Selected Solutions
Beginning Functional Analysis

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Appendix A

A.1 (a) Multiplying the numerator and denominator by the complex conjugate produces the necessary simplification.

$$\frac{2-5i}{1+i}\frac{1-i}{1-i} = \frac{2-5i-2i-5}{1+1} = -\frac{3}{2} - \frac{7}{2}i$$

(b) We are able to show this equality simply by multiplying by 1.

$$\frac{1}{i} = \frac{1}{i}\frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$$

A.2 For this exercise, we write $w = a + ib$ and $z = c + id$. The first four of these will be done by direct calculation from the definitions.

(a) $\overline{(w+z)} = \overline{a+ib+c+id} = \overline{a+c+i(b+d)} = a+c-i(b+d) = a-ib+c-id = \overline{a+ib}+\overline{c+id} = \overline{w}+\overline{z}$

(b) $\overline{w\cdot z} = \overline{(a+ib)(c+id)} = \overline{ac-bd+i(ad+bc)} = ac-bd-i(ad+bc) = (a-ib)(c-id) = \overline{(a+ib)}\,\overline{(c+id)} = \overline{w}\cdot\overline{z}$

(c) Using part (d) and part (b).
$|wz| = \sqrt{wz\overline{wz}} = \sqrt{w\cdot z\cdot\overline{w}\cdot\overline{z}} = \sqrt{w\overline{w}z\overline{z}} = \sqrt{w\overline{w}}\sqrt{z\overline{z}} = |w||z|$

(d) $z\overline{z} = (c+id)(c-id) = c^2+d^2 = \sqrt{c^2+d^2}^2 = |c+id|^2 = |z|^2$

(e) This inequality is an equality for $n = 1$. We will prove $n \geq 2$ by induction. The statement holds for $n = 2$.

$$\begin{aligned}
|z_1+z_2|^2 &= (z_1+z_2)\overline{(z_1+z_2)} \\
&= (z_1+z_2)(\overline{z_1}+\overline{z_2}) \\
&= z_1\overline{z_1}+z_2\overline{z_2}+z_1\overline{z_2}+z_2\overline{z_1} \\
&\leq z_1\overline{z_1}+z_2\overline{z_2}+2\sqrt{z_1\overline{z_1}z_2\overline{z_2}} \\
&= |z_1|^2+|z_2|^2+2|z_1||z_2| \\
&= (|z_1|+|z_2|)^2
\end{aligned}$$

Thus $|z_1+z_2| \leq |z_1|+|z_2|$. Now assume $N > 2$ and that the statement holds for all $n < N$. Since addition of complex numbers is associative (and using the base case above), we can conclude.

$$\begin{aligned}
|z_1+z_2+\cdots+z_N| &\leq |z_1+z_2+\cdots+z_{N-1}|+|z_N| \\
&\leq |z_1|+|z_2|+\cdots+|z_{N-1}|+|z_N|
\end{aligned}$$

Therefore, by the strong principle of mathematical induction,

$$|z_1+z_2+\cdots+z_n| \leq |z_1|+|z_2|+\cdots+|z_n|$$

is true for all n.

(f) Since $|z|^2 = x^2 + y^2$, we can conclude that $|x| \leq |z|$ and $|y| \leq |z|$. Let $m = \max(|x|, |y|)$. Then $m \leq |z|$ (because both $|x|$ and $|y|$ are) and

$$|z|^2 = |x|^2 + |y|^2 \leq m^2 + m^2 = 2m^2 \Rightarrow |z| \leq \sqrt{2}m$$

Therefore, $m \leq |z| \leq \sqrt{2}m$.

(g) This can be done by direct calculation.

$$\frac{z + \overline{z}}{2} = \frac{a + ib + a - ib}{2} = \frac{2a}{2} = a = \mathrm{re}(z)$$

$$\frac{z - \overline{z}}{2i} = \frac{a + ib - a - ib}{2i} = \frac{2ib}{2i} = b = \mathrm{im}(z)$$

A.3 (a) Here we set $w = a + ib$ and $z = c + id$ and use Euler's formula for $e^{i\theta}$. Then we can directly calculate the desired equality.

$$\begin{aligned}
e^{w+z} &= e^{a+c+i(b+d)} \\
&= e^{a+c}(\cos(b+d) + i\sin(b+d)) \\
&= e^a e^c(\cos(b)\cos(d) - \sin(b)\sin(d) + i(\cos(b)\sin(d) + \sin(b)\cos(d))) \\
&= e^a e^c(\cos(b) + i\sin(b))(\cos(d) + i\sin(d)) \\
&= e^a(\cos(b) + i\sin(b))e^c(\cos(d) + i\sin(d)) \\
&= e^{a+ib}e^{c+id} \\
&= e^w e^z
\end{aligned}$$

Thus $e^{w+z} = e^w e^z$ for any complex numbers w and z.

(b) We can prove this by contradiction. Assume there exists a complex number $z = a + ib$ such that $e^z = 0$. Then

$$e^z = e^a(\cos(b) + i\sin(b)) = 0$$

Since a is a real number, $e^a \neq 0$, so

$$\cos(b) + i\sin(b) = 0 \Rightarrow \cos(b) = 0 \text{ and } \sin(b) = 0$$

But there does not exist a real number b such that both $\cos(b)$ and $\sin(b)$ are equal to zero. Therefore, e^z cannot equal zero.

(c) Using Euler's formula, we can prove this with direction calculation.

$$|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = \sqrt{1} = 1$$

Thus $|e^{i\theta}| = 1$.

(d) Using Euler's formula and exponent rules, we can prove this statement for any real number n by direct calculation.

$$(\cos(\theta) + i\sin(\theta))^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

(e) $e^z = e^w$ does not imply that $z = w$ (like it does for real numbers). Counter-example: Let $z = a+ib$ and $w = a + i(b + 2\pi)$. Clearly, $z \neq w$, but

$$e^w = e^a(\cos(b + 2\pi) + i\sin(b + 2\pi)) = e^a(\cos(b) + i\sin(b)) = e^z$$

Therefore, the implication does not hold.

A.4

$$\frac{(\sqrt{3}+i)^6}{(1-i)^{10}} = \frac{(2e^{i\pi/6})^6}{(\sqrt{2}e^{i\pi/4})^{10}} = \frac{2^6 e^{i\pi}}{2^5 e^{i5\pi/2}} = 2e^{-i3\pi/2} = 2e^{i\pi/2} = 0 + 2i$$

So the real part is 0 and the imaginary part is 2.

A.5 This equality is not true. Counterexample: Let $z = w = i$, then

$$\text{re}(z \cdot w) = \text{re}(-1) = -1$$

But

$$\text{re}(z) \cdot \text{re}(w) = 0 \cdot 0 = 0 \neq -1$$

A.6 An ordered field must satisfy two properties for all elements a, b, and c in the field. (In addition to the axioms of a linear order and a field.)

1. If $a \leq b$ then $a + c \leq b + c$.
2. If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$.

The second property leads directly to all squares being positive. More specifically, if a is a member of an ordered field, then $a < 0$ or $a \geq 0$. If $a \geq 0$, then it follows directly from the second property that $a^2 \geq 0$. If $a < 0$, then $-a > 0$ and $(-a)^2 = (-1)^2 a^2 = a^2 > 0$. Thus $a^2 \geq 0$ for any a in the ordered field. But the complex numbers have i as an element and $i^2 = -1 < 0$, so the complex numbers cannot be ordered to give an ordered field.

Appendix B

B.1 De Morgan's Laws can be proven by following the definitions of the sets. Let J be an the index set for a collection of sets $\{A_i\}$.

$$\begin{aligned}
\left(\bigcup_{i \in J} A_i\right)^C &= \{x : x \in A_i \text{ for some } i \in J\}^C \\
&= \{x : x \notin A_i \text{ for all } i \in J\} \\
&= \left\{x : x \in A_i^C \text{ for all } i \in J\right\} \\
&= \bigcap_{i \in J} A_i^C
\end{aligned}$$

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&= \left\{x : x \in A_i^C \text{ for some } i \in J\right\} \\
&= \bigcup_{i \in J} A_i^C
\end{aligned}$$

B.2 Define $h : \mathbb{Q} \to \mathbb{N}$ as

$$h\left(\frac{a}{b}\right) = \begin{cases} 2^a 3^b & a \geq 0 \\ 2^{|a|} 3^b 5 & a < 0 \end{cases}$$

where $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ are used to represent a general element in $\mathbb{Q}$ in its lowest form. This function is injective because each element in $\mathbb{Q}$ maps to a unique positive integer (due to prime factorizations). Therefore $\mathbb{Q}$ is countable because it has an injective map into $\mathbb{N}$.

B.3 Assume that $\{A_i : i \in J\}$ is a collection of countable sets A_i with a countable index set J. Assume that the A_i are disjoint. (If not, then define $A'_1 = A_1$, $A'_2 = A_2 \setminus A'_1$, $\ldots$, $A'_n = A_n \setminus A'_{n-1}$. Then $\bigcup_{i\in J} A'_i = \bigcup_{i\in J} A_i$ and the A'_i are disjoint.) Label each element such that a_{ij} is the jth element in A_i. (This enumeration can be done because each individual set is countable by assumption.) Then define $g : \bigcup_{i\in J} A_i \to \mathbb{N}$ by

$$g(a_{ij}) = 2^i 3^j$$

which makes g injective. We have found an injective map from $\bigcup_{i\in J} A_i$ to $\mathbb{N}$; therefore, $\bigcup_{i\in J} A_i$ is countable.

B.4 Assume that $\mathbb{R}$ is countable. Then every subset is countable, specifically, $[0, 1]$. Assume $f : [0, 1] \to \mathbb{N}$ is the injection between $[0, 1]$ and $\mathbb{N}$. Define $h : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ by $h(k) = (k + 1) \mod 10$. Let a_i be the ith decimal place of $f^{-1}(i)$. Then the real number

$$0.h(a_1)h(a_2)h(a_3)\ldots$$

is an element of $[0, 1]$ and cannot be equal to any member of the domain of f by construction. Therefore f is not an injection and we have found a contradiction. Thus $\mathbb{R}$ is not countable.

Chapter 1

1.1.1 (a) We will verify each norm by showing that it satisfies each of the required conditions using the numbering scheme in the text. These verifications are mostly simple algebra and small logical steps. Let $u, v \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$.

Let $\|\cdot\|_1 : \mathbb{R}^2 \to \mathbb{R}$ be defined as $\|u\|_1 = |u_1| + |u_2|$.

(i) $\|u\|_1 = |u_1| + |u_2| \geq 0$ because the sum of two positive numbers is always positive.

(ii) $\|u\|_1 = 0 \iff |u_1| + |u_2| = 0 \iff u_1 = 0$ and $u_2 = 0 \iff u = 0$

(iii) $\|\lambda u\|_1 = |\lambda u_1| + |\lambda u_2| = |\lambda|(|u_1| + |u_2|) = |\lambda| \|u\|_1$

(iv) $\|u + v\|_1 = |u_1 + v_1| + |u_2 + v_2| \leq |u_1| + |v_1| + |u_2| + |v_2| = \|u\|_1 + \|v\|_1$

Let $\|\cdot\|_2 : \mathbb{R}^2 \to \mathbb{R}$ be defined as $\|u\|_2 = \sqrt{u_1^2 + u_2^2}$.

(i) $\|u\|_2 = \sqrt{u_1^2 + u_2^2} \geq 0$ because the sum of two positive numbers is always positive.

(ii) $\|u\|_2 = 0 \iff \sqrt{u_1^2 + u_2^2} = 0 \iff u_1^2 + u_2^2 = 0 \iff u_1 = 0$ and $u_2 = 0 \iff u = 0$

(iii) $\|\lambda u\|_2 = \sqrt{(\lambda u_1)^2 + (\lambda u_2)^2} = \sqrt{\lambda^2(u_1^2 + u_2^2)} = |\lambda|\sqrt{u_1^2 + u_2^2} = |\lambda| \|u\|_2$

(iv) $\|u + v\|_2 = \sqrt{(u_1 + v_1)^2 + (u_2 + v_2)^2} = \sqrt{u_1^2 + v_1^2 + 2u_1v_1 + u_2^2 + v_2^2 + 2u_2v_2} \leq \sqrt{u_1^2 + u_2^2} + \sqrt{v_1^2 + v_2^2} = \|u\|_2 + \|v\|_2$

Let $\|\cdot\|_\infty : \mathbb{R}^2 \to \mathbb{R}$ be defined as $\|u\|_\infty = \max(|u_1|, |u_2|)$.

(i) $\|u\|_\infty \geq 0$ because both $|u_1| \geq 0$ and $|u_2| \geq 0$.

