Graph Theory

Module 1

Section 5 : Paths and Connectedness

August 3, 2023

Walk

Definition (walk)

A walk is an alternating finite sequence $W : v_0 e_1 v_1 e_2 v_2 \ldots e_p v_p$ where v_{j-1} and v_j are end vertices of e_j for $j = 1, 2, \ldots$.

The length of a walk is the number of edges in it.

- **origin** is the vertex v_0.
- **terminus** is the vertex v_p,
- **closed** A walk is closed if $v_0 = v_p$.
- **trail** is a walk in which every edges are distinct.
- **path** is a walk in which every vertex is distinct.
- **cycle** is a closed trail.

Path

C_k is a cycle of length k.

P_k is a path on k vertices.

Path, P Let $P : v_1, e_1, v_2, \ldots, e_p, v_p$ be a path in G. Then we omit edges and write $P : v_1, v_2, \ldots, v_p$.

Inverse of P The path $P' : v_p, v_{p-1}, \ldots, v_1$ is the inverse of P.

section of P Let $P : v_1, v_2, \ldots, v_p$ be a path in G. A subsequence $v_j, v_{j+1}, \ldots, v_k$ is $v_j - v_k$ section of P.

Connected

Definition

Two vertices u, v are connected in a graph G if there is a $u - v$ path in G.

Definition

A graph G is connected if every pair of vertices in G are connected.

Equivalence Relation : Connectedness

reflexive	Every vertex is connected to itself by trivial path.
symmetric	every $u - v$ path is a $v - u$ path
transitive	$u - v$ path followed by $v - w$ path contains a $u - w$ path.

Definition (component)

Let $V_1, V_2, \ldots, V_\omega$ be the equivalence classes of the relation 'connectedness' in G. Then the induced subgraphs $G[V_1], G[V_2], \ldots, G[V_\omega]$ are the component of G.

- If $\omega = 1$, then G is connected.
- If $\omega \geq 2$, then G is disconnected.

Metric Space

Definition

Let d be a function on the vertex set of a graph G defined as $d(u, v)$ is the length of the shortest $u - v$ path in G.

Function d is a metric on $V(G)$

1. $d(u, v) \geq 0$ and $d(u, v) = 0 \iff u = v$.
2. $d(u, v) = d(v, u)$
3. $d(u, v) \leq d(u, w) + d(w, v)$.

Minimum Degree of a Simple, Connected Graph

Proposition

If G is simple and $\delta(G) \geq \frac{n-1}{2}$, then G is connected.

Proof.

- Let G be disconnected graph of order n.
- Suppose G has at least two components.
- Number of vertices in each component is at least $\frac{n-1}{2} + 1$.
- Number of vertices in G is $n + 1$.(contradiction)

$\square$

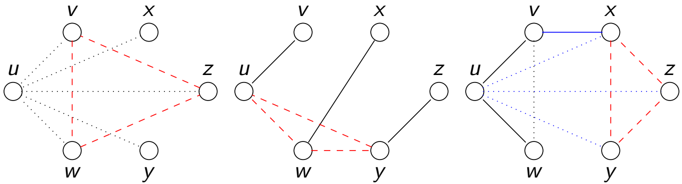
Exercises : Minimum Degree

- There exists non-simple disconnected graph G with $\delta(G) \geq (n-1)/2$.
  - Draw a graph with two components.
  - Use parallel edge or loops to increase minimum degree of the graph.
- $\delta(G) \geq (n-2)/2$ does not imply that G is connected.
  - If $\delta(G) = 1$, then we have a graph of order 4, say $2K_2$
  - In general, $2K_{\delta(G)+1}$ has order $n = 2\delta(G) + 2$.

Ramsey Number, $R(3,3) = 6$

In a group of six people, there must be three people who are mutually acquainted or three people are not mutually acquainted.

- Let $u \in V(G)$. Then $deg(u) \neq 0$
- $\Delta(G) \neq 1$.
- Let $deg(u) = 2$. Let $uv, uw \in E(G)$.
- $vw \notin E(G)$ otherwise (u, v, w) forms $\triangle$.
- WLOG $vx \in E(G)$, otherwise (v, w, x) forms $\triangle$.
- $ux \notin E(G)$, otherwise (u, v, x) forms $\triangle$.
- Similarly, $uy, uz \notin E(G)$.
- $xy \in E(G)$, otherwise (u, x, y) forms $\triangle$.
- Similarly, $yz, zx \in E(G)$.
- (x, y, z) forms $\triangle$ (contradiction)



Complement of a disconnected Graph

Theorem

If a simple graph G is disconnected, then G^c is connected.

Proof.

- Let G be a disconnected graph.
- Let G_1 and G_2 be two components of G.
  - Let $u \in V(G_1)$ and $v \in V(G_2)$.
    $uv \notin G \implies uv \in G^c$.
  - Let $u, w \in V(G_1)$ and $v \in V(G_2)$.
    $uv, vw \notin G \implies u, v, w$ is a $u - w$ path in G^c.

$\square$

Characterisation of Self complementary Graph

If G is self complementary, then $n(G) \cong 0$ or $1 \pmod 4$.

