Part I

ME010202 Advanced Topology

Chapter 7

Separation Axioms

7.1 Compactness and Separation Axioms

7.2 The Urysohn Characterisation of Normality

7.3 Tietze Characterisation of Normality

Chapter 8

Products and Coproducts

8.1 Cartesian Products of Families of Sets

8.2 The Product Topology

8.3 Productive Properties

Chapter 9

Embedding and Metrisation

9.1 Evaluation Functions into Products

9.2 Embedding Lemma and Tychonoff Embedding

9.3 The Urysohn Metrisation Theorem

Chapter 10

Nets and Filters

10.1 Definition and Convergence of Nets

Chapter 11

Compactness

11.1 Variations of Compactness

In this chapter, we have two other notions of compactness - countable compactness and sequential compactness.[1]

Compact A topological space is compact iff every open cover of it has a finite subcover. ([Joshi,] 6.1.1) [Heine-Borel]

Countably Compact A topological space is countably compact iff every countable, open cover of it has a finite subcover. ([Joshi,] 11.1.1)

Sequentially Compact A topological space is sequentially compact iff every sequence in it has a convergent subsequence. ([Joshi,] 11.1.8) [Bolzano-Weierstrass]

Countable compactness is a weaker notion compared to compactness.[2] However, sequentially compact and compact are not necessarily comparable.[3].

We have seen earlier that compactness has the following properties 1. compactness is weakly hereditary([Joshi,] 6.1.10) 2. compactness is preserved under continuous functions([Joshi,] 6.1.8) 3. every continuous real functions on compact space is bounded and attains its extrema([Joshi,] 6.1.6) 4. every continuous real function on a compact, metric space is uniformly continuous by Lebesgue covering lemma.([Joshi,] 6.1.7)

Countably compact spaces, Sequentially compact spaces have all the four properites listed above.

11.1.1 Countable compactness

Weakly hereditary property

A subspace $(A, \mathcal{T}_{/A})$ being countably compact doesn't imply that $(X, \mathcal{T})$ is countably compact. However, if $(X, \mathcal{T})$ is a countably compact space and A

[1]For $\mathbb{R}$, Compactness & Sequentially compactness are equivalent to the completeness axiom.
[2]Every compact space is countably compact.
[3]$\mathcal{T}_1, \mathcal{T}_2$ are non-comparable, if $\mathcal{T}_1 \not\subset \mathcal{T}_2$ and $\mathcal{T}_2 \not\subset \mathcal{T}_1$.([Joshi,] 4.2.1)

is a closed subset of X, then $(A, \mathcal{T}_{/A})$ is also a countably compact space. In other words, countably compactness is weakly hereditary.

Theorem 11.1. *Countable compactness is weakly hereditary.([Joshi,] 11.1.3)*

Synopsis. *Let A be a closed subset of countably compact space, X. If A has a countable open cover $\mathcal{U}$, then we can obtain a respective countable, open cover for X by attaching $X - A$ to the extensions of members of $\mathcal{U}$ to X. This cover has a finite subcover. Then restricting them to A, we get a finite subcover of $\mathcal{U}$.*

Proof. Suppose X is a countably compact space. And A is a closed subset of X. We need to show that A is countably compact. Without loss of generality,[4] assume that A is a proper subset of X. Then $X - A$ is a non-empty, open subset of X.

Let $\mathcal{U}$ be a countable open cover of A. Then $\mathcal{U} = \{U_1, U_2, \cdots\}$ where each element $U_k \in \mathcal{U}$ is an open subset of A. Since A is a subspace of X, every open set U_k in A is of the form $G \cap A$ for some open set G in X. Therefore, there exists open sets $V(U_k)$ for each U_k such that $A \cap V(U_k) = U_k$.[5]

Define $\mathcal{V} = \{X - A, V(U_1), V(U_2), \cdots\}$. Clearly, $\mathcal{V}$ is a countable open cover[6] of X. We have X is countably compact, thus $\mathcal{V}$ has a finite subcover, say $\mathcal{V}'$. Without loss of generality assume[7] that $X - A \in \mathcal{V}'$. Suppose $X - A \notin \mathcal{V}'$, then we can define another finite subcover $\mathcal{V}' \cup \{X - A\}$. Thus $\mathcal{V}' = \{X - A,\ V(U_{n_1}),\ V(U_{n_2}), \cdots,\ V(U_{n_k})\}$.

Then the corresponding subcover $\mathcal{U}' = \{U_{n_1}, U_{n_2}, \cdots, U_{n_k}\}$ is a finite subcover of $\mathcal{U}$. Since countable open cover $\mathcal{U}$ and closed subset A are arbitrary, every closed subset of X with relative topology is countably compact. Therefore, countable compactness is weakly hereditary. $\square$

Remark. *Proof depends on the following,*

1. *There is an extension map, $\psi : P(A) \to P(X)$ that preserve open sets (and closed sets). This ψ is an open map which not a true inverse of the restriction, $r : P(X) \to P(A)$, defined by $r(G) = G \cap A$ for every subset G of X.*
2. *Also we rely on the subset A being closed. Suppose X have many countable open covers, but X has only uncountable open covers corresponding to a particular countable open cover of A. In such a case, X being countably compact is insufficient for A to be countably compact.*

The behaviour of countinous functions

We will now study the nature of continuous functions defined on countably compact spaces. Suppose X, Y are topological space and function $f : X \to Y$ is continuous. If X is countably compact, then $f(X)$ is also countably compact.

[4] Suppose A is not a proper subset of X. Then $X = A$ and A is countably compact.
[5] Relative topology, $\mathcal{T}_{/A} = \{G \cap A : G \in \mathcal{T}\}$
[6] $X - A$ is open in X. If $y \notin A$, then $y \in X - A$. If $y \in A$, then $y \in U_k$ for some k.
[7] Otherwise, you will have to consider two cases: $X - A \in \mathcal{V}'$ and $X - A \notin \mathcal{V}'$

Continuous images of countably compact spaces are countably compact. In other words, countable compactness is preserved under continuous functions.[8]

Theorem 11.2. *Countable compactness is preserved under continuous functions. ([Joshi,] 11.1.2)*

Synopsis. *Let X be countably compact and $f : X \to Y$ be continuous. Suppose $\mathcal{U}$ is a countable cover of $f(X)$, then X has a countable cover $\mathcal{V}$ obtained by taking inverse images. Since X is countably compact, $\mathcal{V}$ has a finite subcover $\mathcal{V}'$. Now taking images of members of $\mathcal{V}'$, we get a finite subcover $\mathcal{U}'$ of $f(X)$.*

Proof. Suppose X is a countably compact space, Y is a topological space and $f : X \to Y$ is a continuous function. Let $\mathcal{U} = \{U_1, U_2, \cdots\}$ be a countable cover of $f(X)$ by set open in $f(X)$. We have to show that $\mathcal{U}$ has a finite subcover.

