Part I

ME010202 Advanced Topology

Chapter 7

Separation Axioms

7.1 Compactness and Separation Axioms

Proposition 7.1. *Let X be a T_2 space, $x \in X$ and F is a compact subset of X not containing x. Then there exist opensets U, V such that $x \in U$, $F \subset V$ & $U \cap V = \phi$.*

Corollary 7.1.1. *A compact subset in a T_2 space is closed.*

Corollary 7.1.2. *Every map from a compact space into a T_2 space is closed. And its range is a quotient space of the domain.*

Corollary 7.1.3. *A continuous bijection from a compact space onto a T_2 sspace is a homeomorphism.*

Corollary 7.1.4. *Every continuous, one-to-one function from a compact space into a T_2 space is an embedding.*

Theorem 7.2. *Every compact T_2 space is a T_3 space.*

Proposition 7.3. *Let X be a regular space, C a closed subset of X and F a compact subset of X, such that $C \cap F = \phi$. Then there exist open sets U, V such that $C \subset U$, $F \subset V$ and $U \cap V = \phi$.*

Theorem 7.4. *Every regular, Lindeloff space is normal.*

Corollary 7.4.1. *Every regular, second countable space is normal.*

Corollary 7.4.2. *Every compact T_2 space is T_4.*

7.2 The Urysohn Characterisation of Normality

Proposition 7.5. *Let A B be subsets of a space X and suppose there exists a continuous function $f : X \to [0,1]$, such that $f(x) = 0,\ \forall x \in A$ and $f(x) = 1,\ \forall x \in B$. Then there exists disjoint open sets U, V such that $A \subset U$ and $B \subset V$.*

Corollary 7.5.1. *If X has the property that for any disjoint closed subsets A, B of X, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0,\ \forall x \in A$ and $f(x) = 1,\ \forall x \in B$, then X is normal.*

Theorem 7.6. *A topological space X is normal iff it has the property that for every mutually disjoint, closed subsets A, B of X, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$*

Lemma 7.7. *Let $f : X \to [0,1]$ be continuous. For each $t \in \mathbb{R}$ let $F_t\{x \in X : f(x) < t\}$. Then the indexed family $\{F_t : t \in \mathbb{R}\}$ has the following properties*

1. *F_t is an open subset of X for each $t \in \mathbb{R}$*
2. *$F_t = \phi$ for $t < 0$*
3. *$F_t = X$ for $t > 1$*
4. *For any $s,\ t \in \mathbb{R},\ s < t \implies \overline{F_s} \subset F_t$.*

Moreover, for each $x \in X,\ f(x) = \inf\{t \in \mathbb{Q} : x \in F_t\}$.

Lemma 7.8. *Let X be a topological space and suppose $\{F_t : t \in \mathbb{Q}\}$ is a family of sets in X such that*

1. *F_t is open in X for each $t \in \mathbb{Q}$*
2. *$F_t = \phi$ for $t \in \mathbb{Q},\ t < 0$*
3. *$F_t = X$ for $t \in \mathbb{Q},\ t > 1$*
4. *$\overline{F_s} \subset F_t$ for $s,\ t \in \mathbb{Q},\ s < t$*

For $x \in X$, let $f(x) = \inf\{t \in \mathbb{Q} : x \in F_t\}$. Then f is a continuous real-valued function on X and it takes values in the unit interval $[0,1]$.

Corollary 7.8.1. *All T_4 spaces are completely regular and hence Tychonoff.*

Proof. Let $x \in X$ and D be closed subset not containing x. We have X is a T_4 space. Therefore X is T_1 as well as normal. Now the singleton set, $\{x\}$ is closed, since X is a T_1 space. And by Urysohn's lemma for disjoint, closed subsets $\{x\},\ D$ there exists a continuous, real-valued function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in D$. Therefore the space X is completely regular and hence Tychonoff. $\square$

Remark (Urysohn function)**.** *The function whose existence is asserted by Urysohn's lemma is called a Urysohn function*

7.3 Tietze Characterisation of Normality

Proposition 7.9. *Let A be a subset of a space X and let $f : A \to \mathbb{R}$ be continuous. Then any two extensions of f to X agree on $\overline{A}$. In other words, if at all an extension of f exists its values on $\overline{A}$ are uniquely determined by values of f on A.*

Proposition 7.10. *Suppose a topological space X has the property that for every closed subset A of X, every continuous real valued function on A has a continuous extension to X. Then X is normal.*

Definitions 7.11 (Pointwise Convergence)**.** *Let X be a topological space and (Y, d) a metric space. Then a sequence of functions $\{f_n\}$ from X to Y converges pointwise to f if for every $x \in X$ the sequence $\{f_n(x)\}$ converges to $f(x)$ in Y.*

In other words, given a very small value, $\epsilon > 0$, there exists some $\delta > 0$ such that for every $x \in X$ there exists $N_x \in \mathbb{N}$. This N_x may be different for different values of x and for every $n > N_x$, $d(f(x), f_n(x)) < \delta$.

Definitions 7.12 (Uniform Convergence)**.** *Let X be a topological space and (Y, d) a metric space. Then a sequence of functions $\{f_n\}$ from X to Y converges uniformly to f if given a small $\epsilon > 0$, there exists $\delta > 0$ such that there exists $N \in \mathbb{N}$. This N is independent of the value of x and for every $n > N$, $d(f(x), f_n(x)) < \delta$.*

Proposition 7.13. *Let X, (Y, d), $\{f_n\}$ and f be as above and suppose $\{f_n\}$ converges to f uniformly. If each f_n is continuous, then f is continuous.*

Definitions 7.14 (Uniform Convergence of Series)**.** *Let X be a topological space and (Y, d) be a metric space. Then a series of function $\sum_{n=1}^{\infty} f_n$ converges uniformly to f if the sequence of partial sums converges uniformly to f.*

In other words, let $g_m = \sum_{n=1}^{m} f_n$. Then $\sum_{n=1}^{\infty} f_n$ converges to f uniformly if the sequence of partial sums $\{g_m\}$ converges to f uniformly.

Proposition 7.15. *Let $\sum_{n=1}^{\infty} M_n$ be a convergent series of non-negative real numbers. Suppose $\{f_n\}$ is a sequence of real valued functions on a space X such that for each $x \in X$ and $n \in \mathbb{N}$, $|f_n(x)| \leq M_n$. Then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to a real valued function on X.*

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Chapter 8

Products and Coproducts

8.1 Cartesian Products of Families of Sets

8.2 The Product Topology

8.3 Productive Properties

Chapter 9

Embedding and Metrisation

9.1 Evaluation Functions into Products

9.2 Embedding Lemma and Tychonoff Embedding

9.3 The Urysohn Metrisation Theorem

Chapter 10

Nets and Filters

10.1 Definition and Convergence of Nets

Definitions 10.1 (Directed Set)**.** *[Joshi, 1983, 10.1.1]*
A directed set D is a pair $(D, \geq)$ where D is a nonempty set and ge is a binary relation on D such that

1. *The relation 'follows'($\geq$) is transitive. ie, $m \geq n,\ n \geq p \implies m \geq p$*
2. *The relation 'follows'($\geq$) is reflexive. ie, For every $m \in D,\ m \geq m$*
3. *For any $m, n \in D$, there exists $p \in D$ such that $p \geq m$ and $p \geq n$.*

sequence in a set X is a function f from the set of all integers into X.

Definitions 10.2 (Net)**.** *[Joshi, 1983, 10.1.2]*
A net in a set X is a function S from a directed set D into the set X.

Remark. *The set $\mathbb{N}$ together with the relation 'less than or equal to'($\leq$) is a directed set. Clearly, the relation 'less than or equal to' is reflexive and trasitive. And the third condition is true iff every finite subset E of D has an element $p \in E$ such that p follows each element of E. This is a weaker notion compared to the well ordering principle[1] of the set of all integers. Thus $\mathbb{N}$ is a directed set and every sequence in X is also a net in X.*

Remark (Significance of Net)**.** *A net on a set is a generalisation of 'a sequence on a set' obtained by simplifying the domain of the sequence into a directed set. The notion directed set is derived by assuming a few properties of $\mathbb{N}$.*

The convergence of sequence is not strong enough to characterise topologies as the limit of convergent sequences are unique for both Hausdorff and Co-countable spaces. The notion of Net allows us to differentiate between Hausdorff spaces from Co-countable spaces in terms of convergence of nets. The limit of a convergent net on a topological space is unique iff it is a Hausdorff space. ie, We have removed a few restrictions, so that we will have some convergent nets (which are obviously not sequences) with multiple limit points for Co-countable spaces.

[1] Well-ordering principle : Every subset of $\mathbb{N}$ has a least element in it.

Remark. *Examples of Directed Sets*

1. *Let X be a topological space and $x \in X$. Then the neighbourhood system $\mathcal{N}_x$ is a directed set with the binary relation $\subset$ (subset/inclusion).*
   (a) *Let $U,\ V,\ W$ be any three neighbourhoods of $x \in X$ such that $U \subset V$ and $V \subset W$. Then, clearly $U \subset W$.*
   Therefore, $U \geq V,\ V \geq W \implies U \geq W$.
   (b) *Let U be any neighbourhood of $x \in X$, then $U \subset U$.*
   Therefore, $U \geq U$.
   (c) *Let U, V be any two neighbourhoods of $x \in X$, then there exists their intersection $W = U \cap V$, which is a neighbourhood of x. Clearly $W \subset U$ and $W \subset V$.*
   Therefore $\forall U, V \in \mathcal{N}_x, \exists W \in \mathcal{N}_x$ such that $W \geq U$ and $W \geq V$.
2. *Let $\mathcal{P}$ be the set of all partitions on closed unit interval $[0, 1]$. A partition $P \in \mathcal{P}$ is a refinement of $Q \in \mathcal{P}$ if every subinterval in P is contained in some subinterval of Q. Then $\mathcal{P}$ with the binary relation refinement is a directed set.*

   For example, let $P = \{0,\ 0.3,\ 0.7,\ 1\}$. Then the subintervals in P are $[0, 0.3]$, $[0.3, 0.7]$ and $[0.7, 1]$. Let $Q = \{0,\ 0.3,\ 0.5,\ 1\}$ and $R = \{0,\ 0.3,\ 0.5,\ 0.7,\ 1\}$. Then R is a refinement of P, but Q is not a refinement of P since there is a subinterval $[0.5, 1]$ in Q which is not properly contained in any subinterval of P. However, R is a refinement of Q as well.
   (a) *Suppose $P,\ Q,\ R$ are three partitions of $[0, 1]$ such that P is a refinement of Q and Q is a refinement of R, then clearly P is a refinement of R since each subinterval of P is contained some subinterval of Q, which is contained in some subinterval of R.*
   Therefore, $P \geq Q,\ Q \geq R \implies P \geq R$
   (b) *Suppose P is a partition of $[0, 1]$. Then trivialy, P is a refinement of itself since every subinterval of P is contained in the same subinterval of P.*
   Therefore, $\forall P \in \mathcal{P},\ P \geq P$
   (c) *Suppose $P,\ Q$ be any two partition of $[0, 1]$. Then $R = P \cup Q$ is a refinement of both the partitions.*
   Therefore $\forall P, Q \in \mathcal{P},\ \exists R \in \mathcal{P}$ such that $R \geq P$ and $R \geq Q$

Remark. *Examples of Nets*

1. *Let X be a topological space and $x \in X$. Let $\mathcal{N}_x$ be the set of all neighbourhoods of x. Let $D = (\mathcal{N}_x, X)$ be the directed set given by $(N, y) \in (\mathcal{N}_x, X)$ if $N \in \mathcal{N}_x$ and $y \in N$ and $(N, y) \geq (M, z)$ if $N \subset M$. Then the function $S : (\mathcal{N}_x, X) \to X$ given by $S(N, y) = y$ is a net on X.*

   For example, let $X = \{a, b, c, d\}$ and $\mathcal{T} = \{\{a\},\ \{a, b\},\ \{a, b, c\},\ \{a, b, c, d\}\}$. Also let $S : (\mathcal{N}_b, X) \to X$ defined by $S(N, y) = y$. Suppose $C = \{a, b, c\}$. Then $C \in \mathcal{N}_b$. ie, C is a neighbourhood of b. Then $S(C, c) = c$.

2. *Riemann Net - Let $D = (\mathcal{P}, \xi)$ where $\mathcal{P}$ is the set of all partitions on $[0, 1]$ and ξ is a finite sequence in $[0, 1]$ such that consecutive terms belongs to consecutive subintervals of the partition. The set $(\mathcal{P}, \xi)$ is directed set with $\geq$ given by $(P, \eta) \geq (Q, \psi)$ iff P is a refinement of Q.*

   For example, let $P \in \mathcal{P}$ is given by $P = \{\ 0,\ 0.3,\ 0.7,\ 1\ \}$ and $\eta = \{\ 0.2,\ 0.6,\ 0.9\ \}$. Then $(P, \eta) \in (\mathcal{P}, \xi)$.

