Some Characterizations of TTC in Multiple-Object Reallocation Problems*

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Abstract

This paper considers reallocation of indivisible objects when agents are endowed with and can consume any bundles. We obtain characterizations of generalized versions of the *Top Trading Cycles (TTC) rule* on several preference domains. On the lexicographic domain, the TTC rule is uniquely determined by *balancedness*, *Pareto efficiency*, the *worst endowment lower bound*, and either *truncation-proofness* or *drop strategy-proofness*. On the more general responsive domain, the TTC rule is the unique *individual-good-based* rule that satisfies *balancedness*, *individual-good efficiency*, *truncation-proofness*, and either *individual rationality* or the *worst endowment lower bound*. On the conditionally lexicographic domain, the augmented TTC rule is characterized by *balancedness*, *Pareto efficiency*, the *worst endowment lower bound*, and *drop strategy-proofness*. The conditionally lexicographic domain is a maximal domain on which *Pareto efficiency* coincides with *individual-good efficiency*. For the housing market introduced by Shapley and Scarf (1974), the TTC rule is characterized by *Pareto efficiency*, *individual rationality*, and *truncation-proofness*.

Keywords: exchange of indivisible objects; Top Trading Cycles; heuristic manipulation; endowment lower bound.

JEL Classification: C78; D47; D71.

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1 Introduction

We consider *multi-object reallocation problems* without monetary transfers. These problems involve a group of *agents*, each endowed with a set of heterogeneous and indivisible *objects* and equipped with strict preferences over *bundles* of objects. An allocation *rule* specifies how objects are reassigned based on the agents' reported preferences. A special case of this framework is when each agent is endowed with a single object, leading to the standard *single-object reallocation problem* (often called a *housing market*) introduced by Shapley and Scarf (1974).

Object reallocation problems are ubiquitous. Firms plan shift schedules months in advance, and employees frequently exchange their assigned shifts with one another (Manjunath and Westkamp (2021)). Universities exchange students in programs such as The Tuition Exchange Program in the United States and the Erasmus Student Exchange Program in Europe (Dur and Ünver (2019), Bloch et al. (2020)), and seats in oversubscribed courses are (re)allocated among university students (Bichler et al. (2021), Budish (2011), Budish and Cantillon (2012)). Living-donor organ exchange programs, which facilitate the reallocation of organs (e.g., kidneys and livers) among patient-donor pairs, are an important example featuring single-object exchange (Roth et al. (2004, 2005), Ergin et al. (2020)).

In contrast with single-object reallocation, which admits many positive results,[1] multi-object reallocation presents significant challenges for both practitioners and theorists. One practical challenge is the vast number of feasible bundles, which makes it difficult or impossible for agents to accurately report (or even know) their preferences over bundles.[2] As Roth (2015, p. 331) explains, "a practical mechanism must simplify the language in which preferences can be reported, and by doing so it will restrict which preferences can be reported." A prominent example is the National Resident Matching Program (NRMP), which matches doctors to hospitals in the United States. Each hospital reports only its rank-order list over individual doctors, even though it may have rather complex preferences over sets of doctors (see, e.g., Roth and Peranson (1999), Milgrom (2009, 2011)).

Motivated by these considerations, we focus on *individual-good-based* rules—allocation rules with a simple reporting language that consists of rank-order lists over individual objects. *Individual-good-based* rules are particularly appealing in environments where objects are substitutable. For instance, if agents have lexicographic preferences,[3] then *individual-good-based* rules are without loss of generality since an agent's preferences can be succinctly represented

[1] A central result pertinent to the present study is that Gale's Top Trading Cycles rule (Shapley and Scarf, 1974) is the unique rule satisfying *Pareto efficiency*, *individual rationality*, and *strategy-proofness* (Ma, 1994).

[2] For instance, in a shift reallocation problem with only 20 shifts, there are $\binom{20}{5} = 15,504$ bundles composed of five shifts.

[3] Preferences are *lexicographic* if the relative ranking between two bundles depends primarily on the most-preferred objects in each of them.

by a rank-order list over individual objects. More generally, when agents have responsive preferences,[4] an agent's preferences over individual objects capture much—though not all—of the relevant information about her preferences over bundles of objects.[5]

A significant theoretical challenge is the inherent conflict among three criteria of interest: efficiency, individual rationality, and strategic robustness. This conflict manifests in various impossibility results. For example, Sönmez (1999) shows that the ideal properties—*Pareto efficiency*, *individual rationality*, and *strategy-proofness*—are incompatible whenever at least one agent is endowed with more than one object. Much of the literature sidesteps this issue by focusing on the *strategy-proof* rules that fulfill only one of the remaining two ideals. Loosely speaking, the only *strategy-proof* allocation rules satisfying *Pareto efficiency* are dictatorial (Pápai (2001), Klaus and Miyagawa (2002), Ehlers and Klaus (2003), Hatfield (2009)), while the only *strategy-proof* rules satisfying *individual rationality* are *Segmented Trading Cycles rules* (Pápai (2003); see also Pápai (2007)). Such rules are not suitable for our purposes, as they may severely compromise the third criterion.

In this paper, we circumvent the incompatibility by making small compromises relative to the ideal notions of efficiency and strategic robustness. We study the *generalized Top Trading Cycles (TTC) rule*, an *individual-good-based* rule that extends Gale's TTC rule (Shapley and Scarf, 1974) to multi-object problems. Despite the challenges posed by the incompatibility of our ideal desiderata, we show that the TTC rule performs remarkably well according to all three criteria. In particular, we provide axiomatic characterizations of the TTC rule based on *individual-good efficiency*, *individual rationality*, and *truncation-proofness*.

On the lexicographic domain, the TTC rule is uniquely determined by *balancedness*, *Pareto efficiency*, the *worst endowment lower bound*, and either *truncation-proofness* (Theorem 1) or *drop strategy-proofness* (Theorem 2). Extending to the responsive preference domain and focusing on *individual-good-based* rules, we find that although full *Pareto efficiency* is unattainable, the TTC rule uniquely satisfies *balancedness*, *individual-good efficiency*, *truncation-proofness*, and either the *weak endowment lower bound* (Theorem 3) or *individual rationality* (Theorem 4). In the special case of the housing market, where each agent is endowed with a single object, we characterize the TTC rule through *Pareto efficiency*, *individual rationality*, and *truncation-proofness* (Theorem 5), generalizing the classic result of Ma (1994). Our key properties are discussed in detail below.

Balancedness posits that a rule assigns the same number of objects to each agent as the initial

[4] Preferences are *responsive* if the relative ranking between two objects does not depend on the other objects they are obtained with.

[5] An obvious limitation of *individual-good-based* rules is that they do not allow agents to express information about complex preferences exhibiting complementarities, as discussed in Budish and Kessler (2022). Recognizing this limitation, we explore an alternative reporting language in Section 6 that accommodates complementarities.

allocation. *Balancedness* is an inviolable constraint in many practical reallocation problems. In a shift reallocation problem, for example, *balancedness* may be imposed so that staff can meet mandatory training requirements (Manjunath and Westkamp (2021)). It is also a typical requirement in applications such as student exchange programs (Dur and Ünver (2019)) and time banks (Andersson et al. (2021), Biró et al. (2022a)). In the absence of strict constraints, *balancedness* captures a limited notion of fairness.

Individual rationality and the *worst endowment lower bound* are participation guarantees that differ in the promises they make to the agents. *Individual rationality* guarantees that each agent enjoys a bundle at least as good as her endowment, whereas the *worst endowment lower bound* ensures that no agent ever receives an object worse than the least-preferred object in her endowment. While the two properties coincide in single-object problems, they are distinct in multi-object problems. Intuitively, the *worst endowment lower bound* allows agents to veto individual objects by ranking them below the worst object in their endowment.

Individual-good efficiency requires that no group of agents can destabilize the allocation by carrying out a simple exchange where each involved agent trades a single object for a better one. Under lexicographic preferences, *individual-good efficiency* coincides with *Pareto efficiency*. However, in the more general setting of responsive preferences, *Pareto efficiency* becomes difficult to achieve and verify in practice. Specifically, verifying full *Pareto efficiency* requires considering an exponentially large number of possible reallocations, making it computationally impractical as the number of agents and objects grows (e.g., De Keijzer et al. (2009), Aziz et al. (2019)). Adding to this challenge, no *individual-good-based* rule simultaneously satisfies *Pareto efficiency* and *individual rationality* under responsive preferences (Manjunath and Westkamp (2024)). In contrast, *individual-good efficiency* is more readily attainable and can be verified relatively easily.[6] This practical advantage makes *individual-good efficiency* a more realistic goal in settings with complex preferences and limited computational resources.

Truncation-proofness is an incentive compatibility requirement that prevents agents from benefiting by misrepresenting their preferences through truncation strategies. In our multi-object setting, a truncation strategy involves an agent dropping all objects owned by other agents and ranked below a certain cutoff object to the bottom of her rank-order list, possibly to avoid being assigned those objects. Truncation strategies are straightforward for agents to implement, as they only need to identify the cutoff point in their preference ordering. These strategies have been extensively studied in the matching literature due to their simplicity and theoretical appeal (Roth and Vande Vate (1991), Roth and Rothblum (1999), Ehlers (2008), Kojima (2013), Coles and Shorrer (2014)). Empirical evidence shows that agents often employ

[6]More precisely, *individual-good efficiency* can be verified in polynomial time, whereas verifying *Pareto efficiency* is *co-NP complete* (e.g., De Keijzer et al. (2009), Aziz et al. (2019)).

such strategies in real-world matching markets (Mongell and Roth (1991)) and object allocation problems (Guillen and Hakimov (2018)).

In addition to *individual-good-based* rules, we explore allocation rules that utilize a richer reporting language capable of expressing complementarities among objects. Specifically, we consider rules defined on the domain of conditionally lexicographic preferences.[7] Despite their added flexibility, these preferences can be described succinctly using "lexicographic preference trees." Furthermore, this domain preserves the desirable equivalence between *individual-good efficiency* and *Pareto efficiency* found under lexicographic preferences.

The *Augmented Top Trading Cycles (ATTC) rule*, introduced by Fujita et al. (2018), extends the TTC rule to the conditionally lexicographic domain. We show that the ATTC rule satisfies generalized versions of our key properties and provide a characterization based on *balancedness*, *Pareto efficiency*, the *worst endowment lower bound*, and *drop strategy-proofness* (Theorem 6). Finally, we demonstrate that the conditionally lexicographic domain is maximal in the sense that *Pareto efficiency* and *individual-good efficiency* coincide only on this domain, highlighting the difficulty in extending our results to broader preference domains.

The remainder of this paper is organized as follows. The next subsection presents an overview of the related literature. In Section 2, we introduce the model details, including the lexicographic and responsive preference domains, key properties of allocation rules, and a formal description of the generalized TTC rule. Our results for the lexicographic domain are presented in Section 3. Section 4 extends our analysis to the more general responsive domain. Our findings for the housing market are discussed in Section 5. In Section 6, we introduce the domain of conditionally lexicographic preferences, discuss the extension of our properties to this domain, and we give a characterization of the ATTC rule. Section 7 concludes. Appendix A contains all proofs omitted from the main text, and we demonstrate the independence of our properties in Appendix B.1. Several other examples are provided in Appendix B.2.

1.1 Related literature

The seminal work of Shapley and Scarf (1974) introduces the *housing market* and the famous TTC rule, attributed to David Gale. They establish that the TTC rule always selects a core allocation. Roth (1982) demonstrates that the TTC rule is *strategy-proof*, and Ma (1994) provides a fundamental characterization, showing that it is the unique rule satisfying *Pareto efficiency*, *individual rationality*, and *strategy-proofness*.

Building upon this foundation, the early literature on multi-object reallocation consists mostly of negative results. Sönmez (1999) shows that no rule satisfies *Pareto efficiency*, *indi-*

[7]Preferences are *conditionally lexicographic* if, for any bundle Y and any nonempty bundle X disjoint from Y, there is an object $o \in X$ that is the "lexicographically best" addition to Y.

vidual rationality, and *strategy-proofness* on the unrestricted preference domain when agents may own multiple objects. Todo et al. (2014) shows that the impossibility holds even on the lexicographic domain, and Konishi et al. (2001) demonstrates that the core can be empty even if agents have additive preferences.

In response to these impossibility results, recent studies have obtained positive findings by restricting to more well-behaved variants of the multi-object reallocation problem. For instance, *Pareto efficiency*, *individual rationality*, and *strategy-proofness* are compatible when agents have "dichotomous" preferences (Andersson et al. (2021)) or "trichotomous" preferences (Manjunath and Westkamp (2021)). Similarly, the ATTC rule always selects an allocation within the core when agents have conditionally lexicographic preferences (Fujita et al. (2018)).

In a related study, Biró et al. (2022a) examine a model where each agent owns multiple copies of a homogeneous, agent-specific object. They characterize the capacity configurations under which *Pareto efficiency*, *individual rationality*, and *strategy-proofness* are compatible. Focusing on *individual-good-based* rules for responsive preferences, they characterize a variant of the TTC rule using *subset drop strategy-proofness*. We build upon their approach when extending our characterization from the lexicographic domain to the responsive domain.

Altuntaş et al. (2023) consider the general multi-object reallocation problem but focus on the lexicographic domain. They show that the TTC rule is *drop strategy-proof*, providing a characterization based on this property. Our Theorem 2 generalizes their result by showing that uniqueness holds under substantially weaker criteria. Unlike the proof in Altuntaş et al. (2023), which is constructive, we proceed by minimal counterexample, borrowing the notions of "size" from Sethuraman (2016) and "similarity" from Ekici (2024). The novelty of our approach lies in simultaneously exploiting both functions to select a profile satisfying a different minimality criterion.

Several papers illustrate the tightness of the incompatibility among *Pareto efficiency*, *individual rationality*, and *strategy-proofness* by characterizing rules that satisfy only two of the three objectives. When agents consume more than one object, combining *Pareto efficiency* and *strategy-proofness* with *non-bossiness* leads to *sequential dictatorships* (Pápai (2001), Ehlers and Klaus (2003), Hatfield (2009), Monte and Tumennasan (2015)). Other studies prioritize *individual rationality* and *strategy-proofness* at the expense of *Pareto efficiency*. For example, on the domain of responsive preferences, the *Segmented Trading Cycles rules* are characterized by *strategy-proofness*, *non-bossiness*, *trade sovereignty*, and *strong individual rationality* (Pápai (2003); see also Pápai (2007), Anno and Kurino (2016)).

In our paper, we resolve the incompatibility by weakening *strategy-proofness* to *truncation-proofness* or *drop strategy-proofness*. This approach is partly justified by several papers emphasizing the difficulty of manipulating the TTC rule (e.g., Fujita et al. (2018), Phan and Purcell

(2022)). There are many other relaxed forms of *strategy-proofness*, such as *rank monotonicity* (Chen and Zhao (2021)), *partial strategy-proofness* (Mennle and Seuken (2021)), *truncation-invariance* (Chen et al. (2024)), *weak truncation robustness* (Hashimoto et al. (2014)), and *convex strategy-proofness* (Balbuzanov (2016)).

Our notion of truncation strategies has its roots in the literature on matching theory (Mongell and Roth (1991), Roth and Vande Vate (1991), Roth and Rothblum (1999), Ehlers (2008)), though the version we consider aligns more closely with the definitions provided by Kojima (2013) and Biró et al. (2022a,b) for models of multi-object (re)allocation.

As we move beyond *individual-good-based* rules and consider more complex reporting languages, *(approximate) competitive equilibrium* becomes an appropriate solution concept. In this approach, agents trade through a pseudo-market procedure, as first proposed by Hylland and Zeckhauser (1979) for the house allocation model. This method has developed significantly in recent years (e.g., Echenique et al. (2021, 2023), Nguyen et al. (2021), Kornbluth and Kushnir (2023), Nguyen and Vohra (2024)). Although this approach is effective at approximating desirable outcomes, it suffers from onerous reporting requirements and considerable computational complexity. Our study of the conditionally lexicographic domain provides a practical alternative that balances expressiveness with tractability.

2 Model

Let $N = \{1, 2, \ldots, n\}$ be a finite set of $n \geq 2$ *agents*. Let O be a finite set of heterogeneous and indivisible *objects* such that $|O| \geq n$. A *bundle* is a subset of O. Let 2^O denote the set of bundles. We denote generic elements of O by lowercase letters (e.g., x, y, z), and generic elements of 2^O by uppercase letters (e.g., X, Y, Z). To simplify notation, when there is no risk of confusion, we identify a singleton set $\{x\}$ with the element x itself. For example, we write $X \cup x$ to denote $X \cup \{x\}$.