(ii) $\|u\|_\infty = 0 \iff u_1 = 0$ and $u_2 = 0 \iff u = 0$

(iii) $\|\lambda u\|_\infty = \max(|\lambda u_1|, |\lambda u_2|) = |\lambda| \max(|u_1|, |u_2|) = |\lambda| \|u\|_\infty$

(iv) $\|u + v\|_\infty = \max(|u_1 + v_1|, |u_2 + v_2|) \leq \max(|u_1| + |v_1|, |u_2| + |v_2|) \leq \max(|u_1|, |u_2|) + \max(|v_1|, |v_2|) = \|u\|_\infty + \|v\|_\infty$

(b) We obtain a metric d from a norm $\|\cdot\|$ by $d(u, v) = \|u - v\|$, so

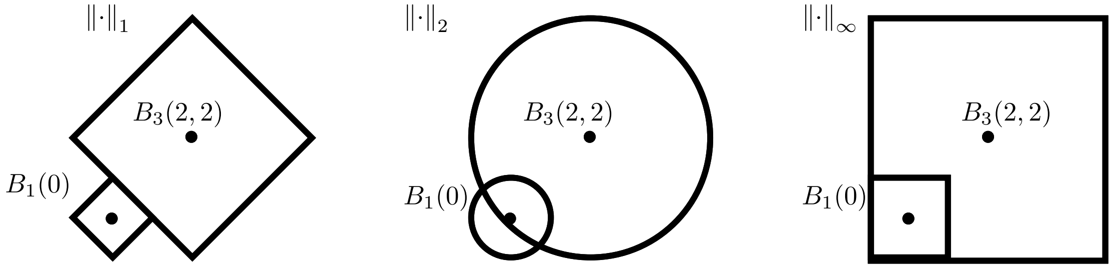
$$d_1((1, 1), (2, 3)) = \|(1, 1) - (2, 3)\|_1 = |1 - 2| + |1 - 3| = 1 + 2 = 3$$

$$d_2((1, 1), (2, 3)) = \|(1, 1) - (2, 3)\|_2 = \sqrt{(1 - 2)^2 + (1 - 3)^2} = \sqrt{1 + 4} = \sqrt{5}$$

$$d_\infty((1, 1), (2, 3)) = \|(1, 1) - (2, 3)\|_\infty = \max(|1 - 2|, |1 - 3|) = \max(1, 2) = 2$$

(c) Below are the example r-balls for the different norms.

1.1.2 (a) Let $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ be defined as $\|v\| = (1/3) \|v\|_1 + (2/3) \|v\|_\infty$. Let $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. In each case, we will use the facts that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms.



(i) $\|x\| = (1/3)\|x\|_1 + (2/3)\|x\|_\infty \geq 0$

(ii) $\|x\| = 0 \iff (1/3)\|x\|_1 + (2/3)\|x\|_\infty = 0 \iff \|x\|_1 = 0$ and $\|x\|_\infty = 0 \iff x = 0$.

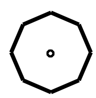
(iii)

$$\begin{aligned}\|\lambda x\| &= (1/3)\|\lambda x\|_1 + (2/3)\|\lambda x\|_\infty = (1/3)|\lambda|\,\|x\|_1 + (2/3)|\lambda|\,\|x\|_\infty \\ &= |\lambda|((1/3)\|x\|_1 + (2/3)\|x\|_\infty) = |\lambda|\,\|x\|\end{aligned}$$

(iv) $\|x + y\| = (1/3)\|x + y\|_1 + (2/3)\|x + y\|_\infty \leq (1/3)(\|x\|_1 + \|y\|_1) + (2/3)(\|x\|_\infty + \|y\|_\infty) = \|x\| + \|y\|$.

Thus $\|\cdot\|$ satisfies the conditions to be a norm on $\mathbb{R}^n$.

(b) Below is the unit ball around the origin for $\|\cdot\|$.



1.1.3 ℓ^1 and ℓ^∞ are linear spaces using element-wise addition as the binary operation and the zero sequence $\mathbf{0} = \{0\}_{j=1}^\infty$ as the zero element. Now we must show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms in their respective spaces. Let $\mathbf{a}, \mathbf{b}$ be elements in the linear space and $\lambda \in \mathbb{R}$.
First, $\|\cdot\|_1$ satisfies the conditions to be a norm:

(i) $\|\mathbf{a}\|_1 = \sum |a_i| \geq 0$ because each element of the sum $|a_i| \geq 0$.

(ii) $\|\mathbf{a}\|_1 = \sum |a_i| = 0 \iff |a_i| = 0$ for all $i \iff \mathbf{a} = \mathbf{0}$.

(iii) $\|\lambda \mathbf{a}\|_1 = \sum |\lambda a_i| = \sum |\lambda||a_i| = |\lambda| \sum |a_i| = |\lambda|\,\|\mathbf{a}\|_1$.

(iv) This condition is satisfied by a central theorem in real analysis, which extends the triangle inequality to absolutely convergent series.

Next, $\|\cdot\|_\infty$ satisfies the conditions to be a norm:

(i) $\|\mathbf{a}\|_\infty = \sup\{|a_i|\} \geq 0$ because each $|a_i| \geq 0$.

(ii) $\|\mathbf{a}\|_\infty = \sup\{|a_i|\} = 0 \iff |a_i| = 0$ for all $i \iff \mathbf{a} = \mathbf{0}$.

(iii) $\|\lambda \mathbf{a}\|_\infty = \sup\{|\lambda a_i|\} = |\lambda| \sup\{|a_i|\} = |\lambda|\,\|\mathbf{a}\|_\infty$.

(iv) $\|\mathbf{a} + \mathbf{b}\|_\infty = \sup\{|a_i + b_i|\} \leq \sup\{|a_i| + |b_i|\} \leq \sup\{|a_i|\} + \sup\{|b_i|\} = \|\mathbf{a}\|_\infty + \|\mathbf{b}\|_\infty$.

Therefore ℓ^1 and ℓ^∞ are both normed linear spaces.

1.1.4 $C([a,b])$ is a linear space with point-wise addition as the binary operation and $o(x) = 0$ for all $x \in [a,b]$ as the zero element. All that is left now is showing that $\|\cdot\|_\infty$ is a norm. Let $f, g \in C([a,b])$ and $\lambda \in \mathbb{R}$.

(i) $\|f\|_\infty = \sup\{|f(x)| : x \in [a,b]\} \geq 0$ because $|f(x)| \geq 0$ for all $x \in [a,b]$.

(ii) $\|f\|_\infty = \sup\{|f(x)| : x \in [a,b]\} = 0 \iff |f(x)| = 0$ for all $x \in [a,b] \iff f = o$.

(iii) $\|\lambda f\|_\infty = \sup\{|\lambda f(x)| : x \in [a,b]\} = \sup\{|\lambda||f(x)| : x \in [a,b]\} = |\lambda| \sup\{|f(x)| : x \in [a,b]\} = |\lambda| \, \|f\|_\infty$.

(iv) $\|f+g\|_\infty = \sup\{|f(x)+g(x)| : x \in [a,b]\} \le \sup\{|f(x)|+|g(x)| : x \in [a,b]\} \le \sup\{|f(x)| : x \in [a,b]\} + \sup\{|g(x)| : x \in [a,b]\} = \|f\|_\infty + \|g\|_\infty$.

1.1.5 Assume $(V, \|\cdot\|)$ is a normed linear space and $d : V \times V \to \mathbb{R}$ is defined as

$$d(v,w) = \|v - w\|$$

Then

(i) $d(v,w) = \|v-w\| \ge 0$ for all $v, w \in V$ because $v - w \in V$ and $\|\cdot\|$ is a norm.

(ii) $d(v,w) = 0 \iff \|v-w\| = 0 \iff v - w = 0 \iff v = w$.

(iii) $d(v,w) = \|v-w\| = \|(-1)(w-v)\| = |-1| \, \|w-v\| = \|w-v\| = d(w,v)$ for all $v,w \in V$.

(iv) $d(v,w) = \|v-w\| = \|v-z+z-w\| \le \|v-z\| + \|z-w\| = d(v,z) + d(z,w)$ for all $v,w,z \in V$.

So d is a metric on V.

1.1.6 Assume $(V, \langle \cdot, \cdot \rangle)$ is an inner product space. Define $\|\cdot\| : V \to \mathbb{R}$ by

$$\|v\| = \sqrt{\langle v, v \rangle}$$

Then

(i) $\|v\| = \sqrt{\langle v,v \rangle} \ge 0$ for all $v \in V$.

(ii) $\|v\| = \sqrt{\langle v,v \rangle} = 0 \iff \langle v,v \rangle = 0 \iff v = 0$.

(iii) $\|\lambda v\| = \sqrt{\langle \lambda v, \lambda v \rangle} = \sqrt{\lambda^2 \langle v,v \rangle} = |\lambda| \sqrt{\langle v,v \rangle} = |\lambda| \, \|v\|$ for all $v \in V$ and $\lambda \in \mathbb{R}$.

(iv) $\|v+w\| = \sqrt{\langle v+w, v+w \rangle} = \sqrt{\langle v,v \rangle + \langle v,w \rangle + \langle w,v \rangle + \langle w,w \rangle} \le \sqrt{\langle v,v \rangle} + \sqrt{\langle w,w \rangle} = \|v\| + \|w\|$ for all $v,w \in V$.

So $\|\cdot\|$ is a norm on V.

1.1.7 In any complex inner product space $(V, \langle \cdot, \cdot \rangle)$, we can construct a series of equalities using Hermitian symmetry and multiplicativity of the inner product. Let $v, w \in V$ and $\lambda \in \mathbb{C}$. Then

$$\langle v, \lambda w \rangle = \overline{\langle \lambda w, v \rangle} = \overline{\lambda} \overline{\langle w, v \rangle} = \overline{\lambda} \langle v, w \rangle$$

So we can conclude $\langle v, \lambda w \rangle = \overline{\lambda} \langle v, w \rangle$.

1.1.8 (a) Assume that $(V, \|\cdot\|)$ is a normed linear space.
($\Rightarrow$) Assume that $\|v\| = \sqrt{\langle v,v \rangle}$ for some inner product $\langle \cdot, \cdot \rangle$. Then (using the properties of an inner product)

$$\begin{aligned}
\|u+v\|^2 + \|u-v\|^2 &= \langle u+v, u+v \rangle + \langle u-v, u-v \rangle \\
&= \langle u,u \rangle + 2\langle u,v \rangle + \langle v,v \rangle + \langle u,u \rangle - 2\langle u,v \rangle + \langle v,v \rangle \\
&= 2\langle u,u \rangle + 2\langle v,v \rangle \\
&= 2\|u\|^2 + 2\|v\|^2
\end{aligned}$$

So $\|\cdot\|$ satisfies the parallelogram equality for an arbitrary pair $u, v \in V$.
($\Leftarrow$) Assume that $\|\cdot\|$ satisfies the parallelogram equality for all $u,v \in V$. Define $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ as

$$\langle u, v \rangle = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 - \frac{1}{2}\|u-v\|^2 = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right)$$

Then

$$\langle v, v \rangle = \frac{1}{4}\|2v\|^2 = \|v\|^2 \ge 0$$

proving (i), and $\langle v, v\rangle = \|v\|^2 = 0$ if and only if $v = 0$ because of the properties of $\|\cdot\|$ — proving (ii). Also, for any $u, v \in V$

$$\begin{aligned}
\langle u, v\rangle &= \frac{1}{4}\left(\|u+v\|^2 + \|u-v\|^2\right) \\
&= \frac{1}{4}\left(\|v+u\|^2 + \|(-1)(v-u)\|^2\right) \\
&= \frac{1}{4}\left(\|v+u\|^2 + \|v-u\|^2\right) \\
&= \langle v, u\rangle
\end{aligned}$$

proving (iv). Now let $u, v, w \in V$. From the parallelogram equality we have

$$2\|u+w\|^2 + 2\|v\|^2 = \|u+v+w\|^2 + \|u-v+w\|^2$$

Re-arranging this gives

$$\begin{aligned}
\|u+v+w\|^2 &= 2\|u+w\|^2 + 2\|v\|^2 - \|u-v+w\|^2 \\
&= 2\|v+w\|^2 + 2\|u\|^2 - \|w+v-u\|^2
\end{aligned}$$

where the second line comes from exchanging (the arbitrary) u and w. Adding the right hand sides together and dividing by two gives us the left hand side again, so

$$\|u+v+w\|^2 = \|u\|^2 + \|w\|^2 + \|u+w\|^2 + \|v+w\|^2 - \frac{1}{2}\|u+v-w\|^2 + \frac{1}{2}\|w+v-u\|^2$$

Replacing w with $-w$ gives

$$\|u+v-w\|^2 = \|u\|^2 + \|v\|^2 + \|u-w\|^2 + \|v-w\|^2 - \frac{1}{2}\|u-v-w\|^2 + \frac{1}{2}\|v-u-w\|^2$$

Combining these calculations, we obtain

$$\begin{aligned}
\langle u+v, w\rangle &= \frac{1}{4}\left(\|u+v+w\|^2 - \|u+v-w\|^2\right) \\
&= \frac{1}{4}\left(\|u+w\|^2 - \|u-w\|^2\right) + \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2\right) \\
&= \langle u, w\rangle + \langle v, w\rangle
\end{aligned}$$

This proves that $\langle\cdot,\cdot\rangle$ satisfies property (v). By our definition for $\langle\cdot,\cdot\rangle$, $\langle\lambda u, v\rangle = \lambda\langle u, v\rangle$ is satisfied for $\lambda = -1$ and — using property (v) and induction — equality holds for $\lambda \in \mathbb{N}$. Thus equality holds for all $\lambda \in \mathbb{Z}$. Let $\lambda \in \mathbb{Q}$, then $\lambda = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ and $q \neq 0$. Then

$$q\langle\lambda u, v\rangle = q\langle p\left(\frac{u}{q}\right), v\rangle = p\langle q\left(\frac{u}{q}\right), v\rangle = p\langle u, v\rangle$$

which means (dividing both sides by q) that equality holds for $\lambda \in \mathbb{Q}$. In order to jump to all $\lambda \in \mathbb{R}$, we make a hand-waving argument that works well when the details are fleshed out. First, note that the map $(u, v) \to \langle u, v\rangle$ is continuous because the norm it is defined with respect to is continuous. For each $r \in \mathbb{R}$ we can define a sequence of $q_i \in \mathbb{Q}$ such that $(q_i) \to r$. Since $\langle\cdot,\cdot\rangle$ is continuous, we can say

$$\lim_{i\to\infty}\langle q_i u, v\rangle = \langle\lambda u, v\rangle \quad \text{and} \quad \lim_{i\to\infty} q_i\langle u, v\rangle = \lambda\langle u, v\rangle$$

And since $\langle q_i u, v\rangle = q_i\langle u, v\rangle$ for each i, we can conclude

$$\langle\lambda u, v\rangle = \lim_{i\to\infty}\langle q_i u, v\rangle = \lim_{i\to\infty} q_i\langle u, v\rangle = \lambda\langle u, v\rangle$$

proving (iii). Thus our definition of $\langle\cdot,\cdot\rangle$ satisfies the requirements to be an inner product on V. We have proved that if $(V, \|\cdot\|)$ is a normed linear space then $\|\cdot\|$ satisfies the parallelogram equality if and only if $\|\cdot\|$ comes from an inner product.

