Part I

ME010202 Advanced Topology

Chapter 7

Separation Axioms

7.1 Compactness and Separation Axioms

7.2 The Urysohn Characterisation of Normality

7.3 Tietze Characterisation of Normality

Chapter 8

Products and Coproducts

8.1 Cartesian Products of Families of Sets

8.2 The Product Topology

8.3 Productive Properties

Chapter 9

Embedding and Metrisation

9.1 Evaluation Functions into Products

9.2 Embedding Lemma and Tychonoff Embedding

9.3 The Urysohn Metrisation Theorem

Chapter 10

Nets and Filters

10.1 Definition and Convergence of Nets

Definitions 10.1. *A directed set D is a set with a binary relation $\geq$('follows') such that*

1. *The relation 'follows'($\geq$) is transitive. ie, $m \geq n,\ n \geq p \implies m \geq p$*
2. *The relation 'follows'($\geq$) is reflexive. ie, For every $m \in D,\ m \geq m$*
3. *For any $m, n \in D$, there exists $p \in D$ such that $p \geq m$ and $p \geq n$.*

sequence in a set X is a function f from the set of all integers into X.

Definitions 10.2. *A net in a set X is a function S from a directed set D into the set X.*

Remark. *The set $\mathbb{N}$ together with the relation 'less than or equal to'($\leq$) is a directed set. Clearly, the relation 'less than or equal to' is reflexive and trasitive. And the third condition is true iff every finite subset E of D has an element $p \in E$ such that p follows each element of E. This is a weaker notion compared to the well ordering principle[1] of the set of all integers. Thus $\mathbb{N}$ is a directed set and every sequence in X is also a net in X.*

Remark (Significance of Net)**.** *A net on a set is a generalisation of 'a sequence on a set' obtained by simplifying the domain of the sequence into a directed set. The notion directed set is derived by assuming a few properties of $\mathbb{N}$.*

The convergence of sequence is not strong enough to characterise topologies as the limit of convergent sequences are unique for both Hausdorff and Co-countable spaces. The notion of Net allows us to differentiate between Hausdorff spaces from Co-countable spaces in terms of convergence of nets. The limit of a convergent net on a topological space is unique iff it is a Hausdorff space. ie, We have removed a few restrictions, so that we will have some convergent nets (which are obviously not sequences) with multiple limit points for Co-countable spaces.

[1] Well-ordering principle : Every subset of $\mathbb{N}$ has a least element in it.

Remark. *Examples of Directed Sets*

1. *Let X be a topological space and $x \in X$. Then the neighbourhood system $\mathcal{N}_x$ is a directed set with the binary relation $\subset$ (subset/inclusion).*
   (a) *Let U, V, W be any three neighbourhoods of $x \in X$ such that $U \subset V$ and $V \subset W$. Then, clearly $U \subset W$.*
   Therefore, $U \geq V,\ V \geq W \implies U \geq W$.
   (b) *Let U be any neighbourhood of $x \in X$, then $U \subset U$.*
   Therefore, $U \geq U$.
   (c) *Let U, V be any two neighbourhoods of $x \in X$, then there exists their intersection $W = U \cap V$, which is a neighbourhood of x. Clearly $W \subset U$ and $W \subset V$.*
   Therefore $\forall U, V \in \mathcal{N}_x, \exists W \in \mathcal{N}_x$ such that $W \geq U$ and $W \geq V$.

2. *Let $\mathcal{P}$ be the set of all partitions on closed unit interval $[0, 1]$. A partition $P \in \mathcal{P}$ is a refinement of $Q \in \mathcal{P}$ if every subinterval in P is contained in some subinterval of Q. Then $\mathcal{P}$ with the binary relation refinement is a directed set.*

   For example, let $P = \{0,\ 0.3,\ 0.7,\ 1\}$. Then the subintervals in P are $[0, 0.3]$, $[0.3, 0.7]$ and $[0.7, 1]$. Let $Q = \{0,\ 0.3,\ 0.5,\ 1\}$ and $R = \{0,\ 0.3,\ 0.5,\ 0.7,\ 1\}$. Then R is a refinement of P, but Q is not a refinement of P since there is a subinterval $[0.5, 1]$ in Q which is not properly contained in any subinterval of P. However, R is a refinement of Q as well.

