Part I

ME800402 Algorithmic Graph Theory

Chapter 1

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Networks

5.1 An Introduction to Networks

Definitions 5.1. *A network N is a digraph D with two special vertices source s and sink t together with a capacity function $c : E(D) \to \mathbb{Z}$ such that for every arc $a = (u, v)$ of the digraph, $c(u, v)$ is non-negative.*

Remark. *Mathematical Modeling using Network,*

1. *There is no restriction on indegree/outdegree of source/sink vertices of the digraph D of a network N.*

2. *Applications of Network : Transportation problem.*

$c(u, v)$ is the capacity of the arc (u, v) of D

$N^+(x) = \{y \in V(D) : (x, y) \in E(D)\}$ is the out-neighbourhood of x.

$N^-(x) = \{y \in V(D) : (y, x) \in E(D)\}$ is the in-neighbourhood of x.

Definitions 5.2. *A flow f in a network N is function $f : E(D) \to \mathbb{Z}$ such that 1. each edge satisfies capacity constraint and 2. each vertex except source and sink satisfies conservation equation.*

capacity constraint

$$0 \leq f(a) \leq c(a) \text{ for every arc } a \in V(D) \tag{5.1}$$

conservation equation

$$\sum_{y \in N^+(x)} f(x, y) = \sum_{y \in N^-(x)} f(y, x), \ \forall \text{vertex } x \in V(D) - \{s, t\} \tag{5.2}$$

net flow out of x

$$\sum_{y \in N^+(x)} f(x, y) - \sum_{y \in N^-(x)} f(y, x)$$

net flow into x

$$\sum_{y \in N^-(x)} f(y, x) - \sum_{y \in N^+(x)} f(x, y)$$

Definitions 5.3. *The flow f in a network N is the net flow out of source s.*

Remark.   *1. net flow out of/into $x \in V(D) - \{s, t\}$ is zero.*

2. Without loss of generality[1], underlying digraph is always assymetric.

$(X, Y) = \{(x, y) \in E(D) : x \in X, \ y \in Y\}$.
    Let X, Y be non-empty subsets of $V(D)$ such that X, Y are disjoint. Then (X, Y) is the set of all arcs from X to Y.

flow from X to Y is the sum of flow on each arc in (X, Y)

$$f(X, Y) = \sum_{(x,y) \in (X,Y)} f(x, y) \tag{5.3}$$

capacity of the partition (X, Y) is the total capacity of arcs in (X, Y)

$$c(X, Y) = \sum_{(x,y) \in (X,Y)} c(x, y) \tag{5.4}$$

cut Let $P \subset V(D)$ such that $s \in P$ and $t \notin P$ and $\bar{P} = V(D) - P$, then $(P, \bar{P})$ is a cut.

flow from P to $\bar{P}$ is the sum of flow on each arc in $(P, \bar{P})$.

$$f(P, \bar{P}) = \sum_{(x,y) \in (P,\bar{P})} f(x, y) \tag{5.5}$$

flow from $\bar{P}$ to P is the sum of flow on each arc in $(\bar{P}, P)$

$$f(\bar{P}, P) = \sum_{(x,y) \in (\bar{P},P)} f(x, y) \tag{5.6}$$

capacity of the cut $(P, \bar{P})$ is the total capacity of the arcs in $(P, \bar{P})$

$$c(P, \bar{P}) = \sum_{(x,y) \in (P,\bar{P})} c(x, y) \tag{5.7}$$

Theorem 5.4. *For any cut $(P, \bar{P})$, the flow in N is $f(N) = f(P, \bar{P}) - f(\bar{P}, P)$.*

Synopsis. *The net flow out of source s is the flow $f(N)$ in the network N. Let $(P, \bar{P})$ be a cut of N, then $s \in P$ and $t \notin P$. Suppose $P = \{s\}$, then the theorem is true. Suppose P is not singleton, then for each vertex $x \in P$, $x \neq s$, the net flow out of x is zero by flow conservation equation. And flow between vertices in P cancels out each other. Thus adding net flow out of each vertex in P, will be same as the net flow out of source which is the flow in the network, $f(N)$.*

