Qubits And Quantum Computers

QuTe

Funded by
the European Union

Ready, set, go!

Grab your *Qubits and Quantum Computers* notes and follow along!

Funded by
the European Union

Bits and Qubits

Funded by
the European Union

Bits

- Two state system

Funded by
the European Union

Bits

- Two state system
- heads or tail

Funded by
the European Union

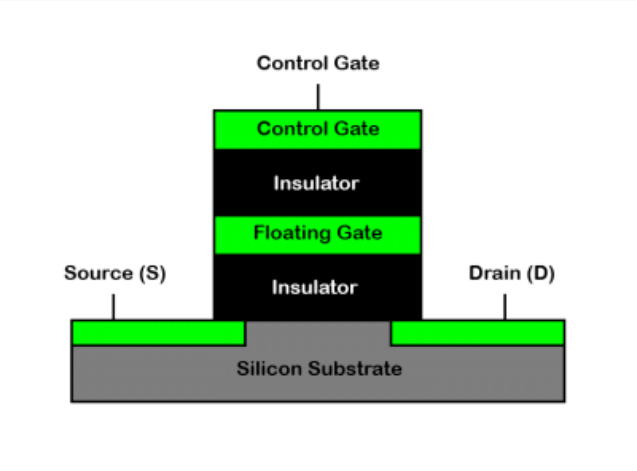
Bits

- Two state system
- heads or tail
- up or down

Funded by
the European Union

Bits

- Two state system
- heads or tail
- up or down
- charge or no charge



Funded by
the European Union

Bits

- Two state system
- heads or tail
- up or down
- charge or no charge
- 0 or 1

0 1

Funded by
the European Union

Bits

Exercise 1 p. 2 in the notes

How many possible combinations are possible with 4 bits? Same question but with 10 bits and with n bits?

Funded by
the European Union

A classical computer

How many of the numbers are greater than 4?

8	2	0	3	1	4	5	8	2	1	7	2	7	3	8	5	5	2	9	0	6	3	1	6	4
0	8	7	3	3	1	9	7	5	2	5	7	6	9	8	0	3	6	2	5	1	2	7	5	2
2	3	3	8	6	1	4	2	4	0	2	6	1	8	9	5	2	6	9	8	3	4	0	1	0
4	7	5	5	6	3	0	7	7	1	9	1	6	1	7	4	1	7	1	3	7	9	3	3	7
1	9	3	9	5	3	4	9	5	5	2	7	5	8	0	3	4	8	8	1	2	7	5	3	4
2	8	7	8	1	4	1	4	9	4	2	4	1	5	2	9	4	6	2	1	5	2	8	1	9
8	4	8	5	1	3	9	6	6	0	7	2	1	9	0	2	0	6	7	0	6	0	1	3	0
0	3	8	8	4	7	5	1	5	1	7	3	4	5	2	0	7	4	7	9	6	6	7	7	4
3	5	3	1	9	3	7	4	9	5	0	2	0	1	4	6	2	5	4	5	8	5	0	9	2
3	4	5	9	5	2	7	9	8	9	0	5	5	8	5	1	7	7	3	5	5	4	7	7	2
4	1	5	3	0	9	1	3	7	2	5	8	7	7	1	3	6	3	9	7	8	7	9	1	7
7	2	9	5	6	7	8	5	4	5	3	4	5	4	1	9	8	6	7	5	7	9	3	1	8
5	9	2	8	9	8	6	4	4	1	5	3	7	7	0	8	0	2	5	6	0	6	1	2	0
1	3	3	3	9	0	5	2	8	7	4	0	9	0	3	7	3	1	7	9	4	5	5	2	8
4	6	0	1	0	8	6	2	1	0	0	5	0	3	1	5	4	9	0	3	7	4	7	0	1
7	7	0	6	6	3	2	8	8	5	8	9	5	6	4	0	5	9	1	8	0	5	4	9	4
3	3	8	5	7	5	7	4	3	4	5	7	9	6	9	5	0	7	7	6	6	8	8	5	9
9	1	7	1	3	6	9	2	9	1	9	4	2	3	3	0	8	1	8	7	7	6	4	7	2
6	2	2	8	0	9	4	5	3	7	2	5	4	6	6	5	6	6	5	0	4	6	5	6	8
1	7	5	9	0	0	2	0	5	6	5	8	5	1	9	5	3	3	7	4	0	5	8	2	4
0	3	9	6	9	4	7	3	5	7	0	6	5	4	7	1	1	8	5	3	2	8	0	9	8

Funded by
the European Union

A classical computer

How many of the numbers are greater than 4?

