Differential Geometry

Module I

Chapter 4 : Surfaces

June 14, 2021

n-Surface

Definition (n-Surface)

A non-empty subset S of $\mathbb{R}^{n+1}$ is an n-surface if it is of the form $S = f^{-1}(c)$ where $f : U \to \mathbb{R}$, U open subset of $\mathbb{R}^{n+1}$ is a smooth function with the property that $\nabla f(p) \neq 0$ for every $p \in S$.

$n = 1$ plane curve

$n = 2$ surface

$n > 2$ hypersurface

The n-surfaces are subspaces of dimension n.

Summary : Surface

Definition (Surface)

- The subset S of the level set of a smooth function f
- The points of the surface S are regular points of f
- Depends on S, but independent of the function f

Why independent of f ?

- Level sets are independent of the function
- Different level sets can have same subset

Example : n-sphere

Definition (n-sphere)

The unit n-sphere $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$ is the level set $f^{-1}(1)$ of the function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$f(x_1, x_2, \cdots, x_{n+1}) = x_1^2 + x_2^2 + \cdots + x_{n+1}^2 \tag{1}$$

n-Sphere as level set

$$f(x_1, x_2, \cdots, x_{n+1}) = x_1^2 + x_2^2 + \cdots + x_{n+1}^2$$

$$\begin{aligned} f^{-1}(1) &= \{(x_1, x_2, \cdots, x_{n+1}) : f(x_1, x_2, \cdots, x_{n+1}) = 1\} \\ &= \{(x_1, x_2, \cdots, x_{n+1}) : x_1^2 + x_2^2 + \cdots x_{n+1}^2 = 1\} \end{aligned}$$

This level set is the set of all points in $\mathbb{R}^{n+1}$ satsifying

$$x_1^2 + x_2^2 + \cdots x_{n+1}^2 = 1 \tag{2}$$

n-Spheres

The n-Sphere is an n-dimensional surface in $\mathbb{R}^{n+1}$.

$n = 1$ unit circle

$$x_1^2 + x_2^2 = 1$$

$n = 2$ unit sphere

$$x_1^2 + x_2^2 + x_3^2 = 1$$

$n > 2$ hypersphere

$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$$

Example : n-plane

Definition (n-plane)

An n-plane $a_1x_1 + a_2x_2 + \cdots + a_{n+1}x_{n+1} = b$ is the level set $f^{-1}(b)$ of the function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$f(x_1, x_2, \cdots, x_{n+1}) = a_1x_1 + a_2x_2 + \cdots + a_{n+1}x_{n+1} \tag{3}$$

n-Plane as level set

$$f(x_1, x_2, \cdots, x_{n+1}) = a_1x_1 + a_2x_2 + \cdots + a_{n+1}x_{n+1}$$

$$\begin{aligned} f^{-1}(b) &= \{(x_1, x_2, \cdots, x_{n+1}) : f(x_1, x_2, \cdots, x_{n+1}) = b\} \\ &= \{(x_1, x_2, \cdots, x_{n+1}) : a_1x_1 + a_2x_2 + \cdots a_{n+1}x_{n+1} = b\} \end{aligned}$$

This level set is the set of all points in $\mathbb{R}^{n+1}$ satsifying

$$a_1x_1 + a_2x_2 + \cdots a_{n+1}x_{n+1} = b \tag{4}$$

n-Planes

The n-Plane is an n-dimensional surface in $\mathbb{R}^{n+1}$.

$n = 1$ line in $\mathbb{R}^2$

$$a_1x_1 + a_2x_2 = b$$

$n = 2$ plane in $\mathbb{R}^3$

$$a_1x_1 + a_2x_2 + a_3x_3 = b$$

$n > 2$ hyperplane in $\mathbb{R}^{n+1}$

$$a_1x_1 + a_2x_2 + \cdots + a_{n+1}x_{n+1} = b$$

Parallel Planes

Definition (Parallel Planes)

Two n-planes are parallel if they are of the form, $f^{-1}(b_1)$ and $f^{-1}(b_2)$ where $f(x_1, x_2, \cdots, x_{n+1}) = a_1x_1 + a_2x_2 + \cdots + a_{n+1}x_{n+1}$ and $b_1, b_2 \in \mathbb{R}$.

Example

The planes P_1 and P_2 are parallel.

$$P_1 : x_1 + 2x_2 - 3x_3 = 1$$

$$P_2 : x_1 + 2x_2 - 3x_3 = 2$$

Cylinder over a Surface

Definition

Let S be an $(n-1)$ surface in $\mathbb{R}^n$, given by $S = f^{-1}(c)$, where $f : U \to \mathbb{R}$ such that $\nabla f(p) \neq 0, \ \forall p \in S$

$$\nabla f(p) = \left(p, \frac{\partial f}{\partial x_1}(p), \frac{\partial f}{\partial x_2}(p), \cdots, \frac{\partial f}{\partial x_n}(p)\right) \neq (p, 0), \ \forall p \in S$$

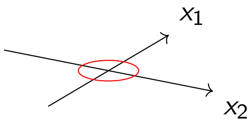
Let $g : U \times \mathbb{R} \to \mathbb{R}$, where $g(x_1, x_2, \cdots, x_{n+1}) = f(x_1, x_2, \cdots, x_n)$.

$$\nabla g(q) = \left(q, \frac{\partial f}{\partial x_1}(q), \frac{\partial f}{\partial x_2}(q), \cdots, \frac{\partial f}{\partial x_n}(q), 0\right) \neq (q, 0), \ \forall q \in g^{-1}(c)$$

This n-surface $g^{-1}(c)$ is the **cylinder over** S.

