Midterm 2

CID: 6265

I promise that I have abided the rules of the exam with complete integrity *[signature]*

(1) A) we don't know if the momentum associated with a variable in the kinetic energy will be linear, hence why we call it the generalized momentum. For instance, variables expressing angular motion may have a generalized momentum that describes the physical angular momentum of the system, not linear.

B) we use Lagrange Multipliers. This consists of writing down the Lagrangian with a constraint force (the normal force on the particle in this case) and solving for that force through the use of a constraint equation $f(x, y) = R^2 = x^2 + y^2 = const.$ where R is the radius of the track. we will have an extra unknown function, $\lambda(t)$, to solve for, but the inclusion of the constraint equation makes the solution possible. Lagrangian will look like this: $\frac{\partial \mathcal{L}}{\partial q} + \lambda(t)\frac{\partial f}{\partial q} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}$

C)

c) The center of Mass is an inertial frame when no external force is present. If we choose our reference frame to be attached to it, $\dot{R} = 0$ (velocity of CM = 0). This makes it disappear from our Lagrangian, so we can focus entirely on the motion of the 2 bodies $\left(\mathcal{L} = \frac{1}{2}\mu\dot{r}^2 - U(r)\right)$ where $\vec{r} = \vec{r}_1 - \vec{r}_2$. $\vec{R}$, representing the position of CM, is ignorable in terms of the Lagrangian. Our 2 body problem can be reduced to one body (fiticious) because of this.

D) The Hamiltonian in most cases we deal with is given by $\mathcal{H} = T + U$. These problems have conservative forces giving the potential U, and thus E_{total} is conserved. Since E_{tot} is just the kinetic energy T plus the potential energy U, $E_{tot} = T + U = \mathcal{H} = \text{const}$, so the hamiltonian is conserved. The Lagrangian on the other hand is given by $T - U$, which is not a conserved quantity

$$\left(\begin{array}{ll} \text{consider} \rightarrow 2+5=7 & \text{vs. } 2-5=-3. \\ \text{now} \rightarrow \underbrace{1+6=7}_{\text{conserved}} & \text{vs. } \underbrace{1-6=-5.}_{\text{not conserved}} \end{array}\right)$$

(2) A) $F = kx$, $m\ddot{x} = kx$ $\quad k > 0$ $\quad U = -\int_0^x F\,dx$

$T = \frac{1}{2}m\dot{x}^2$ $\quad F = -\nabla U$

$U = -\frac{1}{2}kx^2$

$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$

$\frac{\partial \mathcal{L}}{\partial x} = kx$ $\quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$

$\frac{d}{dt}[m\dot{x}] = m\ddot{x}$

$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \rightarrow \boxed{kx = m\ddot{x}}$ matches!

B) $m\ddot{x} = kx \rightarrow \ddot{x} = \frac{k}{m}x$ $\quad r^2 = \frac{k}{m}$ $\quad r = \pm\sqrt{\frac{k}{m}}$

$x(t) = c_1 e^{\sqrt{\frac{k}{m}}t} + c_2 e^{-\sqrt{\frac{k}{m}}t}$ $\quad x(0) = x_0$, $\dot{x}(0) = v_0$

$x(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 = x_0$

$\dot{x}(t) = \sqrt{\frac{k}{m}}c_1 e^{\sqrt{\frac{k}{m}}t} + -\sqrt{\frac{k}{m}}c_2 e^{-\sqrt{\frac{k}{m}}t}$

$\dot{x}(0) = \sqrt{\frac{k}{m}}c_1 e^0 - \sqrt{\frac{k}{m}}c_2 e^0 = \sqrt{\frac{k}{m}}c_1 - \sqrt{\frac{k}{m}}c_2 = v_0$

$c_1 = x_0 - c_2 \longrightarrow \sqrt{\frac{k}{m}}x_0 - \sqrt{\frac{k}{m}}c_2 - \sqrt{\frac{k}{m}}c_2 = v_0$

$+2\sqrt{\frac{k}{m}}c_2 = -v_0 + \sqrt{\frac{k}{m}}x_0$

$c_1 = x_0 + \frac{v_0}{2\sqrt{\frac{k}{m}}} - \frac{x_0}{2}$ $\quad c_2 = \frac{-v_0}{2\sqrt{\frac{k}{m}}} + \frac{x_0}{2}$