8	2	0	3	1	4	5	8	2	1	7	2	7	3	8	5	5	2	9	0	6	3	1	6	4
0	8	7	3	3	1	9	7	5	2	5	7	6	9	8	0	3	6	2	5	1	2	7	5	2
2	3	3	8	6	1	4	2	4	0	2	6	1	8	9	5	2	6	9	8	3	4	0	1	0
4	7	5	5	6	3	0	7	7	1	9	1	6	1	7	4	1	7	1	3	7	9	3	3	7
1	9	3	9	5	3	4	9	5	5	2	7	5	8	0	3	4	8	8	1	2	7	5	3	4
2	8	7	8	1	4	1	4	9	4	2	4	1	5	2	9	4	6	2	1	5	2	8	1	9
8	4	8	5	1	3	9	6	6	0	7	2	1	9	0	2	0	6	7	0	6	0	1	3	0
0	3	8	8	4	7	5	1	5	1	7	3	4	5	2	0	7	4	7	9	6	6	7	7	4
3	5	3	1	9	3	7	4	9	5	0	2	0	1	4	6	2	5	4	5	8	5	0	9	2
3	4	5	9	5	2	7	9	8	9	0	5	5	8	5	1	7	7	3	5	5	4	7	7	2
4	1	5	3	0	9	1	3	7	2	5	8	7	7	1	3	6	3	9	7	8	7	9	1	7
7	2	9	5	6	7	8	5	4	5	3	4	5	4	1	9	8	6	7	5	7	9	3	1	8
5	9	2	8	9	8	6	4	4	1	5	3	7	7	0	8	0	2	5	6	0	6	1	2	0
1	3	3	3	9	0	5	2	8	7	4	0	9	0	3	7	3	1	7	9	4	5	5	2	8
4	6	0	1	0	8	6	2	1	0	0	5	0	3	1	5	4	9	0	3	7	4	7	0	1
7	7	0	6	6	3	2	8	8	5	8	9	5	6	4	0	5	9	1	8	0	5	4	9	4
3	3	8	5	7	5	7	4	3	4	5	7	9	6	9	5	0	7	7	6	6	8	8	5	9
9	1	7	1	3	6	9	2	9	1	9	4	2	3	3	0	8	1	8	7	7	6	4	7	2
6	2	2	8	0	9	4	5	3	7	2	5	4	6	6	5	6	6	5	0	4	6	5	6	8
1	7	5	9	0	0	2	0	5	6	5	8	5	1	9	5	3	3	7	4	0	5	8	2	4
0	3	9	6	9	4	7	3	5	7	0	6	5	4	7	1	1	8	5	3	2	8	0	9	8

Perfect for a classical computer: One thing at a time, very fast!

Funded by
the European Union

Unlock the bike

Exercise 2 p. 2 in the notes

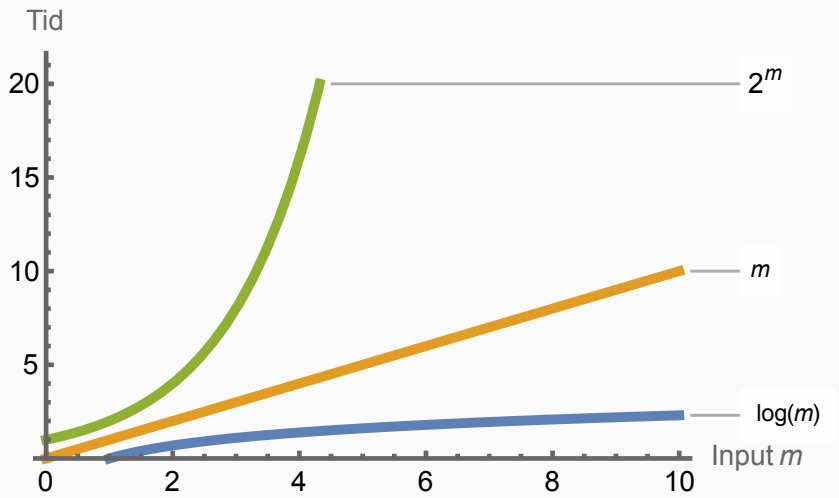
The code cylinder on a bicycle lock is made up of 5 numbers between 0 and 9: How many possible combinations (i.e. how many codes) does the lock have?

Funded by the European Union

Complexity

Definition 1

Complexity of a calculation: the number of operations needed to find a solution to a given problem when the size of hte input is m.

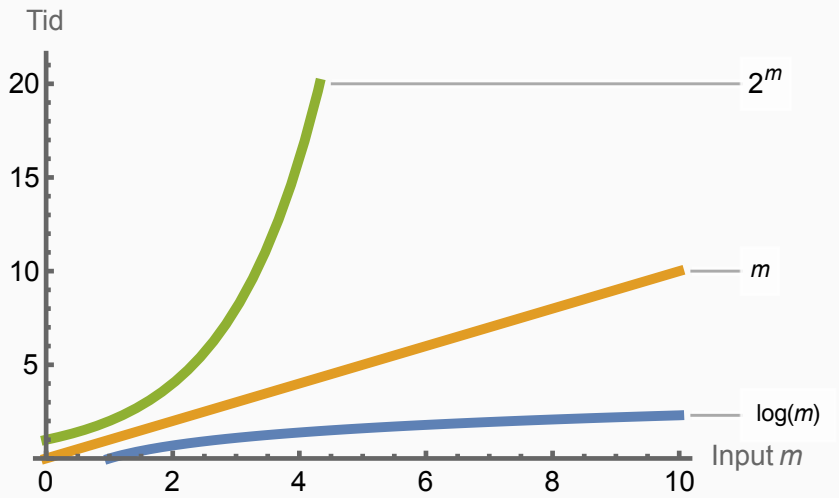


Funded by the European Union

Complexity

Definition 1

Complexity of a calculation: the number of operations needed to find a solution to a given problem when the size of hte input is m.



Often the problems increases *exponentially*.

Funded by the European Union

Exercises

Exercise 3 p. 3 in the notes

A classical supercomputer can perform 10^{17} operation each second. Vi use it to break a code lock like the one before, except with m digits instead of 5. Assume that the classical supercomputer can check 10^{17} different combinations on the lock each second. How large can m be if the classical supercomputer must be able to complete the task in a year?

Exercise 4 p. 3 in the notes

Estimate the time it would take you to try all of the possible combinations of the clothes (including accessories!) you choose between each morning.