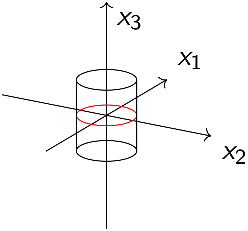
Cylinder over Surface : Example

- $S : x_1^2 + x_2^2 = 1$, unit circle.
- $S = f^{-1}(1)$ where $f(x_1, x_2) = x_1^2 + x_2^2$.
- $\nabla f(x_1, x_2) = (x_1, x_2, 2x_1, 2x_2) \neq (x_1, x_2, 0, 0)$, since $(0, 0) \notin S$

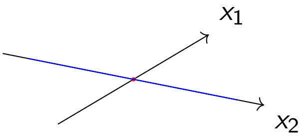


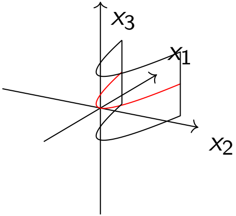
Cylinder over Surface : Example

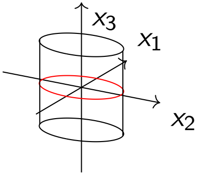
- $S : x_1^2 + x_2^2 = 1$, unit circle.
- $S = f^{-1}(1)$ where $f(x_1, x_2) = x_1^2 + x_2^2$.
- $\nabla f(x_1, x_2) = (x_1, x_2, 2x_1, 2x_2) \neq (x_1, x_2, 0, 0)$, since $(0, 0) \notin S$
- $g(x_1, x_2, x_3) = x_1^2 + x_2^2$
- $g^{-1}(1)$ is the usual cylinder in $\mathbb{R}^3$ and
- $\nabla g(x_1, x_2, x_3) = (x_1, x_2, x_3, 2x_1, 2x_2, 0) \neq (x_1, x_2, x_3, 0, 0, 0)$
  since $(0, 0, z) \notin g^{-1}(1)$

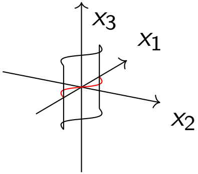


Cylinder over Surface : Examples





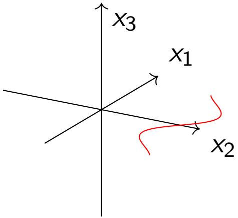




Surface of Revolution

Obtained by rotating a curve about an axis

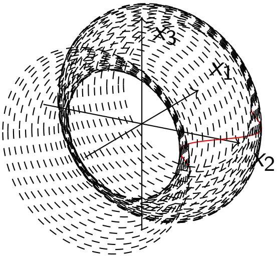
- $f : U \to \mathbb{R}, \ \nabla f(p) \neq 0, \ \forall p \in U$
- $C = f^{-1}(c)$ where $U \subset \mathbb{R}^2$ with $x_2 > 0$
  That is, C is a curve in $\mathbb{R}^2$ not touching the x_1 axis



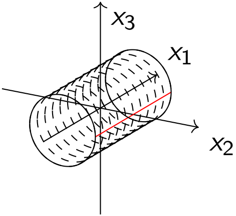
Surface of Revolution

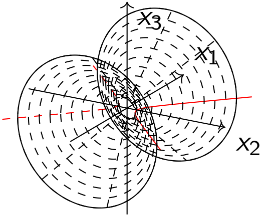
Obtained by rotating a curve about an axis

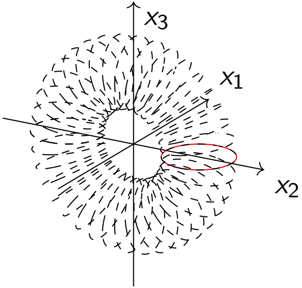
- $f : U \to \mathbb{R},\ \nabla f(p) \neq 0,\ \forall p \in U$
- $C = f^{-1}(c)$ where $U \subset \mathbb{R}^2$ with $x_2 > 0$
  That is, C is a curve in $\mathbb{R}^2$ not touching the x_1 axis
- $g : U \times \mathbb{R} \to \mathbb{R},\ g(x_1, x_2, x_3) = f(x_1, (x_2^2 + x_3^2)^{\frac{1}{2}})$
- $S = g^{-1}(c)$ is a surface of revolution of C about x_1 axis.



Surface of revolution of Curve : Examples







Extreme Points

Definition (Extreme Point)

Let an n-surface $S \subset U$ and $g : U \to \mathbb{R}$ be a smooth function. Then $p \in S$ is an extreme point of g on the surface S if

- $g(p) \leq g(q), \ \forall q \in S$ or
- $g(p) \geq g(q), \ \forall q \in S$.

Theorem

Let S be an n-surface in $\mathbb{R}^{n+1}$, $S = f^{-1}(c)$ where $f : U \to \mathbb{R}$ such that $\nabla f(q) \neq 0, \ \forall q \in S$. Let $g : U \to \mathbb{R}$ be a smooth function and $p \in S$ be an extreme point of g on S. Then there exists a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$.

Extreme Points : Proof

- $v \in S_p \implies v = \dot{\alpha}(t_0),\ \alpha : I \to S,\ \alpha(t_0) = p$
- If p is an extreme point of g,
  then t_0 is an extreme point of $g \circ \alpha$

$$
\begin{aligned}
(g \circ \alpha)'(t_0) &= 0 \\
\nabla g(\alpha(t_0)) \cdot \dot{\alpha}(t_0) &= 0 \\
\nabla g(p) \cdot v &= 0
\end{aligned}
$$

- $\nabla g(p) \in S_p^\perp$
- $[\nabla f(p)]^\perp = S_p$
- If $g(p) \in S_p^\perp$, then $\nabla g(p) = \lambda \nabla f(p)$
  Since, $S_p^\perp$ is 1-dimensional and is spanned by $\nabla f(p)$