Define $\mathcal{V} = \{f^{-1}(U_1), f^{-1}(U_2), \cdots\}$. Then $\mathcal{V}$ is a countable open cover of X, since $f^{-1}(U_k)$ are open subsets of X and,

$$\begin{aligned} \bigcup_{k=1}^{\infty} U_k = f(X) &\implies f^{-1}\left(\bigcup_{k=1}^{\infty} U_k\right) = X \\ &\implies \bigcup_{k=1}^{\infty} f^{-1}(U_k) = X \end{aligned}$$

We have, $\mathcal{V}$ is a countable open cover of X, which is a countably compact space. Therefore $\mathcal{V}$ has a finite subcover $\mathcal{V}' = \{f^{-1}(U_{n_1}),\ f^{-1}(U_{n_2}), \cdots,\ f^{-1}(U_{n_k})\}$.

$$\begin{aligned} \bigcup_{j=1}^{k} f^{-1}(U_{n_j}) = X &\implies f^{-1}\left(\bigcup_{j=1}^{k} U_{n_j}\right) = X \\ &\implies \bigcup_{j=1}^{k} U_{n_j} = f(X) \end{aligned}$$

Clearly $\mathcal{U}' = \{U_{n_1}, U_{n_2}, \cdots, U_{n_k}\}$ is a finite subcover of $\mathcal{U}$. Thus every countable open cover of $f(X)$ by sets open in $f(X)$ has a finite subcover. Therefore, continuous images of countably compact spaces are countably compact. □

Remark. 1. *For a continuous function, $f : X \to Y$ the inverse images of open sets are open in X. The relation $f^{-1} \subset f(X) \times X$ is not a function. However, we may consider a function, $\psi : P(Y) \to P(X)$ such that $\psi(U) = f^{-1}(U)$ for any subset U of Y. This ψ is an open map which maps open subsets of Y to open subsets of X.*

Theorem 11.3. *Every continuous, real-valued function on a countably compact, metric space is bounded and attains its extrema.([Joshi,] 11.1.7)*

[8] A topological property is preserved under continuous functions if whenever a space has that property so does every continuous image of it.([Joshi,] 6.1.9)

Synopsis. *Let X be a countably compact space and function $f : X \to \mathbb{R}$ be continuous. Then $f(X) \subset \mathbb{R}$ is countably compact. Real line $\mathbb{R}$ is metrisable*[9]*. Then $f(X)$ is countably compact, metric space. Therefore $f(X)$ compact*[10]*. The subset $f(X)$ of $\mathbb{R}$ is bounded and closed, since every compact subset of $\mathbb{R}$ is bounded and closed. Thus $f(X)$ contains its supremum and infimum. Therefore, f is bounded and attains its extrema.*

Proof. Let X be a countably compact space and $f : X \to \mathbb{R}$ be continuous, real-valued function on the countably compact space, X. We have to show that f is bounded and attains its extrema.

Since countable compactness is preserved under continuous functions, $f(X)$ is countably compact subset of $\mathbb{R}$. Since, $f(X)$ is a subset of the metric space, $\mathbb{R}$ and metrisability is hereditary, $f(X)$ is again metrisable. (suppose) We have, every countably compact, metric space is compact. Then $f(X)$ is a compact subset of $\mathbb{R}$.

Since every compact subset of $\mathbb{R}$ is bounded and closed, $f(X)$ is bounded and closed. Since every closed subset of $\mathbb{R}$ contains supremem and infimum, $f(X)$ contains its extrema. Therefore, every continuous, real-valued function on a countably compact space is bounded and attains its extrema.

We have assumed that every countably compact, metric space is compact. This result will be proved in the last section of this chapter. $\square$

Remark. *Since countably compact, metric spaces are compact. The above theorem can be used to prove that continuous, real-valued functions on a compact, metric space attains its extrema.*

Due to the Lebesgue covering lemma, next result is quite simple.*

Theorem 11.4. *Every continuous, real-valued function on a countably compact, metric space is uniformly continuous.*

Proposition 11.5. *Let X be a first countable, Hausdorff space. Then every countably compact subset A of X is closed.([Joshi,] Exercises 11.1.7)*

11.1.2 Sequential Compactness

Weakly hereditary property

Theorem 11.6. *Sequential compactness is weakly hereditary.([Joshi,] Exercises 11.1.3)*

The behaviour of countinous functions

Theorem 11.7. *Sequential compactness is preserved under continuous functions. ([Joshi,] Exercises 11.1.4)*

[9] [Joshi,] 4.2 Example 4, $\mathbb{R}$ with usual metric $d : R \to R,\ d(x, y) = |x - y|$

[10] [Joshi,] 11.1.11 On metric spaces, countable compactness $\implies$ compactness.

Synopsis. *Let X be sequentially compact and function $f : X \to Y$ be continuous. Then any sequence, $\{y_k\}$ in $f(X)$ has a sequence, $\{x_k\}$ in X such that $f(x_k) = y_k$. Sequence $\{x_k\}$ has a subsequence $\{x_{n_k}\}$ converging to x, then sequence $\{f(x_n)\}$ in $f(X)$ has the subsequence $\{f(x_{n_k})\}$ converging to $f(x)$.*

Proof. Let X be a sequentially compact space, function $f : X \to Y$ be continuous and $\{y_n\}$ be a sequence in $f(X)$ subset of Y. Construct a sequence $\{x_n\}$ such that $f(x_k) = y_k, \ \forall k$.

Every sequence in X has a convergent subsequence. Thus $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to $x \in X$. The image of this subsequence $\{f(x_{n_k})\}$ is a subsequence of $\{y_k\}$. We claim that, $\{f(x_{n_k})\}$ converges to $f(x) \in f(X)$.

Let U be an open set containing $f(x)$, then $f^{-1}(U)$ is an open set containing x. Since $\{x_{n_k}\}$ converges to x. There exists an integer n such that for every $k \geq n$, $x_k \in f^{-1}(U)$. Clearly, for each $k \geq n$, $f(x_k) \in U$. Since U is arbitrary, $\{f(x_{n_k})\}$ converges to $f(x)$. Therefore, the image of any sequentially compact space is sequentially compact. In other words, sequentially compactness is preserved under continuous functions. □

Remark. *1. Given a sequence $\{y_n\}$ in $f(X)$, there is a sequence of subsets $\{U_n\}$ in $P(Y)$ such that $U_n = f^{-1}(y_n)$. Since each U_n is non-empty, we can construct a sequence $\{x_n\}$ in X using a choice function. The convergent subsequence of $\{y_n\}$ depends on the selection of this choice function.*

Given every sequentially compact, metric space is countably compact. We may assert the properties of countably compact, metric spaces on sequentially compact, metric spaces.

Theorem 11.8. *Every continuous, real-valued function on a sequentially compact, metric space is bounded and attains its extrema.*

Theorem 11.9. *Every continuous, real-valued function on a sequentially compact, metric space is uniformly continuous.([Joshi,] Exercises 11.1.6)*

11.1.3 Countable Compactness on T_1 spaces

In this section, we are going to see four different characterisations of countable compactness in T_1 spaces. The first two characterisations doesn't have anything to do with the T_1 axiom.

