Differential Geometry

Module I

Chapter 5 : Vector Fields on Surfaces, Orientation

June 14, 2021

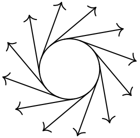
Vector Fields on Surfaces

Definition (Vector Field on a surface)

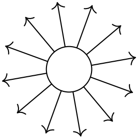
- n-Surface, $S \subset \mathbb{R}^{n+1}$
- Vector Field on S, $\mathbf{X}(p) = (p, X(p)),\ \forall p \in S$
  where $X : S \to \mathbb{R}^{n+1} \implies X(p) \in \mathbb{R}^{n+1} \implies \mathbf{X}(p) \in \mathbb{R}^{n+1}_p$

Vector Field on 1-sphere

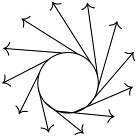
$f(x, y) = x^2 + y^2$ and $S = f^{-1}(1) \subset \mathbb{R}^2$
$\nabla f(a, b) = (a, b, 2a, 2b) \neq (x, y, 0, 0),\ \forall (x, y) \in S$



(a) $X(x, y) = (2y, -2x)$



(b) $X(x, y) = (2x, 2y)$



(c) $X(x, y) = (-2y, 2x + 0.5)$

Figure: Vector Fields over 1-sphere

Smooth Vector Field over a Surface

Vector Field and Surface

- n-surface, $S = f^{-1}(c) \subset \mathbb{R}^{n+1}$
  where $f : U \to \mathbb{R}$ and $\nabla f(p) \neq 0, \ \forall p \in S$
- Vector Field, $\mathbf{X}(p) = (p, X(p))$, where $\mathbf{X}(p) \in \mathbb{R}_p^{n+1}$ and
- Associated function, $X : S \to R^{n+1}$

Definition (Smooth Vector Field over Surface)

$\mathbf{X}$ on S is **smooth** if the associated function X has a smooth extension to an open set containing S

- $\exists$ open set $V, \ S \subset V \subset U \subset \mathbb{R}^{n+1}$ and
- $\exists \tilde{X} : V \to R^{n+1}$ is smooth
  where $\tilde{X} : V \to R^{n+1}, \ X(p) = \tilde{X}(p), \ \forall p \in S$

Unique Integral Curve on Surface

Theorem (Maximal, Integral Curve on Surface)

- *$\mathbf{X}$ smooth, tangent vector field on n-surface S*
- $\forall p \in S$
- $\exists$ *open interval I containing 0 and*
- $\exists$ *parameterised curve $\alpha : I \to S$ such that there exists a unique, maximal, integral curve on S through p*
  - $\alpha(0) = p$
  - $\dot{\alpha}(t) = \mathbf{X}(\alpha(t)), \ \forall t \in I$
  - $\beta : \tilde{I} \to S$ *such that* $\tilde{I} \subset I$ *and* $\beta(t) = \alpha(t), \ \forall t \in \tilde{I}$

Proof : Maximal, Integral Curve on Surface

- Smooth, Vector Field $\mathbf{X}$ on S
  - $\mathbf{X}$ on S is smooth, then extension $\tilde{X}$ smooth on open set V
  - S is n-surface, then we have open set U such that
    $f : U \to \mathbb{R}$, $S = f^{-1}(c) \subset U$, and $\nabla f(p) \neq 0, \ \forall p \in S$
- Smooth, Vector Field on an open set W containing S
  $W = \{q \in U \cap V : \nabla f(q) \neq 0\}$

$$\mathbf{Y}(q) = \tilde{\mathbf{X}}(q) - \frac{\tilde{\mathbf{X}}(q) \cdot \nabla f(q)}{\|\nabla f(q)\|^2} \nabla f(q), \ \forall q \in W$$