An *allocation* is a function $\mu : N \to 2^O$ such that (i) for all $i \in N$, $\mu(i) \neq \emptyset$, (ii) for all $i, j \in N$, $i \neq j$ implies $\mu(i) \cap \mu(j) = \emptyset$, and (iii) $\bigcup_{i \in N} \mu(i) = O$. Thus, an allocation μ can be represented as a profile $(\mu_i)_{i \in N}$ of nonempty, pairwise disjoint bundles satisfying $\bigcup_{i \in N} \mu_i = O$. For each $i \in N$, μ_i is called agent i's *assignment* at μ. Let $\mathcal{A}$ denote the set of allocations.

The *initial allocation*, also referred to as the *endowment allocation*, is denoted by $\omega = (\omega_i)_{i \in N}$. For each $i \in N$, ω_i is called agent i's *endowment*. The initial *owner* of object o, denoted $\omega^{-1}(o)$, is the agent i such that $o \in \omega_i$.

Each agent i has a *(strict) preference* relation P_i on the set of bundles. We assume that P_i belongs to some specified subset $\mathcal{P}_i$ of all strict preference relations on 2^O. In subsequent sections, we impose further structure on the sets $\mathcal{P}_i$. If agent i prefers bundle X to bundle Y, then we write $X \; P_i \; Y$. Let R_i denote the *at least as good as* relation associated with P_i, defined by $X \; R_i \; Y$ if and only if ($X \; P_i \; Y$ or $X = Y$).[8] Given a nonempty bundle $X \in 2^O$, $\max_{P_i}(X)$ denotes the most-preferred object in X at P_i, i.e., $\max_{P_i}(X) = x$ if $x \in X$ and $x \; R_i \; y$ for all $y \in X$. Similarly, $\min_{P_i}(X)$ denotes the least-preferred object in X at P_i, i.e., $\min_{P_i}(X) = x$ if $x \in X$ and $y \; R_i \; x$ for all $y \in X$. A *preference profile* is an indexed family $P = (P_i)_{i \in N}$ of preference relations. The *domain* is the set $\mathcal{P} := \prod_{i \in N} \mathcal{P}_i$, representing all possible preference profiles under consideration.

An object reallocation problem (or simply a *problem*) is a triple (N, ω, P). Since (N, ω) remains fixed throughout, we will identify a problem with its preference profile P. Thus, the domain $\mathcal{P} = \prod_{i \in N} \mathcal{P}_i$ of preference profiles represents the set of all problems.

A *rule (on $\mathcal{P}$)* is a function $\varphi : \mathcal{P} \to \mathcal{A}$ that associates with each preference profile P an allocation $\varphi(P)$. For each $i \in N$, $\varphi_i(P)$ denotes agent i's assignment at $\varphi(P)$.

Individual-good-based rules

In most of this paper, we focus on rules that can be implemented with a simple reporting language consisting of linear orders over individual objects. Such rules are often desired in practice because their relatively modest informational requirements streamline their implementation and reduce the burden on participants. A landmark example is the NRMP, which matches doctors to hospitals across the United States and has become a model for market-design interventions worldwide (e.g., Roth and Peranson (1999), Milgrom (2009, 2011)).[9]

To formalize this concept, we need some notation. Given a preference relation P_i on 2^O, the induced *marginal preference* relation over individual objects is the strict linear order $\succ^{P_i}$ on O such that, for all $x, y \in O$, $x \succ^{P_i} y$ if and only if $x \; P_i \; y$.[10] For all $x, y \in O$, $x \succeq^{P_i} y$ means that ($x \succ^{P_i} y$ or $x = y$). We shall often represent a marginal preference $\succ^{P_i}$ as an ordered list of objects; for example, $\succ^{P_i}$: $x_1, x_2, \ldots, x_{|O|}$ means that $x_1 \succ^{P_i} x_2 \succ^{P_i} \cdots \succ^{P_i} x_{|O|}$, and $\succ^{P_i}$: $x_1, x_2, \ldots, x_k, \ldots$ means that $x_1 \succ^{P_i} x_2 \succ^{P_i} \cdots \succ^{P_i} x_k \succ^{P_i} o$ for all $o \in O \backslash \{x_1, x_2, \ldots, x_k\}$.

For a preference profile $P = (P_i)_{i \in N}$, the induced *marginal preference profile* over individual objects is the profile $\succ^P = (\succ^{P_i})_{i \in N}$. A rule is *individual-good-based* if it depends solely on the agents' marginal preferences.

Definition 1. A rule φ is *individual-good-based* if

$$\text{for all } P, P' \in \mathcal{P}, \quad \succ^P = \succ^{P'} \implies \varphi(P) = \varphi(P'). \tag{1}$$

[8] Formally, R_i is a linear order (i.e., a *complete*, *transitive*, and *antisymmetric* binary relation) on 2^O , and P_i is the strict (i.e., *irreflexive* and *asymmetric*) part of R_i.

[9] In the NRMP, each hospital reports only a "rank-order list" over individual doctors together with the number of positions it would like to fill.

[10] In other words, $\succ^{P_i}$ is the restriction of P_i to singleton subsets of O.

Individual-good-based rules are particularly well suited to environments in which the objects are substitutes, as agents' preferences over bundles can be effectively summarized by their marginal preferences over individual objects. Moreover, even in settings where complementarities exist, focusing on *individual-good-based* rules can be practical and effective. As noted by Roth and Peranson (1999) and Roth (2002), the desirable theoretical properties of the NRMP, guaranteed for simple models without complementarities, tend to hold approximately even in more complex real-world markets.

2.1 Preference domains

In the following, we introduce two domains of preferences in which *individual-good-based* rules are especially effective: the lexicographic and responsive preference domains.

An agent has lexicographic preferences if, when evaluating distinct bundles X and Y, she prefers the bundle containing the most-preferred object in $X \cup Y$; if the most-preferred object in $X \cup Y$ is common to X and Y, then she prefers the bundle containing the second-most-preferred object in $X \cup Y$, and so on. This decision-making process reflects heuristics that are typical in human behavior.[11] Formally, a preference relation P_i on 2^O is *lexicographic* if, for any two distinct bundles X and Y,

$$X \; P_i \; Y \iff \max_{P_i} (X \triangle Y) \in X, \tag{2}$$

where $X \triangle Y = (X \backslash Y) \cup (Y \backslash X)$ is the symmetric difference of X and Y. For each $i \in N$, let $\mathcal{L}_i$ be the set of lexicographic preferences on 2^O. Then $\mathcal{L} \coloneqq \prod_{i \in N} \mathcal{L}_i$ is called the *lexicographic domain*.

Although it is rather restrictive, the lexicographic domain is a natural starting point in our analysis because any rule defined on it is automatically *individual-good-based*. This is because each lexicographic preference $P_i \in \mathcal{L}_i$ is uniquely determined by its marginal preference $\succ^{P_i}$ over individual objects; that is, for all $P_i, P_i' \in \mathcal{L}_i$, $\succ^{P_i} = \succ^{P_i'}$ implies $P_i = P_i'$. Consequently, we identify each $P_i \in \mathcal{L}_i$ with its associated marginal preference $\succ^{P_i}$ and write $P_i : x_1, x_2, \ldots, x_{|O|}$ if $x_1 \; P_i \; x_2 \; P_i \cdots P_i \; x_{|O|}$.

We are primarily interested in the domain of responsive preferences, a more general domain first studied by Roth (1985) for many-to-one matching models. An agent has responsive preferences if, for any two bundles that differ in one object, she prefers the bundle containing the more-preferred object. Formally, a preference relation P_i on 2^O is *responsive* if, for any bundle X and any objects $y, z \in O \backslash X$, $y \, P_i \, z$ if and only if $(X \cup y) \, P_i \, (X \cup z)$. Intuitively, responsiveness rules out complementarities, as the relative ranking between any two objects is independent of

[11] The "Take The Best" heuristic, where individuals make choices by considering the most important attribute first and proceeding sequentially, performs surprisingly well in real-world inferential tasks (Gigerenzer and Goldstein (1996); Gigerenzer and Todd (1999)).

the other objects they are obtained with. For each $i \in N$, let $\mathcal{R}_i$ denote the set of responsive preferences on 2^O. Then $\mathcal{R} \coloneqq \prod_{i \in N} \mathcal{R}_i$ is called the *responsive domain*.

Finally, it is convenient to define the domain of monotonic preferences. A preference relation P_i on 2^O is *monotonic* if, for any bundles X and Y, $X \, R_i \, Y$ whenever $X \supseteq Y$. For each $i \in N$, let $\mathcal{M}_i$ be the set of monotonic preferences on 2^O. Then $\mathcal{M} \coloneqq \prod_{i \in N} \mathcal{M}_i$ is called the *monotonic domain*.

Note that every lexicographic preference relation is both responsive and monotonic, but the converse is false whenever $|O| \geq 3$. Moreover, while the sets of responsive and monotonic preferences overlap, neither set is entirely contained within the other. Specifically, for any $i \in N$, we have $\mathcal{L}_i \subseteq \mathcal{R}_i \cap \mathcal{M}_i$ (with strict inclusion for $|O| \geq 3$), $\mathcal{R}_i \nsubseteq \mathcal{M}_i$, and $\mathcal{M}_i \nsubseteq \mathcal{R}_i$.

2.2 Properties of allocation rules

This section introduces several desirable properties of allocation rules. All properties are defined for any arbitrary domain $\mathcal{P}$ in which each agent has strict preferences.

Our first requirement is that exchange be "balanced" in the sense that each agent ends up with the same number of objects as initially endowed. Formally, an allocation μ is *balanced* if, for each agent $i \in N$, the cardinality of her assignment equals that of her endowment: $|\mu_i| = |\omega_i|$.

Definition 2. A rule φ is ***balanced*** if, for each $P \in \mathcal{P}$, $\varphi(P)$ is *balanced*.

Balancedness is often a key consideration—and sometimes a strict constraint—in practical reallocation problems. For example, in shift reallocation, *balancedness* may be desired to prevent overwork or underemployment. It might also be imposed for training purposes or because employment contracts specify a certain number of shifts per week (Manjunath and Westkamp, 2021). Similarly, in student and tuition exchange programs, *balancedness* is desired (at least in the long run) to maintain reciprocal relationships and prevent education costs from increasing at popular schools (Andersson et al. (2021); Biró et al. (2022a); Dur and Ünver (2019)).[12] In the absence of strict constraints, *balancedness* captures a limited notion of equity: it ensures the gains from trade are shared fairly among the agents, at least regarding the number of objects exchanged.

2.2.1 Efficiency

An allocation $\overline{\mu}$ *Pareto-dominates* another allocation μ at a preference profile P if (i) for all $i \in N$, $\overline{\mu}_i \, R_i \, \mu_i$, and (ii) for some $i \in N$, $\overline{\mu}_i \, P_i \, \mu_i$. An allocation μ is *Pareto efficient* at P if it is

[12] Indeed, Dur and Ünver (2019) document several cases in which such long-run imbalances led to the failure of an exchange program.

not Pareto-dominated at P by any other allocation. The strongest efficiency property that we consider is the following.

Definition 3. A rule φ is ***Pareto efficient*** if, for each $P \in \mathcal{P}$, $\varphi(P)$ is *Pareto efficient* at P.

If an allocation is not *Pareto efficient*, then a group of agents could, in principle, destabilize it by carrying out a Pareto-improving exchange. However, such exchanges are generally complex and difficult to coordinate, potentially involving intricate trades of multiple objects among many agents. Indeed, the problem of verifying whether a given allocation is *Pareto efficient* is computationally intractable: even when agents have "additive" preferences (a subclass of responsive preferences), determining *Pareto efficiency* is *coNP-complete* (e.g., De Keijzer et al. (2009), Aziz et al. (2019)).

Given these challenges, we consider a weaker notion of efficiency that is more readily attainable. *Individual-good efficiency* rules out destabilizing Pareto-improving exchanges that can be easily coordinated due to their relatively simple structure. Toward a formalization, we say that an allocation μ admits a *Pareto-improving single-object exchange* at a preference profile P if there is a cycle $C = (o_1, i_1, o_2, i_2, o_3, \ldots, i_k, o_{k+1} = o_1)$ of objects and agents such that, for all $\ell \in \{1, \ldots, k\}$,

$$i_\ell \in N, \quad o_\ell \in \mu_{i_\ell}, \quad \text{and} \quad (\mu_{i_\ell} \cup o_{i_{\ell+1}}) \backslash o_{i_\ell} \, P_{i_\ell} \, \mu_{i_\ell}. \tag{3}$$

An allocation μ is called *individual-good efficient (ig-efficient)* at P if it does not admit a Pareto-improving single-object exchange at P.[13]

Definition 4. A rule φ is ***individual-good efficient (ig-efficient)*** if, for each $P \in \mathcal{P}$, $\varphi(P)$ is *ig-efficient* at P.

Clearly, *Pareto efficiency* implies *ig-efficiency* on an arbitrary domain of strict preferences. Although the two properties are equivalent under lexicographic preferences (Aziz et al., 2019),[14] *ig-efficiency* is substantially weaker than *Pareto efficiency* under more general preference domains. However, a practical advantage of *ig-efficiency* is that it can be verified in polynomial time (e.g., Cechlárová et al. (2014), Aziz et al. (2019)).

2.2.2 Participation guarantees

The following is a standard participation guarantee which ensures that no agent is harmed by the reallocation. An allocation μ is called *individually rational* at a preference profile P if, for each $i \in N$, $\mu_i \, R_i \, \omega_i$.

[13]The terminology "*ig-efficiency*" is borrowed from Biró et al. (2022a). Similar properties are studied in Aziz et al. (2019), Caspari (2020), and Coreno and Balbuzanov (2022).

[14]We show that this equivalence extends to "conditionally lexicograhic" preferences (Proposition 5).

Definition 5. A rule φ is ***individually rational*** if, for each $P \in \mathcal{P}$ and each $i \in N$, $\varphi_i(P) R_i \omega_i$.

We now introduce another participation guarantee that depends exclusively on the agents' rankings over individual objects. Specifically, it ensures that no agent receives an object that is worse than her least-preferred object in her own endowment. An allocation μ satisfies the *worst endowment lower bound* at a preference profile P if, for each agent $i \in N$, and each of her assigned objects $o \in \mu_i$, $o \, R_i \, \min_{P_i}(\omega_i)$.

Definition 6. A rule φ satisfies the ***worst endowment lower bound*** if, for each $P \in \mathcal{P}$, $\varphi(P)$ satisfies the *worst endowment lower bound* at P.

Individual rationality and the *worst endowment lower bound* coincide on the class of single-unit reallocation problems. However, for general multi-object problems, the two properties are independent. Crucially, *individual rationality* allows for an agent to be assigned any unfavorable object as long as it is part of a desirable bundle, whereas the *worst endowment lower bound* ensures that agents are not assigned highly unfavorable objects. Thus, a rule satisfying the *worst endowment lower bound* effectively allows agents to veto certain objects owned by the other agents. This veto right can be viewed as a minimal participation guarantee and is particularly meaningful in certain applications. In shift reallocation, for example, a worker may have prior commitments that prevent her from fulfilling certain shifts. By ranking these shifts below every shift in her endowment, she effectively vetoes them.

Under certain mild conditions, *individual rationality* is a stronger requirement than the *worst endowment lower bound*. For example, if we restrict attention to the class of *individual-good-based* and *balanced* rules defined on the responsive domain, the main focus of the present paper, then *individual rationality* implies the *worst endowment lower bound*.[15]

Lemma 1. *On the responsive domain, if an individual-good-based rule is balanced and individually rational, then it satisfies the worst endowment lower bound.*

2.2.3 Incentive properties

Let $P = (P_i)_{i \in N}$ be a preference profile, and let P'_i be a preference relation for agent i. We use the standard notation (P'_i, P_{-i}) to denote the preference profile in which agent i's preference relation is P'_i and, for each agent $j \in N \backslash \{i\}$, agent j's preference relation remains P_j. Given a

[15] Alternatively, we could consider a natural domain under which any bundle that violates the *worst endowment lower bound* is "unacceptable." Formally, for each $i \in N$ and each $P_i \in \mathcal{P}_i$, any bundle that intersects $\{o \in O \mid o \, R_i \, \min_{P_i}(\omega_i)\}$ is worse than any bundle that does not. Clearly, any rule on $\mathcal{P}^*$ that satisfies *individual rationality* also satisfies the *worst endowment lower bound*. Furthermore, if we assume preferences P_i to be lexicographic (or responsive) when restricted to $\{o \in O \mid o \, R_i \, \min_{P_i}(\omega_i)\}$, then our subsequent characterization results would remain valid.

rule φ, we say that agent i can *manipulate* φ at P by misreporting P_i' if $\varphi_i(P_i', P_{-i})\, P_i\, \varphi_i(P)$. A rule is *strategy-proof* if no agent can manipulate it by misreporting any preference relation.