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[1]If underlying digraph of a network is symmetric, then by replacing an arc (u, v) with a new vertex w and two arcs $(u, w), (w, v)$ gives an assymetric digraph.[Gray Chartrand,]pp.131

Proof.

$$\text{Flow, } f = \sum_{y \in N^+(s)} f(s,y) - \sum_{y \in N^-(s)} f(y,s) \tag{5.8}$$

By conservation equation, we have $\forall x \in P, \ x \neq s,$

$$\sum_{y \in N^+(x)} f(x,y) - \sum_{y \in N^-(x)} f(y,x) = 0 \tag{5.9}$$

By above equations,

$$\text{Flow, } f = \sum_{x \in P} \sum_{y \in N^+(x)} f(x,y) - \sum_{x \in P} \sum_{y \in N^-(x)} f(y,x)$$
$$= \sum_{(x,y) \in (P,\bar{P})} f(x,y) - \sum_{(y,x) \in (\bar{P},P)} f(y,x) \tag{5.10}$$

$\square$

Corollary 5.4.1. *Flow cannot exceed the capacity of any cut $(P,\bar{P})$. Further, $f(N) \leq \min c(P,\bar{P})$.*

Synopsis. *Let $(P,\bar{P})$ be a cut in network N, then by theorem the flow $f(N) = $ flow from P to $\bar{P}$ - flow from $\bar{P}$ to P. Since the flow from $\bar{P}$ to P is non-negative, $f(N) \leq$ flow from P to $\bar{P}$. Clearly, $f(x,y) \leqq c(x,y)$ by the capacity constaint. Thus $f(N) \leq f(P,\bar{P}) \leq c(P,\bar{P}) \leq \min c(P,\bar{P})$.*

Proof.

$$f(N) = \sum_{(x,y) \in (P,\bar{P})} f(x,y) - \sum_{(y,x) \in (\bar{P},P)} f(y,x)$$
$$\leq \sum_{(x,y) \in (P,\bar{P})} f(x,y) = f(P,\bar{P})$$
$$\leq \sum_{(x,y) \in (P,\bar{P})} c(x,y) = c(P,\bar{P}), \quad \because \forall x,y \in V(D), \ f(x,y) \leq c(x,y)$$
$$\leq \min c(P,\bar{P})$$

$\square$

Corollary 5.4.2. *In a network N flow is the net flow into the sink of N.*

Synopsis. *Let $\bar{P} = \{t\}$, then by theorem $f(N)$ is the net flow into the sink.*

Proof. Suppose $P = V(D) - \{t\}$. Then by theorem, we have

$$f(N) = \sum_{(x,y) \in (P,\bar{P})} f(x,y) - \sum_{(y,x) \in (\bar{P},P)} f(y,x)$$
$$= \sum_{x \in N^-(t)} f(x,t) - \sum_{x \in N^+(t)} f(t,x)$$

$\square$

Remark. *Exercise 5.1*

    4. Let N be a network with underlying digraph D which has a vertex $v \in V(D) - \{s,t\}$ with zero indegree. Clearly the flow into v is zero. Thus flow out of v is also zero by flow conservation equation. Let N' be the network obtained from N by deleting the vertex v. Then $f(N) = f(N')$.

