Algorithmic Graph Theory

Module III

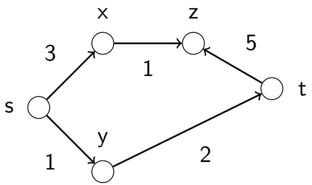
Chapter 5 : Networks

June 29, 2023

Networks

Network, N

1. Digraph D with two special vertices source and sink, $s \& t$.
2. Capacity Function $c : E(D) \rightarrow \mathbb{Z}^{+}$.



Applications

1. Logistics - Transportation Problem
2. Flow Management - Oil Pipelines

Networks - Graph Theory

Out-neighbourhood is the set of all out-neighbours.

$$N^+(x) = \{y \in V(D) : (x, y) \in E(D)\}$$

In-neighbourhood is the set of all in-neighbours.

$$N^-(x) = \{y \in V(D) : (y, x) \in E(D)\}$$

Indegree $id\ x = |N^-(x)|$.

Outdegree $od\ x = |N^+(x)|$.

Flow in a Network

Flow is a function $f : E(D) \to \mathbb{Z}$ satisfying

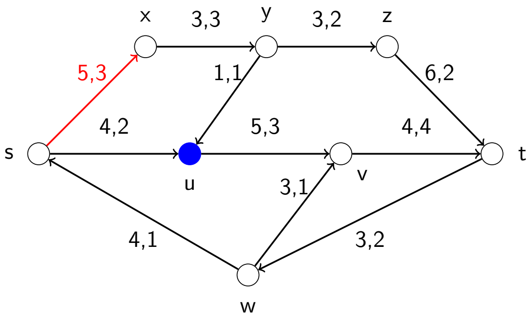
1. Capacity Constraints

$$0 \leq f(a) \leq c(a), \ \forall a \in E(D)$$

2. Conservation of Flow

$$\sum_{y \in N^+(x)} f(x,y) = \sum_{y \in N^-(x)} f(y,x), \ \forall x \in V(D) - \{s,t\}$$

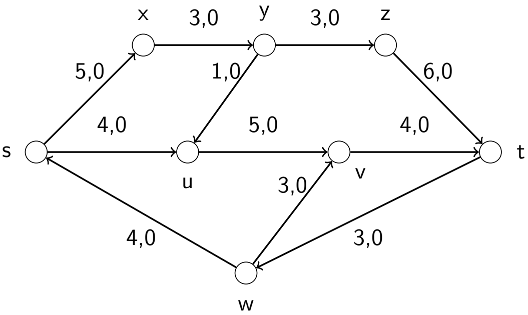
Network with a Flow



- $c(s, x) = 5$ and $f(s, x) = 3$.
- $f(s, u) + f(y, u) = f(u, v)$.

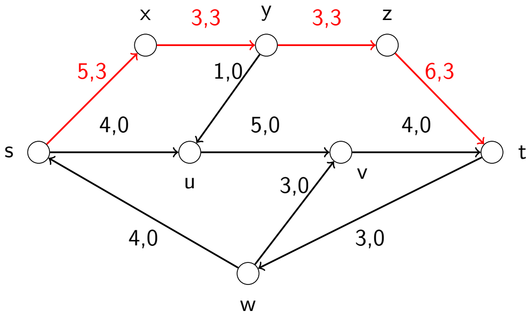
Flow and Cut

- Flow in a Network, $f(N)$ is the flow out of source s.
- $f(N) = 0$.



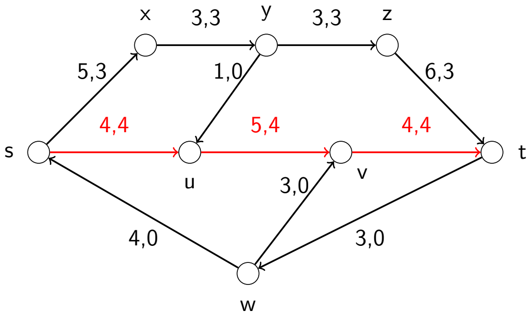
Flow and Cut

- Flow in a Network, $f(N)$ is the flow out of source s.
- $f(N) = 3$.