Definition 7 (Strategy-proofness)**.** A rule φ is ***strategy-proof*** if, for each $P \in \mathcal{P}$, each $i \in N$, and each $P_i' \in \mathcal{P}_i$, $\varphi_i(P)\, R_i\, \varphi_i(P_i', P_{-i})$.

Even on the relatively narrow lexicographic domain, no allocation rule simultaneously satisfies *strategy-proofness*, *individual-good efficiency*, and *individual rationality* (Todo et al., 2014). Given this incompatibility, we explore relaxations of *strategy-proofness* by restricting the set of manipulation strategies that agents might employ. In particular, we focus on "truncation strategies" and "drop strategies." These strategies involve agents misrepresenting their marginal preferences in straightforward ways, making them intuitively appealing and easy to implement.

Truncation strategies and drop strategies are both special kinds of "subset drop strategies" (Biró et al. (2022a,b), Altuntaş et al. (2023)). Loosely speaking, a subset drop strategy is a manipulation whereby an agent drops a subset of the other agents' endowments to the bottom of her marginal preference list, possibly to avoid being assigned those objects. Truncation strategies are obtained by dropping a "tail subset" of objects, i.e., a subset consisting of the agent's least-preferred objects, whereas drop strategies are obtained by dropping a singleton subset. Because these strategies involve agents misrepresenting their marginal preferences, they are particularly meaningful when dealing with *individual-good-based* rules.

Truncation strategies have a well-established history in matching literature, particularly in models where agents are matched to single objects (e.g., Mongell and Roth (1991), Roth and Vande Vate (1991), Roth and Rothblum (1999), Ehlers (2008)). In our multi-object reallocation setting, the notion of truncation strategies aligns more closely with the definitions provided by Kojima (2013) and Biró et al. (2022a,b). Empirical evidence suggests that agents employ truncation strategies in practice (see Mongell and Roth (1991) and Guillen and Hakimov (2018)). Similarly, drop strategies have been studied by Altuntaş et al. (2023) in the context of multi-object reallocation under lexicographic preferences.

Given an agent i and a preference relation $P_i \in \mathcal{P}_i$, we say that $P_i' \in \mathcal{P}_i$ is a *subset drop strategy* for P_i if there exists $X \subseteq O \backslash \omega_i$ such that:

1. for all $x \in X$ and $y \in O \backslash X$, $y\, P_i'\, x$; and

2. for all $x, y \in X$, $x\, P_i'\, y$ if and only if $x\, P_i\, y$; and

3. for all $x, y \in O \backslash X$, $x\, P_i'\, y$ if and only if $x\, P_i\, y$.

In this case, we say that P_i' is obtained from P_i by *dropping* X (and that $\succ^{P_i'}$ is obtained from $\succ^{P_i}$ by dropping X). Furthermore, P_i' is a *drop strategy* for P_i if it is obtained by dropping a singleton subset, i.e., $|X| = 1$. Finally, P_i' is a *truncation strategy* for P_i if either $X = O\backslash\omega_i$ or $X = \{o \in O\backslash\omega_i \mid x\ P_i\ o\}$ for some $x \in O$. In the latter case, i.e., when $X = \{o \in O\backslash\omega_i \mid x\ P_i\ o\}$, P_i' is called a *truncation of P_i at x*.[16] Let $\mathcal{S}_i(P_i)$ $(\subseteq \mathcal{P}_i)$ denote the set of all subset drop strategies for P_i. Similarly, $\mathcal{D}_i(P_i)$ and $\mathcal{T}_i(P_i)$ denote, respectively, the sets of drop strategies and truncation strategies for P_i.

A rule is *truncation-proof* if it cannot be manipulated through truncation strategies. *Drop strategy-proofness* and *subset drop strategy-proofness* are defined analogously.

Definition 8. A rule φ is

- ***truncation-proof*** if, for each $P \in \mathcal{P}$, each $i \in N$, and each $P_i' \in \mathcal{T}_i(P_i)$, $\varphi_i(P)\ R_i\ \varphi_i(P_i', P_{-i})$.
- ***drop strategy-proof*** if, for each $P \in \mathcal{P}$, each $i \in N$, and each $P_i' \in \mathcal{D}_i(P_i)$, $\varphi_i(P)\ R_i\ \varphi_i(P_i', P_{-i})$.
- ***subset drop strategy-proof*** if, for each $P \in \mathcal{P}$, each $i \in N$, and each $P_i' \in \mathcal{S}_i(P_i)$, $\varphi_i(P)\ R_i\ \varphi_i(P_i', P_{-i})$.

Subset drop strategy-proofness entails both *truncation-proofness* and *drop strategy-proofness*, as it defends against a larger set of manipulation strategies. However, *truncation-proofness* and *drop strategy-proofness* are independent; a rule can satisfy one without satisfying the other.

Before illustrating these definitions with examples (Example 1), we offer two remarks to clarify certain technical points.

Remark 1. For any preference $P_i \in \mathcal{P}_i$ and any subset $X \subseteq O\backslash\omega_i$, there is a unique marginal preference, say $\succ^{P_i'}$, obtained from $\succ^{P_i}$ by dropping X to the bottom of the preference list. However, there may be multiple preference relations $P_i'' \in \mathcal{P}_i$ that share this marginal preference $\succ^{P_i'}$, and each such P_i'' is a subset drop strategy obtained from P_i by dropping X. Despite this multiplicity, any *individual-good-based* rule will choose the same allocation for all such P_i'', so this technical detail does not play a substantive role in our analysis.

Remark 2. By employing some subset drop strategy for P_i (or successively employing drop strategies), an agent can push any object in $\{o \in O\backslash\omega_i \mid o\ P_i\ \min_{P_i}(\omega_i)\}$ to the top of her marginal preference list. This is not possible with truncation strategies, as every truncation strategy P_i^* for P_i must agree with P_i on $O\backslash\omega_i$ (see footnote 16).

Example 1. Suppose $\omega_i = \{x, y\}$, and let $P_i \in \mathcal{P}_i$ be such that $\succ^{P_i}$: $a, b, \underline{x}, c, d, \underline{y}, e$ (agent i's endowment is underlined for emphasis). Then:

[16] Equivalently, P_i' is a *truncation of P_i at x* if (i) $\succ^{P_i'}$ agrees with $\succ^{P_i}$ on ω_i; (ii) $\succ^{P_i'}$ agrees with $\succ^{P_i}$ on $O\backslash\omega_i$; and (iii) for each $y \in O\backslash\omega_i$ with $x\ P_i\ y$, $\min_{P_i'}(\omega_i)\ P_i'\ y$.

- any $P_i^1 \in \mathcal{P}_i$ with $\succ^{P_i^1}$: $b, \underline{x}, c, d, \underline{y}, e, a$ is a drop strategy obtained from P_i by dropping object a. Note that any such P_i^1 is not a truncation strategy for P_i.
- any $P_i^2 \in \mathcal{P}_i$ with $\succ^{P_i^2}$: $a, b, \underline{x}, c, \underline{y}, d, e$ is a truncation strategy obtained from P_i from P_i by dropping $\{d, e\}$ (or successively dropping d then e).
- any $P_i^3 \in \mathcal{P}_i$ with $\succ^{P_i^3}$: $a, \underline{x}, \underline{y}, b, c, d, e,$ is a truncation strategy obtained from P_i by dropping $\{b, c, d, e\}$ (or successively dropping $b, c, d,$ then e). $\diamond$

2.3 Top Trading Cycles

A *cycle* is a circular sequence

$$C = \left(o_{i_1}, i_1, o_{i_2}, i_2, o_{i_3}, \ldots, i_{k-1}, o_{i_k}, i_k, o_{i_{k+1}} = o_{i_1}\right)$$

consisting of k (≥ 1) distinct objects and k distinct agents such that (i) each object on the cycle precedes (or "points to") its owner, and (ii) each agent on the cycle points to an object. The sets of agents and objects on C are denoted by $N(C) = \{i_1, i_2, \ldots, i_k\}$ and $O(C) = \{o_{i_1}, o_{i_2}, \ldots, o_{i_k}\}$, respectively. An allocation μ is said to *execute* the cycle C if it assigns to each agent in $N(C)$ the object she points to within C; that is, for each $i_\ell \in N(C)$, $o_{i_{\ell+1}} \in \mu_{i_\ell}$.

We study an extension of Gale's Top Trading Cycles algorithm (Shapley and Scarf, 1974) from single-object to multi-object reallocation problems. At each step of this procedure, every agent points to her most-preferred unassigned object, and every unassigned object points to its owner. There exists at least one cycle, and each agent involved in a cycle is assigned the object to which she points. All objects involved in a cycle are then removed. If unassigned objects remain, then the procedure continues to the next step; otherwise, it terminates with the corresponding allocation.

For our proofs, we consider a modified version of this procedure that executes only one cycle at each step. Specifically, if multiple cycles arise, then this modified procedure executes only the cycle containing the *minimum agent*—the agent with the smallest label among those involved in cycles. It is well known that this modified procedure yields the same allocation as the standard procedure, which executes all prevailing cycles at each step.

We now formalize the *(generalized) Top Trading Cycles (TTC) rule*, which we denote by φ^{TTC}. Given a preference profile $P \in \mathcal{P}$, the allocation $\varphi^{\text{TTC}}(P)$ is determined by running the *(generalized) TTC algorithm* at P. We denote this specific instance as $\text{TTC}(P)$. The algorithm $\text{TTC}(P)$ is defined as follows.

Algorithm: TTC (P)

Input: A preference profile $P \in \mathcal{P}$.

Output: An allocation $\varphi^{\mathrm{TTC}}(P)$.

Initialization: Set $\mu^0 := (\emptyset)_{i \in N}$ and $O^1 := O$.

Step $t \geq 1$:

1. **(Graph construction)** Construct a bipartite directed graph with independent vertex sets N and O^t, and edge sets defined as follows:
   (a) For each agent $i \in N$, there is a directed edge from i to $\max_{P_i}(O^t)$.
   (b) For each object $o \in O^t$, there is a directed edge from o to its owner, $\omega^{-1}(o)$.
2. **(Cycle selection)** Because there are finitely many vertices, each with an outgoing edge, there is at least one cycle.
   (a) Let $\mathcal{C}_t(P)$ denote the set of cycles that arise at Step t.
   (b) Let $C_t(P)$ be the cycle in $\mathcal{C}_t(P)$ containing the minimum agent:
   i.e., $\min N(C_t(P)) \leq \min N(C)$ for all $C \in \mathcal{C}_t(P)$.
3. **(Assignment)** Assign to each agent $i \in N(C_t(P))$ the object $\max_{P_i}(O^t)$. That is, let $\mu^t = (\mu_i^t)_{i \in N}$ be such that
   (a) for all $i \in N(C_t(P))$, $\mu_i^t = \mu_i^{t-1} \cup \{\max_{P_i}(O^t)\}$, and
   (b) for all $i \in N \backslash N(C_t(P))$, $\mu_i^t = \mu_i^{t-1}$.
4. **(Removal)** Let $O^{t+1} := O^t \backslash O(C_t(P))$ be the set of objects remaining at Step $t+1$.
   (a) If $O^{t+1} \neq \emptyset$, then proceed to Step $t+1$.
   (b) If $O^{t+1} = \emptyset$, then proceed to Termination.

Termination: Because O is finite and $|O^1| > |O^2| > \cdots > |O^t|$, the algorithm terminates at some step T. Return the allocation $\varphi^{\mathrm{TTC}}(P) := \mu^T$.

The following example illustrates the TTC algorithm.

Example 2. Suppose $N = \{1, 2, 3\}$, $O = \{a, b, c, d\}$, and $\omega = (\{a, b\}, \{c\}, \{d\})$. Consider a preference profile $P = (P_1, P_2, P_3)$, where $\succ^{P_1}$: c, a, d, b, $\succ^{P_2}$: a, b, c, d, and $\succ^{P_3}$: a, c, b, d. The algorithm TTC(P) works as follows.

Step 1: Each agent points to her most-preferred object, and each object points to its owner. There is a cycle $C_1(P) = (1, c, 2, a, 1)$. We set $\mu^1 = (\{c\}, \{a\}, \emptyset)$ and $O^2 = \{b, d\}$. Since $O^2 \neq \emptyset$, we proceed to Step 2.

Step 2: Each agent points to her most-preferred object among $O^2 = \{b, d\}$, and each object in O^2 points to its owner. There is a cycle $C_2(P) = (1, d, 3, b, 1)$. We set $\mu^2 = (\{c, d\}, \{a\}, \{b\})$ and $O^3 = \emptyset$. Since $O^3 = \emptyset$, we stop and return the allocation $\varphi^{\text{TTC}}(P) = (\{c, d\}, \{a\}, \{b\})$. $\diamond$

Several key properties of the TTC rule hold universally across any domain of strict preferences. The following proposition outlines these properties.

Proposition 1. *For any domain $\mathcal{P}$, the TTC rule is individual-good-based, balanced, and satisfies the worst endowment lower bound.*

Proof. At each step of the TTC algorithm, each agent points to her most-preferred unassigned object. Since this choice depends only on her marginal preferences over individual objects, the TTC rule is *individual-good-based*. *Balancedness* and the *worst endowment lower bound* follow from the fact that whenever an agent relinquishes an object from her endowment, she receives an object in return that is weakly preferred according to her marginal preferences. $\square$

Whether the TTC rule satisfies other properties—such as *Pareto efficiency*, *truncation-proofness*, or *drop strategy-proofness*—depends on the specific domain of preferences being considered.

3 Results for Lexicographic Preferences

Our analysis begins on the domain of lexicographic preferences, a natural starting point because of its relative tractability. As mentioned previously, the lexicographic domain has the following desirable features: (i) any rule defined on it is *individual-good-based*, and (ii) *Pareto efficiency* is equivalent to *individual-good efficiency*, which can be verified in polynomial time (Cechlárová et al. (2014), Aziz et al. (2019)).

Within the lexicographic domain, the TTC rule exhibits several desirable properties. Fujita et al. (2018) demonstrate that it is *core selecting*, which entails both *Pareto efficiency* and *individual rationality*. Furthermore, Altuntaş et al. (2023) establish that the TTC rule is *subset drop strategy-proof*, implying that it is also *truncation-proof* and *drop strategy-proof*. These key properties are formally stated in the following proposition.

Proposition 2. *On the lexicographic domain, the TTC rule satisfies Pareto efficiency, individual rationality, and subset drop strategy-proofness.*

We now present our main characterization of the TTC rule for the lexicographic domain. The following theorem states that the TTC rule is uniquely determined by the combination of *balancedness*, *ig-efficiency*, the *worst endowment lower bound*, and *truncation-proofness*.

Theorem 1. *On the lexicographic domain, a rule satisfies*

- *balancedness,*
- *ig-efficiency (or Pareto efficiency),*
- *the worst endowment lower bound, and*
- *truncation-proofness*

if and only if it is the TTC rule.

The proof proceeds by minimal counterexample and exploits both a "similarity" function (Ekici, 2024) and a "size" function (Sethuraman, 2016). Our main technical innovation lies in combining both functions to establish the result. Suppose φ is a rule that satisfies the properties but differs from φ^{TTC}. Call a preference profile $P \in \mathcal{P}$ a *conflict profile* if $\varphi(P) \neq \varphi^{\mathrm{TTC}}(P)$. For each conflict profile P, let $\rho(P) \coloneqq t$ denote the earliest step t of $\mathrm{TTC}(P)$ at which $\varphi(P)$ does not execute $C_t(P)$, the cycle executed at step t of $\mathrm{TTC}(P)$. In this way, we define the "similarity" function ρ from conflict profiles to the natural numbers (Ekici, 2024), measuring the earliest point of divergence between φ and φ^{TTC}. Among all conflict profiles minimizing ρ, we select a profile P that further minimizes the "size" function $s(P) \coloneqq \sum_{i \in N} |\{o \in O \mid o \; R_i \min_{P_i}(\omega_i)\}|$ (Sethuraman, 2016). We then demonstrate that, given our choice of P, the allocation $\varphi(P)$ cannot be *ig-efficient* at P, a contradiction.