- Maximal Integral Curve α on $\mathbf{Y}$ through p (Ref : Chap. 2)
  $\alpha : I \to W, \ \dot{\alpha}(t) = \mathbf{Y}(\alpha(t)), \forall t \in I$ and $\alpha(0) = p$
- Maximal Integral Curve α on S through p
  $(f \circ \alpha)'(t) = \nabla f(\alpha(t)) \cdot \dot{\alpha}(t) = \nabla f(\alpha(t)) \cdot \mathbf{Y}(\alpha(t)) = 0$
  $p \in f^{-1}(c) \implies f(\alpha(0)) = c \implies f \circ \alpha(t) = c$
  $\implies \alpha(I) \subset f^{-1}(c) = S \implies \alpha : I \to S$
  $\dot{\alpha}(t) = \mathbf{Y}(\alpha(t)) = \mathbf{X}(\alpha(t)) \implies \dot{\alpha}(t) = \mathbf{X}(\alpha(t))$

Connectedness and Components

Definition (Connectedness)

Subset S of $\mathbb{R}^{n+1}$ is connected if for any $p, q \in S$, there a continuous function $\alpha : [a, b] \to S$ such that $\alpha(a) = p$ and $\alpha(b) = q$. That is, there a path connecting any two points.

Definition (Connected Component)

The equivalence classes of S under connectedness are the connected components of an n-surface, S.

Orientation

Theorem (Oriented n-surface)

- $\forall$ *connected n-surface, S*
- $\exists$ *two unit normal vector fields* $\mathbf{N_1}, \mathbf{N_2}$ *and*
- $\mathbf{N_2}(p) = -\mathbf{N_1}(p), \ \forall p \in S$

Proof.

- n-surface S
  $\implies S = f^{-1}(c), \ f : U \to \mathbb{R}, \ \nabla f(p) \neq 0, \ \forall p \in S$

$$\mathbf{N_1}(p) = \frac{\nabla f(p)}{\|\nabla f(p)\|}, \ \forall p \in S$$

- $\mathbf{N_2}(p) = -\mathbf{N_1}(p)$
- $\exists \mathbf{N_3} \in S_p^{\perp} \implies \mathbf{N_3}(p) = g(p)\mathbf{N_1}(p) = \pm\mathbf{N_1}(p)$

$\square$

Orientation

Definition (Orientation)

A smooth, unit normal vector field on an n-surface S is an **orientation** of S

Definition (Oriented Surface)

- n-surface, S
- an orientation of S, $\mathbf{N}$

Möbius Band is not an n-surface

- Möbius Band, B doesn't have two orientations
- There doesn't exists a smooth function f such that $B = f^{-1}(c),\ f : U \to \mathbb{R},\ \nabla f(p) \neq 0,\ \forall p \in B$

Positive Tangent Direction

Definition (direction)

A unit vector in $\mathbb{R}^{n+1}_p$ is a **direction** at p

Definition (Positive Tangent Direction)

- 1-surface/Plane Curve, C
- Orientation, N
- **Positive Tangent Direction** at p is obtained by rotating orientation at p by $-\pi/2$ in anticlockwise direction.
- If $N(p) = (x, y)$, then the positive tangent direction at p is $T(p) = (-y, x)$

Positive θ-Rotation

Definition (Positive θ-Rotation)

- 2-surface, S
- Orientation, $\mathbf{N}$
- Positive θ-rotation, $R_\theta : S_p \to S_p$
  $R_\theta(\mathbf{v}) = \cos\theta\mathbf{v} + \sin\theta\mathbf{N}(p) \times \mathbf{v}$
- R_θ is the Right-handed rotation about $\mathbf{N}(p)$ through Angle θ

Consistent Basis

Definition (Consistent Basis)

- 3-surface S
- Orientation $\mathbf{N}$
- $\mathcal{B} = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ be ordered basis for S_p
- Consistent Basis, $\mathcal{B}$ if

$$\det \begin{pmatrix} \mathbf{e_1} \\ \mathbf{e_2} \\ \mathbf{e_3} \\ \mathbf{N}(p) \end{pmatrix} \text{ is positive}$$

- Inconsistent basis if determinant is negative.

Thank You