T_1 Space A topological space X satisfy T_1 axiom if for any two distinct points $x, y \in X$, there exists an open set $U \subset X$ containing x but not y.([Joshi,] 7.1.2)

countable compactness A topological space is countably compact if every countable open cover has a finite subcover.([Joshi,] 11.1.1)

finite intersection property A family $\mathcal{F}$ of subsets of X has finite intersection property(f.i.p.) if every finite subfamily of $\mathcal{F}$ has a non-empty intersection.([Joshi,] 10.2.6)

accumulation point A point $x \in X$ is accumulation point of a subset $A \subset X$ if every open set containing x has atleast one point of A other than x.([Joshi,] 5.2.7)

limit point A point $x \in X$ is a limit point of a sequence $< x_k >$ in X if for every open set U containing x, there exists an integer $N \in \mathbb{N}$ such that $x_k \in U$ for every $k \geq N$.([Joshi,] 4.1.7)

cluster point A point $x \in X$ is a cluster point of a sequence $< x_k >$ in X if for any neighbourhood V of x, the sequence $< x_k >$ assumes a point in V infinitely many times.[11]

Countable compactness in T_1 spaces

Theorem 11.10. *In a T_1 space X, following statements are equivalent,*

1. *X is countably compact*
2. *Every countably family of closed subsets of X with finite intersection property have non-empty intersection.*
3. *Every infinite subset $A \subset X$ has an accumulation point.*[12]
4. *Every sequence $< x_k >$ in X has a cluster point.*
5. *Every infinite open cover of X has a proper subcover.[Arens-Dugundji]*

Proof. $1 \implies 2$
Suppose X is countably compact. Let $\mathcal{C} = \{C_1, C_2, \cdots\}$ be a countable family of closed subsets of X with empty intersection. Define $\mathcal{U} = \{X - C_1, X - C_2, \cdots\}$ is a family of open subsets of X. By de Morgan's law, [13]

$$\bigcap_{k=1}^{\infty} C_k = \phi, \text{ then } X = X - \left(\bigcap_{k=1}^{\infty} C_k\right) = \bigcup_{k=1}^{\infty} (X - C_k)$$

We have $\mathcal{U}$ is a countable cover of X and X is countably compact space. Thus $\mathcal{U}$ has a finite subcover $\mathcal{U}' = \{X - C_{n_1}, X - C_{n_2}, \cdots, X - C_{n_k}\}$.

$$\mathcal{U}' \text{ is a cover of } X, \text{ then } X = \bigcup_{j=1}^{k} \left(X - C_{n_j}\right)$$

$$X - \bigcup_{j=1}^{k} \left(X - C_{n_j}\right) = \bigcap_{j=1}^{k} \left(X - \left(X - C_{n_j}\right)\right) = \bigcap_{j=1}^{k} C_{n_j} = \phi$$

Now $\mathcal{C}' = \{C_{n_1}, C_{n_2}, \cdots, C_{n_k}\}$ has empty intersection. This is a contradiction to the finite intersection property of $\mathcal{C}$. Thus $\mathcal{C}$ has non-empty intersection. Therefore, every countably family of closed subsets of X have non-empy intersection.

[11] x is a cluster point of $< x_k >$ if for every integer N, there exists $k > N$ such that $x_k \in V$.In other words, $< x_k >$ is frequently in V.([Joshi,] 10.1.9)
[12] Every infinite subset of $\mathbb{R}$ has a limit point is equivalent to the completeness axiom.
[13] Complement of Intersection = Union of complements, $X - (C \cap D) = (X - C) \cup (X - D)$,

$2 \implies 1$

Let $\mathcal{U} = \{U_1, U_2, \cdots\}$ be a countable cover of X. Then $\mathcal{C} = \{X - U_1, X - U_2, \cdots\}$ is a countable family of closed subsets of X.

Let $\mathcal{U}' = \{U_{n_1}, U_{n_2}, \cdots, U_{n_k}\}$ be any finite subfamily of $\mathcal{U}$. Suppose X is not countably compact, then $\mathcal{U}$ doesn't have a finite subcover. Therefore, $\mathcal{U}'$ is not a cover of X. And $\mathcal{C}$ is a family of closed sets with finite intersection property.

Therefore by assumption, the countable family of closed sets $\mathcal{C}$ has a non-empty intersection.

$$\bigcap_{k=1}^{\infty} C_k \neq \phi, \text{ then } \bigcap_{k=1}^{\infty} C_k = \bigcap_{k=1}^{\infty} (X - U_k) = X - \left(\bigcup_{k=1}^{\infty} U_k \right) \neq \phi$$

Then $\mathcal{U}$ is not a cover of X as well. This is a contradiction, therefore X is countably compact.

$1 \implies 3$
Suppose X is countably compact. Let A be an infinite subset of X. Suppose A doesn't have an accumulation point.

Let B be a countably infinite subset of A. Then B also doesn't have any accumulation point. Therefore, the derived set B' is empty. Thus B is a closed subset of X. Since countable compactness is weakly hereditary, subspace B is again countably compact.

For each point $b \in B$, there is an open set V_b such that $V_b \cap B = \{b\}$, since $b \in B$ is not an accumulation point. Thus $\mathcal{U} = \{V_b \cap B : b \in B\}$ is a countable open cover of B. Clearly, $\mathcal{U}$ doesn't have any finite subcover.

This is a contradiction to B being countably compact. Therefore, A has an accumulation point. □

11.1.4 Variations of Compactness on Metric Spaces

In this document, we will see that from metric space point of view these two notions were equivalent to the compactness and were used alternatively. For example : in functional analysis (semester 3), you will find definitions like 'a normed space is compact iff every sequence in it has a convergent subsequence', which is clearly sequential compactness for a topologist.

Lindeloff A topological space is Lindeloff iff every open cover has a countable subcover.

First countable A topological space is first countable iff every point in it has a countable local base.

Second countable A topological space is second countable iff it has a countable base.

Base A family of subsets $\mathcal{B}$ of X is a base of a topological space if every open set can be expressed as union of some members of $\mathcal{B}$

Base Characterisation A family of subsets $\mathcal{B}$ of X is a base of a topological space iff for every $x \in X$, and for every neighbourhood U of x, there is a member $B \in \mathcal{B}$ such that $x \in B \subset U$.

Local Base A family of subsets $\mathcal{L}$ of X is a local base at point $x \in X$ if for every neighbourhood U of x, there is a member $L \in \mathcal{L}$ such that $x \in L \subset U$.

Equivalence

We are going to see when these three notions: compactness, countable compactness and sequentially compactness are equivalent.

Theorem 11.11. *Countably compact, metric spaces are second countable.*

Synopsis. *For every positive real number r, there exists a non-empty maximal subsets A_r with every pair of points atleast r distance apart. A_r are finite. The union of maximal subsets $A_{\frac{1}{n}}$ for each natural number n is a countable, dense subset D of X. Thus countably compact, metric spaces are separable. The family $\mathcal{B}$ of all open balls with center at $d \in D$ and rational radius is a countable, base for X. Thus countably compact, metric spaces are second countable.*

Proof. Let $(X; d)$ be a countably compact,, metric space. For each positive real number $r \in \mathbb{R},\ r > 0$ construct a family of subsets $A_r \subset X$ such that it is a maximal set of points which are atleast r distances apart.

Then A_r is finite for every positive real number r. Suppose A_r is infinite for some real number $r > 0$, then A_r has a accumulation point, say x by the Characterisation of countable compactness of X.

