Some Characterizations of TTC in Multiple-Object Exchange Problems*

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Abstract

This paper considers exchange of indivisible objects when agents are endowed with and desire bundles of objects. Agents are assumed to have lexicographic preferences over bundles. We show that *Top Trading Cycles (TTC)* is characterized by *efficiency*, the *weak endowment lower bound*, *balancedness*, and *truncation-proofness*. In the classic Shapley–Scarf Economy, TTC is characterized by *efficiency*, *individual rationality*, and *truncation-proofness*. These results strengthen the uniqueness results of Ma (1994) and, more recently, Altuntaş et al. (2023). In a model with variable endowments, TTC is susceptible to various forms of *endowment manipulation*. However, no rule is *core-selecting* and *hiding-proof*.

Keywords: exchange of indivisible objects; Top Trading Cycles; heuristic manipulation; truncation-proof; endowment manipulation.

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1 Introduction

This paper is concerned with exchange of indivisible goods. There is a group of *agents*, each of whom is initially endowed with some set of heterogeneous and indivisible *objects*. Each agent

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has preferences over *bundles* of objects. A social planner can redistribute the objects among the agents, but monetary transfers are forbidden.

There are many examples of such problems. Employees at a firm may exchange tasks, shifts, or equipment. University students exchange course schedules, while universities exchange students for semester-long exchange programs. Teams in professional sports leagues trade players' contract rights.[1] Collectors trade collectibles, such as Pokémon cards. The classic Shapley–Scarf economy, in which each agent is endowed with a single object (called a *house*), is an important special case (Shapley and Scarf, 1974).[2]

We assume that agents have lexicographic preferences over bundles of objects. That is, each agent has strict preferences over objects, and these preferences are extended to bundles as follows: for any two distinct bundles, X and Y, bundle X is preferred to Y if and only if the best object that belongs to exactly one of the two bundles is in X. Lexicographic preferences are natural in some settings. For example, students may strongly prefer certain classes to others and, hence, they may have lexicographic preferences over course schedules. Many single-object problems can be reinterpreted as a multiple-object problem by identifying each object with its set of attributes. For example, in kidney exchange problems, each kidney is identified by its characteristics (e.g., compatibility, age of donor, etc.), which can be ranked lexicographically.[3]

We are interested in allocation rules that satisfy appealing properties, such as *(Pareto) efficiency* and *individual rationality*, and which eliminate incentives for agents to misrepresent their preferences. For the Shapley–Scarf economy, the only possible rule is *Top Trading Cycles (TTC)* (Ma, 1994). Unfortunately, for multiple-object problems, no rule is *efficient*, *individually rational*, and *strategy-proof*, even on the lexicographic domain (Todo et al., 2014). Altuntaş et al. (2023) show that a variant of TTC, generalized to multiple-object problems, is immune to manipulation via *drop strategies*, in which agents misrepresent their preferences by simply *dropping* some object to the bottom of their preference list; furthermore, they show that TTC is the unique rule satisfying (i) *efficiency*, (ii) the *strong endowment lower bound*, and (iii) *drop*

[1] Of course, money is a feature of professional sports leagues. However, the presence of salary caps means that prices are not the main rationing instrument.

[2] Throughout this paper, we shall distinguish between *single-object problems*, in which agents are endowed with a single object, and *multiple-object problems*, in which agents are endowed with bundles.

[3] Lexicographic preferences are obviously restrictive. We make this restriction for two main reasons. First, lexicographic preferences have a compact representation: each lexicographic preference relation defined on the set of bundles is completely specified by a strict preference relation on the set of objects. Thus, the social planner only needs to elicit a minimal amount of information about the agents' preferences. This is particularly useful if the set of objects is large, so that agents could not feasibly rank all bundles. The second reason is technical: the lexicographic domain allows us to get around some—though not all—of the impossibility results that arise on larger preference domains. For instance, Konishi et al. (2001) show that the core can be empty under additive preferences, while Fujita et al. (2018) show that (conditionally) lexicographic preferences guarantee a nonempty core.

strategy-proofness.[4]

As in Altuntaş et al. (2023), we search for rules that prohibit only certain kinds of manipulation heuristics. We focus on *truncation* strategies, which can be employed in various environments in which agents have some outside option and they are facing an *individually rational* rule.[5] Loosely speaking, a truncation strategy involves reporting a smaller "acceptable set" but truthfully reporting the ranking of all acceptable objects. Truncation strategies are most easily understood in the Shapley–Scarf economy, where each agent i is initially endowed a single object, say o_i. Starting from her true preference relation, say $P_i = a, b, c, o_i, d, \ldots$, agent i can "push up" her own object o_i in her preference list, whilst leaving the relative ranking of all other objects unchanged. For example, $P'_i = a, b, o_i, c, d, \ldots$ is the *truncation of* P_i *at* b. If the rule is *individually rational*, i.e., it always assigns agent i an object weakly better than o_i, then misreporting P'_i can be interpreted as vetoing object c, even though it is *acceptable* at agent i's true preference relation, P_i. This interpretation extends to truncations in general multiple-object exchange economies.

Our main result is that, in multiple-object exchange economies, TTC is the unique rule satisfying the following properties: (i) *efficiency*, (ii) the *weak endowment lower bound*, (iii) *balancedness*, and (iv) *truncation-proofness*. The *weak endowment lower bound* requires that no agent ever receives an object which is worse than the worst object in her endowment. It is a participation requirement which is equivalent to *individual rationality* in single-object problems.[6] More generally, it can be interpreted as a property that allows agents to veto some of the other agents' objects.[7] A *balanced* rule assigns the same number of objects to each agent as the initial allocation. *Balancedness* implies that agents are rewarded with more objects if they bring more objects to the market.

Our main result yields several other interesting results as corollaries. Because every rule satisfying the *weak endowment lower bound* and *drop strategy-proofness* is *truncation-proof*, it follows that TTC is the unique rule satisfying (i) *efficiency*, (ii) the *weak endowment lower bound*, (iii) *balancedness*, and (iv) *drop strategy-proofness*. In Shapley–Scarf economies, TTC is the unique rule satisfying (i) *efficiency*, (ii) *individual rationality*, and (iii) *truncation-proofness*. Thus, a social planner interested in other desiderata (e.g., fairness), in addition to *efficiency* and *individual rationality*, cannot improve upon TTC if she insists on *truncation-proofness* (and

[4] Biró et al. (2022) prove a similar result in a related setting in which each agent is endowed with a set of agent-specific homogeneous objects.

[5] Roth and Rothblum (1999) study truncation strategies in two-sided matching markets, such as the NRMP. They interpret *truncation strategies* as (p. 23) "strategies in which applicants restrict the number of positions for which they apply, but faithfully transmit their true preferences about those positions for which they do apply."

[6] The *weak endowment lower bound* is independent of *individual rationality* in multiple-object problems.

[7] There are some settings in which *individual rationality* is unnecessarily demanding. For example, in professional sports leagues, some players at team i may express their desire to be traded to another team. There is no reason to expect the resulting reallocation to benefit team i.

permits other, more complex, kinds of manipulation).

Finally, we consider agents' incentives to misrepresent their endowment. As in Postlewaite (1979) and Atlamaz and Klaus (2007), we consider three kinds of endowment manipulation. An agent could manipulate by (i) *hiding* part of her endowment for her own personal consumption, (ii) *destroying* part of her endowment so no agent can consume it, or (iii) *transferring* part of her endowment to another agent who is not made worse off by the transfer. Unfortunately, TTC is susceptible to all three kinds of endowment manipulation. We show that no rule is both *core-selecting* and *hiding-proof*.

2 Economies with fixed endowments

2.1 Preliminaries

Let $N = \{1, 2, \ldots, n\}$ be a finite set of *agents*. Let O be a finite set of heterogeneous and indivisible *objects*. A *bundle* is a subset of O. Let 2^O denote the set of bundles.

An *allocation* is a function $\mu : N \to 2^O$ such that (i) for all $i, j \in N$, $i \neq j$ implies $\mu(i) \cap \mu(j) = \emptyset$, and (ii) $\bigcup_{i \in N} \mu(i) = O$. We shall represent each allocation μ as a profile $(\mu_i)_{i \in N}$ of disjoint bundles with $\bigcup_{i \in N} \mu_i = O$. For each $i \in N$, μ_i denotes agent i's *assignment* at μ. Let $\mathcal{M}$ denote the set of allocations.

Let $\omega = (\omega_i)_{i \in N}$ denote the *initial allocation*. For each $i \in N$, bundle ω_i is agent i's *endowment*. We assume that each ω_i is nonempty. With some abuse of notation, we denote the initial *owner* of object o by $\omega^{-1}(o)$.

Each agent i has a *strict preference* relation P_i over the set 2^O of all bundles. If agent i prefers bundle X to bundle Y, then we write XP_iY. Let R_i denote the *at least as good as* relation associated with P_i. Then XR_iY means that (XP_iY or $X = Y$).[8] Let $P = (P_i)_{i \in N}$ denote a preference *profile*, where P_i is agent i's preference relation.

When there is no danger of confusion, we abuse notation and denote singletons $\{x\}$ by x. For example, xP_iy means $\{x\} P_i \{y\}$, and $\omega_i \cup x$ means $\omega_i \cup \{x\}$.

Given a preference relation P_i on 2^O and a nonempty bundle $X \in 2^O$, let $\top_{P_i}(X)$ denote the top-ranked object in X, i.e., $\top_{P_i}(X) = x$, where $x \in X$ and xR_iy for all $y \in X$. Similarly, let $\bot_{P_i}(X)$ denote the bottom-ranked object in X, i.e., $\bot_{P_i}(X) = x$, where $x \in X$ and yR_ix for all $y \in X$. Given $k \in \{1, \ldots, |O|\}$, $\top_{P_i}(X, k)$ denotes the bundle consisting of the best k objects in X according to P_i.

We assume that agents have *lexicographic preferences* on 2^O. A preference relation P_i on 2^O

[8]Formally, R_i is a *complete*, *transitive*, and *antisymmetric* binary relation on 2^O, and P_i is the *asymmetric* part of R_i.

is *lexicographic* if, for all distinct bundles $X, Y \in 2^O$,

$$XP_iY \iff \top_{P_i}(X \triangle Y) \in X, \tag{1}$$

where $\triangle$ denotes the *symmetric difference*, i.e., $A \triangle B = (A \backslash B) \cup (B \backslash A)$. The set of lexicographic preference relations on 2^O is in bijection with the set of strict linear orders on O. Therefore, we will identify lexicographic preference relations with their associated strict linear orders. In particular, we frequently represent lexicographic preferences as ordered list of objects, e.g., $P_i = x_1, x_2, \ldots, x_m$ means that $x_1 P_i x_2 P_i \cdots P_i x_m$, with the understanding that all remaining relations among bundles in 2^O can be deduced from (1). Let $\mathcal{P}$ denote the set of all lexicographic preference relations on 2^O (or, equivalently, their associated strict linear orders on O). Let $\mathcal{P}^N$ denote the set of all preference profiles.