$c_1 = \frac{x_0}{2} + \frac{v_0}{2\sqrt{\frac{k}{m}}}$

$$x(t) = \left[\frac{x_0}{2} + \frac{v_0}{2\sqrt{\frac{k}{m}}}\right]e^{\sqrt{\frac{k}{m}}t} + \left[\frac{-v_0}{2\sqrt{\frac{k}{m}}} + \frac{x_0}{2}\right]e^{-\sqrt{\frac{k}{m}}t}$$

c) $P_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$ thus $\dot{x} = \frac{P_x}{m}$

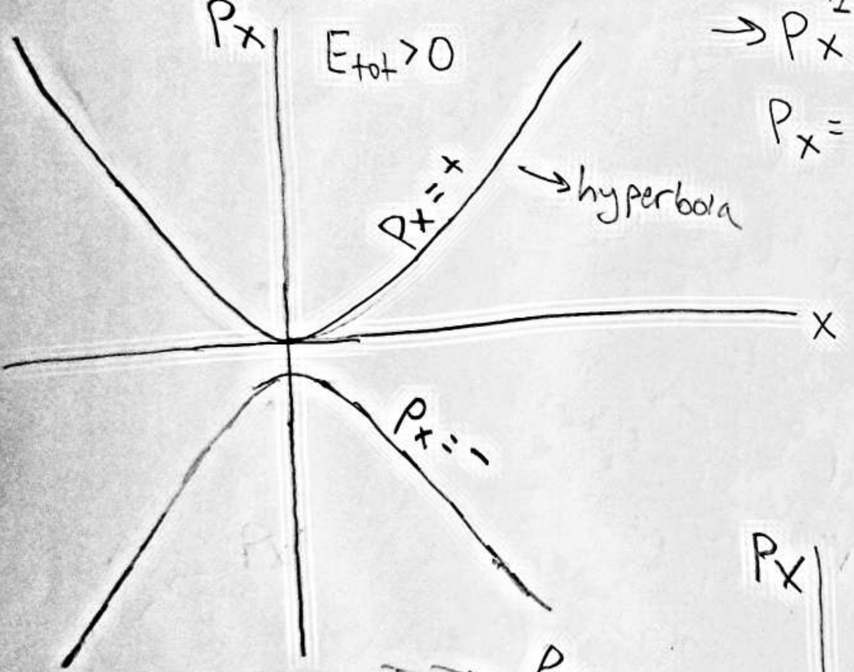
$T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} \cancel{m} \frac{P_x^2}{m^{\cancel{2}}} = \frac{P_x^2}{2m}$

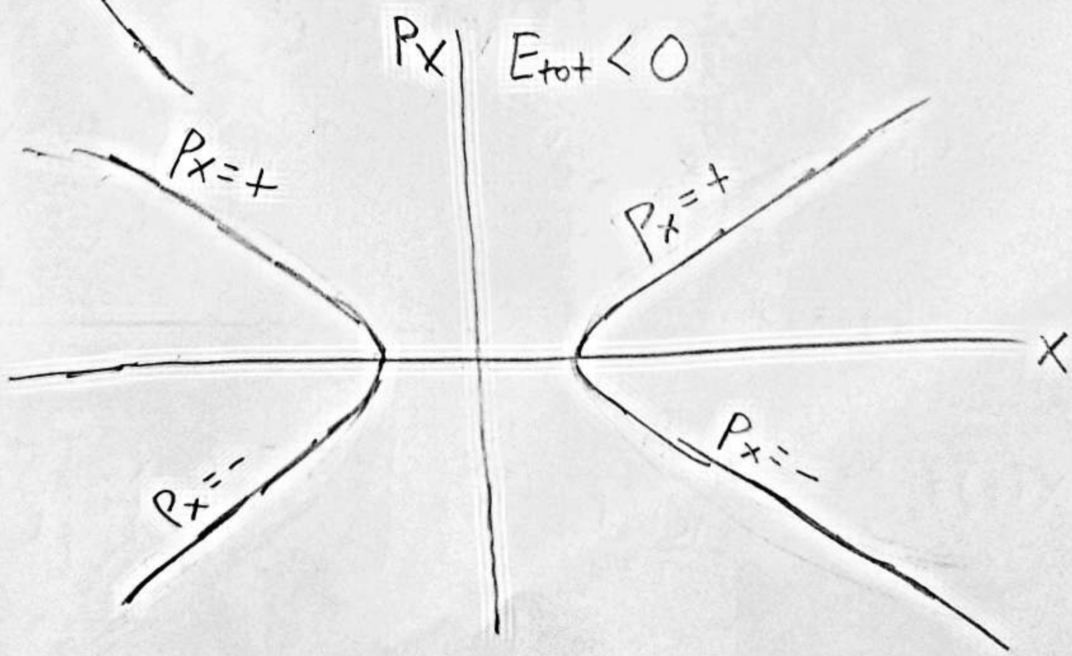
$\mathcal{H} = \frac{P_x^2}{2m} - \frac{1}{2} k x^2$

