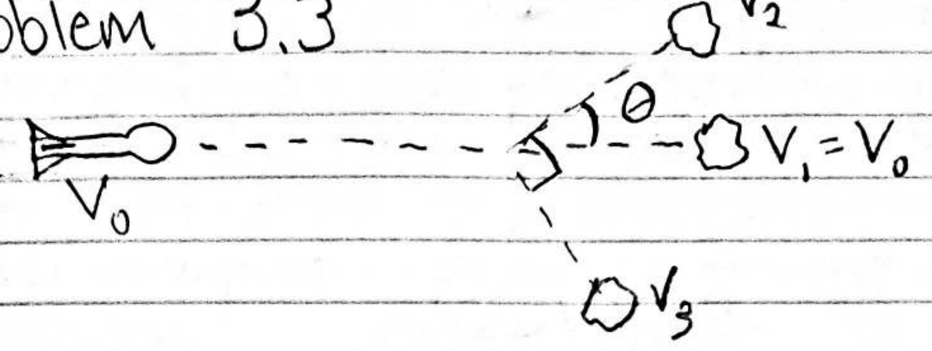
HW 5

① Problem 3.3



$V_2 = V_3$ $\quad M_s$ = shell mass

$$V_0 M_s = V_0 M_1 + V_2 M_2 \cos(\theta) + V_3 M_3 \cos(90-\theta)$$

$$V_0 (M_s - M_1) = V_2 M_2 \cos\theta + V_3 M_3 \sin(\theta)$$

$$M_1 = M_2 = M_3 = M_p$$

$$V_0 (M_s - M_p) = V_2 M_p \cos\theta + V_3 M_p \sin(\theta)$$

$$M_s = 3 M_p$$

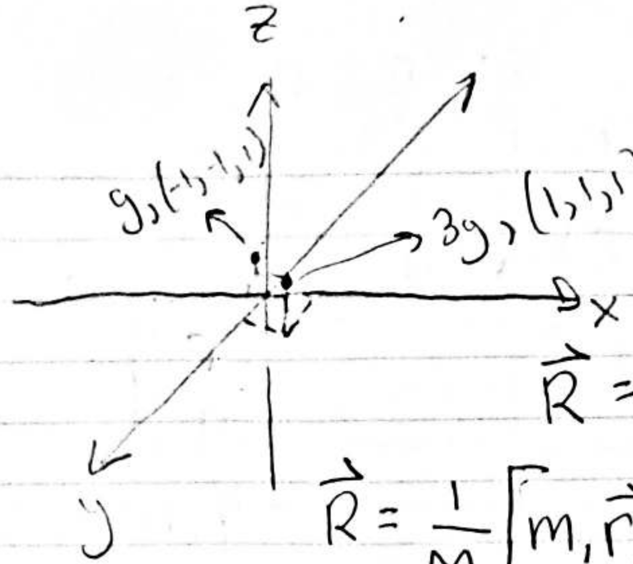
$$2V_0 = V_2 \cos\theta + V_3 \sin\theta$$

$$V_3 \sin\theta = V_2 \sin\theta$$

$$2V_0 = V_2 (\cos\theta + \sin\theta)$$

$$\boxed{V_2 = \frac{2V_0}{\cos\theta + \sin\theta} = V_3}$$

(3) A)



$$\vec{R} = \frac{1}{M}\sum_i m_i \vec{r}_i$$

$$\vec{R} = \frac{1}{M}\left[m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + m_4\vec{r}_4\right]$$

$$M = 3+1+1+1 = 6$$

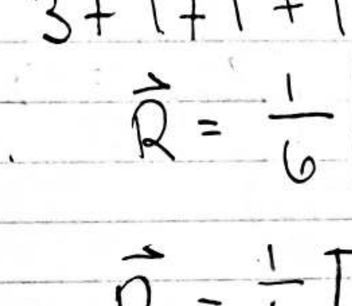
$$\vec{R} = \frac{1}{6}\left[\langle 3,3,3\rangle + \langle -1,-1,1\rangle + \langle -1,1,-1\rangle + \langle 1,-1,-1\rangle\right]$$

$$\vec{R} = \frac{1}{6}\left[\langle 3-1 \cancel{-1+1}, 3-1+\cancel{1-1}, 3+\cancel{1-1}-1\rangle\right]$$

$$\vec{R} = \frac{1}{6}\left[\langle 2, 2, 2\rangle\right]$$

$$\boxed{\vec{R} = \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\rangle}$$