Given a preference relation P_i and an object $x \in O$, we denote $U(P_i, x) = \{o \in O \mid oP_ix\}$ and $U(R_i, x) = \{o \in O \mid oR_ix\}$. If $X \subseteq O$, then $P_i|_X$ denotes the restriction of P_i to X (and, more generally, bundles in 2^X). Given a preference profile $P = (P_i)_{i \in N}$ and a relation P_i' for agent i, $\left(P_i', P_{N \backslash i}\right)$ denotes the preference profile in which agent i's preference relation is P_i' and, for each agent $j \in N \backslash \{i\}$, agent j's preference relation is P_j.

A *rule* is a function $\varphi : \mathcal{P}^N \to \mathcal{M}$. That is, a rule φ associates to each profile P an allocation $\varphi(P)$. For each $i \in N$, $\varphi_i(P)$ denotes agent i's assignment at $\varphi(P)$.

2.2 Properties

An allocation μ is *efficient* at P if there is no allocation $\overline{\mu} \in \mathcal{M}$ such that (i) for all $i \in N$, $\overline{\mu}_i R_i \mu_i$, and (ii) for some $i \in N$, $\overline{\mu}_i P_i \mu_i$.

Definition 1 (Efficiency)**.** A rule φ is ***efficient*** if, for each profile $P \in \mathcal{P}^N$, the allocation $\varphi(P)$ is efficient at P.

An allocation μ is *pair-efficient* at P if there is no allocation $\overline{\mu} \in \mathcal{M}$ and no pair $\{i, j\} \subseteq N$ of agents such that $\mu_j P_i \mu_i$ and $\mu_i P_j \mu_j$.

Definition 2 (Pair-efficiency)**.** A rule φ is ***pair-efficient*** if, for each profile $P \in \mathcal{P}^N$, the allocation $\varphi(P)$ is pair-efficient at P.

An allocation μ is *individually rational* at P if, for each $i \in N$, $\mu_i R_i \omega_i$.

Definition 3 (Individual rationality)**.** A rule φ is ***individually rational*** if, for each profile $P \in \mathcal{P}^N$, the allocation $\varphi(P)$ is individually rational at P.

An allocation μ satisfies the *weak endowment lower bound* at P if, for each $i \in N$, $\mu_i \subseteq U(R_i, \perp_{P_i}(\omega_i))$. That is, no agent is assigned an object ranked below the worst object in her endowment.

Definition 4 (Weak endowment lower bound)**.** A rule φ satisfies the ***weak endowment lower bound*** if, for each profile $P \in \mathcal{P}^N$, the allocation $\varphi(P)$ satisfies the weak endowment lower bound at P.

An allocation μ satisfies the *strong endowment lower bound* at P if, for each $i \in N$, μ_i *pairwise dominates* ω_i, i.e., there is a injection $f : \omega_i \to \mu_i$ such that, for each $o \in \omega_i$, $f(o)\, R_i o$.

Definition 5 (Strong endowment lower bound)**.** A rule φ satisfies the ***strong endowment lower bound*** if, for each profile $P \in \mathcal{P}^N$, the allocation $\varphi(P)$ satisfies the strong endowment lower bound at P.

A rule is *balanced* if it assigns the same number of objects to each agent as the initial allocation.

Definition 6 (Balancedness)**.** A rule φ is ***balanced*** if, for all $P \in \mathcal{P}^N$ and all $i \in N$, $|\varphi_i(P)| = |\omega_i|$.

Remark 1. Any rule satisfying the *strong endowment lower bound* is also *individually rational*, *balanced*, and satisfies the *weak endowment lower bound*. However, the converse is false: there are rules satisfying *individual rationality*, *balancedness*, and the *weak endowment lower bound* which violate the *strong endowment lower bound*. Moreover, *balanced* rules satisfying the *weak endowment lower bound* need not even be *individually rational*.

Example 1. Suppose $N = \{1, 2\}$, $\omega = (\{a, b, c\}, \{d, e, f\})$, and P is given by

$$P_1 = d, a, e, b, f, c \quad \text{and} \quad P_2 = a, d, b, e, c, f.$$

Consider the allocation $\mu = (\{d, f, c\}, \{a, b, e\})$. Clearly, μ satisfies *individual rationality*, *balancedness*, and the *weak endowment lower bound*. However, μ violates the *strong endowment lower bound* because μ_1 does not pairwise dominate ω_1. In particular, the second-best object in ω_1, b, is better than the second-best object in μ_1, f.

Now consider the allocation $\nu = (\{b, f, c\}, \{a, d, e\})$. Then ν satisfies *balancedness* and the *weak endowment lower bound*. However, ν is not *individually rational* because $\omega_1 P_1 \nu_1$.

2.2.1 Preference manipulation

We say that agent i can *manipulate* φ at P by misreporting $P_i' \in \mathcal{P}$ if $\varphi_i\left(P_i', P_{N\backslash i}\right) P_i \varphi_i(P)$. A rule φ is *strategy-proof* if no agent i can manipulate it at any profile P by misreporting *any* $P_i' \in \mathcal{P}$.

Definition 7 (Strategy-proofness)**.** A rule φ is ***strategy-proof*** if, for each profile $P \in \mathcal{P}^N$, and each $i \in N$, there is no $P_i' \in \mathcal{P}$ such that $\varphi_i\left(P_i', P_{N\backslash i}\right) P_i \varphi_i(P)$.

A strategy-proof rule φ prohibits each agent i from manipulating the rule at any P by misreporting any preference relation in $\mathcal{P}$. A natural weakening of strategy-proofness is to prohibit each agent i from manipulating the rule at any P by reporting any preference relation in some subset $\mathcal{S}(P_i)$ of $\mathcal{P}$. The set $\mathcal{S}(P_i)$ could consist of misrepresentations which are, in some sense, more robust, or whose consequences are simpler to understand. To this end, we define three manipulation heuristics: *drop strategies*, *subset drop strategies*, and *truncations*. We then define the associated incentive properties, namely *drop strategy-proofness*, *subset drop strategy-proofness*, and *truncation-proofness*.

Manipulation heuristics. Let $P_i, P_i' \in \mathcal{P}$. We say that P_i' is a *drop strategy* for P_i if there exists $x \in O\backslash\omega_i$ such that:

1. for each $o \in O$, $oR_i'x$;
2. $P_i' \mid_{O\backslash\{x\}} = P_i \mid_{O\backslash\{x\}}$.

In this case, we say that P_i' is obtained from P_i by *dropping* object x. Let $\mathcal{D}(P_i)$ denote the set of all drop strategies for P_i.

We say that P_i' is a *subset drop strategy* for P_i if there exists $X \subseteq O\backslash\omega_i$ such that:

1. for each $x \in X$ and $o \in O\backslash X$, $oR_i'x$;
2. $P_i' \mid_{O\backslash X} = P_i \mid_{O\backslash X}$.
3. $P_i' \mid_X = P_i \mid_X$.

In this case, we say that P_i' is obtained from P_i by *dropping* the subset X. Let $\mathcal{SD}(P_i)$ denote the set of all subset drop strategies for P_i. Clearly, every drop strategy is a subset drop strategy, i.e., $\mathcal{D}(P_i) \subseteq \mathcal{SD}(P_i)$.

We say that P_i' is a *truncation* of P_i if either:

1. P_i' is obtained from P_i by dropping the subset $O\backslash\omega_i$; or

2. there exists $x \in O\backslash\omega_i$ such that P'_i is obtained from P_i by dropping the subset $\{o \in O\backslash\omega_i \mid xP_io\}$.

If (1) holds, P'_i is called the *complete truncation* of P_i. If (2) holds, we say that P'_i is the *truncation of* P_i *at* x. Let $\mathcal{T}(P_i)$ denote the set of truncations of P_i. Clearly, every truncation is a subset drop strategy, i.e., $\mathcal{T}(P_i) \subseteq \mathcal{SD}(P_i)$.

For example, suppose $\omega_i = \{o_i^1, o_i^2\}$ and $P_i = a, b, o_i^1, c, d, o_i^2, e$. Then:

- $P_i^1 = b, o_i^1, c, d, o_i^2, e, a$ is a drop strategy for P_i; it is obtained by dropping object a.
- $P_i^2 = b, o_i^1, c, o_i^2, e, a, d$ is a subset drop strategy for P_i; it is obtained by dropping the subset $\{a, d\}$ of objects. Note that P_i^2 is not a drop strategy for P_i.
- $P_i^3 = a, b, o_i^1, c, o_i^2, d, e$ and $P_i^4 = a, o_i^1, o_i^2, b, c, d, e$, are the truncations of P_i at c and a, respectively; that is, P_i^3 is obtained from P_i by dropping the subset $\{d, e\}$, while P_i^4 is obtained from P_i by dropping the subset $\{b, c, d, e\}$.

Definition 8. A rule φ is[9]

- ***drop strategy-proof*** if, for each profile $P \in \mathcal{P}^N$, and each $i \in N$, there is no $P'_i \in \mathcal{D}(P_i)$ such that $\varphi_i\left(P'_i, P_{N\backslash i}\right) P_i \varphi_i(P)$.
- ***subset drop strategy-proof*** if, for each profile $P \in \mathcal{P}^N$, and each $i \in N$, there is no $P'_i \in \mathcal{SD}(P_i)$ such that $\varphi_i\left(P'_i, P_{N\backslash i}\right) P_i \varphi_i(P)$.
- ***truncation-proof*** if, for each profile $P \in \mathcal{P}^N$, and each $i \in N$, there is no $P'_i \in \mathcal{T}(P_i)$ such that $\varphi_i\left(P'_i, P_{N\backslash i}\right) P_i \varphi_i(P)$.

Subset drop strategies are considered in Altuntaş et al. (2023) and Biró et al. (2022). Our definition of a *subset drop strategy* for P_i is identical to that of a *subset total drop strategy* in Altuntaş et al. (2023), except for our additional requirement that $P'_i|_X = P_i|_X$.

An important distinction between *truncations* and other *subset drop strategies* is that, in general, subset drop strategies allow an agent i to misrepresent her top-ranked object in $O\backslash\omega_i$, while truncation strategies do not. In particular, for any preference relation P_i of agent i, and any $o \in O\backslash\omega_i$, there exists $P'_i \in \mathcal{SD}(P_i)$ such that $\top_{P'_i}(O\backslash\omega_i) = o$; moreover, for any $o \in U(R_i, \bot_{P_i}(\omega_i))\backslash\omega_i$, there exists $P'_i \in \mathcal{SD}(P_i)$ such that $\top_{P'_i}(O) = o$. On the other hand, any $P'_i \in \mathcal{T}(P_i)$ must agree with P_i on $O\backslash\omega_i$.