   (a) *Suppose P, Q, R are three partitions of $[0, 1]$ such that P is a refinement of Q and Q is a refinement of R, then clearly P is a refinement of R since each subinterval of P is contained some subinterval of Q, which is contained in some subinterval of R.*
   Therefore, $P \geq Q,\ Q \geq R \implies P \geq R$
   (b) *Suppose P is a partition of $[0, 1]$. Then trivialy, P is a refinement of itself since every subinterval of P is contained in the same subinterval of P.*
   Therefore, $\forall P \in \mathcal{P},\ P \geq P$
   (c) *Suppose P, Q be any two partition of $[0, 1]$. Then $R = P \cup Q$ is a refinement of both the partitions.*
   Therefore $\forall P, Q \in \mathcal{P},\ \exists R \in \mathcal{P}$ such that $R \geq P$ and $R \geq Q$

Remark. *Examples of Nets*

1. *Let X be a topological space and $x \in X$. Let $\mathcal{N}_x$ be the set of all neighbourhoods of x. Let $D = (\mathcal{N}_x, X)$ be the directed set given by $(N, y) \in (\mathcal{N}_x, X)$ if $N \in \mathcal{N}_x$ and $y \in N$ and $(N, y) \geq (M, z)$ if $N \subset M$. Then the function $S : (\mathcal{N}_x, X) \to X$ given by $S(N, y) = y$ is a net on X.*

   For example, let $X = \{a, b, c, d\}$ and $\mathcal{T} = \{\{a\},\ \{a, b\},\ \{a, b, c\},\ \{a, b, c, d\}\}$. Also let $S : (\mathcal{N}_b, X) \to X$ defined by $S(N, y) = y$. Suppose $C = \{a, b, c\}$. Then $C \in \mathcal{N}_b$. ie, C is a neighbourhood of b. Then $S(C, c) = c$.

2. *Riemann Net - Let $D = (\mathcal{P}, \xi)$ where $\mathcal{P}$ is the set of all partitions on $[0,1]$ and ξ is a finite sequence in $[0,1]$ such that consecutive terms belongs to consecutive subintervals of the partition. The set $(\mathcal{P}, \xi)$ is directed set with $\geq$ given by $(P, \eta) \geq (Q, \psi)$ iff P is a refinement of Q.*

   For example, let $P \in \mathcal{P}$ is given by $P = \{\ 0,\ 0.3,\ 0.7,\ 1\ \}$ and $\eta = \{\ 0.2,\ 0.6,\ 0.9\ \}$. Then $(P, \eta) \in (\mathcal{P}, \xi)$.

   Let $f : \mathbb{R} \to \mathbb{R}$ be any function, then the function,

$$S : (\mathcal{P}, \xi) \to \mathbb{R} \text{ defined by } S(P, \eta) = \sum_{j=1}^{k} f(\eta_k)(a_k - a_{k-1})$$

   where $P = \{a_0,\ a_1, \cdots,\ a_k\ \}$ is the Riemann Net with respect to the real function f.

   For example, let $f(x) = x^2$ and P, η are same as above example, then $S(P, \eta) = 0.2^2(0.3 - 0) + 0.6^2(0.7 - 0.3) + 0.9(1 - 0.7) = 3.99$

Definitions 10.3. *A net $S : D \to X$ converges to a point $x \in X$ if for any nbd U of x, there exists $m \in D$ such that $n \in D,\ n \geq m \implies S(n) \in U$*

Remark. *The choice of m depends on the choice of neighbourhood U.*

$S : D \to X,\ S \to x \iff (\forall U \in \mathcal{N}_x,\ \exists m_U \in D,\ \text{such that } n \geq m \implies S(n) \in U)$

Theorem 10.4. *A topological space is Hausdorff iff limits of all nets in it are unique.*

Proof. Let X be a Hausdorff space. Suppose $S : D \to X$ is net on X such that S converges to two distinct points $x, y \in X$. Since X is a Hausdorff space and $x \neq y$, there exists open sets U, V such that $x \in U,\ y \in V, U \cap V = \phi$.

