Differential Geometry

Module I

Chapter 2 : Vector Fields

June 14, 2021

Vector at a point

Definition

A vector $\mathbf{v}$ at a point $p \in \mathbb{R}^{n+1}$ is a pair $\mathbf{v} = (p, v)$ where $v \in \mathbb{R}^{n+1}$.

Operations

vector addition $\mathbf{v} + \mathbf{w} = (p, v) + (p, w) = (p, v + w)$.

scalar multiplication Let $c \in \mathbb{R}$, then $c\mathbf{v} = c(p, v) = (p, cv)$.

dot product $\mathbf{v} \cdot \mathbf{w} = (p, v) \cdot (p, w) = v \cdot w$

cross product $\mathbf{v} \times \mathbf{w} = (p, v) \times (p, w) = (p, v \times w)$

Dot Product

Angle

Angle θ between $\mathbf{v}$ and $\mathbf{w}$ is given by,

$$\cos\theta = \mathbf{v}\cdot\mathbf{w} = (p, v)\cdot(p, w) = v.w \tag{1}$$

And the length of a vector $\mathbf{v}$ is given by,

$$\|\mathbf{v}\| = (\mathbf{v}\cdot\mathbf{v})^{\frac{1}{2}} = ((p, v)\cdot(p, v))^{\frac{1}{2}} = (v\cdot v)^{\frac{1}{2}} = \|v\| \tag{2}$$

Vector Space $\mathbb{R}^{n+1}_p$

Definition (Vector Space)

Let $c \in \mathbb{R}$ and $p \in \mathbb{R}^{n+1}$. Let $\mathbf{v}, \mathbf{w}$ be two vectors at p. That is, $\mathbf{v} = (p, v)$ and $\mathbf{w} = (p, w)$ for some $v, w \in \mathbb{R}^{n+1}$. Then the set of all vectors at p is a vector space with vector addition $\mathbf{v} + \mathbf{w} = (p, v + w)$ and scalar multiplication $c\mathbf{v} = (p, cv)$. This vector space is denoted by $\mathbb{R}^{n+1}_p$.

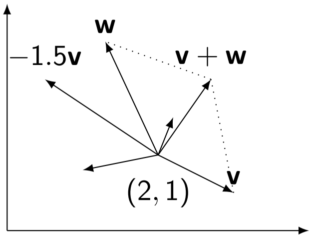

Figure: The vector space of all vectors at $(2, 1)$, $\mathbb{R}^2_{(2,1)}$

Vector Field

Definition (Vector Field)

The vector field $\mathbf{X}$ on $\mathbb{R}^{n+1}$ is a function which assigns to each point of $\mathbb{R}^{n+1}$ a vector at that point. That is, $\mathbf{X}(p) = (p, X(p))$.

For example, $\mathbf{X}(p) = (p, X(p))$ where the associated function of the vector field, $X : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $X(p) = (1, 2)$ assigns a constant vector $(1, 2)$ at every vector in $\mathbb{R}^2$.

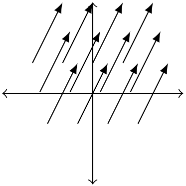

Figure: Vector field with associated function $X(p) = (1, 2)$

Smooth Vector Field

Definition (smooth)

A function $f : \mathbb{R} \to \mathbb{R}$ is smooth if its partial derivatives of all orders exists and are continuous. A function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is smooth if its component functions $f = (f_1, f_2, \cdots, f_{n+1})$ are smooth. A vector field **X** is smooth if the associated function $X(p)$ is smooth.

Gradient of a function

Definition (Gradient)

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$. Then the gradient of f at p is,

$$\nabla f(p) = \left(p, \frac{\partial f}{\partial x_1}(p), \frac{\partial f}{\partial x_2}(p), \cdots, \frac{\partial f}{\partial x_{n+1}}(p) \right) \tag{3}$$

$\nabla f(p)$

If f is a smooth function, then the gradient of f at p is a smooth vector field.

Example

For example, $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = 2x_1x_2$ is a smooth function. We have, $\frac{\partial f}{\partial x_1} = 2x_2$ and $\frac{\partial f}{\partial x_2} = 2x_1$. And gradient of f at (x_1, x_2) is $(x_1, x_2, 2x_2, 2x_1)$. That is, $(2x_2, 2x_1)$ at (x_1, x_2).