Observe that, if P'_i is a obtained from P_i by dropping the subset X, then P'_i is obtained from a sequence of $|X|$ drop strategies. Indeed, suppose $X = \{x_1, x_2, \ldots, x_k\}$ and $x_1 P_i x_2 P_i \cdots P_i x_k$.

[9] Our definition of *drop strategy-proofness* is precisely the weakening of *drop strategy-proofness* and *subset total drop strategy-proofness* discussed in footnote 31 of Altuntaş et al. (2023).

Then $P_i' = P_i^k$, where $P_i^0 = P_i$ and $P_i^1, \ldots, P_i^k$ are such that, for each $\ell \in \{1, \ldots, k\}$, P_i^ℓ is obtained from $P_i^{\ell-1}$ by dropping object x_ℓ.

Because drop strategies and truncations are subset drop strategies, any *subset drop strategy-proof* rule is also *drop strategy-proof* and *truncation-proof*. The converse is false. However, one can show that any *drop strategy-proof* rule satisfying the *weak endowment lower bound* is *subset drop strategy-proof* and, hence, *truncation-proof*.

Proposition 1. *If a rule φ satisfies **drop strategy-proofness** and the **weak endowment lower bound**, then it satisfies **subset drop strategy-proofness** (and, hence, **truncation-proofness**).*

In Example 2 of Section 3.2, we demonstrate that *truncation-proofness* and the *weak endowment lower bound* do not jointly imply *drop strategy-proofness*.

It is straightforward to show that *balanced* rules satisfying the *weak endowment lower bound* and *truncation-proofness* are *individually rational*.

Proposition 2. *If a rule φ satisfies **balancedness**, the **weak endowment lower bound**, and **truncation-proofness**, then it is **individually rational**.*

Remark 2. Another plausible definition of *truncation* strategies is as follows. Let $P_i, P_i' \in \mathcal{P}$. We say that P_i' is a $*$-*truncation* of P_i if the following properties are satisfied:

1. for each $o \in \omega_i$, $U(R_i', o) \subseteq U(R_i, o)$;
2. $P_i' \mid_{O \setminus \omega_i} = P_i \mid_{O \setminus \omega_i}$;
3. $P_i' \mid_{\omega_i} = P_i \mid_{\omega_i}$.

Let $\mathcal{T}^*(P_i)$ denote the set of all $*$-truncations of P_i, and call a rule φ $*$-*truncation-proof* if no agent i can manipulate it at any profile P by reporting misreporting any $P_i' \in \mathcal{T}^*(P_i)$. It is easy to see that TTC, defined in Section 2.3, is not $*$-*truncation-proof*.

For example, suppose $N = \{1, 2\}$, $\omega = (\{a, b\}, \{c\})$, $P_1 = c, a, b$ and $P_2 = a, b, c$. Then $\varphi^{\text{TTC}}(P) = (\{c, b\}, \{a\})$. However, $P_1' = a, c, b$ is a $*$-truncation of P_1 such that $\varphi^{\text{TTC}}(P) = (\{a, c\}, \{b\})$. Thus, agent 1 prefers to misreport $P_1' \in \mathcal{T}^*(P_1)$ at P.

2.3 Top Trading Cycles

A *cycle* is a sequence

$$C = \left(i_1, o_{i_2}, i_2, o_{i_3}, \ldots, i_{k-1}, o_{i_k}, i_k, o_{i_{k+1}}, i_{k+1} = i_1\right)$$

of $k \geq 1$ distinct agents and objects such that each agent on the cycle points to her favorite object among some subset of O and each object points to its owner. The set of agents on C is denoted by $N(C) = \{i_1, i_2, \ldots, i_k\}$. Similarly, the set of objects on C is denoted by $O(C)$.

An allocation μ *executes* the cycle C if μ assigns each agent in $N(C)$ the object to which she points in C, i.e., for each $i_\ell \in N(C)$, $o_{i_{\ell+1}} \in \mu_{i_\ell}$.

We consider a generalized version of the *Top Trading Cycles (TTC)* procedure. At each step, every agent points to her favorite remaining object, and every remaining object points to its owner. At least one cycle obtains. Our proofs rely on a variant of TTC that executes only a single cycle at each step. If multiple cycles obtain, then we exectue only the *smallest cycle*, i.e., the cycle containing the agent identified by the smallest natural number. That is, we assign each agent on the smallest cycle the object to which she points, then clear all objects on the cycle. It is well-known that this procedure is equivalent to the variant of TTC that, at each step, executes *all* prevailing cycles.

Formally, for each profile $P \in \mathcal{P}^N$, TTC selects the allocation $\varphi^{\text{TTC}}(P)$ determined by the following procedure, which we call $\text{TTC}(P)$.

Algorithm: TTC (P)

Input: A preference profile $P \in \mathcal{P}^N$.

Output: An allocation $\varphi^{\text{TTC}}(P)$.

Step 0. Define $\mu^0 := (\emptyset)_{i \in N}$ and $O^1 := O$.

Step 1. Construct a bipartite directed graph with independent vertex sets N and O^1. For each agent $i \in N$, there is a directed edge from i to agent i's most-preferred object in O^1, $\top_{P_i}(O^1)$. For each object $o \in O^1$, there is a directed edge from o to its owner, $\omega^{-1}(o)$. At least one cycle exists. Let $C_1^1, C_2^1, \ldots, C_k^1$ denote the cycles that obtain at Step 1, and assume that $\min N(C_1^1) < \min N(C_2^1) < \cdots < \min N(C_k^1)$. Assign each agent $i \in N(C_1^1)$ the object to which she points on C_1^1, namely $\top_{P_i}(O^1)$; that is, let $\mu^1 = (\mu_i^1)_{i \in N}$ be the allocation such that (i) for all $i \in N(C_1^1)$, $\mu_i^1 = \mu_i^0 \cup \{\top_{P_i}(O^1)\}$, and (ii) for all $i \in N \backslash N(C_1^1)$, $\mu_i^1 = \mu_i^0$. Clear all objects in $O(C_1^1)$; that is, let $O^2 := O^1 \backslash O(C_1^1)$. If $O^2 \neq \emptyset$, then proceed to Step 2; otherwise, proceed to Termination.

Step $t \geq 2$. Construct a bipartite directed graph with independent vertex sets N and O^t. For each agent $i \in N$, there is a directed edge from i to agent i's most-preferred object in O^1, $\top_{P_i}(O^t)$. For each object $o \in O^t$, there is a directed edge from o to its owner, $\omega^{-1}(o)$. At least one cycle exists. Let $C_1^t, C_2^t, \ldots, C_k^t$ denote the cycles that obtain at Step t, and assume that $\min N(C_1^t) < \min N(C_2^t) < \cdots < \min N(C_k^t)$. Assign each agent

$i \in N(C_1^t)$ the object to which she points on C_1^t, namely $\top_{P_i}(O^t)$; that is, let $\mu^t = (\mu_i^t)_{i \in N}$ be the allocation such that (i) for all $i \in N(C_1^t)$, $\mu_i^t = \mu_i^{t-1} \cup \{\top_{P_i}(O^t)\}$, and (ii) for all $i \in N \backslash N(C_1^t)$, $\mu_i^t = \mu_i^{t-1}$. Clear all objects in $O(C_1^t)$; that is, let $O^{t+1} \coloneqq O^t \backslash O(C_1^t)$. If $O^{t+1} \neq \emptyset$, then proceed to Step $t+1$; otherwise, proceed to Termination.

Termination. Because O is finite, and $|O^1| > |O^2| > \cdots > |O^t|$, the procedure terminates at some step T. Return the allocation $\varphi^{\text{TTC}}(P) \coloneqq \mu^T$.

2.4 Characterizations of TTC

Fujita et al. (2018) show that TTC is *core-selecting*.[10] It is therefore *efficient* and *individually rational*. TTC satisfies the *strong endowment lower bound*; hence, it is *balanced* and satisfies the *weak endowment lower bound*. Although TTC is not *strategy-proof*, Altuntaş et al. (2023) show that φ^{TTC} is *subset drop strategy-proof*; it follows that TTC is *drop strategy-proof* and *truncation-proof*.[11]

Our main result shows that TTC is the unique *balanced* rule satisfying *efficiency*, the *weak endowment lower bound*, and *truncation-proofness*.

Theorem 1. *Only TTC satisfies* ***efficiency****, the* ***weak endowment lower bound****,* ***balancedness****, and* ***truncation-proofness****.*

The proof of Theorem 1 is in the Appendix. In Sections 3.2 and 3.3, we state and prove a special case of Theorem 1 for Shapley–Scarf Economies.

Theorem 1 and Proposition 1 imply the following corollaries.

Corollary 1. *Only TTC satisfies:*

1. ***efficiency****, the* ***weak endowment lower bound****,* ***balancedness****, and* ***drop strategy-proofness****.*
2. ***efficiency****, the* ***weak endowment lower bound****,* ***balancedness****, and* ***subset drop strategy-proofness****.*

By Remark 1, we obtain the following corollaries.

[10] See Section 4 for a formal definition of this property.

[11] Actually, Altuntaş et al. (2023) show that TTC satisfies a property which is stronger than our version of *drop strategy-proofness*. In particular, they show that no agent i can manipulate by dropping any object in O (including objects in her endowment, ω_i).

Corollary 2. *Only TTC satisfies **efficiency**, the **strong endowment lower bound**, and **truncation-proofness**.*

Corollary 3 (Altuntaş et al., 2023)**.** *Only TTC satisfies:*

1. ***efficiency**, the **strong endowment lower bound**, and **drop strategy-proofness**.*
2. ***efficiency**, the **strong endowment lower bound**, and **subset drop strategy-proofness**.*

2.4.1 Independence of the Properties in Theorem 1

We now demonstrate that the properties in Theorem 1 are independent. For each property, we give an example of a rule $\varphi \neq \varphi^{\text{TTC}}$ that violates the given property but satisfies each of the remaining properties.

Efficiency. Let φ be the *no trade rule*, defined for each $P \in \mathcal{P}^N$ by $\varphi(P) = (\omega_i)_{i \in N}$. Clearly, φ is not *efficient*. However, φ is *drop strategy-proof* and it satisfies the *strong endowment lower bound*.

