Differential Geometry

Module II

Chapter 6 : The Gauss Map

July 8, 2021

Gauss Map and Spherical Image

Oriented n-Surface

- n-surface, S
- Orientation, $\mathbf{N}(p) = (p, N(p))$

Definition (Gauss Map)

The associated funcion of the smooth, unit normal vector field $\mathbf{N}$ on the n-surface S is the **Gauss Map**, $N : S \to S^n$

Definition (Spherical Image)

The image of the Gauss map $N(S) = \{q \in S^n : q = N(p),\ p \in S\}$ is the **spherical image** of the oriented n-surface S.

Compact, Connected, Oriented n-Surface

Theorem (Spherical Image of Oriented n-Surface)

- *Compact, Connected, Oriented n-Surface S*
- *The Gauss Map $N : S \to S^n$ is surjective.*
- *The Spherical Image is $N(S) = S^n$, unit n-sphere.*

Importance of Compactness

- Counter-example : n-Plane S
- $N(S) = \left\{ \frac{-\nabla f(p)}{\|\nabla f(p)\|} : p \in S \right\}$ is singleton
- If oriented n-surface S is compact and connected, then S divides $\mathbb{R}^{n+1}$ into two parts - inside and outside
- Compute the distance of $q \in \mathbb{R}^{n+1}$ from an n-surface S ?

Proof : Spherical Image of Oriented n-Surface

Step 1 : Lagrange multiplier theorem

- $v \in S^n$
- $g : \mathbb{R}^{n+1} \to \mathbb{R},\ g(p) = p \cdot v$
- Level Sets $g^{-1}(c)$ are n-planes parallel to $v^\perp$
- By Lagrange multiplier theorem, the restriction of g to n-surface S attains maximum and minimum at p, q
  - $(p, v) = \nabla g(p) = \lambda \nabla f(p) = \lambda \|\nabla f(p)\| \mathbf{N}(p)$
  - $(q, -v) = \nabla g(q) = \lambda \nabla f(q) = \lambda \|\nabla f(q)\| \mathbf{N}(q)$

    $\implies N(p) = \pm v$ and $N(q) = \pm v$
- By intermediate value theorem, if there exists a continuous function $\alpha : [0, 1] \to \mathbb{R}^{n+1}$ such that
  - $\alpha(0) = p$, $\alpha(1) = q$, $\dot{\alpha}(0) = (p, v)$, $\dot{\alpha}(1) = (q, v)$
  - $\alpha(t) \notin S,\ 0 < t < 1$

  then $N(p) \neq N(q) \implies N(p) = v$ OR $N(q) = v$

Proof : Spherical Image of Oriented n-Surface

Step 2 : Construction of α

- $\exists S_1$ such that $S \subset S_1$ since S is bounded(compact)
- $0 < x < y < 1$
- $\alpha_1 : [0, x] \to \mathbb{R}^{n+1}, \ \alpha_1(t) = p + tv$
- $\alpha_2 : [y, 1] \to \mathbb{R}^{n+1}, \ \alpha_2(t) = q + (t-1)v$
- $\alpha_3 : [x, y] \to S_1$ such that
  - $\alpha_3(x) = \alpha_1(x) = p + xv$
  - $\alpha_3(y) = \alpha_2(y) = q + (y-1)v$
- $$\alpha(t) = \begin{cases} \alpha_1(t) & t \in [0, x) \\ \alpha_3(t) & t \in [x, y] \\ \alpha_2(t) & t \in (y, 1] \end{cases}$$

Proof : Spherical Image of Oriented n-Surface

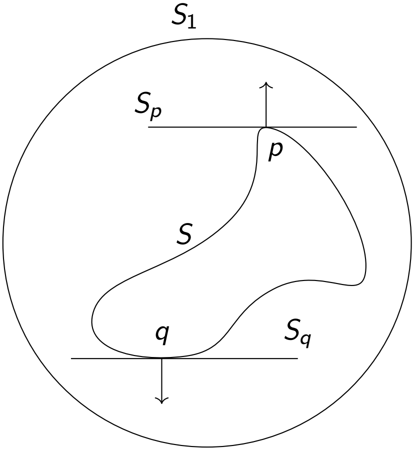

Figure: Construction of α

Proof : Spherical Image of Oriented n-Surface

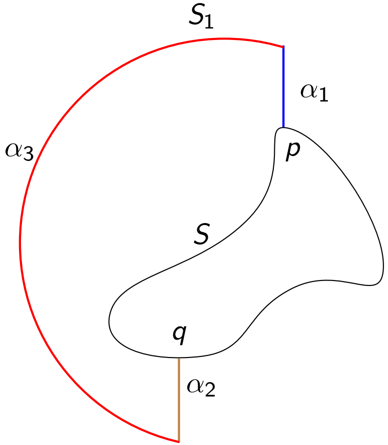

Figure: Construction of α

Proof : Spherical Image of Oriented n-Surface

Step 3 : $N(p) \neq N(q)$

- n-Surface $S = f^{-1}(c) \implies f(p) = c, \; \forall p \in S$
- $f \circ \alpha(0) = c$ and $f \circ \alpha(1) = c$
- $\dot{\alpha}(0) = v$ and $\dot{\alpha}(1) = v$
- $(f \circ \alpha)'(t) = \nabla f(\alpha(t)) \cdot \dot{\alpha}(t)$ (by Chain Rule)
  $(f \circ \alpha)'(0) = \|\nabla f(p)\| \mathbf{N}(p) \cdot (p, v) = \|\nabla f(p)\| N(p) \cdot v$
  $(f \circ \alpha)'(1) = \|\nabla f(q)\| \mathbf{N}(q) \cdot (q, v) = \|\nabla f(q)\| N(q) \cdot v$
- Suppose $N(p) = N(q)$
  $\implies (f \circ \alpha)'(0)$ and $(f \circ \alpha)'(1)$ are of the same sign
- Case 1 : $f \circ \alpha$ is increasing at both 0 and 1
  - For $\epsilon > 0$, $f \circ \alpha(\epsilon) > c$ and $f \circ \alpha(1 - \epsilon) < c$
  - $\exists t$ such that $t \in (0, 1)$ and $f \circ \alpha(t) = c$
  - $\exists t \in (0, 1)$ such that $\alpha(t) \in S$ contradicts $\alpha(t) \notin S, t \in (0, 1)$
- Case 2 : $f \circ \alpha$ is decreasing at both 0 and 1
  - For $\epsilon > 0$, $f \circ \alpha(\epsilon) < c$ and $f \circ \alpha(1 - \epsilon) > c$

Thank You