The net S converges to $x \in X$, therefore $\exists m_x \in D$ such that $n \geq m_x \implies S(n) \in U$ And, the net S converges to $y \in X$, therefore $\exists m_y \in D$ such that $n \geq m_y \implies S(n) \in V$.

Since D is a directed set and $m_x, m_y \in D$, there exists $p \in D$ such that $p \geq m_x$ and $p \geq m_y$. Now, $n \geq p \implies n \geq m_x,\ n \geq m_y$, since $\geq$ is transitive. (ie, $n \geq p,\ p \geq m_x \implies n \geq m_x$, and $n \geq p,\ p \geq m_y \implies n \geq m_y$).

We have $n \geq p \implies n \geq m_x$ and $n \geq m_x \implies S(n) \in U$. Therefore, $n \geq p \implies S(n) \in U$. Similarly, $n \geq p \implies n \geq m_y \implies S(n) \in V$. Therefore $S(n) \in U \cap V$ which is a contradiction, since $U \cap V = \phi$. Therefore, if a net S converges to two points x, y, then $x = y$. That is, if a net S in a Hausdorff space X is convergent then its limit is unique.

Conversely, suppose that X is a topological space and every convergent net in X has a unique limit. Suppose X is not a Haudorff space. Then there exists at least two distinct points $x, y \in X$ such that every neighbourhood of x

intersects with every neighbourhood of y. Now consider the set $D = \mathcal{N}_x \times \mathcal{N}_y$ and relation $\geq$ on D such that $(U_1, V_1) \geq (U_2, V_2)$ if $U_1 \subset U_2$ and $V_1 \subset V_2$.–to be continued– $\square$

Chapter 11

Compactness

11.1 Variations of Compactness

In this chapter, we have two other notions of compactness - countable compactness and sequential compactness.[1]

Compact A topological space is compact iff every open cover of it has a finite subcover. ([Joshi, , 6.1.1]) [Heine-Borel]

Countably Compact A topological space is countably compact iff every countable, open cover of it has a finite subcover. [Joshi, , 11.1.1]

Sequentially Compact A topological space is sequentially compact iff every sequence in it has a convergent subsequence. [Joshi, , 11.1.8] [Bolzano-Weierstrass]

Countable compactness is a weaker notion compared to compactness.[2] However, sequentially compact and compact are not necessarily comparable.[3].

We have seen earlier that compactness has the following properties 1. compactness is weakly hereditary.[Joshi, , 6.1.10] 2. compactness is preserved under continuous functions.[Joshi, , 6.1.8] 3. every continuous real functions on compact space is bounded and attains its extrema.[Joshi, , 6.1.6] 4. every continuous real function on a compact, metric space is uniformly continuous by Lebesgue covering lemma.[Joshi, , 6.1.7]

Countably compact spaces, Sequentially compact spaces have all the four properites listed above.

11.1.1 Countable compactness

Weakly hereditary property

A subspace $(A, \mathcal{T}_{/A})$ being countably compact doesn't imply that $(X, \mathcal{T})$ is countably compact. However, if $(X, \mathcal{T})$ is a countably compact space and A

[1]For $\mathbb{R}$, Compactness & Sequentially compactness are equivalent to the completeness axiom.
[2]Every compact space is countably compact.
[3]$\mathcal{T}_1, \mathcal{T}_2$ are non-comparable, if $\mathcal{T}_1 \not\subset \mathcal{T}_2$ and $\mathcal{T}_2 \not\subset \mathcal{T}_1$.[Joshi, , 4.2.1]

is a closed subset of X, then $(A, \mathcal{T}_{/A})$ is also a countably compact space. In other words, countably compactness is weakly hereditary.