Weak endowment lower bound. Let φ be the *balanced serial dictatorship rule*, defined recursively as follows. For each $P \in \mathcal{P}^N$, $\varphi_1(P) = \top_{P_1}(O, |\omega_1|)$ and, for each $i \in \{2, \ldots, n\}$,

$$\varphi_i(P) = \top_{P_i}\left(O \setminus \bigcup_{j=1}^{i-1} \varphi_j(P), |\omega_i|\right).$$

Then φ is *efficient*, *balanced*, and *strategy-proof*. However, φ violates the *weak endowment lower bound* at, say, any profile P in which all agents have the same preferences.

Balancedness. Let φ be the *serial dictatorship rule*, defined recursively as follows. For each $P \in \mathcal{P}^N$, $\varphi_1(P) = U(R_1, \perp_{P_1}(\omega_1))$ and, for each $i \in \{2, \ldots, n\}$,

$$\varphi_i(P) = U(R_i, \perp_{P_i}(\omega_i)) \setminus \bigcup_{j=1}^{i-1} \varphi_j(P).$$

Clearly, φ is not *balanced* because $\varphi_1(P) = O$ whenever P_1 ranks every member of $O \setminus \omega_1$ above every member of ω_1. However, φ is *efficient*, *strategy-proof*, and satisfies the *weak endowment lower bound*.

Truncation-proofness. For each $i \in N$, let $\omega_i = \left\{o_1^i, o_2^i, \ldots, o_{|\omega_i|}^i\right\}$. Let $P' \in \mathcal{P}^N$ be a profile satisfying the following properties:

1. $P'_1 = x^n, o_1^1, o_2^1, \ldots, o_{|\omega_1|}^1, \ldots$, where $x^n = \top_{P'_n}(\omega_n)$.

2. for all $i \in \{2, \ldots, n-1\}$, $P'_i = x^{i+1}, o^i_1, o^i_2, \ldots, o^i_{|\omega_i|}, \ldots$, where $x^{i+1} = \top_{P'_{i+1}}(\omega_{i+1})$.
3. $P'_n = x^2, x^1, o^n_1, o^n_2, \ldots, o^n_{|\omega_n|}, \ldots$, where $x^2 = \top_{P'_2}(\omega_2)$ and $x^1 = \top_{P'_1}(\omega_1)$.[12]

Consider the rule φ defined as follows. For all $P \in \mathcal{P}^N$,

$$\varphi(P) = \begin{cases} \varphi^{\mathrm{TTC}}(P), & \text{if } P \neq P' \\ \left((\omega_1 \cup x^n) \setminus x^1, \omega_2, \ldots, \omega_{n-1}, (\omega_n \cup x^1) \setminus x^n\right), & \text{if } P = P'. \end{cases}$$

Observe that $\varphi(P')$ is obtained from ω by letting agents 1 and n swap x^1 and x^n; otherwise, all agents retain all other objects in their endowments. On the other hand, $\varphi^{\mathrm{TTC}}(P')$ is obtained from ω by first executing the cycle[13]

$$C^1 = \left(2, x^3, 3, x^4, \ldots, x^{n-1}, n-1, x^n, n, x^2, 2\right),$$

then assigning each agent the remaining objects in their endowment. That is,

$$\varphi^{\mathrm{TTC}}(P') = \left(\omega_1, \left(\omega_2 \cup x^3\right) \setminus x^2, \ldots, (\omega_{n-1} \cup x^n) \setminus x^{n-1}, \left(\omega_n \cup x^2\right) \setminus x_n\right).$$

It is straightforward to show that φ is *efficient*, *balanced*, and it satisfies the *weak endowment lower bound*. To see that φ is not *truncation-proof*, suppose that $P_1 \in \mathcal{P}$ is such that $P_1 \neq P'_1$ and P'_1 is a truncation of P_1.[14] Then $\varphi_1\left(P_1, P'_{N\setminus 1}\right) = \varphi^{\mathrm{TTC}}_1\left(P_1, P'_{N\setminus 1}\right) = \omega_1$ and $\varphi_1\left(P'_1, P'_{N\setminus 1}\right) = \varphi_1(P') = (\omega_1 \cup x^n) \setminus x^1$. Hence, $\varphi_1\left(P'_1, P'_{N\setminus 1}\right) P_1 \varphi_1\left(P_1, P'_{N\setminus 1}\right)$, so agent 1 can manipulate φ at $\left(P_1, P'_{N\setminus 1}\right)$ by misreporting $P'_1 \in \mathcal{T}(P_1)$.

3 Shapley–Scarf Economies

The classic Shapley–Scarf Economy, in which each agent is endowed with a single object and has the need for a single object, can be viewed as a special case of our model (Shapley and Scarf, 1974). To see this, let $O = \{o_1, \ldots, o_n\}$ and suppose that, for each $i \in N$, the owner of object o_i is $\omega^{-1}(o_i) = i$. Hence, the initial allocation is $\omega = (o_i)_{i \in N}$.

In line with convention, this section considers only *balanced allocations*, i.e., allocations that assign exactly one object to each agent. That is, an *allocation* as a bijection $\mu : N \to O$ or, equivalently, a profile $\mu = (\mu_i)_{i \in N}$ of distinct objects. We also assume that agents have strict

[12] If $n = 2$, we let $P'_n = o^n_1, x^1, o^n_2, \ldots, o^n_{|\omega_n|}, \ldots$.

[13] Or $C^1 = \left(2, x^2, 2\right)$ if $n = 2$.

[14] Such a P_1 exists. Indeed, if $P'_1 = x^n, o^1_1, o^1_2, \ldots, o^1_{|\omega_1|}, y_1, \ldots, y_m$, then $P_1 = x^n, y_1, \ldots, y_m, o^1_1, o^1_2, \ldots, o^1_{|\omega_1|}$ satisfies the stated properties.

preferences P_i on O.[15] For each $i \in N$, μ_i denotes agent i's *assignment* at μ. Let $\mathcal{M}$ denote the set of allocations. Let $\mathcal{P}$ denote the set of all strict preferences on O, and $\mathcal{P}^N$ the set of all strict preference profiles. A *rule* is a function $\varphi : \mathcal{P}^N \to \mathcal{M}$.

3.1 Properties

Using the slightly modified definitions of $\mathcal{P}$ and $\mathcal{M}$, and noting that $\omega_i = o_i$ for all $i \in N$, the definitions of the properties in Section 2.2 carry over exactly to this setting.

All rules are *balanced* by definition. Consequently, the *strong endowment lower bound*, the *weak endowment lower bound*, and *individual rationality* are all equivalent. The definitions of *drop strategies*, *subset drop strategies*, and *truncations*, as well as the associated incentive properties, are exactly the same as before. Nevertheless, it is helpful to reformulate the definitions of *truncation strategies* and *truncation-proofness* as follows.

Let $P_i, P_i' \in \mathcal{P}$. We say that P_i' is a *truncation* of P_i if the following is satisfied:

1. $U(R_i', o_i) \subseteq U(R_i, o_i)$.
2. $P_i' |_{O\setminus\{o_i\}} = P_i |_{O\setminus\{o_i\}}$.

Intuitively, P_i' is obtained from P_i by "pushing up" agent i's own object o_i, whilst leaving the relative rankings of all other objects unchanged. Let $\mathcal{T}(P_i)$ denote the set of all truncations of P_i. Note that the definition from Section 2.2 is equivalent to this definition when each agent is endowed with a single object.

A rule φ is *truncation-proof* if no agent i can manipulate it at any profile P by misreporting any $P_i' \in \mathcal{T}(P_i)$.

3.2 Characterizations of TTC

Roth (1982) shows TTC is *strategy-proof*. Ma (1994) shows that TTC is the unique rule satisfying *efficiency*, *individual rationality*, and *strategy-proofness*. The following result is the analog of Theorem 1 for Shapley–Scarf economies.

Theorem 2. *Only TTC is* ***efficient***, ***individually rational***, *and* ***truncation-proof***.

[15] As in Section 2.1, one could define an *allocation* as a function $\overline{\mu} : N \to 2^O$ such that (i) for all $i, j \in N$, $i \neq j$ implies $\overline{\mu}(i) \cap \overline{\mu}(j) = \emptyset$, and (ii) $\bigcup_{i\in N} \overline{\mu}(i) = O$. However, for any profile P of lexicographic preferences on 2^O, any allocation $\overline{\mu}$ which is *individually rational* at P must give exactly one object to each agent. Since we will be studying *individually rational* rules, the restriction to bijections $\mu : N \to O$ and preferences P_i on O is not a substantive one.

We defer the proof of Theorem 2 until Section 3.3.

Chen et al. (2021) show that TTC is characterized by *efficiency*, *individual rationality*, and *truncation invariance*.[16] They consider a weaker version of *truncation-proofness* and show that there are other rules that satisfy this property together with *efficiency* and *individual rationality* (Example 4, Chen et al., 2021).[17]

By Proposition 1, *drop strategy-proofness* and *individual rationality* imply *subset drop strategy-proofness* (and, hence, *truncation-proofness*). We therefore obtain the following corollary.

Corollary 4 (Biró et al., 2022; Altuntaş et al., 2023)**.** *Only TTC is:*

1. ***efficient**, **individually rational**, and **drop strategy-proof**;*
2. ***efficient**, **individually rational**, and **subset drop strategy-proof**.*

The following example shows that *truncation-proofness* and *individual rationality* (or, equivalently, *truncation-proofness* and the *weak endowment lower bound*) do not jointly imply *drop strategy-proofness*.

Example 2. For simplicity, consider three agents, $N = \{1, 2, 3\}$, and suppose $\omega = (o_1, o_2, o_3)$. Let $P^* \in \mathcal{P}^N$ be such that

$$P_1^* = o_2, o_3, o_1, \quad P_2^* = o_3, o_1, o_2, \quad P_3^* = o_1, o_2, o_3.$$

Then $\omega = (o_1, o_2, o_3)$ is *individually rational* at P^*. Consider the rule φ defined for all $P \in \mathcal{P}^N$ by

$$\varphi(P) = \begin{cases} \omega, & \text{if } P \in \mathcal{T}(P_1^*) \times \mathcal{T}(P_2^*) \times \mathcal{T}(P_3^*) \\ \varphi^{\text{TTC}}(P), & \text{otherwise.} \end{cases}$$

That is, $\varphi(P) = \omega$ whenever P is such that, for all $i \in N$, P_i is a truncation of P_i^*; otherwise, $\varphi(P) = \varphi^{\text{TTC}}(P)$. It is straightforward to show that φ is *truncation-proof*. To see that φ is not *drop strategy-proof*, consider $P_1' = o_3, o_1, o_2$. Then P_1' is obtained from P_1^* by dropping object $o_2 \in O \backslash \omega_1$. Because

$$\varphi_1\left(P_1', P_{N\backslash 1}^*\right) = \varphi_1^{\text{TTC}}\left(P_1', P_{N\backslash 1}^*\right) = o_3 P_1^* o_1 = \varphi_1(P^*),$$

agent 1 can manipulate at P^* by misreporting $P_1' \in \mathcal{D}(P_1^*)$. Note that P_1' is not a truncation of P_1^* because $P_1' |_{O\backslash\{o_1\}} \neq P_1 |_{O\backslash\{o_1\}}$.

