# Graph Theory

Module 1

Section 5 : Paths and Connectedness

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# Walk

## Definition (walk)

A walk is an alternating finite sequence $W : v_0 e_1 v_1 e_2 v_2 \ldots e_p v_p$ where $v_{j-1}$ and $v_j$ are end vertices of $e_j$ for $j = 1, 2, \ldots$.

The length of a walk is the number of edges in it.

- **origin** is the vertex $v_0$.
- **terminus** is the vertex $v_p$,
- **closed** A walk is closed if $v_0 = v_p$.
- **trail** is a walk in which every edges are distinct.
- **path** is a walk in which every vertex is distinct.
- **cycle** is a closed trail.

# Path

$C_k$ is a cycle of length $k$.

$P_k$ is a path on $k$ vertices.

Path, $P$ Let $P : v_1, e_1, v_2, \ldots, e_p, v_p$ be a path in $G$. Then we omit edges and write $P : v_1, v_2, \ldots, v_p$.

Inverse of $P$ The path $P' : v_p, v_{p-1}, \ldots, v_1$ is the inverse of $P$.

section of $P$ Let $P : v_1, v_2, \ldots, v_p$ be a path in $G$. A subsequence $v_j, v_{j+1}, \ldots, v_k$ is $v_j - v_k$ section of $P$.

# Connected

**Definition**

Two vertices $u, v$ are connected in a graph $G$ if there is a $u - v$ path in $G$.

**Definition**

A graph $G$ is connected if every pair of vertices in $G$ are connected.

# Equivalence Relation : Connectedness

reflexive Every vertex is connected to itself by trivial path.

symmetric every $u - v$ path is a $v - u$ path

transitive $u - v$ path followed by $v - w$ path contains a $u - w$ path.

### Definition (component)

Let $V_1, V_2, \dots, V_\omega$ be the equivalence classes of the relation 'connectedness' in $G$. Then the induced subgraphs $G[V_1], G[V_2], \dots, G[V_\omega]$ are the component of $G$.

- If $\omega = 1$, then $G$ is connected.
- If $\omega \geq 2$, then $G$ is disconnected.

# Metric Space

## Definition

Let $d$ be a function on the vertex set of a graph $G$ defined as $d(u, v)$ is the length of the shortest $u - v$ path in $G$.

## Function $d$ is a metric on $V(G)$

1. $d(u, v) \geq 0$ and $d(u, v) = 0 \iff u = v$.
2. $d(u, v) = d(v, u)$
3. $d(u, v) \leq d(u, w) + d(w, v)$.

# Minimum Degree of a Simple, Connected Graph

### Proposition

*If $G$ is simple and $\delta(G) \geq \frac{n-1}{2}$, then $G$ is connected.*

### Proof.

- Let $G$ be disconnected graph of order $n$.
- Suppose $G$ has at least two components.
- Number of vertices in each component is at least $\frac{n-1}{2} + 1$.
- Number of vertices in $G$ is $n + 1$.(contradiction)

$\square$

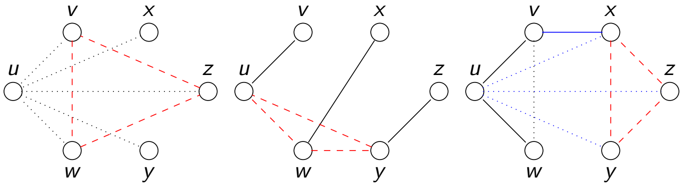
# Exercises : Minimum Degree

- There exists non-simple disconnected graph $G$ with $\delta(G) \geq (n-1)/2$.
  - Draw a graph with two components.
  - Use parallel edge or loops to increase minimum degree of the graph.
- $\delta(G) \geq (n-2)/2$ does not imply that $G$ is connected.
  - If $\delta(G) = 1$, then we have a graph of order 4, say $2K_2$
  - In general, $2K_{\delta(G)+1}$ has order $n = 2\delta(G) + 2$.

# Ramsey Number, $R(3,3) = 6$

In a group of six people, there must be three people who are mutually acquainted or three people are not mutually acquainted.

- Let $u \in V(G)$. Then $deg(u) \neq 0$
- $\Delta(G) \neq 1$.
- Let $deg(u) = 2$. Let $uv, uw \in E(G)$.
- $vw \notin E(G)$ otherwise $(u, v, w)$ forms $\triangle$.
- WLOG $vx \in E(G)$, otherwise $(v, w, x)$ forms $\triangle$.
- $ux \notin E(G)$, otherwise $(u, v, x)$ forms $\triangle$.
- Similarly, $uy, uz \notin E(G)$.
- $xy \in E(G)$, otherwise $(u, x, y)$ forms $\triangle$.
- Similarly, $yz, zx \in E(G)$.
- $(x, y, z)$ forms $\triangle$ (contradiction)



# Complement of a disconnected Graph

## Theorem

*If a simple graph $G$ is disconnected, then $G^c$ is connected.*

## Proof.

- Let $G$ be a disconnected graph.
- Let $G_1$ and $G_2$ be two components of $G$.
  - Let $u \in V(G_1)$ and $v \in V(G_2)$.
    $uv \notin G \implies uv \in G^c$.
  - Let $u, w \in V(G_1)$ and $v \in V(G_2)$.
    $uv, vw \notin G \implies u, v, w$ is a $u - w$ path in $G^c$.

$\square$

## Characterisation of Self complementary Graph

If $G$ is self complementary, then $n(G) \cong 0$ or $1 \pmod 4$.