   Let $f : \mathbb{R} \to \mathbb{R}$ be any function, then the function,

$$S : (\mathcal{P}, \xi) \to \mathbb{R} \text{ defined by } S(P, \eta) = \sum_{j=1}^{k} f(\eta_k)(a_k - a_{k-1})$$

   where $P = \{a_0,\ a_1, \cdots,\ a_k\ \}$ is the Riemann Net with respect to the real function f.

   For example, let $f(x) = x^2$ and P, η are same as above example, then $S(P, \eta) = 0.2^2(0.3 - 0) + 0.6^2(0.7 - 0.3) + 0.9(1 - 0.7) = 3.99$

Definitions 10.3 (Convergence of a Net)**.** *[Joshi, 1983, 10.1.3]*
A net $S : D \to X$ converges to a point $x \in X$ if for any nbd U of x, there exists $m \in D$ such that $n \in D,\ n \geq m \implies S(n) \in U$. And x is a limit of the net S.

Remark. *The choice of m depends on the choice of neighbourhood U.*

$S : D \to X,\ S \to x \iff (\forall U \in \mathcal{N}_x,\ \exists m_U \in D,\ \text{such that } n \geq m_U \implies S(n) \in U)$

Theorem 10.4 (Net characterisation of Hausdorff space)**.** *[Joshi, 1983, 10.1.4]*
A topological space is Hausdorff iff limits of all nets in it are unique.

Proof. Let X be a Hausdorff space. Suppose $S : D \to X$ is net on X such that S converges to two distinct points $x, y \in X$. Since X is a Hausdorff space and $x \neq y$, there exists open sets U, V such that $x \in U,\ y \in V, U \cap V = \phi$.

The net S converges to $x \in X$, therefore $\exists m_x \in D$ such that $n \geq m_x \implies S(n) \in U$ And, the net S converges to $y \in X$, therefore $\exists m_y \in D$ such that $n \geq m_y \implies S(n) \in V$.

Since D is a directed set and $m_x, m_y \in D$, there exists $p \in D$ such that $p \geq m_x$ and $p \geq m_y$. Now, $n \geq p \implies n \geq m_x,\ n \geq m_y$, since $\geq$ is transitive. (ie, $n \geq p,\ p \geq m_x \implies n \geq m_x$, and $n \geq p,\ p \geq m_y \implies n \geq m_y$).

We have $n \geq p \implies n \geq m_x$ and $n \geq m_x \implies S(n) \in U$. Therefore, $n \geq p \implies S(n) \in U$. Similarly, $n \geq p \implies n \geq m_y \implies S(n) \in V$. Therefore $S(n) \in U \cap V$ which is a contradiction, since $U \cap V = \phi$. Therefore, if a net S converges to two points x, y, then $x = y$. That is, if a net S in a Hausdorff space X is convergent then its limit is unique.

Conversely, suppose that X is a topological space and every convergent net in X has a unique limit. Suppose X is not a Haudorff space. Then there exists at least two distinct points $x, y \in X$ such that every neighbourhood of x intersects with every neighbourhood of y. Now consider the set $D = \mathcal{N}_x \times \mathcal{N}_y$ and relation $\geq$ on D such that $(U_1, V_1) \geq (U_2, V_2)$ if $U_1 \subset U_2$ and $V_1 \subset V_2$.

By the axiom of choice, a function $S : D \to X$ such that $S(U, V) \in U \cap V$ is well defined, since every nbd of x intersects every nbd of y. Thus, S is a net in X. We claim that S converges to both x and y.

Let U be a nbd of x. Then $S(U', V') \in U' \cap V'$. We have $(U, X) \in D$ such that $(U', V') \geq (U, X) \implies U' \subset U$. Then, $S(U', V') \in U' \cap V' \subset U \cap X = U$. Thus, for any nbd U containing x, we have $(U, X) \in D$ such that $(U', V') \geq (U, X) \implies S(U', V') \in U$. Therefore, S converges to $x \in X$.

Similarly, Let V be a nbd of y. Then for any nbd V containing y, we have $(X, V) \in D$ such that $(U', V') \geq (X, V) \implies S(U', V') \in V$, since $S(U', V') \in U' \cap V' \subset X \cap V = V$. Therefore, S converges to $y \in X$ as well, where $x \neq y$. This is a contradition to the assumption that every convergent net in X has a unique limit. Therefore, for any two points $x, y \in X$, there should be some nbd of x that doesn't intersect some nbd of y. Therefore, X is a Hausdorff space. $\square$

Definitions 10.5 (Eventual Subset). *[Joshi, 1983, 10.1.5]*
A subset E of a directed set D is an eventual subset of D if there exists $m \in D$ such that $n \geq m \implies n \in E$.

Remark. *Let E be an eventual subset of D such that $n \geq m \implies n \in E$. Then $p \in E \not\Longrightarrow p \geq m$. ie, Subset E may contain elements that doesn't follow the above m.*

Remark. *[Joshi, 1983, 10.1.6]*
Let E be an eventual subset of D, then E is a directed set.

1. $m, n, p \in E,\ m \geq n,\ n \geq p \implies m, n, p \in D,\ m \geq n,\ n \geq p \implies m \geq p$
2. $m \in E \implies m \in D \implies m \geq m$
3. $m, n \in E \implies m, n \in D \implies \exists p \in D$ *such that* $p \geq m$ *and* $\geq n$.
   Since E is eventual, $\exists m' \in D$ such that $n' \geq m' \implies n' \in E$.
   Also, $p, m' \in D$, $\exists p' \in D$ such that $p' \geq p$ and $p' \geq m'$.(Condition 3)
   Since, E is eventual subset of D with respect to m', $p' \geq m' \implies p' \in E$.
   And since $p' \geq p,\ p \geq m \implies p' \geq m$ and $p' \geq p,\ p \geq n \implies p' \geq n$.
   Therefore $\forall m, n \in E,\ \exists p' \in E$ such that $p' \geq m$ and $p' \geq n$.

Definitions 10.6 (Net eventually in A). *[Joshi, 1983, 10.1.5]*
Let $S : D \to X$ be a net in a topological space X. Then S is eventually in subset A of X if $S^{-1}(A)$ is an eventual subset of D.

Remark. *Let $S : D \to X$ be a net in X. Then S converges to $x \in X$ if S is eventually in each nbd U of x.*

Definitions 10.7 (Cofinal subset). *[Joshi, 1983, 10.1.7]*
A subset F of a directed D is a cofinal subset of D if for any $m \in D$, there exists $n \in F$ such that $n \geq m$.

Remark. *Let X be a topological space and $x \in X$. Let $\mathcal{N}_x$ be the set of all neighbourhood of x and $\mathcal{L}$ be a local base of X at x. We have, $(\mathcal{N}_x, \geq)$ is a directed set where $\forall U, V \in \mathcal{N}_x,\ U \geq V \iff U \subset V$, then $\mathcal{L}$ is cofinal in $\mathcal{N}_x$.*

Remark. *[Joshi, 1983, 10.1.8]*
Let F be a cofinal subset of D, then F is a directed set.

1. $m, n, p \in F,\ m \geq n,\ n \geq p \implies m, n, p \in D,\ m \geq n,\ n \geq p \implies m \geq p$
2. $m \in F \implies m \in D \implies m \geq m$
3. $m, n \in F \implies m, n \in D \implies \exists p \in D$ *such that* $p \geq m$ *and* $\geq n$.
   Since E is cofinal, $p \in D \implies \exists p' \in F$ *such that* $p' \geq p$.
   And since $p' \geq p,\ p \geq m \implies p' \geq m$ *and* $p' \geq p,\ p \geq n \implies p' \geq n$.
   Therefore $\forall m, n \in F,\ \exists p' \in F$ *such that* $p' \geq m$ *and* $p' \geq n$.

Definitions 10.8 (Net frequently in A)**.** *[Joshi, 1983, 10.1.7]*
Let $S : D \to X$ be a net in a topological space X. Then S is frequently in subset B of X if $S^{-1}(B)$ is a cofinal subset of D.

Proposition 10.9. *[Joshi, 1983, 10.1.6]*
Let $S : D \to X$ be a net in a topological space X. Let E be an eventual subset of D. Then, S converges to x iff $S_{/_E}$ converges to x.[Joshi, 1983, 10.1.6]

Proof. Let $S : D \to X$ be a net in X, E be an eventual subset of D, and $x \in X$. Then, $S_{/_E} : E \to X$ is defined by $n \in E \implies S_{/_E}(n) = S(n)$

Suppose S converges to x. Let U be a nbd of x, then S is eventually in U. ie, $S^{-1}(U)$ is an eventual subset of D. Then $\exists m \in D$ such that $n \geq m \implies n \in S^{-1}(U) \implies S(n) \in U$. Since set E is eventual subset of D, $\exists m' \in D$ such that $n \geq m' \implies n \in E$.

Since E is a directed set, $S_{/_E} : E \to X$ is a net in X. And $m, m' \in D \implies \exists p \in D$ such that $p \geq m$ and $p \geq m'$. We have, $p \geq m' \implies p \in E$. And $n \geq' p \implies n \geq p,\ p \geq m \implies n \geq m \implies S(n) \in U \implies S_{/_E}(n) \in U$. Therefore, $n \geq' p \implies S_{/_E}(n) \in U$. Since U is arbitrary, $S_{/_E}$ converges to x.

Conversely, suppose that $S_{/_E}$ converges to x. Let U be a nbd of x, then $S_{/_E}$ is eventually in U. ie, $S_{/_E}^{-1}(U)$ is an eventual subset of D. ie, $\exists m \in D$ such that $n \geq m \implies n \in S_{/_E}^{-1}(U) \implies S_{/_E}(n) \in U \implies S(n) \in U$. Therefore, $n \geq m \implies S(n) \in U$. Since, U is arbitrary, S converges to every nbd of x. ie, S converges to x. □

Proposition 10.10. *[Joshi, 1983, 10.1.8]*
Let $S : D \to X$ be a net in a topological space X. Let F be a cofinal subset of D. If S converges to x, then $S_{/_F}$ converges to x.

Proof. Let $S : D \to X$ be a net in X and S converges to $x \in X$. Also let F be a cofinal subset of D. Then $S_{/_F}$ is also a net in X, since $(F, \geq')$ is a directed set where $\forall m, n \in F,\ m \geq n \implies m \geq' n$.

Since S converges to x, for any nbd U of x, $\exists m \in D$, such that $n \geq m \implies S(n) \in U$. Since F is cofinal, $\exists p \in F$ such that $p \geq m$. Thus $n \geq' p \implies n \geq$

p, $p \geq m \implies n \geq m \implies S(n) \in U \implies S_{/_F}(n) \in U$. Therefore, $\exists p \in F$ such that $n \geq' p \implies S_{/_F}(n) \in U$. Since U is arbitrary, $S_{/_F}$ is eventually in every nbd of x. ie, $S_{/_F}$ converges to x. □

Remark. *But converse of the above is not true. $S_{/_F}$ converges to x does not imply that S converges to x, since cofinal subset F not necessarily contain every element following a particular m.*

Definitions 10.11 (Cluster point)**.** *[Joshi, 1983, 10.1.9]*
Let $S : D \to X$ be a net in a topological space X. Then $x \in X$ is a cluster point of S, if S is frequently in each nbd U of x in X.

Proposition 10.12. *[Joshi, 1983, 10.1.10]*
Let $S : D \to X$ be a net in a topological space X. Then $x \in X$ is a cluster points of X, if $S_{/_F}$ converges to x for some cofinal subset F of D.

Proof. Let $S : D \to X$ be a net in X and $(F, \geq')$ be a cofinal subset of $(D, \geq)$. Then $S_{/_F}$ is also a net in X. Suppose $S_{/_F}$ converges to $x \in X$. Let U be a nbd of x, then $\exists m \in F$ such that $n \geq' m \implies S_{/_F}(n) \in U$.

Let $m' \in D$. Then $\exists p' \in F$ such that $p' \geq m'$, since F is a cofinal subset of D. We have, $m, p' \in F$, then $\exists p \in F$ such that $p \geq' m$ and $p \geq' p'$. Since $F \subset D$, we have $p, m \in F \implies p, m \in D$ and $p \geq' m \implies p \geq m$.

Also $p \geq' m \implies S_{/_F}(p) \in U \implies S(p) \in U$. Therefore, $\forall m' \in D$, $\exists p \in D$ such that $p \geq m'$ and $S(p) \in U$. Since U, m' are arbitrary, S is frequently in every nbd of x. ie, x is a cluster point of S. □

Definitions 10.13 (Subnet)**.** *[Joshi, 1983, 10.1.11]*
Let $S : D \to X$ be a net in a topological space X. Then a net $T : E \to X$ in X, is a subnet of S if there exists a function $N : E \to D$ such that $S \circ N = T$ and $\forall m \in D$, $\exists p \in E$ such that $n \geq' p \implies N(n) \geq m$.

Remark. *A net $T : E \to X$ is a subnet of $S : D \to X$ if $\exists N : E \to D$ such that $S \circ N = T$ and S is frequently in $T(E)$.*
A net $T : E \to X$ is a subnet of $S : D \to X$. If T eventually in A subset of X, then S is frequently in A.

Proposition 10.14. *[Joshi, 1983, 10.1.12]*
Let $S : D \to X$ be a net in a topological space X. Then $x \in X$ is a cluster point of S iff there exists a subnet of S which converges to x.