[16] A rule φ is *truncation invariant* if, for each profile $P \in \mathcal{P}^N$, and each $i \in N$, if P_i' is the truncation of P_i at $\varphi_i(P)$, then $\varphi_i(P_i', P_{-i}) = \varphi_i(P)$.

[17] A rule is *truncation-proof* in the sense of Chen et al. (2021) if, for each profile $P \in \mathcal{P}^N$, and each agent $i \in N$, if P_i' is the truncation of P_i at $\varphi_i(P)$, then $\varphi_i(P) R_i \varphi_i\left(P_i', P_{N\backslash i}\right)$.

Ekici (2023) shows that TTC is characterized by *pair-efficiency*, *individual rationality*, and *strategy-proofness*. One might wonder whether this result can be strengthened by replacing *strategy-proofness* with, e.g., *subset drop strategy-proofness*. The following example illustrates that the answer is negative.

Example 3. For simplicity, consider three agents, $N = \{1, 2, 3\}$, and suppose $\omega = (o_1, o_2, o_3)$. $\omega = (a, b, c)$. Let $P^* \in \mathcal{P}^N$ be such that

$$P_1^* = o_2, o_1, o_3, \quad P_2^* = o_3, o_2, o_1, \quad P_3^* = o_1, o_3, o_2.$$

Then $\omega = (o_1, o_2, o_3)$ is *pair-efficient* at P^*; it is not *efficient* at because it is Pareto-dominated by $\varphi^{\text{TTC}}(P^*) = (o_2, o_3, o_1)$ at P^*. Consider the rule φ defined for all $P \in \mathcal{P}^N$ by

$$\varphi(P) = \begin{cases} \omega, & \text{if } P = P^* \\ \varphi^{\text{TTC}}(P), & \text{otherwise.} \end{cases}$$

Clearly, φ is *pair-efficient* and *individually rational*. It is straightforward to show that φ is *subset drop strategy-proof*.

The previous examples can be extended to any set N of agents with $|N| \geq 3$. No such examples exist for $|N| = 2$; in this case, any *drop strategy* is a *truncation strategy*, and *pair-efficiency* coincides with *efficiency*.

3.3 Proof of Theorem 2

3.3.1 Proof Technique.

We give a proof by minimum counterexample, utilizing the methodological framework developed in Sethuraman (2016), Ekici (2023), and Ekici and Sethuraman (2024). The novelty of our contribution comes from the application of their ingenious technique to the problem at hand.

Following Sethuraman (2016), denote the *size* of a profile P by $s(P) = \sum_{i \in N} |\{o \in O \mid oR_i o_i\}|$. As in Ekici (2023), we define a TTC-similarity function ρ which measures the similarity between an arbitrary rule and TTC at a given profile. Given a rule φ and a profile P, $\rho(\varphi, P) \in \{0, 1, \ldots, \infty\}^2$ is defined as follows. Consider running the procedure $\text{TTC}(P)$.

- At Step 1, let C^1 be the cycle that is executed. If $\varphi(P)$ does not execute C^1, then $\rho(\varphi, P) = 1$. Suppose $\varphi(P)$ executes C^1. If $\text{TTC}(P)$ terminates at Step 1, then $\rho(\varphi, P) = \infty$; otherwise, proceed to Step 2.

- At Step t, let C^t be the cycle that is executed. If $\varphi(P)$ does not execute C^1, then $\rho(\varphi, P) = t$. Suppose $\varphi(P)$ executes C^t. If $\mathrm{TTC}(P)$ terminates at Step t, then $\rho(\varphi, P) = \infty$; otherwise, proceed to Step $t+1$.

Note that $\rho(\varphi, P) = t < \infty$ means the following. At profile P, φ assigns objects first by running $\mathrm{TTC}(P)$ and executing the smallest cycles, say $C^1, C^2, \ldots, C^{t-1}$, that arise in Steps $1, 2, \ldots, t-1$. But then φ does not execute the smallest cycle, say C^t, that arises at Step t. Specifically, there is some agent in $i \in N(C^t)$ who is not assigned the object to which she points in C^t.

For each rule φ, denote $\rho(\varphi) = \min_{P \in \mathcal{P}^N} \rho(\varphi, P)$. Note that, if $\rho(\varphi) = \infty$, then $\rho(\varphi, P) = \infty$ for all $P \in \mathcal{P}^N$, and consequently $\varphi = \varphi^{\mathrm{TTC}}$.

We are now ready to present the proof.

3.3.2 Proof of Theorem 2

Clearly, φ^{TTC} satisfies the properties. To establish uniqueness, suppose toward contradiction that φ satisfies the properties but $\varphi \neq \varphi^{\mathrm{TTC}}$. Then $\rho(\varphi) = t$ for some $t < \infty$, and the set $\left\{P \in \mathcal{P}^N \mid \rho(\varphi, P) = t\right\}$ is nonempty. Among all profiles in $\left\{P \in \mathcal{P}^N \mid \rho(\varphi, P) = t\right\}$, let P be one whose size is smallest.

Consider running $\mathrm{TTC}(P)$. Let $C^1, C^2, \ldots, C^t$ be, in order, the cycles that are executed at steps $1, 2, \ldots, t$ of $\mathrm{TTC}(P)$.

Since $\rho(\varphi, P) = t$, $\varphi(P)$ executes the cycles $C^1, C^2, \ldots, C^{t-1}$ but not C^t. Suppose that

$$C^t = \left(i_1, o_{i_2}, i_2, o_{i_3}, \ldots, i_{k-1}, o_{i_k}, i_k, o_{i_{k+1}}, i_{k+1} = i_1\right).$$

Let $S \coloneqq \{i_1, i_2, \ldots, i_k\}$ and $O^t \coloneqq O \backslash \bigcup_{s=1}^{t-1} O(C^s)$. Note that, for each $i_\ell \in S$, $o_{i_{\ell+1}} = \mathrm{top}_{P_{i_\ell}}(O^t)$.

Because $\varphi(P)$ does not execute C^t, there is some agent i_ℓ such that $o_{i_{\ell+1}} P_{i_\ell} \varphi_{i_\ell}(P)$. Without loss of generality, let $i_\ell = i_k$. Then *individual rationality* implies that $o_{i_1} P_{i_k} \varphi_{i_k}(P) R_{i_k} o_{i_k}$.

Claim 1. $\varphi_{i_k}(P) = o_{i_k}$.

Proof of Claim 1. Suppose otherwise. Then $o_{i_1} P_{i_k} \varphi_{i_k}(P) P_{i_k} o_{i_k}$. Let $P'_{i_k} = \ldots, o_{i_1}, o_{i_k}, \ldots$ be the truncation of P_{i_k} at o_{i_1}. Because $\varphi^{\mathrm{TTC}}_{i_k}(P) = o_{i_1}$ and $P'_{i_k} |_{U\left(R_i, o_{i_1}\right)} = P_{i_k} |_{U\left(R_i, o_{i_1}\right)}$, the directed graph at any step t of $\mathrm{TTC}\left(P'_{i_k}, P_{N \backslash i_k}\right)$ is identical to the corresponding directed graph at step t of $\mathrm{TTC}(P)$; hence, $\varphi^{\mathrm{TTC}}\left(P'_{i_k}, P_{N \backslash i_k}\right) = \varphi^{\mathrm{TTC}}(P)$. Because φ is *truncation-proof*, $\varphi_{i_k}(P) R_{i_k} \varphi_{i_k}\left(P'_{i_k}, P_{N \backslash i_k}\right)$. Consequently, $\varphi_{i_k}\left(P'_{i_k}, P_{N \backslash i_k}\right) = o_{i_k}$ by *individual rationality*. Thus, $\varphi\left(P'_{i_k}, P_{N \backslash i_k}\right)$ does not execute C^t either. But then $\rho\left(\varphi, \left(P'_{i_k}, P_{N \backslash i_k}\right)\right) \leq t$ and $s\left(\left(P'_{i_k}, P_{N \backslash i_k}\right)\right) < s(P)$, which contradicts the choice of P! ■

Because $\varphi_{i_k}(P) = o_{i_k}$ and φ is *individually rational*, we must have $o_{i_k} P_{i_{k-1}} \varphi_{i_{k-1}}(P) R_{i_{k-1}} o_{i_{k-1}}$. The same reasoning as in the proof of Claim 1 demonstrates that $\varphi_{i_{k-1}}(P) = o_{i_{k-1}}$. Hence, by *individual rationality*, $o_{i_{k-1}} P_{i_{k-2}} \varphi_{i_{k-2}}(P) R_{i_{k-2}} o_{i_{k-2}}$.

Proceeding by induction, one can show that $\varphi_{i_\ell}(P) = o_{i_\ell}$ for each $\ell \in \{1, \ldots, k\}$. But then the agents in S can benefit by trading along the cycle C^t; that is, $\varphi(P)$ is not *efficient*. ■

4 Economies with variable endowments

This section considers environments in which the set of agents is fixed, but the endowments of the agents can vary. The interpretation is that allocation rules should allow agents to report not only their preferences but also their endowments.

Let $N = \{1, 2, \ldots, n\}$ be a fixed set of *agents*. Let $\mathbb{O}$ be a set of heterogeneous and indivisible *potential objects*. Let $\mathcal{O} := \{O \subseteq \mathbb{O} \mid |O| < \infty\}$ be the set of all finite subsets of $\mathbb{O}$.

Given $O \in \mathcal{O}$, an *O-allocation* is a profile $\mu = (\mu_i)_{i \in N}$ of disjoint bundles with $\bigcup_{i \in N} \mu_i = O$. Let $\mathcal{M}(O)$ denote the set of all O-allocations. An *allocation* is an O-allocation for some $O \in \mathcal{O}$. Let $\mathcal{M} = \bigcup_{O \in \mathcal{O}} \mathcal{M}(O)$ denote the set of all allocations. Given a subset $I \subseteq N$ and an allocation μ, we denote $\mu_I = (\mu_i)_{i \in I}$ and $\mu(I) = \bigcup_{i \in I} \mu_i$. Hence, if μ is an O-allocation, then $\mu(N) = O$.

