Some Characterizations of TTC in Multi-Object Reallocation Problems

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Example: Shift Exchange

- Alice, Bob, and Carol are initially assigned the following schedules:

	Mon	Tue	Wed	Thu	Fri
am	Alice	Carol	Bob	Carol	Bob
pm	Bob	Alice	Alice	Alice	Carol

- Each worker has strict preferences over all schedules.
  - Communicating these preferences is difficult.[1]
  - What preference information should we elicit?
- Reallocating the shifts can make all workers happier.
  - But how exactly should we do it?

[1] Even with only 10 shifts, there are $\binom{10}{3} = 120$ three-shift schedules for Bob / Carol and $\binom{10}{4} = 210$ four-shift schedules for Alice. Ranking them all is not feasible.

Reallocation problems

Shift Exchange is one instance of the multi-object reallocation problem:

- a group of agents.
- each agent owns a set of *heterogeneous* and *indivisible* objects.
- each agent has strict preferences over *bundles* of objects.
- a planner can redistribute objects.
- no monetary transfers.

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Other examples:

- workers exchange tasks/equipment (Yu and Zhang, 2020).
- "tuition exchange programs" in the US (Dur and Ünver, 2019).
- the "housing market" is a special case (Shapley and Scarf, 1974).

Desiderata

We want allocation rules that are

- "simple" to implement.[2]
- efficient.
- strategically robust.
- and that provide a welfare guarantee to participants.

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Pareto efficiency + strategy-proofness + individual rationality,

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are incompatible (Sönmez, 1999; Konishi et al., 2001; Todo et al., 2014).

We therefore consider relaxed notions of strategy-proofness.

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Preview of the results

- We provide characterizations of generalized Top Trading Cycles (TTC) on three domains:
  - lexicographic preferences
  - responsive preferences
  - conditionally lexicographic preferences
- Informal result: TTC is characterized by
  - individual-good efficiency
  - balancedness
  - the endowment lower bound
  - truncation-proofness
- We obtain a new characterization for the Shapley-Scarf model.[3]
- The lexicographic and conditionally lexicographic preferences are *maximal domains* on which our two efficiency notions coincide.

[3] Only TTC is Pareto efficient, individually rational, and truncation-proof.

Outline

Model: Preliminaries

A (reallocation) problem consists of:

- a set $N = \{1, 2, \ldots, n\}$ of agents.
- a set O of *heterogeneous* and *indivisible* objects, with $|O| \geq n$.
- an initial allocation $\omega = (\omega_i)_{i \in N}$ of objects to agents.
  - an "indexed partition" of O.
  - ω_i is agent i's endowment.
- a profile $P = (P_i)_{i \in N}$ of strict preferences over bundles, 2^O.
  - each P_i belongs to some domain $\mathcal{P}$.
  - R_i is the "*at least as good as*" relation associated with P_i.[4]

[4]That is, $X \; R_i \; Y$ if ($X \; P_i \; Y$ or $X = Y$).

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Since N, O, ω will remain fixed, we identify a problem with its profile P.

Thus, $\mathcal{P}^N$ is the set of all problems.

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Model: Allocations and rules

- An allocation $\mu = (\mu_i)_{i \in N}$ is a (re)assignment of objects to agents.
  - an "indexed partition" of O.
  - $\mathcal{A}$ denotes the set of allocations.
  - note that $\omega \in \mathcal{A}$.

Model: Allocations and rules

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  - an "indexed partition" of O.
  - $\mathcal{A}$ denotes the set of allocations.
  - note that $\omega \in \mathcal{A}$.
- A rule is a systematic procedure for reallocating the objects, i.e., a function $\varphi : \mathcal{P}^N \to \mathcal{A}$.
  - for example, the "no-trade rule" $\varphi \equiv \omega$.

Outline

Lexicographic Preferences

- Our focus on "simple" rules is *without loss* if agents have lexicographic preferences.