Funded by the European Union

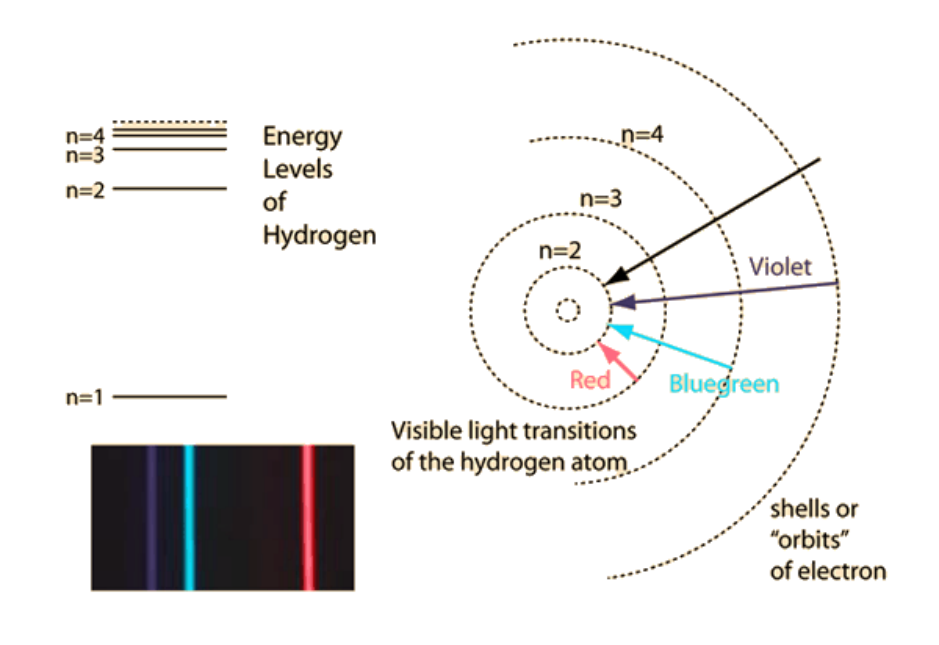
Qubits

- Qubits are *quantum systems*

Funded by
the European Union

Qubits

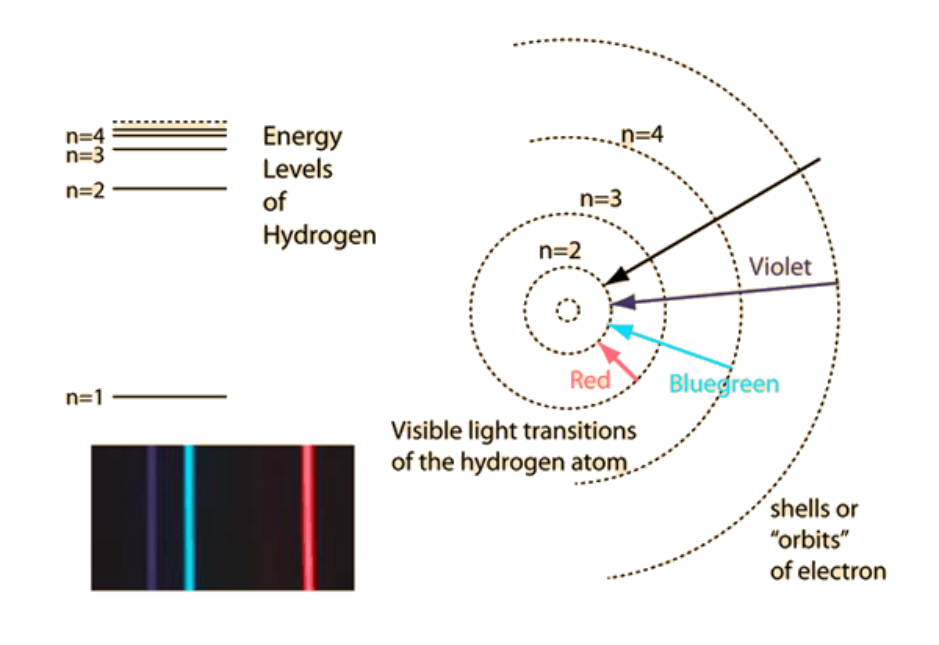
- Qubits are *quantum systems*
- Like Bohr's atomic model



Funded by the European Union

Qubits

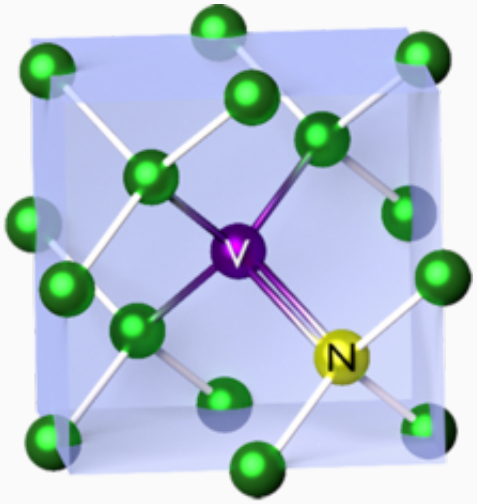
- Qubits are *quantum systems*
- Like Bohr's atomic model
- Qubits are "simple". *Only 2 states*

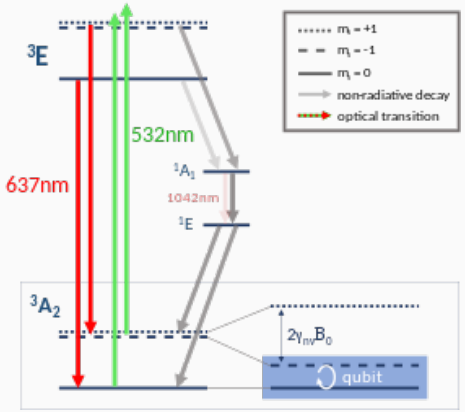


Funded by the European Union

Qubits

- Qubits are *quantum systems*
- Like Bohr's atomic model
- Qubits are "simple". *Only 2 states*
- You know *real simple* like NV Centers...