Synopsis. *Let $(D, \geq)$, $(E, \geq')$ be two directed sets.*

If T converges to x, then T is eventually in each nbd U of x. And since T is a subnet of S, there exists $N : E \to D$ such that $N(E)$ is a cofinal subset of D. Therefore, S is frequently in each nbd U of x. Thus, x is a cluster point of S.

Proof. Let $S : D \to X$ be a net in X. Suppose there exists a subnet $T : E \to X$ that converges to $x \in X$. By the definition of subnet, we have $\exists N : E \to D$ such that $S \circ N = T$ and S is frequently in $T(E)$.

We have, T convergers to x, thus for any neighbourhood U of x, there exists $m' \in E$ such that $n' \geq' m' \implies T(n') \in U$.

Also we have, T is a subnet of S. Then $\exists N : E \to D$ such that $\forall m \in D$, $\exists p' \in E$ such that $n' \geq' p' \implies N(n') \geq m$.

Now, for any $m \in D$, we have $m', p' \in E$. Since E is a directed set, there exists $n' \in E$ such that $n' \geq' m'$ and $n' \geq' p'$.

Then, $n' \geq m' \implies T(n') \in U$ and $n' \geq' p' \implies N(n') \geq m$.

Thus for any $m \in D$, there exists $N(n') = n \in D$ such that $S(n) = S(N(n')) = T(n') \in U$.

Thus S is frequently in any neighbourhood U of x. Therefore, x is a cluster point of S.

Conversely, suppose that x is a cluster point of S. We have to construct a directed set $(E, \geq')$ and a function $N : E \to D$ such that T is a subnet of S and T converges to x.

Consider $E = D \times \mathcal{N}(x)$ and define $\geq'$ by $(n, U) \geq' (m, V)$ if $n \geq m$ and $U \subset V$. Trivially, $(n, U) \geq' (m, V) \geq' (p, W) \implies (n, U) \geq' (p, W)$ and $(n, U) \geq' (n, U)$. Also, for any $(n, U), (m, V) \in E$, we have $n, m \in D$ and $U, V \in \mathcal{N}(x)$. Since D is a directed set, $\exists p \in D$ such that $p \geq n$ and $p \geq m$. And $U \cap V \in \mathcal{N}(x)$ such that $U \cap V \subset U$ and $U \cap V \subset V$. Thus $\exists (p, U \cap V) \in E$ such that $(p, U \cap V) \geq' (n, U)$ and $(p, U \cap V) \geq' (m, V)$. Therefore, $(E, \geq')$ is a directed set.

Define $N : E \to D$ by $N(n, U) = n$. Again for any $(m, V) \in E$, there exists $m \in D$ such that $(n, U) \geq' (m, U)$ implies there exists $n \in D$ such that $N(n, U) = n$ and $n \geq m$. Now, we have $T : E \to X$ defined by $T(n, U) = S(N(n, U)) = S(n)$. Therefore, T is a subset of S. – to be continued page 234– $\square$

Remark. *A proof that doesn't work : If x is a cluster point of a net S in X, then S is frequently in some cofinal subset of D. Thus, there exists a cofinal subset $E \subset D$ which is a direct set with $\geq$ restricted to E. Then $N : E \to D$ defined by $N(n) = n$ gives a subset $T : E \to X$ of S. However, this subnet need not converge to x. The strongest statement, we can make on T is that 'x is a cluster point of T'.*

$N : D \times \mathcal{N}(x) \to D,\ N(n) = n$ *is completely independent of* U.

Chapter 11

Compactness

11.1 Variations of Compactness

In this chapter, we have two other notions of compactness - countable compactness and sequential compactness.[1]

Compact A topological space is compact iff every open cover of it has a finite subcover. ([Joshi, 1983, 6.1.1]) [Heine-Borel]

Countably Compact A topological space is countably compact iff every countable, open cover of it has a finite subcover. [Joshi, 1983, 11.1.1]

Sequentially Compact A topological space is sequentially compact iff every sequence in it has a convergent subsequence. [Joshi, 1983, 11.1.8] [Bolzano-Weierstrass]

Countable compactness is a weaker notion compared to compactness.[2] However, sequentially compact and compact are not necessarily comparable.[3].

We have seen earlier that compactness has the following properties 1. compactness is weakly hereditary.[Joshi, 1983, 6.1.10] 2. compactness is preserved under continuous functions.[Joshi, 1983, 6.1.8] 3. every continuous real functions on compact space is bounded and attains its extrema.[Joshi, 1983, 6.1.6] 4. every continuous real function on a compact, metric space is uniformly continuous by Lebesgue covering lemma.[Joshi, 1983, 6.1.7]

Countably compact spaces, Sequentially compact spaces have all the four properites listed above.

11.1.1 Countable compactness

Weakly hereditary property

A subspace $(A, \mathcal{T}_{/A})$ being countably compact doesn't imply that $(X, \mathcal{T})$ is countably compact. However, if $(X, \mathcal{T})$ is a countably compact space and A

[1] For $\mathbb{R}$, Compactness & Sequentially compactness are equivalent to the completeness axiom.
[2] Every compact space is countably compact.
[3] $\mathcal{T}_1, \mathcal{T}_2$ are non-comparable, if $\mathcal{T}_1 \not\subset \mathcal{T}_2$ and $\mathcal{T}_2 \not\subset \mathcal{T}_1$.[Joshi, 1983, 4.2.1]

is a closed subset of X, then $(A, \mathcal{T}_{/A})$ is also a countably compact space. In other words, countably compactness is weakly hereditary.

Theorem 11.1. *Countable compactness is weakly hereditary.[Joshi, 1983, 11.1.3]*

Synopsis. *Let A be a closed subset of countably compact space, X. If A has a countable open cover $\mathcal{U}$, then we can obtain a respective countable, open cover for X by attaching $X - A$ to the extensions of members of $\mathcal{U}$ to X. This cover has a finite subcover. Then restricting them to A, we get a finite subcover of $\mathcal{U}$.*

Proof. Suppose X is a countably compact space. And A is a closed subset of X. We need to show that A is countably compact. Without loss of generality,[4] assume that A is a proper subset of X. Then $X - A$ is a non-empty, open subset of X.

Let $\mathcal{U}$ be a countable open cover of A. Then $\mathcal{U} = \{U_1, U_2, \cdots\}$ where each element $U_k \in \mathcal{U}$ is an open subset of A. Since A is a subspace of X, every open set U_k in A is of the form $G \cap A$ for some open set G in X. Therefore, there exists open sets $V(U_k)$ for each U_k such that $A \cap V(U_k) = U_k$.[5]

Define $\mathcal{V} = \{X - A, V(U_1), V(U_2), \cdots\}$. Clearly, $\mathcal{V}$ is a countable open cover[6] of X. We have X is countably compact, thus $\mathcal{V}$ has a finite subcover, say $\mathcal{V}'$. Without loss of generality assume[7] that $X - A \in \mathcal{V}'$. Suppose $X - A \notin \mathcal{V}'$, then we can define another finite subcover $\mathcal{V}' \cup \{X - A\}$. Thus $\mathcal{V}' = \{X - A,\ V(U_{n_1}),\ V(U_{n_2}), \cdots,\ V(U_{n_k})\}$.

Then the corresponding subcover $\mathcal{U}' = \{U_{n_1}, U_{n_2}, \cdots, U_{n_k}\}$ is a finite subcover of $\mathcal{U}$. Since countable open cover $\mathcal{U}$ and closed subset A are arbitrary, every closed subset of X with relative topology is countably compact. Therefore, countable compactness is weakly hereditary. □

Remark. *Proof depends on the following,*

1. *There is an extension map, $\psi : P(A) \to P(X)$ that preserve open sets (and closed sets). This ψ is an open map which not a true inverse of the restriction, $r : P(X) \to P(A)$, defined by $r(G) = G \cap A$ for every subset G of X.*
2. *Also we rely on the subset A being closed. Suppose X have many countable open covers, but X has only uncountable open covers corresponding to a particular countable open cover of A. In such a case, X being countably compact is insufficient for A to be countably compact.*

The behaviour of countinous functions

We will now study the nature of continuous functions defined on countably compact spaces. Suppose X, Y are topological space and function $f : X \to Y$ is continuous. If X is countably compact, then $f(X)$ is also countably compact.

[4]Suppose A is not a proper subset of X. Then $X = A$ and A is countably compact.
[5]Relative topology,$\mathcal{T}_{/A} = \{G \cap A : G \in \mathcal{T}\}$
[6]$X - A$ is open in X. If $y \notin A$, then $y \in X - A$. If $y \in A$, then $y \in U_k$ for some k.
[7]Otherwise, you will have to consider two cases: $X - A \in \mathcal{V}'$ and $X - A \notin \mathcal{V}'$

Continuous images of countably compact spaces are countably compact. In other words, countable compactness is preserved under continuous functions.[8]

Theorem 11.2. *Countable compactness is preserved under continuous functions.[Joshi, 1983, 11.1.2]*

Synopsis. *Let X be countably compact and $f : X \to Y$ be continuous. Suppose $\mathcal{U}$ is a countable cover of $f(X)$, then X has a countable cover $\mathcal{V}$ obtained by taking inverse images. Since X is countably compact, $\mathcal{V}$ has a finite subcover $\mathcal{V}'$. Now taking images of members of $\mathcal{V}'$, we get a finite subcover $\mathcal{U}'$ of $f(X)$.*

Proof. Suppose X is a countably compact space, Y is a topological space and $f : X \to Y$ is a continuous function. Let $\mathcal{U} = \{U_1, U_2, \cdots\}$ be a countable cover of $f(X)$ by set open in $f(X)$. We have to show that $\mathcal{U}$ has a finite subcover.

Define $\mathcal{V} = \{f^{-1}(U_1), f^{-1}(U_2), \cdots\}$. Then $\mathcal{V}$ is a countable open cover of X, since $f^{-1}(U_k)$ are open subsets of X and,

$$\begin{aligned}\bigcup_{k=1}^{\infty} U_k = f(X) &\implies f^{-1}\left(\bigcup_{k=1}^{\infty} U_k\right) = X \\ &\implies \bigcup_{k=1}^{\infty} f^{-1}(U_k) = X\end{aligned}$$

We have, $\mathcal{V}$ is a countable open cover of X, which is a countably compact space. Therefore $\mathcal{V}$ has a finite subcover $\mathcal{V}' = \{f^{-1}(U_{n_1}),\ f^{-1}(U_{n_2}), \cdots,\ f^{-1}(U_{n_k})\}$.

$$\begin{aligned}\bigcup_{j=1}^{k} f^{-1}(U_{n_j}) = X &\implies f^{-1}\left(\bigcup_{j=1}^{k} U_{n_j}\right) = X \\ &\implies \bigcup_{j=1}^{k} U_{n_j} = f(X)\end{aligned}$$

Clearly $\mathcal{U}' = \{U_{n_1}, U_{n_2}, \cdots, U_{n_k}\}$ is a finite subcover of $\mathcal{U}$. Thus every countable open cover of $f(X)$ by sets open in $f(X)$ has a finite subcover. Therefore, continuous images of countably compact spaces are countably compact. $\square$

Remark.

1. *For a continuous function, $f : X \to Y$ the inverse images of open sets are open in X. The relation $f^{-1} \subset f(X) \times X$ is not a function. However, we may consider a function, $\psi : P(Y) \to P(X)$ such that $\psi(U) = f^{-1}(U)$ for any subset U of Y. This ψ is an open map which maps open subsets of Y to open subsets of X.*

Theorem 11.3. *Every continuous, real-valued function on a countably compact, metric space is bounded and attains its extrema.[Joshi, 1983, 11.1.7]*

[8] A topological property is preserved under continuous functions if whenever a space has that property so does every continuous image of it.[Joshi, 1983, 6.1.9]

Synopsis. *Let X be a countably compact space and function $f : X \to \mathbb{R}$ be continuous. Then $f(X) \subset \mathbb{R}$ is countably compact. Real line $\mathbb{R}$ is metrisable*[9]*. Then $f(X)$ is countably compact, metric space. Therefore $f(X)$ compact.*[10]*. The subset $f(X)$ of $\mathbb{R}$ is bounded and closed, since every compact subset of $\mathbb{R}$ is bounded and closed. Thus $f(X)$ contains its supremum and infimum. Therefore, f is bounded and attains its extrema.*

Proof. Let X be a countably compact space and $f : X \to \mathbb{R}$ be continuous, real-valued function on the countably compact space, X. We have to show that f is bounded and attains its extrema.

Since countable compactness is preserved under continuous functions, $f(X)$ is countably compact subset of $\mathbb{R}$. Since, $f(X)$ is a subset of the metric space, $\mathbb{R}$ and metrisability is hereditary, $f(X)$ is again metrisable. (suppose) We have, every countably compact, metric space is compact. Then $f(X)$ is a compact subset of $\mathbb{R}$.

Since every compact subset of $\mathbb{R}$ is bounded and closed, $f(X)$ is bounded and closed. Since every closed subset of $\mathbb{R}$ contains supremem and infimum, $f(X)$ contains its extrema. Therefore, every continuous, real-valued function on a countably compact space is bounded and attains its extrema.

