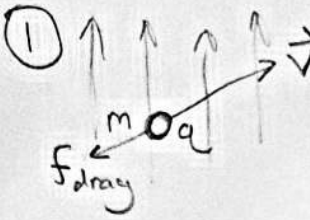
Phys 321 Final

CID 6265

"I promise that I have abided the rules of the exam with complete integrity"

① 

$\vec B = B\hat z \qquad F_B = q\vec v \times \vec B$

$\vec B = \langle 0, 0, B\rangle$

$\vec v = \langle \dot x, \dot y, \dot z\rangle \qquad \hat r = \dfrac{\vec r}{r}$

$$m\ddot{\vec r} = q\dot{\vec r}\times\vec B - mg\hat z - b\dot r\hat r$$

$$m\ddot{\vec r} = q\dot{\vec r}\times\vec B - mg\hat z - b\dot{\vec r}$$

$$\langle \dot x, \dot y, \dot z\rangle \rightarrow \langle \dot x, \dot y, \dot z\rangle \times \langle 0, 0, B\rangle = \langle B\dot y, -B\dot x, 0\rangle$$

$$\boxed{\begin{aligned} m\ddot x &= B\dot y - b\dot x \\ m\ddot y &= -B\dot x - b\dot y \\ m\ddot z &= -mg - b\dot z \end{aligned}}$$

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② 1) You can derive the Lagrangian $\mathcal{L} = T - U$ and do: $\dfrac{\partial\mathcal{L}}{\partial q_i} - \dfrac{d}{dt}\dfrac{\partial\mathcal{L}}{\partial\dot q_i} = 0$ for each q_i in your system

2) You can also use Hamiltonian $\sum_{i=1}^{n} p_i\dot q_i - \mathcal{L}$ which gives us Hamilton's equations $\dot q_i = \dfrac{\partial\mathcal{H}}{\partial p_i}$ and $\dot p_i = -\dfrac{\partial\mathcal{H}}{\partial q_i}$ $[i = 1, \dots, n]$ which can solved for equations of motion (which are often subbed into one another in a way to reveal more newtonian-esque equations).

(2) continued...

3) You can use the relationship of $\dot{T} + \dot{U} = 0$ (due to conservation of energy). However, this strategy usually only works for simple systems like one-dimensional.

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(3) points of equilibrium $\rightarrow \frac{dU}{d\theta} = 0$

(A) $\frac{dU}{d\theta} = mg\left[-(r+b)\sin\theta + r(\theta\cos\theta + \sin\theta)\right] = 0$

if θ is small, then $\cos\theta \approx 1 - \frac{\theta^2}{2}$ $\sin\theta \approx \theta$

$$\frac{dU}{d\theta} = mg\left[\cancel{-r\sin\theta} - b\sin\theta + r\theta\cos\theta + \cancel{r\sin\theta}\right] = 0$$

$$\frac{dU}{d\theta} = \cancel{mg}\left[-b\theta + r\theta\left(1 - \frac{\theta^2}{2}\right)\right] = 0$$

$$\frac{dU}{d\theta} = -b\theta + r\theta - \frac{r\theta^3}{2} = 0$$

solved in Mathematica... $\theta \rightarrow 0, \theta \rightarrow \frac{-\sqrt{2}\sqrt{-b+r}}{\sqrt{r}}, \theta \rightarrow \frac{\sqrt{2}\sqrt{-b+r}}{\sqrt{r}}$

we choose $\boxed{\theta = 0}$ since it is the smallest magnitude.

(B) we want $U''(\theta) > 0$

$$U''(\theta) = -b + r - \frac{r\theta^2 3}{2} > 0$$

where $\theta = 0 \rightarrow -b + r > 0$

$$\boxed{r > b}$$

$$U''(\theta) = mg\left[-(r+b)\cos\theta + r\left(-\theta\sin\theta + \cos\theta + \cos\theta\right)\right]$$

$$= mg\left[-b\cos\theta - r\theta\sin\theta + r\cos\theta\right]$$

$$= mg\left[-b + \frac{b\theta^2}{2} - r\theta^2 + r - \frac{r\theta^2}{2}\right]$$

$$= \cancel{mg}\left[-b + r\right] > 0 \quad r > b$$

(4) One situation where this was helpful was when dealing with the 2-body central force problem. When we transform to the CM frame, our problem simplifies to a one-body problem, where quantities measured for this ficticious body apply to the original system, such as total angular momentum.

Another case is when trying to measure the total linear momentum of a system of particles. Here, when moving to the CM frame, this total momentum is zero, making it trivial to solve for individual unknown momentums.

