Differential Geometry

Module II

Chapter 8 : Parallel Transport

June 14, 2021

Covariant Derivative $\mathbf{X}'$

- n-surface S (with Orientation $\mathbf{N}$)‡[1]
- parametrised curve α in surface S, $\alpha : I \to S$
- smooth vector field $\mathbf{X}$ along α, tangent to S, $\mathbf{X}(\alpha(t)) \in S_{\alpha(t)}$
- Derivative $\dot{\mathbf{X}} = \left(\alpha(t), \frac{d}{dt}X(t)\right)$
- But, $\dot{\mathbf{X}}(\alpha) \notin S_{\alpha(t)}$

Definition (Covariant Derivative)

The orthogonal projection of the ordinary derivative $\dot{\mathbf{X}}$

$$\mathbf{X}'(t) = \dot{\mathbf{X}}(t) - \left[\dot{\mathbf{X}}(t) \cdot \mathbf{N}(\alpha(t))\right] \mathbf{N}(\alpha(t))$$

Simply, the component of derivative $\dot{\mathbf{X}}$ in tangent space $S_{\alpha(t)}$.
Clearly, $\mathbf{X}'(t) \in S_{\alpha(t)}, \ \forall t \in I$

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[1] $\mathbf{X}'$ is independent of the choice of orientation $\mathbf{N}$

Properties

We have, $\dot{\mathbf{X}} = \mathbf{X}' + (\dot{\mathbf{X}} \cdot (\mathbf{N} \dot{\circ}\, \alpha)\mathbf{N} \circ \alpha$

Properties of Covariant Derivative

- $(\mathbf{X} + \mathbf{Y})' = \mathbf{X}' + \mathbf{Y}'$
- $(f\mathbf{X})' = f'\mathbf{X} + f\mathbf{X}'$
- $(\mathbf{X} \cdot \mathbf{Y})' = \mathbf{X}' \cdot \mathbf{Y} + \mathbf{X} \cdot \mathbf{Y}'$

Euclidean vs Levi-Civita Parallel

- *n*-surface S
- Parametrised curve α
- Smooth Vector Field $\mathbf{X}$ along α

Definition (Parallel)

- $\mathbf{X}$ is Euclidean Parallel along α if $\dot{\mathbf{X}} = \mathbf{0}$
  - All the assigned vectors are parallel
  - $\mathbf{X}(\alpha(t)) = (\alpha(t), 1, 2)$
- $\mathbf{X}$ is (Levi-Civita) Parallel along α if $\mathbf{X}' = \mathbf{0}$
  - From the surface, all the assigned vectors looks parallel

Thank You