We have assumed that every countably compact, metric space is compact. This result will be proved in the last section of this chapter. □

Remark. *Since countably compact, metric spaces are compact. The above theorem can be used to prove that continuous, real-valued functions on a compact, metric space attains its extrema.*

Due to the Lebesgue covering lemma, next result is quite simple.*

Theorem 11.4. *Every continuous, real-valued function on a countably compact, metric space is uniformly continuous.*

Proposition 11.5. *Let X be a first countable, Hausdorff space. Then every countably compact subset A of X is closed.[Joshi, 1983, Exercises 11.1.7]*

11.1.2 Sequential Compactness

Weakly hereditary property

Theorem 11.6. *Sequential compactness is weakly hereditary.[Joshi, 1983, Exercises 11.1.3]*

The behaviour of countinous functions

Theorem 11.7. *Sequential compactness is preserved under continuous functions.[Joshi, 1983, Exercises 11.1.4]*

[9] [Joshi, 1983, 4.2 Example 4], $\mathbb{R}$ with usual metric $d : R \to R$, $d(x, y) = |x - y|$

[10] [Joshi, 1983, 11.1.11] On metric spaces, countable compactness $\implies$ compactness.

Synopsis. *Let X be sequentially compact and function $f : X \to Y$ be continuous. Then any sequence, $\{y_k\}$ in $f(X)$ has a sequence, $\{x_k\}$ in X such that $f(x_k) = y_k$. Sequence $\{x_k\}$ has a subsequence $\{x_{n_k}\}$ converging to x, then sequence $\{f(x_n)\}$ in $f(X)$ has the subsequence $\{f(x_{n_k})\}$ converging to $f(x)$.*

Proof. Let X be a sequentially compact space, function $f : X \to Y$ be continuous and $\{y_n\}$ be a sequence in $f(X)$ subset of Y. Construct a sequence $\{x_n\}$ such that $f(x_k) = y_k, \ \forall k$.

Every sequence in X has a convergent subsequence. Thus $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to $x \in X$. The image of this subsequence $\{f(x_{n_k})\}$ is a subsequence of $\{y_k\}$. We claim that, $\{f(x_{n_k})\}$ converges to $f(x) \in f(X)$.

Let U be an open set containing $f(x)$, then $f^{-1}(U)$ is an open set containing x. Since $\{x_{n_k}\}$ converges to x. There exists an integer n such that for every $k \geq n$, $x_k \in f^{-1}(U)$. Clearly, for each $k \geq n$, $f(x_k) \in U$. Since U is arbitrary, $\{f(x_{n_k})\}$ converges to $f(x)$. Therefore, the image of any sequentially compact space is sequentially compact. In other words, sequentially compactness is preserved under continuous functions. □

Remark. *1. Given a sequence $\{y_n\}$ in $f(X)$, there is a sequence of subsets $\{U_n\}$ in $P(Y)$ such that $U_n = f^{-1}(y_n)$. Since each U_n is non-empty, we can construct a sequence $\{x_n\}$ in X using a choice function. The convergent subsequence of $\{y_n\}$ depends on the selection of this choice function.*

Given every sequentially compact, metric space is countably compact. We may assert the properties of countably compact, metric spaces on sequentially compact, metric spaces.

Theorem 11.8. *Every continuous, real-valued function on a sequentially compact, metric space is bounded and attains its extrema.*

Theorem 11.9. *Every continuous, real-valued function on a sequentially compact, metric space is uniformly continuous.[Joshi, 1983, Exercises 11.1.6]*

11.1.3 Countable Compactness on T_1 spaces

In this section, we are going to see four different characterisations of countable compactness in T_1 spaces. The first two characterisations doesn't have anything to do with the T_1 axiom.

T_1 **Space** A topological space X satisfy T_1 axiom if for any two distinct points $x, y \in X$, there exists an open set $U \subset X$ containing x but not y.[Joshi, 1983, 7.1.2]

countable compactness A topological space is countably compact if every countable open cover has a finite subcover.[Joshi, 1983, 11.1.1]

finite intersection property A family $\mathcal{F}$ of subsets of X has finite intersection property(f.i.p.) if every finite subfamily of $\mathcal{F}$ has a non-empty intersection.[Joshi, 1983, 10.2.6]

accumulation point A point $x \in X$ is accumulation point of a subset $A \subset X$ if every open set containing x has atleast one point of A other than x.[Joshi, 1983, 5.2.7]

limit point A point $x \in X$ is a limit point of a sequence $< x_k >$ in X if for every open set U containing x, there exists an integer $N \in \mathbb{N}$ such that $x_k \in U$ for every $k \geq N$.[Joshi, 1983, 4.1.7]

cluster point A point $x \in X$ is a cluster point of a sequence $< x_k >$ in X if for any neighbourhood V of x, the sequence $< x_k >$ assumes a point in V infinitely many times.[11]

Countable compactness in T_1 spaces

Theorem 11.10. *In a T_1 space X, following statements are equivalent,*

1. *X is countably compact*
2. *Every countably family of closed subsets of X with finite intersection property have non-empty intersection.*
3. *Every infinite subset $A \subset X$ has an accumulation point.*[12]
4. *Every sequence $< x_k >$ in X has a cluster point.*
5. *Every infinite open cover of X has a proper subcover.[Arens-Dugundji]*

Proof. $1 \implies 2$
Suppose X is countably compact. Let $\mathcal{C} = \{C_1, C_2, \cdots\}$ be a countable family of closed subsets of X with empty intersection. Define $\mathcal{U} = \{X - C_1, X - C_2, \cdots\}$ is a family of open subsets of X. By de Morgan's law, [13]

$$\bigcap_{k=1}^{\infty} C_k = \phi, \text{ then } X = X - \left(\bigcap_{k=1}^{\infty} C_k\right) = \bigcup_{k=1}^{\infty} (X - C_k)$$

We have $\mathcal{U}$ is a countable cover of X and X is countably compact space. Thus $\mathcal{U}$ has a finite subcover $\mathcal{U}' = \{X - C_{n_1}, X - C_{n_2}, \cdots, X - C_{n_k}\}$.

$$\mathcal{U}' \text{ is a cover of } X, \text{ then } X = \bigcup_{j=1}^{k} \left(X - C_{n_j}\right)$$

$$X - \bigcup_{j=1}^{k} \left(X - C_{n_j}\right) = \bigcap_{j=1}^{k} \left(X - \left(X - C_{n_j}\right)\right) = \bigcap_{j=1}^{k} C_{n_j} = \phi$$

Now $\mathcal{C}' = \{C_{n_1}, C_{n_2}, \cdots, C_{n_k}\}$ has empty intersection. This is a contradiction to the finite intersection property of $\mathcal{C}$. Thus $\mathcal{C}$ has non-empty intersection. Therefore, every countably family of closed subsets of X have non-empy intersection.

---

[11] x is a cluster point of $< x_k >$ if for every integer N, there exists $k > N$ such that $x_k \in V$.In other words, $< x_k >$ is frequently in V.[Joshi, 1983, 10.1.9]

[12] Every infinite subset of $\mathbb{R}$ has a limit point is equivalent to the completeness axiom.

[13] Complement of Intersection = Union of complements, $X - (C \cap D) = (X - C) \cup (X - D)$,

$2 \implies 1$

Let $\mathcal{U} = \{U_1, U_2, \cdots\}$ be a countable cover of X. Then $\mathcal{C} = \{X - U_1, X - U_2, \cdots\}$ is a countable family of closed subsets of X.

Let $\mathcal{U}' = \{U_{n_1}, U_{n_2}, \cdots, U_{n_k}\}$ be any finite subfamily of $\mathcal{U}$. Suppose X is not countably compact, then $\mathcal{U}$ doesn't have a finite subcover. Therefore, $\mathcal{U}'$ is not a cover of X. And $\mathcal{C}$ is a family of closed sets with finite intersection property.

Therefore by assumption, the countable family of closed sets $\mathcal{C}$ has a non-empty intersection.

$$\bigcap_{k=1}^{\infty} C_k \neq \phi, \text{ then } \bigcap_{k=1}^{\infty} C_k = \bigcap_{k=1}^{\infty} (X - U_k) = X - \left(\bigcup_{k=1}^{\infty} U_k\right) \neq \phi$$

Then $\mathcal{U}$ is not a cover of X as well. This is a contradiction, therefore X is countably compact.

$1 \implies 3$
Suppose X is countably compact. Let A be an infinite subset of X. Suppose A doesn't have an accumulation point.

Let B be a countably infinite subset of A. Then B also doesn't have any accumulation point. Therefore, the derived set B' is empty. Thus B is a closed subset of X. Since countable compactness is weakly hereditary, subspace B is again countably compact.

For each point $b \in B$, there is an open set V_b such that $V_b \cap B = \{b\}$, since $b \in B$ is not an accumulation point. Thus $\mathcal{U} = \{V_b \cap B : b \in B\}$ is a countable open cover of B. Clearly, $\mathcal{U}$ doesn't have any finite subcover.

This is a contradiction to B being countably compact. Therefore, A has an accumulation point. □

11.1.4 Variations of Compactness on Metric Spaces

In this document, we will see that from metric space point of view these two notions were equivalent to the compactness and were used alternatively. For example : in functional analysis (semester 3), you will find definitions like 'a normed space is compact iff every sequence in it has a convergent subsequence', which is clearly sequential compactness for a topologist.

Lindeloff A topological space is Lindeloff iff every open cover has a countable subcover.

First countable A topological space is first countable iff every point in it has a countable local base.

Second countable A topological space is second countable iff it has a countable base.

Base A family of subsets $\mathcal{B}$ of X is a base of a topological space if every open set can be expressed as union of some members of $\mathcal{B}$

Base Characterisation A family of subsets $\mathcal{B}$ of X is a base of a topological space iff for every $x \in X$, and for every neighbourhood U of x, there is a member $B \in \mathcal{B}$ such that $x \in B \subset U$.

Local Base A family of subsets $\mathcal{L}$ of X is a local base at point $x \in X$ if for every neighbourhood U of x, there is a member $L \in \mathcal{L}$ such that $x \in L \subset U$.

Equivalence

We are going to see when these three notions: compactness, countable compactness and sequentially compactness are equivalent.

Theorem 11.11. *Countably compact, metric spaces are second countable.*

Synopsis. *For every positive real number r, there exists a non-empty maximal subsets A_r with every pair of points atleast r distance apart. A_r are finite. The union of maximal subsets $A_{\frac{1}{n}}$ for each natural number n is a countable, dense subset D of X. Thus countably compact, metric spaces are separable. The family $\mathcal{B}$ of all open balls with center at $d \in D$ and rational radius is a countable, base for X. Thus countably compact, metric spaces are second countable.*

Proof. Let $(X; d)$ be a countably compact,, metric space. For each positive real number $r \in \mathbb{R},\ r > 0$ construct a family of subsets $A_r \subset X$ such that it is a maximal set of points which are atleast r distances apart.

Then A_r is finite for every positive real number r. Suppose A_r is infinite for some real number $r > 0$, then A_r has a accumulation point, say x by the Characterisation of countable compactness of X.

Then every neighbourhood of x must intersect A_r at infinitely many points, since every metric space is a T_1 space. Consider $B(x, \frac{r}{2})$. Since any two points of $B(x, \frac{r}{2})$ are less than r distances apart, the intersection $B(x, \frac{r}{2}) \cap A_r$ can have atmost one point in it. Thus for every positive real number r, A_r is finite.

Define $D = \cup_{n=1}^{\infty} A_{\frac{1}{n}}$. We claim that D is a countable, dense subset of X.

Let $x \in X$ and $B(x, r)$ be an open ball containing x, then there exists integer $n \in \mathbb{N}$ such that $\frac{1}{n} < r$.[14]

Then $B(x, r) \cap D \neq \phi$, since $B(x, r) \cap A_{\frac{1}{n}} \neq \phi$. Suppose $B(x, r) \cap A_{\frac{1}{n}} = \phi$, then $A_{\frac{1}{n}}$ is not maximal. Since, x is atleast $r > \frac{1}{n}$ distance apart from each points of $A_{\frac{1}{n}}$. Therefore, D intersects with every open set and thus dense in X.

We have have a countable, dense subset D of X. Therefore, X is separable. Now define $\mathcal{B} = \{B(x, r) : r \in \mathbb{Q},\ x \in D\}$. Clearly, $\mathcal{B}$ is a countable base for X. By the construction of $\mathcal{B}$, X is second countable.[15] □

[14] By archimedean property of integers, we have $\forall r \in \mathbb{R},\ r > 0,\ \exists n \in \mathbb{N}$ such that $nr > 1$.

[15] Every separable, metric space is second countable.