Funded by the European Union

Bits and Qubits

Bits

- Denoted 0 and 1
- If a bit is in state 0 we are sure to measure 0
- If a bit is in state 1 we are sure to measure 1
- Can *only* be in either state 0 or 1

Qubits

- Denoted $|0\rangle$ and $|1\rangle$
- If a bit is in state $|0\rangle$ we are sure to measure 0
- If a bit is in state $|1\rangle$ we are sure to measure 1
- Can be in either state $|0\rangle$, $|1\rangle$ **or in a superposition of them.**

Funded by the European Union

Bits and Qubits

Bits

- Denoted 0 and 1
- If a bit is in state 0 we are sure to measure 0
- If a bit is in state 1 we are sure to measure 1
- Can *only* be in either state 0 or 1

Qubits

- Denoted $|0\rangle$ and $|1\rangle$
- If a bit is in state $|0\rangle$ we are sure to measure 0
- If a bit is in state $|1\rangle$ we are sure to measure 1
- Can be in either state $|0\rangle$, $|1\rangle$ **or in a superposition of them.**

Funny notation

A quantum system i described by a state which we denote $|\ldots\rangle$.

It is called a *ket*.

Funded by the European Union

Lingual oddities

- Together with a ket, $|\ldots\rangle$, there is a

Funded by
the European Union

Lingual oddities

- Together with a ket, $|\ldots\rangle$, there is a
- *bra*, $\langle\ldots|$

Funded by
the European Union

Lingual oddities

- Together with a ket, $|\ldots\rangle$, there is a
- *bra*, $\langle\ldots|$
- Combining them we get a *bra(c)ket*

Funded by
the European Union

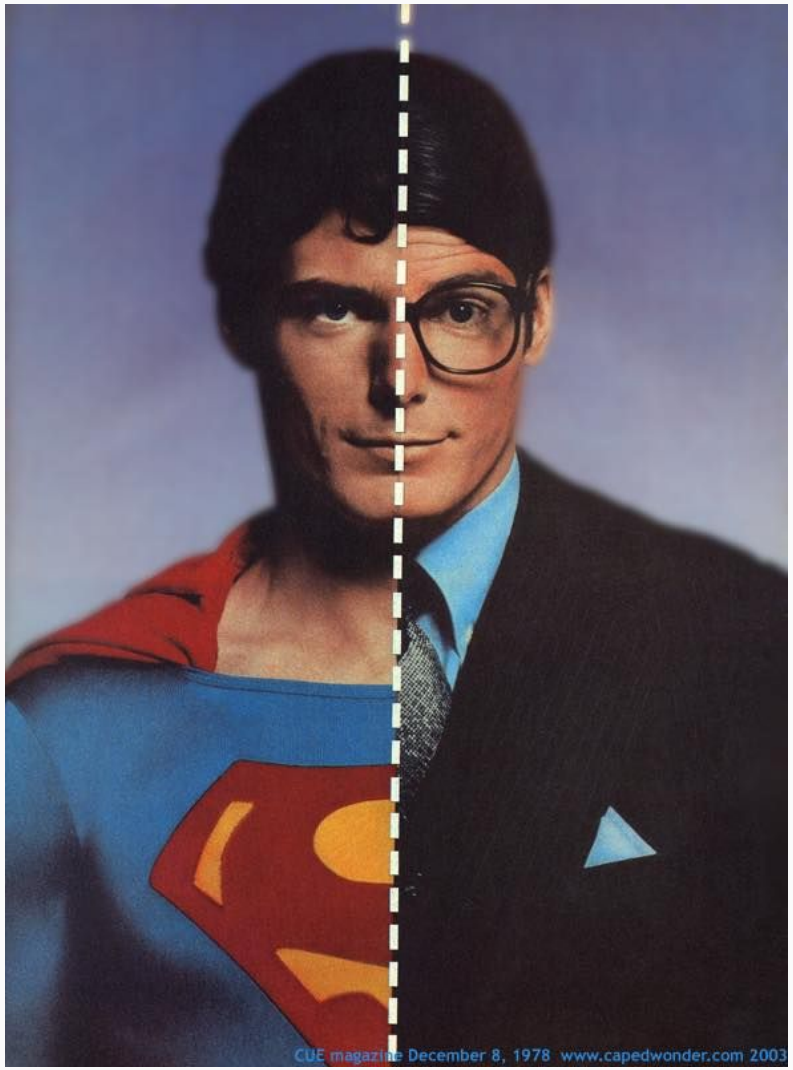
Lingual oddities

- Together with a ket, $|\ldots\rangle$, there is a
- *bra*, $\langle\ldots|$
- Combining them we get a *bra(c)ket*
- or bra(c)ket, $\langle\ldots|\ldots\rangle$, like in *parenthesis*

[ ]
{ }
( )
⟨ ⟩
<>

Funded by the European Union

Superposition



Funded by
the European Union

Single Qubit Superposition

A single qubit can be in either:

