Midterm 3 CID 6265

"I promise that I have abided the rules of the exam with complete integrity."

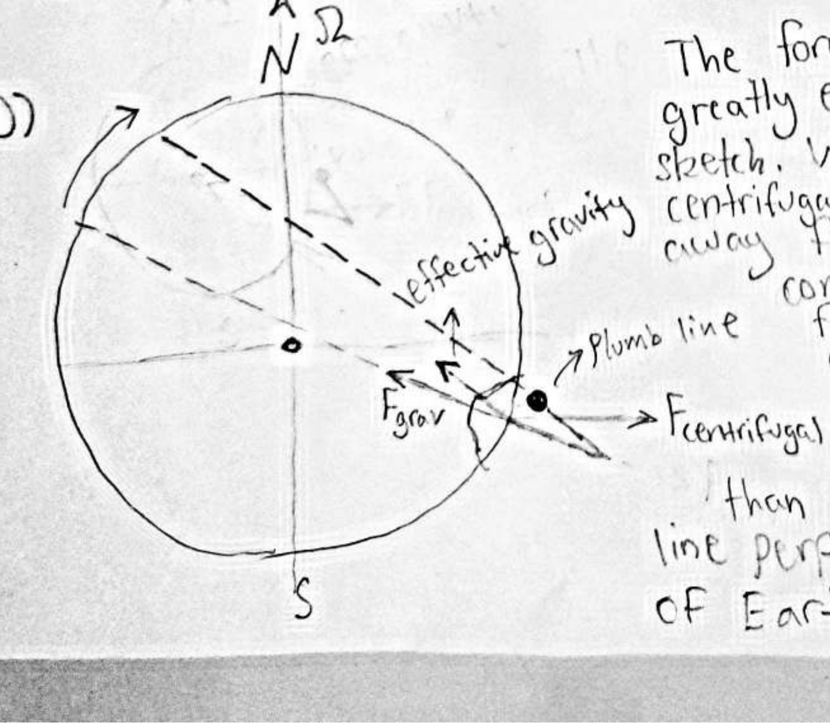
(1) The Puck will have both a centrifugal and coriolis force on it. These forces combine to make it appear as though the Puck is moving around the table in the rotating reference frame.

A) ↗

B) When we consider only small-angle oscillations, the complicated Kinetic Energy and Potential Energy equations for this system reduce to homogeneous quadratic functions. When we differentiate these, we receive homogeneous linear functions which are easily solved, reminiscent of a spring-cart system.

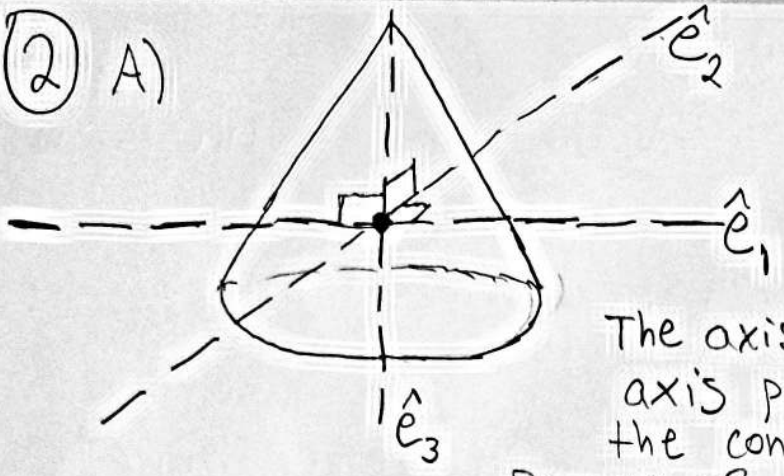
C) When the inertial tensor is evaluated with respect to the principal axes, we get a diagonal inertial tensor. Rotation about one of these axes results in $\vec{L}$ being parallel to $\vec{\omega}$.

D)



The forces have been greatly exaggerated in this sketch. We see that the centrifugal force, pointing directly away from the rotation axis, combines with the actual force of gravity to change how the plumb line hangs. You will end up further North than if you dug along the line perpendicular to the surface of Earth.

(2) A)



The axis $\hat{e}_3$ is a symmetry axis passing straight down the cone through the CM. Because of this, any two other axes perpendicular to themselves and the symmetry axis serve as principal axes. My choice is shown above, with all three passing through the CM.