Theorem 11.1. *Countable compactness is weakly hereditary.[Joshi, , 11.1.3]*

Synopsis. *Let A be a closed subset of countably compact space, X. If A has a countable open cover $\mathcal{U}$, then we can obtain a respective countable, open cover for X by attaching $X - A$ to the extensions of members of $\mathcal{U}$ to X. This cover has a finite subcover. Then restricting them to A, we get a finite subcover of $\mathcal{U}$.*

Proof. Suppose X is a countably compact space. And A is a closed subset of X. We need to show that A is countably compact. Without loss of generality,[4] assume that A is a proper subset of X. Then $X - A$ is a non-empty, open subset of X.

Let $\mathcal{U}$ be a countable open cover of A. Then $\mathcal{U} = \{U_1, U_2, \cdots\}$ where each element $U_k \in \mathcal{U}$ is an open subset of A. Since A is a subspace of X, every open set U_k in A is of the form $G \cap A$ for some open set G in X. Therefore, there exists open sets $V(U_k)$ for each U_k such that $A \cap V(U_k) = U_k$.[5]

Define $\mathcal{V} = \{X - A, V(U_1), V(U_2), \cdots\}$. Clearly, $\mathcal{V}$ is a countable open cover[6] of X. We have X is countably compact, thus $\mathcal{V}$ has a finite subcover, say $\mathcal{V}'$. Without loss of generality assume[7] that $X - A \in \mathcal{V}'$. Suppose $X - A \notin \mathcal{V}'$, then we can define another finite subcover $\mathcal{V}' \cup \{X - A\}$. Thus $\mathcal{V}' = \{X - A,\ V(U_{n_1}),\ V(U_{n_2}), \cdots,\ V(U_{n_k})\}$.

Then the corresponding subcover $\mathcal{U}' = \{U_{n_1}, U_{n_2}, \cdots, U_{n_k}\}$ is a finite subcover of $\mathcal{U}$. Since countable open cover $\mathcal{U}$ and closed subset A are arbitrary, every closed subset of X with relative topology is countably compact. Therefore, countable compactness is weakly hereditary. $\square$

Remark. *Proof depends on the following,*

1. *There is an extension map, $\psi : P(A) \to P(X)$ that preserve open sets (and closed sets). This ψ is an open map which not a true inverse of the restriction, $r : P(X) \to P(A)$, defined by $r(G) = G \cap A$ for every subset G of X.*
2. *Also we rely on the subset A being closed. Suppose X have many countable open covers, but X has only uncountable open covers corresponding to a particular countable open cover of A. In such a case, X being countably compact is insufficient for A to be countably compact.*

The behaviour of countinous functions

We will now study the nature of continuous functions defined on countably compact spaces. Suppose X, Y are topological space and function $f : X \to Y$ is continuous. If X is countably compact, then $f(X)$ is also countably compact.

[4] Suppose A is not a proper subset of X. Then $X = A$ and A is countably compact.
[5] Relative topology, $\mathcal{T}_{/A} = \{G \cap A : G \in \mathcal{T}\}$
[6] $X - A$ is open in X. If $y \notin A$, then $y \in X - A$. If $y \in A$, then $y \in U_k$ for some k.
[7] Otherwise, you will have to consider two cases: $X - A \in \mathcal{V}'$ and $X - A \notin \mathcal{V}'$

Continuous images of countably compact spaces are countably compact. In other words, countable compactness is preserved under continuous functions.[8]

Theorem 11.2. *Countable compactness is preserved under continuous functions.[Joshi, , 11.1.2]*

Synopsis. *Let X be countably compact and $f : X \to Y$ be continuous. Suppose $\mathcal{U}$ is a countable cover of $f(X)$, then X has a countable cover $\mathcal{V}$ obtained by taking inverse images. Since X is countably compact, $\mathcal{V}$ has a finite subcover $\mathcal{V}'$. Now taking images of members of $\mathcal{V}'$, we get a finite subcover $\mathcal{U}'$ of $f(X)$.*

Proof. Suppose X is a countably compact space, Y is a topological space and $f : X \to Y$ is a continuous function. Let $\mathcal{U} = \{U_1, U_2, \cdots\}$ be a countable cover of $f(X)$ by set open in $f(X)$. We have to show that $\mathcal{U}$ has a finite subcover.