- state $|0\rangle$
- state $|1\rangle$

Funded by
the European Union

Single Qubit Superposition

A single qubit can be in either:

- state $|0\rangle$
- state $|1\rangle$

Funded by
the European Union

Single Qubit Superposition

A single qubit can be in either:

- state $|0\rangle$
- state $|1\rangle$

Funded by
the European Union

Single Qubit Superposition

A single qubit can be in either:

- state $|0\rangle$
- state $|1\rangle$
- or in a **superposition** $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

Funded by the European Union

Single Qubit Superposition

Or with a coin analogy

- state $|0\rangle$
- state $|1\rangle$
- or in a **superposition**
  $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

Funded by
the European Union

Single Qubit Superposition

Or with a coin analogy

- state $|0\rangle$
- state $|1\rangle$
- or in a **superposition**
  $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

Funded by
the European Union

Single Qubit Superposition

Or with a coin analogy

- state $|0\rangle$
- state $|1\rangle$
- or in a **superposition**
  $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

Funded by
the European Union

Single Qubit Superposition

Or with a coin analogy

- state $|0\rangle$
- state $|1\rangle$
- or in a **superposition** $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

Funded by
the European Union

Superposition and normalisation

Definition: Superposition

$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

α and β can be choosen arbitrarily as long it is **normalised**.

Definition: Normalisation

The qubit state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ is normalised if $\alpha^2 + \beta^2 = 1$.

Funded by
the European Union

Superposition and normalisation

Definition: Superposition

$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

α and β can be choosen arbitrarily as long it is **normalised**.

Definition: Normalisation

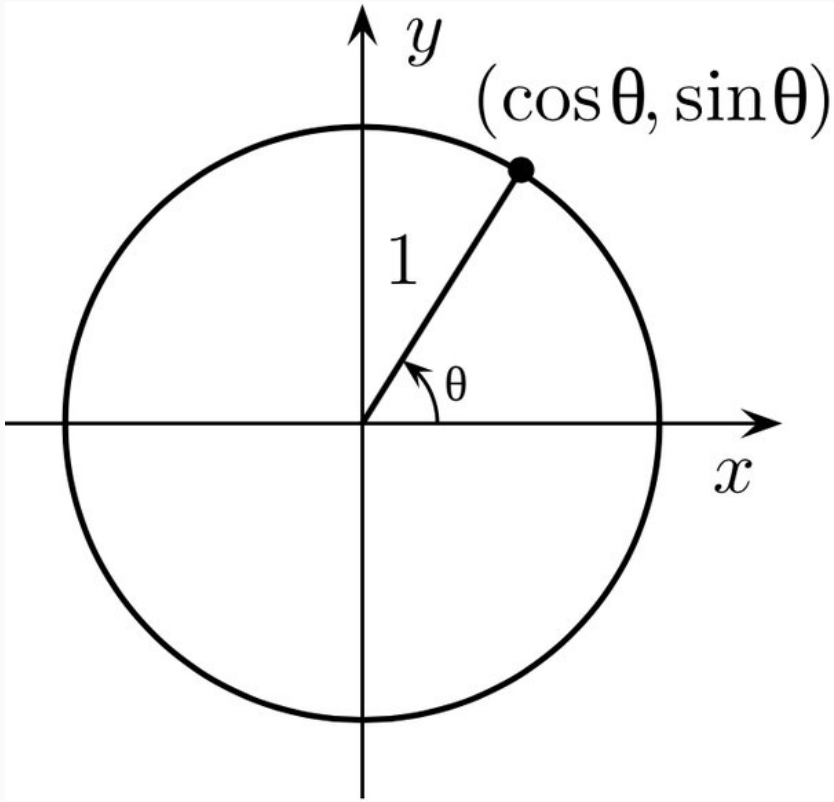
The qubit state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ is normalised if $\alpha^2 + \beta^2 = 1$.

Read through the example and do exercise 5 and 6 on page 4 in the notes.

Funded by
the European Union

A unit vector

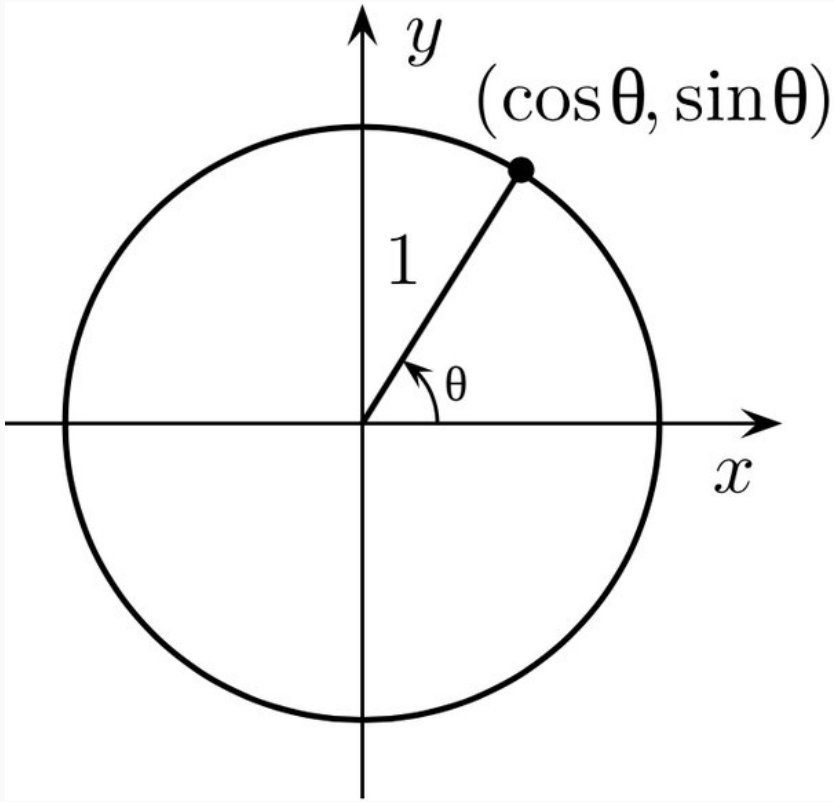
- $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Unit vector along x-axis.
- $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Unit vector along y-axis.
- $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ Unit vector pointing from (0,0) to (α, β) on the unit circle.