B) The component along the axis $\hat{e}_3$ is always conserved in this case, where we have two equal principal moments.

$$\lambda_1 \dot{\omega}_1 = (\lambda_2 - \lambda_3)\omega_2\omega_3 \qquad \lambda_1 = \lambda_2 \text{ in this case}$$
$$\lambda_2 \dot{\omega}_2 = (\lambda_3 - \lambda_1)\omega_3\omega_1$$
$$\lambda_3 \dot{\omega}_3 = (\lambda_1 - \lambda_2)\omega_1\omega_2$$

Thus $\lambda_1 \dot{\omega}_1 = (\lambda_2 - \lambda_3)\omega_2\omega_3$

$$\lambda_2 \dot{\omega}_2 = (\lambda_3 - \lambda_1)\omega_3\omega_1$$

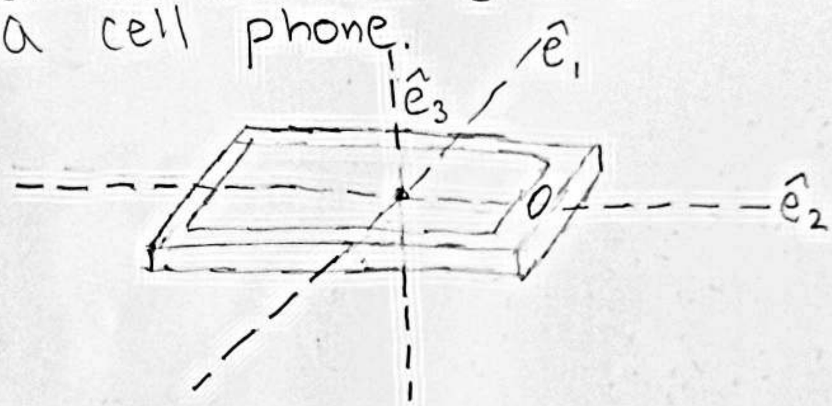
$\lambda_3 \dot{\omega}_3 = 0$ meaning that $\omega_3 = \text{constant}$

we know that $L_3 = \lambda_3 \omega_3 \hat{e}_3$, meaning that the component of $\vec{L}$ along $\hat{e}_3$ will be constant.

c) The angular momentum in the Space frame is completely conserved, fixed. This is because there are no torques in the inertial frame to change the angular momentum.

D)

(B) No component of the angular momentum is conserved in the body frame. This is because we have 3 unique principal moments with a cell phone.



through the CM

These are the only choice of principal axes. Thus

where

$\lambda_1 \neq \lambda_2 \neq \lambda_3$

$$\lambda_1 \dot{\omega}_1 = (\lambda_2 - \lambda_3)\omega_2\omega_3$$
$$\lambda_2 \dot{\omega}_2 = (\lambda_3 - \lambda_1)\omega_3\omega_1$$
$$\lambda_3 \dot{\omega}_3 = (\lambda_1 - \lambda_2)\omega_1\omega_2$$

none of ω_1, ω_2, or ω_3 are constant as seen above, since none of the right sides are zero. Thus each component L_1, L_2, L_3 are not conserved in the body frame. The only way for a component of angular momentum to be conserved is if the cell phone was rotating about one of the principal axes, such as $\hat{e}_3$. As seen above, this means that $\omega_1 = \omega_2 = 0$, so the right sides of all three equations are zero. ω_1 and ω_2 remain zero and ω_3 is constant then.

D)

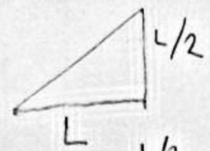
(c) yes, the angular momentum is completely conserved, all components are fixed.

This is because, once again, there are no torques in the inertial frame, so we have nothing to change $\vec{L}$.

