Fourier Series

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Abstract

This is the project for Differential Equations.

1 Description

Fourier series are expansions of periodic functions in terms of an infinite sum of sines and cosines. It is a way to describe functions in a combination of simpler functions and a way to make analysis of physical systems easier.

2 Methods of Studying Fourier Series

Fast Fourier Transform, Discrete Fourier Transform, Discrete-time Fourier Transform, Z Transform, Convolution

2.1 Fast Fourier Transformations

2.2 Discrete Fourier Transformation

2.3 Discrete-time Fourier Transform

2.4 Z Transform

2.5 Convolution

Definition 1 *A* semi-quaver *is defined to be half a quaver.*

If that definition is not enough, here is another:

Definition 2 *The* order *of a note n in a quaver Q, $O(n,Q)$, is defined by the equation*

$$O(n,Q) = \int_0^\infty \sin(n^2 t)/(1-Qn)dt.$$

Some equations one writes inline, such as *Pythagoras' Theorem,* $c^2 = a^2 + b^2$, while others are better off as displayed equations, like the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which solves the quadratic $ax^2 + bx + c = 0$. If the quadratic formula is written inline, $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, it is readable but not very nice. This form, $x = (-b \pm \sqrt{b^2 - 4ac})/2a$, is harder to read as a fraction, but better because of the larger type.

Every inline equation must be part of a sentence: Since $x < 1/2$ we have $x + y < x + 1/2$. Inline fractions, such as $x < \frac{1}{2}$, are discouraged but not prohibited.

You can use formulae in theorems, as in the following.

Theorem 1 *If $f(x)$ is defined by the equation*

$$f(x) = \begin{cases} x^2, & \textit{for } x \geq 0. \\ -x^2, & \textit{for } x < 0 \end{cases} \tag{1}$$

Then $f(x)$ is continuous at $x = 0$.

PROOF: Since $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -x^2 = 0$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$ it follows that $\lim_{x\to 0} f(x) = 0 = f(x)$, as required. QED

Did you notice the grammatical error in Theorem 1? The sentence leading into equation 1 is never completed. The following theorem is worded correctly.

Theorem 2 *If $f(x)$ is defined by the equation*

$$f(x) = \begin{cases} x^2, & \textit{for } x \geq 0 \\ -x^2, & \textit{for } x < 0 \end{cases}, \tag{2}$$

then $f(x)$ is continuous at $x = 0$.

One rule of thumb of mathematical composition is to use mathematical notation inside sentences only for nouns. For example, one writes that "R is the the radius of a circle", but not that "the radius of the circle ='s the side length of the square". According to this rule it is correct to write that "$(x > 0) \Rightarrow (x^3 > 0)$" since the double arrow is part of an equation, but not to write "x is positive $\Rightarrow$ x^3 is positive", since the double arrow is acting as a verb.

Here is another kind of common structure:

Theorem 3 *The following are equivalent.*

1. $a \leq b$
2. $b \geq a$
3. $a = b$ *or* $a < b$

And here is a typical matrix

$$\begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \alpha & -\alpha & -\alpha & 0 & 0 \\ \alpha & \alpha & 0 & 0 & 0 & 0 & -\alpha & -\alpha \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

For instance in Figure 1 the picture of the parabola was produced in Maple and saved as an epsf (eps) file, (pure ascii mode, no preview, no thumbnail). When the document is processed, the .eps file must reside in the same directory

Figure 1: $y = x^2$

as the .tex file. Also, the command `\usepackage{graphicx}` should occur near the top of the document. Notice that this figure has been scaled so that the overall size is convenient (width 2.5 inches), but now the text is far too small. To avoid this problem it is often best to generate a figure which is approximately the same size as it will appear in the document.

A complete source of information on writing documents in LaTeXis [1]. (Look in the source so see how to produce that citation.) Last of all is the style of the ΠME biliography: Author's names in small caps, journal article titles uncapitalized and in italics, book titles capitalized and in quotes.

3 Applications of Fourier Series

It is used in proving the Nyquist-Shannon sampling theorem, studying harmonic oscillations, waveforms, signal processing, diffractions, interference, and Young's double slit experiment, and Radiation from surface currents, supernovae simulations, partial differential equations by separating variables, solving the heat equation.

3.1 Nyquist-Shannon Sampling Theorem

Theorem 4 *If a time-varying signal is periodically sampled at a rate of at least twice the frequency of the highest sinusoidal component contained within the signal, then the original time-varying signal can be exactly recovered from the periodic samples.*

The theorem also explains why aliasing occurs if the discrete sampling frequency is not sufficiently high enough to capture all the changes in the signal. Fourier series comes into play in this theorem through breaking a signal into its component frequencies.

3.2 Signal Processing

In signal processing, the Fourier series, and especially the Fourier transform, is used to determine what frequencies are present in a signal, and in what proportions. The magnitude squared of the Fourier series of a given signal gives the amount of power the signal has at that particular frequency. The Fourier series also makes it easier for specific frequencies present in a signal to be blocked out, or nullified.

3.3 Astrophysical Simulations

As part of research I am conducting with Dr. James Imamura, where we simulate stars as they accumulate mass and become unstable, the Fourier series coefficients are graphed and examined to see how the simulation is going. As the star becomes more unstable, the first few terms grow larger and fluctuate more dramatically.

3.4 Heat Equation

Solving the heat equation was one of the first applications of the Fourier series, and is what Joseph Fourier proposed when first coming up with the Fourier series.

3.5 Quantum Physics

In quantum physics, the Fourier transform can be used with wavefunctions, as well as with the uncertainty principle to show that completely knowing the momentum of a particle means that the location is completely unknown, as well as the opposite.

References

[1] Lamport, L., "LaTeX- A Document Preparation System", Addison-Wesley, 1998.

[2] Fillioque R. and Heliotrope, B., *Why Fermat's last theorem is really a lemma,* American Mathematical Weekly, Vol. 7, No. 1, pp 115-116, 1998.