- That is, agent i's preferences over bundles are *completely determined* by her ranking over individual objects as follows: for distinct bundles X and Y,[5]

$$X \, P_i \, Y \iff \text{top}_{P_i}(X \triangle Y) \in X. \tag{1}$$

[5] Here, $X \triangle Y = (X \backslash Y) \cup (Y \backslash X)$. Equivalently,

- if agent i prefers the best object in X to that in Y, then $X \, P_i \, Y$.
- if these objects are the same, then i compares the second-best object in X to that in Y, and so on.

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- We identify a lexicographic P_i with its ranking over objects:
  - e.g., $P_i : o_1, o_2, \ldots, o_m$ means $o_1 \; P_i \; o_2 \; P_i \cdots P_i \; o_m$ and all other relations between bundles are deduced from (1).

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Top Trading Cycles

Our proofs use the following variant of the Top Trading Cycles (TTC) procedure. The associated rule is denoted φ^{TTC}.

TTC(P)

Step 0. Let $O^1 := O$.

Step $t \geq 1$.

1. Each agent $i \in N$ points to her top-ranked object in O^t.
2. Each object $o \in O^t$ points to its owner.
3. There exists a cycle. Let $C_t\left(P\right) = (i_0, o_1, i_1, o_2, \ldots, o_k, i_k = i_0)$ be the one involving the "smallest agent."
4. Assign each agent on $C_t\left(P\right)$ the object to which she points.
5. Remove all objects (but not the agents) on $C_t\left(P\right)$.
   Let $O^{t+1} := O^t \backslash \left\{o_1, \ldots, o_k\right\}$ be the objects remaining at Step $t+1$.
6. If $O^{t+1} \neq \emptyset$, proceed to Step $t+1$; otherwise, return the allocation.

Properties: I

A rule φ satisfies

1. Pareto efficiency if, for each profile P, $\varphi(P)$ is Pareto efficient.

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3. individual rationality if, for each profile P and each agent i,

$$\varphi_i(P)\ R_i\ \omega_i.$$

4. the endowment lower bound if, for each profile P and each agent i,

$$\varphi_i(P) \subseteq \{o \in O \mid o\ R_i\ \text{bottom}_{P_i}(\omega_i)\}.$$

   - e.g., if $P_i : a, b, x, c, d, y, e$ and $\omega_i = \{x, y\}$, then $\varphi_i(P)$ does not contain e.

Properties: II (Strategic robustness)

Given agent i's true preference P_i, we say that

- P_i' is a drop strategy if it is obtained by dropping an object in $O\backslash\omega_i$ to the bottom.
- P_i^* is a truncation strategy if it is obtained by dropping a "tail subset" of $O\backslash\omega_i$ to the bottom.[6]

[6] i.e., a subset X such that if $x \in X$, $y \in O\backslash\omega_i$, and $x \, P_i \, y$, then $y \in X$.

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Example

Suppose $P_i : a, b, x, c, d, y, e$ and $\omega_i = \{x, y\}$. Then:

- $P_i' : b, x, c, d, y, e, a$ is obtained by dropping object a.
- $P_i^* : a, b, x, c, y, d, e$ is obtained by "truncating at c" i.e., dropping the set $\{o \in O \backslash \omega_i \mid c\, P_i\, o\} = \{d, e\}$.
- $P_i^\circ : a, x, y, b, c, d, e$ is obtained by "truncating at a" i.e., dropping the set $\{o \in O \backslash \omega_i \mid a\, P_i\, o\} = \{b, c, d, e\}$.

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1. strategy-proof if no agent can manipulate via *any strategy*.
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Properties of TTC

Proposition

TTC satisfies

1. *Pareto efficiency*,[a]
2. *balancedness*,
3. *individual rationality*,
4. *the endowment lower bound*,
5. *truncation-proofness*,
6. *drop strategy-proofness (Altuntaş et al., 2023).*

[a] In fact, it is "core-selecting" (Fujita et al., 2018).

