ME010101 Abstract Algebra

Jacob Antony

February 9, 2021

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First Isomorphism Theorem

Theorem

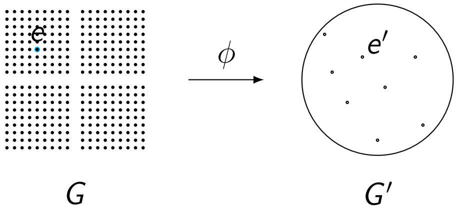
Let $\phi : G \to G'$ be a group homomorphism with kernel K. And let $\gamma_K : G \to G/K$ be the canonical homomorphism. Then there is a unique isomorphism $\mu : G/K \to \phi[G]$ such that $\phi(x) = \mu(\gamma_K(x))$.

$$\begin{tikzcd}
G \arrow[rr, "\phi"] \arrow[dr, "\gamma_K"'] & & \phi[G] \\
& G/K \arrow[ur, dotted, dash, "\mu"'] &
\end{tikzcd}$$

Figure: First Isomorphism Theorem

First Isomorphism Theorem

Let G, G' be groups. And $\phi : G \to G'$ be a group homomorphism.
Let $g \in G$. Let $K = \ker(\phi)$.
That is, $\phi[K] = e'$ or $\forall k \in K,\ \phi(k) = e'$.
Then $\phi[gK] = \phi(g)$, since $\phi(gk) = \phi(g)\phi(k) = \phi(g)$.
Let $\gamma_K : G \to G/K$ be a canonical homomorphism.
Then $\mu : G/K \to \phi[G]$ defined by $\mu(gK) = \phi(g)$ is an isomorphism.



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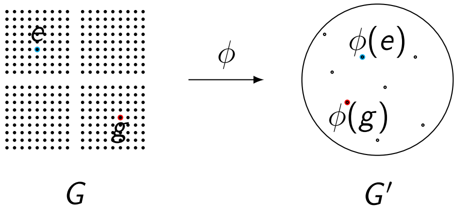

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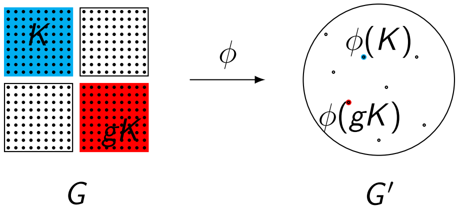

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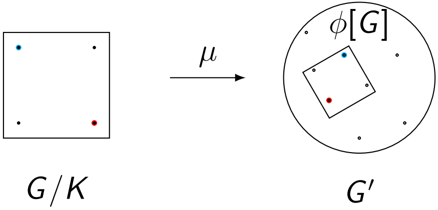

Second Isomorphism Theorem

Theorem

Let H be a subgroup of G and let N be a normal subgroup of G. Then $(HN)/N \simeq H/(H \cap N)$.

Lemma

Let N be a normal subgroup of G and let $\gamma : G \to G/N$ be the canonical homomorphism. Then the map ϕ from the set of normal subgroups of G containing N to the set of normal subgroup of G/N given by $\phi(L) = \gamma[L]$ is one-to-one and onto.

Lemma

If N is a normal subgroup of G, then $H \cap N = HN = NH$. Furthermore, if H is also normal in G, then HN is normal in G.

Lemma 1

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Let N be a normal subgroup of G and let $\gamma : G \to G/N$ be the canonical homomorphism. Then the map ϕ from the set of normal subgroups of G containing N to the set of normal subgroup of G/N given by $\phi(L) = \gamma[L]$ is one-to-one and onto.

Lemma 2

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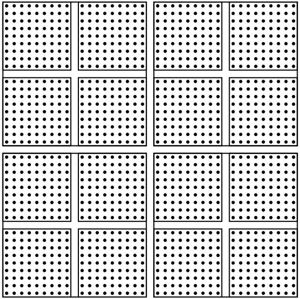
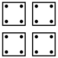




Third Isomorphism Theorem

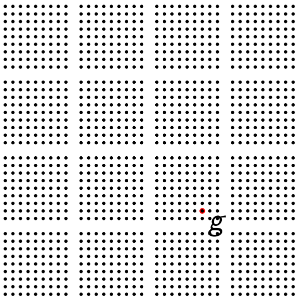
Theorem

Let H and K be normal subgroup of G with $K \leq H$. Then $G/H \simeq (G/K)/(H/K)$.

$$\begin{tikzcd}
G \arrow[rr, "\gamma_H"] \arrow[d, "\gamma_K"] & & G/H \arrow[d, dotted, dash, "\mu"] \\
G/K \arrow[rr, "\gamma_{H/K}"] & & (G/K)/(H/K)
\end{tikzcd}$$

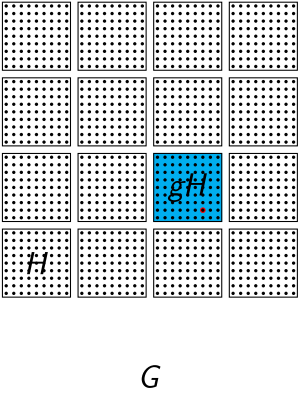
Figure: Third Isomorphism Theorem

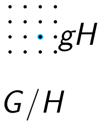
Third Isomorphism Theorem



G

Third Isomorphism Theorem





Third Isomorphism Theorem