Differential Geometry

Module II

Chapter 7 : Geodesics
Straight Lines on an n-Surface

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Vector Field along Parameterised Curve

Vector Field(Ref : Chapter 2)

- $\mathbf{X}(p) = (p, X(p))$ where $X : U \to \mathbb{R}^{n+1}$, $U \underset{\text{open}}{\subset} \mathbb{R}^{n+1}$
- For each point p in U, a unique vector $X(p)$ is assigned

Vector Field along Curve α

- $\mathbf{X}(\alpha(t)) = (\alpha(t), X(t))$ where $\alpha : I \to U$, $X : I \to \mathbb{R}^{n+1}$, $I \underset{\text{open}}{\subset} \mathbb{R}$ and $U \underset{\text{open}}{\subset} \mathbb{R}^{n+1}$
- For each point on the curve α, vectors are assigned depending on the value of the parameter t

Function along Parameterised Curve

Definition (Function along α)

Let $\alpha : I \to \mathbb{R}^{n+1}$. Function along α is a real-valued function defined on the same parameter interval I. That is, $f : I \to \mathbb{R}$

$$\mathbf{X}(\alpha(t)) = (\alpha(t), X(t))$$

Remark : For vector field $\mathbf{X}$ along α, the component functions X_j of the associated function X are functions along α.

$$\mathbf{X}(\alpha(t)) = (\alpha(t), X_1(t), X_2(t), \cdots, X_{n+1}(t))$$

Velocity and Speed of a Curve

Let $\alpha : I \to \mathbb{R}^{n+1}$ be a parametrised curve.

Definition (Velocity)

Velocity of α is a vector field $\dot{\boldsymbol{\alpha}}$ along α defined by

$$\dot{\boldsymbol{\alpha}}(t) = \left(\alpha(t), \frac{d}{dt}\alpha(t)\right)$$

Definition (Speed)

Speed of α is a function $\|\dot{\alpha}\|$ along α defined by

$$\|\dot{\alpha}\| : I \to \mathbb{R}, \ \|\dot{\alpha}\|(t) = \|\dot{\alpha}(t)\|$$

Example : $\alpha(t) = (t, t^2)$ (Ref : Exercise 7.1a)

Velocity, $\dot{\boldsymbol{\alpha}}(t) = (t, t^2, 1, 2t), \ \forall t \in I$

Speed, $\|\dot{\alpha}\|(t) = \|(1, 2t)\| = \sqrt{1 + 4t^2}, \ \forall t \in I$

Acceleration of a Curve

Definition (Acceleration)

Acceleration of α is the vector field $\ddot{\alpha}(t)$ along α is defined by

$$\ddot{\boldsymbol{\alpha}}(t) = \left(\alpha(t), \frac{d^2}{dt^2}\alpha(t)\right)$$

Example : Acceleration of the parametrised curve, α
$\alpha : I \to \mathbb{R}^2, \ \alpha(t) = (t, t^2)$ is $\ddot{\boldsymbol{\alpha}}(t) = (t, t^2, 0, 2)$

Differentiation along a Curve

Definition (Differentiating Vector Field along Curve)

Let $\mathbf{X}(\alpha(t))$ be a vector field along α.
The derivative of $\mathbf{X}$ along α is $\dot{\mathbf{X}}$ along α (or simply $\dot{\mathbf{X}}$),

$$\dot{\mathbf{X}}(\alpha(t)) = \left(\alpha(t), \frac{d}{dt}X(t)\right)$$

Properties of Differentiation

1. $\dot{\overline{\mathbf{X} + \mathbf{Y}}} = \dot{\mathbf{X}} + \dot{\mathbf{Y}}$

$$\frac{d}{dt}(X(t) + Y(t)) = \frac{d}{dt}X(t) + \frac{d}{dt}Y(t)$$

2. $\dot{\overline{f\mathbf{X}}} = f'\mathbf{X} + f\dot{\mathbf{X}}$

$$\frac{d}{dt}f(t)X(t) = \left(\frac{d}{dt}f(t)\right)X(t) + f(t)\left(\frac{d}{dt}X(t)\right)$$

3. $(\mathbf{X} \cdot \mathbf{Y})' = \dot{\mathbf{X}} \cdot \mathbf{Y} + \mathbf{X} \cdot \dot{\mathbf{Y}}$

$$\frac{d}{dt}X_1(t)Y_1(t) = \left(\frac{d}{dt}X_1(t)\right)Y_1(t) + X_1(t)\left(\frac{d}{dt}Y_1(t)\right)$$

Geodesic

Definition (Geodesic)

A geodesic is an n-surface S is a parametrised curve $\alpha : I \to S$ with acceleration orthogonal to S.