Countable Compactness, Lindeloff $\iff$ Compactness

Theorem 11.12. *A topological space X is compact iff it is countably compact, Lindeloff space.*

Proof. Let X be a compact space. Let $\mathcal{U}$ be a countable open cover of X, then $\mathcal{U}$ has a finite subcover $\mathcal{U}'$. Therefore, every compact space is countably compact.[16]

Conversely, suppose X is a countably compact, Lindeloff space. Since X is Lindeloff, every open cover $\mathcal{U}$ has a countable subcover $\mathcal{U}'$. Since X countably compact, every countable open cover $\mathcal{U}'$ has a finite subcover $\mathcal{U}''$. Thus every open cover $\mathcal{U}$ has a finite subcover $\mathcal{U}''$. Therefore every countably compact, Lindeloff space is compact. $\square$

Countable Compactness, First Countable $\implies$ Seq. Compactness

Theorem 11.13. *Every countably compact, first countable space is Sequentially compact.*

Proof. Let X be a countably compact, first countable space. Let $\{x_n\}$ be a sequence in X. By, equivalent conditions[17] of countably compact spaces, every sequence in countably compact space X has a cluster point, say x. We have, X is first countable. Therefore, X has a countable local base $\mathcal{L}$ at $x \in X$. How to construct a subsequence of $\{x_n\}$ converging to x ?$\star$[18] $\square$

Remark. *Every sequentially compact space is countably compact.*$\star$

Theorem 11.14. *In a second countable space, all the three forms of compactness are equivalent.[Joshi, 1983, 11.1.10]*

Proof. Every second countable space is both first countable and Lindeloff. Every countably compact, Lindeloff space is countably compact. Therefore every countably compact, second countable space compact. Again, every countably compact, first countable space is sequentially compact. Therefore every countably compact, second countable space is sequentially compact. Conversely, every compact space is countably compact and every sequentially compact space is countably compact.[19] $\square$

Theorem 11.15. *In a metric space, all the three forms of compactness are equivalent.[Joshi, 1983, 11.1.11]*

Proof. In a metric space each form of compactness implies second countability. And in second countable spaces, they are all equivalent. $\square$

[16] Countable compactness is a weaker notion than compactness.

[17] [Joshi, 1983, 11.1] Conditions 1,2, and 4 are equivalent. 2 $\implies$ 4 without T_1 axiom is out of scope.

[18] [Joshi, 1983, Exercises 10.1.11]

[19] Countable compactness is a weaker notion than sequential compactness as well.

Part II

ME010203 Numerical Analysis with Python

Chapter 12

Expressions

Chapter 13

Calculus

Chapter 14

Interpolation & Curve Fitting

Definitions 14.1. *Given* $(n+1)$ *data points* $(x_k, y_k),\ k = 0, 1, \cdots, n$, *the problem of estimating* $y(x)$ *using a function* $y : \mathbb{R} \to \mathbb{R}$ *that satisfy the data points is the interpolation problem. ie,* $y(x_k) = y_k,\ k = 0, 1, \cdots, n$.

Definitions 14.2. *Given* $(n+1)$ *data points* $(x_k, y_k),\ k = 0, 1, \cdots, n$, *the problem of estimating* $y(x)$ *using a function* $y : \mathbb{R} \to \mathbb{R}$ *that is sufficiently close to the data points is the curve-fitting problem.*
ie, Given $\epsilon > 0,\ |y(x_k) - y_k| < \epsilon,\ k = 0, 1, \cdots, n$.

Remark. *The data could be from scientific experiments or computations on mathematical models. The interpolation problem assumes that the data is accurate. But, curve-fitting problem assumes that there are some errors involved which are sufficiently small.*

Definitions 14.3. *Given* $(n+1)$ *data points* $(x_k, y_k),\ k = 0, 1, \cdots, n$, *the problem of estimating* $y(x)$ *using a polynomial function of degree* n *that satisfy the data points is the polynomial interpolation problem.*

Remark. *Polynomial is the simplest interpolant.[Kiusalaas, 2013, 3.2]*

14.1 Polynomial Interpolation

There exists a unique polynomial of degree n that satisfy $(n+1)$ distinct data points. There are a few methods to find this polynomial : 1. Lagrange's method 2. Newton's method 3. Neville's method. The Neville's method is out of scope.

14.1.1 Lagrange's Method

Interpolation polynomial[1] is given by,

$$P(x) = \sum_{i=0}^{n} y_i l_i(x), \text{ where } l_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_i}{x_j - x_i} \tag{14.1}$$

[1]Using P_n to represent some polynomial of degree n. It is quite a confusing a notation when it comes to Newton's method as author construct a psuedo-recursive definition.

Remark. *Lagrange's cardinal functions l_i, are polynomials of degree n and*

$$l_i(x_j) = \delta_{ij} = \begin{cases} 0, & i = j \\ 1, & i \neq j \end{cases}$$

Proposition 14.4. *Error in polynomial interpolation is given by*

$$f(x) - P(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_n)}{(n+1)!} f^{(n+1)}(\xi) \tag{14.2}$$

where $\xi \in (x_0, x_n)$

Remark. *The error increases as x moves away from the unknown value ξ.*

14.1.2 Newton's Method

The interpolation polynomial is given by,

$$P(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \tag{14.3}$$

where $a_i = \nabla^i y_i,\ i = 0, 1, \cdots, n$.

Remark. *For Newton's Method, it is assumed that $x_0 < x_1 < \cdots < x_n$.*

Remark. *Lagrange's method is conceptually simple. But, Newton's method is computationaly more efficient than Lagrange's method.*

Computing coefficients a_i of the polynomial

The coefficients are given by,

$$a_0 = y_0,\ a_1 = \nabla y_1,\ a_2 = \nabla^2 y_2,\ a_3 = \nabla^3 y_3, \cdots, a_n = \nabla^n y_n \tag{14.4}$$

Remark. *The divided difference $\nabla^i y_i$ are computed as follows:*

$$\begin{aligned} &\nabla y_1 = \frac{y_1 - y_0}{x_1 - x_0} \\ &\nabla y_2 = \frac{y_2 - y_1}{x_2 - x_1} && \nabla^2 y_2 = \frac{\nabla y_2 - \nabla y_1}{x_2 - x_1} \\ &\nabla y_3 = \frac{y_3 - y_2}{x_3 - x_2} && \nabla^2 y_3 = \frac{\nabla y_3 - \nabla y_2}{x_3 - x_2} && \nabla^3 y_3 = \frac{\nabla^2 y_3 - \nabla^2 y_2}{x_3 - x_2} \end{aligned}$$

x_0	y_0				
x_1	y_1	∇y_1			
x_2	y_2	∇y_2	$\nabla^2 y_2$		
$\dots$	$\dots$	$\dots$	$\dots$	$\ddots$	
x_n	y_n	∇y_n	$\nabla^2 y_n$	$\dots$	$\nabla^n y_n$

Table 14.1: The $\nabla^i y_i$ Computation Table

Remark. *Practise Problems*
Find interpolation polynomial for the following data points :

1. $\{(0, 7),\ (2, 11),\ (3, 28)\}$
   [Kiusalaas, 2013, Example 3.1]

2. $\{(-2, -1),\ (1, 2),\ (4, 59),\ (-1, 4),\ (3, 24),\ (-4, -53)\}$
   [Kiusalaas, 2013, Example 3.2]

3. $\{(-1.2, -5.76),\ (0.3, -5.61),\ (1.1, -3.69)\}$
   [Kiusalaas, 2013, Problem Set 3.1.1]

4. $\{(-2, -1),\ (1, 2),\ (4, 59),\ (-1, 4),\ (3, 24),\ (-4, -53)\}$
   [Kiusalaas, 2013, Problem Set 3.1.6]

5. $\{(-3, 0),\ (2, 5),\ (-1, -4),\ (3, 12),\ (1, 0)\}$
   [Kiusalaas, 2013, Problem Set 3.1.7]

6. $\{(0, 1.225), (3, 0.905), (6, 0.652)\}$
   [Kiusalaas, 2013, Problem Set 3.1.9]

Remark. *In Lagrange's Method, we can interpolate at the given point even without computing the polynomial. In Newton's method, we have to compute polynomial and then interpolate for the given point.*

That is, evaluate the value of cardinal polynomials at the point and substitute in Equation 14.1 as shown in Section 3.2.[Kiusalaas, 2013, Example 3.1]

14.1.3 Implementation of Newton's Method

Program 14.5. *Computing Coefficients*

```
def coefficients(xData,yData):
  m = len(xData)
  a = yData.copy()
  for k in range(1,m):
    a[k:m] = (a[k:m]-a[k-1])/(xData[k:m]-xData[k-1])
  return a
```

Line 1 : ***Defines a function which takes two arguments/parameters, named*** *xData* ***and*** *yData.* *In [Kiusalaas, 2013, 3.2], you will find coeffts which I have changed to coefficients. xData,yData are numpy array objects. xData is a array with values* $x_0, x_1, \cdots, x_n$*. And yData is array with values* $y_0, y_1, \cdots, y_n$*. For example, the value of* x_3 *can be accessed as* $xData[3]$.

Line 2 : ***The functon*** $len()$ ***is extended by numpy to give the length of array objects.*** *In this context,* $len(xData)$ *will return the value* $n+1$*, since there are* $n+1$ *values in xData array.*

Line 3 : We need a copy of yData to work with. Unlike other programming languages like java, in python $a = yData$ *will assign a new label a to the same memory location and manipulating a will corrupt the original data in yData as well. In order to avoid this, we are* ***making a copy of the array object***

using the array method provided by the numpy library.

Line 4 : ***This is a python loop statement. This ask python interpreter to repeat the following sub-block*** $m-1$ ***times.***[2] *In this context, Line 5 will be executed* n *times, since the range*$(1, m)$ *object is a list-type object with values* $1, 2, \cdots, m-1$. *And intepreter executes Line 5 for each values in the range() object, ie,* $k = 1, 2, \cdots, m-1$ *before interpreting Line 6.*

Line 5 : This is very nice feature available in python. ***This statement, evaluates*** $m-k$ ***values in a single step.ie,*** $a[k], a[k+1], \cdots, a[m]$. ***This calculation corresponds to subsequent columns of the divided difference table, that we are familiar with.*** *For example, executing Line 5 with* $k = 3$ *is same as evaluating the* $\nabla^3 y_j$ *column. Note that the value* $a[0]$ *is never updated and similarly* $a[2]$ *changes when Line 5 is executed with* $k = 1, 2$. *From column 3 onward,* $a[2]$ *is not updated. Therefore,* ***after completing*** n***th executing of the Line 5, we have*** $a[0] = y_0,\ a[1] = \nabla y_1,\ a[2] = \nabla^2 y_2, \cdots,\ a[n] = \nabla^n y_n$.

Line 6 : ***This returns the array*** a ***which is the array of coefficients.***

The logic of this program is in Line 4 and Line 5. So they need more explanation/understanding than anything else.

Program 14.6. *Interpolating using Newton's Method*

```
def interpolate(a,xData,x):
  n = len(xData)-1
  p = a[n]
  for k in range(1,n+1):
    p = a[n-k]+(x-xData[n-k])*p
  return p
```

The logic this program is in Line 3, Line 4 and Line 5.

Line 3 : We initialize the polynomial with the coefficient $a[n] = \nabla^n y_n = a_n$.

Line 4 : We are going define the polynomial recursively. This takes exactly n *steps further. So we use a loop which repeats* n *times.*

Line 5 : The value of p *and* k *changes each time Line 5 is executed. Let* P_j *be the value in* p *after executing Line 5 with* $k = j$. *Then,*

$P_0 = p = a[n]$

$P_1 = a[n-1] + (x - x_{n-1})P_0$

$P_2 = a[n-2] + (x - x_{n-2})P_1$

$\vdots$

$P_n = a[0] + (x - x_0)P_{n-1}$.

Clearly, P_n *is the unique* n *degree polynomial given by the Newton's method.*

Program 14.7. *How to interpolate ?*

[2]Python block is a group of statement with same level of indentation. A sub-block is a block with an additional indentation.

```
from numpy import array
xData = array([-2,1,4,-1,3,-4]) #change as needed
yData = array([-1,2,59,4,24,-53]) #change as needed
a = coefficients(xData,yData)
print(interpolate(a,xData,2))
```

You will have to define both the functions (coefficients, interpolate) before doing this.

Line 1 : For defining array objects, we need to import them from numpy library.

Line 2 : You can change this line according to the first component of the given data points.

Line 3 : You can change this line according to the second component of the given data points.

Line 4 : Call function `coefficients` *and store the array returned into a*

Line 5 : Call function `interpolate` *to interpolate at* $x = 2$ *and print the value* $P(2)$

Program 14.8 (Just for Fun)**.** *We can do more using sympy !*

```
from numpy import array
from sympy import Symbol
xData = array([-2,1,4,-1,3,-4])
yData = array([-1,2,59,4,24,-53])
a = coefficients(xData,yData)
x = Symbol('x')
p = interpolate(a,xData,x)
p.subs({x:2})
```

Remark. *Programming Problems*

1. $\{(0.15, 4.79867),\ (2.30, 4.49013),\ (3.15, 4.2243),\ (4, 85, 3.47313),$ $(6.25, 2.66674),\ (7.95, 1.51909)\}$ *[Kiusalaas, 2013, Example 3.4]*

2. $\{(0, -0.7854),\ (0.5, 0.6529),\ (1, 1.7390),\ (1.5, 2.2071),\ (2, 1.9425)\}$ *[Kiusalaas, 2013, Problem Set 3.1.5]*

14.1.4 Limitations of Polynomial Interpolation

1. Inaccuracy - The error in interpolation increases as the point moves away from most of the data points.

2. Oscilation - As the number of data points considered for polynomial interpolation increases, the degree of the polynomial increases. And the graph of the interpolant tend to oscilate excessively. In such cases, the error in interpolation is quite high.