Two characterizations

Theorem

Only TTC satisfies

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- *and truncation-proofness.*

Lemma

If a rule satisfies drop strategy-proofness and the endowment lower bound, then it is truncation-proof.

Corollary

Only TTC satisfies Pareto efficiency, balancedness, the endowment lower bound, and drop strategy-proofness.

Discussion: Properties

Balancedness: for each profile P and each agent i, $|\varphi_i(P)| = |\omega_i|$.

- an inviolable constraint in many practical problems:
  - in shift allocation, it is often imposed for training reasons (e.g., for medical residents.)
  - a requirement in student exchange programs (e.g., Erasmus in Europe and "tuition exchange programs" in the US.)
- in the absence of constraints, it has some normative appeal:
  - "simplicity": all balanced allocations are obtained by executing single-object exchanges.
  - "endowment monotonicity": agents are rewarded with *more* objects when they bring more objects.

Discussion: Properties

The endowment lower bound: for each profile P and each agent i, $\varphi_i(P) \subseteq \{o \in O \mid o\, R_i\, \text{bottom}_{P_i}(\omega_i)\}$.

- allows agents to explicitly veto some of other agents' objects
  - the right to veto is a minimal requirement.
- disciplines the set of objects an agent can receive in any bundle
  - under individual rationality, an agent can be assigned *any object* if part of a desirable bundle.
- agrees with individual rationality for single-object problems:
  - thus, one possible extension of individual rationality to multi-object problems.
  - its role in proofs is analogous to that of individual rationality in single-object problems.

Discussion: Properties

Truncation-proofness: no agent can manipulate via *truncation strategies*.

- coupled with endowment lower bound, it ensures agents cannot benefit by vetoing objects they do not own.
- truncations are "intuitively appealing and simple for agents to implement" (Castillo and Dianat, 2016):
  - very close to true preferences (they agree on $O \backslash \omega_i$ and on ω_i).
  - agents may only consider manipulations similar to their true preferences (Mennle et al., 2015)
  - in many settings, they are the *only* manipulations that are profitable (Roth and Rothblum, 1999; Ehlers, 2008; Kojima and Pathak, 2009; Kojima, 2013).
  - hence, a minimal incentive requirement.

Outline

The Shapley-Scarf Model

- The Shapley-Scarf model is the special case in which each agent owns and receives one object.
- In this model:
  - only TTC is Pareto efficient, individually rational, and strategy-proof (Ma, 1994).
  - all allocations are balanced.
  - the endowment lower bound coincides with individual rationality.

[7] e.g., consider an employer reallocating tasks among employees.

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Corollary

Only TTC is Pareto efficient, individually rational, and truncation-proof.

- Though a planner with a stake in the outcome[7] may consider relaxing strategy-proofness to truncation-proofness ...
- ... we show that this relaxation does not give rise to any new rules.

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Proof Sketch. Step 1: Select a "minimal profile"

- Toward contradiction, suppose $\varphi \neq \varphi^{\mathsf{TTC}}$ is Pareto efficient, individually rational, and truncation-proof.
- We select a profile P which is "minimal" according to some criteria—for that we need some notation.

Proof Sketch. Step 1: Select a "minimal profile"

- Toward contradiction, suppose $\varphi \neq \varphi^{\mathsf{TTC}}$ is Pareto efficient, individually rational, and truncation-proof.
- We select a profile P which is "minimal" according to some criteria—for that we need some notation.
- For each profile P, let
  - $C_t(P)$ be the cycle executed at step t of TTC(P).
  - $s(P) = \sum_{i \in N} |\{o \in O \mid o \, R_i \, o_i\}|$ be the size of P, where $\omega_i = \{o_i\}$.
- Define the similarity $\rho : \mathcal{P}^N \to \mathbb{N} \cup \{\infty\}$ as follows:
  - if $\varphi(P) = \varphi^{\mathsf{TTC}}(P)$, then $\rho(P) = \infty$;
  - otherwise, $\rho(P) = \min\{t \in \mathbb{N} \mid \varphi(P) \text{ does not execute } C_t(P)\}$.
- Let $t \coloneqq \min_{P \in \mathcal{P}^N} \rho(P)$; then $\varphi \neq \varphi^{\mathsf{TTC}}$ implies $t < \infty$.
- Among all profiles in $\left\{P' \in \mathcal{P}^N \mid \rho(P') = t\right\}$, let P be one that minimizes $s(P)$.