Then every neighbourhood of x must intersect A_r at infinitely many points, since every metric space is a T_1 space. Consider $B(x, \frac{r}{2})$. Since any two points of $B(x, \frac{r}{2})$ are less than r distances apart, the intersection $B(x, \frac{r}{2}) \cap A_r$ can have atmost one point in it. Thus for every positive real number r, A_r is finite.

Define $D = \cup_{n=1}^{\infty} A_{\frac{1}{n}}$. We claim that D is a countable, dense subset of X.

Let $x \in X$ and $B(x, r)$ be an open ball containing x, then there exists integer $n \in \mathbb{N}$ such that $\frac{1}{n} < r$.[14]

Then $B(x, r) \cap D \neq \phi$, since $B(x, r) \cap A_{\frac{1}{n}} \neq \phi$. Suppose $B(x, r) \cap A_{\frac{1}{n}} = \phi$, then $A_{\frac{1}{n}}$ is not maximal. Since, x is atleast $r > \frac{1}{n}$ distance apart from each points of $A_{\frac{1}{n}}$. Therefore, D intersects with every open set and thus dense in X.

We have have a countable, dense subset D of X. Therefore, X is separable. Now define $\mathcal{B} = \{B(x, r) : r \in \mathbb{Q},\ x \in D\}$. Clearly, $\mathcal{B}$ is a countable base for X. By the construction of $\mathcal{B}$, X is second countable.[15] $\square$

[14] By archimedean property of integers, we have $\forall r \in \mathbb{R},\ r > 0,\ \exists n \in \mathbb{N}$ such that $nr > 1$.

[15] Every separable, metric space is second countable.

Countable Compactness, Lindeloff $\iff$ Compactness

Theorem 11.12. *A topological space X is compact iff it is countably compact, Lindeloff space.*

Proof. Let X be a compact space. Let $\mathcal{U}$ be a countable open cover of X, then $\mathcal{U}$ has a finite subcover $\mathcal{U}'$. Therefore, every compact space is countably compact.[16]

Conversely, suppose X is a countably compact, Lindeloff space. Since X is Lindeloff, every open cover $\mathcal{U}$ has a countable subcover $\mathcal{U}'$. Since X countably compact, every countable open cover $\mathcal{U}'$ has a finite subcover $\mathcal{U}''$. Thus every open cover $\mathcal{U}$ has a finite subcover $\mathcal{U}''$. Therefore every countably compact, Lindeloff space is compact. □

Countable Compactness, First Countable $\implies$ Seq. Compactness

Theorem 11.13. *Every countably compact, first countable space is Sequentially compact.*

Proof. Let X be a countably compact, first countable space. Let $\{x_n\}$ be a sequence in X. By, equivalent conditions[17] of countably compact spaces, every sequence in countably compact space X has a cluster point, say x. We have, X is first countable. Therefore, X has a countable local base $\mathcal{L}$ at $x \in X$. How to construct a subsequence of $\{x_n\}$ converging to x ?⋆[18] □

Remark. *Every sequentially compact space is countably compact.⋆*

Theorem 11.14. *In a second countable space, all the three forms of compactness are equivalent.([Joshi,] 11.1.10)*

Proof. Every second countable space is both first countable and Lindeloff. Every countably compact, Lindeloff space is countably compact. Therefore every countably compact, second countable space compact. Again, every countably compact, first countable space is sequentially compact. Therefore every countably compact, second countable space is sequentially compact. Conversely, every compact space is countably compact and every sequentially compact space is countably compact.[19] □

Theorem 11.15. *In a metric space, all the three forms of compactness are equivalent.([Joshi,] 11.1.11)*

Proof. In a metric space each form of compactness implies second countability. And in second countable spaces, they are all equivalent. □

[16] Countable compactness is a weaker notion than compactness.
[17] [Joshi,] 11.1 Conditions 1,2, and 4 are equivalent. 2 $\implies$ 4 without T_1 axiom is out of scope.
[18] [Joshi,] Exercise 10.1.11
[19] Countable compactness is a weaker notion than sequential compactness as well.

Part II

ME010203 Numerical Analysis with Python

Chapter 12

Expressions

Chapter 13

Calculus

Chapter 14

Interpolation & Curve Fitting

Chapter 15

Matrix Operations

Part III

ME800402 Algorithmic Graph Theory

Chapter 16

An Introduction to Graphs

Chapter 17

An Introduction to Algorithms

17.1 Algorithmic Complexity

17.2 Serach Algorithms

17.3 Sorting Algorithms

17.5 Greedy Algorithms

17.6 Representing Graphs in a Computer

Chapter 18

Trees

Chapter 19

Paths and Distance in Graphs

19.1 Distance in Graphs

19.2 Distance in Weighted Graphs

19.3 The Center and Median of a Graph

19.4 Activity Digraphs and Critical Paths

Chapter 20

Networks

20.1 An Introduction to Networks

Definitions 20.1. *A network N is a digraph D with two special vertices source s and sink t together with a capacity function $c : E(D) \to \mathbb{Z}$ such that for every arc $a = (u, v)$ of the digraph, $c(u, v)$ is non-negative.*

Remark. *Mathematical Modeling using Network,*

1. *There is no restriction on indegree/outdegree of source/sink vertices of the digraph D of a network N.*

2. *Applications of Network : Transportation problem.*

$c(u, v)$ is the capacity of the arc (u, v) of D

$N^+(x) = \{y \in V(D) : (x, y) \in E(D)\}$ is the out-neighbourhood of x.

$N^-(x) = \{y \in V(D) : (y, x) \in E(D)\}$ is the in-neighbourhood of x.

Definitions 20.2. *A flow f in a network N is function $f : E(D) \to \mathbb{Z}$ such that 1. each edge satisfies capacity constraint and 2. each vertex except source and sink satisfies conservation equation.*

capacity constraint

$$0 \le f(a) \le c(a) \text{ for every arc } a \in V(D) \tag{20.1}$$

conservation equation

$$\sum_{y \in N^+(x)} f(x, y) = \sum_{y \in N^-(x)} f(y, x), \ \forall \text{vertex } x \in V(D) - \{s, t\} \tag{20.2}$$

net flow out of x

$$\sum_{y \in N^+(x)} f(x, y) - \sum_{y \in N^-(x)} f(y, x)$$

net flow into x

$$\sum_{y \in N^-(x)} f(y, x) - \sum_{y \in N^+(x)} f(x, y)$$

Definitions 20.3. *The flow f in a network N is the net flow out of source s.*

Remark. *1. net flow out of/into $x \in V(D) - \{s,t\}$ is zero.*

2. Without loss of generality[1], underlying digraph is always assymetric.

(X,Y) $= \{(x,y) \in E(D) : x \in X,\ y \in Y\}$.
Let X, Y be non-empty subsets of $V(D)$ such that X, Y are disjoint. Then (X,Y) is the set of all arcs from X to Y.

flow from X **to** Y is the sum of flow on each arc in (X,Y)

$$f(X,Y) = \sum_{(x,y)\in(X,Y)} f(x,y) \tag{20.3}$$

capacity of the partition (X,Y) is the total capacity of arcs in (X,Y)

$$c(X,Y) = \sum_{(x,y)\in(X,Y)} c(x,y) \tag{20.4}$$

cut Let $P \subset V(D)$ such that $s \in P$ and $t \notin P$ and $\bar{P} = V(D) - P$, then $(P,\bar{P})$ is a cut.