Theorem 2 allows us to derive an alternative characterization of the TTC rule as a corollary. To see how, we first state a useful lemma.[17]

Lemma 2. *On the lexicographic domain, if a rule satisfies drop strategy-proofness and the worst endowment lower bound, then it is subset drop strategy-proof.*

Suppose φ is a rule satisfying *balancedness*, *ig-efficiency*, the *worst endowment lower bound*, and *drop strategy-proofness*. Such a rule exists by Propositions 1 and 2. Lemma 2 implies that φ is *subset drop strategy-proof*, hence *truncation-proof*. Therefore, φ satisfies the properties in Theorem 1, which means it must equal the TTC rule. This result is stated in the following theorem, which strengthens the characterization provided by Altuntaş et al. (2023, Theorem 1).

[17]Example 7 in Appendix B.2 demonstrates that *truncation-proofness* and the *worst endowment lower bound* do not jointly imply *drop strategy-proofness*. Example 8 shows that *truncation-proofness* is not implied by *drop strategy-proofness* alone.

Theorem 2. *On the lexicographic domain, a rule satisfies*

- *balancedness,*
- *ig-efficiency (or Pareto efficiency),*
- *the worst endowment lower bound, and*
- *drop strategy-proofness*

if and only if it is the TTC rule.

We cannot substitute the *worst endowment lower bound* with *individual rationality* in Theorems 1 and 2. On the lexicographic domain, there exist rules other than the TTC rule that satisfy *balancedness*, *Pareto efficiency*, *individual rationality*, *truncation-proofness*, and *drop strategy-proofness*. The following example illustrates this point.

Example 3. Suppose that $N = \{1, 2, 3\}$, $O = \{a, b, c, d\}$, and $\omega = (\{a, b\}, \{c\}, \{d\})$. Let φ be the rule defined as follows. For all $P \in \mathcal{L}$,

$$\varphi(P) = \begin{cases} \varphi^{(1231)}(P), & \text{if } \max_{P_1}(O) = c \text{ and } \max_{P_2}(O) \in \{a, b\} \\ \varphi^{(1321)}(P), & \text{if } \max_{P_1}(O) = d \text{ and } \max_{P_3}(O) \in \{a, b\} \\ \varphi^{\text{TTC}}(P), & \text{otherwise,} \end{cases}$$

where $\varphi^{(ijk\ell)}$ is the sequential priority rule which, at each step, assigns to an agent her most-preferred unassigned object, proceeding in the order (i, j, k, ℓ).

One can show that φ satisfies *balancedness*, *ig-efficiency*, *individual rationality*, *truncation-proofness*, and *drop strategy-proofness*. To see that φ violates the *worst endowment lower bound*, let $P \in \mathcal{L}$ satisfy $P_1 : c, b, a, d$ and $P_2 = P_3 : a, b, c, d$. Then $\varphi(P) = (\{c, d\}, \{a\}, \{b\})$, which violates the *worst endowment lower bound* because object d is assigned to agent 1 although $\min_{P_1}(\omega_1) = a \; P_1 \; d$. ◇

The issue illustrated in Example 3 arises from a crucial difference between the two participation guarantees. While *individual rationality* allows an agent to be assigned any object as part of a desirable bundle, the *worst endowment lower bound* restricts which objects can be included in an agent's bundle. Interestingly, this issue does not occur on the responsive domain, where *individual rationality* suffices for the characterization (Theorem 4).[18]

[18]Similarly, this issue does not arise on the domain described in footnote 15, as *individual rationality* implies the *worst endowment lower bound* on that domain.

4 Results for Responsive Preferences

In this section, we extend our analysis to the domain of responsive preferences, which presents several challenges compared to the lexicographic domain.

One major hurdle is that achieving *Pareto efficiency* becomes more difficult, as it generally requires coordinating intricate exchanges involving multiple objects among many agents. As mentioned previously, the problem of verifying whether an allocation is *Pareto efficient* is coNP-complete (e.g., De Keijzer et al. (2009), Aziz et al. (2019)), making it computationally challenging. Adding to this complexity, Manjunath and Westkamp (2024) show that no *individual-good-based* rule satisfies both *Pareto efficiency* and *individual rationality*. Since the TTC rule is an *individual-good-based* rule that satisfies *individual rationality*, it follows that the TTC rule cannot be *Pareto efficient* in this setting.

To illustrate this incompatibility, consider the following example.

Example 4 (Manjunath and Westkamp (2024))**.** Suppose that $N = \{1, 2\}$, $O = \{a, b, c, d\}$, and $\omega = (\{a, d\}, \{b, c\})$. Toward contradiction, let φ be an *individual-good-based* rule on $\mathcal{R}$ satisfying *individual rationality* and *Pareto efficiency*.

Let $P = (P_1, P_2)$ be the lexicographic preference profile where both agents rank individual objects in the order $\succ^{P_1} = \succ^{P_2}$: a, b, c, d. Since φ is *Pareto efficient* and *individually rational*, it must assign $\varphi(P) = (\{a, d\}, \{b, c\})$. Now let $P' = (P'_1, P'_2)$ be a responsive preference profile with the same marginal preferences such that each agent prefers the other's endowment; that is,

$$\succ^{P'_1} = \succ^{P'_2}\colon a, b, c, d, \quad \{b, c\}\ P'_1\ \{a, d\}, \quad \text{and} \quad \{a, d\}\ P'_2\ \{b, c\}.$$

Because $\succ^{P'} = \succ^{P}$ and φ is *individual-good-based*, we have $\varphi(P') = \varphi(P) = (\{a, d\}, \{b, c\})$. However, $\varphi(P')$ is Pareto-dominated by $(\{b, c\}, \{a, d\})$ at P', violating *Pareto efficiency*.

In this example, the TTC rule assigns the allocation $(\{a, d\}, \{b, c\})$ at both P and P', which directly shows that it is not *Pareto efficient*. $\diamond$

Despite this limitation, the TTC rule remains *ig-efficient* on the responsive domain. To see this, suppose $\varphi^{\text{TTC}}(P)$ were not *ig-efficient* at some responsive preference profile P. Consider the corresponding lexicographic preference profile P' with $\succ^{P'} = \succ^{P}$. Since the TTC rule is *individual-good-based*, we have $\varphi^{\text{TTC}}(P') = \varphi^{\text{TTC}}(P)$. However, this would imply that $\varphi^{\text{TTC}}(P')$ is not *Pareto efficient* at P', contradicting the *Pareto efficiency* of the TTC rule on the lexicographic domain.

It turns out that the TTC rule is not *drop strategy-proof* when there are three or more agents.[19]

[19] If there are only two agents, then the TTC rule is *drop strategy-proof* as each agent can only drop objects from the other agent's endowment.

Example 5. Let $N = \{1, 2, 3\}$ and $\omega = (\{a, b\}, \{c, d\}, \{e, f\})$. Let P_2 and P_3 be lexicographic preference relations such that $P_2 : e, f, b, \ldots$ and $P_3 : a, c, d, \ldots$.

Let P_1 be a responsive preference relation such that $\succ^{P_1}$: c, f, e, d, a, b and $\{d, f\}\ P_1\ \{c, b\}$. Under the TTC rule, the allocation is $\varphi^{\mathrm{TTC}}(P) = (\{c, b\}, \{e, f\}, \{a, d\})$. Now, suppose agent 1 employs a drop strategy P'_1 obtained from P_1 by dropping object c, resulting in the marginal preferences $\succ^{P'_1}$: f, e, d, a, b, c. The TTC rule then yields $\varphi^{\mathrm{TTC}}(P'_1, P_{-1}) = (\{d, f\}, \{e, b\}, \{a, c\})$. Because $\varphi_1^{\mathrm{TTC}}(P'_1, P_{-1}) = \{d, f\}\ P_1\ \{c, b\} = \varphi_1^{\mathrm{TTC}}(P)$, agent 1 benefits from the drop strategy. Thus, the TTC rule is not *drop strategy-proof* on $\mathcal{R}$. ◇

Although agents may benefit by employing drop strategies, they cannot benefit by truncating their preferences. Indeed, we find that the TTC rule is *truncation-proof* on the responsive domain. This distinction highlights the independence between the two types of manipulation heuristics.

The positive properties of the TTC rule on $\mathcal{R}$ are summarized in the following proposition.

Proposition 3. *On the responsive domain, the TTC rule satisfies ig-efficiency, individual rationality, and truncation-proofness.*

We extend our characterization of the TTC rule (Theorem 1) from the lexicographic domain to the responsive domain by adapting a standard technique (see, e.g., Biró et al. (2022a)).

Let φ be any *individual-good-based* rule on $\mathcal{R}$ satisfying *balancedness*, *ig-efficiency*, the *worst endowment lower bound*, and *truncation-proofness*. An argument similar to the proof of Theorem 1 demonstrates that φ must coincide with the TTC rule on $\mathcal{L}$. For any $P' \in \mathcal{R}$, let $P \in \mathcal{L}$ be the unique lexicographic preference profile such that $\succ^P = \succ^{P'}$. Since φ and the TTC rule are *individual-good-based*, we have

$$\varphi(P') = \varphi(P) = \varphi^{\mathrm{TTC}}(P) = \varphi^{\mathrm{TTC}}(P').$$

Therefore, the characterization extends to all of $\mathcal{R}$.

Theorem 3. *On the responsive domain, an individual-good-based rule satisfies*

- *balancedness,*
- *ig-efficiency,*
- *the worst endowment lower bound, and*
- *truncation-proofness*

if and only if it is the TTC rule.

It turns out that the *worst endowment lower bound* can be replaced with *individual rationality* in the statement of Theorem 3. The following theorem is an immediate consequence of Theorem 3, Lemma 1, and the fact that the TTC rule is *individually rational*.

Theorem 4. *On the responsive domain, an individual-good-based rule satisfies*

- *balancedness,*
- *ig-efficiency,*
- *individual rationality, and*
- *truncation-proofness*

if and only if it is the TTC rule.

Our characterizations establish the TTC rule as the only *individual-good-based* rule satisfying a set of desirable properties. However, relaxing the *individual-good-based* assumption allows for alternative rules. There exist rules that are not *individual-good-based* but still satisfy all the other properties we consider and, moreover, some of these rules can Pareto-dominate the TTC rule; that is, at every preference profile, they either coincide with the TTC rule or provide a Pareto improvement. We present an example of such a rule below.

Example 6 (A non-*individual-good-based* rule)**.** Let $N = \{1, 2\}$ and $\omega = (\{a, b\}, \{c, d\})$. Define the allocation $\mu \coloneqq (\{c, d\}, \{a, b\})$. Consider the rule φ on $\mathcal{R}$ defined by

$$\varphi(P) = \begin{cases} \mu, & \text{if } \varphi^{\text{TTC}}(P) = \omega, \text{ and } \mu \text{ Pareto-dominates } \omega \text{ at } P; \\ \varphi^{\text{TTC}}(P), & \text{otherwise.} \end{cases}$$

This rule satisfies *balancedness*, *ig-efficiency*, the *worst endowment lower bound*, and *truncation-proofness*. Moreover, for any preference profile $P \in \mathcal{R}$, either $\varphi(P) = \varphi^{\text{TTC}}(P)$ or $\varphi(P)$ Pareto-dominates $\varphi^{\text{TTC}}(P)$ at P. However, it is not *individual-good-based*, illustrating that the property is essential for our characterization.

We note that φ coincides with φ^{TTC} on the lexicographic domain, since when P is lexicographic with $\varphi^{\text{TTC}}(P) = \omega$, μ does not Pareto-dominate ω. ⋄

The rule φ in Example 6, which is not *individual-good-based*, satisfies all our desired properties and Pareto-dominates the TTC rule. This highlights an important trade-off between efficiency and ease of implementation. While *individual-good-based* rules are attractive in practice due to their simplicity and transparency, such simplicity may come at the expense of efficiency.

5 Single-object reallocation (The Shapley-Scarf model)

Our characterization of the TTC rule on the lexicographic domain (Theorem 1) leads to a new characterization in the classic single-object reallocation problem (Shapley and Scarf, 1974), where each agent is endowed with a single object. In this section, we formally introduce the single-object reallocation problem and present our main result: the TTC rule is the unique rule satisfying *Pareto efficiency*, *individual rationality*, and *truncation-proofness* in this setting.

Recall that $N = \{1, 2, \ldots, n\}$ is the set of agents, and O is the set of objects, with $|O| = n$. Without loss of generality, we assume that $O = \{o_1, \ldots, o_n\}$ and that each agent $i \in N$ is endowed with object $\omega_i = o_i$. Each agent has (strict) preferences P_i over individual objects O, and $\mathcal{P} = \prod_{i \in N} \mathcal{P}_i$ denotes the domain of preference profiles in which each agent i has strict preferences.[20] An allocation is a bijection $\mu : N \to O$, represented as $\mu = (\mu_i)_{i \in N}$, where μ_i is the object assigned to agent i.

In this setting, all allocations are *balanced* by definition since each agent receives exactly one object. Moreover, the *worst endowment lower bound* is equivalent to *individual rationality*, and *Pareto efficiency* coincides with *individual-good efficiency*.

While the definitions of *drop strategy-proofness* and *truncation-proofness* remain the same, it is helpful to reformulate the concept of truncation strategies in this context. Given a preference relation $P_i \in \mathcal{P}_i$, we say that $P'_i \in \mathcal{P}_i$ is a *truncation strategy* for P_i if

1. $\{x \in O \mid x \, R'_i \, o_i\} \subseteq \{x \in O \mid x \, R_i \, o_i\}$, and
2. for each $x, y \in O \backslash \{o_i\}$, $x \, P'_i \, y$ if and only if $x \, P_i \, y$.

Intuitively, a truncation strategy involves agent i promoting her own object in their preference list while preserving the original ordering of other objects. Let $\mathcal{T}_i(P_i)$ denote the set of all truncation strategies for P_i.

We now present our main result for the single-object reallocation problem.

Theorem 5. *In the single-object reallocation problem, a rule satisfies*

- *Pareto efficiency,*
- *individual rationality, and*
- *truncation-proofness*

if and only if it is the TTC rule.

[20] In this section, we adopt the standard convention that each agent has preferences P_i defined on individual objects O. This minor departure plays no role in the analysis. If instead we maintained the assumption that each agent has lexicographic preferences on 2^O, then our subsequent result would be a direct corollary of Theorem 1.

This theorem refines several characterizations of the TTC rule for the single-object reallocation problem. For example, Ma (1994) characterizes the TTC rule using *Pareto efficiency*, *individual rationality*, and *strategy-proofness*, while Altuntaş et al. (2023) show that *strategy-proofness* can be replaced with *drop strategy-proofness*. In light of Remark 2 and Lemma 2, which carry over to this single-object environment, we generalize these results by establishing uniqueness under even weaker criteria.

Given that the TTC rule also satisfies *strategy-proofness* in this environment, Theorem 5 is especially relevant in applications where full *strategy-proofness* is unnecessarily strong. For example, in Paired Kidney Exchange with strict preferences, patients' preferences over donors' kidneys are often common knowledge because they are based on publicly verifiable criteria (Ashlagi and Roth (2014), Nicoló and Rodríguez-Álvarez (2012)). In such settings, concerns about strategic manipulation may be limited to specific forms like truncation. A patient might misrepresent her willingness to remain on dialysis (and retain her donor's kidney), effectively truncating her preference list. Alternatively, a donor might condition her participation on the expected outcome for the patient she is paired with, agreeing to donate only if the patient receives a sufficiently suitable kidney.

Due to the interest in characterizations of the TTC rule for the Shapley-Scarf model, we provide a direct proof of Theorem 5 in Appendix A. The proof is concise and shares key ideas with the proof of Theorem 1, offering insights into the proof of the more general result.

6 Extension: Conditionally Lexicographic Preferences

In this section, we introduce conditionally lexicographic preferences, a generalization of purely lexicographic preferences. Unlike responsive preferences, conditionally lexicographic preferences allow for the relative ranking of two objects to depend on the other objects they are obtained with. For example, an agent might prefer drinking Champagne to Bordeaux when paired with oysters, but Bordeaux to Champagne otherwise. Thus, conditionally lexicographic preferences are flexible enough to accommodate complementarity among objects. Despite this added flexibility, conditionally lexicographic preferences retain some of the appealing features of lexicographic preferences. Notably, they have a compact representation, which makes them appealing from an implementation perspective. Furthermore, as we shall see, *Pareto efficiency* is equivalent to *ig-efficiency* on this domain. Conditionally lexicographic preferences have been widely studied in computer science, particularly in artificial intelligence (e.g., Booth et al. (2010), Domshlak et al. (2011), Pigozzi et al. (2016)).