(3)

A)

$$I = \begin{bmatrix} J_{yy}+J_{zz} & -J_{xy} & -J_{xz} \\ -J_{xy} & J_{zz}+J_{xx} & -J_{yz} \\ -J_{xz} & -J_{yz} & J_{xx}+J_{yy} \end{bmatrix}$$



$$y = \frac{L/2}{L}x + 0 = \frac{1}{2}x$$

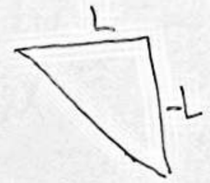
$$J_{xy} = \int xy\rho \, dV \qquad \sigma = \text{const.} = \frac{M}{A} \qquad dA = dx\,dy$$

$$A = \frac{\frac{L}{2}L}{2} = \frac{L^2}{4} + \frac{L \cdot L}{2} = \frac{L^2}{2} \Big\} \frac{3L^2}{4}$$

$$\sigma = \frac{M4}{3L^2}$$

$$J_{xy} = \int xy \frac{M4}{3L^2} dx\,dy$$

$$= \frac{M4}{3L^2}\int_0^L \left[x \cdot \int_{-x}^{1/2 x} y \, dy \right] dx$$



$$y = -x$$

$$= -\frac{3L^4}{32}\left(\frac{M4}{3L^2}\right) = -\frac{L^2 M}{8}$$

$$J_{xx} = \frac{M4}{3L^2}\int_0^L \left[\int_{-x}^{1/2 x} x^2 \, dy \right] dx = \frac{L^2 M}{2}$$

$$J_{yy} = \frac{M4}{3L^2}\int_0^L \left[\int_{-x}^{1/2 x} y^2 \, dy \right] dx = \frac{L^2 M}{8}$$

$$J_{zz} = 0 \qquad J_{yz} = 0$$

$$J_{xz} = 0$$

$$I = \begin{bmatrix} \frac{L^2 M}{8} & \frac{L^2 M}{8} & 0 \\ \frac{L^2 M}{8} & \frac{L^2 M}{2} & 0 \\ 0 & 0 & \frac{5L^2 M}{8} \end{bmatrix}$$

B) $I = \begin{bmatrix} \frac{L^2M}{8} & \frac{L^2M}{8} & 0 \\ \frac{L^2M}{8} & \frac{L^2M}{2} & 0 \\ 0 & 0 & \frac{5L^2M}{8} \end{bmatrix} = \mu \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

where $\mu = \frac{L^2M}{8}$

$\vec{I} - \lambda \vec{1} = \begin{bmatrix} \mu-\lambda & \mu & 0 \\ \mu & 4\mu-\lambda & 0 \\ 0 & 0 & 5\mu-\lambda \end{bmatrix}$

$\det(\vec{I} - \lambda \vec{1}) = (\mu-\lambda)(4\mu-\lambda)(5\mu-\lambda) - \mu(\mu)(5\mu-\lambda) = 0$

$\lambda_1 = 5\mu \quad \lambda_2 = \frac{1}{2}(5\mu - \sqrt{13}\mu) \quad \lambda_3 = \frac{1}{2}(5\mu + \sqrt{13}\mu)$

$\underline{\lambda_1}$

$(\vec{I} - \lambda_1 \vec{1})\vec{\omega} = \mu \begin{bmatrix} 1-5 & 1 & 0 \\ 1 & 4-5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = 0$

$-4\omega_x + \omega_y = 0$ $\qquad \omega_x = \omega_y$

$\omega_x - \omega_y = 0$ $\qquad$ however $4\omega_x = \omega_y$

Thus $\omega_x = \omega_y = 0$

This means that our first principal axis is in the direction $\langle 0, 0, 1 \rangle$ with a moment of inertia $\lambda_1 = 5\mu = \frac{5L^2M}{8}$

$\underline{\lambda_2}$

$$(\vec{I} - \lambda_2 \vec{1})\vec{\omega} = M \begin{bmatrix} 1-\frac{1}{2}(5-\sqrt{13}) & 1 & 0 \\ 1 & 4-\frac{1}{2}(5-\sqrt{13}) & 0 \\ 0 & 0 & 5-\frac{1}{2}(5-\sqrt{13}) \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = 0$$

$$\left(1-\tfrac{1}{2}(5-\sqrt{13})\right)\omega_x + \omega_y = 0$$

$$\omega_x + \left(4-\tfrac{1}{2}(5-\sqrt{13})\right)\omega_y = 0$$

$$\left(5-\tfrac{1}{2}(5-\sqrt{13})\right)\omega_z = 0$$

means that $\omega_z = 0$

$\omega_y = -\frac{1}{2}(\sqrt{13}-3)\,\omega_x$ which satisfies

$$\omega_x + \left(-\tfrac{1}{2}(\sqrt{13}-3)\right)\left(4-\tfrac{1}{2}(5-\sqrt{13})\right)\omega_x = 0$$