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(5) $m\ddot{\vec{r}} = \vec{T} + m\vec{g} + m(\vec{\Omega}\times\vec{r})\times\vec{\Omega} + 2m\dot{\vec{r}}\times\vec{\Omega}$

(A) $T_z = T\cos(\beta) \approx T$ for small β

and $T_z \approx mg \approx T$

$T_x = -\frac{mgx}{L}$ $T_y = -\frac{mgy}{L}$

$$\boxed{m\ddot{x} = -\frac{mgx}{L} + \Omega^2 Rm}$$

$$\boxed{m\ddot{y} = -\frac{mgy}{L}}$$

$\vec{\Omega} = \langle 0, 0, \Omega\rangle$

$\vec{r} = \langle R, 0, 0\rangle$

$\dot{\vec{r}} = \langle 0, 0, 0\rangle$

$2m\dot{\vec{r}}\times\vec{\Omega} = \langle 0,0,0\rangle \times \langle 0,0,\Omega\rangle = \langle 0,0,0\rangle$

$\vec{\Omega}\times\vec{r} = \langle 0,0,\Omega\rangle \times \langle R,0,0\rangle = \langle 0,\Omega R,0\rangle$

$(\vec{\Omega}\times\vec{r})\times\vec{\Omega} = \langle 0,\Omega R,0\rangle \times \langle 0,0,\Omega\rangle = \langle \Omega^2 R,0,0\rangle$

(B) $x_2 \to x - \frac{RL\Omega^2}{g}$

$\ddot{x} = -\frac{g}{L}\left[x_2 + \frac{RL\Omega^2}{g}\right] + \Omega^2 R$

$\ddot{x} = -\frac{g}{L}x_2 - R\Omega^2 + \Omega^2 R$

$\ddot{x} = -\frac{g}{L}x_2$

$\ddot{y} = -\frac{g}{L}y$

The pendulum has simple harmonic motion in both the x and y directions

(6) (A) Total angular momentum, total linear momentum, total Energy.

(B) since CM is Stationary, and total momentum is represented by CM,

$\vec{P}_{tot} = 0 = \vec{P}_1 + \vec{P}_2$

$m_1 = 1\,kg \quad m_2 = 1.5\,kg$

$\vec{P}_2 = 1.5\langle -6, 2.5, 0\rangle = \langle -9\frac{kgm}{s}, 3.75\frac{kgm}{s}, 0\frac{kgm}{s}\rangle$

$\vec{P}_1 = -\vec{P}_2 = \langle 9\frac{kgm}{s}, -3.75\frac{kgm}{s}, 0\frac{kgm}{s}\rangle$

$p = m\vec{v} \quad \vec{v}_1 = \frac{\vec{P}_1}{m_1} = \boxed{\langle 9\,m/s, -3.75\,m/s, 0\,m/s\rangle = \vec{V}_1}$

$\vec{r}_1 = \frac{-m_2}{m_1}\vec{r}_2 = -1.5\langle 2.5m, -1.3m, 0m\rangle$

$= \boxed{\langle -3.75m, 1.95m, 0m\rangle = \vec{r}_1}$

$U = \frac{-\gamma}{r} \quad r = |\vec{r}_2 - \vec{r}_1| = |\langle 6.25, -3.25, 0\rangle|$

$= \sqrt{6.25^2 + 3.25^2 + 0^2}$

$= 7.0445m$

$U = \frac{-1000\,Jm}{7.0445\,m}$

$= -141.9547J$

$|\vec{V}_1| = \sqrt{9^2 + 3.75^2 + 0^2} = 9.75\,m/s$

$|\vec{V}_2| = \sqrt{6^2 + 2.5^2 + 0^2} = 6.5\,m/s$

$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}(9.75)^2 + \frac{1}{2}(1.5)(6.5)^2 = 79.218\,J$

Total Energy $= T + U = 79.218 - 141.9547 = \boxed{-62.736\,J = E_{tot}}$

$\vec{L}_{tot} = \vec{r}_1 \times \vec{P}_1 + \vec{r}_2 \times \vec{P}_2 = \langle -3.75, 1.95, 0\rangle \times \langle 9, -3.75, 0\rangle + \langle 2.5, -1.3, 0\rangle \times \langle -9, 3.75, 0\rangle$

$= \langle 0, 0, -3.4875\rangle + \langle 0, 0, -2.325\rangle$

$\boxed{= \langle 0, 0, -5.8125\rangle = \vec{L}_{tot}}$

(7)

(A)