- Acceleration of geodesics is orthogonal to the Surface S.

$$\ddot{\boldsymbol{\alpha}}(t) \in S^{\perp}_{\alpha(t)}, \ \forall t \in I$$

- Velocity of geodesics is tangent to the Surface S.

$$\dot{\boldsymbol{\alpha}}(t) \in S_{\alpha(t)}, \ \forall t \in I$$

- Geodesics have constant speed, since $\dot{\alpha}(t) \cdot \ddot{\alpha}(t) = 0$.

$$(\dot{\alpha}(t) \cdot \dot{\alpha}(t))' = \ddot{\alpha}(t) \cdot \dot{\alpha}(t) + \dot{\alpha}(t) \cdot \ddot{\alpha}(t) = 2\dot{\alpha}(t) \cdot \ddot{\alpha}(t)$$

$$\frac{d}{dt}\|\dot{\alpha}(t)\|^2 = 2\frac{d}{dt}\dot{\alpha}(t) \cdot \ddot{\alpha}(t) = 0$$

Maximal Geodesic

Theorem (Maximal Geodesic)

- *n-surface S*
- $p \in S$ *(through a point p)*
- $\mathbf{v} = (p, v) \in S_p$ *(with constant/starting velocity v)*
- $\exists$ *open interval I containing* 0 *and*
- $\exists$ *unique, maximal geodesic* $\alpha : I \to S$
  - $\alpha(0) = p \in S$
  - $\dot{\alpha}(0) = \mathbf{v} = (p, v) \in S_p$ *and*
  - α *is maximal (and unique)*
    If there is another geodesic $\beta : \tilde{I} \to S$ *with* $\beta(0) = p$ *and* $\dot{\beta}(0) = \mathbf{v}$, *then* $\tilde{I} \subset I$ *and* $\beta(t) = \alpha(t), \ \forall t \in \tilde{I}$

Proof : Maximal Geodesic

Step 1 : Conditions for Geodesic

- n-surface S, $S = f^{-1}(c)$, smooth $f : U \to \mathbb{R}$, $U \underset{\text{open}}{\subset} \mathbb{R}^{n+1}$
  and $\nabla f(p) \neq 0$, $p \in S$
- WLOG $\nabla f(p) \neq 0$, $\forall p \in U$
  If not, restrict U to such an open set containing S
- Vector Field $\mathbf{N}$ on U ($\mathbf{N}$ restricted to S is an orientation)
  $\mathbf{N}(p) = (p, N(p))$, where $N(p) = \dfrac{\nabla f(p)}{\|\nabla f(p)\|}$, $\forall p \in U$
  $\mathbf{N}$ is well-defined since U is open and $\|\nabla f(p)\| \neq 0$
- $S_p^\perp$ is one-dimensional and $\mathbf{N}(p) \in S_p^\perp$, $\mathbf{N}(p) \neq 0$, $\forall p \in S$
  Thus, $\ddot{\alpha}$ is a scalar multiple of $\mathbf{N}$
- Parametrised Curve $\alpha : I \to S$ is geodesics if and only if
  $\ddot{\alpha}(t) = g(t)\mathbf{N}(\alpha(t))$ where $g : I \to \mathbb{R}$ (scalar depends on t)

Proof : Maximal Geodesic

Step 2 : α geodesic $\iff \ddot{\boldsymbol{\alpha}} + (\dot{\boldsymbol{\alpha}} \cdot \mathbf{N} \dot{\circ} \alpha)(\mathbf{N} \circ \alpha) = 0$

$$\ddot{\boldsymbol{\alpha}} = g(\mathbf{N} \circ \alpha)$$
$$\ddot{\boldsymbol{\alpha}} \cdot \mathbf{N} \circ \alpha = g(\mathbf{N} \circ \alpha) \cdot (\mathbf{N} \circ \alpha) = g$$

