HW 19

① position of pendulum $\longrightarrow$ $x_2 = x + \ell \sin\phi$

$y_2 = \ell - \ell\cos\phi$

A) $KE_p = \frac{1}{2}m(\dot{x} + \ell\cos\phi\dot{\phi})^2 + \frac{1}{2}m(\ell\sin\phi\dot{\phi})^2$

$= \frac{1}{2}m[\dot{x}^2 + 2\ell\dot{x}\dot{\phi}\cos\phi + \ell^2\cos^2\phi\dot{\phi}^2] + \frac{1}{2}m\ell^2\sin^2\phi\dot{\phi}^2$

$KE_p = \frac{1}{2}m[\dot{x}^2 + 2\ell\dot{x}\dot{\phi}\cos\phi + \ell^2\dot{\phi}^2]$ since $\cos^2\phi + \sin^2\phi = 1$

$KE_s = \frac{1}{2}M\dot{x}^2$

$PE_p = mg\ell(1-\cos\phi)$

$PE_s = \frac{1}{2}kx^2$

$$\boxed{\mathcal{L} = \frac{1}{2}m[\dot{x}^2 + 2\ell\dot{x}\dot{\phi}\cos\phi + \ell^2\dot{\phi}^2] + \frac{1}{2}M\dot{x}^2 - \frac{1}{2}kx^2 - mg\ell(1-\cos\phi)}$$

B) if $k=0$, x will be ignorable. This means that the generalized momentum corresponding to x is conserved, or, in this problem, that the momentum in the x direction of the system is conserved. If we were to displace the bob of the pendulum, the motion of the mass would be opposite the pendulum's swing in order to cancel x direction momentum, leaving the system with net 0 momentum in this direction like it started with at rest.

C)

c) $\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \qquad \frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$

$$\frac{\partial}{\partial \phi}\mathcal{L} = \frac{\partial}{\partial \phi}\left[ml\dot{x}\dot{\phi}\cos\phi + mgl\cos\phi\right] = -ml\dot{x}\dot{\phi}\sin\phi - mgl\sin\phi$$

$$\frac{\partial}{\partial \dot{\phi}}\mathcal{L} = \frac{\partial}{\partial \dot{\phi}}\left[ml\dot{x}\dot{\phi}\cos\phi + \tfrac{1}{2}ml^2\dot{\phi}^2\right] = ml\dot{x}\cos\phi + l^2\dot{\phi}m$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{d}{dt}\left[ml\dot{x}\cos\phi + l^2\dot{\phi}\right] = -ml\dot{x}\dot{\phi}\sin\phi + ml\ddot{x}\cos\phi + l^2\ddot{\phi}m$$

thus $\cancel{-ml\dot{x}\dot{\phi}\sin\phi} - \cancel{m}gl\sin\phi = \cancel{-ml\dot{x}\dot{\phi}\sin\phi} + \cancel{m}l\ddot{x}\cos\phi + l^2\ddot{\phi}\cancel{m}$

$$-g\cancel{l}\sin\phi = \cancel{l}\ddot{x}\cos\phi + l^{\cancel{2}}\ddot{\phi}$$

$$-g\sin\phi - \ddot{x}\cos\phi - l\ddot{\phi} = 0$$

$$\boxed{\frac{g}{l}\sin\phi + \frac{\ddot{x}}{l}\cos\phi + \ddot{\phi} = 0} \quad \checkmark$$

$$\frac{\partial}{\partial x}\mathcal{L} = \frac{\partial}{\partial x}\left[-\tfrac{1}{2}kx^2\right] = -kx$$

$$\frac{\partial}{\partial \dot{x}}\mathcal{L} = \frac{\partial}{\partial \dot{x}}\left[\tfrac{1}{2}m\dot{x}^2 + ml\dot{x}\dot{\phi}\cos\phi + \tfrac{1}{2}M\dot{x}^2\right] = m\dot{x} + ml\dot{\phi}\cos\phi + M\dot{x}$$

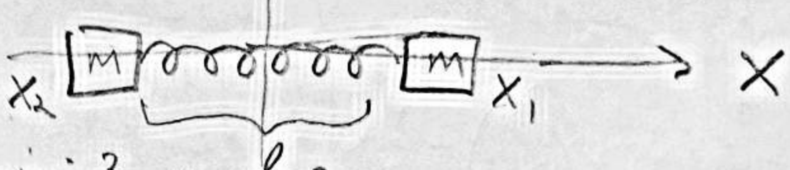
$$\frac{d}{dt}\left[m\dot{x} + ml\dot{\phi}\cos\phi + M\dot{x}\right] = m\ddot{x} - ml\dot{\phi}^2\sin\phi + ml\ddot{\phi}\cos\phi + M\ddot{x}$$

$$-kx = m\ddot{x} - ml\dot{\phi}^2\sin\phi + ml\ddot{\phi}\cos\phi + M\ddot{x}$$

$$ml\left(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi\right) = -kx \underbrace{- m\ddot{x} - M\ddot{x}}_{-\ddot{x}(M+m)}$$

$$\boxed{\text{thus} \quad \ddot{x} + \frac{ml}{m+M}\left(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi\right) + \frac{kx}{M+m} = 0} \quad \checkmark$$

7.8 $m_1 = m_2 = m$



a) $KE_{masses} = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$

$PE_{spring} = \frac{1}{2} k (x_1 - x_2 - \ell)^2$

$$\boxed{\mathcal{L} = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - \frac{1}{2} k (x_1 - x_2 - \ell)^2}$$

see next page for parts b and C

b) $X = \frac{1}{2}(x_1 + x_2)$ $\quad \mathcal{L} = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2)$

