Abstract Algebra

Module 4

Section 24 : Noncommutative Examples

June 14, 2021

$End(A)$: Noncommutative Ring

$\langle End(A), +, \circ \rangle$ is well-defined only if A is an abelian group.

Let $\theta, \phi, \psi \in End(A),\ x \in A$

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   6.1 Multiplication is left-distributive over Addition.
   $\theta(\phi + \psi) = \theta\phi + \theta\psi$
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   $(\phi + \psi)\theta = \phi\theta + \psi\theta$

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8. Multiplication is non-commutative. $\theta\phi \neq \phi\theta$

Functions - Graduate Level Notions

well-defined function

Let $f : A \to B$ be a function. Then,

1. $\forall x \in A, \; f(x) \in B$
2. If $x_1 \xrightarrow{f} y_1$, and $x_2 \xrightarrow{f} y_2$, then $x_1 = x_2 \implies y_1 = y_2$

Example

- $f(x) = \sqrt{x}$

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- $f : \mathbb{R} \to \mathbb{R}, \; f(x) = \sqrt{x}$ is not well-defined. Since, $f(-1) \notin \mathbb{R}$

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Example

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- $f : \mathbb{R} \to \mathbb{R},\ f(x) = \sqrt{x}$ is not well-defined. Since, $f(-1) \notin \mathbb{R}$
- $f : \mathbb{R}^+ \to \mathbb{R},\ f(x) = \sqrt{x}$ is not well-defined. Since, $f(1) =?$

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Example

- $f(x) = \sqrt{x}$ is not appreciated.
- $f : \mathbb{R}^+ \to \mathbb{R}^+,\ f(x) = \sqrt{x}$ is preferred.
- $f : \mathbb{R} \to \mathbb{R},\ f(x) = \sqrt{x}$ is not well-defined. Since, $f(-1) \notin \mathbb{R}$
- $f : \mathbb{R}^+ \to \mathbb{R},\ f(x) = \sqrt{x}$ is not well-defined. Since, $f(1) =?$
- $f : \mathbb{R}^+ \to \mathbb{R}^+,\ f(x) = \sqrt{x}$ is well-defined.

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$\langle End(A), +, \circ \rangle$ is well-defined ?

Given $\langle A, * \rangle$ is an abelian group with idenitty e.

1. $End(A)$ is a non-empty set, since $0 \in End(A)$ where $0 : A \to A$ is defined by the relation $0(x) = e$.
2. Addition is well-defined.
   2.1 $\phi + \psi$ is an endomorphism of A
   2.2 $(\phi + \psi)(x)$ is uniquely defined for every $x \in A$
3. Multiplication/composition is well-defined.
   3.1 $\phi\psi$ is an endomorphism of A
   3.2 $(\phi\psi)(x)$ is uniquely defined for every $x \in A$.

Note : An endomorphism of A is a homomorphism from A into A.

Step 1 : $\phi + \psi \in End(A)$

Let $\phi, \psi \in End(A)$, and $x \in A$.

1. function $\phi + \psi : A \to A$ is well-defined,
   $(\phi + \psi)(x) = \phi(x) + \psi(x) \in A$
   1.1 ϕ, ψ are well-defined. $\phi(x), \psi(x) \in A$
   1.2 Group addition is closed.
2. $\phi + \psi : A \to A$ is a homomorphism
   $(\phi + \psi)(x + y) = (\phi + \psi)(x) + (\phi + \psi)(y)$

$$
\begin{aligned}
(\phi + \psi)(x + y) &= \phi(x + y) + \psi(x + y) && (1)\\
&= (\phi(x) + \phi(y)) + (\psi((x) + \psi(y)) && (2)\\
&= \phi(x) + ((\phi(y) + \psi(x)) + \psi(y)) && (3)\\
&= \phi(x) + ((\psi(x) + \phi(y)) + \psi(y)) && (4)\\
&= (\phi(x) + \psi(x)) + (\phi(y) + \psi(y)) && (5)\\
&= (\phi + \psi)(x) + (\phi + \psi)(y) && (6)
\end{aligned}
$$

Step 2 : $(\phi + \psi)(x)$ is uniquely defined

Let $\phi, \psi \in End(A)$, and $x \in A$.

1. $\phi(x), \psi(x)$ are uniquely defined in A.
   $\forall x \in A,\ \forall \phi \in End(A),\ \exists$ unique $\phi(x) \in A$.
2. Group addition is uniquely defined in A.
   $\phi(x), \psi(x) \in A \implies \exists$ unique $\phi(x) + \psi(x) \in A$

Step 3 : $\phi\psi \in End(A)$

Let $\phi, \psi \in End(A)$, and $x \in A$.

1. function $\phi\psi : A \to A$ is well-defined, $\phi\psi(x) = \phi(\psi(x))$
   1.1 $\phi(\psi(x)) = \phi(y) \in A$ where $y = \psi(x) \in A$
   1.2 $\psi(x), \phi(y) \in A$ where $y \in \psi(x) \in A$.
2. $\phi\psi = \phi \circ \psi$ is a homomorphism.

$$
\begin{aligned}
(\phi\psi)(x+y) &= \phi(\psi(x+y)) && (7)\\
&= \phi(\psi(x) + \psi(y)) && (8)\\
&= \phi(\psi(x)) + \phi(\psi(y)) && (9)\\
&= \phi\psi(x) + \phi\psi(y) && (10)
\end{aligned}
$$

Step 4 : $(\phi\psi)(x)$ is uniquely defined

Let $\phi, \psi \in End(A)$, and $x, y \in A$.

1. $\psi(x) = y$ is uniquely defined in A.
2. $\psi(x), \phi(y)$ are uniquely defined in A.

$\forall x \in A, \ \exists$ unique $(\phi\psi)(x) = \phi(\psi(x)) \in A$