flow from P **to** $\bar{P}$ is the sum of flow on each arc in $(P,\bar{P})$.

$$f(P,\bar{P}) = \sum_{(x,y)\in(P,\bar{P})} f(x,y) \tag{20.5}$$

flow from $\bar{P}$ **to** P is the sum of flow on each arc in $(\bar{P},P)$

$$f(\bar{P},P) = \sum_{(x,y)\in(\bar{P},P)} f(x,y) \tag{20.6}$$

capacity of the cut $(P,\bar{P})$ is the total capacity of the arcs in $(P,\bar{P})$

$$c(P,\bar{P}) = \sum_{(x,y)\in(P,\bar{P})} c(x,y) \tag{20.7}$$

Theorem 20.4. *For any cut $(P,\bar{P})$, the flow in N is $f(N) = f(P,\bar{P}) - f(\bar{P},P)$.*

Synopsis. *The net flow out of source s is the flow $f(N)$ in the network N. Let $(P,\bar{P})$ be a cut of N, then $s \in P$ and $t \notin P$. Suppose $P = \{s\}$, then the theorem is true. Suppose P is not singleton, then for each vertex $x \in P,\ x \neq s$, the net flow out of x is zero by flow conservation equation. And flow between vertices in P cancels out each other. Thus adding net flow out of each vertex in P, will be same as the net flow out of source which is the flow in the network, $f(N)$.*

[1] If underlying digraph of a network is symmetric, then by replacing an arc (u,v) with a new vertex w and two arcs $(u,w),(w,v)$ gives an assymetric digraph.[Gray Chartrand,]pp.131

Proof.

$$\text{Flow, } f = \sum_{y \in N^+(s)} f(s,y) - \sum_{y \in N^-(s)} f(y,s) \tag{20.8}$$

By conservation equation, we have $\forall x \in P,\ x \neq s$,

$$\sum_{y \in N^+(x)} f(x,y) - \sum_{y \in N^-(x)} f(y,x) = 0 \tag{20.9}$$

By above equations,

$$\begin{aligned} \text{Flow, } f &= \sum_{x \in P} \sum_{y \in N^+(x)} f(x,y) - \sum_{x \in P} \sum_{y \in N^-(x)} f(y,x) \\ &= \sum_{(x,y) \in (P,\bar{P})} f(x,y) - \sum_{(y,x) \in (\bar{P},P)} f(y,x) \end{aligned} \tag{20.10}$$

□

Corollary 20.4.1. *Flow cannot exceed the capacity of any cut $(P,\bar{P})$. Further, $f(N) \leq \min c(P,\bar{P})$.*

Synopsis. *Let $(P,\bar{P})$ be a cut in network N, then by theorem the flow $f(N) =$ flow from P to $\bar{P}$ - flow from $\bar{P}$ to P. Since the flow from $\bar{P}$ to P is non-negative, $f(N) \leq$ flow from P to $\bar{P}$. Clearly, $f(x,y) \leq c(x,y)$ by the capacity constaint. Thus $f(N) \leq f(P,\bar{P}) \leq c(P,\bar{P}) \leq \min c(P,\bar{P})$.*

Proof.

$$\begin{aligned} f(N) &= \sum_{(x,y) \in (P,\bar{P})} f(x,y) - \sum_{(y,x) \in (\bar{P},P)} f(y,x) \\ &\leq \sum_{(x,y) \in (P,\bar{P})} f(x,y) = f(P,\bar{P}) \\ &\leq \sum_{(x,y) \in (P,\bar{P})} c(x,y) = c(P,\bar{P}), \quad \because \forall x,y \in V(D),\ f(x,y) \leq c(x,y) \\ &\leq \min c(P,\bar{P}) \end{aligned}$$

□

Corollary 20.4.2. *In a network N flow is the net flow into the sink of N.*

Synopsis. *Let $\bar{P} = \{t\}$, then by theorem $f(N)$ is the net flow into the sink.*

Proof. Suppose $P = V(D) - \{t\}$. Then by theorem, we have

$$\begin{aligned} f(N) &= \sum_{(x,y) \in (P,\bar{P})} f(x,y) - \sum_{(y,x) \in (\bar{P},P)} f(y,x) \\ &= \sum_{x \in N^-(t)} f(x,t) - \sum_{x \in N^+(t)} f(t,x) \end{aligned}$$

□

Remark. *Exercise 5.1*

4. *Let N be a network with underlying digraph D which has a vertex $v \in V(D) - \{s,t\}$ with zero indegree. Clearly the flow into v is zero. Thus flow out of v is also zero by flow conservation equation. Let N' be the network obtained from N by deleting the vertex v. Then $f(N) = f(N')$.*

20.2 The Max-Flow Min-Cut Theorem

Definitions 20.5. maximum flow *A flow f in network N is maximum flow in N, if $f(N) \geq f'(N)$ for each flow f' in N.*

minimum cut *A cut $(P, \bar{P})$ in network N is minimum cut of N, if $c(P, \bar{P}) \leq c(X, \bar{X})$ for each cut $(X, \bar{X})$ in N.*

f-unsaturated *Let f be a flow in network N with underlying digraph D, and $Q = u_0, a_1, u_1, a_2, \cdots, u_{n-1}, a_n, u_n$ be a semipath in D such that every forward arc $a_i = (u_{i-1}, u_i)$ has flow not upto its capacity, $f(a_i) < c(a_i)$ and every reverse arc $a_i = (u_i, u_{i-1})$ has some positive flow in it, $f(a_i) > 0$*

f-augmenting semipath *Let f be a flow in a network N with underlying digraph D. Suppose semipath $Q = s, a_1, u_1, a_2, \cdots, u_{n-1}, a_n, t$ (from source to sink) is f-unsaturated, then Q is an f-augmenting semipath.*

Theorem 20.6. *Let f be a flow in a network N with underlying digraph D. The flow f is maximum in N iff there is no f-augmenting semipath in D.*

Synopsis. *Suppose Q is an f-augmenting semipath in D, then there exists a flow f^* in N such that $f(N) + \Delta = f^*(N)$. Therefore, f is not a maximum flow in N. Suppose there is no f-augmenting semipath in D, then there exists a cut $(P, \bar{P})$ such that $f(a) = c(a)$ $\forall a \in (P, \bar{P})$ and $f(a) = 0$ $\forall a \in (\bar{P}, P)$. Suppose f^* in a maximum flow in N, then $f(N) \leq f^*(N) \leq c(P, \bar{P}) = f(N)$.*

Proof. Let f be a flow in a network N with underlying digraph D and $Q = s, a_1, u_1, a_2, u_2, \cdots, u_{n-1}, a_n, t$ be an f-augmenting semipath in D.

$$\text{define } \Delta_i = \begin{cases} c(a_i) - f(a_i) \text{ for every forward arc } a_i \in Q, \\ f(a_i) \text{ for every reverse arc } a_i \in Q, \end{cases}$$

Define $\Delta = \min\{\Delta_i\}$. Also define $f^* : E(D) \to \mathbb{Z}$ such that

$$f^*(a_i) = \begin{cases} f(a) + \Delta, \text{ for every forward arc } a_i \in Q, \\ f(a) - \Delta, \text{ for every reverse arc } a_i \in Q, \\ f(a_i), \text{ for every arc of } D \text{ which are not in } Q. \end{cases}$$

Since Q is an f-augmenting semipath in D, $\Delta > 0$ and $f(N) + \Delta = f^*(N)$.