1.1.9 We are able to find two functions that do not satisfy the parallelogram equality; therefore, the supremum norm cannot come from an inner product. Let $f, g \in C([a,b])$ be defined as

$$f(x) = \frac{1}{b-a}(x-a) \quad g(x) = 1 - \frac{1}{b-a}(x-a)$$

Then

$$(f+g)(x) = 1 \quad (f-g)(x) = -1 + \frac{2}{b-a}(x-a)$$

And

$$\|f\|_\infty = 1 \quad \|g\|_\infty = 1 \quad \|f+g\|_\infty = 1 \quad \|f-g\|_\infty = 1$$

So

$$2\|f\|_\infty^2 + 2\|g\|_\infty^2 = 2 + 2 = 4$$

But

$$\|f+g\|_\infty^2 + \|f-g\|_\infty^2 = 1 + 1 = 2 \neq 4$$

Thus $\|\cdot\|_\infty$ cannot come from an inner product.

1.1.10 (a)

$$d(f,g) = \|f-g\|_\infty = \sup_{x\in[0,1]}\{|f(x)-g(x)|\} = \sup_{x\in[0,1]}\{|1-x|\} = 1$$

(b)

$$d(f,g) = \sqrt{\langle f-g, f-g\rangle} = \sqrt{\langle 1-x, 1-x\rangle} = \sqrt{\int_0^1 (1-x)^2 dx} = \sqrt{\int_0^1 u^2 du} = \frac{1}{\sqrt{3}}$$

1.3.1 Another basis for $\mathbb{R}^3$ is

$$\{\langle 1,1,1\rangle, \langle 1,1,0\rangle, \langle 1,0,0\rangle\}$$

Proof: Suppose

$$a\langle 1,1,1\rangle + b\langle 1,1,0\rangle + c\langle 1,0,0\rangle = \langle 0,0,0\rangle$$

Then we can see that $a = b = c = 0$, so this set is linearly independent. Let $\langle x,y,z\rangle \in \mathbb{R}^3$. Then

$$\begin{aligned} z\langle 1,1,1\rangle + (y-z)\langle 1,1,0\rangle + (x-y)\langle 1,0,0\rangle &= \langle z,z,z\rangle + \langle y-z, y-z, 0\rangle + \langle x-y, 0, 0\rangle \\ &= \langle x,y,z\rangle \end{aligned}$$

So this set also spans $\mathbb{R}^3$. Thus it is a basis for $\mathbb{R}^3$.

1.3.2 Let x^n be the sequence of all zeros except with 1 at the nth position. Then $x^n \in \ell^1$ for all n because $\sum_i |x_i^n| = 1$. Let $N > 0$ be given and $\mathbf{0}$ be the zero sequence. Suppose

$$\sum_{i=1}^{N} \alpha_i x^i = \mathbf{0}$$

Then we must have $\alpha_i = 0$ for all i because that is the only way to make each position in the sequence zero. Thus $\{x^i : i \leq N\}$ is a linearly independent set. Since N is arbitrary, ℓ^1 is infinite-dimensional. Since $\ell^1 \subseteq \ell^\infty$ (as a linear space), ℓ^∞ must be infinite-dimensional because if a finite basis existed for ℓ^∞, it would also cover ℓ^1 (which we know cannot happen).

1.3.3 Let $f_n : [0,1] \to \mathbb{R}$ be defined as $f_n(x) = x^n$ for all $n \in \mathbb{N}$. Let $N \in \mathbb{N}$ be given. Define $o : [0,1] \to \mathbb{R}$ as $o(x) = 0$. Suppose

$$\sum_{i=0}^{N} \alpha_i f_i = o$$

The summation on the left side is a polynomial of degree N; therefore, it can have (at most) N zeros if it has non-zero coefficients. However, since it equals the zero function, it has an uncountably infinite number of zeros; thus, $\alpha_i = 0$ for all i. Therefore $\{f_n : n \leq N\}$ is a linearly independent set and since N is arbitrary, $C([0,1])$ is infinite-dimensional.

Chapter 2

2.1.1 Let $E \subseteq (M,d)$ a metric space, and define $E^C = M \setminus E$.
($\Rightarrow$) Assume E is open. Let x be a limit point of E^C, then either $x \in E$ or $x \in E^C$. If $x \in E$, then there exists $r > 0$ such that $B_r(x) \subseteq E$ since E is open. But $B_r(x) \cap E^C = \emptyset$. Thus $x \in E^C$. Therefore E^C contains all of its limit points and therefore is closed.
($\Leftarrow$) Assume E^C is closed. Let $x \in E$. Since E^C is closed, there must be an $r > 0$ such that $B_r(x) \cap E^C = \emptyset$. (Otherwise, if no such r exists, then x would be a limit point of E^C that is not in E^C.) Thus $B_r(x) \subseteq \left(E^C\right)^C = E$. Therefore E is open.

2.1.2 We will use the equivalence given in Exercise 2.1.1 for parts (b) and (d).

(a) Assume $\{E_\alpha\}$ is a collection of open sets. Let $x \in \bigcup_\alpha E_\alpha$. Then $x \in E_\alpha$ for some α. Thus there exists an $r > 0$ such that $B_r(x) \subseteq E_\alpha \subseteq \bigcup_\alpha E_\alpha$. Therefore $\bigcup_\alpha E_\alpha$ is open.

(b) Assume $\{F_\alpha\}$ is a collection of closed sets. Then $\{F_\alpha^C\}$ is a collection of open sets. Thus (by part (a)) $\bigcup_\alpha F_\alpha^C$ is open, which means its complement is closed. Thus $\left(\bigcup_\alpha F_\alpha^C\right)^C = \bigcap_\alpha F_\alpha$ is closed.

(c) Assume $\{E_i\}_{i=1}^N$ is a finite collection of open sets. Let $x \in \bigcap_{i=1}^N E_i$. Then $x \in E_i$ for all i, which means there exists r_i such that $B_{r_i}(x) \subseteq E_i$ for all i. Set $r = \min r_i$. Then $B_r(x) \subseteq B_{r_i}(x) \subseteq E_i$ for all i. Thus $B_r(x) \subseteq \bigcap_{i=1}^N E_i$. Therefore $\bigcap_{i=1}^N E_i$ is open.

(d) Assume $\{F_i\}_{i=1}^N$ is a finite collection of closed sets. Then $\{F_i^C\}$ is a finite collection of open sets. Thus (by part (c)), $\bigcap_{i=1}^N F_i^C$ is open, which means its complement is closed. Therefore $\left(\bigcap_{i=1}^N F_i^C\right)^C = \bigcup_{i=1}^N F_i$ is closed.

2.1.3 The collection $\{(-1, 1/n) : n \in \mathbb{N}\}$ is a collection of open sets, but

$$\bigcup_{n=1}^{\infty} \left(-1, \frac{1}{n}\right) = (-1, 0]$$

which is not an open set.
The collection $\{[0, 1 - 1/n] : n \in \mathbb{N}\}$ is a collection of closed sets, but

$$\bigcup_{n=1}^{\infty} \left[0, 1 - \frac{1}{n}\right] = [0, 1)$$

which is not a closed set.

2.1.4 Let (M,d) be a metric space and let $E \subseteq M$. *Note:* We use the claim that E is closed if and only if E^C is open — this is proven in Exercise 2.1.1.
Assume E is compact and let $x \in E^C$. Define $r_e = d(e,x)/2$ for all $e \in E$. Define an open cover of E as

$$\mathcal{F} = \{B_{r_e}(e) : e \in E\}$$

Since E is compact, $\mathcal{F}$ has a finite sub-cover

$$\left\{B_{r_{e_1}}(e_1), \ldots, B_{r_{e_N}}(e_N)\right\}$$

Let $\rho = \min r_{e_1}, \ldots, r_{e_N}$. Suppose $y \in B_\rho(x) \cap E$. Then $y \in B_{r_{e_i}}(x)$ for some i in the finite cover. Then

$$d(x, e_i) \leq d(x,y) + d(y, e_i) < \rho + r_{e_i} \leq 2r_{e_i} = d(x, e_i)$$

which is impossible. So there is no $y \in E$ that is in $B_\rho(x)$. Thus $B_\rho(x) \subseteq E^C$, which means E^C is open. Therefore E is closed.

2.1.5 (a) In this space $(0,1)$ is open and not closed. *Proof:* Let $x \in (0,1)$. Then set $r = \min x, 1 - x$. Then $B_r(x) \subseteq (0,1)$. Thus $(0,1)$ is open. 0 is a limit point of $(0,1)$ and $0 \notin (0,1)$, so $(0,1)$ is not closed.

(b) In this space $(0,1)$ is neither open nor closed. *Proof:* The point $(1/2,1) \in (0,1)$ but any ball around it will include points off the x-axis, so $(0,1)$ cannot be open. The point $(0,0)$ is a limit point of $(0,1)$ that is not in it, so $(0,1)$ cannot be closed.

2.1.6 Generally $\text{int}(A \cup B) \neq \text{int}(A) \cup \text{int}(B)$. Counter-example: Let $A = (0,1]$ and $B = [1,2)$ in $\mathbb{R}$. Then $\text{int}(A \cup B) = (0,2)$ and $\text{int}(A) \cup \text{int}(B) = (0,1) \cup (1,2) \neq (0,2)$.
Contrarily, it is generally true that $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$.
Proof: ($\subseteq$) Let $x \in \text{int}(A \cap B)$. Then there exists $r > 0$ such that $B_r(x) \subseteq A \cap B$. This means $B_r(x) \subseteq A$ and $B_r(x) \subseteq B$. Thus x is in both interiors, meaning $x \in \text{int}(A) \cap \text{int}(B)$.
($\supseteq$) Let $x \in \text{int}(A) \cap \text{int}(B)$. Then there exist r_1, r_2 such that $B_{r_1}(x) \subseteq A$ and $B_{r_2}(x) \subseteq B$. Set $r = \min r_1, r_2$. Then $B_r(x) \subseteq A \cap B$. Thus $x \in \text{int}(A \cap B)$.

2.1.7 If M has the discrete metric, then the singleton sets are open because

$$\{x\} = B_{1/2}(x)$$

Thus if $E \subseteq M$, it can be written as

$$E = \bigcup_{x \in E} \{x\}$$

which shows that E is the union of open sets, making it open. Therefore, all subsets of M are open (The collection of open sets in M is 2^M).

2.1.8 (a) The set of limit points of $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ is $\{0\}$.

(b) Because $\mathbb{Q}$ is dense in $\mathbb{R}$, each point of $\mathbb{R}$ is a limit point of $\mathbb{Q}$.

2.1.9 Define e_i to be the zero sequence with 1 in the ith position. Then the set $E = \{e_i : i \in \mathbb{N}\} \subseteq \ell^\infty$ is closed and bounded, but the open cover $\{B_{1/2}(e_i) : i \in \mathbb{N}\}$ has no finite sub-cover (so E is not compact).

2.1.10 The open cover $\{(-10 + 1/n, 10] : n \in \mathbb{N}\}$ has no finite sub-cover.

2.1.11 Let $C = \bigcap_{n=1}^\infty C_n$ be the Cantor set where the C_n are the sub-divisions of $[0,1]$ in the usual definition of C.

(a) Since $C \subseteq \mathbb{R}$, we can use the Heine-Borel Theorem to show that it is compact by showing that it is closed and bounded. By definition of C (as a subset of $[0,1]$), it is bounded. Since C_n is a finite union of closed intervals (for each n), C_n is closed for all n. Since arbitrary intersections of closed sets are closed, C is closed. Thus C is closed and bounded and therefore compact.

(b) $\text{int}(C) = \emptyset$. Suppose there is an open interval $I = (a,b)$ (equivalently, an open ball) contained in C. Then $I \subseteq C_n$ for all n. The C_n are made up of disjoint intervals of width $(2/3)^n$, so for I to be a subset of all C_n, $0 < b - a < (2/3)^n$ for all n. This is impossible. Therefore, there is no open ball contained in C.

2.1.12 Assume E is totally bounded. Let $\{B_r(x_i)\}_{i=1}^N$ be the finite cover of r-balls. Let $x \in E$. Define the diameter D as

$$D = \max\{2r + d(x_i, x_j) : 1 \leq i, j \leq N\}$$

Claim: $B_D(x) \supseteq E$. Let $y \in E$ be given. Then $x \in B_r(x_i)$ for some i and $y \in B_r(x_j)$ for some j. Then

$$d(x,y) \leq d(x,x_i) + d(x_i,x_j) + d(x_j,y) < r + d(x_i,x_j) + r \leq D$$

Thus $y \in B_D(x)$. Therefore $B_D(x) \supseteq E$.
Any set with an infinite number of elements would not admit a finite cover of r-balls with $r < 1$ under the discrete metric.