5.2 The Max-Flow Min-Cut Theorem

Definitions 5.5. maximum flow *A flow f in network N is maximum flow in N, if $f(N) \geq f'(N)$ for each flow f' in N.*

minimum cut *A cut $(P, \bar{P})$ in network N is minimum cut of N, if $c(P, \bar{P}) \leq c(X, \bar{X})$ for each cut $(X, \bar{X})$ in N.*

f-unsaturated *Let f be a flow in network N with underlying digraph D, and $Q = u_0, a_1, u_1, a_2, \cdots, u_{n-1}, a_n, u_n$ be a semipath in D such that every forward arc $a_i = (u_{i-1}, u_i)$ has flow not upto its capacity, $f(a_i) < c(a_i)$ and every reverse arc $a_i = (u_i, u_{i-1})$ has some positive flow in it, $f(a_i) > 0$*

f-augmenting semipath *Let f be a flow in a network N with underlying digraph D. Suppose semipath $Q = s, a_1, u_1, a_2, \cdots, u_{n-1}, a_n, t$ (from source to sink) is f-unsaturated, then Q is an f-augmenting semipath.*

Theorem 5.6. *Let f be a flow in a network N with underlying digraph D. The flow f is maximum in N iff there is no f-augmenting semipath in D.*

Synopsis. *Suppose Q is an f-augmenting semipath in D, then there exists a flow f^* in N such that $f(N) + \Delta = f^*(N)$. Therefore, f is not a maximum flow in N. Suppose there is no f-augmenting semipath in D, then there exists a cut $(P, \bar{P})$ such that $f(a) = c(a) \ \forall a \in (P, \bar{P})$ and $f(a) = 0 \ \forall a \in (\bar{P}, P)$. Suppose f^* in a maximum flow in N, then $f(N) \leq f^*(N) \leq c(P, \bar{P}) = f(N)$.*

Proof. Let f be a flow in a network N with underlying digraph D and $Q = s, a_1, u_1, a_2, u_2, \cdots, u_{n-1}, a_n, t$ be an f-augmenting semipath in D.

$$\text{define } \Delta_i = \begin{cases} c(a_i) - f(a_i) \text{ for every forward arc } a_i \in Q, \\ f(a_i) \text{ for every reverse arc } a_i \in Q, \end{cases}$$

Define $\Delta = \min\{\Delta_i\}$. Also define $f^* : E(D) \to \mathbb{Z}$ such that

$$f^*(a_i) = \begin{cases} f(a) + \Delta, \text{ for every forward arc } a_i \in Q, \\ f(a) - \Delta, \text{ for every reverse arc } a_i \in Q, \\ f(a_i), \text{ for every arc of } D \text{ which are not in } Q. \end{cases}$$

Since Q is an f-augmenting semipath in D, $\Delta > 0$ and $f(N) + \Delta = f^*(N)$.

Clearly $f(N) < f^*(N)$, and it is enough to show that f^* is a flow in N. f^* is a flow if it satisfies 1. capacity constraint and 2. conservation equation. For any arc $a_i \notin Q$, $f^*(a_i) = f(a_i) \leq c(a_i)$. Suppose $a_i \in Q$. If $a_i = (u_{i-1}, u_i)$, a_i is a forward arc and we have $f^*(a_i) = f(a_i) + \Delta \leq f(a_i) + \Delta_i = f(a_i) + c(a_i) - f(a_i) = c(a_i)$. If $a_i = (u_i, u_{i-1})$, then a_i is a reverse arc and we have $f^*(a_i) = \Delta \leq \min\{\Delta_i\} = \Delta_i = c(a_i)$. Thus f^* satisfies capacity constraint on every arc of D.

Let $x \in V(D) - \{s, t\}$. Suppose $x \notin Q$,

$$\text{Net flow out of } x = \sum_{y \in N^+(x)} f^*(x, y) - \sum_{y \in N^-(x)} f^*(y, x)$$

$$= \sum_{y \in N^+(x)} f(x, y) - \sum_{y \in N^-(x)} f(y, x)$$

$$= 0$$

Suppose $x = u_i \in Q$, then Q has two arc having vertex x say, a_{i-1}, and a_i. There are four possibilities for these two arcs,