Loosely speaking, an agent has conditionally lexicographic preferences if, for any bundle $Y \subsetneq O$, there is an object $o \in O \setminus Y$ that she considers the "lexicographically best" addition to Y.

Formally, conditionally lexicographic preferences are represented by "lexicographic preference trees" on the set O of objects.

Definition 9. A *lexicographic preference tree (LP tree)* on O is a rooted directed tree τ_i such that

- each vertex v is labeled with an object $o(v) \in O$.
- every object appears exactly once on any path from the root to a leaf.
- every internal (non-leaf) vertex has two outgoing edges:
  - an "in edge" labeled $o(v)$, representing the presence of $o(v)$ in a bundle;
  - a "not-in edge" labeled $\neg o(v)$, representing the absence of $o(v)$ from a bundle.

Intuitively, an LP tree represents the conditional preferences of an agent in a hierarchical manner. For any given bundle Y, there is a unique path from the root to a leaf that is consistent with Y: at each vertex, we follow the "in edge" if the corresponding object is in Y and the "not-in edge" if it is not. This path specifies the agent's preference ordering over objects, conditional on receiving Y. Figures 1 and 2 provide a graphical representation of two LP trees, τ_i and τ_i^*, on the set $O = \{a, b, c, d\}$ of objects.

Given an LP tree τ_i and a bundle $X \subseteq O$, let $\tau_i(X)$ be a directed path from the root to a leaf of τ_i containing only edges consistent with X, i.e., edges (v, v') labeled with $o(v)$ if $o(v) \in X$ and $\neg o(v)$ if $o(v) \notin X$. Note that such a path $\tau_i(X)$ is unique. Given bundles A and B with $A \neq B$, let $\tau_i(A, B)$ denote the first vertex v visited by both $\tau_i(A)$ and $\tau_i(B)$ and such that $o(v) \in A \triangle B$, i.e., $o(v)$ belongs to exactly one of A and B. Equivalently, $\tau_i(A, B)$ is the last vertex which is common to both $\tau_i(A)$ and $\tau_i(B)$.

Definition 10. The preference relation P_{τ_i} associated with an LP tree τ_i is defined as follows:

1. for all $A, B \subseteq O$ with $A \neq B$, $[AP_{\tau_i}B \iff o(\tau_i(A, B)) \in A \backslash B]$.
2. for all $A, B \subseteq O$, $[AR_{\tau_i}B \iff (A = B \text{ or } AP_{\tau_i}B)]$.

A preference relation P_i on 2^O is called *conditionally lexicographic* if there exists an LP tree τ_i such that $P_i = P_{\tau_i}$. For each $i \in N$, let $\mathcal{CL}_i$ be the set of conditionally lexicographic preferences on 2^O. Then $\mathcal{CL} := \prod_{i \in N} \mathcal{CL}_i$ is called the *conditionally lexicographic domain.* Given the one-to-one correspondence between LP trees and conditionally lexicographic preferences, we denote the unique LP tree associated with P_i by τ_{P_i}. When context permits, we may refer to a conditionally lexicographic P_i and its corresponding LP tree τ_{P_i} interchangeably.

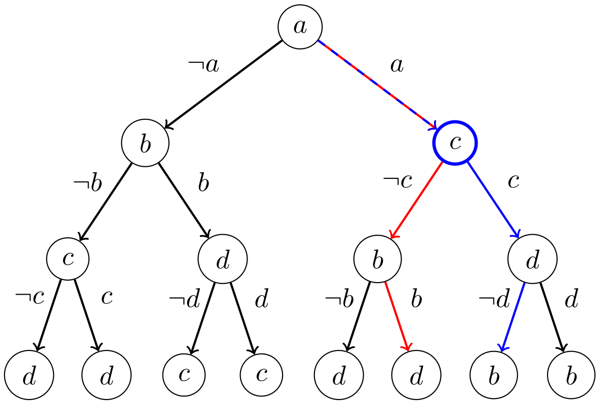

Figure 1: An LP tree τ_i

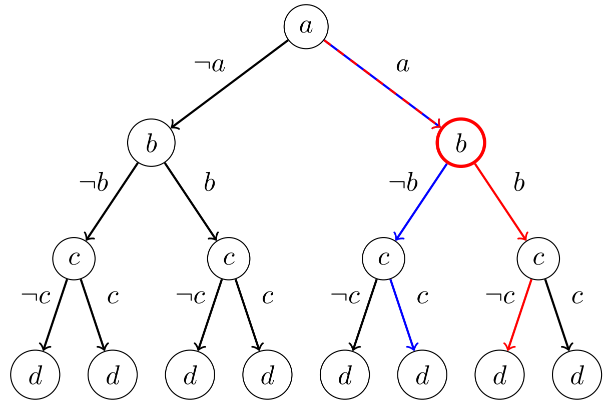

Figure 2: An LP tree τ_i^*

Notes: In both figures, the paths corresponding to the bundles $\{a, b\}$ and $\{a, c\}$ are highlighted in red and blue, respectively. In Figure 1, the last common vertex, $\tau_i(\{a, b\}, \{a, c\})$, is highlighted in blue. Because $o(\tau_i(\{a, b\}, \{a, c\}) = c$ belongs to $\{a, c\}\backslash\{a, b\}$, we have $\{a, c\}\ P_{\tau_i}\ \{a, b\}$. In Figure 2, the last common vertex, $\tau_i^*(\{a, b\}, \{a, c\})$, is highlighted in red. Because $o(\tau_i^*(\{a, b\}, \{a, c\})) = b$ belongs to $\{a, b\}\backslash\{a, c\}$, we have $\{a, b\}\ P_{\tau_i^*}\ \{a, c\}$. Since all paths in τ_i^* use the same object order, $P_{\tau_i^*}$ is purely lexicographic. ⋄

Note that every conditionally lexicographic preference relation is monotonic. While every lexicographic preference relation is conditionally lexicographic, the converse is not true when $|O| \geq 3$. Specifically, a lexicographic preference relation corresponds to an LP tree where all paths from the root to a leaf use the same object order (i.e., all vertices at the same depth are labeled with the same object). Moreover, a preference relation is lexicographic if and only if it is conditionally lexicographic and responsive. In other words, for any $i \in N$, we have $\mathcal{CL}_i \subseteq \mathcal{M}_i$, $\mathcal{L}_i \subseteq \mathcal{CL}_i$ (with strict inclusion for $|O| \geq 3$), and $\mathcal{L}_i = \mathcal{CL}_i \cap \mathcal{R}_i$.

The following proposition provides an alternative characterization of conditionally lexicographic preferences. It states that for any set of objects X not yet in agent i's bundle Y, there is a unique object in X, denoted $\max_{P_i}(X \mid Y)$, which is "lexicographically best" among X conditional on receiving Y. Its proof is straightforward and it is omitted.

Proposition 4. *A preference relation P_i on 2^O is conditionally lexicographic if and only if the following property holds: for all disjoint bundles $X, Y \in 2^O$ with $X \neq \emptyset$, there is a unique object in $x^* \in X$, denoted $x^* = \max_{P_i}(X \mid Y)$, such that*

$$\textit{for all } Z \subseteq X\backslash\{x^*\}, \quad (Y \cup x^*)\ P_i\ (Y \cup Z). \tag{4}$$

That is, $\max_{P_i}(X \mid Y)$ is agent i's (lexicographically) most-preferred object in X conditional on already having Y.[21]

[21] We note that $\max_{P_i}(X \mid Y) = o(\tau_{P_i}(Y, Y \cup X))$.

6.1 Properties of allocation rules

In what follows, we drop the requirement that the allocation depends solely on agents' marginal preferences over individual objects. That is, we consider allocation rules on $\mathcal{CL}$ that are not necessarily *individual-good-based*.

Most of the properties defined in Section 2.2—namely *balancedness*, *Pareto efficiency*, *ig-efficiency*, and *individual rationality*—are defined exactly as before. However, the definitions of the *worst endowment lower bound* and *drop strategy-proofness* require adaptation to accommodate conditionally lexicographic preferences.[22]

To define the *worst endowment lower bound*, we first need some notation. The idea is to ensure that no agent is assigned a bundle containing an object that she considers conditionally worse than "conditionally worst" object in her own endowment, given the bundle she receives.

Given an LP tree $\tau_i \in \mathcal{CL}_i$ and a vertex v of τ_i, let $a(v)$ denote the set of objects labeling the ancestors of v, including v itself. That is, if $(v_1, v_2, \ldots, v_k = v)$ is the unique path from the root of τ_i to v, then $a(v) = \{o(v_1), o(v_2), \ldots, o(v_k)\}$. Given a preference relation $P_i \in \mathcal{CL}_i$ and a bundle $X \in 2^O$, define $w_{P_i}(\omega_i \mid X)$ as the unique vertex on the path $\tau_{P_i}(X)$ that corresponds to the "conditionally worst" object in agent i's endowment ω_i, given X. Formally, $w_{P_i}(\omega_i \mid X)$ is the unique vertex on $\tau_{P_i}(X)$ such that $o(w_{P_i}(\omega_i \mid X)) \in \omega_i$ and $\omega_i \subseteq a(w_{P_i}(\omega_i \mid X))$. This means that $w_{P_i}(\omega_i \mid X))$ is the last vertex on the path $\tau_{P_i}(X)$ to be labeled with an object in ω_i. We illustrate this construction graphically in figures 3 and 4.

An allocation μ satisfies the *worst endowment lower bound* at a preference profile $P \in \mathcal{CL}$ if, for each $i \in N$, $\mu_i \subseteq a(w_{P_i}(\omega_i \mid \mu_i))$. This means that every object in μ_i labels a vertex on the path $\tau_{P_i}(\mu_i)$ that occurs before or at the vertex $w_{P_i}(\omega_i \mid \mu_i)$. Intuitively, no agent i is assigned an object that she considers conditionally worse than the "conditionally worst" object in ω_i, given the bundle μ_i. Note that when agents have purely lexicographic preferences, this definition aligns with the original *worst endowment lower bound*.

Definition 11. A rule φ satisfies the ***worst endowment lower bound*** if, for each $P \in \mathcal{CL}$, $\varphi(P)$ satisfies the *worst endowment lower bound* at P.

The concept of drop strategies extends naturally to conditionally lexicographic preferences. An agent employs a drop strategy by dropping an object she does not own to the bottom of her LP tree. Formally, given a preference relation $P_i \in \mathcal{CL}_i$, we say that a preference relation $P_i' \in \mathcal{CL}_i$ is a *drop strategy* for P_i if there exists $x \in O \backslash \omega_i$ such that:

1. for all nonempty bundles $Y \subseteq O \backslash \{x\}$, $Y \; P_i' \; x$; and

2. for any $Y, Z \subseteq O \backslash \{x\}$, $Y \; P_i' \; Z$ if and only if $Y \; P_i \; Z$.

[22] We do not study *truncation-proofness* in this setting, as the definition becomes rather unwieldy.

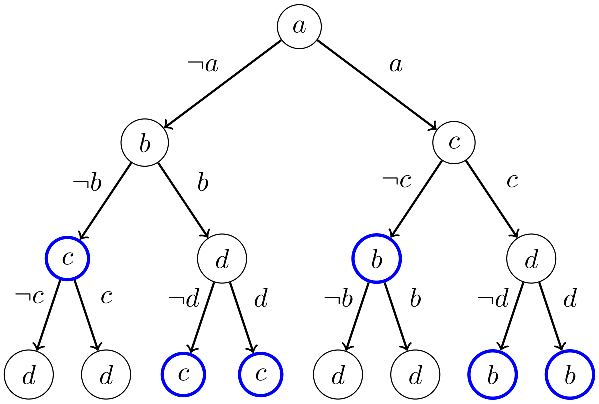

Figure 3: The LP tree τ_i

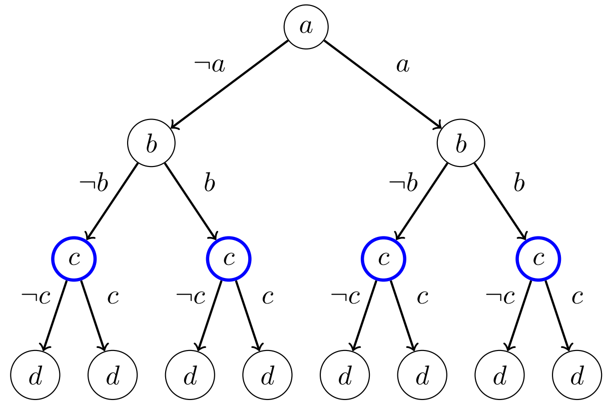

Figure 4: The LP tree τ_i^*

Notes: Figures 3 and 4 illustrate the construction of $w_{P_i}(\omega_i \mid X)$. Suppose agent i's endowment is $\omega_i = \{b, c\}$. In both figures, the vertices corresponding to $w_{P_i}(\omega_i \mid X)$ for various $X \in 2^O$ are highlighted in blue. Notice that on any path from the root to a leaf, there is exactly one such vertex. Under τ_i, the *worst endowment lower bound* requires that agent i is not assigned object d together with any of the bundles $\emptyset, \{a\}, \{a, b\}$, or $\{c\}$; that is, she does not receive a bundle in $\{\{d\}, \{a, d\}, \{a, b, d\}, \{c, d\}\}$. Under τ_i^*, it requires that agent i does not receive any bundle containing object d. $\diamond$

In this case, we say that P_i' is obtained from P_i by dropping object x.[23] Note that this definition aligns with the original one from Section 2.2 whenever P_i is purely lexicographic. Let $\mathcal{D}_i(P_i)$ denote the set of all drop strategies for P_i.

The drop strategy P_i' obtained from P_i by dropping object x can also be represented via LP trees. Let $\tau_{P_i}^{-x}$ denote the LP tree on $O\backslash\{x\}$ representing the restriction of P_i to subsets of $O\backslash\{x\}$.[24] To construct $\tau_{P_i'}$, at each leaf of $\tau_{P_i}^{-x}$ append two child nodes labeled with x, connected by an "in edge" and a "not-in edge" accordingly. This modification places x and the bottom of the LP tree, ensuring that x is the "lexicographically worst" addition to any bundle. Figures 5 and 6 illustrate this construction.

Definition 12. A rule φ is ***drop strategy-proof*** if, for each $P \in \mathcal{CL}$, each $i \in N$, and each $P_i' \in \mathcal{D}_i(P_i)$, $\varphi_i(P)\; R_i\; \varphi_i(P_i', P_{-i})$.

6.2 Augmented Top Trading Cycles

The *Augmented Top Trading Cycles (ATTC) rule*, introduced by Fujita et al. (2018), is the natural extension of the TTC rule from the lexicographic domain to the conditionally lexicographic

[23] As is the case for purely lexicographic preferences (but not responsive preferences), there is exactly one P_i' obtained from P_i by dropping object x.

[24] We can construct $\tau_{P_i}^{-x}$ as follows. For each vertex v of τ_{P_i}, let T_v denote the maximal subtree of τ_{P_i} consisting of a vertex v of τ_{P_i} together with all of its successors, and let v' be the child of v whose incoming edge (v, v') is labeled with $\neg o(v)$. For each vertex v of τ_{P_i} such that $o(v) = x$, simply replace T_v with the subtree $T_{v'}$ (or the empty tree if v is a leaf of τ_{P_i}).