D) $E_{tot} = \frac{P_x^2}{2m} - \frac{1}{2} k x^2 \rightarrow \frac{P_x^2}{2m} = E_{tot} + \frac{1}{2} k x^2$

$\rightarrow P_x^2 = \left[E_{tot} + \frac{1}{2} k x^2\right] 2m$

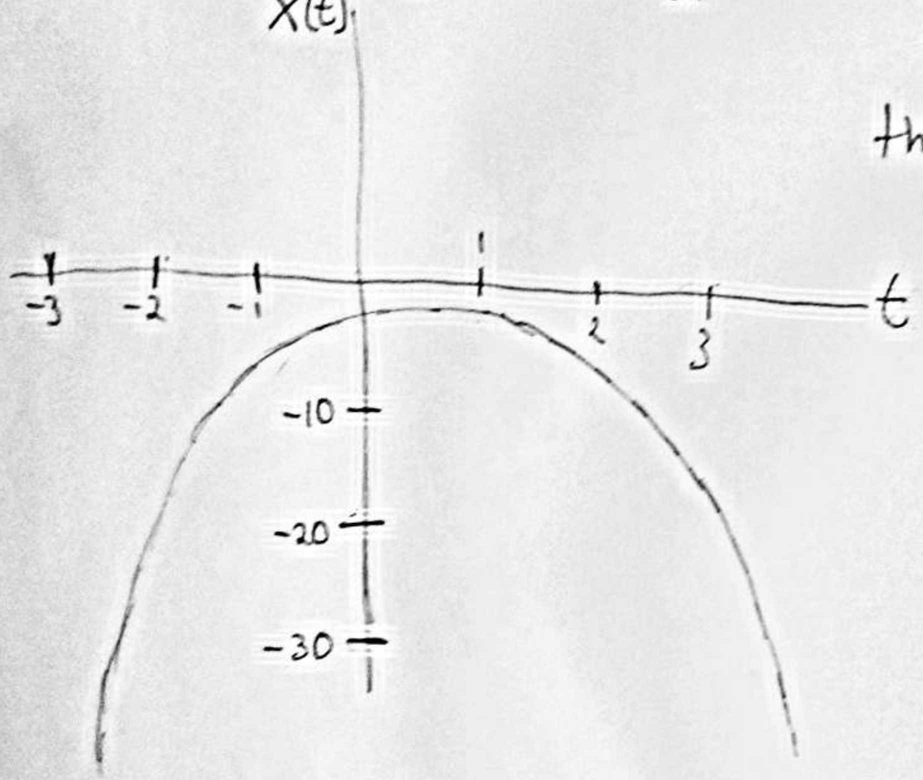
$P_x = \pm\sqrt{\left[E_{tot} + \frac{1}{2} k x^2\right] 2m}$



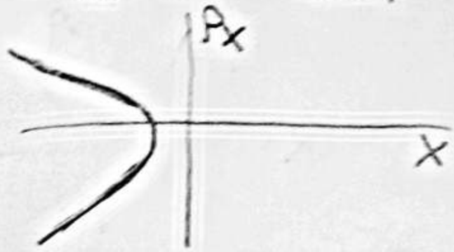


E) $X(t) = \left[\frac{-3}{2} + \frac{2}{2\sqrt{\frac{1}{1}}}\right]e^{t} + \left[\frac{-2}{2\sqrt{\frac{1}{1}}} + \frac{-3}{2}\right]e^{-t}$

$X(t) = -\frac{1}{2}e^{t} - \frac{5}{2}e^{-t}$



This graph matches the phase space diagram where $E_{tot} < 0$ and



(3) A) $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$

r is always const. Thus we have 2 degrees of freedom, ϕ and θ

but, θ is from $-z$... $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = -r\cos\theta$

$U = mgr(1-\cos\theta)$

$\dot{x} = -r\sin\phi\sin\theta\,\dot{\phi} + r\cos\phi\cos\theta\,\dot{\theta}$

$\dot{y} = r\cos\phi\sin\theta\,\dot{\phi} + r\cos\theta\sin\phi\,\dot{\theta}$

$\dot{z} = r\sin\theta\,\dot{\theta}$

$$\mathcal{L} = \frac{1}{2}m\left[(-r\sin\phi\sin\theta\dot{\phi} + r\cos\phi\cos\theta\dot{\theta})^2 + (r\cos\phi\sin\theta\dot{\phi} + r\cos\theta\sin\phi\dot{\theta})^2 + (r\sin\theta\dot{\theta})^2\right] - mgr(1-\cos\theta)$$

simplified via Mathematica...

