Differential Geometry

Module III

Chapter 9 : Weingarten Map

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Directional Derivatives

Suppose,

$$f : \mathbb{R}^{n+1} \to \mathbb{R}$$

$$\boldsymbol{X}(p) = (p, X_1(p), X_2(p), \ldots, X_{n+1}(p))$$

- $\nabla_{\boldsymbol{v}} f$, the derivative of function f with respect to $\boldsymbol{v}$

$$\nabla_{\boldsymbol{v}} f = \nabla f(p) \cdot \boldsymbol{v} \text{ where } \boldsymbol{v} = (p, v)$$

- $\nabla_{\boldsymbol{v}} \boldsymbol{X}$, the derivative vector field $\boldsymbol{X}$ with respect to $\boldsymbol{v}$.

$$\nabla_{\boldsymbol{v}} \boldsymbol{X} = (p, \nabla_{\boldsymbol{v}} X_1, \nabla_{\boldsymbol{v}} X_2, \ldots, \nabla_{\boldsymbol{v}} X_{n+1})$$

- $D_{\boldsymbol{v}} \boldsymbol{X}$, the covariant derivative of $\boldsymbol{X}$ with respect to $\boldsymbol{v}$.

$$D_{\boldsymbol{v}} \boldsymbol{X} = \nabla_{\boldsymbol{v}} \boldsymbol{X} - (\nabla_{\boldsymbol{v}} \boldsymbol{X} \cdot \boldsymbol{N})\, \boldsymbol{N}$$

Properties of $\nabla_{\mathbf{v}} f$

$\nabla_{\mathbf{v}} f$ is a linear map.

1. $\nabla_{\mathbf{v}+\mathbf{w}} f = \nabla_{\mathbf{v}} f + \nabla_{\mathbf{w}} f$.
2. $\nabla_{c\mathbf{v}} f = c\nabla_{\mathbf{v}} f$.

We also have,

1. $\nabla_{\mathbf{v}}(f+g) = \nabla_{\mathbf{v}} f + \nabla_{\mathbf{v}} g$.
2. $\nabla_{\mathbf{v}}(fg) = f(\nabla_{\mathbf{v}} g) + (\nabla_{\mathbf{v}} f)g$.

Properties of $\nabla_{\boldsymbol{v}}\boldsymbol{X}$ and $D_{\boldsymbol{v}}\boldsymbol{X}$

Properties of derivative of $\boldsymbol{X}$ with respect to $\boldsymbol{v}$,

1. $\nabla_{\boldsymbol{v}}(\boldsymbol{X}+\boldsymbol{Y}) = \nabla_{\boldsymbol{v}}\boldsymbol{X} + \nabla_{\boldsymbol{v}}\boldsymbol{Y}$.
2. $\nabla_{\boldsymbol{v}}(f\boldsymbol{X}) = f(\nabla_{\boldsymbol{v}}\boldsymbol{X} + (\nabla_{\boldsymbol{v}}f)\boldsymbol{X}$.
3. $\nabla_{\boldsymbol{v}}(\boldsymbol{X}\cdot\boldsymbol{Y}) = \boldsymbol{X}\cdot\nabla_{\boldsymbol{v}}\boldsymbol{Y} + (\nabla_{\boldsymbol{v}}\boldsymbol{X})\cdot\boldsymbol{Y}$.

Properties of covariant derivative of $\boldsymbol{X}$ with respect to $\boldsymbol{v}$,

1. $D_{\boldsymbol{v}}(\boldsymbol{X}+\boldsymbol{Y}) = D_{\boldsymbol{v}}\boldsymbol{X} + D_{\boldsymbol{v}}\boldsymbol{Y}$.
2. $D_{\boldsymbol{v}}(f\boldsymbol{X}) = f(D_{\boldsymbol{v}}\boldsymbol{X}) + (\nabla_{\boldsymbol{v}}f)\boldsymbol{X}$.
3. $D_{\boldsymbol{v}}(\boldsymbol{X}\cdot\boldsymbol{Y}) = \boldsymbol{X}\cdot D_{\boldsymbol{v}}\boldsymbol{Y} + (D_{\boldsymbol{v}}\boldsymbol{X})\cdot\boldsymbol{Y}$.

Weingarten Map, L_p

Weingarten map of an n-surface S at p,

$$L_p : S_p \to S_p,\; L_p(\boldsymbol{v}) = -\nabla_{\boldsymbol{v}} \boldsymbol{N}$$

Practically, $-\nabla_{\boldsymbol{v}} \boldsymbol{N}(p) = -\nabla_{\boldsymbol{v}} \tilde{\boldsymbol{N}}(p)$.

- n-sphere of radius r, $L_p(\boldsymbol{v}) = \frac{1}{r}\boldsymbol{v}$.

The shape operator, L_p

- Rate of change of the unit normal at each point on S.
- Measure of the turning of tangent space at each point on S.
- Describes the shape of the surface S.

Properties of Weingarten Map

1. The normal component of acceleration at p is the same for all parametrized curves with same velocity at p.

$$\ddot{\alpha}(t_0) \cdot \boldsymbol{N}(p) = L_p(\boldsymbol{v}) \cdot \boldsymbol{v}$$

2. Weingarten Map is self adjoint.

$$L_p(\boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{v} \cdot L_p(\boldsymbol{w})$$

Thank You