3. The best practice is to consider four to six data points nearest to the point of interest and ignore the rest of them.

Remark. *The interpolant obtained by joining cubic polynomials corresponding to four nearest data points each, is a cubic spline*[3].

14.2 Roots of a Function

Definitions 14.9. *Let $f : \mathbb{R} \to \mathbb{R}$, then $x \in \mathbb{R}$ is a root of f if $f(x) = 0$.*

Remark. *Suppose $a < b$ and $f(a), f(b)$ are nonzero and are of different signs. If f is continuous in $[a, b]$, then there is a point $c \in [a, b]$ such that $f(c) = 0$.*

Thus given $a < b$ and $f(a), f(b)$ are nonzero values of different sign, then there may be a bracketed root in $[a, b]$.

Note : There is no guarantee that there exists a root in $[a, b]$ as we are not sure about the continuity of f.

Remark. *Given a bracketed root, we can find it using*

1. Bisection Method or

2. Newton-Raphson Method

14.2.1 Bisection Method

Suppose $a < b$ and $f(a), f(b)$ are nonzero values of different signs. We evaluate $f(c)$ where $c = \frac{a+b}{2}$. If $f(c)$ is a nonzero value, then at least one of the pairs $f(a), f(c)$ or $f(c), f(b)$ are of different signs. WLOG suppose that $f(a), f(c)$ are of different signs, then set $b = c$ and $c = \frac{a+b}{2}$. And continue this process until we get sufficiently accurate value of a root.

Remark. *Suppose $f(x) = x^5 - 2$. Then $f(0) = -2,\ f(1) = -1,\ f(2) = 30$. Since f is known to be continuous, there is a bracketed root in $[1, 2]$. Now*
$f(1.5) > 0 \implies [1, 1.5]$
$f(1.25) > 0 \implies [1, 1.25]$
$f(1.125) < 0 \implies [1.125, 1.25]$
$f(1.1825) > 0 \implies [1.125, 1.1825]$
$f(1.15375) > 0 \implies [1.125, 1.15375]$
$f(1.139375) < 0 \implies [1.139375, 1.15375]$
$f(1.1465625) < 0 \implies [1.1465625, 1.15375]$
$f(1.150156250) > 0 \implies [1.1465625, 1.15015625]$
$f(1.148359375) < 0 \implies [1.1483594, 1.15015625]$
$f(1.149257825) > 0 \implies [1.1483594, 1.14925783]$

Thus, we have 1.14 is a root of f with accuracy upto two decimal points.

[3] Cubic spline is a function, the graph of which is piece-wise cubic

14.2.2 Newton-Raphson Method

Suppose f is differentiable at $x \in \mathbb{R}$ and $f(x) \neq 0$. Then compute $x = x - \frac{f(x)}{df(x)}$ and evaluate $f(x)$. Repeat this process to get more accurate value of a root near x.

Remark. *Suppose* $f(x) = x^5 - 2$. *Then* $df(x) = 5x^4$. *Let* $x = 2$. *Then*
$x = 2 - \frac{30}{80} \implies f(1.625) = 9.330$
$x = 1.625 - \frac{9.330}{34.86} \implies f(1.35735) = 2.6074$
$x = 1.35735 - \frac{2.6074}{16.9721} \implies f(1.20373) = 0.52733$
$x = 1.20373 - \frac{0.52733}{10.4975} \implies f(1.15351) = 0.04224$
$x = 1.15351 - \frac{0.042245}{8.85225} \implies f(1.148738) = 0.00034312$

Thus we have 1.1487 *is quite close to a root of* f.

Chapter 15

Matrix Operations

Part III

ME010303 Multivariate Calculus & Integral Transforms

Chapter 16

Fourier Series and Fourier Integrals

16.1 The Weierstrass Approximation Theorem

Every continuous, real valued function on a compact interval has a polynomial approximation.[apo, 1973, Theorem 11.17]

Theorem 16.1 (Weierstrass)**.** *Let f be a real-valued, continuous function on a compact interval $[a, b]$. Then for every $\epsilon > 0$, there is a polynomial p such that $|f(x) - p(x)| < \epsilon$ for every $x \in [a, b]$.*

Synopsis. *Given a real-valued continuous function on compact interval $[a, b]$, we can construct a real-valued, continous function g on $\mathbb{R}$ which is periodic with period 2π. We have, if $f \in L(I)$ and f is bounded almost everywhere in I, then $f \in L^2(I)$.[apo, 1973, Theorem 10.52].By Fejer's theorem ([apo, 1973, Theorem 11.15]), the fourier series generated by g ([apo, 1973, definition 11.3]) converges to the Cesaro sum ([apo, 1973, Definition 8.47]), which is g itself in this case. Thus for any $\epsilon > 0$, there is a finite sum of trignometric functions. The power series expansions of trignometric functions ([apo, 1973, definition 9.27]) being uniformly convergent, there exists a polynomial p_m which approximates g. And we can construct p (polynomial approximation of g) using p_m.*

Proof. Define $g : \mathbb{R} \to \mathbb{R}$,

$$g(t) = \begin{cases} f(a + (b-a)t/\pi), \ t \in [0, \pi) \\ f(a + (2\pi - t)(b-a)/\pi), \ t \in [\pi, 2\pi] \\ g(t - 2n\pi), \ t > 2\pi, \ n \in \mathbb{N} \\ g(t + 2n\pi), \ t < 0, \ n \in \mathbb{N} \end{cases}$$

Thus g is a continuous, real-valued, periodic function with period 2π such that

$$f(x) = g\left(\frac{\pi(x-a)}{b-a}\right), \ x \in [a, b] \tag{16.1}$$

The fourier series generated by g is given by,

$$g(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt)$$

$$\text{where } a_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos kt\ dt,\ b_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin kt\ dt$$

Let $\{s_n(t)\}$ be the sequence of partial sums of the fourier series generated by g. And $\{\sigma_n(t)\}$ be the sequence of averages of $s_n(t)$ given by,

$$\sigma_n(t) = \frac{1}{n}\sum_{k=1}^{n} s_k(t),\ \text{where } s_k(t) = \frac{a_0}{2} + \sum_{j=1}^{k}(a_j\cos jt + b_j\sin jt)$$

Function $f \in L(I)$ being real-valued continuous function on a compact interval, it is bounded and hence is Lebesgue square integrable.ie, $f \in L^2(I)$. Thus, $g \in L^2(I)$.

Since g is continous on $\mathbb{R}$, the function $s : \mathbb{R} \to \mathbb{R}$ defined by,

$$s(t) = \lim_{h\to 0^+} \frac{g(t+h) - g(t-h)}{2}$$

is well-defined on $\mathbb{R}$ and $s(t) = g(t),\ \forall t \in \mathbb{R}$.

Then by Fejer's Theorem, the sequence $\{\sigma_n(t)\}$ converges uniformly to $g(t)$ for every $t \in \mathbb{R}$. Thus, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall t \in \mathbb{R}$, $|g(t) - \sigma_N(t)| < \frac{\epsilon}{2}$.

We have,

$$\sigma_N(t) = \sum_{k=0}^{N}(A_k\cos kt + B_k\sin kt),\ \text{where } A_k, B_k \in \mathbb{R} \tag{16.2}$$

By the power series expansion of the trignometric functions about origin,

$$\cos kt = \sum_{j=1}^{\infty}\left(\frac{\cos^{(j)} 0}{j!}(kt)^j\right) = \sum_{j=1}^{\infty} A'_j t^j \text{ where } A'_j \in \mathbb{R} \tag{16.3}$$

$$\sin kt = \sum_{j=1}^{\infty}\left(\frac{\sin^{(j)} 0}{j!}(kt)^j\right) = \sum_{j=1}^{\infty} B'_j t^j \text{ where } B'_j \in \mathbb{R} \tag{16.4}$$

Since the above power series expansions of trignometric functions are uniformly convergent, their finite linear combination $\{\sigma_N(t)\}$ is also uniformly convergent. ie, Given $\epsilon > 0$ there exists $m \in \mathbb{N}$ such that for every $t \in \mathbb{R}$

$$\left|\sum_{k=0}^{m} C_k t^k - \sigma_N(t)\right| < \frac{\epsilon}{2} \text{ where } C_k \in \mathbb{R}$$

Therefore, $|p_m(t)-g(t)| \le |p_m(t)-\sigma_N(t)|+|\sigma_N(t)-g(t)| < \epsilon$ where $p_m(t) = \sum_{k=0}^{m} C_k t^k$. Define $p : [a, b] \to \mathbb{R}$ by,

$$p(x) = p_m\left(\frac{\pi(x-a)}{b-a}\right) \tag{16.5}$$

By equations 16.1 and 16.5, $|p(x) - f(x)| < \epsilon$ for every $x \in [a, b]$. $\square$

16.2 Other Forms of Fourier Series

Let $f \in L([0,2\pi])$, then the fourier series generated by f is given by,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\text{where } a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\ dt, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\ dt$$

By Euler's forumula $e^{inx} = \cos nx + i\sin nx$. We have, $\cos nx = \frac{(e^{inx}+e^{-inx})}{2}$ and $\sin nx = \frac{(e^{inx}-e^{-inx})}{2i}$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\alpha_n e^{inx} + \beta_n e^{-inx}\right)$$

$$\text{where } \alpha_n = \frac{(a_n - ib_n)}{2} \qquad \beta_n = \frac{(a_n + ib_n)}{2}$$

Therefore, by assigning $\alpha_0 = a_0/2$, $\alpha_{-n} = \beta_n$, we get the following exponential form of fourier series generated by f,

$$f(x) \sim \sum_{n=-\infty}^{\infty} \alpha_n e^{inx} \text{ where } \alpha_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)\ e^{-int}\ dt$$

Note : If f is periodic with period 2π, then the interval of integration $[0,2\pi]$ can be replaced with any interval of length 2π. eg. $[-\pi,\pi]$

16.2.1 Periodic with period p

Let $f \in L([0,p])$ and f is periodic with period p. Then

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi nx}{p} + b_n \sin\frac{2\pi nx}{p}\right)$$

$$\text{where } a_n = \frac{2}{p}\int_0^{p} f(t)\cos\frac{2\pi nt}{p}\ dt \qquad b_n = \frac{2}{p}\int_0^{p} f(t)\sin\frac{2\pi nt}{p}\ dt$$

Therefore, we have the exponential form of the above fourier series given by,

$$f(x) \sim \sum_{n=-\infty}^{\infty} \alpha_n e^{\frac{2\pi inx}{p}}, \text{ where } \alpha_n = \frac{1}{p}\int_0^{p} f(t)\ e^{\frac{-2\pi int}{p}}\ dt$$

16.3 Fourier Integral Theorem

Theorem 16.2 (Fourier Integral Theorem)**.** *Let $f \in L(-\infty,\infty)$. Suppose $x \in \mathbb{R}$ and an interval $[x-\delta, x+\delta]$ about x such that either*

1. *f is of bounded variation on an interval $[x-\delta, x+\delta]$ about x or*

2. both limits $f(x+)$ and $f(x-)$ exists and both Lebesgue intergrals

$$\int_0^\delta \frac{f(x+t)-f(x+)}{t} dt \text{ and } \int_0^\delta \frac{f(x-t)-f(x-)}{t} dt$$

exists.

Then,

$$\frac{f(x+)+f(x-)}{2} = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(u)\cos v(u-x)\ du\ dv,$$

the integral $\int_0^\infty$ being an improper Riemann integral.

Synopsis.

$$f(x+t)\frac{\sin \alpha t}{\pi t} dt \to f(u)\frac{\sin \alpha(u-x)}{\pi(u-x)} \to \frac{f(u)}{\pi}\int_0^\alpha \cos v(u-x) dv$$

By Riemann-Lebesgue lemma[apo, 1973, Theorem 11.6],

$$f \in L(I) \implies \lim_{\alpha \to +\infty} \int_I f(x)\sin \alpha t\ dt = 0$$

By Jordan's Theorem[apo, 1973, Theorem 10.8], if g is of bounded variation on $[0, \delta]$, then

$$\lim_{\alpha \to +\infty} \frac{2}{\pi}\int_0^\delta g(t)\frac{\sin \alpha t}{t} dt = g(0+)$$

By Dini's Theorem[apo, 1973, Theorem 10.9], if the limit $g(x+)$ exists and Lebesgue integral $\int_0^\delta \frac{g(t)+g(0+)}{t} dt$ exists for some $\delta > 0$, then

$$\lim_{\alpha to +\infty} \frac{2}{\pi}\int_0^\delta g(t)\frac{\sin \alpha t}{t} dt = g(0+)$$

The order of Lebesgue integrals can be interchanged[apo, 1973, Theorem 10.40], Suppose $f \in L(X)$ and $g \in L(Y)$. Then

$$\int_X f(x)\left(\int_Y g(y)k(x,y)dy\right) dx = \int_Y g(y)\left(\int_X f(x)k(x,y)dx\right) dy$$

Proof. Consider $\int_{-\infty}^\infty f(x+t)\frac{\sin \alpha t}{\pi t} dt$. We prove that this integral is equal to the either sides.

$$\int_{-\infty}^\infty f(x+t)\frac{\sin \alpha t}{\pi t} dt = \int_{-\infty}^{-\delta} + \int_{-\delta}^0 + \int_0^{-\delta} + \int_\delta^\infty f(x+t)\frac{\sin \alpha t}{\pi t} dt$$

We have, function $\frac{f(x+t)}{\pi t}$ is bounded on $(-\infty, -\delta) \cup (\delta, \infty)$, hence $\frac{f(x+t)}{\pi t}$ is Lebesgue integrable on $(-\infty, -\delta) \cup (\delta, \infty)$.