Step 2: Agents on $C_t(P)$ retain their endowments

- Because $\rho(P) = t$, $\varphi(P)$ executes cycles $C_1(P), \ldots, C_{t-1}(P)$ but not $C_t(P)$.
- Let $C \coloneqq C_t(P)$, say

$$C = (i_0, o_1, i_1, o_2, \ldots, i_{k-1}, o_k, i_k = i_0).$$

- Because $\varphi(P)$ does not execute C, can assume WLOG that i_k $(= i_0)$ does not receive o_1. Thus, $\varphi^{\mathsf{TTC}}_{i_k}(P) = o_1 \, P_{i_k} \, \varphi_{i_k}(P)$.[8]

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- Thus, the profile P looks as follows (endowments are blue):

$$\begin{array}{ccccc}
P_{i_1} & P_{i_2} & \cdots & P_{i_{k-1}} & P_{i_k} \\
\hline
\vdots & \vdots & \ddots & \vdots & \vdots \\
o_2 & o_3 & \cdots & o_k & o_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\color{blue}{o_1} & \color{blue}{o_2} & \cdots & \color{blue}{o_{k-1}} & \varphi_{i_k}(P)
\end{array}$$

[8] By individual rationality, the number of agents on C is $k \geq 2$.

Step 2: Agents on $C_t(P)$ retain their endowments

- Suppose $\varphi_{i_k}(P) \neq o_k$.
- By individual rationality, the profile P looks as follows:

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\vdots & \vdots & \ddots & \vdots & \vdots \\
 & & & & o_k \\
 & & & & \vdots
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Step 2: Agents on $C_t(P)$ retain their endowments

- Suppose $\varphi_{i_k}(P) \neq o_k$.
- Let P'_{i_k} be the truncation of P_{i_k} at o_1:

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\vdots & \vdots & \ddots & \vdots & \varphi_{i_k}(P) \\
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\end{array}
$$

- Letting $P' := \left(P'_{i_k}, P_{-i_k}\right)$, our choice of P implies that $\varphi(P')$ executes cycles $C_1(P'), \ldots, C_t(P')$ $(= C_1(P), \ldots, C_t(P))$.
- Thus, $\varphi_{i_k}(P') = o_1 \; P_{i_k} \; \varphi_{i_k}(P)$, a violation of truncation-proofness.

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- Thus, $\varphi_{i_k}(P) = o_k$, which means that $o_k \, P_{i_{k-1}} \, \varphi_{i_{k-1}}(P)$.

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\vdots & \vdots & \ddots & \vdots & \vdots
\end{array}$$

- A similar argument shows that $\varphi_{i_{k-1}}(P) = o_{k-1}$.

Step 2: Agents on $C_t(P)$ retain their endowments

- By a recursive argument, the profile P looks as follows:

$$
\begin{array}{ccccc}
P_{i_1} & P_{i_2} & \cdots & P_{i_{k-1}} & P_{i_k} \\
\hline
\vdots & \vdots & \ddots & \vdots & \vdots \\
o_2 & o_3 & \cdots & o_k & o_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\color{blue}{o_1} & \color{blue}{o_2} & \cdots & \varphi_{i_{k-1}}(P) = \color{blue}{o_{k-1}} & \varphi_{i_k}(P) = \color{blue}{o_k} \\
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- ... but then φ is not Pareto efficient! $\square$

Outline

Responsive preferences

- Agent i has responsive preferences if, for any bundle X and any $y, z \in O \backslash X$,

$$y \, P_i \, z \iff (X \cup y) \, P_i \, (X \cup z) \, .$$

  - $\mathcal{L} \subseteq \mathcal{R}$, where $\mathcal{L}$ and $\mathcal{R}$ are the lexicographic and responsive domains.