1. Both a_{i-1}, a_i are forward arcs.

2. Arc a_{i-1} is forward, but arc a_i is reverse.

3. Arc a_{i-1} is reverse, but arc a_i is forward.

4. Both a_{i-1}, a_i are reverse arcs.

Case 1 $a_{i-1} = (u_{i-1}, u_i)$ and $a_i = (u_i, u_{i+1})$.

$$\text{Net flow out of } x = \sum_{y \in N^+(x)} f^*(x, y) - \sum_{y \in N^-(x)} f^*(y, x)$$

$$= \sum_{\substack{y \in N^+(x) \\ y \neq u_{i+1}}} f^*(x, y) + f^*(u_i, u_{i+1}) - \left(\sum_{\substack{y \in N^-(x) \\ y \neq u_{i-1}}} f^*(y, x) + f^*(u_{i-1}, u_i) \right)$$

$$= \sum_{\substack{y \in N^+(x) \\ y \neq u_{i+1}}} f(x, y) + f(u_i, u_{i+1}) + \Delta - \left(\sum_{\substack{y \in N^-(x) \\ y \neq u_{i-1}}} f(y, x) + f(u_{i-1}, u_i) \right) - \Delta$$

$$= \sum_{y \in N^+(x)} f(x, y) - \sum_{y \in N^-(x)} f(y, x)$$

$$= 0$$

Case 2 $a_{i-1} = (u_{i-1}, u_i)$ and $a_i = (u_{i+1}, u_i)$.

$$\text{Net flow out of } x = \sum_{y \in N^+(x)} f^*(x, y) - \sum_{y \in N^-(x)} f^*(y, x)$$

$$= \sum_{\substack{y \in N^+(x) \\ y \neq u_{i+1}, u_{i-1}}} f^*(x, y) + f^*(u_i, u_{i+1}) + f^*(u_i, u_{i-1}) - \sum_{y \in N^-(x)} f^*(y, x)$$

$$= \sum_{\substack{y \in N^+(x) \\ y \neq u_{i+1}, u_{i-1}}} f(x, y) + f(u_i, u_{i+1}) + \Delta + f(u_i, u_{i-1}) - \Delta - \sum_{y \in N^-(x)} f(y, x)$$

$$= \sum_{y \in N^+(x)} f(x, y) - \sum_{y \in N^-(x)} f(y, x)$$

$$= 0$$

Case 3 $a_{i-1} = (u_i, u_{i-1})$ and $a_i = (u_i, u_{i+1})$.

$$
\text{Net flow out of } x = \sum_{y \in N^+(x)} f^*(x,y) - \sum_{y \in N^-(x)} f^*(y,x)
$$

$$
= \sum_{y \in N^+(x)} f^*(x,y) - \left(\sum_{\substack{y \in N^-(x) \\ y \neq u_{i-1}, u_{i+1}}} f^*(y,x) + f^*(u_{i-1}, u_i) + f^*(u_{i+1}, u_i) \right)
$$

$$
= \sum_{y \in N^+(x)} f^*(x,y) - \left(\sum_{\substack{y \in N^-(x) \\ y \neq u_{i-1}, u_{i+1}}} f(y,x) + f(u_{i-1}, u_i) + \Delta + f(u_{i+1}, u_i) - \Delta \right)
$$

$$
= \sum_{y \in N^+(x)} f(x,y) - \sum_{y \in N^-(x)} f(y,x)
$$

$$
= 0
$$

Case 4 $a_{i-1} = (u_i, u_{i-1})$ and $a_i = (u_{i+1}, u_i)$.