Calculations :

p	(x_1, x_2)	$(0,0)$	$(1,0)$	$(0,1)$	$(-1,0)$	$(0,-1)$
$X(p)$	$(2x_2, 2x_1)$	$(0,0)$	$(0,2)$	$(2,0)$	$(0,-2)$	$(-2,0)$

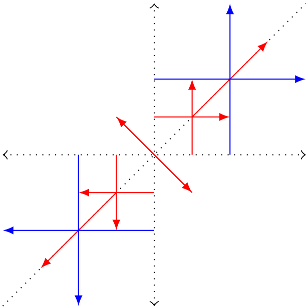

Figure: The gradient of $f(x_1, x_2) = 2x_1x_2$

Parameterised Curve

Definition

A parameterised curve is a function, $\alpha : I \to \mathbb{R}^{n+1}$ where I is some open interval in $\mathbb{R}$. The velocity vector of a parameterised curve $\alpha : I \to \mathbb{R}^{n+1}$ at a point $\alpha(t)$ is the tangent to the curve at that point.

$$\dot{\alpha}(t) = \left(\alpha(t), \frac{d\alpha}{dt}(t)\right) \tag{4}$$

For example, $\alpha : I \to \mathbb{R}^2$ defined by $\alpha(t) = (2t, t^2)$ is a parameterised curve. We have, $\frac{d\alpha}{dt} = (\frac{dx_1}{dt}(t), \frac{dx_2}{dt}(t)) = (2, 2t)$ where $\alpha(t) = (x_1(t), x_2(t))$. The velocity vector at $t = 3$ is $\dot{\alpha}(3) = (\alpha(t), \frac{d\alpha}{dt}) = (6, 9, 2, 6)$.

Integral Curve

Definition

Let $\mathbf{X}$ be a vector field and let U be an open subet of $\mathbb{R}^{n+1}$. An integral curve α on U is a parameterised curve, $\alpha : I \to \mathbb{R}^{n+1}$ such that for each $\alpha(t) = p \in U$, the velocity vector $\dot{\alpha}(t)$ is the associated vector $\mathbf{X}(p)$ of the vector field $\mathbf{X}$ at that point. Thus, for each $t \in I$, $\dot{\alpha}(t) = \mathbf{X}(\alpha(t))$.

$$\left(\alpha(t), \frac{d\alpha}{dt}(t)\right) = (\alpha(t), X(\alpha(t))) \tag{5}$$

Let $X(p) = (X_1(p), X_2(p), \cdots, X_{n+1}(p))$ and $\alpha(t) = (x_1(t), x_2(t), \cdots, x_{n+1}(t))$. Then, comparing components of the vector at $\alpha(t)$ we get the following system of equations,

$$\frac{dx_j}{dt}(t) = X_j(\alpha(t)),\ j = 1, 2, \cdots, (n+1) \tag{6}$$

Example

For example, Consider $\alpha : (2, 3) \to \mathbb{R}^2$ defined by $\alpha(t) = (t, t^2)$. Then α is a parameterised curve in vector field, $\mathbf{X}$ which has the associated function $X(x_1, x_2) = (1, 2x_1)$. Then, $\mathbf{X}(x_1, x_2) = (x_1, x_2, 1, 2x_1)$. And

$$\dot{\alpha}(t) = \left(\alpha(t), \frac{d\alpha}{dt}(t)\right) = \left(x_1(t), x_2(t), \frac{dx_1}{dt}(t), \frac{dx_2}{dt}(t)\right) = (t, t^2, 1, 2t)$$

Clearly, α is an integral curve of $\mathbf{X}$ as $\dot{\alpha}(t) = X(\alpha(t))$ for every $t \in (2, 3)$.

Integral Curve

Calculations:

p	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$X(p)$	$(1,0)$	$(2,2)$	$(1,1)$	$(2,3)$
$(-1,0)$	$(0,-1)$	$(-1,1)$	$(1,-1)$	$(-1,-1)$
$(0,-2)$	$(1,1)$	$(0,-1)$	$(2,1)$	$(0,-3)$

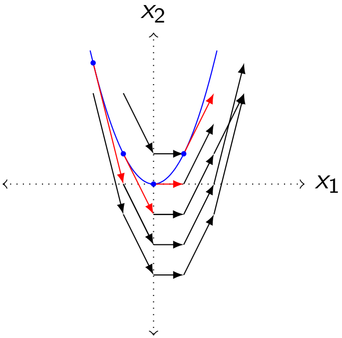

Figure: Integral Curve $\alpha(t) = (t, t^2)$ in $\mathbf{X}$ with $X(x_1, x_2) = (1, 2x_1)$

