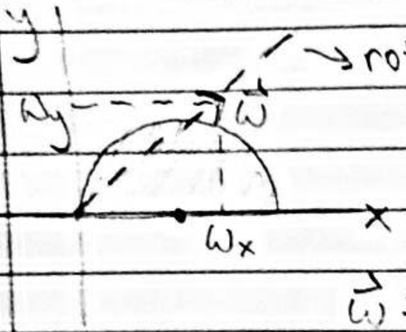
HW #32 problem 1



$$\vec{L} = \vec{I}\,\vec{\omega}$$

$$\vec{\omega} = \omega \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \qquad \omega_x = \omega/\sqrt{2} \quad \omega_y = \omega/\sqrt{2} \quad \omega_z = 0$$

$$I = \begin{pmatrix} J_{yy} & -J_{xy} & 0 \\ -J_{xy} & J_{xx} & 0 \\ 0 & 0 & J_{yy}+J_{xx} \end{pmatrix}$$ since we have no z coordinate

$$J_{xy} = \iint xy\sigma \, dx\, dy$$ integrate along x dir from 0 to 2R.

$$\sigma = \frac{M}{A} = \frac{M}{\frac{1}{2}\pi r^2}$$ For y integral, from 0 to the curve.

$$J_{xy} = \int_0^{2R} \int_0^{\sqrt{x}\sqrt{2R-x}} xy\sigma \, dy\, dx = \frac{2R^4\sigma}{3}$$

$(x-R)^2 + y^2 = R^2$

$$J_{yy} = \int_0^{2R} \int_0^{\sqrt{x}\sqrt{2R-x}} y^2 \sigma \, dy\, dx = \frac{1}{8}\pi R^4 \sigma$$

$$J_{xx} = \int_0^{2R} \int_0^{\sqrt{x}\sqrt{2R-x}} x^2 \sigma \, dy\, dx = \frac{5}{8}\pi R^4 \sigma$$

Thus

$$I = \begin{bmatrix} \frac{1}{2}\pi R^4\sigma & -\frac{2R^4\sigma}{3} & 0 \\ -\frac{2R^4\sigma}{3} & \frac{5}{8}\pi R^4\sigma & 0 \\ 0 & 0 & \frac{6}{8}\pi R^4\sigma \end{bmatrix}$$

$$\vec{L} = I\vec{\omega}$$

$$\vec{L} = \begin{bmatrix} \; \end{bmatrix} \begin{pmatrix} \omega/\sqrt{2} \\ \omega/\sqrt{2} \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} \frac{\omega\pi R^4\sigma}{2\sqrt{2}} - \frac{2R^4\omega\sigma}{3\sqrt{2}} \\ -\frac{2\omega R^4\sigma}{3\sqrt{2}} + \frac{5\omega\pi R^4\sigma}{\sqrt{2}\,8} \\ 0 \end{pmatrix}} = \vec{L} \quad \begin{matrix} x \\ y \\ z \end{matrix}$$

(2) In the body frame, $\vec{L}$ is "attached" to $\vec{\omega}$, with both precessing about $\hat{e}_3$ at some rate Ω. $\hat{e}_3$ stays constant in this frame.
In the space frame, $\hat{e}_3$ is no longer constant, but moves around a fixed $\vec{L}$ along w/ $\vec{\omega}$ at some rate Ω_s. So, in both frames $\vec{\omega}$ is not constant.

(3) Euler's Equations refer to axes fixed in the body frame. The axes are awkward to work with. The Lagrangian is set up to specify orientation of the body to a non-rotating frame, which is a lot easier to deal with.