2.1.14 Assume $A \subseteq M$ is compact and $U \subseteq M$ is open. Let $\mathcal{E} = \{E_\alpha\}$ be an open cover of $A \setminus U$. Then $\{U\} \cup \{E_\alpha\}$ is an open cover of A. This open cover has a finite sub-cover $\mathcal{F}$ because A is compact. Then $\mathcal{F} \setminus \{U\}$ still covers $A \setminus U$, and is still finite. Thus $\mathcal{E}$ has a finite subcover, making $A \setminus U$ compact.

2.1.15 (a) Let $f : M_1 \to M_2$.

($\Rightarrow$) Let $U \subseteq M_2$ be open and let f be continuous. Let $y \in M_1$ such that $f(y) \in U$. Since U is open, there exists $\epsilon > 0$ such that $B_\epsilon(f(y)) \subseteq U$. Since f is continuous, there exists $\delta > 0$ such that if $x \in B_\delta(y)$ then $f(x) \in B_\epsilon(f(y)) \subseteq U$. Thus $B_\delta(y) \subseteq f^{-1}(U)$ which means $f^{-1}(U)$ is open.

($\Leftarrow$) If $U \subseteq M_2$ is open, then $f^{-1}(U) \subseteq M_1$ is open. Let $\epsilon > 0$ be given. Then $B_\epsilon(f(x)) \subseteq M_2$ is open, which means $f^{-1}(B_\epsilon(f(x))) \subseteq M_1$ is open. Thus there exists a $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x)))$. Therefore f is continuous.

(b) Let $f : M \to \mathbb{R}$ be continuous and let $A \subseteq M$. Assume $f(A)$ is unbounded. Define $\{y_i\}_{i=1}^\infty \subseteq f(A)$ to be a sequence that diverges to infinity. Then $\{x_i : f(x_i) = y_i\}_{i=1}^\infty \subseteq A$ is a sequence that has no convergent subsequence because otherwise $\{y_i\}_{i=1}^\infty$ wouldn't diverge. Thus A is not sequentially compact and therefore not compact.

Equivalently, if A is compact, then $f(A)$ is bounded.

2.2.2 Let $\mathbf{q} = (q, 0, 0, \dots) \in \ell^1$. The set $Q = \{\mathbf{q} : q \in \mathbb{Q}\}$ is countable and dense in ℓ^1. Q is in one-to-one correspondence with the countable set $\mathbb{Q}$ under the mapping $\mathbf{q} \to q$.

Let $\mathbf{a} \in \ell^1$ and $\epsilon > 0$ be given. Choose $q \in \mathbb{Q}$ such that $\|\mathbf{a}\|_1 < q < \|\mathbf{a}\|_1 + \epsilon$. Then

$$\begin{aligned} \|\mathbf{a} - \mathbf{q}\|_1 &= \|\mathbf{a}\|_1 - (|a_1| - |a_1 - q|) \\ &= \|\mathbf{a}\|_1 - |q| \\ &< \epsilon \end{aligned}$$

Thus $\mathbf{q} \in Q \cap B_\epsilon(\mathbf{a})$. Thus $\mathbf{a}$ is a limit point of Q. Therefore, since $\mathbf{a}$ is arbitrary, all the points of ℓ^1 are limit points of Q.

Thus Q is a countable dense subset of ℓ^1 making ℓ^1 separable.

2.2.3 Now define $\mathbf{q} = (q, q, q, \dots) \in \ell^\infty$ for all $q \in \mathbb{Q}$. The set $Q = \{\mathbf{q} : q \in \mathbb{Q}\}$ is countable because it is in one-to-one correspondence with $\mathbb{Q}$.

Let $\mathbf{a} \in \ell^\infty$ and $\epsilon > 0$ be given. Choose $q \in \mathbb{Q}$ such that $\|\mathbf{a}\|_\infty < q < \|\mathbf{a}\|_\infty + \epsilon$. Then

$$\|\mathbf{a} - \mathbf{q}\|_\infty = \sup\{|a_i - q|\} \le |\sup\{|a_i|\} - q| = |\,\|\mathbf{a}\|_\infty - q| < \epsilon$$

Thus $\mathbf{q} \in Q \cap B_\epsilon(\mathbf{a})$ which means $\mathbf{a}$ is a limit point of Q. Therefore Q is countable and dense in ℓ^∞, meaning ℓ^∞ is separable.

2.3.1 Assume $X = \{x_n\}_{n=1}^\infty \subseteq M$ is Cauchy. Define $N \in \mathbb{N}$ such that $d(x_i, x_j) < 1$ for all $i, j \ge N$ — possible since X is Cauchy. Now define $R = \max\{1, d(x_1, x_N), d(x_2, x_N), \dots, d(x_{N-1}, x_N)\}$. Then $d(x_i, x_N) \le R$ for all i because of the definition of R and N. Let $n \in \mathbb{N}$ be given. For any i,

$$d(x_i, x_n) \le d(x_i, x_N) + d(x_N, x_n) \le 2R$$

which means $X \subseteq B_{2R}(x_n)$. Therefore X is bounded.

2.3.2 Assume there exists $a > 0$ and $b > 0$ such that

$$a\,\|x\| \le |||x||| \le b\,\|x\| \quad \text{for all } x \in X$$

($\Rightarrow$) Assume $(X, \|\cdot\|)$ is complete. Let $\{x_n\}_{n=1}^\infty$ be Cauchy in $(X, |||\cdot|||)$ and let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $|||x_n - x_m||| < a\epsilon$ for all $n, m \ge N$. Then

$$\|x_n - x_m\| \le \frac{1}{a}|||x_n - x_m||| < \epsilon \quad \text{for all } n, m \ge N$$

Thus $\{x_n\}_{n=1}^\infty$ is Cauchy in $(X, \|\cdot\|)$ and therefore converges to some $x \in X$. Let $\delta > 0$ be given. Choose $M \in \mathbb{N}$ such that $\|x_n - x\| < \delta/b$ for all $n \ge M$. Then

$$|||x_n - x||| \le b\,\|x_n - x\| < \delta \quad \text{for all } n \ge M$$

Thus $\{x_n\}_{n=1}^{\infty}$ converges to x in $(X, |||\cdot|||)$. Therefore $(X, |||\cdot|||)$ is complete.
$(\Leftarrow)$ Because $a > 0$ and $b > 0$ we are able to conclude

$$a\,\|x\| \leq |||x||| \leq b\,\|x\| \quad \Longrightarrow \quad \frac{1}{b}|||x||| \leq \|x\|\,\frac{1}{a}|||x|||$$

Then we use the preceding argument to conclude that $(X, \|\cdot\|)$ is complete if $(X, |||\cdot|||)$ is complete.

2.3.4 If d is the discrete metric, then (M, d) is complete. The only way for $\{x_n\}_{n=1}^{\infty}$ to be Cauchy is for $x_n = x$ for all $n \geq N$ for some N. This would mean that $\{x_n\}_{n=1}^{\infty}$ would converge to $x \in M$ as well.

2.3.5 Define $\|\cdot\|_1 : C([a, b]) \to \mathbb{R}$ by

$$\|f\|_1 = \int_a^b |f(x)|dx$$

Let $o(x) = 0$ for all $x \in [a, b]$. We say two functions are equal if they are equal almost everywhere. Then

(i) $\|f\|_1 \geq 0$ for all $f \in C([a, b])$.

(ii) $\|f\|_1 = 0 \iff |f(x)| = 0$ for almost all $x \in [a, b] \iff f = o$.

(iii) $\|\lambda f\|_1 = \int_a^b |\lambda f(x)|dx = |\lambda| \int_a^b |f(x)|dx = |\lambda|\,\|f\|_1$.

(iv) $\|f + g\|_1 = \int_a^b |f(x) + g(x)|dx \leq \int_a^b |f(x)| + |g(x)|dx = \|f\|_1 + \|g\|_1$.

Thus $\|\cdot\|_1$ is a norm on $C([a, b])$.
This norm does not come from an inner product. The functions defined below do not satisfy the parallelogram equality.

$$f(x) = \frac{1}{b-a}(x - a) \quad g(x) = 1 - \frac{1}{b-a}(x - a)$$

The normed linear space $(C([a, b]), \|\cdot\|_1)$ is also not a Banach space because it is not complete.
Define $f_n : [0, 1] \to \mathbb{R}$ as $f_n(x) = x^n$ for all n. Let $\epsilon > 0$ be given. Choose N such that $1/N < \epsilon$. Then if $n, m \geq N$ we have

$$\|f_n - f_m\|_1 = \int_0^1 |x^n - x^m|dx = \left|\frac{1}{n+1} - \frac{1}{m+1}\right| < \frac{1}{N} < \epsilon$$

So $\{f_n\}_{n=1}^{\infty}$ is Cauchy, but it converges point-wise to

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

which is not a continuous function.

2.3.6 Define f_n in the same way as 2.3.5 above. Let $\epsilon > 0$ be given. Choose N such that $2/N < \epsilon^2$. Then if $n, m \geq N$ we have

$$\begin{aligned}\|f_n - f_m\|_2^2 &= \int_0^1 (x^n - x^m)^2 dx = \int_0^1 (x^{2n} + x^{2m} - 2x^{n+m})dx = \frac{1}{2n+1} + \frac{1}{2m+1} - \frac{2}{n+m+1} \\ &< \frac{1}{N} + \frac{1}{N} < \epsilon^2\end{aligned}$$

So $\{f_n\}_{n=1}^{\infty}$ is Cauchy, but it converges point-wise to the function f above which is not continuous.

2.3.7 Let $\{a^n\}_{n=1}^{\infty} \in \ell^1$ be Cauchy. Let $\epsilon > 0$ be given. Choose N such that if $n, m \geq N$ we have

$$\|a^n - a^m\|_1 < \epsilon$$

Then using the definition of $\|\cdot\|_1$ we know

$$\sum_{i=1}^{\infty} |a_i^n - a_i^m| < \epsilon \quad \Longrightarrow \quad |a_i^n - a_i^m| < \epsilon \text{ for all } i$$

Thus, for each $i \in \mathbb{N}$, $\{a_i^n\}_{n=1}^{\infty}$ is a Cauchy sequence of reals. Since $\mathbb{R}$ is complete, $\{a_i^n\}_{n=1}^{\infty}$ converges to some real, call it a_i.
Define $a = \{a_i\}_{i=1}^{\infty}$, then

$$\|a\|_1 = \lim_{J\to\infty} \sum_{i=1}^{J} |a_i| = \lim_{J\to\infty} \sum_{i=1}^{J} \lim_{n\to\infty} |a_i^n|$$

We can inter-change the limits and sums because the terms are positive reals, so

$$\|a\|_1 = \lim_{n\to\infty} \lim_{J\to\infty} \sum_{i=1}^{J} |a_i^n| = \lim_{n\to\infty} \|a^n\|_1$$

For some M and each n, we know $\|a^n\|_1 < M$, thus

$$\|a\|_1 < \lim_{n\to\infty} M = M$$

Thus $a \in \ell^1$.
Now we need to show that $\{a^n\}_{n=1}^{\infty}$ converges to a. Let $\epsilon > 0$ be given. Choose N such that if $n, m \geq N$ we have

$$\|a^n - a^m\|_1 < \epsilon$$

Then for any J and $n, m \geq N$ we have

$$\sum_{i=1}^{J} |a_i^n - a_i^m| \leq \sum_{i=1}^{\infty} |a_i^n - a_i^m| = \|a^n - a^m\|_1 < \epsilon$$

Fix $n > N$ and J, let $m \to \infty$, then

$$\sum_{i=1}^{J} |a_i^n - a_i| = \lim_{m\to\infty} \sum_{i=1}^{J} |a_i^n - a_i^m| < \epsilon$$

This is true for arbitrary J, so

$$\|a^n - a\|_1 \leq \epsilon$$

showing that $\{a^n\}_{n=1}^{\infty}$ converges to $a \in \ell^1$.

2.3.8 Let $\{a^n\}_{n=1}^{\infty} \in \ell^{\infty}$ be Cauchy. Let $\epsilon > 0$ be given. Choose N such that if $n, m \geq N$ we have

$$\|a^n - a^m\|_{\infty} < \epsilon$$

Then using the definition of $\|\cdot\|_{\infty}$ we know

$$\sup\{|a_i^n - a_i^m|\} < \epsilon \quad \Longrightarrow \quad |a_i^n - a_i^m| < \epsilon \text{ for all } i$$

Thus, for each $i \in \mathbb{N}$, $\{a_i^n\}_{n=1}^{\infty}$ is a Cauchy sequence of reals. Since $\mathbb{R}$ is complete, $\{a_i^n\}_{n=1}^{\infty}$ converges to some real, call it a_i.
Define $a = \{a_i\}_{i=1}^{\infty}$, then

$$\|a\|_{\infty} = \sup_i\{|a_i|\} = \sup_i \left\{ \left| \lim_{n\to\infty} a_i^n \right| \right\} = \lim_{n\to\infty} \sup_i\{|a_i^n|\} < \infty$$

So $a \in \ell^{\infty}$.
Now we must show that $\{a^n\}_{n=1}^{\infty}$ converges to a. Let $\epsilon > 0$ be given. Choose N such that if $n, m \geq N$ we have

$$\|a^n - a^m\|_{\infty} < \epsilon$$

Then letting $n > N$ we have

$$\|a^n - a\|_{\infty} = \sup_i\{|a_i^n - a_i|\} = \lim_{m\to\infty} \sup_i\{|a_i^n - a_i^m|\} = \lim_{m\to\infty} \|a^n - a^m\|_{\infty} < \lim_{m\to\infty} \epsilon = \epsilon$$

Thus $\{a^n\}_{n=1}^{\infty}$ converges to a.