Unique Integral Curve in Smooth Vector Field

Theorem

Let $\mathbf{X}$ be a smooth vector field on an open set $U \subset \mathbb{R}^{n+1}$ and let $p \in U$. Then there exists an open interval I containing 0 and an integral curve $\alpha : I \to U$ such that

1. $\alpha(0) = p$
2. *If $\beta : \tilde{I} \to U$ is any other integral curve with $\beta(0) = p$, then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$, for all $t \in \tilde{I}$.*

Proof

Proof.
Let $\mathbf{X}$ be a smooth vector field. Suppose α be an integral curve in $\mathbf{X}$. Then, $\dot{\alpha}(t) = \mathbf{X}(\alpha(t))$. Let $x_j(t)$ be the components of $\alpha(t)$ and $X_j(p)$ be the components of $X(p)$.

$$\begin{aligned}
\dot{\alpha}(t) &= \left(\alpha(t), \frac{d\alpha}{dt}(t)\right) \\
&= \left(x_1(t), \cdots, x_{n+1}(t), \frac{dx_1}{dt}(t), \cdots, \frac{dx_{n+1}}{dt}(t)\right) \\
\mathbf{X}(\alpha(t)) &= (\alpha(t), X(\alpha(t))) \\
&= (x_1(t), \cdots, x_{n+1}(t), X_1(\alpha(t)), \cdots, X_{n+1}(\alpha(t)))
\end{aligned}$$

$\square$

Proof

Proof.

Thus, we a system of $n+1$ first order differential equations in $n+1$ unknowns satisfying the initial condition $\alpha(0) = p$.

$$\begin{aligned}
\frac{dx_1}{dt}(t) &= X_1(\alpha(t)) \\
\frac{dx_2}{dt}(t) &= X_2(\alpha(t)) \\
&\vdots \\
\frac{dx_{n+1}}{dt}(t) &= X_{n+1}(\alpha(t))
\end{aligned}$$

□

Proof

Proof.

By the theorem on solution of systems of first order ordinary differential equations, there exists an interval I containing 0 and a solution — a family of functions $\{x_1(t), x_2(t), \cdots, x_{n+1}(t)\}$ satisfying the above system of equations satisfying the initial condition $\alpha(0) = p$.

Define $\alpha : I \to U$ using the component functions of α as x_js in the above solution. Then, we have a integral curve of the vector field $\mathbf{X}$ satifying the initial condition $\alpha(0) = p$.

Let $\beta : \tilde{I} \to U$ be another integral curve with $\beta(0) = p$. Then by the uniqueness of the solution for the system of first order ordinary differential equations with an initial condition, $\beta(t) = \alpha(t)$ for every $t \in I \cup \tilde{I}$. □

Proof

Proof.

Let $\{\beta_1, \beta_2, \cdots\}$ be the family of integral curves with $\beta_j : I_j \to U$ satisfying $\beta_j(0) = p$. Consider $I = \bigcup_{j \in \mathbb{N}} I_j$.

Define $\alpha : I \to U$ by $\alpha(t) = \beta_j(t)$ where $t \in I_j$ for some $j \in \mathbb{N}$. Then α is well-defined and is a maximal integral curve in $\mathbf{X}$ such that $\alpha(0) = p$. $\square$

Complete Vector Field

Definition

A smooth vector field $\mathbf{X}$ on $U \subset \mathbb{R}^{n+1}$ is **complete** if for every $p \in U$, the maximal integral curve through p has domain equal to $\mathbb{R}$.

Divergence of a Vector Field

Definition

The **divergence** of a smooth vector field $\mathbf{X}$ on $U \subset \mathbb{R}^{n+1}$ is the function $div\ \mathbf{X} : U \to \mathbb{R}$ defined by

$$div\ X(x_1, x_2, \cdots, x_{n+1}) = \sum_{i=1}^{n+1} \frac{\partial X_i}{\partial x_i}$$

where X_i are the component function of the associated function X of the vector field $\mathbf{X}$.

For example, Consider $\mathbf{X}$ with associated function $X : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $X(x_1, x_2) = (2x_1, x_1x_2)$. Then $div\ \mathbf{X}(x_1, x_2) = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} = 2 + x_1$.

Local 1-parameter Group

Definition (Local 1-parameter Group)

Let $\mathbf{X}$ be a smooth vector field on U, $U \subset \mathbb{R}^{n+1}$. Let $\phi_t(p) = \alpha_p(t)$ where α_t is the maximal integral curve of $\mathbf{X}$ through p. Then ϕ_t together with function composition is a group. And is called the local 1-parameter group associated with $\mathbf{X}$.