Clearly $f(N) < f^*(N)$, and it is enough to show that f^* is a flow in N. f^* is a flow if it satisfies 1. capacity constraint and 2. conservation equation. For any arc $a_i \notin Q$, $f^*(a_i) = f(a_i) \leq c(a_i)$. Suppose $a_i \in Q$. If $a_i = (u_{i-1}, u_i)$, a_i is a forward arc and we have $f^*(a_i) = f(a_i) + \Delta \leq f(a_i) + \Delta_i = f(a_i) + c(a_i) - f(a_i) = c(a_i)$. If $a_i = (u_i, u_{i-1})$, then a_i is a reverse arc and we have $f^*(a_i) = \Delta \leq \min\{\Delta_i\} = \Delta_i = c(a_i)$. Thus f^* satisfies capactity constraint on every arc of D.

Let $x \in V(D) - \{s,t\}$. Suppose $x \notin Q$,

$$\begin{aligned}\text{Net flow out of } x &= \sum_{y\in N^+(x)} f^*(x,y) - \sum_{y\in N^-(x)} f^*(y,x)\\ &= \sum_{y\in N^+(x)} f(x,y) - \sum_{y\in N^-(x)} f(y,x)\\ &= 0\end{aligned}$$

Suppose $x = u_i \in Q$, then Q has two arc having vertex x say, a_{i-1}, and a_i. There are four possibilities for these two arcs,

1. Both $a_{i-1},\ a_i$ are forward arcs.
2. Arc a_{i-1} is forward, but arc a_i is reverse.
3. Arc a_{i-1} is reverse, but arc a_i is forward.
4. Both $a_{i-1},\ a_i$ are reverse arcs.

Case 1 $a_{i-1} = (u_{i-1}, u_i)$ and $a_i = (u_i, u_{i+1})$.

$$\begin{aligned}\text{Net flow out of } x &= \sum_{y\in N^+(x)} f^*(x,y) - \sum_{y\in N^-(x)} f^*(y,x)\\ &= \sum_{\substack{y\in N^+(x)\\ y\neq u_{i+1}}} f^*(x,y) + f^*(u_i,u_{i+1}) - \left(\sum_{\substack{y\in N^-(x)\\ y\neq u_{i-1}}} f^*(y,x) + f^*(u_{i-1},u_i)\right)\\ &= \sum_{\substack{y\in N^+(x)\\ y\neq u_{i+1}}} f(x,y) + f(u_i,u_{i+1}) + \Delta - \left(\sum_{\substack{y\in N^-(x)\\ y\neq u_{i-1}}} f(y,x) + f(u_{i-1},u_i)\right) - \Delta\\ &= \sum_{y\in N^+(x)} f(x,y) - \sum_{y\in N^-(x)} f(y,x)\\ &= 0\end{aligned}$$

Case 2 $a_{i-1} = (u_{i-1}, u_i)$ and $a_i = (u_{i+1}, u_i)$.

$$\begin{aligned}\text{Net flow out of } x &= \sum_{y\in N^+(x)} f^*(x,y) - \sum_{y\in N^-(x)} f^*(y,x)\\ &= \sum_{\substack{y\in N^+(x)\\ y\neq u_{i+1},u_{i-1}}} f^*(x,y) + f^*(u_i,u_{i+1}) + f^*(u_i,u_{i-1}) - \sum_{y\in N^-(x)} f^*(y,x)\\ &= \sum_{\substack{y\in N^+(x)\\ y\neq u_{i+1},u_{i-1}}} f(x,y) + f(u_i,u_{i+1}) + \Delta + f(u_i,u_{i-1}) - \Delta - \sum_{y\in N^-(x)} f(y,x)\\ &= \sum_{y\in N^+(x)} f(x,y) - \sum_{y\in N^-(x)} f(y,x)\\ &= 0\end{aligned}$$

Case 3 $a_{i-1} = (u_i, u_{i-1})$ and $a_i = (u_i, u_{i+1})$.

$$
\begin{aligned}
\text{Net flow out of } x &= \sum_{y \in N^+(x)} f^*(x,y) - \sum_{y \in N^-(x)} f^*(y,x) \\
&= \sum_{y \in N^+(x)} f^*(x,y) - \left(\sum_{\substack{y \in N^-(x) \\ y \neq u_{i-1}, u_{i+1}}} f^*(y,x) + f^*(u_{i-1}, u_i) + f^*(u_{i+1}, u_i) \right) \\
&= \sum_{y \in N^+(x)} f^*(x,y) - \left(\sum_{\substack{y \in N^-(x) \\ y \neq u_{i-1}, u_{i+1}}} f(y,x) + f(u_{i-1}, u_i) + \Delta + f(u_{i+1}, u_i) - \Delta \right) \\
&= \sum_{y \in N^+(x)} f(x,y) - \sum_{y \in N^-(x)} f(y,x) \\
&= 0
\end{aligned}
$$

Case 4 $a_{i-1} = (u_i, u_{i-1})$ and $a_i = (u_{i+1}, u_i)$.

$$
\begin{aligned}
\text{Net flow out of } x &= \sum_{y \in N^+(x)} f^*(x,y) - \sum_{y \in N^-(x)} f^*(y,x) \\
&= \sum_{\substack{y \in N^+(x) \\ y \neq u_{i-1}}} f^*(x,y) + f^*(u_i, u_{i-1}) - \left(\sum_{\substack{y \in N^-(x) \\ y \neq u_{i+1}}} f^*(y,x) + f^*(u_{i+1}, u_i) \right) \\
&= \sum_{\substack{y \in N^+(x) \\ y \neq u_{i-1}}} f(x,y) + f(u_i, u_{i-1}) - \Delta - \left(\sum_{\substack{y \in N^-(x) \\ y \neq u_{i+1}}} f(y,x) + f(u_{i+1}, u_i) \right) + \Delta \\
&= \sum_{y \in N^+(x)} f(x,y) - \sum_{y \in N^-(x)} f(y,x) \\
&= 0
\end{aligned}
$$

Therefore, f^* is a flow on N. We have $f(N) < f^*(N)$. Thus f is not maximum flow in N due to the existence of an f-augmenting semipath in D.

Conversely, assume that there is no f-augmenting semipath in D. Now, we construct a cut $(P, \bar{P})$ of N. Let P be the set of all vertices $x \in V(D)$ such that there is an f-unsaturated $s - x$ semipath in D. Trivially, $s \in P$. And $t \notin P$ since there are no f-augmenting semipath in D.[2] Clearly, $(P, \bar{P})$ is a cut of the network N.

We claim that $c(P, \bar{P}) = f(N)$. Suppose there is a forward arc $(x, y) \in (P, \bar{P})$, then flow in it is saturated. If $f(x, y) < c(x, y)$, then there is an f-unsaturated $s - y$ semipath in D. ie, $s - x$ semipath + arc (x, y). Thus every forward arc $(x, y) \in (P, \bar{P})$ is saturated. Suppose there is a reverse arc $(y, x) \in$

[2] An f-augmenting semipath is an f-unsaturated $s - t$ semipath in D.