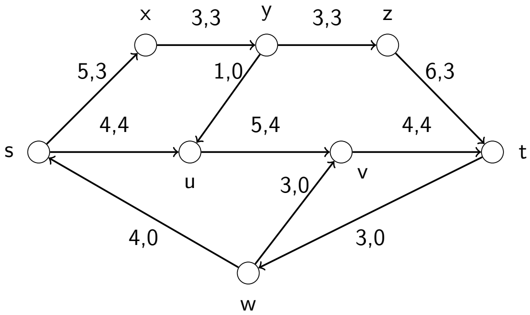
Flow and Cut

- Flow in a Network, $f(N)$ is the flow out of source s.
- $f(N) = 7$.



Flow and Cut

- Flow in a Network, $f(N)$ is the flow out of source s.
- $f(N) = 7$ is the maximum flow.



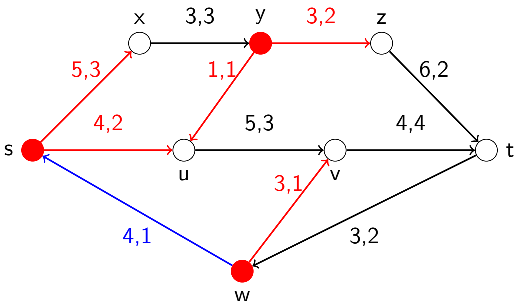
Flow and Capacity between two Partitions

- Let $X, Y \subset V(D)$ such that $X \cap Y = \phi$ and $X \neq \phi$, $Y \neq \phi$.

$$f(X, Y) = \sum_{(x,y)\in(X,Y)} f(x, y)$$

$$c(X, Y) = \sum_{(x,y)\in(X,Y)} c(x, y)$$

- For $X = \{s, y, w\}$, $f(X, Y) = 9$ and $c(X, Y) = 16$.



Theorem : Flow is restricted by the minimum cut

Cut $(P, \bar{P})$ is a partition of $V(D)$ such that $s \in P$ and $t \in \bar{P}$.

Theorem

Let N be a network with flow $f(N)$ and $(P, \bar{P})$ be a cut in N. Then,

$$f(N) = f(P, \bar{P}) - f(\bar{P}, P)$$

Corollary

Let N be a network with flow $f(N)$. Then,

$$f(N) \leq \min c(P, \bar{P})$$

Corollary

Let N be a network with flow $f(N)$. Then,

$$f(N) = \sum_{x \in N^-(t)} f(x, t) - \sum_{x \in N^+(t)} f(t, x)$$

Max Flow Min Cut Theorem

- A semipath is f-unsaturated if all forward arcs have more capacity than flow and all reverse arcs have positive flow.
- An $s - t$ semipath is f-augmenting if it is f-unsaturated.

Theorem (f-augmenting semipath)

A flow in a network $f(N)$ is maximum if and only if there is no f-augmenting semipath in D.

Theorem (Max Flow Min Cut)

Let N be a network with maximum flow $f(N)$. Then $f(N)$ is equal to the capacity of the minimum cut $(P, \bar{P})$.

Proof : f-augmenting semipath

If there is an f-augmenting semipath Q in N then flow $f(N)$ can be further augmented. Thus, flow $f(N)$ is not maximum.

- Compute Δ, maximum value that can be augmented along Q.
- f^* with augmentation is a flow in network N.
  - Case 0 : $x \notin Q$.
  - Case 1 & 2 : Both arcs enters/leaves x.
  - Case 3 : One arc enters x and Other leaves x

Suppose $f(N)$ has no f-augmenting path. And f^* is maximum flow.

- $P = \{x \in V(D) :$ there exists an f-augmenting path $s - x\ \}$.
- $t \in \bar{P}$.
- $(P, \bar{P})$ is a cut in network N.
- $f(N) = f(P, \bar{P}) - f(\bar{P}, P) = f(P, \bar{P}) = c(P, \bar{P})$
- If f^* is maximum flow and $(X, \bar{X})$ is minimum cut. Then $f(N) \leq f^*(N) \leq c(X, \bar{X}) \leq C(P, \bar{P})$.

Thank You