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  - $\mathcal{L} \subseteq \mathcal{R}$, where $\mathcal{L}$ and $\mathcal{R}$ are the lexicographic and responsive domains.
- Given $P_i \in \mathcal{R}$, let $\succ^{P_i}$ denote the associated ordering over O.
- There are many responsive extensions of an ordering $\succ^{P_i}$ over O, i.e.,
$$\succ^{P_i} = \succ^{P'_i} \;\not\Rightarrow\; P_i = P'_i .$$
  ... but the lexicographic extension is *unique*.

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... but the lexicographic extension is *unique*.

Example

We may have

$$\{a, d\} \, P_i \, \{b, c\} \text{ and } \{b, c\} \, P_i' \, \{a, d\}$$

even though P_i and P_i' both rank objects in the order

$$\succ^{P_i} = \succ^{P_i'} : a, b, c, d.$$

Simple rules

- We focus on rules that depend only on the orderings $\succ^P = (\succ^{P_i})_{i \in N}$ associated with a profile $P = (P_i)_{i \in N}$.
- Formally, a rule φ is individual-good-based if

$$\text{for all } P, P' \in \mathcal{R}^N, \ \ \succ^P = \succ^{P'} \implies \varphi(P) = \varphi(P'). \tag{2}$$

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- One interpretation is that the rule elicits only $\succ^P$, but agents evaluate allocations based on their underlying preferences P.
- This assumption is common in theory (e.g., Aziz et al., 2019; Biró et al., 2022) and in practice.
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  - e.g., in the National Resident Matching Program which matches doctors to hospitals in the US, hospitals report only their rankings over individual doctors (Milgrom, 2009, 2011).
- Note that TTC is an individual-good-based rule.

Properties

Our properties are defined as before, with the understanding that *drop strategies* and *truncation strategies* for P_i are defined wrt $\succ^{P_i}$.

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Example

Suppose P_i is such that $\succ^{P_i}$: a, b, x, c, d, y, e and $\omega_i = \{x, y\}$. Then:

- *any* P_i' with $\succ^{P_i'}$: b, x, c, d, y, e, a is obtained by dropping object a.
- *any* P_i^* with $\succ^{P_i^*}$: a, b, x, c, y, d, e is obtained by "truncating at c."

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We also consider a weak version of efficiency: a rule φ is

- individual-good efficient (ig-efficient) if, for each profile P, there is no "Pareto-improving *single-object* exchange" at $\varphi(P)$.

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- individual-good efficient (ig-efficient) if, for each profile P, there is no "Pareto-improving *single-object* exchange" at $\varphi(P)$.[9]

Proposition

TTC satisfies

1. *ig-efficiency (but not Pareto efficiency)*
2. *truncation-proofness (but not drop strategy-proofness).*

---

[9] i.e., no cycle $C = (i_0, o_1, i_1, \ldots, i_{k-1}, o_k, i_k = i_0)$ such that, for all $\ell \in \{1, \ldots, k\}$,

$$o_\ell \in \varphi_{i_\ell}(P) \quad \text{and} \quad (\varphi_{i_\ell}(P) \cup o_{\ell+1}) \setminus o_\ell \; P_{i_\ell} \; \varphi_{i_\ell}(P).$$

Two characterizations

Theorem

An individual-good-based rule satisfies

1. *ig-efficiency,*
2. *balancedness,*
3. *the endowment lower bound, and*
4. *truncation-proofness*

if and only if it is TTC.

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Proof.