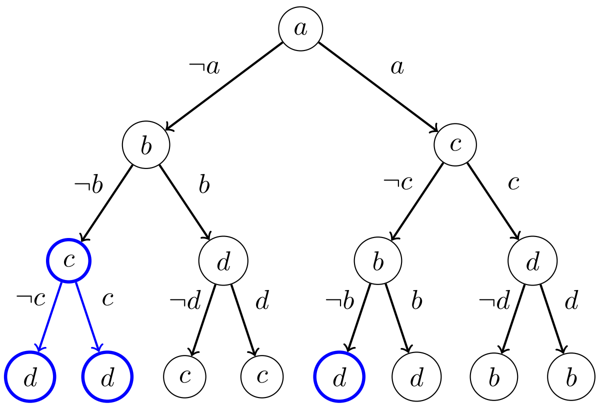

Figure 5: The LP tree τ_i

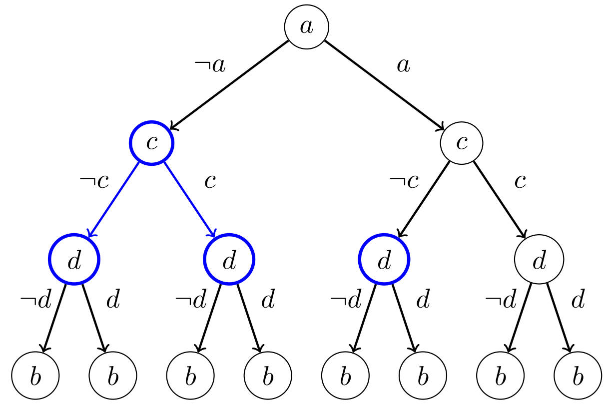

Figure 6: The LP tree τ_i' obtained by dropping b

Notes: Figure 5 displays the original LP tree from Figures 1 and 3. To illustrate the drop strategy where agent i drops object b, we construct the LP tree τ_i' shown in Figure 6. At each internal vertex v labeled b, we do the following: (i) remove the subtree consisting of v and all of its descendants, and (ii) append the subtree, shown in blue, consisting of the child of v reached via the "not-in edge" together with all of its descendants. Then, at each leaf of this modified tree, we append two child nodes labeled b, connected via an "in edge" and a "not-in edge" accordingly. ⋄

domain.

At each step of the ATTC algorithm, every agent points to her most-preferred unassigned object, *conditional on the objects already assigned to her*, and every unassigned object points to its owner. There exists at least one cycle and, as in the TTC algorithm, we execute only the cycle containing the *minimum agent*. All objects involved in this cycle are then removed. If unassigned objects remain, then the procedure continues to the next step; otherwise, it terminates with the corresponding allocation.

Formally, given a preference profile $P \in \mathcal{P}$, the *Augmented Top Trading Cycles (ATTC) rule* selects the allocation $\varphi^{\text{ATTC}}(P)$ determined by the *ATTC algorithm* at P. We denote this algorithm as ATTC(P).

Algorithm: ATTC(P)

Input: A preference profile $P \in \mathcal{CL}$.

Output: An allocation $\varphi^{\text{ATTC}}(P)$.

Initialization: Set $\mu^0 := (\emptyset)_{i \in N}$ and $O^1 := O$.

Step $t \geq 1$: 1. **(Graph construction)** Construct a bipartite directed graph with independent vertex sets N and O^t, and edge sets defined as follows:

(a) For each agent $i \in N$, there is a directed edge from i to $\max_{P_i}(O^t \mid \mu_i^{t-1})$.

(b) For each object $o \in O^t$, there is a directed edge from o to its owner, $\omega^{-1}(o)$.

2. **(Cycle selection)** There is at least one cycle.

   (a) Let $\mathcal{C}_t(P)$ denote the set of cycles that arise at Step t.

   (b) Let $C_t(P)$ be the cycle in $\mathcal{C}_t(P)$ containing the minimum agent: i.e., $\min N(C_t(P)) \leq \min N(C)$ for all $C \in \mathcal{C}_t(P)$.

3. **(Assignment)** Assign to each agent $i \in N(C_t(P))$ the object $\max_{P_i}(O^t \mid \mu_i^{t-1})$. That is, let $\mu^t = (\mu_i^t)_{i \in N}$ be such that

   (a) for all $i \in N(C_t(P))$, $\mu_i^t = \mu_i^{t-1} \cup \left\{\max_{P_i}(O^t \mid \mu_i^{t-1})\right\}$, and

   (b) for all $i \in N \backslash N(C_t(P))$, $\mu_i^t = \mu_i^{t-1}$.

4. **(Removal)** Let $O^{t+1} \coloneqq O^t \backslash O(C_t(P))$ be the set of objects remaining at Step $t+1$.

   (a) If $O^{t+1} \neq \emptyset$, then proceed to Step $t+1$.

   (b) If $O^{t+1} = \emptyset$, then proceed to Termination.

Termination: Because O is finite and $|O^1| > |O^2| > \cdots > |O^t|$, the algorithm terminates at some step T. Return the allocation $\varphi^{\text{ATTC}}(P) \coloneqq \mu^T$.

6.3 Results for Conditionally Lexicographic Preferences

Despite its additional flexibility, the conditionally lexicographic domain retains some of the desirable features of the lexicographic domain. In particular, *Pareto efficiency* and *ig-efficiency* are equivalent on this domain, making *Pareto efficient* allocations relatively easy to identify. This equivalence is formalized in the following proposition.

Proposition 5. *On the conditionally lexicographic domain, an allocation is ig-efficient if and only if it is Pareto efficient.*

This equivalence implies that myopic procedures like the ATTC algorithm, which greedily execute Pareto-improving single-object exchanges until no such exchange remains, will always result in *Pareto efficient* allocations on this domain. Moreover, Fujita et al. (2018) establish that the ATTC rule is *core selecting*, a property stronger than both *Pareto efficiency* and *individual rationality*. They also show that it is *NP-hard* for an agent to manipulate the ATTC rule, and even if a beneficial manipulation is found, the gains from manipulating are bounded. Specifically, no manipulation can yield an object that the agent prefers over her most-preferred object obtained by truth-telling. Complementing the results of Fujita et al. (2018), we establish that the ATTC rule is *drop strategy-proof* on the conditionally lexicographic domain.

Proposition 6. *On the conditionally lexicographic domain, the ATTC rule is drop strategy-proof.*

Proposition 6 extends the work of Altuntaş et al. (2023), who established that the TTC rule is *drop strategy-proof* on the lexicographic domain. Unlike in their paper, we consider a weak notion of *drop strategy-proofness* that defends only against manipulations where agents drop objects they do not own. Nonetheless, the ATTC rule actually satisfies the stronger notion, which also defends against manipulations where agents drop objects they *do* own. However, for our characterization result, the weak version suffices.

Theorem 6. *On the conditionally lexicographic domain, a rule satisfies*

- *balancedness,*
- *ig-efficiency (or Pareto efficiency),*
- *the worst endowment lower bound, and*
- *drop strategy-proofness*

if and only if it is the ATTC rule.

Theorem 6 states that the ATTC rule is categorically determined by *balancedness*, *Pareto efficiency*, the *worst endowment lower bound*, and *drop strategy-proofness*, addressing an open question posed by Fujita et al. (2018, p. 531). This effectively extends our previous characterization of the TTC rule (Theorem 2) from the lexicographic domain to the broader conditionally lexicographic domain.

Beyond Conditionally Lexicographic Preferences

We conclude this section by highlighting the difficulty in extending our analysis beyond the conditionally lexicographic domain. On any domain comprising monotonic preferences that is larger than the conditionally lexicographic domain, the equivalence between *Pareto efficiency* and *ig-efficiency* breaks down.[25]

Proposition 7. *Within the domain of monotonic preferences, the conditionally lexicographic domain is a maximal domain on which ig-efficiency and Pareto efficiency are equivalent.*

(That is, if $\mathcal{CL} \subsetneq \mathcal{P} \subseteq \mathcal{M}$, then there is a set N of agents, a preference profile $P \in \mathcal{P}$, and an allocation $\mu \in \mathcal{A}$ such that μ is ig-efficient but not Pareto efficient at P.)

[25] It is straightforward to strengthen Proposition 7 if we broaden our definition of allocations to allow agents to be assigned empty bundles. In this case, the conditionally lexicographic domain is a maximal domain *among all strict preferences* on which the two notions coincide.

7 Conclusion

In this paper, we provide several characterizations of generalized versions of Gale's TTC rule for multi-object reallocation problems. Specifically, we present the first characterizations of the TTC rule on the responsive preference domain and the first characterization of the ATTC rule on the conditionally lexicographic preference domain.

Our analysis sheds light on the trade-offs involved in multi-object (re)allocation with responsive preferences. Given the inherent incompatibility among efficiency, individual rationality, and strategic robustness, other prominent allocation rules typically fulfill only two of the three objectives, often neglecting the third. Although the TTC rule does not meet the most stringent efficiency and incentive requirements, it satisfies *ig-efficiency* and *truncation-proofness*, in addition to *individual rationality*. Furthermore, within the class of *balanced* and *individual-good-based* rules, there is no other rule that meets these criteria. Thus, the TTC rule performs remarkably well according to all three objectives.

Our work highlights the potential of the TTC and ATTC rules in settings where agents have complex preferences but practical considerations necessitate simple reporting languages.

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A Omitted proofs

A.1 Proof of Lemma 1

Suppose φ is an *individual-good-based* rule satisfying *balancedness* and *individual rationality*.

Let $P \in \mathcal{R}$ be a profile of responsive preferences. Toward contradiction, suppose $\mu \coloneqq \varphi(P)$ violates the *worst endowment lower bound* at P. Then there is some agent $i \in N$ such that $\mu_i \nsubseteq \{o \in O \mid o \; R_i \; \min_{P_i}(\omega_i)\}$. By *balancedness*, we must have $|\mu_i| = |\omega_i|$. Because $\min_{P_i}(\mu_i)$ is worse than $\min_{P_i}(\omega_i)$ at P_i, there is a responsive preference relation $P'_i \in \mathcal{R}_i$ such that $\succ^{P'_i} = \succ^{P_i}$ and $\omega_i \; P'_i \; \mu_i$. By *individual rationality*, we must have $\varphi_i(P'_i, P_{-i}) \neq \mu_i$. However, $\varphi(P'_i, P_{-i}) = \varphi(P) = \mu$ because φ is *individual-good-based*. This is a contradiction. ■

A.2 Proof of Lemma 2

Suppose φ satisfies *drop strategy-proofness* and the *worst endowment lower bound*. Let P'_i be a subset drop strategy obtained from P_i by dropping some subset $X \subseteq O \backslash \omega_i$. Suppose that $X = \{x_1, x_2, \ldots, x_k\}$, where $x_1 \; P_i \; x_2 \; P_i \cdots P_i \; x_k$. Then P'_i is obtained from P_i by successively performing k drop strategies. That is, $P'_i = P_i^k$, where $P_i^0 = P_i$ and $P_i^1, \ldots, P_i^k$ are such that, for each $\ell \in \{1, \ldots, k\}$, P_i^ℓ is obtained from $P_i^{\ell-1}$ by dropping object x_ℓ.

Claim 1. For each $\ell \in \{1, \ldots, k\}$, $\varphi_i(P_i^{\ell-1}, P_{-i}) \; R_i \; \varphi_i(P_i^\ell, P_{-i})$.

Proof of Claim 1. The proof is by induction on ℓ. Clearly, $\varphi_i(P) = \varphi_i(P_i^0, P_{-i}) \; R_i \; \varphi_i(P_i^1, P_{-i})$ by *drop strategy-proofness*. For the inductive step, suppose that $\ell \in \{1, \ldots, k-1\}$ is such that

$$\varphi_i(P) = \varphi_i(P_i^0, P_{-i}) \; R_i \; \varphi_i(P_i^1, P_{-i}) \; R_i \cdots R_i \; \varphi_i(P_i^\ell, P_{-i}).$$

It suffices to show that $\varphi_i(P_i^\ell, P_{-i}) \; R_i \; \varphi_i(P_i^{\ell+1}, P_{-i})$. By *drop strategy-proofness*, we have $\varphi_i(P_i^\ell, P_{-i}) \; R_i^\ell \; \varphi_i(P_i^{\ell+1}, P_{-i})$. Moreover, the *worst endowment lower bound* implies that

$$\varphi_i(P_i^\ell, P_{-i}) \subseteq O \backslash \{x_1, \ldots, x_\ell\} \quad \text{and} \quad \varphi_i(P_i^{\ell+1}, P_{-i}) \subseteq O \backslash \{x_1, \ldots, x_{\ell+1}\} \subseteq O \backslash \{x_1, \ldots, x_\ell\}.$$

Because $\succ^{P_i^\ell}$ agrees with $\succ^{P_i}$ on $O \backslash \{x_1, \ldots, x_\ell\}$, and each of P_i and P_i^ℓ are lexicographic, we have $\varphi_i(P_i^\ell, P_{-i}) \; R_i \; \varphi_i(P_i^{\ell+1}, P_{-i})$, as desired. □

It follows from Claim 1 that

$$\varphi_i(P) = \varphi_i(P_i^0, P_{-i}) \; R_i \; \varphi_i(P_i^1, P_{-i}) \; R_i \cdots R_i \; \varphi_i(P_i^k, P_{-i}) = \varphi_i(P'_i, P_{-i}).$$

Therefore, $\varphi_i(P) \; R_i \; \varphi_i(P'_i, P_{-i})$. ■

A.3 Proof of Theorem 1

Toward contradiction, suppose that φ satisfies the properties but $\varphi \neq \varphi^{\mathrm{TTC}}$.

For each $P \in \mathcal{P}^N$, and each $t \in \mathbb{N}$, let $\mathcal{C}_t(P)$ be the set of cycles that obtain at step t of TTC(P). Similarly, let $C_t(P) \in \mathcal{C}_t(P)$ be the cycle that is executed at step t of TTC(P).

Denote the *size* of a profile P by $s(P) = \sum_{i\in N} |\{o \in O \mid oR_i \min_{P_i}(\omega_i)\}|$.

The *similarity* between φ and φ^{TTC} is a function $\rho : \mathcal{P}^N \to \mathbb{N}$ such that, for all $P \in \mathcal{P}$, $\rho(P)$ is defined as follows.

- Step 1. If $\varphi(P)$ does not execute $C_1(P)$, then $\rho(P) = 1$. Suppose $\varphi(P)$ executes $C_1(P)$. If TTC(P) terminates at Step 1, then $\rho(P) = |O| + 1$; otherwise, proceed to Step 2.
- Step t (≥ 2). If $\varphi(P)$ does not execute $C_t(P)$, then $\rho(P) = t$. Suppose $\varphi(P)$ executes $C_t(P)$. If TTC(P) terminates at Step t, then $\rho(P) = |O| + 1$; otherwise, proceed to Step $t + 1$.

Note that, for all $P \in \mathcal{P}^N$, (i) $\rho(P) \leq |O| + 1$, and (ii) $\rho(P) = |O| + 1$ if and only if $\varphi(P) = \varphi^{\mathrm{TTC}}(P)$.

Let P be a profile such that for any other profile $\tilde{P} \neq P$, either

(1) $\rho(P) < \rho(\tilde{P})$, or

(2) $\rho(P) = \rho(\tilde{P})$ and $s(P) \leq s(\tilde{P})$.

Let $t \coloneqq \rho(P)$. Since $\rho(P) = t$, $\varphi(P)$ executes the cycles $C_1(P), C_2(P), \ldots, C_{t-1}(P)$ but not $C_t(P)$. Let $C \coloneqq C_t(P)$ and $O^t \coloneqq O \setminus \bigcup_{\tau=1}^{t-1} O(C_\tau(P))$. Suppose that

$$C = \left(i_1, o_{i_2}, i_2, o_{i_3}, \ldots, i_{k-1}, o_{i_k}, i_k, o_{i_{k+1}}, i_{k+1} = i_1\right).$$

Because $\varphi(P)$ does not execute C, there is an agent $i_\ell \in N(C)$ such that, although agent i_ℓ points to $o_{i_{\ell+1}}$ on C, she does not receive $o_{i_{\ell+1}}$ at $\varphi(P)$, i.e., $o_{i_{\ell+1}} \notin \varphi_i(P)$. Note that, by the definition of TTC(P), $o_{i_{\ell+1}}$ is agent i_ℓ's most-preferred object in O^t at P_{i_ℓ}, i.e., $o_{i_{\ell+1}} = \max_{P_{i_\ell}}(O^t)$. Without loss of generality, let $i_\ell = i_k$. Thus, $o_{i_1} \notin \varphi_{i_k}(P)$.

Let X_{i_k} be the set of all objects in $O \setminus O^t$ that are assigned to agent i_k, i.e., $X_{i_k} = \varphi_{i_k}(P) \setminus O^t$. We next show that, apart from the objects in X_{i_k}, agent i_k only receives the remainder of her endowment.

Claim 2. $\varphi_{i_k}(P) \cap O^t = \omega_{i_k} \cap O^t$.

Proof of Claim 2. Suppose otherwise. By *balancedness* and the fact that $|\varphi_{i_k}(P) \setminus O^t| = |\omega_{i_k} \setminus O^t|$, we must have $|\varphi_{i_k}(P) \cap O^t| = |\omega_{i_k} \cap O^t|$. Consequently, $\varphi_{i_k}(P) \cap O^t \neq \omega_{i_k} \cap O^t$ implies there is an object $o' \in \varphi_{i_k}(P) \cap O^t$ with $o' \notin \omega_{i_k}$.

