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HW #17

① ignorable coordinates are coordinates which do not show up in the Lagrangian. This means that the generalized momentum associated with the coordinate in question will be constant.

The 2 coordinates for this problem are θ and ϕ. The Lagrangian has θ in it, along with $\dot{\theta}$ and $\dot{\phi}$, but no ϕ. Thus, ϕ is an ignorable coordinate. Thus, the generalized momentum associated with ϕ is consered (angular in this case).

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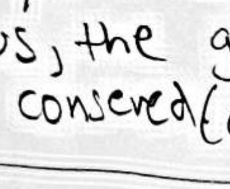
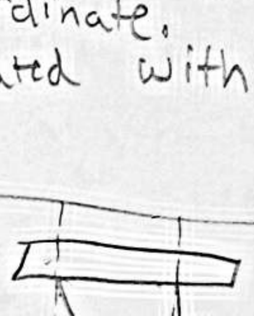
② 

Each pendulum can be described by the angle it makes with equilibrium



If the oscillations are small, we can describe the stretch of the spring by the x components of the two pendulums.

$$PE_{spring} = \frac{1}{2}k(l\sin\theta_1 - l\sin\theta_2)^2$$

$$PE_{pendulums} = (l - l\cos\theta_1)mg + (l - l\cos\theta_2)mg$$

$$T = \frac{1}{2}m(l\dot{\theta}_1)^2 + \frac{1}{2}m(l\dot{\theta}_2)^2$$

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\dot{\theta}_2^2 - \frac{1}{2}k(l\sin\theta_1 - l\sin\theta_2)^2 - (l - l\cos\theta_1)mg - (l - l\cos\theta_2)mg$$

$\sin\theta = \theta \quad \cos\theta = 1 - \frac{1}{2}\theta^2$

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\dot{\theta}_2^2 - \frac{1}{2}k(l\theta_1 - l\theta_2)^2 - \left(\frac{l}{2}\theta_1^2\right)mg - \left(\frac{l}{2}\theta_2^2\right)mg$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = ml^2\dot{\theta}_1 \qquad \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = ml^2\dot{\theta}_2$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -glm\theta_1 - kl(l\theta_1 - l\theta_2) \qquad \frac{\partial \mathcal{L}}{\partial \theta_2} = -gml\theta_2 + kl(l\theta_1 - l\theta_2)$$

(2) continued...

$$-gml\Theta_2 + kl(l\Theta_1 - l\Theta_2) = ml^2\ddot{\Theta}_2$$
$$-gml\Theta_1 - kl(l\Theta_1 - l\Theta_2) = ml^2\ddot{\Theta}_1$$

} Lagrange equations of motion

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(3) a) 7.14

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega^2$$

$$\omega = \frac{v}{R}$$

$$\frac{1}{4}m\cancel{R^2}\frac{v^2}{\cancel{R^2}} = \frac{1}{4}m\dot{x}^2$$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{4}m\dot{x}^2 = \frac{3}{4}m\dot{x}^2$$

$$U = -mgx$$

$$\mathcal{L} = \frac{3}{4}m\dot{x}^2 + mgx$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \rightarrow mg = \frac{d}{dt}\left[\frac{3}{2}m\dot{x}\right] \rightarrow \cancel{m}g = \frac{3}{2}\cancel{m}\ddot{x}$$

$$\boxed{\ddot{x} = \frac{2g}{3}}$$

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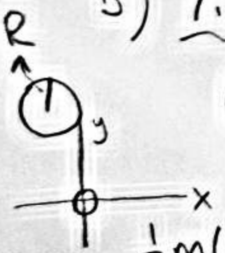
b) 7.29



$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

where $t=0 \rightarrow$

$\phi = 0$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$T = \frac{1}{2}ml^2\dot{\phi}^2$ without rotating wheel

where $\dot{\Theta} = \omega$

$$x = l\sin\phi - R\cos\Theta \qquad \dot{x} = l\cos\phi\,\dot{\phi} + R\sin\theta\,\dot{\theta}$$

$$y = l - l\cos\phi + R\sin\theta \qquad \dot{y} = l\sin\phi\,\dot{\phi} + R\cos\Theta\,\dot{\Theta}$$

$$U = mg(l - l\cos\phi + R\sin\theta) \qquad T = \frac{1}{2}m\left[l\cos\phi\,\dot{\phi} + R\sin\theta\,\dot{\theta}\right]^2 + \frac{1}{2}m\left[l\sin\phi\,\dot{\phi} + R\cos\theta\,\dot{\theta}\right]^2$$

$$\boxed{\mathcal{L}} = \frac{1}{2}m\left[l\cos\phi\,\dot{\phi} + R\sin\theta\,\dot{\theta}\right]^2 + \frac{1}{2}m\left[l\sin\phi\,\dot{\phi} + R\cos\theta\,\dot{\theta}\right]^2 - mg(l - l\cos\phi + R\sin\theta)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -glm\sin(\phi) + lm\dot{\phi}\cos(\phi)\cdot\left[l\dot{\phi}\sin(\phi) + R\cos(\theta)\cdot\dot{\theta}\right]$$
$$- lm\dot{\phi}\sin(\phi)\left[l\dot{\phi}\cos(\phi) + R\sin(\theta)\dot{\theta}\right]$$

b) continued

If $\omega = 0$, then $\dot{\theta} = 0$ and $\theta = 0$ at $t = 0$, θ is unchanging. Thus

$$\frac{\partial \mathcal{L}}{\partial \phi} = -gml\sin(\phi)$$

let's check this for a stationary pendulum...

$T = \frac{1}{2}ml^2\dot{\phi}^2 \quad U = mgl(1-\cos\phi)$

$\mathcal{L} = \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1-\cos\phi)$

thus

$$\frac{\partial \mathcal{L}}{\partial \phi} = -mgl\sin\phi$$

we have a match! This shows that our lagrangian works for the limiting case of $\omega = 0$.

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = lm\sin\phi\left[l\dot{\phi}\sin\phi + R\cos\theta\,\dot{\theta}\right] + lm\cos\phi\left[l\dot{\phi}\cos\phi + R\sin\theta\,\dot{\theta}\right]$$

Use this and $\frac{\partial \mathcal{L}}{\partial \phi}$ as I found earlier for the equation of motion of angle ϕ...

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right]$$

It's long...