Funded by the European Union

A unit vector

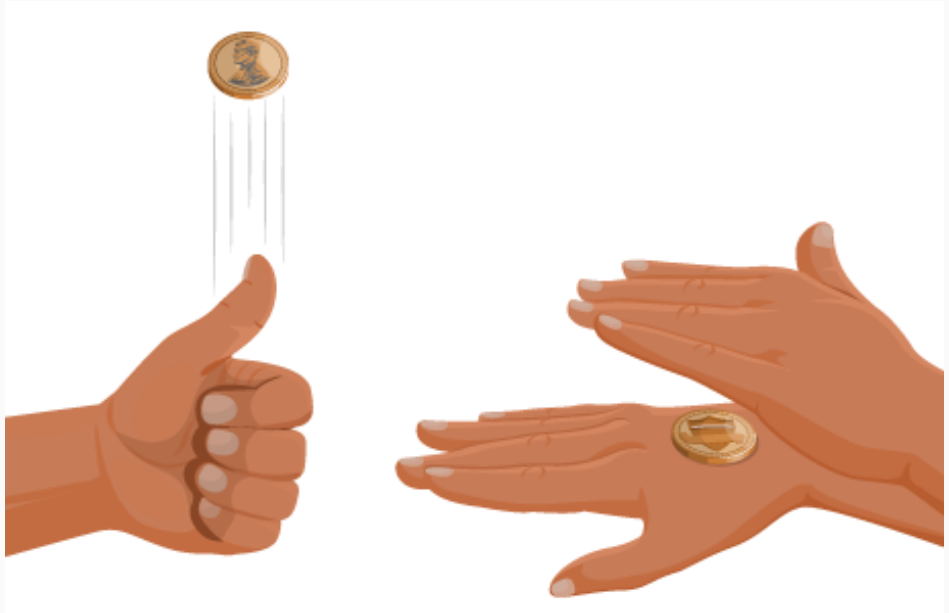
- $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Unit vector along x-axis.
- $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Unit vector along y-axis.
- $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ Unit vector pointing from (0,0) to (α, β) on the unit circle.



Do exercise 7 on page 4 in the notes.

Funded by the European Union

Measurement and Probability



Funded by
the European Union

Measuring on $|\psi\rangle = |0\rangle$

- If the qubit is in the state $|\psi\rangle = |0\rangle$

Funded by
the European Union

Measuring on $|\psi\rangle = |0\rangle$

- If the qubit is in the state $|\psi\rangle = |0\rangle$
- and the quantum computer performs a measurement

Funded by the European Union

Measuring on $|\psi\rangle = |0\rangle$

- If the qubit is in the state $|\psi\rangle = |0\rangle$
- and the quantum computer performs a measurement
- we get 0 with 100% probability.

Funded by
the European Union

Measuring on $|\psi\rangle = |1\rangle$

- If the qubit is in the state $|\psi\rangle = |1\rangle$

Funded by
the European Union

Measuring on $|\psi\rangle = |1\rangle$

- If the qubit is in the state $|\psi\rangle = |1\rangle$
- and the quantum computer performs a measurement

Funded by the European Union

Measuring on $|\psi\rangle = |1\rangle$

- If the qubit is in the state $|\psi\rangle = |1\rangle$
- and the quantum computer performs a measurement
- we get 1 with 100% probability.

Funded by
the European Union

Measuring $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

- If the qubit now is in the state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

Funded by
the European Union

Measuring $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

- If the qubit now is in the state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$
- and the quantum computer performs a measurement.

Funded by the European Union

Measuring $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

- If the qubit now is in the state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$
- and the quantum computer performs a measurement.
- we get **either** 0 or 1 and **never** something in between.

or

Funded by the European Union

Measuring $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

- If the qubit now is in the state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$
- and the quantum computer performs a measurement.
- we get **either** 0 or 1 and **never** something in between.
- We cannot predict the outcome. Only calculate the probabilities!

or

Funded by the European Union

Measuring $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

- If the qubit now is in the state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$
- and the quantum computer performs a measurement.
- we get **either** 0 or 1 and **never** something in between.
- We cannot predict the outcome. Only calculate the probabilities!
- We have to abandon **determinism**.

or

Funded by the European Union

Probabilities

Definition: Probability

If our qubit is in the state $|\psi\rangle$ then the probability of measuring 0 is given by $P_0 = (\langle 0|\psi\rangle)^2$ and the probability of measuring 1 is given by $P_1 = (\langle 1|\psi\rangle)^2$.