By Riemann Lebesgue lemma,

$$\frac{f(x+t)}{\pi t} \in L(-\infty, -\delta) \implies \int_{-\infty}^{-\delta} f(x+t)\frac{\sin \alpha t}{\pi t} dt = 0,$$

$$\frac{f(x+t)}{\pi t} \in L(\delta, \infty) \implies \int_\delta^\infty f(x+t)\frac{\sin \alpha t}{\pi t} dt = 0$$

Case 1 Suppose f is of bounded variation on $[x-\delta, x+\delta]$, put $g(t) = f(x+t)$ then g is of bounded variation on $[-\delta, \delta]$. Thus g is of bounded variation on $[0, \delta]$. Then by Jordan's Theorem

$$\lim_{\alpha\to+\infty} \frac{2}{\pi}\int_0^\delta f(x+t)\frac{\sin \alpha t}{t}dt = \lim_{\alpha\to+\infty} \frac{2}{\pi}\int_0^\delta g(t)\frac{\sin \alpha t}{t}dt = g(0+) = f(x+)$$

Case 2 Suppose both the limits $f(x+)$ and $f(x-)$ exists and both Lebesgue integrals

$$\int_0^\delta \frac{f(x+t)-f(x+)}{t}dt \text{ and } \int_0^\delta \frac{f(x-t)-f(x-)}{t}dt$$

exists.

Thus, we have $f(x+)$ exists and the Lebesgue integral $\int_0^\delta \frac{f(x+t)-f(x+)}{t}dt$ exists. Put $g(t) = f(x+t)$, then $g(0+) = f(x+)$ exists and the Lebesgue integral $\int_0^\delta \frac{g(t)-g(0+)}{t}dt$ exists, then by Dini's Theorem,

$$\lim_{\alpha\to+\infty} \frac{2}{\pi}\int_0^\delta f(x+t)\frac{\sin \alpha t}{t}dt = \lim_{\alpha\to+\infty} \frac{2}{\pi}\int_0^\delta g(t)\frac{\sin \alpha t}{t}dt = g(0+) = f(x+)$$

Similarly, $f(x-)$ exists and the Lebesgue integral $\int_0^\delta \frac{f(x-t)-f(x-)}{t}dt$ exists. Put $g(t) = f(x-t)$, then $g(0+) = f(x-)$ exists and the Lebesgue integral $\int_0^\delta \frac{g(t)-g(0+)}{t}dt$ exists, then by Dini's Theorem,

$$\begin{aligned}\lim_{\alpha\to+\infty} \frac{2}{\pi}\int_{-\delta}^0 f(x+t)\frac{\sin \alpha t}{t}dt &= \lim_{\alpha\to+\infty} \frac{2}{\pi}\int_0^\delta f(x-\tau)\frac{\sin \alpha \tau}{\tau}d\tau \\ &= \lim_{\alpha\to+\infty} \frac{2}{\pi}\int_0^\delta g(\tau)\frac{\sin \alpha \tau}{\tau}d\tau = g(0+) = f(x-)\end{aligned}$$

Then by either cases,

$$\begin{aligned}\lim_{\alpha\to+\infty} \int_{-\infty}^\infty f(x+t)\frac{\sin \alpha t}{\pi t}dt &= \lim_{\alpha\to+\infty} \int_{-\delta}^0 + \int_0^\delta f(x+t)\frac{\sin \alpha t}{\pi t}dt \\ &= \frac{f(x+)+f(x-)}{2}\end{aligned}$$

We have, $\int_0^\alpha \cos v(u-x)dv = \frac{\sin v(u-x)}{u-x}$.

$$\begin{aligned}\lim_{\alpha\to+\infty} \int_{-\infty}^\infty f(x)\frac{\sin \alpha t}{\pi t}dt &= \lim_{\alpha\to+\infty} \int_{-\infty}^\infty f(u)\frac{\sin \alpha(u-x)}{u-x}du, \text{ (put } u = x+t) \\ &= \lim_{\alpha\to+\infty} \int_{-\infty}^\infty f(u)\left(\int_0^\alpha \cos v(u-x)dv\right)du \\ &= \lim_{\alpha\to+\infty} \int_0^\alpha \left(\int_{-\infty}^\infty f(u)\cos v(u-x)du\right)dv,\end{aligned}$$

since, the order of Lebesgue integrals can be reversed.

$$= \int_0^\infty \left(\int_{-\infty}^\infty f(u)\cos v(u-x)du\right)dv$$

where, $\displaystyle\int_0^\infty$ is not a Lebesgue integral, but an improper Riemann integral

Therefore,

$$\begin{aligned}\int_0^\infty \left(\int_{-\infty}^\infty f(u)\cos v(u-x)du\right)dv &= \lim_{\alpha\to+\infty}\int_{-\infty}^\infty f(x)\frac{\sin\alpha t}{\pi t}dt \\ &= \frac{f(x+)+f(x-)}{2}\end{aligned}$$

□

Remark. *If a function f on $(-\infty,\infty)$ is non-periodic, then it may not have a fourier series represenation. In such cases, we have fourier intergral representaion.*

16.4 Exponential form of Fourier Integral Theorem

Let $f \in L(-\infty,\infty)$. Suppose $x \in \mathbb{R}$ and an interval $[x-\delta, x+\delta]$ about x such that either

1. f is of bounded variation on an interval $[x-\delta, x+\delta]$ about x or
2. both limits $f(x+)$ and $f(x-)$ exists and both Lebesgue intergrals

$$\int_0^\delta \frac{f(x+t)-f(x+)}{t}dt \text{ and } \int_0^\delta \frac{f(x-t)-f(x-)}{t}dt$$

exists.

Then,

$$\frac{f(x+)+f(x-)}{2} = \lim_{\alpha\to\infty}\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\left(\int_{-\infty}^{\infty} f(u)e^{iv(u-x)}\, du\right)dv$$

Proof. Let $F(v) = \int_{-\infty}^\infty f(u)\cos v(u-x)du$. Then $F(v) = F(-v)$ and

$$\begin{aligned}\lim_{\alpha\to\infty}\frac{1}{2\pi}\int_{-\alpha}^{\alpha} F(v)dv &= \lim_{\alpha\to\infty}\frac{1}{\pi}\int_0^\alpha\int_{-\infty}^\infty f(u)\cos v(u-x)dudv \\ &= \frac{f(x+)+f(x-)}{2}\end{aligned}$$

Let $G(v) = \int_{-\infty}^\infty f(u)\sin v(u-x)du$. Then $G(v) = -G(-v)$ and

$$\lim_{\alpha\to\infty}\frac{1}{2\pi}\int_{-\alpha}^{\alpha} G(v)dv = 0$$

Thus

$$\lim_{\alpha\to\infty}\frac{1}{2\pi}\int_{-\alpha}^{\alpha} F(v)+iG(v)dv = \frac{f(x+)+f(x-)}{2}$$

□

16.5 Integral Transforms

Definitions 16.3. *Integral transform $g(y)$ of $f(x)$ is a Lebesgue integral or Improper Riemann integral of the form*

$$g(y) = \int_{-\infty}^{\infty} K(x,y)f(x)\ dx$$

, where K is the kernal of the transform. We write $g = \mathscr{K}(f)$.

Remark. *Integral transforms(operators) are linear operators.*
ie, $\mathscr{K}(af_1 + bf_2) = a\mathscr{K}f_1 + b\mathscr{K}f_2$

Remark. *A few commonly used integral transforms,*

1. *Exponential Fourier Transform $\mathscr{F}$,*

$$\mathscr{F}f = \int_{-\infty}^{\infty} e^{-ixy}f(x)\ dx$$

2. *Fourier Cosine Transform $\mathscr{C}$,*

$$\mathscr{C}f = \int_{0}^{\infty} \cos xyf(x)\ dx$$

3. *Fourier Sine Transform $\mathscr{S}$,*

$$\mathscr{S}f = \int_{0}^{\infty} \sin xyf(x)\ dx$$

4. *Laplace Transform $\mathscr{L}$,*

$$\mathscr{L}f = \int_{0}^{\infty} e^{-xy}f(x)\ dx$$

5. *Mellin Transform $\mathscr{M}$,*

$$\mathscr{M}f = \int_{0}^{\infty} x^{y-1}f(x)\ dx$$

Remark. *Suppose $f(x) = 0,\ \forall x < 0$.*

$$\int_{-\infty}^{\infty} e^{-ixy}f(x)\ dx = \int_{0}^{\infty} e^{-ixy}f(x)\ dx = \int_{0}^{\infty} \cos xy\ f(x)\ dx + i\int_{0}^{\infty} \sin xy\ f(x)\ dx$$

$$\mathscr{F}f = \mathscr{C}f + i\mathscr{S}f$$

Therefore Fourier Cosine $\mathscr{C}$ and Sine $\mathscr{S}$ transforms are special cases of fourier integral transform, $\mathscr{F}$ provided f vanishes on negative real axis.

Remark. *Let* $y = u + iv,\ f(x) = 0,\ \forall x < 0.$

$$\int_0^\infty e^{-xy} f(x) = \int_0^\infty e^{-xu} e^{-ixv} f(x)\ dx = \int_0^\infty e^{-ixv} \phi_u(x) dx$$

where $\phi_u(x) = e^{-xu} f(x)$.

$$\mathscr{L} f = \mathscr{F} \phi_u$$

Therefore Laplace transform, $\mathscr{L}$ *is a special case of Fourier integral transform,* $\mathscr{F}$.

Remark. *Let* $g(y) = \mathscr{F} f(x)$.

$$g(y) = \int_{-\infty}^\infty e^{-ixy} f(x)\ dx$$

Suppose f *is continuous at* x*, then by fourier integral theorem,*

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^\infty \left(\int_{-\infty}^\infty f(u) e^{iv(u-x)} du \right) dv \\ &= \int_{-\infty}^\infty e^{-ivx} \left(\frac{1}{2\pi} \int_{-\infty}^\infty e^{ivu} f(u)\ du \right) dv \\ &= \int_{-\infty}^\infty g(v) e^{-ivx} dv = \mathscr{F} g \text{ where } g(v) = \frac{1}{2\pi} \int_{-\infty}^\infty f(u) e^{ivu} du \end{aligned}$$

The above function $g(v)$ *gives the* ***inverse fourier transformation*** *of* f.

Let g *be fourier transform of* f*, then* f *is uniquely determined by its fourier transform* g *by,*

$$f(x) = \mathscr{F}^{-1} g(y) = \frac{1}{2\pi} \lim_{\alpha \to \infty} \int_{-\alpha}^\alpha g(y) e^{ixy} dy$$

6. Inverse Fourier Transform $\mathscr{F}^{-1}$,

$$\mathscr{F}^{-1} f = \int_{-\infty}^\infty \frac{e^{ixy}}{2\pi} f(x)\ dx$$

16.6 Convolutions

Definitions 16.4. *Let* $f, g \in L(-\infty, \infty)$. *Let* S *be the set of all points* x *for which the Lebesgue integral*

$$h(x) = \int_{-\infty}^\infty f(t) g(x - t) dt$$

exists. Then the function $h : S \to \mathbb{R}$ *is a convolution of* f *and* g. *And* $h = f * g$.

Remark. *Convolution operator is commutative.*
ie, $h = f * g = g * f$

Remark. *Suppose* f, g *vanishes on negative real axis, then*

$$h(x) = \int_{-\infty}^\infty f(t)\ g(x-t)\ dt = \int_{-\infty}^0 + \int_0^x + \int_x^\infty f(t)\ g(x-t)\ dt = \int_0^x f(t)\ g(x-t)\ dt$$

Remark. *Singularity is a point at which the convolution integral fails to exists.*

Theorem 16.5. *Let $f, g \in L(\mathbb{R})$ and either f or g is bounded in $\mathbb{R}$. Then the convoluton integral*

$$h(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt$$

exists for every $x \in \mathbb{R}$ and the function h so defined is bouned in $\mathbb{R}$. In addition, if the bounded function is continuous on $\mathbb{R}$, then h is continuous and $h \in L(\mathbb{R})$.

Synopsis.

Proof. □

Remark. *If f, g are both unbounded, the convolution integral may not exist.*

$$\text{eg: } f(t) = \frac{1}{\sqrt{t}}, \; g(t) = \frac{1}{\sqrt{1-t}}$$

Theorem 16.6. *Let $f, g \in L^2(\mathbb{R})$. Then the convolution integral $f * g$ exists for each $x \in \mathbb{R}$ and the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = f * g(x)$ is bounded in $\mathbb{R}$.*

Synopsis.