Thus a second principal axis could be in the direction $\langle 1, -\frac{1}{2}(\sqrt{13}-3), 0\rangle$ with moment of inertia $\lambda_2 = \frac{1}{2}(5M - \sqrt{13}M)$

$$= \frac{1}{16}(5-\sqrt{13})L^2 M$$

$\underline{\lambda_3}$

$$(\vec{I} - \lambda_3 \vec{1})\vec{\omega} = \begin{bmatrix} 1-\frac{1}{2}(5+\sqrt{13}) & 1 & 0 \\ 1 & 4-\frac{1}{2}(5+\sqrt{13}) & 0 \\ 0 & 0 & 5-\frac{1}{2}(5+\sqrt{13}) \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = 0$$

$$\left(1-\tfrac{1}{2}(5+\sqrt{13})\right)\omega_x + \omega_y = 0$$

$$\omega_x + \left(4-\tfrac{1}{2}(5+\sqrt{13})\right)\omega_y = 0$$

$$\left(5-\tfrac{1}{2}(5+\sqrt{13})\right)\omega_z = 0$$

means that $\omega_z = 0$

$\omega_y = \frac{1}{2}(3+\sqrt{13})\,\omega_x$

which satisfies

$$\omega_x + \tfrac{1}{2}(3+\sqrt{13})\left(4-\tfrac{1}{2}(5+\sqrt{13})\right)\omega_x = 0$$

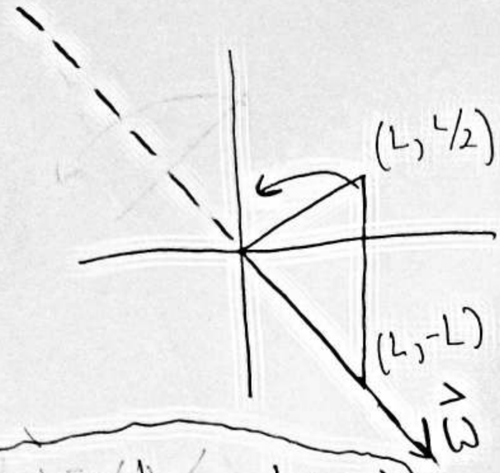
Our third principal axis will be in the direction $\langle 1, \frac{1}{2}(3+\sqrt{13}), 0 \rangle$ with moment of inertia $\lambda_3 = \frac{1}{2}(5\mu + \sqrt{13}\,\mu)$

$$= \frac{1}{16}(5+\sqrt{13})L^2M$$

And now...

$$\vec{I}' = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \frac{5L^2M}{8} & 0 & 0 \\ 0 & \frac{1}{16}(5-\sqrt{13})L^2M & 0 \\ 0 & 0 & \frac{1}{16}(5+\sqrt{13})L^2M \end{bmatrix}$$

C)



Thus

$$\vec{\omega} = \omega \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right\rangle$$

Thus $\vec{\omega} = \omega \hat{\omega}$

$$\hat{\omega} = \frac{\langle 1, -1, 0 \rangle}{|\langle 1, -1, 0 \rangle|}$$

$$= \frac{\langle 1, -1, 0 \rangle}{\sqrt{2}}$$

$$= \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right\rangle$$

D) $\vec{L} = \vec{I}\vec{\omega}$

$$\vec{L} = \begin{bmatrix} \frac{L^3 M}{8} & \frac{L^2 M}{8} & 0 \\ \frac{L^2 M}{8} & \frac{L^2 M}{2} & 0 \\ 0 & 0 & \frac{5L^2 M}{8} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \omega$$

$$\vec{L} = \omega \begin{bmatrix} \frac{L^2 M}{\sqrt{2}\,8} - \frac{L^2 M}{\sqrt{2}\,8} + 0 \\ +\frac{L^2 M}{\sqrt{2}\,8} - \frac{L^2 M}{\sqrt{2}\,2} + 0 \\ 0 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-3L^2 M}{8\sqrt{2}} \\ 0 \end{bmatrix} \omega$$

