Fourier Series

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Abstract

This is the project for Differential Equations.

1 Description

Fourier series are expansions of periodic functions in terms of an infinite sum of sines and cosines. It is a way to describe functions in a combination of simpler functions and a way to make analysis of physical systems easier.

Fourier Series generally take the form of

$$s(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

For odd functions a_n becomes zero and the Fourier series becomes composed of only sine functions, while if the function is even, b_n becomes zero and the Fourier series becomes composed of only cosine functions.

2 Methods of Studying Fourier Series

There are multiple methods to studying Fourier Series, including the Fast Fourier Transform, Discrete-Time Fourier Transform, the Discrete Fourier Transform, and the Convolution.

2.1 Fast Fourier Transformations

The Fast Fourier Transform is a way of transforming a function of time into a function of frequency. It is useful when studying time-dependent phenomena. In comparison to the Discrete Fourier Transform described below, which takes

$$O(n^2)$$

computational steps, the Fast Fourier Transform can be computed in

$$O(nlog(n))$$

time, a considerable speedup when n becomes large.

There are multiple different Fast Fourier Transform algorithms, including the Cooley-Tukey algorithm, among others. The Cooley-Tukey algorithm, named after J.W. Cooley of IBM and John Tukey of Princeton, breaks the Discrete Fourier Transform into smaller Discrete Fourier Transforms of arbitrary size recursively.

2.2 Discrete-Time Fourier Transform

Definition 1 *The* Discrete-time Fourier Transform (DTFT) *can be defined as:*

$$X_{1/T}(f) = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} x[n] \cdot \delta(t - nT)\right.$$

2.3 Discrete Fourier Transformation

Differing from the Discrete-time Fourier Transform in that both the input and the output functions are both finite.

2.4 Convolution

Theorem 1 *If $f(x)$ is defined by the equation*

$$f(x) = \begin{cases} x^2, & \text{for } x \geq 0. \\ -x^2, & \text{for } x < 0 \end{cases} \tag{1}$$

Then $f(x)$ is continuous at $x = 0$.

Theorem 2 *The following are equivalent.*

1. $a \leq b$
2. $b \geq a$
3. $a = b$ *or* $a < b$

3 Applications of Fourier Series

It is used in proving the Nyquist-Shannon sampling theorem, studying harmonic oscillations, waveforms, signal processing, diffractions, interference, and Young's double slit experiment, and Radiation from surface currents, supernovae simulations, partial differential equations by separating variables, solving the heat equation.

3.1 Nyquist-Shannon Sampling Theorem

Theorem 3 *If a time-varying signal is periodically sampled at a rate of at least twice the frequency of the highest sinusoidal component contained within the signal, then the original time-varying signal can be exactly recovered from the periodic samples.*

The theorem also explains why aliasing occurs if the discrete sampling frequency is not sufficiently high enough to capture all the changes in the signal. Fourier series comes into play in this theorem through breaking a signal into its component frequencies.

3.2 Signal Processing

In signal processing, the Fourier series, and especially the Fourier transform, is used to determine what frequencies are present in a signal, and in what proportions. The magnitude squared of the Fourier series of a given signal gives the amount of power the signal has at that particular frequency. The Fourier series also makes it easier for specific frequencies present in a signal to be blocked out, or nullified.

3.3 Astrophysical Simulations

As part of research I am conducting with Dr. James Imamura, where we simulate stars as they accumulate mass and become unstable, the Fourier series coefficients are graphed and examined to see how the simulation is going. As the star becomes more unstable, the first few terms grow larger and fluctuate more dramatically. It is also used in the calculation of Rossby Wave Instabilities.

3.4 Heat Equation

Solving the heat equation was one of the first applications of the Fourier series, and is what Joseph Fourier proposed when first coming up with the Fourier series.

3.5 Quantum Physics

In quantum physics, the Fourier transform can be used with wavefunctions, as well as with the uncertainty principle to show that completely knowing the momentum of a particle means that the location is completely unknown, as well as the opposite.

The Fourier transform is equivalent to splitting light into its component spectrum. It is also used to understand phenomena in optics and with light, such as Young's double slit experiment. For the double slit experiment, the Fourier transform of the transmission function gives the frequency content of the resultant wave, while the square of the Fourier transform gives the intensity of the pattern on the screen. The transform then describes the maxima and minima that occur from the double slit experiment correctly.

References

[1] LAMPORT, L., "LaTeX- A Document Preparation System", Addison-Wesley, 1998.

[2] FILLIOQUE R. and HELIOTROPE, B., *Why Fermat's last theorem is really a lemma,* American Mathematical Weekly, Vol. 7, No. 1, pp 115-116, 1998.