$(\bar{P}, P)$, then there is no flow in it(saturated reversed arc). If $f(y,x) > 0$, then there is an f-unsaturated $s-y$ semipath in D. ie, $s-x$ semipath + arc (y,x). Thus every reverse arc $(y,x) \in (\bar{P}, P)$ is saturated. And we have,

$$\sum_{(x,y)\in(P,\bar{P})} f(x,y) = \sum_{(x,y)\in(P,\bar{P})} c(x,y)$$

$$\sum_{(y,x)\in(\bar{P},P)} f(y,x) = 0$$

$$\begin{aligned} f(N) &= \sum_{(x,y)\in(P,\bar{P})} f(x,y) - \sum_{(y,x)\in(\bar{P},P)} f(y,x) \\ &= \sum_{(x,y)\in(P,\bar{P})} c(x,y) \\ &= c(P,\bar{P}) \end{aligned}$$

Suppose f^* is maximum flow in network N and $(X, \bar{X})$ is minimum cut of N. Then $f(N) \leq f^*(N)$. Thus we have, $f(N) \leq f^*(N) \leq c(X,\bar{X}) \leq c(P,\bar{P}) = f(N)$. Therefore, $f(N) = f^*(N)$. ie, the flow f is maximum in network N if there are no f-augmenting semipaths in D. □

Theorem 20.7 (maximum-flow, min-cut)**.** *In every network, the value of maximum flow equals capacity of minimum cut.*

Proof. Suppose flow f in network N in maximum, then by previous theorem there is no f-augmenting semipath in D. And $f(N) \leq c(X,\bar{X})$ for any cut $(X,\bar{X})$ in N. We can construct a cut $(P,\bar{P})$ in N such that $f(N) = c(P,\bar{P})$. Let P be the set of all vertices x in D such that there is an f-unsaturated $s-x$ semipath in D. Clearly $s \in P$ and $t \notin P$. Also $f(P,\bar{P}) = c(P,\bar{P})$ and $f(\bar{P},P) = 0$. Then the cut $(P,\bar{P})$ is minimum cut of N. Suppose there is a cut $(X,\bar{X})$ such that $c(X,\bar{X}) < c(P,\bar{P})$. Then $f(N) = f(P,\bar{P}) - f(\bar{P},P) = c(P,\bar{P}) < c(X,\bar{X})$ which is a contradiction. Therefore, the value of maximum flow equals capacity of minimum cut. □

Remark. *Exercise 5.2*

1. *Suppose $(X,\bar{X})$ is a cut of N such that $f(a) = c(a)$, $\forall a \in (X,\bar{X})$ and $f(a) = 0$, $\forall a \in (\bar{X},X)$. By the definition of cut, $s \in X$ and $t \in \bar{X}$. Thus there is no f-augmenting semipath in D. Suppose there is an f-augmenting semipath Q in D, then there is either (a) a forward arc $(x,y) \in (X,\bar{X})$ such that $f(x,y) < c(x,y)$ or (b) a reverse arc $(y,x) \in (\bar{X},X)$ such that $f(y,x) > 0$ which is a contradition. Therefore, the flow $f(N)$ is maximum and the given cut $(X,\bar{X})$ is minimum as shown in the proof of the maximum-flow min-cut theorem.*

3. *The algorithm suggested in the hint of this exercise won't work if two subnetworks have a common arc such that the direction of flow in which is not consistent. Suppose, the generalized network is not supposed to have any common arcs. Then construct subnetworks for each pair (s,t) with all those arcs which are on some $s-t$ semipath. Define subnetwork capacity function $c'(a) = c(a)$ for every arc in N'.*

Let N be a generalized network with set of sources S and set of sinks T. A flow in N is maximum if there is not f-augmenting $s-t$ semipath for each pair $(s,t) \in S \times T$. Does there exists a generalised network where max-flow min-cut algorithm is inconsistent ? $\star$ I don't know

20.3 A max-flow min-cut algorithm

Theorem 20.8. *Let N be a network with underlying digraph D, source s, sink t, capacity function c and flow f. Let D' be the digraph with same vertex set as D and arc set defined by $E(D') = \{(x,y) : (x,y) \in E(D),\ c(x,y) > f(x,y)$ or $(y,x) \in E(D),\ f(y,x) > 0\}$. ie, D' has only the unsaturated arcs of D. Then D' has an $s-t$ directed path iff D has an f-augmenting semipath. Moreover, shortest $s-t$ path in D' has the same length as shortest f-augmenting semipath in D.*

Synopsis. *Each directed $s-t$ path in D' has respective f-augmenting semipath in D and vice versa. Clearly, they have the same length.*

Proof. Let N be a network with underlying digraph D, capacity c and flow f. Let D' be the digraph with vertex set $V(D') = V(D)$ and arc set $E(D') = \{(x,y) : \text{either } (x,y) \text{ or } (y,x) \text{ is unsaturated in } N\}$.

Suppose D' has a directed $s-t$ path $Q' : s, u_1, u_2, \cdots, u_{n-1}, t$. Then by the construction of D', for each $u_i \in Q$, there exists an f unsaturated arc a_i in D. ie, either forward arc $a_i = (u_{k-1}, u_k)$ such that $f(u_{k-1}, u_k) < c(u_{k-1}, u_k)$ or reverse arc $a_i = (u_k, u_{k-1})$ such that $f(u_k, u_{k-1}) > 0$. Therefore, we have an $s-t$ semipath $Q : s, a_1, u_1, a_2, \cdots, u_{n-1}, a_n, t$ in D such that Q is an f-augmenting semipath since every arc in Q is f-unsaturated. Clearly, Q, Q' are of the same length.

Conversely, suppose that the digraph D has an f-augmenting semipath $Q : s, a_1, u_1, a_2, \cdots, u_{n-1}, a_n, t$. Then each arc $a_i \in Q$ are f-unsaturated and by the construction of D', there exists a directed $s-t$ path $Q' = s, u_1, u_2, \cdots, u_{n-1}, t$ in D'. And Q, Q' are of the same length.

There is a one-one correspondence betweeen the directed $s-t$ paths in D' and f-augmenting semipaths in D. Clearly, they have the same length. Thus shortest directed $s-t$ path in D and shortest f-augmenting semipath in D' are of the same length. □

saturation arc of N with respect to the flow f is an arc a_j in an f-augmenting semipath Q with $\Delta_j = \Delta$.

augmentation path is an f-augmenting semipath Q in D.