$\dot{X} = \frac{1}{2}(\dot{x}_1 + \dot{x}_2)$

$\dot{X}^2 = \frac{1}{4}[\dot{x}_1^2 + 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2]$

$-\frac{1}{2}\dot{x}_1\dot{x}_2 + \dot{X}^2 = \frac{1}{4}[\dot{x}_1^2 + \dot{x}_2^2]$

$2X - x_2 = x_1$

$(2\dot{X} - \dot{x}_2)^2 = 4\dot{X}^2 - 4\dot{X}\dot{x}_2 + \dot{x}_2^2$

$(2\dot{X} - \dot{x}_1)^2 = 4\dot{X}^2 - 4\dot{X}\dot{x}_1 + \dot{x}_1^2$

$KE_{masses} = \frac{1}{2} m \dot{X}^2$ $\quad \mathcal{L} = \frac{1}{2} m \dot{X}^2 - \frac{1}{2} k X^2$

$PE_{spring} = \frac{1}{2} k (X)^2$

$\frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{X}} \rightarrow 0 = \frac{d}{dt}[m\dot{X}]$

$\boxed{0 = m\ddot{X}}$ no change in momentum along axis!

$\frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{X}}\right]$

$\boxed{-kX = 0}$

b) $x_1 = X + \frac{1}{2}x + \frac{1}{2}l$ $\quad x_2 = X - \frac{1}{2}x - \frac{1}{2}l$ $\quad \dot{x}_1 = \dot{X} + \frac{1}{2}\dot{x}$ $\quad \dot{x}_2 = \dot{X} - \frac{1}{2}\dot{x}$

$$\mathcal{L} = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k(x_1 - x_2 - l)^2$$

becomes

$$\mathcal{L} = \frac{1}{2}m\left(2\dot{X}^2 + \frac{1}{2}\dot{x}^2\right) - \frac{1}{2}kx^2$$

$$\frac{\partial \mathcal{L}}{\partial X} = 0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} \longrightarrow \frac{d}{dt}\left[2m\dot{X}\right] = 2m\ddot{X} = 0$$

$2m\ddot{X} = 0 \rightarrow 2m = M \rightarrow \boxed{M\ddot{X} = 0}$ big X

no change in momentum along axis

$F = \dot{p}$

$$\frac{\partial \mathcal{L}}{\partial x} = -kx \qquad \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{1}{2}m\dot{x}$$

$$\frac{d}{dt}\left[\frac{1}{2}m\dot{x}\right] = \frac{1}{2}m\ddot{x}$$

so $\boxed{\frac{1}{2}m\ddot{x} = -kx}$ little x

$\ddot{x} = -\frac{2k}{m}x$ $\qquad r^2 = -\frac{2k}{m}$

$r = \pm i\sqrt{2}\,\omega$

c)

$M\ddot{X} = 0$

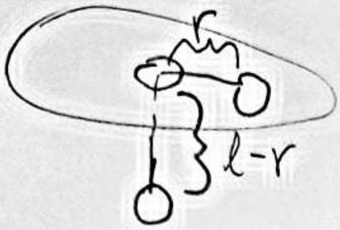
so $\boxed{X = C_3 t + C_4}$ big X

thus $\boxed{x = C_1\cos(\sqrt{2}\,\omega t) + C_2\sin(\sqrt{2}\,\omega t)}$ little x

From these results we know that the center of mass will move at a constant velocity along the axis. We also see that the length of the spring takes on Simple Harmonic motion.

7.37

a)



$T = \frac{1}{2}m\dot{r}^2\phi^2 + \frac{1}{2}m\dot{\phi}^2 r^2 + \frac{1}{2}m\dot{r}^2$

position of table mass ϕr

$\frac{d}{dt}[\phi r] = \dot{\phi} r + \dot{r}\phi$

$PE = mgr$

$$\boxed{\mathcal{L} = \frac{1}{2}m\dot{r}^2\phi^2 + \frac{1}{2}m\dot{\phi}^2 r^2 + \frac{1}{2}m\dot{r}^2 - mgr}$$

b) $\frac{\partial \mathcal{L}}{\partial \phi} = m\dot{r}^2\phi$ $\quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi}$ $\quad \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right] = m2r\dot{r}\dot{\phi} + mr^2\ddot{\phi}$

$$\boxed{m\dot{r}^2\phi = m2r\dot{r}\dot{\phi} + mr^2\ddot{\phi}}$$

$\frac{\partial \mathcal{L}}{\partial r} = m\dot{\phi}^2 r - mg$ $\quad \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}\phi^2 + m\dot{r}$

$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{r}}\right] = m\ddot{r}\phi^2 + 2m\dot{r}\phi\dot{\phi} + m\ddot{r}$

$$m\dot{\phi}^2 r - mg = m\ddot{r}\phi^2 + 2m\dot{r}\phi\dot{\phi} + m\ddot{r}$$

c) $\dot{\phi} = \frac{v}{r}$ $\quad r^2\dot{\phi} = vr$ $\quad \ell = mvr$ $\quad \ell = mr^2\dot{\phi}$ $\quad \dot{\phi} = \frac{\ell}{mr^2}$

$$\boxed{m\frac{\ell^3}{m^2r^3} - mg = m\ddot{r}\phi^2 + 2m\dot{r}\phi\frac{\ell}{mr^2} + m\ddot{r}}$$

$\dot{r} = \text{const.}$ $\quad \ddot{r} = 0$ $\quad m\frac{\ell^2}{m^2r^3} - mg = 2m\phi\frac{\ell}{mr^2}$

$2m\phi\frac{\ell r^2}{m} + mgr^4 = \frac{m\ell^2}{m^2}$

r