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) \qquad z = Ax^2$$

$$\dot{z} = 2A\dot{x}x$$

$$T = \frac{1}{2}m(\dot{x}^2 + (2A\dot{x}x)^2)$$

$$U = mgz = mgAx^2$$

$$\mathcal{L} = T - U = \frac{1}{2}m(\dot{x}^2 + 4A^2\dot{x}^2x^2) - mgAx^2$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} + 4A^2x^2\dot{x}m \qquad \dot{x} = \frac{p}{m(1+4A^2x^2)}$$

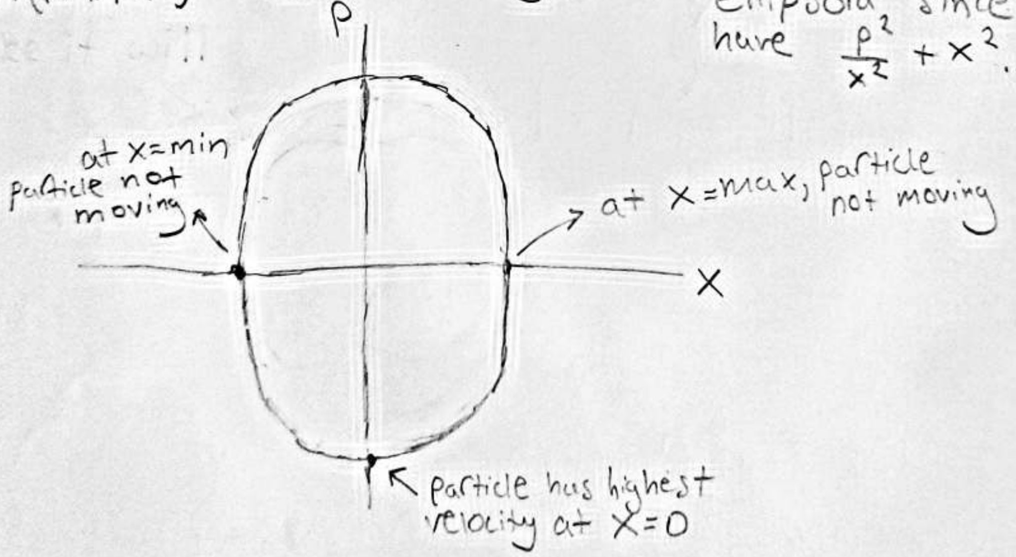
$$\mathcal{H} = p\dot{x} - \mathcal{L} = \frac{p^2}{m(1+4A^2x^2)} - \left[\frac{p^2}{2m(1+4A^2x^2)} - mgAx^2\right]$$

$$\boxed{\mathcal{H} = p\dot{x} - \mathcal{L} = \frac{p^2}{2m(1+4A^2x^2)} + mgAx^2}$$

(B)

$$\frac{p^2}{2m(1+4A^2x^2)} + mgAx^2 = mgH$$

can't be quite ellipsoid since we have $\frac{p^2}{x^2} + x^2 \ldots$



(8) $\lambda(s) = \frac{3m}{2L}\left(1-(s/L)^2\right)$

(A) First, the bob... $I = mr^2 = mL^2$

now the rod...

$$I = \int r_\perp^2 \, dm \qquad dm = \lambda(s)\,ds$$

$$I = \int r^2_\perp \lambda(s)\,ds = \int_0^L r_\perp^2\left(\frac{3m}{2L}\left(1-(s/L)^2\right)\right)ds$$

$$r_\perp = s$$

$$= \int_0^L s^2\left[\frac{3m}{2L} - \frac{s^2}{L^2}\frac{3m}{2L}\right]ds$$

$$= \frac{3m}{2L}\int_0^L \left(s^2 - \frac{s^4}{L^2}\right)ds$$

$$\frac{3m}{2L}\int_0^L s^2\,ds = \frac{m}{2L}s^3\Big|_0^L = \frac{L^3 m}{2L} = \frac{L^2 m}{2}$$

$$\frac{3m}{2L}\int \frac{s^4}{L^2}ds = \frac{3m}{10L^3}s^5\Big|_0^L = \frac{3mL^5}{10L^3} = \frac{3mL^2}{10}$$

$$\frac{L^2 m}{2} - \frac{3mL^2}{10} = \frac{mL^2}{5}$$

Thus $I_{rod+bob} = \frac{mL^2}{5} + mL^2 = \boxed{\frac{6mL^2}{5}}$

(B) $\vec{R} = \frac{1}{M}\int \vec{r}\,dm + \frac{L}{M}m \qquad M = m+m = 2m$

$$dm = \lambda(s)\,ds$$

$$\vec{R} = \frac{1}{2m}\int \vec{r}\,dm + \frac{L}{2}$$

$$\vec{R} = \frac{1}{2m}\int_0^L s\left(\frac{3m}{2L}\left(1-(s/L)^2\right)\right)ds + \frac{L}{2}$$