Proof. □

16.7 The Convolution Theorem for Fourier Tranforms

Theorem 16.7. *Let $f, g \in L(\mathbb{R})$ and either f or g is continuous and bounded on $\mathbb{R}$. Let $h = f * g$. Then for every real u,*

$$\int_{-\infty}^{\infty} h(x)e^{-ixu}dx = \left(\int_{-\infty}^{\infty} f(t)e^{-itu}dt\right)\left(\int_{-\infty}^{\infty} g(y)e^{-iyu}dy\right)$$

The integral on the left exists both as a Lebesgue integral and an improper Riemann integral.

Synopsis.

Proof. □

Remark (Application of Convolution Theorem)**.**

$$B(p,q) = \frac{\Gamma p \Gamma q}{\Gamma p + q}, \text{ where } B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \; \Gamma p = \int_0^{\infty} t^{p-1}e^{-t}dt$$

Chapter 17

Multivariate Differential Calculus

In this chapter, we deal with real functions of several variables. Instead of **c**, we write $\overline{c} \in \mathbb{R}^n$, then $\overline{c} = (c_1, c_2, \dots, c_n)$ where $c_j \in \mathbb{R}$ for every $j = 1, 2, \dots, n$. Again, suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ and $f(\overline{x}) = \overline{y}$, then $\overline{y} = (y_1, y_2, \dots, y_m)$ where each y_k is real. The unit co-ordinate vector, $\overline{u_k}$ is given by $u_{kj} = \delta_{j,k}$

17.1 Directional Derivative

Motivation : The existence of all partial derivatives of a multivariate real function f at a point $\overline{c}$ doesn't imply the continuity of f at $\overline{c}$. Thus, we need a suitable generalisation for the partial derivative which could characterise continuity. And directional derivative is such an attempt.

Definitions 17.1 (Directional Derivative)**.** *Let $S \subset \mathbb{R}^n$ and $f : S \to \mathbb{R}^m$. Let $\overline{c}$ be an interior points of S and $\overline{u} \in \mathbb{R}^n$, then there exists an open ball $B(\overline{c}, r)$ in S. Also for some $\delta > 0$ the line segment $\alpha : [0, \delta] \to S$ given by $\alpha(t) = \overline{c} + t\overline{u}$ lie in $B(\overline{c}, r)$.*

Then the Directional derivative of f at an interior point $\overline{c}$ in the direction $\overline{u}$ is given by

$$f'(\overline{c}, \overline{u}) = \lim_{h \to 0} \frac{f(\overline{c} + h\overline{u}) - f(\overline{c})}{h}$$

Remark. *The direction derivative of f at an interior point $\overline{c}$ in the direction $\overline{u}$ exists only if the above limit exists.*

Remark. *Example, [apo, 1973, Exercise 12.2a]*
Suppose $\overline{x}, \overline{a}, \overline{c}, \overline{u} \in \mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(\overline{x}) = \overline{a} \cdot \overline{x}$. Then

$$f'(\overline{c}, \overline{u}) = \lim_{h \to 0} \frac{\overline{a} \cdot (\overline{c} + h\overline{u}) - \overline{a} \cdot \overline{c}}{h} = \overline{a} \cdot \overline{u}$$

Remark (Properties)**.** *Let $f : S \to \mathbb{R}^m$, where $S \subset \mathbb{R}^n$*

1. $f'(\overline{c}, \overline{0}) = \overline{0}$
   Note : The zero vectors belongs to $\mathbb{R}^n, \mathbb{R}^m$ respectively.

2. *$f'(\overline{c}, \overline{u_k}) = \frac{\partial f}{\partial u_k}(\overline{c}) = D_k f(\overline{c})$, the k^{th} partial derivative of f.*

3. *Let $f = (f_1, f_2, \cdots, f_m)$, such that $f(\overline{c}) = (f_1(\overline{c}), f_2(\overline{c}), \ldots, f_m(\overline{c}))$. Then,*

$$\exists f'(\overline{c}, \overline{u}) \iff \forall k, \exists f'_k(\overline{c}, \overline{u}) \text{ and } f'(\overline{c}, \overline{u}) = (f'_1(\overline{c}, \overline{u}), f'_2(\overline{c}, \overline{u}), \ldots, f'_m(\overline{c}, \overline{u}))$$

ie, Directional derivative of f exists iff directional derivative of each component function f_k exists. And the components of the directional derivatives of f are the directional derivaties of the components of f.
Thus $D_k f(\overline{c}) = (D_k f_1(\overline{c}), D_k f_2(\overline{c}), \ldots, D_k f_m(\overline{c}))$ holds.

4. *Let $F(t) = f(\overline{c} + t\overline{u})$, then $F'(0) = f'(\overline{c}, \overline{u})$ and $F'(t) = f'(\overline{c} + t\overline{u}, \overline{u})$*

5. *Let $f(\overline{c}) = \overline{c} \cdot \overline{c} = \|\overline{c}\|^2$, and $F(t) = f(\overline{c} + t\overline{u})$, then $F'(t) = 2\overline{c} \cdot \overline{u} + 2t\|\overline{u}\|^2$ and $F'(0) = f'(\overline{c}, \overline{u}) = 2\overline{c} \cdot \overline{u}$*

6. *Let f be linear, then $f'(\overline{c}, \overline{u}) = f(\overline{u})$*

7. *Existence of all partial derivatives doesn't imply existence of all directional derivatives.*

$$f(x, y) = \begin{cases} x + y & \text{if } x = 0 \text{ or } y = 0 \\ 1 & \text{otherwise} \end{cases}$$

For above f, directional derivatives exists only along the co-ordinates (ie, partial derivatives).

8. *Existence of all directional derivatives doesn't imply continuity.*

$$f(x, y) = \begin{cases} xy^2(x^2 + y^4) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Above f is discontinuous at $(0, 0)$, however all directional derivatives exists and has finite value.

17.2 Total Derivative

We may define a total derivative $T_c(h) = hf'(c)$ in the case of real-functions of single variable as follows :-

$$\text{Let } E_c(h) = \begin{cases} \frac{f(c+h) - f(c)}{h} - f'(c), & h \neq 0 \\ 0, & h = 0 \end{cases}$$

Then, $f(c + h) = f(c) + hf'(c) + hE_c(h)$ and as $h \to 0$, $E_c(h) \to 0$. Also $T_c(h) = f'(c)h$ is a linear function of h. ie, $T_c(ah_1 + bh_2) = aT_c(h_1) + bT_c(h_2)$. Now, we will define a total derivative of multivariate function that has these two properties.

Definitions 17.2 (Total Derivative)**.** *The function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\overline{c}$ if there exists a **linear** function $T_{\overline{c}} : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(\overline{c} + \overline{v}) = f(\overline{c}) + T_{\overline{c}}(\overline{v}) + \|\overline{v}\| E_{\overline{c}}(\overline{v})$ where $E_{\overline{c}}(\overline{v}) \to \overline{0}$ as $\overline{v} \to \overline{0}$.*

The linear function $T_{\overline{c}}$ is the total derivative of f at $\overline{c}$, $T_{\overline{c}}(\overline{0}) = \overline{0}$ and the condition above gives the First Order Taylor's Formula for $f(\overline{c}+\overline{v}) - f(\overline{c})$.

Remark (Properties). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $f'(\overline{c})(\overline{v}) = T_{\overline{c}}(\overline{v})$ be the total derivative of f at $\overline{c}$ evaluated at $\overline{v}$. Then,*

1. $f'(\overline{c})(\overline{v}) = f'(\overline{c}, \overline{u})$
2. *If f is differentiable at $\overline{c}$, then f is continuous at $\overline{c}$.*
3. $f'(\overline{c})(\overline{v}) = v_1 D_1 f(\overline{c}) + v_2 D_2 f(\overline{c}) + \cdots + v_n D_n f(\overline{c})$

Note. *The above f' is a function from $\mathbb{R}^n$ to the set of all linear functions $\mathscr{L} = \{h : \mathbb{R}^n \to \mathbb{R}^m\}$. $f'(\overline{c})$ is a linear function (in fact, total derivative $T_{\overline{c}}$) which maps $\overline{v}$ into the directional derivatives of f at $\overline{c}$ in the direction $\overline{v}$. This notation generalises f' for univariate f as well.(put $n = m = 1$)*

In this subject, we use the following notations,

$D_k f(\overline{c})$ *partial derivative*

$f'(\overline{c}, \overline{v})$ *directional derivative*

$f'(\overline{c})(\overline{v})$ *total derivative*

Theorem 17.3. *If f is differentiable at $\overline{c}$ with total derivative $T_{\overline{c}}$, then for every $\overline{u} \in \mathbb{R}^n$, $T_{\overline{c}}(\overline{u}) = f'(\overline{c}, \overline{u})$. (ie, $f'(\overline{c})(\overline{v}) = f'(\overline{c}, \overline{v})$)*

Proof. For $\overline{v} = \overline{0}$, we have $T_{\overline{c}}(\overline{0}) = 0 = f'(\overline{c}, \overline{0})$.

Suppose $\overline{v} \neq \overline{0}$, then put $\overline{v} = h\overline{u}$. Since f is differentiable at $\overline{c}$, f has total derivative at $\overline{c}$. That is, there exists a linear function $T_{\overline{c}}$ such that $f(\overline{c} + h\overline{u}) = f(\overline{c}) + T_{\overline{c}}(h\overline{u}) + \|h\overline{u}\| E_{\overline{c}}(h\overline{u})$ where $E_{\overline{c}}(h\overline{u}) \to \overline{0}$ as $h\overline{u} \to \overline{0}$.

$$\begin{aligned}
&\implies f(\overline{c}+h\overline{u}) = f(\overline{c}) + hT_{\overline{c}}(\overline{u}) + |h|\|\overline{u}\|E_{\overline{c}}(h\overline{u}),\ E_{\overline{c}}(h\overline{u}) \to \overline{0} \text{ as } h\overline{u} \to \overline{0}\\
&\implies \frac{f(\overline{c}+h\overline{u}) - f(\overline{c})}{h} = T_{\overline{c}}(\overline{u}) + \frac{|h|\|\overline{u}\|E_{\overline{c}}(h\overline{u})}{h},\ E_{\overline{c}}(h\overline{u}) \to \overline{0} \text{ as } h \to 0\\
&\implies \lim_{h\to 0} \frac{f(\overline{c}+h\overline{u}) - f(\overline{c})}{h} = T_{\overline{c}}(\overline{u}) + \lim_{h\to 0} \frac{|h|\|\overline{u}\|E_{\overline{c}}(h\overline{u})}{h}\\
&\implies f'(\overline{c}, \overline{u}) = T_{\overline{c}}(\overline{u})
\end{aligned}$$

□

Note. *$T_{\overline{c}}$ is linear, however $E_{\overline{c}}$ is not linear. Thus $E_{\overline{c}}(h\overline{u}) \neq hE_{\overline{c}}(\overline{u})$.*

As $h \to 0$, $h\overline{u} \to \overline{0}$ and $E_{\overline{c}}(h\overline{u}) \to \overline{0}$. Since the order of the function $E_{\overline{c}}(h\overline{u})$ is much smaller than that of h, the limit on the right converges to 0.

Theorem 17.4. *If f is differentiable at $\overline{c}$, then f is continuous at $\overline{c}$.*

Proof. Let $\overline{v} \neq 0$, then

$$\begin{aligned}
\overline{v} &= v_1\overline{u_1} + v_2\overline{u_2} + \cdots + v_n\overline{u_n},\\
\overline{v} \to \overline{0} &\implies \forall j,\ v_j \to 0\\
T \text{ is linear } &\implies T_{\overline{c}}(\overline{v}) = v_1 T_{\overline{c}}(\overline{u_1}) + v_2 T_{\overline{c}}(\overline{u_2}) + \cdots + v_n T_{\overline{c}}(\overline{u_n})\\
&\text{Thus, } T_{\overline{c}}(\overline{v}) \to \overline{0} \text{ as } \overline{v} \to 0
\end{aligned}$$

Since f differentiable at $\overline{c}$, there exists linear function $T_{\overline{c}}$ such that

$$\begin{aligned} f(\overline{c}+\overline{v}) &= f(\overline{c}) + T_{\overline{c}}(\overline{v}) + \|v\|E_{\overline{c}}(\overline{v}) \\ \implies \lim_{\overline{v}\to\overline{0}} f(\overline{c}+\overline{v}) &= f(\overline{c}) + \lim_{\overline{v}\to\overline{0}} T_{\overline{c}}(\overline{v}) + \lim_{\overline{v}\to\overline{0}} \|v\|E_{\overline{c}}(\overline{v}) \\ \implies \lim_{\overline{v}\to\overline{0}} f(\overline{c}+\overline{v}) &= f(\overline{c}) \end{aligned}$$

□

Theorem 17.5. *Let $S \subset \mathbb{R}^n$ and $f : S \to \mathbb{R}^m$* *–continue page 345–*

Proof. □

Chapter 18

Implicit Functions and Extremum Problems

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