Advanced Abstract Algebra

Module I

Section 29 : Extension Fields

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Contents

Basic Goal - Kronecker's Theorem

- Field F
- Non-constant Polynomial, $f(x) \in F[x]$
- Existence of field extension E such that
  - $F \underset{\text{field}}{\leq} E$
  - $\exists \alpha \in E,\ f(\alpha) = 0$

Proof.

- $p(x)$ irreducible factor of $f(x)$
- $\langle p(x) \rangle$ is maximal ideal
- $E = F[x]/\langle p(x) \rangle$
- $\psi : F \to E,\ \psi(a) = a + \langle p(x) \rangle \implies F \simeq \psi[F] \underset{\text{field}}{\leq} E$
- $\alpha = x + \langle p(x) \rangle \implies p(\alpha) = \langle p(x) \rangle$ † [1]

□

[1] $0 + \langle p(x) \rangle$ is the additive identity of $F[x]/\langle p(x) \rangle$

Algebraic Number and its Degree

algebraic over F $\alpha \in E$ is algebraic over F if there exists $f(x) \in F[x]$ such that $f(\alpha) = 0$
ex. $\pi \in \mathbb{C}$ is algebraic over $\mathbb{R}$, $\because x - \pi \in \mathbb{R}[x]$

algebraic number A complex number $\alpha \in \mathbb{C}$ is an algebraic number if there exists $f(x) \in \mathbb{Q}$ such that $f(\alpha) = 0$
ex. $\sqrt{-1} \in \mathbb{C}$ is algebraic number, $\because x^2 + 1 \in \mathbb{Q}[x]$

transcendental $=$ not algebraic
ex. $\pi \in \mathbb{C}$ is transcendental over $\mathbb{Q}$

$irr(\alpha, F)$ is the **unique** monic, irreducible polynomial $p(x) \in F[x]$ such that $p(\alpha) = 0$
ex. $irr(\sqrt{3}, \mathbb{Q}) = x^2 - 3$

$deg(\alpha, F)$ is the degree of the monic, irreducible polynomial $p(x) \in F[x]$ such that $p(\alpha) = 0$
ex. $deg(\sqrt{3}, \mathbb{Q}) = 2$

Uniqueness of Irreducible Polynomial $irr(\alpha, F)$

- Existence by Kronecker's Theorem
- $\phi_\alpha : F[x] \to E$ has $\ker(\phi_\alpha) = \langle p(x) \rangle$ for some $p(x) \in F[x]$
- $f(x) \in \ker(\phi_\alpha) \implies p(x) | f(x)$
- $p(x)$ is irreducible over F since E has no zero divisors
  - $p(x) = r(x)s(x) \implies p(\alpha) = r(\alpha)s(\alpha) = 0$
  - $r(\alpha) = 0$ OR $s(\alpha) = 0$ which is not possible

Simple Extension

- There exists field E containing F and α (Kronecker)
- $F(\alpha)$ is the smallest subfield of E containing both F and α. (hint : $F(\alpha) = \phi_\alpha[F[x]]$ where $\phi_\alpha : F[x] \to E$)
- $irr(\alpha, F) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n$
- $\beta \in F(\alpha) \implies \beta = b_0 + b_1 \alpha + \cdots + b_{n-1} \alpha^{n-1}$

Thank You