$$\rightarrow \frac{3}{4L}\int_0^L s\,ds = \frac{3}{8L}s^2\Big|_0^L = \frac{3L}{8} \qquad \frac{3L}{8} - \frac{3L}{16} = \frac{3L}{16}$$

$$\frac{3}{4L}\int_0^L \frac{s^3}{L^2} = \frac{3}{16L^3}s^4\Big|_0^L = \frac{3L}{16} \qquad \frac{3L}{16} + \frac{L}{2} = \boxed{\frac{11L}{16}}$$

distance of CM from pivot point

(8) continued...

(c) Use a Lagrangian w/ Center of Mass...

$T = \frac{1}{2} m l^2 \dot{\phi}^2$ $\quad l = \frac{11L}{16}$

$U = mgl(1-\cos\phi)$

$\mathcal{L} = \frac{1}{2} m \left(\frac{11L}{16}\right)^2 \dot{\phi}^2 - mg \frac{11L}{16}(1-\cos\phi)$

let's just use l for now...

$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \rightarrow -\cancel{m}gl\sin\phi = \frac{d}{dt}(ml^2\dot{\phi}) = \cancel{m}l^2\ddot{\phi}$

$l^2\ddot{\phi} = -gl\sin\phi$

$\ddot{\phi} = -\frac{g}{l}\sin\phi$ $\quad$ at small oscillations $\sin\phi \approx \phi$

$\approx \ddot{\phi} = -\frac{g}{l}\phi$ $\quad \omega = \sqrt{\frac{g}{l}}$ $\quad l = \frac{11L}{16}$

$\boxed{\omega = \sqrt{\frac{16g}{11L}}}$

(9) There should be 8 normal modes to match the number of degrees of freedom.
The mode with the lowest frequency should have all masses moving uniformly around the circle together.
Then, the highest frequency mode will have pairs of masses (so masses right next to each other) oscillating out of phase with respect to each other, but with the same amplitude. In other words, every other mass moves in phase with one another, but masses next to each other move out of phase, all w/ equal amplitude.

(10) $T = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m(\dot{x}_1 + \dot{x}_2)^2$

$U = \frac{1}{2}Kx_2^2$

$\mathcal{L} = T - U = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m(\dot{x}_1 + \dot{x}_2)^2 - \frac{1}{2}Kx_2^2$

$\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = M\dot{x}_1 + m(\dot{x}_1 + \dot{x}_2)$ $\quad \frac{\partial \mathcal{L}}{\partial x_1} = 0$ which means that the corresponding generalized momentum to x_1 is conserved.

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) = 0$ so

$\frac{\partial \mathcal{L}}{\partial \dot{x}_1}$ is conserved

Then $\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m(\dot{x}_1 + \dot{x}_2)$ $\quad \frac{\partial \mathcal{L}}{\partial x_2} = -Kx_2$

$\frac{d}{dt}\left(m(\dot{x}_1 + \dot{x}_2)\right) = -Kx_2 \rightarrow m(\ddot{x}_1 + \ddot{x}_2) = -Kx_2$

$\ddot{x}_1 + \ddot{x}_2 = \frac{-K}{m}x_2 \rightarrow$ harmonic motion

and

$M\ddot{x}_1 + m(\ddot{x}_1 + \ddot{x}_2) = 0$

in other words $\underbrace{M\dot{x}_1}_{\text{lin momentum of block } M} + \underbrace{m(\dot{x}_1 + \dot{x}_2)}_{\text{lin momentum of block } m} = \text{const.}$

Since x_1 does not appear in the Lagrangian, and x_1 is the linear coordinate of the system, the linear momentum is conserved.

(11) For motion to be considered chaotic, the period needs to be infinite, or in other words, it is essentially non-periodic. When we look at graph A, we see that the system is indeed in the non-linear regime where the phase space is not an ellipses (it is not the shape we expect from a linear oscillator). However, this non-linearity does not gaurentee chaos. The graph still shows that the system in question is periodic, just that its period is a multiple of the first, since we can see that the motion does repeat itself over time. Thus, graph A does not represent chaos. Now graph B definitely does, because the motion can be seen to never repeat itself (the trajectory never returns to its original place).

(12) Done

(13) I learned about the true nature of 2 body orbitals. The ficticious particle makes so much sense in the contex of orbits around the Sun in our solar system. In the case of Earth and the Sun, the Earth basically plays the role of our ficticious body since it is so much smaller than the Sun.