$$\mathcal{L} = \frac{1}{4}mr(-4g + \dot{\phi}^2 r + 2r\dot{\theta}^2 + 4g\cos\theta - \dot{\phi}^2 r\cos(2\theta))$$

B) $P_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{1}{2}mr^2\dot{\phi} - \frac{1}{2}mr^2\dot{\phi}\cos(2\theta)$

$P_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} \rightarrow$ angular momentum in θ direction

$T = -\frac{1}{4}mr^2(-\dot{\phi}^2 - 2\dot{\theta}^2 + \dot{\phi}^2\cos(2\theta))$

$\dot{\phi}\left[\frac{1}{2}mr^2 - \frac{1}{2}mr^2\cos(2\theta)\right] = P_\phi \qquad \dot{\phi} = \frac{-2P_\phi}{mr^2(-1+\cos(2\theta))}$

$\dot{\theta} = \frac{P_\theta}{mr^2}$

using mathematica... (I plugged $\dot{\phi}$ and $\dot{\theta}$ into T using /. rule)

$$T = \frac{P_\theta^2 + P_\phi^2 \csc^2(\theta)}{2mr^2}$$

thus $\boxed{\mathcal{H} = \frac{P_\theta^2 + P_\phi^2 \csc^2(\theta)}{2mr^2} + mgr(1-\cos\theta)}$

☆ Note: I just realized that we could have noticed that $V_\theta = r\dot{\theta}$ and $V_\phi = r\dot{\phi}\sin\theta$ instead of taking the derivative of x, y, z.

Thus, $T = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta)$

which <u>is</u> equal to the T I found earlier

$$T = -\frac{1}{4}mr^2(-\dot{\phi}^2 - 2\dot{\theta}^2 + \dot{\phi}^2\cos(2\theta))$$

since $\cos(2\theta) = 1 - 2\sin^2\theta$, which simplifies my T to the other T above.

This means that $\mathcal{L} = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) - mgr(1-\cos\theta)$

and $P_\phi = mr^2\dot{\phi}\sin^2\theta$ and $P_\theta = mr^2\dot{\theta}$

thus $\dot{\theta}^2 = \frac{P_\theta^2}{m^2r^4}$ and $\dot{\phi}^2 = \frac{P_\phi^2}{m^2r^4\sin^4\theta}$

so $T = \frac{1}{2}m\left(\frac{P_\theta^2}{m^2r^2} + \frac{P_\phi^2}{m^2r^2\sin^2\theta}\right) = \frac{P_\theta^2 + P_\phi^2\csc^2(\theta)}{2mr^2}$

$$\rightarrow \boxed{\mathcal{H} = \frac{1}{2mr^2}\left(P_\theta^2 + \frac{P_\phi^2}{\sin^2\theta}\right) + mgr(1-\cos\theta)}$$

c) ϕ is an ignorable coordinate.
This means that the generalized momentum or conjugate momentum associated with ϕ is conserved. This momentum would be the angular momentum in the ϕ direction for this problem. This makes sense because we have no component of force in the ϕ direction when considering gravity. Thus, our pendulum bob never accelerates in the ϕ direction.

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d) $$\frac{\partial H}{\partial \phi} = -\dot{p}_\phi = 0 , \quad \frac{\partial H}{\partial p_\phi} = \dot{\phi} = \frac{p_\phi}{mr^2 \sin^2\theta}$$

$$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta = \frac{-p_\phi^2 \cot(\theta)\csc^2(\theta)}{mr^2} + mgr\sin(\theta)$$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{p_\theta}{mr^2}$$

thus $\boxed{\dot{p}_\phi = 0}$ $\boxed{\dot{\phi} = \frac{p_\phi}{mr^2 \sin^2\theta}}$

$$\boxed{\dot{p}_\theta = \frac{p_\phi^2 \cot(\theta)\csc^2(\theta)}{mr^2} - mgr\sin(\theta)}$$

$\boxed{\dot{\theta} = \frac{p_\theta}{mr^2}}$ where $\dot{p}_\theta = \ddot{\theta} mr^2$

$$\ddot{\theta} mr^2 = \frac{p_\phi^2 \cot(\theta)\csc^2(\theta)}{mr^2} - mgr\sin\theta$$