$$
\text{Net flow out of } x = \sum_{y \in N^+(x)} f^*(x,y) - \sum_{y \in N^-(x)} f^*(y,x)
$$

$$
= \sum_{\substack{y \in N^+(x) \\ y \neq u_{i-1}}} f^*(x,y) + f^*(u_i, u_{i-1}) - \left(\sum_{\substack{y \in N^-(x) \\ y \neq u_{i+1}}} f^*(y,x) + f^*(u_{i+1}, u_i) \right)
$$

$$
= \sum_{\substack{y \in N^+(x) \\ y \neq u_{i-1}}} f(x,y) + f(u_i, u_{i-1}) - \Delta - \left(\sum_{\substack{y \in N^-(x) \\ y \neq u_{i+1}}} f(y,x) + f(u_{i+1}, u_i) \right) + \Delta
$$

$$
= \sum_{y \in N^+(x)} f(x,y) - \sum_{y \in N^-(x)} f(y,x)
$$

$$
= 0
$$

Therefore, f^* is a flow on N. We have $f(N) < f^*(N)$. Thus f is not maximum flow in N due to the existence of an f-augmenting semipath in D.

Conversely, assume that there is no f-augmenting semipath in D. Now, we construct a cut $(P, \bar{P})$ of N. Let P be the set of all vertices $x \in V(D)$ such that there is an f-unsaturated $s - x$ semipath in D. Trivially, $s \in P$. And $t \notin P$ since there are no f-augmenting semipath in D.[2] Clearly, $(P, \bar{P})$ is a cut of the network N.

We claim that $c(P, \bar{P}) = f(N)$. Suppose there is a forward arc $(x, y) \in (P, \bar{P})$, then flow in it is saturated. If $f(x, y) < c(x, y)$, then there is an f-unsaturated $s - y$ semipath in D. ie, $s - x$ semipath + arc (x, y). Thus every forward arc $(x, y) \in (P, \bar{P})$ is saturated. Suppose there is a reverse arc $(y, x) \in$

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[2]An f-augmenting semipath is an f-unsaturated $s - t$ semipath in D.

$(\bar{P}, P)$, then there is no flow in it(saturated reversed arc). If $f(y, x) > 0$, then there is an f-unsaturated $s - y$ semipath in D. ie, $s - x$ semipath + arc (y, x). Thus every reverse arc $(y, x) \in (\bar{P}, P)$ is saturated. And we have,

$$\sum_{(x,y)\in(P,\bar{P})} f(x,y) = \sum_{(x,y)\in(P,\bar{P})} c(x,y)$$

$$\sum_{(y,x)\in(\bar{P},P)} f(y,x) = 0$$

$$f(N) = \sum_{(x,y)\in(P,\bar{P})} f(x,y) - \sum_{(y,x)\in(\bar{P},P)} f(y,x)$$

$$= \sum_{(x,y)\in(P,\bar{P})} c(x,y)$$

$$= c(P, \bar{P})$$

Suppose f^* is maximum flow in network N and $(X, \bar{X})$ is minimum cut of N. Then $f(N) \leq f^*(N)$. Thus we have, $f(N) \leq f^*(N) \leq c(X, \bar{X}) \leq c(P, \bar{P}) = f(N)$. Therefore, $f(N) = f^*(N)$. ie, the flow f is maximum in network N if there are no f-augmenting semipaths in D. $\qquad\square$

Theorem 5.7 (maximum-flow, min-cut). *In every network, the value of maximum flow equals capacity of minimum cut.*

Proof. Suppose flow f in network N in maximum, then by previous theorem there is no f-augmenting semipath in D. And $f(N) \leq c(X, \bar{X})$ for any cut $(X, \bar{X})$ in N. We can construct a cut $(P, \bar{P})$ in N such that $f(N) = c(P, \bar{P})$. Let P be the set of all vertices x in D such that there is an f-unsaturated $s - x$ semipath in D. Clearly $s \in P$ and $t \notin P$. Also $f(P, \bar{P}) = c(P, \bar{P})$ and $f(\bar{P}, P) = 0$. Then the cut $(P, \bar{P})$ is minimum cut of N. Suppose there is a cut $(X, \bar{X})$ such that $c(X, \bar{X}) < c(P, \bar{P})$. Then $f(N) = f(P, \bar{P}) - f(\bar{P}, P) = c(P, \bar{P}) < c(X, \bar{X})$ which is a contradiction. Therefore, the value of maximum flow equals capacity of minimum cut. $\qquad\square$