Rules

$$\langle 0|0\rangle = \langle 1|1\rangle = 1 \quad \text{and} \quad \langle 1\rangle = \langle 1|0\rangle = 0$$

Funded by
the European Union

Probabilities

Definition: Probability

If our qubit is in the state $|\psi\rangle$ then the probability of measuring 0 is given by $P_0 = (\langle 0|\psi\rangle)^2$ and the probability of measuring 1 is given by $P_1 = (\langle 1|\psi\rangle)^2$.

Rules

$$\langle 0|0\rangle = \langle 1|1\rangle = 1 \quad \text{and} \quad \langle 1\rangle = \langle 1|0\rangle = 0$$

If you are into vectors, our rules above correspond to taking the dot product of two vectors. The rules $\langle 0|0\rangle = \langle 1|1\rangle = 1$ mean that the vectors $|0\rangle$ and $|1\rangle$ both have length 1, and the rules $\langle 0|1\rangle = \langle 1|0\rangle = 0$ mean that $|1\rangle$ and $|0\rangle$ are perpendicular.

Funded by the European Union

Time to work

Read through the examples, remarks and do the exercises on page 7 and 8 in the notes.

Funded by
the European Union

The measurement affects the state

- A qubit is in a state $|\psi\rangle$

Funded by
the European Union

The measurement affects the state

- A qubit is in a state $|\psi\rangle$
- We do a measurement

Funded by
the European Union

The measurement affects the state

- A qubit is in a state $|\psi\rangle$
- We do a measurement
- and get 0

Funded by
the European Union

The measurement affects the state

- A qubit is in a state $|\psi\rangle$
- We do a measurement
- and get 0
- We do the same all over again

Funded by
the European Union

The measurement affects the state

- A qubit is in a state $|\psi\rangle$
- We do a measurement
- and get 0
- We do the same all over again
- A qubit is in a state $|\psi\rangle$

Funded by
the European Union

The measurement affects the state

- A qubit is in a state $|\psi\rangle$
- We do a measurement
- and get 0
- We do the same all over again
- A qubit is in a state $|\psi\rangle$
- We do a measurement

Funded by
the European Union

The measurement affects the state

- A qubit is in a state $|\psi\rangle$
- We do a measurement
- and get 0
- We do the same all over again
- A qubit is in a state $|\psi\rangle$
- We do a measurement
- and get 1

Funded by
the European Union

The measurement affects the state

- A qubit is in a state $|\psi\rangle$
- We do a measurement
- and get 0
- We do the same all over again
- A qubit is in a state $|\psi\rangle$
- We do a measurement
- and get 1
- What can we say about the state $|\psi\rangle$?

Funded by
the European Union

The measurement affects the state

- A qubit is in a state $|\psi\rangle$
- We do a measurement
- and get 0
- We do the same all over again
- A qubit is in a state $|\psi\rangle$
- We do a measurement
- and get 1
- What can we say about the state $|\psi\rangle$?
- Only that the probabilities P_0 and P_1 are non zero!

Funded by
the European Union

Collapse of the state

- A qubit is in a state $|\psi\rangle$

Funded by
the European Union

Collapse of the state

- A qubit is in a state $|\psi\rangle$
- We do a measurement

Funded by
the European Union

Collapse of the state

- A qubit is in a state $|\psi\rangle$
- We do a measurement
- and get 0

Funded by
the European Union

Collapse of the state

- A qubit is in a state $|\psi\rangle$
- We do a measurement
- and get 0
- We do a measurement again

Funded by
the European Union

Collapse of the state

- A qubit is in a state $|\psi\rangle$
- We do a measurement
- and get 0
- We do a measurement again
- and get 0

Funded by
the European Union

Collapse of the state

- A qubit is in a state $|\psi\rangle$
- We do a measurement
- and get 0
- We do a measurement again
- and get 0
- and again

Funded by
the European Union

Collapse of the state

- A qubit is in a state $|\psi\rangle$
- We do a measurement
- and get 0
- We do a measurement again
- and get 0
- and again
- and again

Funded by
the European Union

Collapse of the state

- A qubit is in a state $|\psi\rangle$
- We do a measurement
- and get 0
- We do a measurement again
- and get 0
- and again
- and again
- The initial state is collapsed.

Funded by
the European Union

Collapse of the state

- A qubit is in a state $|\psi\rangle$
- We do a measurement
- and get 0
- We do a measurement again
- and get 0
- and again
- and again
- The initial state is collapsed.

Definition

If we measure and the result is 0, then the state of the qubit immediately after the measurement is the state $|0\rangle$. Likewise, if the measurement result is 1, then the state of the qubits immediately after the measurement is the state $|1\rangle$.

Funded by the European Union

Time for some exercise

Do exercise 10 on page 8 in the notes.

Exercise 10

Show that the state $\frac{1}{5}(3\,|0\rangle + 4\,|1\rangle)$ is normalised and determine the probability, P_1, of measuring 1 if the qubit is initially in the state $|\psi\rangle = \frac{1}{5}(3\,|0\rangle + 4\,|1\rangle)$.

Funded by the European Union

Operators and Diagrams

Funded by
the European Union

Single Qubit operators

We want to do something with the qubits! We do that with **operators**.

Definition: Operator

An operator acts on a state and returns a state.

Funded by
the European Union

Doing nothing

$$|0\rangle \text{ ———— } |0\rangle$$

- A single qubit is drawn as a horizontal line.
- Each end indicate the state of the qubit.
- Read from left to right.

