionals on a vector space V over the field F, then g is a scalar multiple of f iff null space of g contains null space of f.*

Theorem 13.17. *Let $g, f_1, f_2, \cdots, f_n$ be linear functionals on a vector space V over the field F and $N, N_1, N_2, \cdots, N_n$ be the respective null spaces, then g is a linear combination of f_js iff N contains $\cap_{j=1}^{n} N_j$.*

Theorem 13.18. *Let V, W be vector spaces over the field F, then for each linear transformation, $T : V \to W$ there exists a unique linear transformation, $T^t : W^* \to V^*$ such that for each $v \in V$ and $g \in W^*$, $T^t g(v) = g(Tv)$.*

Definitions 13.19. *Let V, W be vector spaces over the field F and $T : V \to W$ be a linear transformation, then the linear transformation, $T^t : W^* \to V^*$ such that $T_t g(v) = g(Tv),\ \forall v \in V,\ \forall g \in W^*$ is the transpose/adjoint of T.*

Theorem 13.20. *Let V, W be vector spaces over the field F and $T : V \to W$ be a linear transformation, then the null space of T^t is the annihilator of the range of T.*

Corollary 13.20.1. *Let V, W be finite dimensional vector spaces over the field F, then $rank(T^t) = rank(T)$ and $R(T^t) = N(T)^0$*

Theorem 13.21. *Let V, W be finite dimensional vector spaces over the field F with ordered bases B, B' and B^*, B'^* are respective dual bases for V^*, W^* and $T : V \to W$ be a linear transformation with transpose of T, $T^t : W^* \to V^*$, then $[T]_{BB'}$ is the transpose of the matrix $[T^t]_{B'^*B^*}$.*

13.3 Canonical Forms

13.3.1 Diagonal Forms

13.3.2 Triangular Forms

13.3.3 Jordan Forms

Chapter 14

Inner Product Spaces

14.1 Inner Product Spaces

14.2 Orthonormal Basis

Chapter 15

Quadratic Forms

15.1 Quadratic Forms

15.2 Reduction & Classification of Quadratic Forms

Part III

Topology

Chapter 16

Connectedness and Compactness

16.1 Metric Space

Definitions 16.1. *A function $d : X \times X \to \mathbb{R}$ is a metric on X if it satisfies*

1. $d(x, y) \geq 0,\ d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) = d(x, z) + d(z, y)$

Definitions 16.2. *A set X together with a metric on X is a metric space.*

neighbourhood A subset N is a neighbourhood of a point x if there exists a positive real number r such that $y \in N$ for every y satisfying $d(x, y) < r$.

limit point A point x is a limit point of a subset A if every neighbourhood of x has some another point from the subset A.

isolated point A point which is not a limit point of the subset A.

closed A subset A of X is closed if it has every limit point of it. $\overline{A} = A$

interior point A point x is an interior point the subset A if there exists some neighbourhood of x, which is contained in the subset A. $x \in A^0$

open A subset A is open if each point of it is an interior point in it.

complement The subset of all points which are not in the subset A is the complement of A.

perfect A subset A is perfect if it is closed and has each point of it as its limit point.

bounded A subset A is bounded if there is a point x and a positive real number r such that for each point $y \in A$, $d(x, y) < r$.

dense A subset A is dense if every point of X is either in A or is a limit point of A.

boundary A point x is a boundary point of subset A of X if every neighbourhood of x has some point from A as well as $X - A$. $x \in \partial(A)$

Remark. *1. $\overline{A} \cap \overline{B} \neq \overline{A \cap B}$ eg. $\overline{(1,2)} \cap \overline{(2,3)} = \{2\}$, $\overline{(1,2) \cap (2,3)} = \{\}$*

2. There exists countable, dense subsets of $\mathbb{R}$ with empty interior. eg. $\mathbb{Q}$

3. There exists uncountable, dense subsets of $\mathbb{R}$ with empty interior.

4. $A \times B$ is open(or closed) iff both A, B are open(or closed).

5. For Cantor set $\mathcal{C}$, $\mathcal{C}^0 = \phi$, $\overline{\mathcal{C}} = \mathcal{C}$, $\partial(\mathcal{C}) = \mathcal{C}$

6. $\mathcal{C}$ can't be expresses as countable union of closed intervals.

7. $\mathbb{R} - \mathcal{C}$ can be expresses as countable union of open intervals.

Remark. *1. For any bounded subset A, there exists a point x such that $d(x, y) < r$ for every $y \in A$. Then for any point $z \in A$, $d(z, y) < 2r$ for every $y \in A$.*

Definitions 16.3. radius *The radius of a bounded subset A is the smallest real number r such that for a particular $x \in X$, $d(x, y) < r$ for every point $y \in A$.*

16.2 Topological Space

Definitions 16.4. *A set X together with a family $\mathcal{T}$ of subsets of $X$is a topological space if it satisfies*

1. $\mathcal{T}$ is closed under arbitrary unions

2. $\mathcal{T}$ is closed under finite intersections

usual $d(x, y) = |x - y|$

discrete $d(x, y) = \delta_{xy}$

taxicab $d(x, y) = |x| + |y|$

Definitions 16.5. *A set is separable if it has a countable dense subset.*

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