Define $\mathcal{V} = \{f^{-1}(U_1), f^{-1}(U_2), \cdots\}$. Then $\mathcal{V}$ is a countable open cover of X, since $f^{-1}(U_k)$ are open subsets of X and,

$$\begin{aligned}\bigcup_{k=1}^{\infty} U_k = f(X) &\implies f^{-1}\left(\bigcup_{k=1}^{\infty} U_k\right) = X \\ &\implies \bigcup_{k=1}^{\infty} f^{-1}(U_k) = X\end{aligned}$$

We have, $\mathcal{V}$ is a countable open cover of X, which is a countably compact space. Therefore $\mathcal{V}$ has a finite subcover $\mathcal{V}' = \{f^{-1}(U_{n_1}),\ f^{-1}(U_{n_2}), \cdots,\ f^{-1}(U_{n_k})\}$.

$$\begin{aligned}\bigcup_{j=1}^{k} f^{-1}(U_{n_j}) = X &\implies f^{-1}\left(\bigcup_{j=1}^{k} U_{n_j}\right) = X \\ &\implies \bigcup_{j=1}^{k} U_{n_j} = f(X)\end{aligned}$$

Clearly $\mathcal{U}' = \{U_{n_1}, U_{n_2}, \cdots, U_{n_k}\}$ is a finite subcover of $\mathcal{U}$. Thus every countable open cover of $f(X)$ by sets open in $f(X)$ has a finite subcover. Therefore, continuous images of countably compact spaces are countably compact. □

Remark. *1. For a continuous function, $f : X \to Y$ the inverse images of open sets are open in X. The relation $f^{-1} \subset f(X) \times X$ is not a function. However, we may consider a function, $\psi : P(Y) \to P(X)$ such that $\psi(U) = f^{-1}(U)$ for any subset U of Y. This ψ is an open map which maps open subsets of Y to open subsets of X.*

Theorem 11.3. *Every continuous, real-valued function on a countably compact, metric space is bounded and attains its extrema.[Joshi, , 11.1.7]*

[8] A topological property is preserved under continuous functions if whenever a space has that property so does every continuous image of it.[Joshi, , 6.1.9]

Synopsis. *Let X be a countably compact space and function $f : X \to \mathbb{R}$ be continuous. Then $f(X) \subset \mathbb{R}$ is countably compact. Real line $\mathbb{R}$ is metrisable[9]. Then $f(X)$ is countably compact, metric space. Therefore $f(X)$ compact.[10]. The subset $f(X)$ of $\mathbb{R}$ is bounded and closed, since every compact subset of $\mathbb{R}$ is bounded and closed. Thus $f(X)$ contains its supremum and infimum. Therefore, f is bounded and attains its extrema.*

Proof. Let X be a countably compact space and $f : X \to \mathbb{R}$ be continuous, real-valued function on the countably compact space, X. We have to show that f is bounded and attains its extrema.

Since countable compactness is preserved under continuous functions, $f(X)$ is countably compact subset of $\mathbb{R}$. Since, $f(X)$ is a subset of the metric space, $\mathbb{R}$ and metrisability is hereditary, $f(X)$ is again metrisable. (suppose) We have, every countably compact, metric space is compact. Then $f(X)$ is a compact subset of $\mathbb{R}$.

Since every compact subset of $\mathbb{R}$ is bounded and closed, $f(X)$ is bounded and closed. Since every closed subset of $\mathbb{R}$ contains supremem and infimum, $f(X)$ contains its extrema. Therefore, every continuous, real-valued function on a countably compact space is bounded and attains its extrema.

We have assumed that every countably compact, metric space is compact. This result will be proved in the last section of this chapter. □

Remark. *Since countably compact, metric spaces are compact. The above theorem can be used to prove that continuous, real-valued functions on a compact, metric space attains its extrema.*

Due to the Lebesgue covering lemma, next result is quite simple.*

Theorem 11.4. *Every continuous, real-valued function on a countably compact, metric space is uniformly continuous.*

Proposition 11.5. *Let X be a first countable, Hausdorff space. Then every countably compact subset A of X is closed.[Joshi, , Exercises 11.1.7]*

11.1.2 Sequential Compactness

Weakly hereditary property

Theorem 11.6. *Sequential compactness is weakly hereditary.[Joshi, , Exercises 11.1.3]*

The behaviour of countinous functions

Theorem 11.7. *Sequential compactness is preserved under continuous functions.[Joshi, , Exercises 11.1.4]*

[9] [Joshi, , 4.2 Example 4], $\mathbb{R}$ with usual metric $d : R \to R,\ d(x, y) = |x - y|$

[10] [Joshi, , 11.1.11] On metric spaces, countable compactness $\implies$ compactness.