Chapter 3

3.1.1 Suppose $\mathcal{B}$ is countable. Then enumerate the elements according to the one-to-one correspondence with $\mathbb{N}$.

$$\begin{aligned} a_1 &= \{a_{11}, a_{12}, a_{13}, \dots\} \\ a_2 &= \{a_{21}, a_{22}, a_{23}, \dots\} \\ a_3 &= \{a_{31}, a_{32}, a_{33}, \dots\} \end{aligned}$$

Define a new sequence in $b \in \mathcal{B}$ as

$$b_n = \begin{cases} 1 & \text{if } a_{nn} = 0 \\ 0 & \text{if } a_{nn} = 1 \end{cases}$$

But then b cannot be represented in the enumeration above. Thus $\mathcal{B}$ cannot be countable.

3.1.2 $\mathcal{B}_T$ is countable because each member $\{a_i\} \in \mathcal{B}_T$ can be identified with a terminating series

$$\{a_i\} \to \sum_{i=1}^{\infty} \frac{a_i}{2^i}$$

which means each element of $\mathcal{B}_T$ is identified with a unique rational. Thus $\mathcal{B}_T$ is in one-to-one correspondence with a subset of $\mathbb{Q}$.
Now, each number $r \in (0,1)$ can be written as a non-terminating binary expansion

$$r = \sum_{i=1}^{\infty} \frac{a_i}{2^i} \quad \text{where } a_i \in \{0,1\}$$

Then these $\{a_i\}$ represent the members of $\mathcal{B} \setminus \mathcal{B}_T$ (because they are non-terminating). Thus $\mathcal{B} \setminus \mathcal{B}_T$ is uncountable which means $\mathcal{B}$ is also uncountable.

3.1.4 Let each member $r \in (0,1]$ be written in a binary expansion $r = 0.a_1a_2a_3\dots$. Define $\phi : (0,1] \to (0,1]$ as

$$\phi(0.a_1a_2a_3\dots) = 0.a_1 11 a_2 11 a_3 11\dots$$

ϕ is one-to-one. Suppose $\phi(0.a_1a_2a_3\dots) = \phi(0.b_1b_2b_3\dots)$, then

$$0.a_1 11 a_2 11 a_3 11 \dots = 0.b_1 11 b_2 11 b_3 11 \dots \quad \Longrightarrow \quad a_i = b_i \text{ for all } i \in \mathbb{N}$$

Thus $0.a_1a_2a_3\dots = 0.b_1b_2b_3\dots$.
Let $r \in \phi((0,1])$. Then $r = 0.r_1 11 r_2 11 r_3 11\dots$ which means

$$\lim_{n\to\infty} \frac{s_n(r)}{n} \to \infty$$

So $r \in (0,1] \setminus S$. Thus $\phi((0,1]) \subseteq (0,1] \setminus S$, which means $(0,1] \setminus S$ is uncountable.

3.2.1 First we will prove a necessary set identity.

$$\begin{aligned} A_1 \setminus \bigcup_{n=1}^{\infty} (A_1 \setminus A_n) &= A_1 \cap \left(\bigcup_{n=1}^{\infty} (A_1 \cap A_N^C) \right)^C \\ &= A_1 \cap \bigcap_{n=1}^{\infty} (A_1^C \cup A_n) \\ &= \bigcap_{n=1}^{\infty} (A_1 \cap (A_1^C \cup A_n)) \\ &= \bigcap_{n=1}^{\infty} A_n \end{aligned}$$

Now let $\mathcal{R}$ be a σ-ring. Suppose $A_n \in \mathcal{R}$ for all $n \in \mathbb{N}$. Then $A_1 \setminus A_n \in \mathcal{R}$ for all $n \in \mathbb{N}$. Thus $\bigcup_{n=1}^{\infty}(A_1 \setminus A_n) \in \mathcal{R}$. Therefore

$$\bigcap_{n=1}^{\infty} A_n = A_1 \setminus \bigcup_{n=1}^{\infty}(A_1 \setminus A_n) \in \mathcal{R}$$

3.2.2 Let μ be a non-negative and additive function on a ring $\mathcal{R}$.

(a) Let $A, B \in \mathcal{R}$ with $A \subseteq B$, then

$$B = A \cup (B \setminus A)$$

Thus, using additivity and non-negativity, we can conclude

$$\mu(A) \leq \mu(A) + \mu(B \setminus A) = \mu(A \cup (B \setminus A)) = \mu(B)$$

Therefore μ is monotone.

(b) We can apply induction to part (a) above. Assume

$$\mu\Big(\bigcup_{k=1}^{n} A_k\Big) \leq \sum_{k=1}^{n} \mu(A_k)$$

Then, using part (a) in the first inequality,

$$\begin{aligned}
\mu\Big(\bigcup_{k=1}^{n+1} A_k\Big) &\leq \mu\Big(\bigcup_{k=1}^{n} A_k\Big) + \mu(A_{n+1}) \\
&\leq \sum_{k=1}^{n} \mu(A_k) + \mu(A_{n+1}) = \sum_{k=1}^{n+1} \mu(A_k)
\end{aligned}$$

3.2.3 Assume μ is countably additive on the ring $\mathcal{R}$. First, we prove a helpful identity. If $A \subseteq B$ inside $\mathcal{R}$, then

$$\mu(A) + \mu(B \setminus A) = \mu(A \cup B \setminus A) = \mu(B)$$

Thus

$$\mu(B \setminus A) = \mu(B) - \mu(A)$$

Now, suppose $A_n, A \in \mathcal{R}$ satisfy

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \quad \text{and} \quad A = \bigcup_{n=1}^{\infty} A_n$$

Define $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for all $n \geq 2$. Then $B_n \in \mathcal{R}$, $A = \bigcup_{n=1}^{\infty} B_n$, and $B_i \cap B_j = \emptyset$ if $i \neq j$. Thus, for any $N \in \mathbb{N}$ (and using our helpful identity from above)

$$\begin{aligned}
\mu\Big(\bigcup_{n=1}^{N} B_n\Big) &= \sum_{n=1}^{N} \mu(B_n) \\
&= \mu(A_1) + \mu(A_2 \setminus A_1) + \mu(A_3 \setminus A_2) + \cdots + \mu(A_N \setminus A_{N-1}) \\
&= \mu(A_1) + \mu(A_2) - \mu(A_1) + \mu(A_3) - \mu(A_2) + \cdots + \mu(A_N) - \mu(A_{N-1}) \\
&= \mu(A_N)
\end{aligned}$$

Since μ is countably additive, we can exchange μ and the limit with respect to N. Thus,

$$\begin{aligned}
\mu(A) &= \mu\Big(\lim_{N\to\infty} \bigcup_{n=1}^{N} B_n\Big) \\
&= \lim_{N\to\infty} \mu\Big(\bigcup_{n=1}^{N} B_n\Big) \\
&= \lim_{N\to\infty} \mu(A_N)
\end{aligned}$$

proving the desired conclusion.

3.2.7 (a) We calculate $D(A,B)$ as the area of $S(A,B)$.

$$D(A,B) = m([0,1]\times[0,1]) + m([1,4]\times[1,2]) + m([0,4]\times[2,10]) = 36$$

(b) First, we show that $S(A,B) = S(B,A)$.

$$S(A,B) = (A\setminus B)\cup(B\setminus A) = (B\setminus A)\cup(A\setminus B) = S(B,A)$$

Then the desired conclusion follows quickly.

$$D(A,B) = m^*(S(A,B)) = m^*(S(B,A)) = D(B,A)$$

(c) $D(A,B) = 0$ does *not* imply that $A = B$. For example in $\mathbb{R}$, $D([0,1],[0,1)) = 0$ but $[0,1] \neq [0,1)$.

(d) Let $x \in A\setminus C$. Then $x \in A$ and $x \notin C$. If $x \in B$, then $x \in B\setminus C$. If $x \notin B$, then $x \in A\setminus B$. Thus $x \in (A\setminus B)\cup(B\setminus C)$. Similarly, if $x \in C\setminus A$, $x \in (C\setminus B)\cup(B\setminus A)$. Then we can conclude

$$x \in (A\setminus C)\cup(C\setminus A) \implies x \in (A\setminus B)\cup(B\setminus C)\cup(C\setminus B)\cup(B\setminus A) = (A\setminus B)\cup(B\setminus A)\cup(B\setminus C)\cup(C\setminus B)$$

Which then (applying the definition of symmetric difference) means

$$S(A,C) \subseteq S(A,B)\cup S(B,C)$$

Thus, since m^* is monotone and finitely additive,

$$\begin{aligned} D(A,C) &= m^*(S(A,C)) \\ &\leq m^*(S(A,B)\cup S(B,C)) \\ &= m^*(S(A,B)) + m^*(S(B,C)) \\ &= D(A,B) + D(B,C) \end{aligned}$$

(e) We can conclude this containment using basic set operations.

$$\begin{aligned} (A_1\cup A_2)\setminus(B_1\cup B_2) &= (A_1\cup A_2)\cap(B_1^C\cap B_2^C) \\ &= (A_1\cap B_1^C\cap B_2^C)\cup(A_2\cap B_1^C\cap B_2^C) \\ &\subseteq (A_1\cap B_1^C)\cup(A_2\cap B_2^C) \\ &= (A_1\setminus B_1)\cup(A_2\setminus B_2) \end{aligned}$$

which means

$$S(A_1\cup A_2, B_1\cup B_2) \subseteq S(A_1,B_1)\cup S(A_2,B_2)$$

Thus, since m^* is monotone and finitely additive,

$$\begin{aligned} D(A_1\cup A_2, B_1\cup B_2) &= m^*(S(A_1\cup A_2, B_1\cup B_2)) \\ &\leq m^*(S(A_1,B_1)\cup S(A_2,B_2)) \\ &= m^*(S(A_1,B_1)) + m^*(S(A_2,B_2)) \\ &= D(A_1,B_1) + D(A_2,B_2) \end{aligned}$$

3.2.8 We will not go through displaying that the binary operations defined in this way follow all the requirements of a commutative ring. (Associativity and Distributivity are particularly tedious tasks in set manipulation) However, we would like to point out a few interesting notes.

- The additive identity in this ring is the empty set because $S(A,\emptyset) = A$ for all A.
- The additive inverse of each element is itself because $S(A,A) = \emptyset$ for all A.
- The multiplicative identity in this ring is $\mathbb{R}^n$ because $A\cap\mathbb{R}^n = A$ for all A.
- No element has a multiplicative inverse (expect the identity) because $A\cap B \subseteq A$ for all A, B.

– Moreover, each element (expect the $\mathbb{R}^n$) has several (indeed infinite) zero divisors. Any $B \subseteq A^C$ is a zero divisor of A because $A \cap B \subseteq A \cap A^C = \emptyset$.

These notes show that $2^{(\mathbb{R}^n)}$ under intersection and symmetric difference is a commutative ring with unity (but not an integral domain).

3.2.9 (a) Since $\mathbb{R}^n$ is finite-dimensional, we can use any norm we wish (because all norms are equivalent). We choose $\|\cdot\|_\infty$ because it has r-balls that are equivalent to intervals as defined in the text. Then, if $x \in \mathbb{R}^n$ and $r > 0$, $B_r(x) \in \mathcal{E} \subseteq \mathcal{M}_F$. Let $U \subseteq \mathbb{R}^n$ be open. Define $U_Q = U \cap \mathbb{Q}^n$. For each $q \in U_Q$, define

$$r_q = \sup\{r > 0 : B_r(q) \subseteq U\}$$

Then $B_{r_q}(q) \subseteq U$ for each q and

$$\bigcup_{q \in U_Q} B_{r_q}(q) \subseteq U$$

Now let $x \in U$. Then there exists $r > 0$ such that $B_r(x) \subseteq U$. Choose $q \in \mathbb{Q}^n$ such that

$$\|x - q\|_\infty < \frac{r}{2}$$

Then

$$x \in B_{r/2}(q) \subseteq B_{r_q}(q) \subseteq \bigcup_{q \in U_Q} B_{r_q}(q)$$

showing that

$$U \subseteq \bigcup_{q \in U_Q} B_{r_q}(q)$$

Combining these two containments, we obtain

$$U = \bigcup_{q \in U_Q} B_{r_q}(q)$$

Therefore, since $U_Q \subseteq \mathbb{Q}^n$ is countable, we have written U as a countable union of members of $\mathcal{M}_F$, meaning $U \in \mathcal{M}$.

(b) Let $F \subseteq \mathbb{R}^n$ be closed. Then F^C is open and thus (by part (a)) $F^C \in \mathcal{M}$. Since $\mathcal{M}$ is a σ-ring, $F = (F^C)^C \in \mathcal{M}$.

(c) Let $\{E_i\}_{i=1}^\infty$ be a collection of open or closed sets. Then by parts (a) and (b), each E_i is in $\mathcal{M}$, and since $\mathcal{M}$ is a σ-ring,

$$\bigcup_{i=1}^\infty E_i \in \mathcal{M}$$

Because complements of sets in $\mathcal{M}$ are in $\mathcal{M}$,

$$\bigcap_{i=1}^\infty E_i = \Big(\bigcup_{i=1}^\infty E_i^C\Big)^C \in \mathcal{M}$$

Thus countable unions and intersections of open and closed sets are in $\mathcal{M}$.