E) $T = \frac{1}{2}\vec{\omega} \cdot \vec{L}$

$$= \frac{\omega}{2} \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right\rangle \cdot \left\langle 0, \frac{-3L^2 M \omega}{8\sqrt{2}}, 0 \right\rangle$$

$$= \frac{\omega}{2} \left(\frac{1}{\sqrt{2}}\right) \left(\frac{3L^2 M \omega}{8\sqrt{2}}\right) = \boxed{\frac{3}{32} L^2 M \omega^2}$$

(4) A) $T = \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$

$U = \frac{1}{2} K(x_1^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + x_3^2)$

$$\mathcal{L} = \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \frac{1}{2} K(x_1^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + x_3^2)$$

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$$

$$U = \frac{1}{2} K(x_1^2 + x_2^2 - 2x_1x_2 + x_1^2 + x_3^2 - 2x_2x_3 + x_2^2 + x_3^2)$$

$$U = \frac{1}{2} K(\cancel{x_1^2} + \cancel{x_2^2} - x_1x_2 - x_2x_1 + \cancel{x_1^2} + \cancel{x_3^2} - x_2x_3 - x_3x_2 + \cancel{x_2^2} + \cancel{x_3^2})$$

$$U = \frac{1}{2} K(2x_1^2 + 2x_2^2 + 2x_3^2 - x_1x_2 - x_2x_1 - x_2x_3 - x_3x_2)$$

$$K = \begin{bmatrix} 2K & -K & 0 \\ -K & 2K & -K \\ 0 & -K & 2K \end{bmatrix}$$

thus
$$\boxed{\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -2K & K & 0 \\ K & -2K & K \\ 0 & K & -2K \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}}$$

B) $\det(\underbrace{K-\omega^2 M}) = 0 \longrightarrow (2K-\omega^2 m)\left[(2K-\omega^2 m)^2 - K^2\right] - (+K)\left[(-K)(2K-\omega^2 m)\right] = 0$

$$\begin{bmatrix} 2K-\omega^2 m & -K & 0 \\ -K & 2K-\omega^2 m & -K \\ 0 & -K & 2K-\omega^2 m \end{bmatrix}$$

$= 4K^3 - 10K^2 m\omega^2 + 6Km^2\omega^4 - m^3\omega^6 = 0$

$\longrightarrow \omega_1 = \sqrt{\frac{(2-\sqrt{2})K}{m}} \quad \omega_2 = \sqrt{\frac{2K}{m}} \quad \omega_3 = \sqrt{\frac{(2+\sqrt{2})K}{m}}$

c) For ω_1

$$(K-\omega_1^2 M)\vec{a} = 0 = \begin{bmatrix} 2K - \frac{(2-\sqrt{2})K}{m}m, & -K, & 0 \\ -K, & 2K - \frac{(2-\sqrt{2})K}{m}m, & -K \\ 0, & -K, & 2K - \frac{(2-\sqrt{2})K}{m}m \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$= K \begin{bmatrix} 2-2+\sqrt{2} & -1 & 0 \\ -1 & 2-2+\sqrt{2} & -1 \\ 0 & -1 & 2-2+\sqrt{2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$$

$+\sqrt{2}a_1 - a_2 = 0$

$-a_1 + \sqrt{2}a_2 - a_3 = 0$

$-a_2 + \sqrt{2}a_3 = 0$

$a_1 = a_3$

$a_2 = \sqrt{2}a_1$

$\sqrt{2}a_1 = +a_2$

$\sqrt{2}a_1 = \sqrt{2}a_3$

$a_1 = a_3$

Thus we have $\vec{a} = \begin{bmatrix} A \\ \sqrt{2}A \\ A \end{bmatrix} e^{-i\delta}$

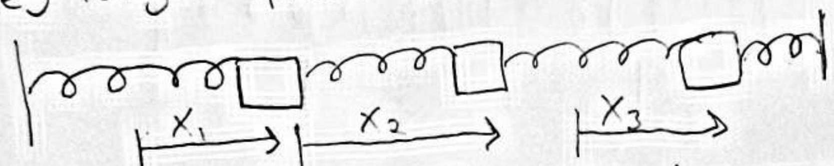
c) continued...

Thus $x_1 = A\cos(\omega_1 t - \delta)$ where $\omega_1 = \sqrt{\dfrac{(2-\sqrt{2})k}{m}}$

$X_2 = \sqrt{2} A\cos(\omega_1 t - \delta)$

$X_3 = A\cos(\omega_1 t - \delta)$

In this first normal mode all blocks are in phase with one another. The middle block in the system simply has an amplitude that is $\sqrt{2}$ times larger than the other two blocks.