An *economy* is a pair (ω, P), where $\omega = (\omega_i)_{i \in N} \in \mathcal{M}$ is the *initial allocation* and $P = (P_i)_{i \in N}$ is a profile of strict preferences on $2^{\omega(N)}$. For each $i \in N$, ω_i is agent i's *endowment* and P_i is agent i's preference relation. As in Section 2.1, we assume that agents have lexicographic preferences. For each $O \in \mathcal{O}$, $\mathcal{P}(O)$ denotes the set of lexicographic preference relations on 2^O (or, equivalently, the set of strict linear orders on O). Let $\mathcal{E} = \left\{(\omega, P) \mid \omega \in \mathcal{M} \text{ and } P \in \mathcal{P}(\omega(N))^N\right\}$ denote the class of all economies.

A *rule* is a function $\varphi : \mathcal{E} \to \mathcal{M}$ such that, for all $(\omega, P) \in \mathcal{E}$, $\varphi(\omega, P) \in \mathcal{M}(\omega(N))$. That is, a rule φ associates to each economy (ω, P) an $\omega(N)$-allocation $\varphi(\omega, P)$. For each $i \in N$, $\varphi_i(\omega, P)$ denotes agent i's assignment at $\varphi(\omega, P)$.

4.1 Properties

An allocation μ is *efficient* at (ω, P) if there is no allocation $\overline{\mu} \in \mathcal{M}(\omega(N))$ such that (i) for all $i \in N$, $\overline{\mu}_i R_i \mu_i$, and (ii) for some $i \in N$, $\overline{\mu}_i P_i \mu_i$.

Definition 9. A rule φ is ***efficient*** if, for each economy $(\omega, P) \in \mathcal{E}$, the allocation $\varphi(\omega, P)$ is efficient at (ω, P).

An allocation μ is *individually rational* at (ω, P) if, for each $i \in N$, $\mu_i R_i \omega_i$.

Definition 10. A rule φ is ***individually rational*** if, for each economy $(\omega, P) \in \mathcal{E}$, the allocation $\varphi(\omega, P)$ is individually rational for (ω, P).

A coalition I is a nonempty subset of N. An allocation μ is *blocked* by a coalition I at (ω, P) if there is some allocation $\overline{\mu} \in \mathcal{M}(\omega(N))$ such that (i) for all $i \in I$, $\overline{\mu}_i R_i \mu_i$, and (ii) for some $i \in I$, $\overline{\mu}_i P_i \mu_i$. In this case, we say that "I blocks μ *via* $\overline{\mu}$" or "μ is blocked by I *via* $\overline{\mu}$" at (ω, P). The *core* of an economy (ω, P), denoted $\mathcal{C}(\omega, P)$, is the set of allocations $\mu \in \mathcal{M}(\omega(N))$ which are not blocked by any coalition I at (ω, P).

Definition 11. A rule φ is ***core-selecting*** if, for each economy $(\omega, P) \in \mathcal{E}$, the allocation $\varphi(\omega, P)$ belongs to the core, $\mathcal{C}(\omega, P)$.

4.1.1 Preference Manipulation

We say that agent i can *manipulate* φ at (ω, P) by misreporting $P'_i \in \mathcal{P}(\omega(N))$ if $\varphi_i\left(P'_i, P_{N\setminus i}\right) P_i \varphi_i(P)$. A rule φ is *strategy-proof* if no agent i can manipulate it at any economy (ω, P) by misreporting *any* $P'_i \in \mathcal{P}(\omega(N))$.

Definition 12. A rule φ is ***strategy-proof*** if, for each economy $(\omega, P) \in \mathcal{E}$, there is no $P'_i \in \mathcal{P}(\omega(N))$ such that $\varphi_i\left(P'_i, P_{N\setminus i}\right) P_i \varphi_i(P)$.

Manipulation heuristics. The preference manipulation heuristics, namely *drop strategies*, *subset drop strategies*, and *truncations*, are defined exactly as in Section 2.2.

Let $\omega \in \mathcal{M}$ and denote $O := \omega(N)$. Let $P_i, P'_i \in \mathcal{P}(O)$. We say that P'_i is a *drop strategy* for (ω_i, P_i) if there exists $x \in O\setminus\omega_i$ such that:

1. for each $o \in O$, $oR'_i x$;
2. $P'_i|_{O\setminus\{x\}} = P_i|_{O\setminus\{x\}}$.

We say that P'_i is a *subset drop strategy* for (ω_i, P_i) if there exists $X \subseteq O\setminus\omega_i$ such that:

1. for each $x \in X$ and $x \in O\setminus X$, $oR'_i x$;
2. $P'_i|_{O\setminus X} = P_i|_{O\setminus X}$;
3. $P'_i|_X = P_i|_X$.

In this case, we say that P'_i is obtained from (ω_i, P_i) by *dropping* the subset X. We say that P'_i is a *truncation* of (ω_i, P_i) if either:

1. P'_i is obtained from (ω_i, P_i) by dropping the subset $O\setminus\omega_i$; or

2. there exists $x \in O \backslash \omega_i$ such that P_i' is obtained from (ω_i, P_i) by dropping the subset $\{o \in O \backslash \omega_i \mid x P_i o\}$.

Let $\mathcal{D}(\omega_i, P_i)$, $\mathcal{S}(\omega_i, P_i)$, and $\mathcal{T}(\omega_i, P_i)$ be the subsets of $\mathcal{P}(O)$ consisting of, respectively, all *drop strategies* for (ω_i, P_i), all *subset drop strategies* for (ω_i, P_i), and all *truncations* of (ω_i, P_i). Clearly, $\mathcal{D}(\omega_i, P_i)$ and $\mathcal{T}(\omega_i, P_i)$ are both subsets of $\mathcal{S}(\omega_i, P_i)$.

Definition 13. A rule φ is

- ***drop strategy-proof*** if, for each economy $(\omega, P) \in \mathcal{E}$, there is no $P_i' \in \mathcal{D}(\omega_i, P_i)$ such that $\varphi_i\left(P_i', P_{N\backslash i}\right) P_i \varphi_i(P)$.
- ***subset drop strategy-proof*** if, for each economy $(\omega, P) \in \mathcal{E}$, there is no $P_i' \in \mathcal{S}(\omega_i, P_i)$ such that $\varphi_i\left(P_i', P_{N\backslash i}\right) P_i \varphi_i(P)$.
- ***truncation-proof*** if, for each economy $(\omega, P) \in \mathcal{E}$, there is no $P_i' \in \mathcal{T}(\omega_i, P_i)$ such that $\varphi_i\left(P_i', P_{N\backslash i}\right) P_i \varphi_i(P)$.

4.1.2 Endowment Manipulation

In the setting with variable endowments, a rule requires each agent i to report not only a preference relation P_i but also an endowment ω_i. Hence, agents may strategize by misrepresenting their true endowment. There are a few simple heuristics that an agent—or, more generally, a group of agents—could employ. First, an agent could report only a subset of her true endowment, *hiding* the remainder for her own personal consumption. Second, an agent could report only a subset of her true endowment, *destroying* the remainder so that no agent can consume it. Third, one agent could *transfer* part of her endowment to another agent, and the pair could report the pair of endowments that results. In order to formalize these three notions of endowment manipulation, we must first introduce some new notation.

Fix an economy $(\omega, P) \in \mathcal{E}$. Given a preference profile $P \in \mathcal{P}(\omega(N))$ and a subset $O' \subseteq \omega(N)$ of objects, let $P\mid_{O'} = (P_i\mid_{O'})_{i \in N}$ denote the *restriction* of P to O'. An economy (ω', P') is called a *subeconomy* of (ω, P) if (i) for all $i \in N$, $\omega_i' \subseteq \omega_i$, and (ii) $P' = P\mid_{\omega'(N)}$. When there is no danger of confusion, we will sometimes denote a subeconomy $\left(\omega', P\mid_{\omega'(N)}\right)$ by (ω', P), i.e., we shall represent the restriction of P to $\omega'(N)$ by P also.

Given an agent $i \in N$ and a subset $\omega_i' \subseteq \mathbb{O} \backslash \omega(N \backslash \{i\})$, let $\left(\omega_i', \omega_{N\backslash i}\right)$ denote the initial allocation in which agent i's endowment is ω_i' and, for each $j \in N \backslash \{i\}$, agent j's endowment is ω_j. Given a pair of distinct agents $\{i, j\} \subseteq N$ and a pair of disjoint subsets $\omega_i', \omega_j' \subseteq \mathbb{O} \backslash \omega(N \backslash \{i, j\})$, let $\left(\omega_i', \omega_j', \omega_{N\backslash\{i,j\}}\right)$ denote the initial allocation in which agent i's endowment is ω_i', agent j's is ω_j', and for all $i^* \in N \backslash \{i, j\}$, agent i^*'s endowment is ω_{i^*}.

Definition 14. A rule φ is ***hiding-proof*** if, for each economy (ω, P), and each agent $i \in N$, there is no $X \subseteq \omega_i$ such that

$$\left[\varphi_i\left(\left(\omega_i \backslash X, \omega_{N \backslash i}\right), P\right) \cup X\right] P_i \varphi_i(\omega, P) .$$

Definition 15. A rule φ is ***destruction-proof*** if, for each economy (ω, P), and each agent $i \in N$, there is no $X \subseteq \omega_i$ such that

$$\varphi_i\left(\left(\omega_i \backslash X, \omega_{N \backslash i}\right), P\right) P_i \varphi_i(\omega, P) .$$

Definition 16. A rule φ is ***weakly destruction-proof*** if, for each economy (ω, P), and each agent $i \in N$, there is no $X \subseteq \omega_i$ such that

$$\left|\varphi_i\left(\left(\omega_i \backslash X, \omega_{N \backslash i}\right), P\right)\right|>\left|\varphi_i(\omega, P)\right| .$$

Definition 17. A rule φ is ***transfer-proof*** if, for each economy (ω, P), each agent $i \in N$, and each $X \subseteq \omega_i$,

$$\varphi_i\left(\left(\omega_i \backslash X, \omega_j \cup X, \omega_i\right), P\right) P_i \varphi_i(\omega, P) \implies \varphi_j(\omega, P) P_j \varphi_j\left(\left(\omega_i \backslash X, \omega_j \cup X, \omega_i\right), P\right) .$$

It is easy to see that any *hiding-proof* rule is *individually rational* and *destruction-proof*.

Postlewaite (1979) considers endowment manipulation when goods are divisible. Atlamaz and Klaus (2007) consider endowment manipulation in economies with indivisibilities, and they prove several impossibility results. Their results do not carry over to our setting, however, because they consider a much larger class of preferences.

4.2 Impossibility Results

The characterization in Theorem 1 still holds in this more general setting. Because *balancedness* implies *weak destruction-proofness*, TTC satisfies *weak destruction-proofness*. However, TTC is susceptible to each of the other three kinds of endowment manipulation.