Synopsis. *Let X be sequentially compact and function $f : X \to Y$ be continuous. Then any sequence, $\{y_k\}$ in $f(X)$ has a sequence, $\{x_k\}$ in X such that $f(x_k) = y_k$. Sequence $\{x_k\}$ has a subsequence $\{x_{n_k}\}$ converging to x, then sequence $\{f(x_n)\}$ in $f(X)$ has the subsequence $\{f(x_{n_k})\}$ converging to $f(x)$.*

Proof. Let X be a sequentially compact space, function $f : X \to Y$ be continuous and $\{y_n\}$ be a sequence in $f(X)$ subset of Y. Construct a sequence $\{x_n\}$ such that $f(x_k) = y_k,\ \forall k$.

Every sequence in X has a convergent subsequence. Thus $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to $x \in X$. The image of this subsequence $\{f(x_{n_k})\}$ is a subsequence of $\{y_k\}$. We claim that, $\{f(x_{n_k})\}$ converges to $f(x) \in f(X)$.

Let U be an open set containing $f(x)$, then $f^{-1}(U)$ is an open set containing x. Since $\{x_{n_k}\}$ converges to x. There exists an integer n such that for every $k \geq n$, $x_k \in f^{-1}(U)$. Clearly, for each $k \geq n$, $f(x_k) \in U$. Since U is arbitrary, $\{f(x_{n_k})\}$ converges to $f(x)$. Therefore, the image of any sequentially compact space is sequentially compact. In other words, sequentially compactness is preserved under continuous functions. □

Remark. *1. Given a sequence $\{y_n\}$ in $f(X)$, there is a sequence of subsets $\{U_n\}$ in $P(Y)$ such that $U_n = f^{-1}(y_n)$. Since each U_n is non-empty, we can construct a sequence $\{x_n\}$ in X using a choice function. The convergent subsequence of $\{y_n\}$ depends on the selection of this choice function.*

Given every sequentially compact, metric space is countably compact. We may assert the properties of countably compact, metric spaces on sequentially compact, metric spaces.

Theorem 11.8. *Every continuous, real-valued function on a sequentially compact, metric space is bounded and attains its extrema.*

Theorem 11.9. *Every continuous, real-valued function on a sequentially compact, metric space is uniformly continuous.[Joshi, , Exercises 11.1.6]*

11.1.3 Countable Compactness on T_1 spaces

In this section, we are going to see four different characterisations of countable compactness in T_1 spaces. The first two characterisations doesn't have anything to do with the T_1 axiom.

T_1 **Space** A topological space X satisfy T_1 axiom if for any two distinct points $x, y \in X$, there exists an open set $U \subset X$ containing x but not y.[Joshi, , 7.1.2]

countable compactness A topological space is countably compact if every countable open cover has a finite subcover.[Joshi, , 11.1.1]

finite intersection property A family $\mathcal{F}$ of subsets of X has finite intersection property(f.i.p.) if every finite subfamily of $\mathcal{F}$ has a non-empty intersection.[Joshi, , 10.2.6]

accumulation point A point $x \in X$ is accumulation point of a subset $A \subset X$ if every open set containing x has atleast one point of A other than x.[Joshi, , 5.2.7]

limit point A point $x \in X$ is a limit point of a sequence $< x_k >$ in X if for every open set U containing x, there exists an integer $N \in \mathbb{N}$ such that $x_k \in U$ for every $k \geq N$.[Joshi, , 4.1.7]

cluster point A point $x \in X$ is a cluster point of a sequence $< x_k >$ in X if for any neighbourhood V of x, the sequence $< x_k >$ assumes a point in V infinitely many times.[11]