Remark. *Exercise 5.2*

1. *Suppose $(X, \bar{X})$ is a cut of N such that $f(a) = c(a)$, $\forall a \in (X, \bar{X})$ and $f(a) = 0$, $\forall a \in (\bar{X}, X)$. By the definition of cut, $s \in X$ and $t \in \bar{X}$. Thus there is no f-augmenting semipath in D. Suppose there is an f-augmenting semipath Q in D, then there is either (a) a forward arc $(x, y) \in (X, \bar{X})$ such that $f(x, y) < c(x, y)$ or (b) a reverse arc $(y, x) \in (\bar{X}, X)$ such that $f(y, x) > 0$ which is a contradition. Therefore, the flow $f(N)$ is maximum and the given cut $(X, \bar{X})$ is minimum as shown in the proof of the maximum-flow min-cut theorem.*

3. *The algorithm suggested in the hint of this exercise won't work if two sub-networks have a common arc such that the direction of flow in which is not consistent. Suppose, the generalized network is not supposed to have any common arcs. Then construct subnetworks for each pair (s, t) with all those arcs which are on some $s - t$ semipath. Define subnetwork capacity function $c'(a) = c(a)$ for every arc in N'.*

Let N be a generalized network with set of sources S and set of sinks T. A flow in N is maximum if there is not f-augmenting $s - t$ semipath for each pair $(s, t) \in S \times T$. Does there exists a generalised network where max-flow min-cut algorithm is inconsistent ? $\star$ I don't know

5.3   A max-flow min-cut algorithm

Theorem 5.8. *Let N be a network with underlying digraph D, source s, sink t, capacity function c and flow f. Let D' be the digraph with same vertex set as D and arc set defined by $E(D') = \{(x, y) : (x, y) \in E(D),\ c(x, y) > f(x, y)$ or $(y, x) \in E(D),\ f(y, x) > 0\}$. ie, D' has only the unsaturated arcs of D. Then D' has an $s - t$ directed path iff D has an f-augmenting semipath. Moreover, shortest $s - t$ path in D' has the same length as shortest f-augmenting semipath in D.*

Synopsis. *Each directed $s - t$ path in D' has respective f-augmenting semipath in D and vice versa. Clearly, they have the same length.*

Proof. Let N be a network with underlying digraph D, capacity c and flow f. Let D' be the digraph with vertex set $V(D') = V(D)$ and arc set $E(D') = \{(x, y) : \text{either } (x, y) \text{ or } (y, x) \text{ is unsaturated in } N\}$.

Suppose D' has a directed $s - t$ path $Q' : s, u_1, u_2, \cdots, u_{n-1}, t$. Then by the construction of D', for each $u_i \in Q$, there exists an f unsaturated arc a_i in D. ie, either forward arc $a_i = (u_{k-1}, u_k)$ such that $f(u_{k-1}, u_k) < c(u_{k-1}, u_k)$ or reverse arc $a_i = (u_k, u_{k-1})$ such that $f(u_k, u_{k-1}) > 0$. Therefore, we have an $s - t$ semipath $Q : s, a_1, u_1, a_2, \cdots, u_{n-1}, a_n, t$ in D such that Q is an f-augmenting semipath since every arc in Q is f-unsaturated. Clearly, Q, Q' are of the same length.

Conversely, suppose that the digraph D has an f-augmenting semipath $Q : s, a_1, u_1, a_2, \cdots, u_{n-1}, a_n, t$. Then each arc $a_i \in Q$ are f-unsaturated and by the construction of D', there exists a directed $s - t$ path $Q' = s, u_1, u_2, \cdots, u_{n-1}, t$ in D'. And Q, Q' are of the same length.