- Let φ be an individual-good-based rule satisfying properties (1)-(4).
- By our theorem for lexicographic prefs., φ agrees with φ^{TTC} on $\mathcal{L}^N$.
- Let $P \in \mathcal{R}^N$, and let $P' \in \mathcal{L}^N$ be such that $\succ^{P'} = \succ^{P}$.
- Because φ and φ^{TTC} are individual-good-based,

$$\varphi(P) = \varphi(P') = \varphi^{\mathsf{TTC}}(P') = \varphi^{\mathsf{TTC}}(P). \quad \square$$

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If an individual-good-based rule is balanced and individually rational, then it satisfies the endowment lower bound.

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Lemma

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Outline

Related Literature

Two closely related papers:

- reallocation with lexicographic preferences: Altuntaș et al. (2023).
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(Re)allocation with responsive preferences and different desiderata:

- *Segmented Trading Cycles* retain strategy-proofness + individual rationality, but lose ig-efficiency (Pápai, 2003).
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Other relaxations of strategy-proofness:

- rank monotonicity (Chen and Zhao, 2021), truncation-invariance (Chen et al., 2024; Hashimoto et al., 2014), etc.

Outline

Conditionally lexicographic preferences

- Agent i has conditionally lexicographic preferences P_i if, given any bundle $Y \subsetneq O$ and any nonempty $X \subseteq O \backslash Y$, there is an object $\text{top}_{P_i}(X \mid Y) \in X$ which is "lexicographically best among X conditional on receiving Y."
  - $\mathcal{CL} \cap \mathcal{R} = \mathcal{L}$, where $\mathcal{CL}$ denotes the conditionally lexicographic domain.

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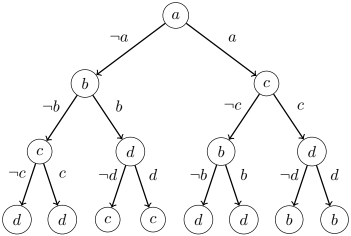
  - $\mathcal{CL} \cap \mathcal{R} = \mathcal{L}$, where $\mathcal{CL}$ denotes the conditionally lexicographic domain.
- Conditionally lexicographic preferences
  - permit *complementarity* between objects.
  - compact "tree representation" makes them simple to elicit.



Properties

Our properties are the same, except for two modifications:

- the endowment lower bound posits that, for each profile P and each agent i, $\varphi_i(P)$ does not contain objects that are "conditionally worse" than all objects in her endowment (conditional on receiving $\varphi_i(P)$).
- drop strategy-proofness posits that no agent can manipulate by "*dropping an object to the bottom of her lexicographic preference tree.*"

A characterization

- The extension of TTC to the conditionally lexicographic domain is called Augmented Top Trading Cycles (ATTC) (Fujita et al., 2018)
  - at step t, agent i points to $\text{top}_{P_i}\left(O^t \mid \mu_i^{t-1}\right)$, where O^t is the set of remaining objects and μ_i^{t-1} is i's assignment after step $t-1$.
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Theorem

Only ATTC satisfies

- *Pareto efficiency*
- *balancedness*
- *the endowment lower bound, and*
- *drop strategy-proofness.*

Maximal domain results

- It is known that ig-efficiency = Pareto efficiency on the lexicographic domain (Aziz et al., 2019).

- Our focus on the lexicographic and conditionally lexicographic domains is justified by the following.

Maximal domain results

- It is known that ig-efficiency = Pareto efficiency on the lexicographic domain (Aziz et al., 2019).
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Proposition

1. *ig-efficiency = Pareto efficiency on the conditionally lexicographic domain.*
2. *$\mathcal{L}$ is a maximal subdomain of $\mathcal{R}$ on which ig-efficiency = Pareto efficiency.*
3. *$\mathcal{CL}$ is a maximal domain on which ig-efficiency = Pareto efficiency.*

Conclusion

- Our axiomatic analysis helps us to better understand the trade-offs involved in multi-object reallocation.
- Although it is manipulable, TTC performs reasonably well according to three criteria of interest: efficiency, individual rationality, and strategic robustness.
- Our characterizations suggest that TTC is a compelling rule in general environments.

Thank you!

☺

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