Real Analysis

Module II

Section 6.15
Properties of Riemann Stieltjes Integral

December 4, 2021

6.15(a) Linearity

$f_1 + f_2$

$$f_1, f_2 \in \mathscr{R}(\alpha) \implies f_1 + f_2 \in \mathscr{R}(\alpha)$$

cf

$$f \in \mathscr{R}(\alpha) \implies cf \in \mathscr{R}(\alpha)$$

Riemann Stieltjes integral is a linear functional from vector space $\mathscr{R}(\alpha)$ over the field $\mathbb{R}$ into $\mathbb{R}$ itself.

Linearity : Proof

Criterion for Integrability

$$f \in \mathscr{R}(\alpha) \iff \forall \varepsilon > 0,\ \exists P \in \mathscr{P}[a,b],\ U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

Fix $\varepsilon > 0$

$$f_1 \in \mathscr{R}(\alpha) \implies \exists P_1 \in \mathscr{P},\ U(P_1,f_1,\alpha) - L(P_1,f_1,\alpha) < \varepsilon$$

$$f_2 \in \mathscr{R}(\alpha) \implies \exists P_2 \in \mathscr{P},\ U(P_2,f_2,\alpha) - L(P_2,f_2,\alpha) < \varepsilon$$

Linearity : Proof

Refinement, $P = P_1 \cup P_2$

The inequalities are true for their common refinement.

$$U(P, f_1, \alpha) - L(P, f_1, \alpha) < \varepsilon$$

$$U(P, f_2, \alpha) - L(P, f_2, \alpha) < \varepsilon$$

Linearity : Proof

Minimum & Maximum $f_1 + f_2$

$$\min f_1 + f_2 \geq \min f_1 + \min f_2$$

$$\max f_1 + f_2 \leq \max f_1 + \max f_2$$

$$\begin{aligned} L(P, f_1, \alpha) + L(P, f_1, \alpha) &\leq L(P, f_1 + f_2, \alpha) \\ &\leq U(P, f_1 + f_2, \alpha) \\ &\leq U(P, f_1, \alpha) + U(P, f_2, \alpha) \end{aligned}$$

Linearity : Proof

$f \in \mathscr{R}(\alpha)$

$$\begin{aligned} U(P, f_1 + f_2, \alpha) - L(P, f_1 + f_2, \alpha) &\leq U(P, f_1, \alpha) - L(P, f_1, \alpha) \\ &\quad + U(P, f_2, \alpha) - L(P, f_2, \alpha) \\ &< 2\varepsilon \end{aligned}$$

Thus,

$$f_1 + f_2 \in \mathscr{R}(\alpha)$$

Linearity : Proof

Value of Integral

$$U(P, f_1, \alpha) - \int_a^b f_1 \; d\alpha < \varepsilon$$

$$U(P, f_2, \alpha) - \int_a^b f_2 \; d\alpha < \varepsilon$$

$$\begin{aligned} U(P, f_1 + f_2, \alpha) &\leq U(P, f_1, \alpha) + U(P, f_2, \alpha) \\ &\leq \int_a^b f_1 \; d\alpha + \int_a^b f_2 \; d\alpha + 2\varepsilon \end{aligned}$$

Linearity : Proof

We have,

$$\int_a^b (f_1 + f_2)\ d\alpha \leq \int_a^b f_1\ d\alpha + \int_a^b f_2\ d\alpha$$

Considering $-f_1, -f_2$,

$$\int_a^b (-f_1 - f_2)\ d\alpha \leq \int_a^b -f_1\ d\alpha + \int_a^b -f_2\ d\alpha$$

$$\implies \int_a^b (f_1 + f_2)\ d\alpha \geq \int_a^b f_1\ d\alpha + \int_a^b f_2\ d\alpha$$

Thank You