There is a one-one correspondence betweeen the directed $s - t$ paths in D' and f-augmenting semipaths in D. Clearly, they have the same length. Thus shortest directed $s - t$ path in D and shortest f-augmenting semipath in D' are of the same length. $\square$

saturation arc of N with respect to the flow f is an arc a_j in an f-augmenting semipath Q with $\Delta_j = \Delta$.

augmentation path is an f-augmenting semipath Q in D.

Algorithm 5.9 (max-flow min-cut). *An algorithm to find maximum flow and minimum cut of a network N with underlying digraph D, source s, sink t, capacity function c and initial flow f.*

1. *Construct digraph D' with vertex set $V(D') = V(D)$ and arc set $E(D') = \{(x, y) : (x, y) \in E(D)\&f(x, y) < c(x, y) \text{ or } (y, x) \in E(D)\&f(y, x) > 0\}$*

2. *Find (shortest) $s-t$ directed path in D' using Moore's breadth first search(BFS) algorithm. If D' doesn't have an $s-t$ path, then proceed to step 5. Otherwise, let $Q' : s, u_1, u_2, \cdots, u_{n-1}, t$ be a (shortest) $s-t$ path in D'.*

3. *Let $Q : s, a_1, u_1, a_2, \cdots, u_{n-1}, a_n, t$ be the respective semipath in D such that $f(a_j) < c(a_j)$ for forward arcs and $f(a_i) > 0$ for reverse arcs. Let $\Delta_j = c(a_j) - f(a_j)$ for forward arcs and $\Delta_j = f(a_j)$ for reverse arcs. And let $\Delta = \min\{\Delta_j\}$. And augment flow f by Δ ie, $f(a_j) \leftarrow f(a_j) + \Delta$ for forward arcs and $f(a_j) \leftarrow f(a_j) - \Delta$ for reverse arcs.*

4. *Goto step 1 (Proceed with new flow f and find whether there are any directed $s-t$ paths in D'. If any, augment the flow along the new augmentation path Q by saturating the flow along the saturation arc.)*

5. *There is no $s-t$ directed path in D'. Thus there is no f-augmenting semipath in D. Therefore the flow f in N is maximum. Let P be the set of all vertices in D' with non-zero breadth first index(bfi) from Moore's BFS algorithm applied in step 2. $(P, \bar{P})$ is minimum cut of N.*

Remark. *Validity of the algorithm is proved in the previous theorem.*

5.5   Connectivity and Edge-Connectivity

edge cutset is the set U subset of $E(G)$ such that $G - U$ is disconnected.

vertex cutset is the set S subset of $V(G)$ such that $G - S$ is disconnected.

edge connectivity $\lambda(G)$ is the minimum cardinality of all edge cutsets of G.

connectivity $\kappa(G)$ is the minimum cardinality of all vertex cutsets of G.

Theorem 5.10. *For every graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$*

Proof. Suppose graph G is disconnected then $\kappa(G) = \lambda(G) = \delta(G) = 0$. Let G be a connected graph. Then G has at least one vertex v with degree $\delta(G)$. Therefore $\lambda(G) \leq \delta(G)$ since edges incident with v form an edge cutset of G and $\lambda(G)$ is the cardinality of all edge cutsets.

Let G be a graph with edge connectivity $\lambda(G) = c$. Let U be a edge cutset with cardinality c and let edge $uv \in U$. Construct a set of vertices $S \subset V(G)$ such that (S is of minimal cardinality and) for each edge in U other uv, S has a vertex incident with it. Cardinality of S is atmost $c - 1$, since we can select one vertex each for each edge in U other than u, v. If $G - S$ is a disconnected graph, then $\kappa(G) < \lambda(G)$. Suppose $G - S$ is a connected graph, then delete a non-pendent vertex u or v from $G - S$, say v. Since $G - S$ is a connected graph with a singleton edge cutset, $\{uv\}$. We have a vertex cutset $S \cup \{v\}$ of G. Therefore, $\kappa(G) \leq c = \lambda(G)$. $\qquad\square$

Theorem 5.11. *If G is a graph of diameter 2, then $\lambda(G) = \delta(G)$*

n-edge connected G is n-edge connected if $\lambda(G) \geq n$.

n connected G is n-connected if $\kappa(G) \geq n$.