3.2.10 The usual definition of the Cantor set is

$$C = \bigcap_{i=1}^\infty \Big(\bigcup_{j=1}^{2^i} I_j^i\Big)$$

where the $I_j^i \subseteq [0,1]$ are the intervals where the middle third is removed from the ith set to produce the $(i+1)$th step. These I_j^i are disjoint in the j coordinate, specifically

$$I_m^i \cap I_n^i = \emptyset \quad \text{if } m \neq n$$

Thus C is a countable intersection of sets that are finite unions of disjoint intervals, which means $C \in \mathcal{M}$.
Since C has no intervals contained in it, C has zero measure.

3.3.1 Let $a \in \mathbb{R}$ be given. The set $(a, \infty) \subseteq \mathbb{R}$ is open. Since $f : \mathbb{R}^n \to \mathbb{R}$ is continuous,

$$f^{-1}((a, \infty)) = \{x : f(x) > a\}\mathbb{R}^n$$

is open and therefore measurable by 3.2.9(a). Since a is arbitrary, f is a measurable function.

3.3.2 Let $A \subseteq \mathbb{R}^n$ be not measurable (exists by the Vitali Theorem). Then define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ -1 & \text{otherwise} \end{cases}$$

Then $A = \{x : f(x) > 0\}$ is not measurable because A is not measurable. Thus f is not measurable. But $|f|$ is a constant function, so it is measurable.

3.3.3 Characteristic functions are integrable if and only if the set it characterizes is measurable. The characteristic function of the rationals is integrable because $\mathbb{Q} \cap [0, 1]$ is measurable.

$$\int_{[0,1]} \chi_{\mathbb{Q}} dm = m([0, 1] \cap \mathbb{Q}) = 0$$

3.3.4 Assume f is measurable. Then $L_a = \{x : f(x) < a\}$ and $G_a = \{x : f(x) > a\}$ are measurable for all a. Let $a \in \mathbb{R}$ be given. We have

$$|f(x)| > a \iff f(x) > a \text{ or } f(x) < -a$$

Then

$$\{x : |f(x)| > a\} = \{x : f(x) > a\} \cup \{x : f(x) < -a\} = G_a \cup L_{-a}$$

which is measurable because the union of two measurable sets is measurable. Since a is arbitrary, $|f|$ is measurable.

3.3.7 By definition, $f \in \mathcal{L}(\mathbb{R}^n)$ if and only if

$$\int_{\mathbb{R}^n} f_+ dm < \infty \text{ and } \int_{\mathbb{R}^n} f_- dm < \infty$$

Since both of these integrals are finite, there sum is finite, so (using Theorem 3.11(a))

$$\int_{\mathbb{R}^n} |f| dm = \int_{\mathbb{R}^n} (f_+ + f_-) dm = \int_{\mathbb{R}^n} f_+ dm + \int_{\mathbb{R}^n} f_- dm < \infty$$

which proves the desired conclusion.

3.4.1 Define $E_k = [-1/2k, 1/2k] \subseteq \mathbb{R}$ and $s_k : [-1, 1] \to \mathbb{R}$ as

$$s_k(x) = k\chi_{E_k}(x)$$

Then

$$\int_{[-1,1]} s_k dm = 1 \quad \text{for all } k \in \mathbb{N}$$

which means $\liminf_{k\to\infty} \int_{[-1,1]} s_k dm = 1$. Also $s = \liminf_{k\to\infty} s_k$ is equal to the zero function almost everywhere, which means

$$\int_{[-1,1]} s dm = 0$$

Thus

$$\int_{[-1,1]} s dm = 0 < 1 = \liminf_{k\to\infty} \int_{[-1,1]} s_k dm$$

which is an example of strict inequality holding in Fatou's Lemma.

3.4.2 Use the function sequence defined above. It is shown in 3.4.1 that the limit of integrals of this sequence is not equal to the integral of the limit. This sequence does not have an integrable function g that bounds it.

3.6.3 Let $y \geq 0$ and $1 < p < \infty$ be fixed. Define $f : [0, \infty] \to \mathbb{R}$ as

$$f(x) = xy - \frac{x^p}{p}$$

Then $f'(x) = y - x^{p-1}$, which is zero when $x = y^{1/(p-1)}$. Thus f attains a maximum at $y^{1/(p-1)}$ meaning $f(x) \leq f(y^{1/(p-1)})$ for all $x \geq 0$. Thus

$$xy - \frac{x^p}{p} \leq y^{1+\frac{1}{p-1}} - \frac{1}{p} y^{\frac{p}{p-1}}$$

Rearranging (noting that the Hölder conjugate of p is given by $1/q = 1 - 1/p$) gives

$$xy \leq \frac{x^p}{p} + \left(1 - \frac{1}{p}\right) y^{\frac{1}{1-1/p}} = \frac{x^p}{p} + \frac{y^q}{q}$$

3.6.4 Let $f \in L^p(\mu)$ and $1 < p < \infty$. Let q be the Hölder conjugate of p. Then

$$(p-1)q = (p-1)\frac{p}{p-1} = p$$

So following the definition of $\|\cdot\|_p$ we have

$$\begin{aligned}
\left\| |f|^{p-1} \right\|_q &= \left(\int_X |f|^{(p-1)q} d\mu \right)^{\frac{1}{q}} \\
&= \Big(\int_X |f|^{(p-1)q} d\mu \Big)^{\frac{p-1}{(p-1)q}} = \big(\|f\|_{(p-1)q} \big)^{p-1} \\
&= \Big(\int_X |f|^p d\mu \Big)^{\frac{p-1}{p}} = \big(\|f\|_p \big)^{p-1}
\end{aligned}$$

Since $f, g \in L^p(\mu)$ means that $f + g \in L^p(\mu)$ we can conclude

$$\left\| |f+g|^{p-1} \right\|_q = \big(\|f+g\|_{(p-1)q} \big)^{p-1} = \big(\|f+g\|_p \big)^{p-1}$$

3.6.5 Let $f, g \in L^\infty$ and suppose $M > 0$ satisfies $|f(x)| \leq M$ almost everywhere and $N > 0$ satisfies $|g(x)| \leq N$ almost everywhere. Then if λ is a scalar, $|\lambda f(x)| = |\lambda||f(x)| \leq |\lambda| M$ almost everywhere, so $\lambda f \in L^\infty$. Also, $|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq M + N$ almost everywhere, so $f + g \in L^\infty$. Thus L^∞ is a linear space.
Now we will show that $\|\cdot\|_\infty$ satisfies the requirements to be a norm.

(i) $\|f\|_\infty \geq 0$ for all f because $|f(x)| \geq 0$ for all f and all x.

(ii) $\|f\|_\infty = 0$ if and only if $0 \leq |f(x)| \leq 0$ almost everywhere which means f is in the same equivalence class as the zero function.

(iii) $\|\lambda f\|_\infty = \inf\{M : |\lambda f(x)| \leq M\} = |\lambda| \inf\{M : |f(x)| \leq M\} = |\lambda| \, \|f\|_\infty$

(iv) $\|f + g\|_\infty = \inf\{M : |f(x) + g(x)| \leq M\} \leq \inf\{M : |f(x)| + |g(x)| \leq M\} \leq \inf\{M : |f(x)| \leq M\} + \inf\{M : |g(x)| \leq M\} = \|f\|_\infty + \|g\|_\infty$

3.6.9 We will use the fact that if a function is both Lebesgue and Riemann integrable, then the value of the two integrals are the same.

(a) These functions are simple functions to integrate.

$$\|f\|_p^p = \int_{-1}^{1} |1+x|^p dx = \int_0^2 u^p du = \frac{2^{p+1}}{p+1}$$
$$\|g\|_p^p = \int_{-1}^{1} |1-x|^p dx = \int_{-1}^{1} |x-1|^p dx = \int_0^2 u^p du = \frac{2^{p+1}}{p+1}$$
$$\|f+g\|_p^p = \int_{-1}^{1} |1+x+1-x|^p dx = \int_{-1}^{1} 2^p dx = 2^{p+1}$$
$$\|f-g\|_p^p = \int_{-1}^{1} |1+x-1+x|^p dx = \int_{-1}^{1} |2x|^p dx = \frac{2^{p+1}}{p+1}$$

(b) We can calculate this conclusion simply from the parallelogram equality.

$$\|f+g\|_p^2 + \|f-g\|_p^2 = 2\|f\|_p^2 + 2\|g\|_p^2$$
$$\left(2^{p+1}\right)^{\frac{2}{p}} + \left(\frac{2^{p+1}}{p+1}\right)^{\frac{2}{p}} = 2\left(\frac{2^{p+1}}{p+1}\right)^{\frac{2}{p}} + 2\left(\frac{2^{p+1}}{p+1}\right)^{\frac{2}{p}}$$
$$3\left(\frac{2^{p+1}}{p+1}\right)^{\frac{2}{p}} = \left(2^{p+1}\right)^{\frac{2}{p}}$$
$$3\left(2^{p+1}\right)^{\frac{2}{p}} = \left(2^{p+1}(p+1)\right)^{\frac{2}{p}}$$
$$3 = (p+1)^{\frac{2}{p}}$$

This equality is satisfied with $p=2$.

$$(2+1)^{\frac{2}{2}} = 3^1 = 3$$

(c) Note that for $p \geq 1$,

$$\frac{\ln(p+1)}{p} > \frac{\ln(p+1)}{(p+1)\ln(p+1)} = \frac{1}{p+1}$$

Let $f:[1,\infty) \to \mathbb{R}$ be defined as $f(p) = (p+1)^{2/p} - 3$. Then

$$f'(p) = \frac{2(p+1)^{2/p}}{p}\left(\frac{1}{p+1} - \frac{\ln(p+1)}{p}\right)$$

and for $p \geq 1$, we have $f'(p) < 0$. Thus f is a strictly decreasing function in p and therefore can only have one zero: $p=2$.

(d) Choose $r > 0$ and c such that $B_r(c) \subseteq I$. Define $f, g : I \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 + \frac{1}{r}(x-c) & x \in B_r(c) \\ 0 & \text{otherwise} \end{cases}$$
$$g(x) = \begin{cases} 1 - \frac{1}{r}(x-c) & x \in B_r(c) \\ 0 & \text{otherwise} \end{cases}$$

Then f and g yield the same results as those previously defined for $[-1,1]$.

$$\|f\|_p^p = \int_{c-r}^{c+r} |1 + \frac{1}{r}(x-c)|^p dx = r\int_{-1}^{1} |1+x|^p dx = r\frac{2^{p+1}}{p+1}$$
$$\|g\|_p^p = \int_{c-r}^{c+r} |1 - \frac{1}{r}(x-c)|^p dx = r\int_{-1}^{1} |1-x|^p dx = r\frac{2^{p+1}}{p+1}$$
$$\|f+g\|_p^p = \int_{c-r}^{c+r} |1 + \frac{1}{r}(x-c) + 1 - \frac{1}{r}(x-c)|^p dx = \int_{c-r}^{c+r} 2^p dx = r2^{p+1}$$
$$\|f-g\|_p^p = \int_{c-r}^{c+r} |1 + \frac{1}{r}(x-c) - 1 + \frac{1}{r}(x-c)|^p dx = r\int_{-1}^{1} |2x|^p dx = r\frac{2^{p+1}}{p+1}$$

which, combined with the parallelogram equality, produces the same restriction on p:

$$(p+1)^{\frac{2}{p}} = 3$$

Then parts (b) and (c) follow.

Chapter 4

4.1.2 (a) We simply need to calculate the coefficients as defined at the beginning of section 4.1.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{0} dx = 1 \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{0} \cos(kx)dx = \frac{1}{\pi}\left[\frac{1}{k}\sin(kx)\right]_{-\pi}^{0} = 0 \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{0} \sin(kx)dx = \frac{1}{\pi}\left[-\frac{1}{k}\cos(kx)\right]_{-\pi}^{0} = \frac{1}{\pi}\left(\frac{(-1)^k-1}{k}\right) \end{aligned}$$

Putting these calculations together with the expansion given, we have

$$f(x) = \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k-1}{k}\sin(kx)$$

(b) Since f defined as given is a square integrable function ($f \in L^2([-\pi,\pi])$) and the classical trigonometric system is complete in $L^2([-\pi,\pi])$, we can conclude by Theorem 4.5 and 4.1 that the series above converges in mean to f.

(c) $\sin(n0) = 0$ for all n, so the series gives a value of $1/2$ at $x = 0$, but $f(0) = 0$. So this series does not converge pointwise. Since pointwise convergence implies uniform convergence, this series does not converge uniformly either.

(d) f is a vertical shift (of $1/2$) from an odd function. All the cosine terms are even functions so they cannot contribute to making the series "get closer" to f.

4.1.4 (a) Assume $\{f_n\}_{n=1}^{\infty}$ converges to f uniformly. Let $\epsilon > 0$ and $x \in [a,b]$ be given. Choose $N \in \mathbb{N}$ such that if $n \geq N$ and $y \in [a,b]$, then

$$|f_n(y) - f(y)| < \epsilon$$

Thus we have (for a particular y)

$$|f_n(x) - f(x)| < \epsilon$$

Therefore $\{f_n\}_{n=1}^{\infty}$ converges to f pointwise.

(b) Define $\{f_n\}_{n=1}^{\infty}$ as

$$f_n(x) = \frac{1}{nx}$$

This sequence converges to $f(x) = 0$ on $(0,1]$ pointwise (but not uniformly).

(c) Assume $\{f_n\}_{n=1}^{\infty}$ converges to f uniformly. Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that if $n \geq N$ and $x \in [a,b]$ we have

$$|f_n(x) - f(x)| < \sqrt{\frac{\epsilon}{b-a}}$$

Then

$$\int_{[a,b]} [f_n(x) - f(x)]^2 dm < \int_{[a,b]} \left(\sqrt{\frac{\epsilon}{b-a}}\right)^2 dm = \frac{\epsilon}{b-a}(b-a) = \epsilon$$

Therefore $\{f_n\}_{n=1}^{\infty}$ converges to f in mean.