By the definition of TTC(P), we know that o_{i_1} is agent i_k's most-preferred object in O^t, i.e., for each $o \in O^t$, $o_{i_1} \; R_{i_k} \; o$. Because $o' \in \varphi_{i_k}(P) \cap O^t$ and $o_{i_1} \notin \varphi_{i_k}(P)$, we have $o_{i_1} \; P_{i_k} \; o'$. By the *worst endowment lower bound*,

$$o' \; P_{i_k} \; w_{i_k}(P_{i_k}).$$

Let P'_{i_k} be the truncation of P_{i_k} at o_{i_1}. That is, P'_{i_k} is obtained from P_{i_k} by dropping the set $\{o \in O \backslash \omega_{i_k} \mid o_{i_1} \; P_{i_k} \; o\}$.[26] Let $P' \coloneqq (P'_{i_k}, P_{-i_k})$.

By the definition of TTC(P), we see that until step t, all top trading cycles that are executed via TTC at P and P' are the same, i.e., $C_\tau(P) = C_\tau(P')$ for $\tau = 1, \ldots, t$. By the selection of P, we know that $\rho(P) = t \leq \rho(P')$. Thus, all cycles in $\{C_1(P), \ldots, C_{t-1}(P)\}$ are executed at $\varphi(P')$. Therefore,

$$X_{i_k} \subseteq \varphi_{i_k}(P').$$

Since φ is *truncation-proof*, we know that i_k cannot be better off by misreporting P'_{i_k} at P. Together with $X_{i_k} \subseteq \varphi_{i_k}(P')$ and $o_{i_1} \notin \varphi_{i_k}(P)$, we conclude that i_k cannot receive o_{i_1} at $\varphi(P')$, i.e., $o_{i_1} \notin \varphi_{i_k}(P')$. Therefore, C is not executed at $\varphi(P')$, which means that $\rho(P') \leq t$. As a result, we find that $\rho(P') = t$. However, since P'_{i_k} is obtained from P_{i_k} by dropping the subset $\{o \in O \backslash \omega_{i_k} \mid o_{i_1} \; P_{i_k} \; o\}$, and $o_{i_1} \; P_{i_k} \; o' \; P_{i_k} \; w_{i_k}(P_{i_k})$, we see that $s(P') < s(P)$. This contradicts the choice of P. $\square$

Next, we show that at least two agents are involved in C.

Claim 3. $|N(C)| > 1$.

Proof. Toward contradiction, suppose that $|N(C)| = 1$, that is to say, $i_1 = i_k$ and $C = (i_1, o_{i_1}, i_1)$. Then $o_{i_1} \in \omega_{i_k}$ and Claim 2 implies that $o_{i_1} \in \varphi_{i_k}(P)$, a contradiction. $\square$

By Claim 2 and Claim 3, agent i_{k-1} points to o_{i_k} on C, yet she does not receive o_{i_k}. An argument similar to the proof of Claim 2 implies that $\varphi_{i_{k-1}}(P') \cap O^t = \omega_{i_{k-1}} \cap O^t$.

Proceeding by induction, we conclude that for each agent i_ℓ that is involved in C, we have $\varphi_{i_\ell}(P) \cap O^t = \omega_{i_\ell} \cap O^t$.

However, this means that φ is not *ig-efficient* (or *Pareto efficient*), as agents in $N(C)$ can benefit by execution of C. This contradiction completes the proof of Theorem 1. $\blacksquare$

A.4 Proof of Proposition 3

We only show that the TTC rule is *truncation-proof* as other properties can be easily verified.

Let $P \in \mathcal{R}$ be any profile of responsive preferences. Let $i \in N$, $x \in O \setminus \omega_i$, and let P'_i be a truncation of P_i at x. Let $P' \coloneqq (P'_i, P_{-i})$. Let $U \coloneqq \{o \in O \mid o \; R_i \; x\}$. There are two cases.

[26] If we use *drop strategy-proofness*, we can consider another preference P'_{i_k} obtained from P_{i_k} by dropping object o'.

Case 1: Suppose $\varphi_i^{\text{TTC}}(P) \subseteq U$. Then, by the definition of TTC, we know that all top trading cycles that are obtained at P and P' are exactly the same. As a result, $\varphi_i^{\text{TTC}}(P) = \varphi_i^{\text{TTC}}(P')$. So agent i is not better off by misreporting P_i'.

Case 2: Suppose $\varphi_i^{\text{TTC}}(P) \nsubseteq U$. Then agent i receives at least one object in $O \setminus U$ at $\varphi^{\text{TTC}}(P)$. Assume that at $\text{TTC}(P)$, the earliest step at which agent i points to an object in $O \setminus U$ is t.

Let $O_t \subseteq O$ be the set of objects that are remaining at step t of $\text{TTC}(P)$. Since $\succ^{P_i}$ and $\succ^{P_i'}$ agree on U,[27] we know that the top trading cycles that are executed at steps $1, \ldots, t-1$ of $\text{TTC}(P)$ and $\text{TTC}(P')$ are the same. Thus, at step t of $\text{TTC}(P')$, the set of remaining objects is also O_t. Moreover, we have

$$\varphi_i^{\text{TTC}}(P) \setminus O_t = \varphi_i^{\text{TTC}}(P') \setminus O_t. \tag{5}$$

By the definition of P_i', we know that all objects in $O \setminus U = \{o \in O \setminus \omega_i \mid x \; P_i \; o\}$ are ranked below i's least-preferred object in ω_i at $\succ^{P_i'}$ (i's least-preferred object in ω_i is the same at P_i and P_i'). Thus, by the definition of TTC, from step t of $\text{TTC}(P')$, agent i only points to (and hence receives) her endowed objects. That is,

$$\varphi_i^{\text{TTC}}(P') \cap O_t = \omega_i \cap O_t. \tag{6}$$

Again, by the definition of TTC, from step t of $\text{TTC}(P)$, agent i only points to (and hence receives) some objects that are weakly better (with respect to $\succ^{P_i}$) than her endowed objects. In other words, there is a bijection $\sigma : \omega_i \cap O_t \to \varphi_i^{\text{TTC}}(P) \cap O_t$ such that for each $o \in \omega_i \cap O_t$, $\sigma(o) \succeq^{P_i} o$. Together with (5) and (6), we conclude that there is a bijection $\pi : \varphi_i^{\text{TTC}}(P') \to \varphi_i^{\text{TTC}}(P)$ such that

$$\text{for each } o \in \varphi_i^{\text{TTC}}(P'), \pi(o) \succeq^{P_i} o. \tag{7}$$

Finally, recall that i's original preference relation P_i is responsive. Therefore, we must have $\varphi_i^{\text{TTC}}(P) \; R_i \; \varphi_i^{\text{TTC}}(P')$.[28] Thus, agent i is not better off by misreporting P_i'. ■

A.5 Proof of Proposition 6

Let $P \in \mathcal{CL}^N$, $i \in N$, and let $\overline{P}_i \in \mathcal{D}(P_i)$. Let $\overline{P} := \left(\overline{P}_i, P_{-i}\right)$ and suppose that $\varphi_i^{\text{ATTC}}\left(\overline{P}\right) R_i \varphi_i^{\text{ATTC}}(P)$. It suffices to show that $\varphi_i^{\text{ATTC}}\left(\overline{P}\right) = \varphi_i^{\text{ATTC}}(P)$.

Let $\varphi_i^{\text{ATTC}}(P) = \{x_1, \ldots, x_m\} =: X$. By relabeling the objects if necessary, we can assume

[27]Formally, $\succ^{P_i} \cap (U \times U) = \succ^{P_i'} \cap (U \times U)$.

[28]Starting from $\varphi_i^{\text{TTC}}(P')$, replace each object $o \in \varphi_i^{\text{TTC}}(P')$ with object $\pi(o)$, one at a time, and apply the definition of responsiveness.

that, for each $k \in \{1, \ldots, m\}$, x_k is the kth object assigned to agent i during $\text{ATTC}(P)$. Let $\tau_i = \tau_{P_i}$ denote the LP tree associated with P_i. For each $k \in \{1, \ldots, m\}$, let v_k denote the unique vertex of $\tau_i(X)$ whose label is x_k. This means that

$$\text{for each } k \in \{1, \ldots, m\}, \quad X \cap a(v_k) = \{x_1, \ldots, x_k\}.$$

Let $\overline{X} := \varphi_i^{\text{ATTC}}\left(\overline{P}\right)$. We show by induction that

$$\text{for each } k \in \{1, \ldots, m\}, \quad \overline{X} \cap a(v_k) = \{x_1, \ldots, x_k\}.$$

Base case: $k = 1$. Consider any object $x \in a(v_1) \setminus \{x_1\}$. That is, x labels one of the (strict) ancestors of v_1 in $\tau_i(X)$, and $x \notin \varphi_i^{\text{ATTC}}(P)$. This means that, although agent i may point to x during $\text{ATTC}(P)$, agent i is not on the cycle of $\text{ATTC}(P)$ that contains object x. It follows that agent i does not belong to the cycle of $\text{ATTC}\left(\overline{P}\right)$ that contains object x either. Hence, $x \notin \varphi_i^{\text{ATTC}}\left(\overline{P}\right)$.

Now consider object x_1. Because $\varphi_i^{\text{ATTC}}\left(\overline{P}\right) R_i \varphi_i^{\text{ATTC}}(P)$ and $\overline{P}_i \in \mathcal{CL}$, we must have $x_1 \in \varphi_i^{\text{ATTC}}\left(\overline{P}\right)$. Consequently, $\overline{X} \cap a(v_1) = \{x_1\}$.

Inductive step. Let $k \in \{1, \ldots, m-1\}$ be such that for each $\ell \in \{1, \ldots, k\}$, $\overline{X} \cap a(v_\ell) = \{x_1, \ldots, x_\ell\}$. We show that $\overline{X} \cap a(v_{k+1}) = \{x_1, \ldots, x_{k+1}\}$.

Consider any object $x \in (a(v_{k+1}) \setminus \{x_{k+1}\}) \setminus a(v_k)$. That is, x labels a vertex that appears *after* v_k and *before* v_{k+1} on $\tau_i(X)$, and $x \notin \varphi_i^{\text{ATTC}}(P)$. This means that, although agent i may point to x during $\text{ATTC}(P)$, agent i is not on the cycle of $\text{ATTC}(P)$ that contains object x. It follows that agent i does not belong to the cycle of $\text{ATTC}\left(\overline{P}\right)$ that contains object x either. Hence, $x \notin \varphi_i^{\text{ATTC}}\left(\overline{P}\right)$.

Now consider object x_{k+1}. Because $\varphi_i^{\text{ATTC}}\left(\overline{P}\right) R_i \varphi_i^{\text{ATTC}}(P)$, we must have $x_{k+1} \in \varphi_i^{\text{ATTC}}\left(\overline{P}\right)$. Consequently, $\overline{X} \cap a(v_{k+1}) = \{x_1, \ldots, x_{k+1}\}$.

By the principle of induction, it then follows that $\overline{X} \cap a(v_m) = \{x_1, \ldots, x_m\}$. Because $\varphi_i^{\text{ATTC}}\left(\overline{P}\right)$ and $\varphi_i^{\text{ATTC}}(P)$ contain the same number of objects, we must have $\varphi_i^{\text{ATTC}}\left(\overline{P}\right) = \varphi_i^{\text{ATTC}}(P)$, as we needed to show. ■

A.6 Proof of Theorem 5

It is known that TTC satisfies all properties. Therefore, it suffices to prove the uniqueness. Toward contradiction, suppose that φ satisfies the properties but $\varphi \neq \varphi^{\text{TTC}}$.

The notions of *size* s and *similarity* ρ are defined exactly as in the proof of Theorem 1.

Recall that, for all $P \in \mathcal{P}^N$, (i) $\rho(P) \leq |O| + 1$, and (ii) $\rho(P) = |O| + 1$ if and only if $\varphi(P) = \varphi^{\mathrm{TTC}}(P)$.

Suppose that $\min_{P \in \mathcal{P}^N} \rho(P) = t$. Then $\varphi \neq \varphi^{\mathrm{TTC}}(P)$ implies that $t \leq |O|$. Among all profiles in $\{P \in \mathcal{P}^N \mid \rho(P) = t\}$, let P be one whose *size* is smallest. Hence, for any profile $\tilde{P} \in \mathcal{P}^N$, we have

(1) $\rho(P) < \rho(\tilde{P})$, or

(2) $\rho(P) = \rho(\tilde{P})$ and $s(P) \leq s(\tilde{P})$.

Since $\rho(P) = t$, $\varphi(P)$ executes the cycles $C_1(P), C_2(P), \ldots, C_{t-1}(P)$ but not $C_t(P)$. Let $\mu \coloneqq \varphi^{\mathrm{TTC}}(P)$, $C \coloneqq C_t(P)$, and $O^t \coloneqq O \setminus \bigcup_{\tau=1}^{t-1} O(C_\tau(P))$. Thus, there is an agent $i \in N(C)$ such that, although agent i points to μ_i on C, she does not receive μ_i at $\varphi(P)$, i.e., $\mu_i \neq \varphi_i(P)$. Note that, by the definition of TTC(P), μ_i is agent i's most-preferred object in O^t at P_i, i.e., $\mu_i = \max_{P_i}(O^t)$. Thus,

$$\mu_i \; P_i \; \varphi_i(P). \tag{8}$$

If $|N(C)| = 1$, then $C = (i, o_i, i)$ and $\mu_i = o_i \; P_i \; \varphi_i(P)$, which violates *individual rationality*. Thus, $|N(C)| \geq 2$.

Next, we show that $\varphi_i(P) = o_i$. Toward contradiction, suppose that $\varphi_i(P) \neq o_i$. By (8) and *individual rationality*, $\mu_i \; P_i \; \varphi_i(P) \; P_i \; o_i$.

Let P'_i be the truncation of P_i at μ_i, i.e., $P'_i : \ldots, \mu_i, o_i, \ldots$[29] Let $P' \coloneqq (P'_i, P_{-i})$. Then $s_i(P'_i) < s_i(P_i)$ and $s(P') < s(P)$.

By the definition of TTC, for each step $\tau \in \{1, \ldots, t\}$, the procedures TTC$(P)$ and TTC(P') generate and execute the same cycles, i.e., for each τ, $C_\tau(P) = C_\tau(P')$. Moreover, the choice of P implies that $\rho(P') \geq \rho(P) = t$. Thus, $\varphi(P')$ executes all cycles in $\{C_1(P), \ldots, C_{t-1}(P)\}$. Thus, $\varphi_i(P') \in O^t$. Moreover, since φ is *truncation-proof*, we know that i cannot be better of by misreporting P'_i at P. Together with (8), we conclude that $\mu_i \neq \varphi_i(P')$. Consequently, $\varphi(P')$ does not execute $C_t(P')$, which means that $\rho(P') = t$. But then we have $\rho(P') = \rho(P)$ and $s(P') < s(P)$, which contradicts the choice of P! Thus, we conclude that $\varphi_i(P) = o_i$.

Let j be the agent who points to o_i on C, i.e., $j \in N(C)$ is such that $\mu_j = o_i$. Then $\varphi_i(P) = o_i = \mu_j$ implies that $\varphi_j(P) \neq \mu_j$. Thus, $\mu_j \; P_j \; \varphi_j(P)$. A similar argument shows that $\varphi_j(P) = o_j$.

By repeating the same argument for each agent in $N(C)$, we can show that each agent in $N(C)$ is assigned her endowment at $\varphi(P)$, i.e., for each $i' \in N(C)$, $\varphi_{i'}(P) = o_{i'}$. But then φ is not *Pareto efficient*, because the agents in $N(C)$ can benefit by executing C. This contradiction completes the proof of Theorem 5.

[29] A similar argument using *drop strategy-proofness* applies if P'_i is obtained from P_i by dropping object $\varphi_i(P)$.

A.7 Proof of Theorem 6

The proof here is almost identical to the proof of Theorem 1. The main difference is that, since we focus on conditionally lexicographic preferences, any statements that pertained to an agent's most-preferred object in Theorem 1 now pertain to an agent's most-preferred object *conditional on receiving some set of objects*.

Toward contradiction, suppose that φ satisfies the properties but $\varphi \neq \varphi^{ATTC}$.