All oscillate in phase, middle larger amp

For ω_2

$$(K - \omega_2^2 M)\vec{a} = 0 = \begin{bmatrix} 2k - \frac{2k}{m}m & -k & 0 \\ -k & 2k - \frac{2k}{m}m & -k \\ 0 & -k & 2k - \frac{2k}{m}m \end{bmatrix} \vec{a}$$

$$= k\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$$

$-a_2 = 0$ $\qquad a_2 = 0$

$-a_1 - a_3 = 0$ $\qquad -a_1 = a_3$

$-a_2 = 0$

Thus we have $\vec{a} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-i\delta}$

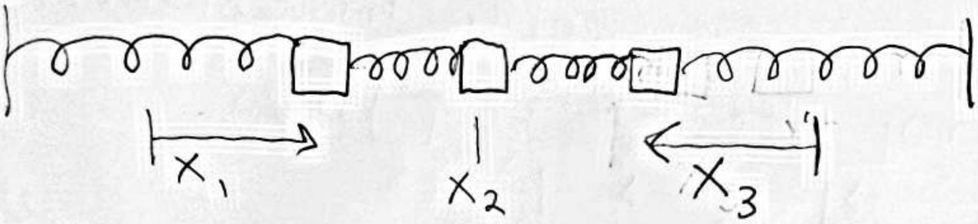
c) continued...

Thus $x_1 = -A\cos(\omega_2 t - \delta)$ where

$x_2 = 0$ $\quad \omega_2 = \sqrt{\frac{2K}{m}}$

$x_3 = A\cos(\omega_2 t - \delta)$

Here, x_2 never changes. The two outside blocks oscillate with the same amplitude but exactly out of phase...



outside blocks out of phase, middle block remains at rest

For ω_3

$$(K - \omega_3^2 M)\vec{a} = 0 = \begin{bmatrix} 2K - \frac{(2+\sqrt{2})K}{m} m & -K & 0 \\ -K & 2K - \frac{(2+\sqrt{2})K}{m} m & -K \\ 0 & -K & 2K - \frac{(2+\sqrt{2})}{m} Km \end{bmatrix} \vec{a}$$

$$= K \begin{bmatrix} 2-2+\sqrt{2} & -1 & 0 \\ -1 & 2-2-\sqrt{2} & -1 \\ 0 & -1 & 2-2-\sqrt{2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$$

$-\sqrt{2}a_1 - a_2 = 0$ $\quad a_2 = -\sqrt{2}a_1$

$-a_1 - \sqrt{2}a_2 - a_3 = 0$

$-a_2 - \sqrt{2}a_3 = 0$

$\sqrt{2}a_1 = -a_2$

$\sqrt{2}a_1 = \sqrt{2}a_3$

$a_1 = a_3$

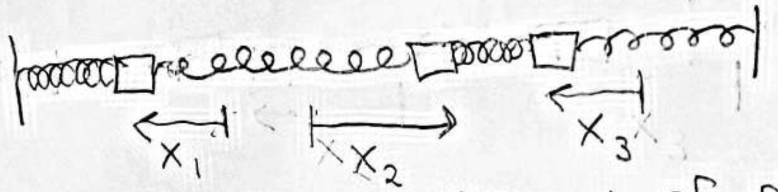
Thus we have $\vec{a} = \begin{bmatrix} A \\ -\sqrt{2}A \\ A \end{bmatrix} e^{-i\delta}$

c) continued...

Thus we have $x_1 = A\cos(\omega_3 t - \delta)$ where $\omega_3 = \sqrt{\frac{(2+\sqrt{2})k}{m}}$

$$x_2 = -\sqrt{2}A\cos(\omega_3 t - \delta)$$
$$x_3 = A\cos(\omega_3 t - \delta)$$

For the third normal mode, The two outer blocks Oscillate in phase with the same amplitude. The middle block oscillates exactly out of phase to the outer blocks, and w/ an amplitude $\sqrt{2}$ times larger.



The two outer are in phase, middle out of phase, also larger amp for middle.

☆ I used Eigensystem$[M^{-1}K]$ to check my work