- $m(G) = m(G^c) = m(K_n)/2$.
- $m = n(n-1)/4$.
- $m \cong 0$ or $1 \pmod 4$.

A self complementary graph with one pendant vertex must have at least another pendant vertex.

- Let $G$ be a self complementary graph with pendant vertex $u$.
- $G^c$ has a pendant vertex $v$.
- $v$ is another pendant vertex in $G$.

# Upper bound for the size of a Simple Graph

## Theorem

*The size $m$ of simple graph of order $n$ with $\omega$ components cannot exceed $(n-\omega)(n-\omega+1)/2$.*

$$m < \frac{(n-\omega)(n-\omega+1)}{2}$$

## Proof.

- Let $G$ be a graph of order $n$, size $m$, and components $\omega$.
- Let $G_1, G_2, \ldots, G_\omega$ be the components of $G$.
- Let $n_i, m_i$ be the order, size of $G_i$ for each $i$.
  - $n_i \leq (n-\omega+1)$.
  - $m_i \leq n_i(n_i-1)/2$.
- $m = \sum n_i(n_i-1)/2 < (n-\omega+1)\sum(n_i-1)$.
- $m < (n-\omega+1)(n-\omega)/2$.

□

# Local Connectedness

## Definition

A graph $G$ is locally connected if for each vertex $v$ in $G$, the subgraph induced by the open neighbourhood $N_G(v)$ is connected.


Figure: $G$ is locally connected at $x$, $u$ and $w$, but not at $v$

# Characterisation of Bipartite Graph

## Theorem

*A graph is bipartite if and only if it has no odd cycles.*

## Proof.

- Suppose $G$ is bipartite.
  - Let $u \in V(G)$.
  - Let $C : v_1, e_1, v_2, \ldots, v_k, e_k, e_1$ be a cycle in $G$.
  - Length of cycle $k$ is even, since $v_k$ adjacent to $v_1$.
- Suppose $G$ has no odd cycle.
  - Suppose $G$ is connected.
  - $X = \{v \in V(G) : d(u, v) \text{ is even.}\}$
  - $Y = \{v \in V(G) : d(u, v) \text{ is odd.}\}$
  - Let $v, w \in X$. Then $u - v$ path $P$ and $u - w$ path $Q$.
  - Let $w_1$ be a common vertex $P, Q$ such that $w_1 - w$ section of $P$ and $w_1 - v$ section of $Q$ has no other common vertices.
  - Suppose $v$ is adjacent to $w$.
  - $w - w_1 + w_1 - v + vw$ is an odd cycle.(contradiction)
  - If $G$ has components. Then $X = \cup X_i$ and $Y = \cup Y_i$.

A simple nontirivial graph $G$ is connected if and only if for any partition $(V_1, V_2)$ of $V(G)$ there is an edge joining a vertex of $V_1$ and a vertex of $V_2$.

- Suppose $G$ is connected.
  - Let $V_1, V_2$ be a partition of $V(G)$.
  - Let $u \in V_1$ and $v \in V_2$.
  - Since $G$ is connected, there exists $u - v$ path, say $P$.
  - $P$ contains an edge joining $V_1$ and $V_2$.
- Suppose that for every partition $(V_1, V_2)$, there exists an edge joining $V_1$ and $V_2$.
  - Let $u, v \in V(G)$.
  - Let $V_1 = \{u\}$. There exists $uw \in E(G)$ where $w \in V_2$.
  - Let $V_1 = \{u, w\}$. There exists $ux$ or $wx$ where $x \in V_2$.
  - Continuing like this, we get $V_1$ containing both $u$ and $v$.

# Longest paths does intersect

In a connected graph $G$ with at least three vertices, any two longest paths have a vertex in common.

- Let $u_1 - u_k$ and $v_1 - v_k$ be two longest paths, say $P, Q$.
- $u_1 - v_1$ path has three sections : $u_1 - u_r, u_r - v_s, v_s - v_1$.
- WLOG suppose $u_1 - u_r$ and $v_1 - v_s$ are at least half as long as longest paths.
- Then $(u_1 - u_r) + (u_r - v_s) + (v_s - v_1)$ is longer. (contradiction)

# Union of Disjoint Paths

Union of two disjoint paths joining two distinct vertices contains a cycle.

- Let $P, Q$ be two distinct $u - v$ paths.
- Let $P', Q'$ be disjoint sections of $P, Q$ with common end vertices.
- $P' + Q'$ is a cycle.

Union of two distinct walks joining two distinct vertices need not contain a cycle.

- Two walks are distinct if one walk visits an edge one more time compared to the other.

# Characterisation of an incomplete Graph

If a simple graph $G$ is not complete, there exists three vertices $u, v, w$ such that $uv, vw$ are edge of $G$ but $uw$ is not an edge of $G$.

# Generalised Petersen Graph

## Definition (Generalised Petersen Graph)

Generalised Petersen Graph is defined by

$$V(P(n,k)) = \{a_i, b_i : 0 \le i \le n-1\}$$

$$E(P(n,k)) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+k}, 0 \le i \le n-1\}$$

If $n$ is even and $k$ is odd, then generalised Petersen Graph $P(n,k)$ is bipartite.

If $G$ is simple and $\delta(G) \geq k$, then $G$ contains a path of length at least $k$.

- Each component of $G$ has at least $k+1$ vertices.
- Apply finite mathematical induction on $\delta(G)$.
  - $k = 1$ is trivial
  - Suppose graph of $\delta(G) = k-1$ has a path of length $k-1$.
  - Suppose $\delta(G) = k$.
  - Additional vertex which is adjacent to one of the end vertices.

Thank You