We have, $(\dot{\boldsymbol{\alpha}} \cdot \mathbf{N} \circ \alpha)' = \ddot{\boldsymbol{\alpha}} \cdot \mathbf{N} \circ \alpha + \dot{\boldsymbol{\alpha}} \cdot \mathbf{N} \dot{\circ} \alpha$ (by property 3)

$$\implies \ddot{\boldsymbol{\alpha}} \cdot \mathbf{N} \circ \alpha = (\dot{\boldsymbol{\alpha}} \cdot \mathbf{N} \circ \alpha)' - \dot{\boldsymbol{\alpha}} \cdot \mathbf{N} \dot{\circ} \alpha \qquad (1)$$

$$\begin{aligned}(1) \implies g &= \ddot{\boldsymbol{\alpha}} \cdot \mathbf{N} \circ \alpha \\ &= (\dot{\boldsymbol{\alpha}} \cdot \mathbf{N} \circ \alpha)' - \dot{\boldsymbol{\alpha}} \cdot \mathbf{N} \dot{\circ} \alpha \\ &= -\dot{\boldsymbol{\alpha}} \cdot \mathbf{N} \dot{\circ} \alpha, \text{ since } \dot{\boldsymbol{\alpha}} \perp \mathbf{N} \circ \alpha\end{aligned}$$

$$\ddot{\boldsymbol{\alpha}} + (\dot{\boldsymbol{\alpha}} \cdot \mathbf{N} \dot{\circ} \alpha)(\mathbf{N} \circ \alpha) = 0 \qquad (2)$$

Proof : Maximal Geodesic

Step 3 : Existence and Uniqueness of α

$$\alpha(t) = (x_1(t), x_2(t), \cdots, x_{n+1}(t)) \tag{3}$$

$$\begin{aligned}\ddot{\boldsymbol{\alpha}}(t) &= \left(\alpha(t), \frac{d^2}{dt^2}\alpha(t)\right) \\ &= \left(x_1(t), \cdots, x_{n+1}(t), \frac{d^2}{dt^2}x_1(t), \cdots, \frac{d^2}{dt^2}x_{n+1}(t)\right)\end{aligned} \tag{4}$$

$$\begin{aligned}\mathbf{N} \circ \alpha(t) &= (\alpha(t), N(\alpha(t))) \\ &= (\alpha(t), N_1(\alpha(t)), N_2(\alpha(t)), \cdots, N_{n+1}(\alpha(t)))\end{aligned} \tag{5}$$

$$\begin{aligned}\mathbf{N} \dot{\circ}\, \alpha(t) &= \left(\alpha(t), \frac{d}{dt}N \circ \alpha(t)\right) \\ &= \left(\alpha(t), \frac{d}{dt}N_1(\alpha(t)), \frac{d}{dt}N_2(\alpha(t)), \cdots, \frac{d}{dt}N_{n+1}(\alpha(t))\right)\end{aligned} \tag{6}$$

Proof : Maximal Geodesic

$$
\begin{aligned}
\frac{d}{dt}N_1(\alpha(t)) =& \frac{\partial}{\partial x_1}N_1(x_1, x_2, \cdots, x_{n+1})\frac{d}{dt}x_1(t) \\
&+ \frac{\partial}{\partial x_2}N_1(x_1, x_2, \cdots, x_{n+1})\frac{d}{dt}x_2(t) \\
&\cdots \\
&+ \frac{\partial}{\partial x_{n+1}}N_1(x_1, x_2, \cdots, x_{n+1})\frac{d}{dt}x_{n+1}(t) \qquad (7)
\end{aligned}
$$

$$
\frac{d}{dt}N_1(\alpha(t)) = \sum_{k=1}^{n+1}\frac{\partial}{\partial x_k}N_1(x_1, x_2, \cdots, x_{n+1})\frac{d}{dt}x_k(t) \qquad (8)
$$

Proof : Maximal Geodesic

$$
\begin{aligned}
\dot{\boldsymbol{\alpha}} \cdot (\mathbf{N} \dot{\circ} \alpha) =& \frac{d}{dt}x_1(t)\ \frac{d}{dt}N_1(\alpha(t)) + \frac{d}{dt}x_2(t)\ \frac{d}{dt}N_2(\alpha(t)) + \\
& \cdots + \frac{d}{dt}x_{n+1}(t)\ \frac{d}{dt}N_{n+1}(\alpha(t)) && (9) \\
=& \sum_{j=1}^{n+1} \frac{d}{dt}x_j(t)\ \sum_{k=1}^{n+1} \frac{\partial}{\partial x_k} N_j(x_1, x_2, \cdots, x_{n+1}) \frac{d}{dt}x_k(t) \\
=& \sum_{j,k=1}^{n+1} \frac{\partial N_j}{\partial x_k}\ \frac{dx_k}{dt}\ \frac{dx_j}{dt} && (10)
\end{aligned}
$$