Proposition 3. *TTC violates* ***hiding-proofness****,* ***destruction-proofness*** *and* ***transfer-proofness****.*

Example 4. For simplicity, suppose that $N = \{1, 2, 3, 4\}$. Let $\omega = (\{a, b, x\}, \{c, d\}, \{e\}, \{f\})$ and let P be such that

$$P_1 = c, x, d, a, b, \ldots; \qquad P_3 = f, d, e, \ldots;$$
$$P_2 = f, e, a, c, b, d, \ldots; \qquad P_4 = x, e, f, \ldots.$$

One can show that $\varphi^{\mathrm{TTC}}(\omega, P) = (\{a, b, c\}, \{e, f\}, \{d\}, \{x\})$.

Suppose that agent 1 *destroys* object x. Then, at the resulting economy $\left(\left(\omega_1 \backslash x, \omega_{N\backslash 1}\right), P\right)$, TTC selects the allocation

$$\varphi^{\mathrm{TTC}}\left(\left(\omega_1 \backslash x, \omega_{N\backslash 1}\right), P\right) = (\{c, d\}, \{a, b\}, \{f\}, \{e\}).$$

Because $\{c, d\} \, P_1 \, \{a, b, c\}$, φ^{TTC} is not *destruction-proof*. It follows that φ^{TTC} is not *hiding-proof* either.

Suppose that agent 1 *transfers* object x to agent 4. Then, at the resulting economy $((\omega_1 \backslash x, \omega_2, \omega_3, \omega_4 \cup x), P)$, TTC selects the allocation

$$\varphi^{\mathrm{TTC}}((\omega_1 \backslash x, \omega_2, \omega_3, \omega_4 \cup x), P) = (\{c, d\}, \{a, b\}, \{f\}, \{e, x\}).$$

Because $\{c, d\} \, P_1 \, \{a, b, c\}$ and $\{e, x\} \, P_4 \, \{x\}$, φ^{TTC} is not *transfer-proof*.

The previous examples can be embedded in an economy with $N = \{1, 2, \ldots, n\}$, where $n \geq 5$, by endowing each agent $i \geq 5$ with a single object, say o_i, and letting her rank o_i above all objects in $O \backslash \{o_i\}$.

The following result shows that every *core-selecting* rule is manipulable via *hiding*.

Theorem 3. *No rule is **core-selecting** and **hiding-proof**.*

Atlamaz and Klaus (2007) show that *efficiency* and *hiding-proofness* are incompatible on the domain of additive preferences, while *efficiency*, *individual rationality*, and *destruction-proofness* are incompatible on the domain of separable preferences. While they consider weaker properties, our results are logically independent because we consider a much narrower preference domain. It remains an open question whether similar results hold on the lexicographic domain.

A Proofs omitted from the main text

A.1 Proof of Theorem 1

A.1.1 Proof Technique

The proof uses the same ideas as the proof of Theorem 2. Denote the *size* of a profile P by $s(P) = \sum_{i \in N} |\{o \in O \mid oR_i \bot_{P_i}(\omega_i)\}|$. We define a TTC-similarity function ρ which measures the similarity between an arbitrary rule and TTC at a given profile. Given a rule φ and a profile P, $\rho(\varphi, P) \in \{0, 1, \ldots, \infty\}^2$ is defined as follows. Consider running the procedure $\mathrm{TTC}(P)$.

- At Step 1, let C^1 be the cycle that is executed. If $\varphi(P)$ does not execute C^1, then $\rho(\varphi, P) = 1$. Suppose $\varphi(P)$ executes C^1. If $\mathrm{TTC}(P)$ terminates at Step 1, then $\rho(\varphi, P) = \infty$; otherwise, proceed to Step 2.

- At Step t, let C^t be the cycle that is executed. If $\varphi(P)$ does not execute C^1, then $\rho(\varphi, P) = t$. Suppose $\varphi(P)$ executes C^t. If $\mathrm{TTC}(P)$ terminates at Step t, then $\rho(\varphi, P) = \infty$; otherwise, proceed to Step $t+1$.

For each rule φ, denote $\rho(\varphi) = \min_{P \in \mathcal{P}^N} \rho(\varphi, P)$. Note that, if $\rho(\varphi) = \infty$, then $\rho(\varphi, P) = \infty$ for all $P \in \mathcal{P}^N$, and consequently $\varphi = \varphi^{\mathrm{TTC}}$.

We are now ready to present our proof.

A.1.2 *Proof of Theorem 1*

Clearly, φ^{TTC} satisfies the properties. To establish uniqueness, suppose toward contradiction that φ satisfies the properties but $\varphi \neq \varphi^{\mathrm{TTC}}$. Then $\rho(\varphi) = t$ for some $t < \infty$. Then the set $\left\{P \in \mathcal{P}^N \mid \rho(\varphi, P) = t\right\}$ is nonempty. Among all profiles in $\left\{P \in \mathcal{P}^N \mid \rho(\varphi, P) = t\right\}$, let P be one whose size is smallest.

Consider running $\mathrm{TTC}(P)$. Let $\mathcal{C}^1, \mathcal{C}^2, \ldots, \mathcal{C}^t$ be, in order, the sets of cycles that arise at Steps $1, 2, \ldots, t$. Let $C^1 \in \mathcal{C}^1, C^2 \in \mathcal{C}^2, \ldots, C^t \in \mathcal{C}^t$ be, in order, the cycles that are executed at steps $1, 2, \ldots, t$ of $\mathrm{TTC}(P)$.

Since $\rho(\varphi, P) = t$, $\varphi(P)$ executes the cycles $C^1, C^2, \ldots, C^{t-1}$ but not C^t. Suppose that

$$C^t = \left(i_1, o_{i_2}, i_2, o_{i_3}, \ldots, i_{k-1}, o_{i_k}, i_k, o_{i_{k+1}}, i_{k+1} = i_1\right).$$

Let $S \coloneqq \{i_1, i_2, \ldots, i_k\}$ and $O^t \coloneqq O \setminus \bigcup_{s=1}^{t-1} O(C^s)$. Note that, for each $i_\ell \in S$, $o_{i_{\ell+1}} = \mathrm{top}_{P_{i_\ell}}(O^t)$.

Because $\varphi(P)$ does not execute C^t, there is some agent i_ℓ such that $o_{i_{\ell+1}} \notin \varphi_{i_\ell}(P)$. Without loss of generality, let $i_\ell = i_k$. Then $o_{i_1} \notin \varphi_{i_k}(P)$.

Claim 2. $\varphi_{i_k}(P) \cap O^t = \omega_{i_k} \cap O^t$.

Proof. Suppose otherwise. By *balancedness* and the fact that $|\varphi_{i_k}(P) \cap (O \backslash O^t)| = |\omega_{i_k} \cap (O \backslash O^t)|$, we must have $|\varphi_{i_k}(P) \cap O^t| = |\omega_{i_k} \cap O^t|$. Consequently, $\varphi_{i_k}(P) \cap O^t \neq \omega_{i_k} \cap O^t$ implies there is an object $o \in [\varphi_{i_k}(P) \cap O^t] \setminus [\omega_{i_k} \cap O^t]$. By the *weak endowment lower bound*, $o \in U\left(R_{i_k}, \perp_{P_{i_k}}(\omega_i)\right)$. Because $o_{i_1} \notin \varphi_{i_k}(P)$, we must have $o_{i_1} P_{i_k} o$.

Denote $\omega_{i_k} \cap O^t = \{x_1, \ldots, x_m\}$, where $x_1 P_{i_k} \cdots P_{i_k} x_m$. Let $P'_{i_k} = \ldots, o_{i_1}, x_1, \ldots, x_m, \ldots$ be the truncation of P_{i_k} at o_{i_1} (i.e., P'_{i_k} ranks each member of $\omega_{i_k} \cap O^t$ immediately below o_{i_1}). Then $o \notin U\left(R'_{i_k}, \perp_{P'_{i_k}}(\omega_i)\right)$, so P'_{i_k} is a proper truncation of P_{i_k}.

It is easy to see that $\varphi^{\mathrm{TTC}}\left(P'_{i_k}, P_{N \backslash i_k}\right)$ executes $C^1, C^2, \ldots, C^t$ also. Consequently, $o_{i_1} \in \varphi^{\mathrm{TTC}}\left(P'_{i_k}, P_{N \backslash i_k}\right)$. By the minimality of t, $\varphi\left(P'_{i_k}, P_{N \backslash i_k}\right)$ executes the cycles $C^1, C^2, \ldots, C^{t-1}$ as well. Because φ is *truncation-proof*, $\varphi_{i_k}(P) R_{i_k} \varphi_{i_k}\left(P'_{i_k}, P_{N \backslash i_k}\right)$. Hence, $o_{i_1} \notin \varphi_{i_k}\left(P'_{i_k}, P_{N \backslash i_k}\right)$. It follows that $\varphi\left(P'_{i_k}, P_{N \backslash i_k}\right)$ does not execute C^t either. But then $\rho\left(\varphi, \left(P'_{i_k}, P_{N \backslash i_k}\right)\right) = t$ and $s\left(\left(P'_{i_k}, P_{N \backslash i_k}\right)\right) < s(P)$, which contradicts the choice of P! ■

Because $\varphi_{i_k}(P) \cap O^t = \omega_{i_k} \cap O^t$, we must have $o_{i_k} \notin \varphi_{i_{k-1}}(P)$.

Claim 3. $\varphi_{i_{k-1}}(P) \cap O^t = \omega_{i_{k-1}} \cap O^t$.

Proof. Suppose otherwise. By *balancedness* and the fact that $\left|\varphi_{i_{k-1}}(P) \cap (O \backslash O^t)\right| = \left|\omega_{i_{k-1}} \cap (O \backslash O^t)\right|$, we must have $\left|\varphi_{i_{k-1}}(P) \cap O^t\right| = \left|\omega_{i_{k-1}} \cap O^t\right|$. Consequently, $\varphi_{i_{k-1}}(P) \cap O^t \neq \omega_{i_{k-1}} \cap O^t$ implies there is an object $o \in \left[\varphi_{i_{k-1}}(P) \cap O^t\right] \setminus \left[\omega_{i_{k-1}} \cap O^t\right]$. By the *weak endowment lower bound*, $o \in U\left(R_{i_{k-1}}, \perp_{P_{i_{k-1}}}(\omega_i)\right)$. Because $o_{i_k} \notin \varphi_{i_{k-1}}(P)$, we must have $o_{i_k} P_{i_{k-1}} o$.