(d) Define $\{f_n\}_{n=1}^{\infty}$ as

$$f_n(x) = \begin{cases} n & x \in [-\frac{1}{n}, 0) \cup (0, \frac{1}{n}] \\ 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

This sequence converges pointwise to

$$f_n(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

However

$$\int_{[-1,1]} (f_n - f)^2 dm = \int_{[-1,1]} f_n^2 dm = 1$$

for all n, which does not go to zero as n goes to infinity.

(e) Take the previous series from 4.1.2 as the counterexample. It converges in mean but not pointwise.

4.2.1 Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space.

(a) Suppose $f, g \in V$ and that f and g are orthogonal ($\langle f, g\rangle = 0$). Then we have

$$\begin{aligned} \|f+g\|^2 &= \langle f+g, f+g\rangle \\ &= \langle f, f\rangle + \langle f, g\rangle + \langle g, f\rangle + \langle g, g\rangle \\ &= \langle f, f\rangle + \langle g, g\rangle \\ &= \|f\|^2 + \|g\|^2 \end{aligned}$$

(b) Assume that $\{f_k\}_{k=1}^{\infty}$ is an orthonormal sequence in $(V, \langle\cdot,\cdot\rangle)$. Let $f \in V$ be given.
($\Rightarrow$) Suppose $\{f_k\}$ is complete. Then we write $f = \sum_{k=1}^{\infty}\langle f, f_k\rangle f_k$. Thus

$$\begin{aligned} \|f\|^2 &= \left\|\sum_{k=1}^{\infty}\langle f, f_k\rangle f_k\right\|^2 \\ &= \sum_{k=1}^{\infty} \|\langle f, f_k\rangle f_k\|^2 \\ &= \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \|f_k\|^2 \\ &= \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \end{aligned}$$

($\Leftarrow$) Now suppose $\|f\|^2 = \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2$. Define $s_n = \sum_{k=1}^{n}\langle f, f_k\rangle f_k$. Then, using the argument given in the proof of Theorem 4.2, we have $\|f - s_n\|^2 + \|s_n\|^2 = \|f\|^2$. Therefore

$$\begin{aligned} \lim_{n\to\infty} \|f - s_n\|^2 &= \lim_{n\to\infty} \left(\|f\|^2 - \|s_n\|^2\right) \\ &= \|f\|^2 - \lim_{n\to\infty} \sum_{k=1}^{n} |\langle f, f_k\rangle| \\ &= 0 \end{aligned}$$

Thus we may write $f = \sum_{k=1}^{\infty}\langle f, f_k\rangle f_k$, showing that $\{f_k\}$ is complete.

4.2.3 $\{f_1\}$ is linearly independent ($c_1 f_1 = 0 \iff c_1 = 0$ when $f_1 \neq 0$).
Assume $\{f_i\}_{i=1}^{n-1}$ is linearly independent if it is orthonormal. Then suppose $\{f_i\}_{i=1}^{n}$ is orthonormal. Let $\sum_{i=1}^{n} c_i f_i = 0$. Then

$$\left\langle \sum_{i=1}^{n} c_i f_i, f_n \right\rangle = c_n$$

Thus c_n must equal zero. Therefore, $\sum_{i=1}^{n-1} c_i f_i = 0$ which means all of the c_i are zero (by assumption). Thus $\{f_i\}_{i=1}^{n}$ is linearly independent (when orthonormal) for all n by the Principle of Mathematical Induction.

4.2.4 This conclusion is shown a simple calculation. We point out that $\|g\|^2 = \langle g, g\rangle$ and the definition of two vectors h, k being orthogonal is $\langle h, k\rangle = 0$.

$$\begin{aligned}
\left\langle f - \frac{\langle f,g\rangle}{\langle g,g\rangle} g, \frac{\langle f,g\rangle}{\langle g,g\rangle} g \right\rangle &= \left\langle f, \frac{\langle f,g\rangle}{\langle g,g\rangle} g \right\rangle - \left\langle \frac{\langle f,g\rangle}{\langle g,g\rangle} g, \frac{\langle f,g\rangle}{\langle g,g\rangle} g \right\rangle \\
&= \frac{\overline{\langle f,g\rangle}}{\overline{\langle g,g\rangle}} \langle f,g\rangle - \frac{\langle f,g\rangle \overline{\langle f,g\rangle}}{\langle g,g\rangle \overline{\langle g,g\rangle}} \langle g,g\rangle \\
&= \frac{\langle f,g\rangle \overline{\langle f,g\rangle}}{\overline{\langle g,g\rangle}} - \frac{\langle f,g\rangle \overline{\langle f,g\rangle}}{\overline{\langle g,g\rangle}} \\
&= 0
\end{aligned}$$

4.2.5 (a) Since the function $f(x) = x$ is an odd function, none of the cosine terms will contribute. More specifically, $a_k = 0$ for all $k \in \mathbb{N}$. Now we calculate the b_k.

$$\begin{aligned}
b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(kx) dx \\
&= \frac{1}{\pi} \left(-\frac{x}{k} \cos(kx) \Big|_{-\pi}^{\pi} + \frac{1}{k} \int_{-\pi}^{\pi} \cos(kx) dx \right) \\
&= \frac{1}{\pi} \left(-\frac{\pi}{k} \cos(k\pi) - \frac{\pi}{k} \cos(-k\pi) + 0 \right) \\
&= 2 \frac{(-1)^{k+1}}{k}
\end{aligned}$$

Therefore, on $[-\pi, \pi]$, we have

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

(b) Parsevals's identity states that $\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2$. In part (a) we have shown

$$|\langle f, f_n\rangle|^2 = \left| \frac{2(-1)^{n+1}}{n} \right|^2 = \frac{4}{n^2}$$

And we can calculate $\|f\|^2$

$$\|f\|_2^2 = \int_{-\pi}^{\pi} x^2 dx = \frac{x^3}{3} \Big|_{-\pi}^{\pi} = \frac{2\pi^3}{3}$$

Thus

$$\frac{2\pi^3}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

which is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^3}{6}$$

Chapter 5

5.1.1 Let H be a Hilbert space and $x_0 \in H$ be fixed. Define $T : H \to \mathbb{C}$ as

$$Tx = \langle x, x_0 \rangle$$

Suppose $\alpha, \beta \in \mathbb{C}$ and $x, y \in H$. Then

$$\begin{aligned} T(\alpha x + \beta y) &= \langle \alpha x + \beta y, x_0 \rangle \\ &= \langle \alpha x, x_0 \rangle + \langle \beta y, x_0 \rangle \\ &= \alpha \langle x, x_0 \rangle + \beta \langle y, x_0 \rangle \\ &= \alpha Tx + \beta Ty \end{aligned}$$

which shows that T is a linear operator.

5.1.2 (a) Define $k : [0, b] \times [0, b] \to \mathbb{C}$ as

$$k(s,t) = \begin{cases} 0 & t > s \\ \frac{1}{s} & t \leq s \end{cases}$$

Then $T : C([0, b]) \to C([0, b])$ is

$$Tf(s) = \int_0^b k(s,t) f(t) dt = \int_0^s \frac{1}{s} f(t) dt = \frac{1}{s} \int_0^s f(t) dt$$

(b) Define T as in part (a). Suppose $\alpha, \beta \in \mathbb{C}$ and $f, g \in C([0, b])$. Then

$$\begin{aligned} T(\alpha f + \beta g) &= \frac{1}{s} \int_0^s (\alpha f(t) + \beta g(t)) dt \\ &= \alpha \frac{1}{s} \int_0^s f(t) dt + \beta \frac{1}{s} \int_0^s g(t) dt \\ &= \alpha Tf + \beta Tg \end{aligned}$$

5.1.3 Define the left shift $T : \ell^2 \to \ell^2$ and right shift $S : \ell^2 \to \ell^2$ as given in the text. Then

$$TS(a_1, a_2, a_3, \dots) = T(0, a_1, a_2, a_3, \dots) = (a_1, a_2, a_3, \dots)$$

which shows that $TS = I$; however,

$$ST(a_1, a_2, a_3, \dots) = S(a_2, a_3, a_4, \dots) = (0, a_2, a_3, \dots)$$

which shows that $ST \neq I$, and since T is not onto, it cannot have a left inverse.

5.2.1 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces. $\mathcal{B}(X, Y)$ is a linear space because if $\alpha, \beta \in \mathbb{C}$, $T, S \in \mathcal{B}(X, Y)$, and $x \in X$, then T is bounded by some M_T and S by some M_S and

$$(\alpha T + \beta S)x = \alpha Tx + \beta Sx \leq \alpha M_T x + \beta M_S x$$

which means $\alpha T + \beta S \in \mathcal{B}(X, Y)$.
Define $\|\cdot\|_{\mathcal{B}} : \mathcal{B}(X, Y) \to \mathbb{R}$ as

$$\|T\|_{\mathcal{B}} = \inf_{x \in X} \{M : \|Tx\|_Y \leq M \|x\|_X\}$$

Then let $S, T \in \mathcal{B}(X, Y)$ and $\lambda \in \mathbb{C}$

(i) $\|T\|_{\mathcal{B}} \geq 0$ because $\|Tx\|_Y \geq 0$ for all $x \in X$.

(ii) $\|T\|_{\mathcal{B}} = 0$ if $\|Tx\|_Y = 0$ for all x where $\|x\|_X \neq 0$. In other words, $Tx = 0$ for all x with non zero norm, so T is the zero operator almost everywhere (everywhere except where $\|x\|_X = 0$).

(iii) $\|\lambda T\|_{\mathcal{B}} = \inf_{x\in X}\{M : \|\lambda Tx\|_Y \leq M \|x\|_X\}$ and we can pull out λ because $\|\cdot\|_Y$ is a norm. Thus $\|\lambda T\|_{\mathcal{B}} = |\lambda| \inf_{x\in X}\{M : \|Tx\|_Y \leq M \|x\|_X\} = |\lambda| \|T\|_{\mathcal{B}}$.

(iv) This follows from the fact that $\|\cdot\|_Y$ is a norm and thus follows the triangle inequality as well.

Thus $\mathcal{B}(X, Y)$ is a normed linear space.

5.2.2 First, define $I : (C([0, 1]), \|\cdot\|_\infty) \to (C([0, 1]), \|\cdot\|_1)$ as $If = f$. Then

$$\|f\|_1 = \int_{[0,1]} f dm \leq \int_{[0,1]} \|f\|_\infty dm = \|f\|_\infty$$

Thus $\|If\|_1 \leq 1 \cdot \|f\|_\infty$ which means I is bounded.
Contrarily, define $I : (C([0, 1]), \|\cdot\|_1) \to (C([0, 1]), \|\cdot\|_\infty)$ the same way. Then let $M \geq 1$ be given. Define $f \in C([0, 1])$ as

$$f(x) = \begin{cases} 2M - 2M^2 x & x \in [0, 1/M] \\ 0 & \text{otherwise} \end{cases}$$

Then $\|f\|_1 = 1$ and $\|f\|_\infty = 2M$ which means $\|If\|_\infty > M \|f\|_1$. Thus I is unbounded.

5.2.4 Let $a = \{a_n\}_{n=1}^\infty \in \ell^\infty$. Define $T : \ell^1 \to \mathbb{C}$ as

$$Tx = \sum_{n=1}^\infty x_n a_n$$

Then

$$\begin{aligned} |Tx| &= \left| \sum_{n=1}^\infty x_n a_n \right| \\ &\leq \sum_{n=1}^\infty |x_n a_n| \\ &\leq \sum_{n=1}^\infty |x_n| \|a\|_\infty \\ &= \|a\|_\infty \|x\|_1 \end{aligned}$$

Therefore T is bounded. Define $a = (1, 0, 0, 0, \dots)$ and $x = (1, 0, 0, 0, \dots)$. Then

$$|Tx| = \left| \sum_{n=1}^\infty x_n a_n \right| = 1 = \|a\|_\infty \|x\|_1$$

Therefore $\|T\| = \|a\|_\infty$.

5.2.7 (a) Since a differentiable function is necessarily continuous, $C^1([0, 1]) \subseteq C([0, 1])$. It is a subspace because the scalar multiplication of and the sum of two differentiable functions is a differentiable function. Define $f_n(x) = x^n$ for all n. These f_n are in $C^1([0, 1])$ but there pointwise limit is not in $C^1([0, 1])$ (or event in $C([0, 1])$).

(b) The differential operator is linear because if $\alpha, \beta \in \mathbb{C}$ and $f, g \in C^1([0, 1])$ we have

$$\frac{d}{dx} [\alpha f + \beta g] = \alpha \frac{df}{dx} + \beta \frac{dg}{dx}$$

However, define $g_n(x) = \sin(2\pi n x)$. Then $g_n \in C^([0, 1])$ for all n but

$$\frac{dg_n}{dx} = 2\pi n \cos(2\pi n x) \implies \left\| \frac{dg_n}{dx} \right\|_\infty = 2\pi n$$

which is an unbounded sequence, so the differential operator is unbounded.

5.3.1 Let X be a normed linear space and $S, T \in \mathcal{B}(X)$.

(a) Since $S \in \mathcal{B}(X)$, its norm is finite $\|S\| < \infty$ and

$$\|Sx\|_X \leq \|S\| \, \|x\|_X \qquad \text{for all } x \in X$$

Similarly, $\|Tx\|_X \leq \|T\| \, \|x\|_X$ for all $x \in X$. Thus (since $Tx \in X$), we can conclude

$$\|STx\|_X \leq \|S\| \, \|Tx\|_X \leq \|S\| \, \|T\| \, \|x\|_X \qquad \text{for all } x \in X$$

Thus, $\|ST\| \leq \|S\| \, \|T\|$.