Proof : Maximal Geodesic

$$\ddot{\boldsymbol{\alpha}} + (\dot{\boldsymbol{\alpha}} \cdot (\mathbf{N} \dot{\circ} \alpha))(\mathbf{N} \circ \alpha) = 0$$

Equating components to zero, we get the following system of second order differential equations

$$\begin{aligned}
\frac{d^2}{dt^2}x_1(t) + N_1(\alpha(t)) \sum_{j,k=1}^{n+1} \frac{\partial N_j}{\partial x_k} \frac{dx_k}{dt} \frac{dx_j}{dt} &= 0 \\
\frac{d^2}{dt^2}x_2(t) + N_2(\alpha(t)) \sum_{j,k=1}^{n+1} \frac{\partial N_j}{\partial x_k} \frac{dx_k}{dt} \frac{dx_j}{dt} &= 0 \\
\cdots &= 0 \\
\frac{d^2}{dt^2}x_{n+1}(t) + N_{n+1}(\alpha(t)) \sum_{j,k=1}^{n+1} \frac{\partial N_j}{\partial x_k} \frac{dx_k}{dt} \frac{dx_j}{dt} &= 0
\end{aligned} \tag{11}$$

Proof : Maximal Geodesic

By existence theorem‡[1] for solution of such equations,

- There exists an open interval I containing 0
- There exists solution $\beta : I \to U$ with $\beta_1(0) = p$, $\dot{\beta}_1(0) = (p, v)$ and
- If there exists another solution $\tilde{\beta} : \tilde{I} \to U$ with $\tilde{\beta}(0) = p$, $\dot{\tilde{\beta}}(0) = (p, v)$ then $\beta(t) = \tilde{\beta}(t), \ \forall t \in I \cap \tilde{I}$

Maximal, Unique Solution

- Suppose there exists solutions $\beta_1, \beta_2, \cdots, \beta_k$
- $I = I_1 \cup I_2 \cup \cdots \cup I_k$
- $\alpha : I \to U$ defined by $\alpha(t) = \beta_j(t), \ t \in I_j$
- Then α is maximal (and unique) geodesic on S through p with initial velocity v if it is a curve on S

[1] The proof of existence theroems of differential equations is not required

Proof : Maximal Geodesic

Step 4 : α is a curve on S

$$
\begin{aligned}
(\dot{\boldsymbol{\alpha}} \cdot \mathbf{N} \circ \alpha)' &= \ddot{\boldsymbol{\alpha}} \cdot \mathbf{N} \circ \alpha + \dot{\boldsymbol{\alpha}} \cdot (\mathbf{N} \dot{\circ} \alpha) \\
&= \left[\ddot{\boldsymbol{\alpha}} + (\dot{\boldsymbol{\alpha}} \cdot (\mathbf{N} \dot{\circ} \alpha))(\mathbf{N} \circ \alpha)\right] \cdot (\mathbf{N} \circ \alpha) \\
&= \mathbf{0}.(\mathbf{N} \circ \alpha) = 0 \\
\Longrightarrow \quad (\dot{\boldsymbol{\alpha}} \cdot \mathbf{N} \circ \alpha) &= \text{constant}
\end{aligned}
$$

$$
(\dot{\boldsymbol{\alpha}} \cdot \mathbf{N} \circ \alpha)(0) = \mathbf{v} \cdot \mathbf{N}(p) = 0, \ \text{since } \mathbf{v} \in S_p, \ \mathbf{N}(p) \in S_p^{\perp}
$$

$$
\begin{aligned}
(f \circ \alpha)'(t) = \nabla f(\alpha(t)) \cdot \dot{\boldsymbol{\alpha}}(t) &= \|\nabla f(\alpha(t))\| \mathbf{N}(\alpha(t)) \cdot \dot{\boldsymbol{\alpha}}(t) = 0 \\
\Longrightarrow f \circ \alpha &= \ \text{constant}
\end{aligned}
$$

But, $f(\alpha(0)) = f(p) = c$, since $p \in S = f^{-1}(c)$

Thus, $f \circ \alpha(t) = c \implies \alpha(t) \subset S = f^{-1}(c), \ \forall t \in I$

Thank You