B)



$$Z_{cm} = \frac{1}{M}\int z\,\rho(\vec{r})\,dV$$

$$\rho(\vec{r}) = \text{constant} = C$$

$$Z_{cm} = \frac{C}{M}\int z\,dx\,dy\,dz$$

spherical coord: $z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$Z_{cm} = \frac{C}{M}\int r^3\cos\theta\sin\theta\,dr\,d\theta\,d\phi$$

3b continued

$$Z_{cm} = \frac{C}{M}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^3\cos\theta\sin\theta\, dr\, d\theta\, d\phi$$

$$\frac{r^4}{4}\Big|_0^R = \frac{R^4}{4}$$

$$-\tfrac{1}{2}\cos^2(\theta)\Big|_0^{\pi} = -\tfrac{1}{2}+\tfrac{1}{2} = \boxed{0}$$

Because one of the integrations results in zero, the whole triple integral is zero. $\boxed{Z_{cm}=0}$

C) $\rho(\vec{r}) = \rho(z) = \rho_0(1 - z/b)$

$X_{cm} = 0 = y_{cm}$

$$Z_{cm} = \frac{1}{M}\int z\rho_0\left(1-\frac{z}{b}\right)dx\,dy\,dz$$

$\cancel{dm = V\rho_0(1-z/b)\,dz}$

$\cancel{\int_0^b a^2 b\rho_0 - \frac{a^2 b\rho_0 z}{b}\,dz = a^2 b\rho_0\int_0^b 1-\frac{z}{b}\,dz}$ $\quad z - \frac{1}{2}\frac{z^2}{b}\Big|_0^b$

$b - \frac{1}{2}b = \frac{1}{2}b$

$dV = a^2 dz$

$$\to a^2\rho_0\int_0^b (1 - z/b)\,dz \to z - \frac{1}{2}\frac{z^2}{b}\Big|_0^b$$

$b - \frac{1}{2}b = \frac{1}{2}b$

$\to a^2\rho_0 \cdot \frac{1}{2}b = M$

$$Z_{cm} = \frac{\cancel{\rho_0}}{a^2\cancel{\rho_0}\frac{1}{2}b}\int z - z^2/b\; dx\,dy\,dz$$

$$Z_{cm} = \frac{1}{a^2\frac{1}{2}b}\int_0^b\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} z - z^2/b\; dx\,dy\,dz$$

3c continued

$$\int_{-a/2}^{a/2} dx \rightarrow x\Big|_{-a/2}^{a/2} = \frac{a}{2} + \frac{a}{2} = \underline{a}$$

$$\int_{-a/2}^{a/2} dy \rightarrow y\Big|_{-a/2}^{a/2} = \frac{a}{2} + \frac{a}{2} = \underline{a}$$

$$\int_{0}^{b} z - \frac{z^2}{b}\,dz \rightarrow \frac{z^2}{2} - \frac{z^3}{3b}\Big|_{0}^{b}$$

$$= \frac{b^2}{2} - \frac{b^2}{3} = \underline{\frac{b^2}{6}}$$

Thus $Z_{cm} = \frac{2}{\cancel{a^2} \cancel{b}} \cdot \cancel{a} \cdot \cancel{a} \cdot \frac{b^{\cancel{2}}}{6}$

$$\boxed{Z_{cm} = \frac{2b}{6} = \frac{1}{3}b}$$