Enumerating subgroups

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Prerequisite

The reader is assumed to know the basics of abstract algebra.

1 Introduction

Let G be a finite, abelian group of order n. Then, G is finitely generated.

1.1 Enumerting distinct subgroups

By *fundamental theorem*, for each factor m of the integer n, G has atleast one subgroup of order m. Suppose m is a factor of n. Then, we want to identify *distinct* subgroups of order m (if any).

m is prime power

Suppose m is rth power of prime p. Then by fundamental theorem, there are partitions-of-r number of different groups of order m. Existence of subgroup of each kind can be **verified** by enumerating element of order p and powers of p. For example, suppose G is an abelian group of order 64. Then G has a *klein-4* subgroup only if G has atleast three element of order 2.

m is square-free

Suppose m is a square-free integer. Then by *fundamental theorem*, there is only one abelian group of order m. Thus, subgroups of order m are unique upto isomorphism.

m is not square-free

Clearly, n is a product of power of primes. Thus G has a subgroup of order m only if G has sufficient number of lements of order p and its powers for each prime factor p of m. For example, suppose G is an abelian group of order 60. Then G has an acyclic subgroup of order 12, if G has atleast three elements of order 2.

1.2 Enumerting isomorphic subgroups

Now we want to enumerate subgroups that are isomorphic. This is a **simple** combinatorial problem with algebraic constraints. For example, G is an abelian group of order 72 and there are 4 elements of order 4, say a,b,c and d. Then we have two pairs (a,b) and (c,d) such that an element is always the third power of its pair. Clearly, G has an acyclic subgroups of order 8 for each pair. Thus, G has exactly two acyclic subgroups of order 8.