(b) ST is bounded by part (a), so $ST \in \mathcal{X}$.

5.3.2 By Theorem 2.4, in metric spaces, compactness and sequential compactness are equivalent. Therefore, let $\{y_n\}_{n=1}^{\infty}$ be an arbitrary sequence in $\overline{K(B)}$. Then, for each n, there exists $x_n \in B$ such that $Kx_n = y_n$. Since the sequence $\{x_n\}_{n=1}^{\infty}$ is a subset of B, it is bounded. Since K is a compact operator, the image of this bounded sequence has a convergent subsequence. Thus $\{y_n\}_{n=1}^{\infty}$ has a convergent subsequence. Therefore $\overline{K(B)}$ is sequentially compact (or equivalently, just compact).

5.3.3 Since $\mathcal{B}(X)$ is a unital Banach algebra, we are able to make the following string of equalities. We multiply both sides by U and then use associativity.

$$\begin{aligned} TV &= I \\ U(TV) &= UI \\ (UT)V &= U \\ IV &= U \\ V &= U \end{aligned}$$

5.3.4 In a unital normed algebra, the norm must be submultiplicative:

$$\|ab\| \leq \|a\| \, \|b\| \qquad \text{for all } a, b$$

Let a be any nonzero element in the algebra and let e be the unit. Then

$$\|ae\| \leq \|a\| \, \|e\|$$

And since $ae = a$, $\|ae\| = \|a\|$, so

$$\|a\| \leq \|a\| \, \|e\|$$

Thus

$$\|e\| \geq 1$$

5.3.5 This is a simple case of induction. The base case is $k = 1$:

$$\left\|a^1\right\| = \|a\| = \|a\|^1$$

Now assume that $\left\|a^{k-1}\right\| \leq \|a\|^{k-1}$, then

$$\left\|a^k\right\| = \left\|a^{k-1}a\right\| \leq \left\|a^{k-1}\right\| \|a\| \leq \|a\|^{k-1} \|a\| = \|a\|^k$$

Thus, by the Principle of Mathematical Induction (and since a was an arbitrary element in the algebra)

$$\left\|a^k\right\| \leq \|a\|^k \qquad \text{for all } k \in \mathbb{N} \text{ and } a$$

5.3.6 $(C([0,1]), \|\cdot\|_\infty)$ is a Banach space by Theorem 2.7. Define pointwise multiplication to be the multiplication, then $C([0,1])$ is an algebra. Let $f, g \in C([0,1])$. Then

$$\|fg\|_\infty = \sup_{x\in[0,1]} \{|f(x)g(x)|\} \leq \sup_{x\in[0,1]} \{|f(x)|\} \sup_{x\in[0,1]} \{|g(x)|\} = \|f\|_\infty \|g\|_\infty$$

which means $\|\cdot\|_\infty$ is submultiplicative. Finally, define $e : [0,1] \to \mathbb{R}$ as $e(x) = 1$. Then

$$(fe)(x) = f(x)e(x) = f(x)1 = f(x) = 1f(x) = e(x)f(x) = (ef)(x)$$

which shows that e is the unit.
Thus $(C([0,1]), \|\cdot\|_\infty)$ is a unital Banach algebra.

5.3.7 Both S and T have inverses, so we can write

$$T^{-1}(T-S)S^{-1} = T^{-1}TS^{-1} - T^{-1}SS^{-1} = S^{-1} - T^{-1}$$

Therefore $S^{-1} = T^{-1}(T-S)S^{-1} + T^{-1}$. Using the triangle inequality and submultiplicativity, we can conclude

$$\begin{aligned}\left\|S^{-1}\right\| &\leq \left\|T^{-1}(T-S)S^{-1}\right\| + \left\|T^{-1}\right\| \\ &\leq \left\|T^{-1}\right\| \|T-S\| \left\|S^{-1}\right\| + \left\|T^{-1}\right\|\end{aligned}$$

Then we can re-arrange the above inequality to obtain

$$\left\|S^{-1}\right\| \left(1 - \left\|T^{-1}\right\| \|T-S\|\right) \leq \left\|T^{-1}\right\|$$

Which proves our desired conclusion.

5.3.11 Suppose $T : X \to Y$ is not bounded. Thus, there exists $\{x_n\}_{n=1}^\infty \subseteq X$ that is bounded, but $\{Tx_n\}_{n=1}^\infty$ is unbounded. Since $\{Tx_n\}_{n=1}^\infty$ is unbounded, it cannot have a convergent subsequence. Therefore, T is not compact.

5.3.13 Let $\phi : [0,1] \to \mathbb{R}$ be continuous. Define $M_\phi : L^2([0,1]) \to L^2([0,1])$ as $(M_\phi f)(x) = \phi(x)f(x)$.

(a) Since ϕ is continuous on a compact interval, it attains a finite maximum $\|\phi\|_\infty$ at some $x_0 \in [0,1]$. Thus, for any $f \in L^2([0,1])$, we have

$$\|M_\phi f\|_2^2 = \int_{[0,1]} |\phi f|^2 dm \leq \|\phi\|_\infty^2 \int_{[0,1]} |f|^2 dm = \|\phi\|_\infty^2 \|f\|_2^2$$

which shows that M_ϕ is bounded and therefore $M_\phi \in \mathcal{B}\left(L^2([0,1])\right)$. Let $0 < \epsilon < \|\phi\|_\infty$ be given. Define

$$S_\epsilon = \{x : |\phi(x)| \geq \|\phi\|_\infty - \epsilon\}$$

Then

$$\|M_\phi \chi_{S_\epsilon}\|_2^2 = \int_{[0,1]} |\phi \chi_{S_\epsilon}|^2 dm = \int_{S_\epsilon} |\phi|^2 dm \geq (\|\phi\|_\infty - \epsilon)^2 \int_{S_\epsilon} dm = (\|\phi\|_\infty - \epsilon)^2 \|\chi_{S_\epsilon}\|_2^2$$

Thus $\|M_\phi \chi_{S_\epsilon}\|_2 \geq (\|\phi\|_\infty - \epsilon) \|\chi_{S_\epsilon}\|_2$ for arbitrarily small ϵ. We can then conclude that $\|M_\phi\| = \|\phi\|_\infty$.

(b) Since $\lambda \notin \phi([0,1])$, $\lambda - \phi(x) \neq 0$ for all $x \in [0,1]$. Thus, the function $\mu : [0,1] \to \mathbb{R}$ defined by $\mu(x) = 1/(\lambda - \phi(x))$ is continuous on $[0,1]$. Then, letting $f \in L^2([0,1])$ be given, we have

$$\|M_\mu f\|_2^2 = \int_{[0,1]} |\mu f|^2 \, dm \leq \|\mu\|_\infty^2 \|f\|_2^2$$

Thus $M_\mu = M_{(\lambda-\phi)^{-1}}$ is bounded. Also,

$$(M_\mu(\lambda I - M_\phi)f)(x) = \mu(x)(\lambda f(x) - \phi(x)f(x)) = \frac{1}{\lambda - \phi(x)}(\lambda - \phi(x))f(x) = f(x)$$

Thus $M_\mu(\lambda I - M_\phi) = I$. Similarly, we find $(\lambda I - M_\phi)M_\mu = I$. Therefore, M_μ is the inverse of $\lambda I - M_\phi$.

(c) Any constant function has only eigenvalues (more precisely, eigenvalue) in its spectrum. Define $\phi : [0,1] \to \mathbb{R}$ as $\phi(x) = a$. Then $M_\phi f = af$ for all $f \in L^2([0,1])$ (so a is an eigenvalue) and $\sigma(M_\phi) = \phi([0,1]) = \{a\}$.
Any non-constant function has no eigenvalues in its spectrum. Let $\psi : [0,1] \to \mathbb{R}$ be continuous and non-constant. Then for $(M_\psi f)(x) = \psi(x)f(x) = af(x)$ for some a and for all $x \in [0,1]$, f would have to be the zero function or $\psi(x) = a$ for all x. This contradicts the assumption that ψ wasn't constant.

5.4.1 *Claim:* In $\mathbb{C}^n$, $A = (a_{ij})$ satisfies $a_{ij} = \overline{a_{ji}}$ for all $1 \le i,j \le n$ if and only if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in \mathbb{C}^n$.
Proof: ($\Rightarrow$) Assume that $A = (a_{ij})$ satisfies $a_{ij} = \overline{a_{ji}}$ for all $1 \le i,j \le n$. Let x, y be given. Then

$$\begin{aligned}
\langle Ax, y\rangle &= \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_i\overline{y_j} \\
&= \sum_{i,j=1}^{n} x_i\overline{\overline{a_{ij}}y_j} \\
&= \sum_{i,j=1}^{n} x_i\overline{a_{ji}y_j} \\
&= \sum_{i=1}^{n} x_i\overline{\sum_{j=1}^{n} a_{ji}y_j} \\
&= \langle x, Ay\rangle
\end{aligned}$$

($\Leftarrow$) Now assume that $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all x, y. Let x, y be given, then (using some of the above work)

$$\begin{aligned}
\langle Ax, y\rangle &= \sum_{i,j=1}^{n} x_i\overline{\overline{a_{ij}}y_j} \\
&= \sum_{i=1}^{n} x_i\overline{\sum_{j=1}^{n} \overline{a_{ij}}y_j} \\
\langle x, Ay\rangle &= \sum_{i=1}^{n} x_i\overline{\sum_{j=1}^{n} a_{ji}y_j}
\end{aligned}$$

Thus, since x and y are arbitrary, we must have $\overline{a_{ij}} = a_{ji}$.

5.4.4 Let $y_1, y_2 \in K$ both be the "closest" element in K to x. That is

$$\|x - y_1\| = \inf\{\|x - z\| : z \in K\} = \|x - y_2\|$$

Thus, since we are working in a Hilbert space,

$$\|x - y_1\| = \|x - y_2\| \qquad \Longrightarrow \qquad \langle x - y_1, x - y_1\rangle = \langle x - y_2, x - y_2\rangle$$

We can subtract the right hand side and obtain

$$\langle y_2 - y_1, y_2 - y_1\rangle = \|y_2 - y_1\|^2 = 0$$

which means $y_1 = y_2$.

5.4.6 Let $T \in \mathcal{B}(H)$ be Hermitian. Suppose λ is an eigenvalue of T. Then $Tx = \lambda x$ for some $x \in H$ ($x \neq 0$). Then

$$\begin{aligned}
\langle Tx, x\rangle &= \langle \lambda x, x\rangle = \lambda\langle x, x\rangle \\
\langle x, Tx\rangle &= \langle x, \lambda x\rangle = \overline{\lambda}\langle x, x\rangle
\end{aligned}$$

Since $\langle Tx, x\rangle = \langle x, Tx\rangle$ and $\langle x, x\rangle \neq 0$, we can conclude that $\lambda = \overline{\lambda}$, which means λ is real.

5.4.7 (a) Let $\alpha \in [0,1]$ be given and let $f \in M_\alpha$. Then if $s \in [0,\alpha]$ we have

$$(Tf)(s) = \int_{[0,s]} f dm = \int_{[0,s]} 0 dm = 0$$

Thus $Tf \in M_\alpha$. Therefore M_α is an invariant subspace for T.

(b) Define $N_\alpha = \{g \in L^2([0,1]) : g(t) = 0 \text{ for all } t \in [\alpha, 1]\}$. Let $f \in M_\alpha$ and $g \in N_\alpha$ be given. Then

$$\langle f, g\rangle = \int_{[0,1]} f\overline{g} dm = 0$$

because f is uniformly zero on $[0,\alpha]$ and g is uniformly zero on $[\alpha, 1]$. Thus N_α is the orthogonal complement of M_α.

5.5.1 Let K be a closed subspace of a Hilbert space H. Define $P : H \to H$ as $Px = y$ where $y \in K$ is the unique element in K such that for some $z \in K^\perp$ we have $x = y + z$.
Bounded: Let $x \in H$ be given. Then $x = y + z$ where $y \in K$ and $z \in K^\perp$, so $\langle y, z\rangle = 0$, which means (by Exercise 4.2.1) $\|y\|^2 + \|z\|^2 = \|y + z\|^2$. Thus

$$\|Px\|^2 = \|y\|^2 \leq \|y\|^2 + \|z\|^2 = \|y + z\|^2 = \|x\|^2$$

Therefore, P is bounded (and actually, $\|P\|_{\mathcal{B}(H)} = 1$).
Linear: Let $x, w \in H$ and $a, b \in \mathbb{C}$ be given. Decompose x and w as $x = y_x + z_x$ and $w = y_w + z_w$ for $y_x, y_w \in K$ and $z_x, z_w \in K^\perp$. Since K and $K^\perp$ are closed subspaces of H, we have $ay_x + by_w \in K$ and $az_x + bz_w \in K^\perp$. Thus we decompose $ax + bw$ as

$$ax + bw = a(y_x + z_x) + b(y_w + z_w) = (ay_x + by_w) + (az_x + bz_w)$$

and since decompositions are unique, $ay_x + by_w$ is the image of $ax + bw$, so

$$P(ax + bw) = ay_x + bz_w = aPx + bPw$$

which shows that P is linear.
$P^2 = P$: If $y \in K$, then its unique decomposition with respect to K is $y = y + 0$. Thus for all $y \in K$, $Py = y$. Since $Px \in K$ for all $x \in H$, we have

$$P^2x = P(Px) = Px \qquad \text{for all } x \in H$$