Algorithm 20.9 (max-flow min-cut)**.** *An algorithm to find maximum flow and minimum cut of a network N with underlying digraph D, source s, sink t, capacity function c and initial flow f.*

1. *Construct digraph D' with vertex set $V(D') = V(D)$ and arc set $E(D') = \{(x,y) : (x,y) \in E(D) \& f(x,y) < c(x,y)$ or $(y,x) \in E(D) \& f(y,x) > 0\}$*

2. *Find (shortest) s–t directed path in D' using Moore's breadth first search(BFS) algorithm. If D' doesn't have an $s-t$ path, then proceed to step 5. Otherwise, let $Q' : s, u_1, u_2, \cdots, u_{n-1}, t$ be a (shortest) $s-t$ path in D'.*

3. *Let $Q : s, a_1, u_1, a_2, \cdots, u_{n-1}, a_n, t$ be the respective semipath in D such that $f(a_j) < c(a_j)$ for forward arcs and $f(a_i) > 0$ for reverse arcs. Let $\Delta_j = c(a_j) - f(a_j)$ for forward arcs and $\Delta_j = f(a_j)$ for reverse arcs. And let $\Delta = \min\{\Delta_j\}$. And augment flow f by Δ ie, $f(a_j) \leftarrow f(a_j) + \Delta$ for forward arcs and $f(a_j) \leftarrow f(a_j) - \Delta$ for reverse arcs.*

4. *Goto step 1 (Proceed with new flow f and find whether there are any directed $s-t$ paths in D'. If any, augment the flow along the new augmentation path Q by saturating the flow along the saturation arc.)*

5. *There is no $s-t$ directed path in D'. Thus there is no f-augmenting semipath in D. Therefore the flow f in N is maximum. Let P be the set of all vertices in D' with non-zero breadth first index(bfi) from Moore's BFS algorithm applied in step 2. $(P, \bar{P})$ is minimum cut of N.*

Remark. *Validity of the algorithm is proved in the previous theorem.*

20.5 Connectivity and Edge-Connectivity

edge cutset is the set U subset of $E(G)$ such that $G - U$ is disconnected.

vertex cutset is the set S subset of $V(G)$ such that $G - S$ is disconnected.

edge connectivity $\lambda(G)$ is the minimum cardinality of all edge cutsets of G.

connectivity $\kappa(G)$ is the minimum cardinality of all vertex cutsets of G.

Theorem 20.10. *For every graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$*

Proof. Suppose graph G is disconnected then $\kappa(G) = \lambda(G) = \delta(G) = 0$. Let G be a connected graph. Then G has at least one vertex v with degree $\delta(G)$. Therefore $\lambda(G) \leq \delta(G)$ since edges incident with v form an edge cutset of G and $\lambda(G)$ is the cardinality of all edge cutsets.

Let G be a graph with edge connectivity $\lambda(G) = c$. Let U be a edge cutset with cardinality c and let edge $uv \in U$. Construct a set of vertices $S \subset V(G)$ such that (S is of minimal cardinality and) for each edge in U other uv, S has a vertex incident with it. Cardinality of S is atmost $c - 1$, since we can select one vertex each for each edge in U other than u, v. If $G - S$ is a disconnected graph, then $\kappa(G) < \lambda(G)$. Suppose $G - S$ is a connected graph, then delete a non-pendent vertex u or v from $G - S$, say v. Since $G - S$ is a connected graph with a singleton edge cutset, $\{uv\}$. We have a vertex cutset $S \cup \{v\}$ of G. Therefore, $\kappa(G) \leq c = \lambda(G)$. □

Theorem 20.11. *If G is a graph of diameter 2, then $\lambda(G) = \delta(G)$*

n-edge connected G is n-edge connected if $\lambda(G) \geq n$.

n connected G is n-connected if $\kappa(G) \geq n$.

Theorem 20.12. *Let G be a graph of order p and n be an integer such that $1 \leq n \leq p-1$. If $\delta(G) \geq \frac{p+n-2}{2}$, then G is n-connected.*

connection number $c(G)$ is the smallest integer such that $2 \leq c(G) \leq p$ and every subgraph of order n in G is connected.

l-connectivity $\kappa_l(G)$ is minimum number of vertices whose removal will produce a disconnected graph with at least l components or a graph with fewer than l vertices.

(n,l)-connected A graph G is (n,l)-connected if $\kappa_l(G) \geq n$.

Remark. *Exercises 5.5*

1. $\lambda(K_{m,n}) = \kappa(K_{m,n}) = m$

8. $c(K_p) = 2$, $c(K_{m,n}) = n+1$, $c(C_p) = p-1$
Every two vertices of complete graph of order p are adjacent. For complete bi-partitie graph $K_{m,n}$ such that $1 \leq m \leq n$, there exists a totally disconnected subgraph of order n. Therefore $c(K_{m,n}) \geq n+1$. And with $n+1$ vertices, both partitions have at least two vertices each and therefore the graph is connected and $c(K_{m,n}) \leq n+1$. For cycle C_p, any subgraph is disconnected if two non-adjacent vertices are deleted. Therefore $c(C_p) \geq p-1$. And C_p remains connected even after deletion of any vertex, therefore $c(C_p) \leq p-1$.

9. $$\delta(G) \geq \frac{p+(l-1)(n-2)}{l} \implies \kappa_l(G) \geq n$$

20.6 Menger's Theorem

Theorem 20.13. *For a non-trivial graph G, $\lambda(u,v) = M'(u,v)$ for every pair (u,v) of vertices of G.*

Corollary 20.13.1. *Graph G is n-edge connected iff every two vertices of G are connected by at least n edge disjoint paths.*

Theorem 20.14. *For every pair of non-adjacent vertices u, v in graph G, $\kappa(u,v) = M(u,v)$.*

Corollary 20.14.1. *Graph G is n-connected iff every pair of vertices of G are connected by at least n internally disjoint paths.*

Algorithm 20.15 (connectivity $\kappa(G)$)**.** .

1. *If degree of every vertex is $p-1$, then output $\kappa = p-1$ and stop. Otherwise, continue.*

2. *If G is disconnected, output $\kappa = 0$ and stop. Otherwise, continue.*

3. $\kappa \leftarrow p$

4. $i \leftarrow 0$

5. *If* $i \le \kappa$*, then* $i \leftarrow i + 1$ *and continue. Otherwise, output* κ *and stop.*

6. $j \leftarrow i + 1$

7. (a) *If* $j = p + 1$*, then return to step 5. Otherwise continue.*

   (b) *If* $v_i v_j \notin E(G)$*, construct network* N *with digraph* D *as follows : for each vertex* $v \in V(G)$*, there are two vertices* $v', v'' \in V(D)$ *and an arc* $(v', v'') \in E(D)$*. And for each edge* $uv \in E(G)$*, there are two arcs* $(u'', v), (v'', u) \in E(D)$*. The capacity function is given by,* $c(v', v'') = 1$ *for every* $v \in V(G)$ *and* $c(a) = \infty$ *for every other arc in* D*. Set source* $s = v''_i$ *and sink* $t = v'_j$ *and find maximum flow in* N *using max-flow min-cut algorithm. Otherwise proceed to step 7d*

   (c) *If* $f(N) < \kappa$*, then* $\kappa \leftarrow f(N)$*. Otherwise, continue.*

   (d) $j \leftarrow j + 1$ *and return to step 7a*

Chapter 21

Matchings and Factorizations

21.1 An Introduction to Matching

21.4 Maximum Matching in General Graphs

21.5 Factorizations

21.6 Block Designs

Bibliography

[Gray Chartrand,] Gray Chartrand, O. O. *Applied and Algorithmic Graph Theory.* Tata McGraw Hill Company.

[Joshi,] Joshi, K. D. *Introduction to General Topology.* Wiley Eastern Ltd.