Denote $\omega_{i_{k-1}} \cap O^t = \{x_1, \ldots, x_m\}$, where $x_1 P_{i_{k-1}} \cdots P_{i_{k-1}} x_m$. Let $P'_{i_{k-1}} = \ldots, o_{i_k}, x_1, \ldots, x_m, \ldots$ be the truncation of $P_{i_{k-1}}$ at o_{i_k} (i.e., $P'_{i_{k-1}}$ ranks each member of $\omega_{i_{k-1}} \cap O^t$ immediately below o_{i_k}). Then $o \notin U\left(R'_{i_{k-1}}, \perp_{P'_{i_{k-1}}}(\omega_i)\right)$, so $P'_{i_{k-1}}$ is a proper truncation of $P_{i_{k-1}}$.

It is easy to see that $\varphi^{\mathrm{TTC}}\left(P'_{i_{k-1}}, P_{N \backslash i_{k-1}}\right)$ executes $C^1, C^2, \ldots, C^t$ also. Consequently, $o_{i_1} \in \varphi^{\mathrm{TTC}}\left(P'_{i_{k-1}}, P_{N \backslash i_{k-1}}\right)$. By the minimality of t, $\varphi\left(P'_{i_{k-1}}, P_{N \backslash i_{k-1}}\right)$ executes the cycles $C^1, C^2, \ldots, C^{t-1}$ as well. Because φ is *truncation-proof*, $\varphi_{i_{k-1}}(P) R_{i_{k-1}} \varphi_{i_{k-1}}\left(P'_{i_{k-1}}, P_{N \backslash i_{k-1}}\right)$. Hence, $o_{i_k} \notin \varphi_{i_{k-1}}\left(P'_{i_{k-1}}, P_{N \backslash i_{k-1}}\right)$. It follows that $\varphi\left(P'_{i_{k-1}}, P_{N \backslash i_{k-1}}\right)$ does not execute C^t either. But then $\rho\left(\varphi, \left(P'_{i_{k-1}}, P_{N \backslash i_{k-1}}\right)\right) = t$ and $s\left(\left(P'_{i_{k-1}}, P_{N \backslash i_{k-1}}\right)\right) < s(P)$, which contradicts the choice of P! ■

Proceeding by induction, one can show that $\varphi_{i_\ell}(P) \cap O^t = \omega_{i_\ell} \cap O^t$ for each $\ell \in \{1, \ldots, k\}$. Because $o_{i_1} \in \omega_{i_1} \cap O^t = \varphi_{i_1}(P) \cap O^t$ and $o_{i_1} \notin \varphi_{i_k}(P)$, we have $i_1 \neq i_k$. Consequently, S contains at least two agents, i.e., $k \geq 2$. It follows that the agents in S can benefit by trading along the cycle C^t; that is, $\varphi(P)$ is not *efficient*. ■

A.2 Other Proofs

Proof of Proposition 1. Suppose φ is *drop strategy-proof* and satisfies the *weak endowment lower bound*. Let P_i' be obtained from P_i by dropping some subset $X \subseteq O \backslash \omega_i$. Suppose that $X = \{x_1, x_2, \ldots, x_k\}$, where $x_1 P_i x_2 P_i \cdots P_i x_k$. Then P_i' is obtained from a sequence of k drop strategies. That is, $P_i' = P_i^k$, where $P_i^0 = P_i$ and $P_i^1, \ldots, P_i^k$ are such that, for each $\ell \in \{1, \ldots, k\}$, P_i^ℓ is obtained from $P_i^{\ell-1}$ by dropping object x_ℓ.

Claim 4. For each $\ell \in \{1, \ldots, k\}$, $\varphi_i\left(P_i^{\ell-1}, P_{-i}\right) R_i \varphi_i\left(P_i^\ell, P_{-i}\right)$.

Proof of Claim 4. The proof is by induction on ℓ. Clearly, $\varphi_i(P) = \varphi_i(P_i^0, P_{-i}) R_i \varphi_i(P_i^1, P_{-i})$ by *drop strategy-proofness*. For the inductive step, suppose that $\ell \in \{1, \ldots, k-1\}$ is such that

$$\varphi_i(P) = \varphi_i\left(P_i^0, P_{-i}\right) R_i \varphi_i\left(P_i^1, P_{-i}\right) R_i \cdots R_i \varphi_i\left(P_i^\ell, P_{-i}\right).$$

It suffices to show that $\varphi_i\left(P_i^\ell, P_{-i}\right) R_i \varphi_i\left(P_i^{\ell+1}, P_{-i}\right)$. By *drop strategy-proofness*, we have $\varphi_i\left(P_i^\ell, P_{-i}\right) R_i^\ell \varphi_i\left(P_i^{\ell+1}, P_{-i}\right)$. By the *weak endowment lower bound*, $\varphi_i\left(P_i^\ell, P_{-i}\right) \subseteq O \backslash \{x_1, \ldots, x_\ell\}$ and $\varphi_i\left(P_i^{\ell+1}, P_{-i}\right) \subseteq O \backslash \{x_1, \ldots, x_{\ell+1}\} \subseteq O \backslash \{x_1, \ldots, x_\ell\}$. Because P_i^ℓ agrees with P_i on $O \backslash \{x_1, \ldots, x_\ell\}$, we have $\varphi_i\left(P_i^\ell, P_{-i}\right) R_i \varphi_i\left(P_i^{\ell+1}, P_{-i}\right)$, as desired. ■

It follows from Claim 4 that

$$\varphi_i(P) = \varphi_i\left(P_i^0, P_{-i}\right) R_i \varphi_i\left(P_i^1, P_{-i}\right) R_i \cdots R_i \varphi_i\left(P_i^k, P_{-i}\right).$$

Therefore, $\varphi_i(P) R_i \varphi_i\left(P_i^k, P_{-i}\right)$. ■

Proof of Proposition 2. Suppose φ satisfies the stated properties. Let P be any preference profile, and suppose $i \in N$. Let P_i' be the complete truncation of P_i. By the *weak endowment lower bound*, $\varphi_i(P_i', P_{-i}) \subseteq \omega_i$. Because φ is *balanced*, we must have $|\varphi_i(P_i', P_{-i})| = |\omega_i|$ and, hence, $\varphi_i(P_i', P_{-i}) = \omega_i$. It follows from *truncation-proofness* that $\varphi_i(P) R_i \varphi_i(P_i', P_{-i}) = \omega_i$. ■

Proof of Theorem 3. Consider $N = \{1, 2, 3\}$ and an economy (ω, P), where $\omega = (\{a, b\}, \{c\}, \{d, e\})$ and

$$P_1 = e, c, d, b, a, \quad P_2 = a, b, e, d, c, \quad P_3 = b, e, c, d, a.$$

Toward contradiction, suppose φ is *core-selecting* and *hiding-proof*. We first determine $\varphi(\omega', P)$ for all subeconomies (ω', P) of (ω, P) such that $\omega' \neq \omega$ and, for all $i \in N$, $|\omega_i'| \geq 1$.

1. $\omega^1 = (\{a\}, \{c\}, \{d\})$: Because φ is *core-selecting*, $\varphi(\omega^1, P) = (\{c\}, \{a\}, \{d\})$.

2. $\omega^2 = (\{b\}, \{c\}, \{d\})$: Because φ is *core-selecting*, $\varphi(\omega^2, P) = (\{c\}, \{b\}, \{d\})$.

3. $\omega^3 = (\{a\}, \{c\}, \{e\})$: Because φ is *core-selecting*, $\varphi(\omega^3, P) = (\{c\}, \{a\}, \{e\})$.

4. $\omega^4 = (\{b\}, \{c\}, \{e\})$: Because φ is *core-selecting*, $\varphi(\omega^4, P) = (\{e\}, \{c\}, \{b\})$.

5. $\omega^5 = (\{a, b\}, \{c\}, \{d\})$: Because φ is *core-selecting*,

$$\varphi\left(\omega^5, P\right) \in \{(\{c\}, \{a, b\}, \{d\}), (\{c, d\}, \{a\}, \{b\})\}.$$

By *hiding-proofness*, $\varphi_1(\omega^5, P) R_1 \varphi_1(\omega^1, P) \cup \{b\} = \{b, c\}$. Consequently, $\varphi(\omega^5, P) = (\{c, d\}, \{a\}, \{b\})$.

6. $\omega^6 = (\{a, b\}, \{c\}, \{e\})$: Because φ is *core-selecting*,

$$\varphi\left(\omega^6, P\right) \in \{(\{e\}, \{a\}, \{b, c\}), (\{c, e\}, \{a\}, \{b\})\}.$$

By *hiding-proofness*, $\varphi_1(\omega^6, P) R_1 \varphi_1(\omega^4, P) \cup \{a\} = \{e, a\}$. Consequently, $\varphi(\omega^6, P) = (\{c, e\}, \{a\}, \{b\})$.

7. $\omega^7 = (\{a\}, \{c\}, \{d, e\})$: Because φ is *core-selecting*, $\varphi(\omega^7, P) = (\{c\}, \{a\}, \{d, e\})$.

8. $\omega^8 = (\{b\}, \{c\}, \{d, e\})$: Because φ is *core-selecting*,

$$\varphi\left(\omega^8, P\right) \in \{(\{c, e\}, \{d\}, \{b\}), (\{e\}, \{d\}, \{b, c\})\}.$$

By *hiding-proofness*, $\varphi_3(\omega^8, P) R_1 \varphi_3(\omega^4, P) \cup \{d\} = \{b, d\}$. Consequently, $\varphi(\omega^8, P) = (\{e\}, \{d\}, \{b, c\})$.

Now consider $\varphi(\omega, P)$. Because φ is *core-selecting*, $\varphi(\omega, P)$ is one of the following seven allocations:

$$\begin{aligned}
&\mu_1 = (\{e\}, \{a\}, \{b, c, d\}), \quad \mu_2 = (\{c, e\}, \{a\}, \{b, d\}), \quad \mu_3 = (\{d, e\}, \{a\}, \{b, c\}) \\
&\mu_4 = (\{c, d\}, \{a\}, \{b, e\}), \quad \mu_5 = (\{c, e\}, \{a, d\}, \{b\}), \quad \mu_6 = (\{e\}, \{a, d\}, \{b, c\}) \\
&\mu_7 = (\{c, d, e\}, \{a\}, \{b\}).
\end{aligned}$$

If agent 3 hides e, then she receives $\varphi_3(\omega^5, P) \cup \{e\} = \{b, e\}$. Consequently, $\varphi(\omega^5, P) = \mu_4 = (\{c, d\}, \{a\}, \{b, e\})$. However, by hiding a, agent 1 receives $\varphi_1(\omega^8, P) \cup \{a\} = \{a, e\}$. Since $\{a, e\} P_1 \{c, d\}$, φ is not *hiding-proof*. ■

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