Countable compactness in T_1 spaces

Theorem 11.10. *In a T_1 space X, following statements are equivalent,*

1. *X is countably compact*
2. *Every countably family of closed subsets of X with finite intersection property have non-empty intersection.*
3. *Every infinite subset $A \subset X$ has an accumulation point.*[12]
4. *Every sequence $< x_k >$ in X has a cluster point.*
5. *Every infinite open cover of X has a proper subcover.[Arens-Dugundji]*

Proof. $1 \implies 2$
Suppose X is countably compact. Let $\mathcal{C} = \{C_1, C_2, \cdots\}$ be a countable family of closed subsets of X with empty intersection. Define $\mathcal{U} = \{X - C_1, X - C_2, \cdots\}$ is a family of open subsets of X. By de Morgan's law, [13]

$$\bigcap_{k=1}^{\infty} C_k = \phi, \text{ then } X = X - \left(\bigcap_{k=1}^{\infty} C_k\right) = \bigcup_{k=1}^{\infty} (X - C_k)$$

We have $\mathcal{U}$ is a countable cover of X and X is countably compact space. Thus $\mathcal{U}$ has a finite subcover $\mathcal{U}' = \{X - C_{n_1}, X - C_{n_2}, \cdots, X - C_{n_k}\}$.

$$\mathcal{U}' \text{ is a cover of } X, \text{ then } X = \bigcup_{j=1}^{k} \left(X - C_{n_j}\right)$$

$$X - \bigcup_{j=1}^{k} \left(X - C_{n_j}\right) = \bigcap_{j=1}^{k} \left(X - \left(X - C_{n_j}\right)\right) = \bigcap_{j=1}^{k} C_{n_j} = \phi$$

Now $\mathcal{C}' = \{C_{n_1}, C_{n_2}, \cdots, C_{n_k}\}$ has empty intersection. This is a contradiction to the finite intersection property of $\mathcal{C}$. Thus $\mathcal{C}$ has non-empty intersection. Therefore, every countably family of closed subsets of X have non-empy intersection.

[11] x is a cluster point of $< x_k >$ if for every integer N, there exists $k > N$ such that $x_k \in V$.In other words, $< x_k >$ is frequently in V.[Joshi, , 10.1.9]
[12] Every infinite subset of $\mathbb{R}$ has a limit point is equivalent to the completeness axiom.
[13] Complement of Intersection = Union of complements, $X - (C \cap D) = (X - C) \cup (X - D)$,

$2 \implies 1$

Let $\mathcal{U} = \{U_1, U_2, \cdots\}$ be a countable cover of X. Then $\mathcal{C} = \{X - U_1, X - U_2, \cdots\}$ is a countable family of closed subsets of X.

Let $\mathcal{U}' = \{U_{n_1}, U_{n_2}, \cdots, U_{n_k}\}$ be any finite subfamily of $\mathcal{U}$. Suppose X is not countably compact, then $\mathcal{U}$ doesn't have a finite subcover. Therefore, $\mathcal{U}'$ is not a cover of X. And $\mathcal{C}$ is a family of closed sets with finite intersection property.

Therefore by assumption, the countable family of closed sets $\mathcal{C}$ has a non-empty intersection.

$$\bigcap_{k=1}^{\infty} C_k \neq \phi, \text{ then } \bigcap_{k=1}^{\infty} C_k = \bigcap_{k=1}^{\infty} (X - U_k) = X - \left(\bigcup_{k=1}^{\infty} U_k\right) \neq \phi$$

Then $\mathcal{U}$ is not a cover of X as well. This is a contradiction, therefore X is countably compact.

$1 \implies 3$
Suppose X is countably compact. Let A be an infinite subset of X. Suppose A doesn't have an accumulation point.

Let B be a countably infinite subset of A. Then B also doesn't have any accumulation point. Therefore, the derived set B' is empty. Thus B is a closed subset of X. Since countable compactness is weakly hereditary, subspace B is again countably compact.