- $m(G) = m(G^c) = m(K_n)/2$.
- $m = n(n-1)/4$.
- $m \cong 0$ or $1 \pmod 4$.

A self complementary graph with one pendant vertex must have at least another pendant vertex.

- Let G be a self complementary graph with pendant vertex u.
- G^c has a pendant vertex v.
- v is another pendant vertex in G.

Upper bound for the size of a Simple Graph

Theorem

The size m of simple graph of order n with ω components cannot exceed $(n-\omega)(n-\omega+1)/2$.

$$m < \frac{(n-\omega)(n-\omega+1)}{2}$$

Proof.

- Let G be a graph of order n, size m, and components ω.
- Let $G_1, G_2, \ldots, G_\omega$ be the components of G.
- Let n_i, m_i be the order, size of G_i for each i.
  - $n_i \leq (n-\omega+1)$.
  - $m_i \leq n_i(n_i-1)/2$.
- $m = \sum n_i(n_i-1)/2 < (n-\omega+1)\sum(n_i-1)$.
- $m < (n-\omega+1)(n-\omega)/2$.

□

Local Connectedness

Definition

A graph G is locally connected if for each vertex v in G, the subgraph induced by the open neighbhourhood $N_G(v)$ is connected.

Figure: G is locally connected at x, u and w, but not at v

Characterisation of Bipartite Graph

Theorem

A graph is bipartite if and only if it has no odd cycles.

Proof.

- Suppose G is bipartite.
  - Let $u \in V(G)$.
  - Let $C : v_1, e_1, v_2, \ldots, v_k, e_k, e_1$ be a cycle in G.
  - Length of cycle k is even, since v_k adjacent to v_1.
- Suppose G has no odd cycle.
  - Suppose G is connected.
  - $X = \{v \in V(G) : d(u, v)$ is even.$\}$
  - $Y = \{v \in V(G) : d(u, v)$ is odd.$\}$
  - Let $v, w \in X$. Then $u - v$ path P and $u - w$ path Q.
  - Let w_1 be a common vertex P, Q such that $w_1 - w$ section of P and $w_1 - v$ section of Q has no other common vertices.
  - Suppose v is adjacent to w.
  - $w - w_1 + w_1 - v + vw$ is an odd cycle.(contradiction)
  - If G has components. Then $X = \cup X_i$ and $Y = \cup Y_i$.

A simple nontirivial graph G is connected if and only if for any partition (V_1, V_2) of $V(G)$ there is an edge joining a vertex of V_1 and a vertex of V_2.

- Suppose G is connected.
  - Let V_1, V_2 be a partition of $V(G)$.
  - Let $u \in V_1$ and $v \in V_2$.
  - Since G is connected, there exists $u - v$ path, say P.
  - P contains an edge joining V_1 and V_2.
- Suppose that for every partition (V_1, V_2), there exists an edge joining V_1 and V_2.
  - Let $u, v \in V(G)$.
  - Let $V_1 = \{u\}$. There exists $uw \in E(G)$ where $w \in V_2$.
  - Let $V_1 = \{u, w\}$. There exists ux or wx where $x \in V_2$.
  - Continuing like this, we get V_1 containing both u and v.

Longest paths does intersect

In a connected graph G with at least three vertices, any two longest paths have a vertex in common.

- Let $u_1 - u_k$ and $v_1 - v_k$ be two longest paths, say P, Q.
- $u_1 - v_1$ path has three sections : $u_1 - u_r, u_r - v_s, v_s - v_1$.
- WLOG suppose $u_1 - u_r$ and $v_1 - v_s$ are at least half as long as longest paths.
- Then $(u_1 - u_r) + (u_r - v_s) + (v_s - v_1)$ is longer. (contradiction)

Union of Disjoint Paths

Union of two disjoint paths joining two distinct vertices contains a cycle.

- Let P, Q be two distinct $u - v$ paths.
- Let P', Q' be disjoint sections of P, Q with common end vertices.
- $P' + Q'$ is a cycle.

Union of two distinct walks joining two distinct vertices need not contain a cycle.

- Two walks are distinct if one walk visits an edge one more time compared to the other.

Characterisation of an incomplete Graph

If a simple graph G is not complete, there exists three vertices u, v, w such that uv, vw are edge of G but uw is not an edge of G.

Generalised Petersen Graph

Definition (Generalised Petersen Graph)

Generalised Petersen Graph is defined by

$$V(P(n,k)) = \{a_i, b_i : 0 \leq i \leq n-1\}$$

$$E(P(n,k)) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+k}, 0 \leq i \leq n-1\}$$

If n is even and k is odd, then generalised Petersen Graph $P(n,k)$ is bipartite.

Longest Path and Minimum Degree

If G is simple and $\delta(G) \geq k$, then G contains a path of length at least k.

- Let G be a graph with $\delta(G) \geq k$.
- Let $v_0, v_1, \ldots, v_r$ be the longest path in G.
- v_r is at most adjacent to $v_0, v_1, \ldots, v_{r-1}$.
  - Otherwise $v_0, v_1, \ldots, v_r, w$ is longer. (contradiction)
- $degree(v_r) \leq r \implies k \leq \delta(G) \leq r$.
- Longest path has length at least k.

Thank You