Theorem 5.12. *Let G be a graph of order p and n be an integer such that $1 \leq n \leq p - 1$. If $\delta(G) \geq \frac{p+n-2}{2}$, then G is n-connected.*

connection number $c(G)$ is the smallest integer such that $2 \leq c(G) \leq p$ and every subgraph of order n in G is connected.

l-**connectivity** $\kappa_l(G)$ is minimum number of vertices whose removal will produce a disconnected graph with at least l components or a graph with fewer than l vertices.

(n, l)-**connected** A graph G is (n, l)-connected if $\kappa_l(G) \geq n$.

Remark. *Exercises 5.5*

1. $\lambda(K_{m,n}) = \kappa(K_{m,n}) = m$

8. $c(K_p) = 2$, $c(K_{m,n}) = n + 1$, $c(C_p) = p - 1$
   Every two vertices of complete graph of order p are adjacent. For complete bi-partitie graph $K_{m,n}$ such that $1 \leq m \leq n$, there exists a totally disconnected subgraph of order n. Therefore $c(K_{m,n}) \geq n + 1$. And with $n + 1$ vertices, both partitions have at least two vertices each and therefore the graph is connected and $c(K_{m,n}) \leq n + 1$. For cycle C_p, any subgraph is disconnected if two non-adjacent vertices are deleted. Therefore $c(C_p) \geq p - 1$. And C_p remains connected even after deletion of any vertex, therefore $c(C_p) \leq p - 1$.

9.
$$\delta(G) \geq \frac{p + (l-1)(n-2)}{l} \implies \kappa_l(G) \geq n$$

5.6   Menger's Theorem

Theorem 5.13. *For a non-trivial graph G, $\lambda(u, v) = M'(u, v)$ for every pair (u, v) of vertices of G.*

Corollary 5.13.1. *Graph G is n-edge connected iff every two vertices of G are connected by at least n edge disjoint paths.*

Theorem 5.14. *For every pair of non-adjacent vertices u, v in graph G, $\kappa(u, v) = M(u, v)$.*

Corollary 5.14.1. *Graph G is n-connected iff every pair of vertices of G are connected by at least n internally disjoint paths.*

Algorithm 5.15 (connectivity $\kappa(G)$)**.** .

1. *If degree of every vertex is $p-1$, then output $\kappa = p-1$ and stop. Otherwise, continue.*

2. *If G is disconnected, output $\kappa = 0$ and stop. Otherwise, continue.*

3. *$\kappa \leftarrow p$*

4. *$i \leftarrow 0$*

5. *If $i \leq \kappa$, then $i \leftarrow i + 1$ and continue. Otherwise, output κ and stop.*

6. *$j \leftarrow i + 1$*

7. (a) *If $j = p + 1$, then return to step 5. Otherwise continue.*

   (b) *If $v_i v_j \notin E(G)$, construct network N with digraph D as follows : for each vertex $v \in V(G)$, there are two vertices $v', v'' \in V(D)$ and an arc $(v', v'') \in E(D)$. And for each edge $uv \in E(G)$, there are two arcs $(u'', v), (v'', u) \in E(D)$. The capacity function is given by, $c(v', v'') = 1$ for every $v \in V(G)$ and $c(a) = \infty$ for every other arc in D. Set source $s = v_i''$ and sink $t = v_j'$ and find maximum flow in N using max-flow min-cut algorithm. Otherwise proceed to step 7d*

   (c) *If $f(N) < \kappa$, then $\kappa \leftarrow f(N)$. Otherwise, continue.*

   (d) *$j \leftarrow j + 1$ and return to step 7a*

Chapter 6

Matchings and Factorizations

Bibliography

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