For each point $b \in B$, there is an open set V_b such that $V_b \cap B = \{b\}$, since $b \in B$ is not an accumulation point. Thus $\mathcal{U} = \{V_b \cap B : b \in B\}$ is a countable open cover of B. Clearly, $\mathcal{U}$ doesn't have any finite subcover.

This is a contradiction to B being countably compact. Therefore, A has an accumulation point. □

11.1.4 Variations of Compactness on Metric Spaces

In this document, we will see that from metric space point of view these two notions were equivalent to the compactness and were used alternatively. For example : in functional analysis (semester 3), you will find definitions like 'a normed space is compact iff every sequence in it has a convergent subsequence', which is clearly sequential compactness for a topologist.

Lindeloff A topological space is Lindeloff iff every open cover has a countable subcover.

First countable A topological space is first countable iff every point in it has a countable local base.

Second countable A topological space is second countable iff it has a countable base.

Base A family of subsets $\mathcal{B}$ of X is a base of a topological space if every open set can be expressed as union of some members of $\mathcal{B}$

Base Characterisation A family of subsets $\mathcal{B}$ of X is a base of a topological space iff for every $x \in X$, and for every neighbourhood U of x, there is a member $B \in \mathcal{B}$ such that $x \in B \subset U$.

Local Base A family of subsets $\mathcal{L}$ of X is a local base at point $x \in X$ if for every neighbourhood U of x, there is a member $L \in \mathcal{L}$ such that $x \in L \subset U$.

Equivalence

We are going to see when these three notions: compactness, countable compactness and sequentially compactness are equivalent.

Theorem 11.11. *Countably compact, metric spaces are second countable.*

Synopsis. *For every positive real number r, there exists a non-empty maximal subsets A_r with every pair of points atleast r distance apart. A_r are finite. The union of maximal subsets $A_{\frac{1}{n}}$ for each natural number n is a countable, dense subset D of X. Thus countably compact, metric spaces are separable. The family $\mathcal{B}$ of all open balls with center at $d \in D$ and rational radius is a countable, base for X. Thus countably compact, metric spaces are second countable.*

Proof. Let $(X; d)$ be a countably compact,, metric space. For each positive real number $r \in \mathbb{R},\ r > 0$ construct a family of subsets $A_r \subset X$ such that it is a maximal set of points which are atleast r distances apart.

Then A_r is finite for every positive real number r. Suppose A_r is infinite for some real number $r > 0$, then A_r has a accumulation point, say x by the Characterisation of countable compactness of X.

Then every neighbourhood of x must intersect A_r at infinitely many points, since every metric space is a T_1 space. Consider $B(x, \frac{r}{2})$. Since any two points of $B(x, \frac{r}{2})$ are less than r distances apart, the intersection $B(x, \frac{r}{2}) \cap A_r$ can have atmost one point in it. Thus for every positive real number r, A_r is finite.

Define $D = \cup_{n=1}^{\infty} A_{\frac{1}{n}}$. We claim that D is a countable, dense subset of X.

Let $x \in X$ and $B(x, r)$ be an open ball containing x, then there exists integer $n \in \mathbb{N}$ such that $\frac{1}{n} < r$.[14]

Then $B(x, r) \cap D \neq \phi$, since $B(x, r) \cap A_{\frac{1}{n}} \neq \phi$. Suppose $B(x, r) \cap A_{\frac{1}{n}} = \phi$, then $A_{\frac{1}{n}}$ is not maximal. Since, x is atleast $r > \frac{1}{n}$ distance apart from each points of $A_{\frac{1}{n}}$. Therefore, D intersects with every open set and thus dense in X.

We have have a countable, dense subset D of X. Therefore, X is separable. Now define $\mathcal{B} = \{B(x, r) : r \in \mathbb{Q},\ x \in D\}$. Clearly, $\mathcal{B}$ is a countable base for X. By the construction of $\mathcal{B}$, X is second countable.[15] □

[14] By archimedean property of integers, we have $\forall r \in \mathbb{R},\ r > 0,\ \exists n \in \mathbb{N}$ such that $nr > 1$.
[15] Every separable, metric space is second countable.