For each $P \in \mathcal{P}^N$, and each $t \in \mathbb{N}$, let $\mathcal{C}_t(P)$ be the set of cycles that obtain at step t of ATTC (P). Similarly, let $C_t(P) \in \mathcal{C}_t(P)$ be the cycle that is executed at step t of ATTC (P).

Denote the *size* of a profile P by $\sum_{i \in N} \sum_{X \in 2^O} |a(w_{P_i}(\omega_i \mid X))|$. The *similarity* ρ is defined exactly as in the proof of Theorem 1.

Let P be a profile such that for any other profile $\tilde{P} \neq P$, either

(1) $\rho(P) < \rho(\tilde{P})$, or

(2) $\rho(P) = \rho(\tilde{P})$ and $s(P) \leq s(\tilde{P})$.

Let $t \coloneqq \rho(P)$. Since $\rho(P) = t$, $\varphi(P)$ executes the cycles $C_1(P), C_2(P), \ldots, C_{t-1}(P)$ but not $C_t(P)$. Let $C \coloneqq C_t(P)$ and $O^t \coloneqq O \setminus \bigcup_{\tau=1}^{t-1} O(C_\tau(P))$. Suppose that

$$C = \left(i_1, o_{i_2}, i_2, o_{i_3}, \ldots, i_{k-1}, o_{i_k}, i_k, o_{i_{k+1}}, i_{k+1} = i_1\right).$$

Because $\varphi(P)$ does not execute C, there is an agent $i_\ell \in N(C)$ such that, although agent i_ℓ points to $o_{i_{\ell+1}}$ on C, she does not receive $o_{i_{\ell+1}}$ at $\varphi(P)$, i.e., $o_{i_{\ell+1}} \notin \varphi_i(P)$. Note that, by the definition of ATTC(P), $o_{i_{\ell+1}}$ is agent i_ℓ's most-preferred object in O^t at P_{i_ℓ} conditional on receiving $\varphi_{i_\ell}(P) \setminus O^t$, i.e., $o_{i_{\ell+1}} = \max_{P_{i_\ell}}(O^t \mid \varphi_{i_\ell}(P) \setminus O^t)$. Without loss of generality, let $i_\ell = i_k$. Thus, $o_{i_1} \notin \varphi_{i_k}(P)$.

Let X_{i_k} be the set of all objects in $O \setminus O^t$ that are assigned to agent i_k, i.e., $X_{i_k} = \varphi_{i_k}(P) \setminus O^t$. We next show that, apart from the objects in X_i, agent i_k only receives the remainder of her endowment.

Claim 4. $\varphi_{i_k}(P) \cap O^t = \omega_{i_k} \cap O^t$.

Proof of Claim 4. Suppose otherwise. By *balancedness* and the fact that $|\varphi_{i_k}(P) \setminus O^t| = |\omega_{i_k} \setminus O^t|$, we must have $|\varphi_{i_k}(P) \cap O^t| = |\omega_{i_k} \cap O^t|$. Consequently, $\varphi_{i_k}(P) \cap O^t \neq \omega_{i_k} \cap O^t$ implies there is an object $o' \in \varphi_{i_k}(P) \cap O^t$ with $o' \notin \omega_{i_k}$. By the *worst endowment lower bound*,

$$o' \in a\left(w_{P_{i_k}}(\omega_{i_k} \mid \varphi_{i_k}(P))\right).$$

By the definition of ATTC(P), we know that o_{i_1} is agent i_k's most-preferred object in O^t

conditional on receiving X_{i_k}, i.e., $o_{i_1} = \max_{P_{i_k}}(O^t \mid X_{i_k})$. Because $o' \in \varphi_{i_k}(P) \cap O^t$ and $o_{i_1} \notin \varphi_{i_k}(P)$, we have $o_{i_1} \neq o'$.

Let P'_{i_k} be the drop strategy for P_{i_k} obtained by dropping object x. Let $P' \coloneqq (P'_{i_k}, P_{-i_k})$. By the definition of TTC, we see that until step t, all top trading cycles that are executed via TTC at P and P' are the same, i.e., $C_\tau(P) = C_\tau(P')$ for $\tau = 1, \ldots, t$. By the selection of P, we know that $\rho(P) = t \leq \rho(P')$. Thus, all cycles in $\{C_1(P), \ldots, C_{t-1}(P)\}$ are executed at $\varphi(P')$. Therefore,

$$X_{i_k} \subseteq \varphi_{i_k}(P').$$

Since φ is *weakly drop strategy-proof*, we know that i_k cannot be better off by misreporting P'_{i_k} at P. Together with $X_{i_k} \subseteq \varphi_{i_k}(P')$ and $o_{i_k} \notin \varphi_{i_k}(P)$, we conclude that i_k cannot receive o_{i_1} at $\varphi(P')$, i.e., $o_{i_1} \notin \varphi_{i_k}(P')$. Therefore, C is not executed at $\varphi(P')$, which means that $\rho(P') \leq t$. As a result, we find that $\rho(P') = t$. However, since P'_{i_k} is obtained from P_{i_k} by dropping an object $o' \in a\left(w_{P_{i_k}}(\omega_{i_k} \mid \varphi_{i_k}(P))\right)$, we see that $s(P') < s(P)$. This contradicts the choice of P. □

Next, we show that at least two agents are involved in C.

Claim 5. $|N(C)| > 1$.

Proof. Toward contradiction, suppose that $|N(C)| = 1$, that is to say, $i_1 = i_k$ and $C = (i_1, o_{i_1}, i_1)$. Then $o_{i_1} \in \omega_{i_k}$ and Claim 4 implies that $o_{i_1} \in \varphi_{i_k}(P)$, a contradiction. □

By Claim 4 and Claim 5, agent i_{k-1} points to o_{i_k} on C, yet she does not receive o_{i_k}. An argument similar to the proof of Claim 4 implies that $\varphi_{i_{k-1}}(P') \cap O^t = \omega_{i_{k-1}} \cap O^t$.

Proceeding by induction, we conclude that for each agent i_ℓ that is involved in C, we have $\varphi_{i_\ell}(P) \cap O^t = \omega_{i_\ell} \cap O^t$.

However, this means that φ is not *ig-efficient*, as agents in $N(C)$ can benefit by execution of C. This contradiction completes the proof of Theorem 6. ■

A.8 Proof of Proposition 7

Consider an agent with monotone preferences that are not conditionally lexicographic. Without loss of generality, let this agent be agent 1. That is, $P_1 \in \mathcal{M} \setminus \mathcal{CL}$. By Proposition 4, there exist disjoint subsets $X, Y \in 2^O$ with $X \neq \emptyset$ such that

$$\text{for all } x \in X, \text{ there exists } Z_x \subseteq X \setminus x \text{ such that } (Y \cup Z_x)\ P_1\ (Y \cup x).$$

Because P_1 is *monotone*, $[Y \cup (X \setminus x)] \; P_1 \; (Y \cup Z_x)$ as $Z_x \subseteq X \setminus x$. Thus, we have

$$\text{for all } x \in X, \quad [Y \cup (X \setminus x)] \; P_1 \; (Y \cup x).$$

Note also that $|X| \geq 3$.[30]

Let $x^* \in X$ be such that

$$\text{for all } x \in X, \quad (Y \cup x^*) \; R_1 \; (Y \cup x).$$

Then it follows that

$$\text{for all } x \in X, \quad [Y \cup (X \setminus x)] \; P_1 \; (Y \cup x^*) \; R_1 \; (Y \cup x).$$

Case 1: Suppose $X \cup Y = O$. Let $N = \{1, 2\}$, and let $P_2 \in \mathcal{L}$ be such that (i) $\max_{P_2}(O) = x^*$, and (ii) for all $x \in X$, $x \; P_2 \; Y$. Then, in particular,

$$x^* \; P_2 \; [Y \cup (X \setminus x^*)] \; R_2 \; (X \setminus x^*).$$

Consider the allocations

$$\mu \coloneqq (Y \cup x^*, X \backslash x^*) \quad \text{and} \quad \overline{\mu} \coloneqq (Y \cup (X \backslash x^*), x^*).$$

Then μ is not *Pareto efficient* at P because it is Pareto-dominated by $\overline{\mu}$ at P.

We claim that μ is *ig-efficient* at P. Indeed, consider any single-object exchange identified by the cycle $C = (1, x, 2, y, 1)$, where $x \in \mu_2$ and $y \in \mu_1$. Executing the exchange results in the allocation μ' where

$$\mu'_1 = (Y \cup \{x^*, x\}) \backslash y \quad \text{and} \quad \mu'_2 = (X \backslash \{x^*, x\}) \cup y.$$

If $y \in Y$, then this exchange harms agent 2, i.e., $\mu_2 \; P_2 \; \mu'_2$. On the other hand, if $y = x^*$, then this exchange harms agent 1 because $\mu_1 = (Y \cup x^*) \; P_1 \; (Y \cup x) = \mu'_1$. Hence, μ is *ig-efficient*.

Case 2: Suppose $X \cup Y \subsetneq O$. Let $N = \{1, 2, 3\}$ and denote $\overline{O} \coloneqq O \setminus (X \cup Y)$. Let $P_2 \in \mathcal{L}$ be such that (i) $\max_{P_2}(O) = x^*$, and (ii) for all $x \in X$, $x \; P_2 \; (O \setminus X)$. Let $P_3 \in \mathcal{L}$ be such that (i) for all $z \in \overline{O}$, $z \; P_3 \; (X \cup Y)$.

[30] If X is a singleton, say $X = \{x\}$, then $Y \; P_1 \; (Y \cup x)$, a violation of monotonicity. If X contains two objects, say $X = \{x, y\}$, we would have $(Y \cup x) \; P_1 \; (Y \cup y) \; P_1 \; (Y \cup x)$, a violation of transitivity.

Consider the allocations

$$\mu \coloneqq (Y \cup x^*, X \backslash x^*, \overline{O}) \quad \text{and} \quad \overline{\mu} \coloneqq (Y \cup (X \backslash x^*), x^*, \overline{O}).$$

Then μ is not *Pareto efficient* at P because it is Pareto-dominated by $\overline{\mu}$ at P.

We claim that μ is *ig-efficient* at P. Because μ_3 is agent 3's top-ranked $|\overline{O}|$-subset of O, any single-object exchange involving agent 3 must harm agent 3. Thus, any Pareto-improving single-object exchange must involve only agents 1 and 2. However, there is no such exchange by the argument in Case 1. Hence, μ is *ig-efficient*. ■

B Examples

B.1 Independence of Properties in Theorems 1 and 6

We establish the independence of the properties in Theorem 1 by providing examples of rules, different from TTC / ATTC, that violate exactly one of the properties. For each of the examples below we indicate the property that the rule fails (while it satisfies all remaining properties).

Individual-good efficiency: the no-trade rule that always selects the initial allocation satisfies all properties except for *individual-good efficiency*.

The worst endowment lower bound: the serial dictatorships subject to *balancedness*[31] satisfy all properties except for the *worst endowment lower bound*.

Balancedness: the serial dictatorships subject to the *worst endowment lower bound*[32] satisfy all properties except for *balancedness*.

Truncation-proofness / Drop strategy-proofness: the following example is from Altuntaş et al. (2023). Let $N = \{1, 2, 3, 4\}$, and for each $i \in N$, $\omega_i = o_i$. Let P^* be defined as follows (endowment is underlined).

$$P^*_1 : o_2, o_4, \underline{o_1}, o_3;$$

$$P^*_2 : o_1, o_3, \underline{o_2}, o_4;$$

$$P^*_3 : o_2, o_4, \underline{o_3}, o_1;$$

$$P^*_4 : o_1, o_3, \underline{o_4}, o_2.$$

[31] That is, each dictator is assigned the same number of objects as her endowment.

[32] That is, each dictator i is assigned her best bundle among all subsets X of the remaining objects that would not violate the *worst endowment lower bound* (i.e., such that $X \subseteq \{o \in O \mid o\, R_i \min_{P_i}(\omega_i)\}$, where P_i denotes i's preference relation.)

Let φ be defined as follows.

$$\varphi(P) = \begin{cases} (o_4, o_3, o_2, o_1), & \text{if } P = P^* \\ \varphi^{\text{ATTC}}(P), & \text{otherwise.} \end{cases}$$

Note that for lexicographic preferences, $\varphi^{\text{ATTC}} = \varphi^{\text{TTC}}$. Altuntaş et al. (2023) show that φ satisfies *Pareto efficiency* and the *strong endowment lower bound* (hence *balancedness* and the *worst endowment lower bound*), but violates *drop strategy-proofness* / *truncation-proofness*. To see this, note that agent 1 receives o_4 at $\varphi(P^*)$, and she can receive her best object o_2 by misreporting a drop strategy $P_1' : o_2, \underline{o_1}, o_3, o_4$ or a truncation strategy $P_1'' : o_2, \underline{o_1}, o_4, o_3$.

B.2 Other examples

Example 7 (A rule on $\mathcal{L}$ that satisfies *truncation-proofness*, the *worst endowment lower bound* and *individual rationality*, but not *drop strategy-proofness*.)**.** Suppose $N = \{1, 2, 3\}$, and $\omega = (o_1, o_2, o_3)$. Let $P^* \in \mathcal{P}$ be such that

$$P_1^* : o_2, o_3, o_1, \quad P_2^* : o_3, o_1, o_2, \quad P_3^* : o_1, o_2, o_3.$$

Note that each agent $i \in N$ only has two truncation strategies, i.e.,

$$\hat{P}_i : o_{i+1}, o_i, o_{i-1} \quad \text{and} \quad \dot{P}_i : o_i, o_{i+1}, o_{i-1} \pmod 3.$$

Consider the rule φ defined for all $P \in \mathcal{P}$ by

$$\varphi(P) = \begin{cases} \omega, & \text{if } P \in \mathcal{T}_1(P_1^*) \times \mathcal{T}_2(P_2^*) \times \mathcal{T}_3(P_3^*) \\ \varphi^{\text{TTC}}(P), & \text{otherwise.} \end{cases}$$

That is, $\varphi(P) = \omega$ whenever P is such that, for all $i \in N$, P_i is a truncation of P_i^*; otherwise, $\varphi(P) = \varphi^{\text{TTC}}(P)$. By the definition of φ, it is easy to see that φ is *individually rational*. It is straightforward to show that φ is *truncation-proof*. To see that φ is not *drop strategy-proof*, consider

$$P_1' : o_3, o_1, o_2.$$

Then P_1' is obtained from P_1^* by dropping object $o_2 \in O \backslash \omega_1$. Because

$$\varphi_1\left(P_1', P_{-1}^*\right) = \varphi_1^{\text{TTC}}\left(P_1', P_{-1}^*\right) = o_3 \; P_1^* \; o_1 = \varphi_1(P^*),$$

agent 1 can manipulate at P^* by misreporting $P_1' \in \mathcal{D}_1(P_1^*)$. Note that P_1' is not a truncation of P_1^*. $\diamond$

Example 8 (*Drop strategy-proofness* does not imply *truncation-proofness.*)**.** Suppose $N = \{1, 2, 3\}$ and $\omega = (o_1, o_2, o_3)$. Let $\mathcal{P}$ be any domain of preferences containing $\mathcal{L}$. Let φ be a rule on $\mathcal{P}$ such that, for all $P \in \mathcal{P}$,

$$\varphi(P) = \begin{cases} (o_1, o_2, o_3), & \text{if } o_1 \text{ is agent 1's second-ranked object at } P_1 \\ (o_2, o_1, o_3), & \text{if } o_2 \text{ is agent 1's second-ranked object at } P_1 \\ (o_3, o_2, o_1), & \text{if } o_3 \text{ is agent 1's second-ranked object at } P_1. \end{cases}$$

Then φ is *drop strategy-proof*. Indeed, if P_1' $(\neq P_1)$ is a drop strategy obtained from P_1 by dropping any object, then agent 1 receives her least-preferred object at (P_1', P_{-1}), i.e., $\varphi_1(P_1', P_{-1}) = \min_{P_1}(O)$.

Consider $P_1 \in \mathcal{L}$ such that $\succ^{P_1}: o_2, o_3, o_1$. Note that $\varphi_1(P_1, P_{-1}) = o_3$ for each profile P_{-1} of the other agents. Next, consider a truncation strategy obtained from P_1 by dropping $\{o_2, o_3\}$, i.e., such that $\succ^{P_1'}: o_1, o_2, o_3$. We see that $\varphi_1(P_1', P_{-1}) = o_2$. Since $\varphi_1(P_1', P_{-1}) = o_2 \; P_1 \; o_3 = \varphi_1(P)$, we find that φ is not *truncation-proof*. $\diamond$