Funded by
the European Union

The Z operator

Definition: The operator Z

$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

$$|0\rangle \;\text{——}\; \boxed{Z} \;\text{——}\; |0\rangle$$

$$|1\rangle \;\text{——}\; \boxed{Z} \;\text{——}\; -|1\rangle$$

Funded by
the European Union

The Z operator

Definition: The operator Z

$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

$$|0\rangle \;\text{——}\boxed{Z}\text{——}\; |0\rangle$$

$$|1\rangle \;\text{——}\boxed{Z}\text{——}\; -|1\rangle$$

- An operator acts on a state from the *left*.

Funded by
the European Union

The Z operator

Definition: The operator Z

$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

$$|0\rangle \;\text{——}\boxed{Z}\text{——}\; |0\rangle$$

$$|1\rangle \;\text{——}\boxed{Z}\text{——}\; -|1\rangle$$

- An operator acts on a state from the *left*.
- i.e. $Z|\psi\rangle = (|0\rangle\langle 0| - |1\rangle\langle 1|)|\psi\rangle$

Funded by the European Union

The Z operator

Definition: The operator Z

$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

$$|0\rangle \;\text{——}\boxed{Z}\text{——}\; |0\rangle$$

$$|1\rangle \;\text{——}\boxed{Z}\text{——}\; -|1\rangle$$

- An operator acts on a state from the *left*.
- i.e. $Z|\psi\rangle = (|0\rangle\langle 0| - |1\rangle\langle 1|)|\psi\rangle$
- What does Z do?

Exercise 15 in the notes.

Use the definition of Z, i.e. that $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$, and let it act on $|0\rangle$ and $|1\rangle$. Show that the result is the same as depicted on the figure.

Funded by the European Union

The X operator

Definition: The operator X

$X = |1\rangle\langle 0| + |0\rangle\langle 1|$

$$|0\rangle \;\text{——}\; \boxed{X} \;\text{——}\; |1\rangle$$

$$|1\rangle \;\text{——}\; \boxed{X} \;\text{——}\; |0\rangle$$

Funded by
the European Union

The X operator

Definition: The operator X

$X = |1\rangle\langle 0| + |0\rangle\langle 1|$

$$|0\rangle \;\text{——}\; \boxed{X} \;\text{——}\; |1\rangle$$

$$|1\rangle \;\text{——}\; \boxed{X} \;\text{——}\; |0\rangle$$

What does the X operator do?

Exercise 16 in the notes.

Show what happen when X acts on the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. If you have studied Appendix 2, then note that you have just shown that $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is an eigenstate of X with eigenvalue 1!

Funded by the European Union

The Hadamard operator, H

Definition: The operator H

$$H = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right)$$

$$|0\rangle \;\text{——}\boxed{H}\text{——}\; \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1\rangle \;\text{——}\boxed{H}\text{——}\; \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Funded by
the European Union

The Hadamard operator, H

Definition: The operator H

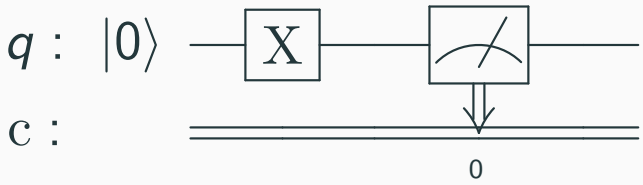
$$H = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right)$$

$$\begin{array}{ccccc} |0\rangle & \text{---} & \boxed{H} & \text{---} & \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |1\rangle & \text{---} & \boxed{H} & \text{---} & \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{array}$$

- Work through the exercises 17, 18, 19 in the notes. Consult the surrounding examples if in doubt or if you need a hint.

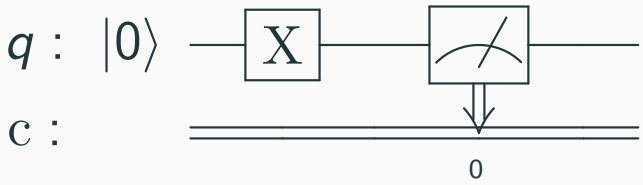
Funded by the European Union

Diagrams with measurements



Funded by
the European Union

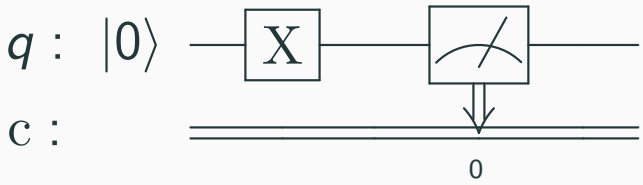
Diagrams with measurements



1. q for **q**ubit and c for **c**lassical bit.

Funded by
the European Union

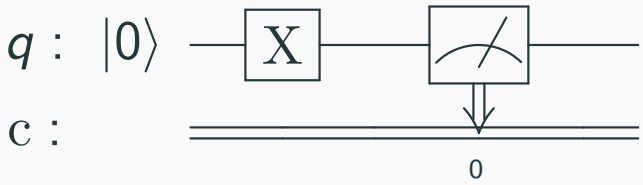
Diagrams with measurements



1. q for **q**ubit and c for **c**lassical bit.
2. qubit starts in state $|0\rangle$

Funded by
the European Union

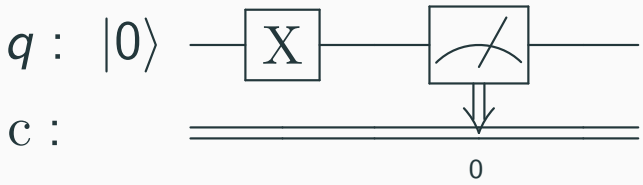
Diagrams with measurements



1. q for **q**ubit and c for **c**lassical bit.
2. qubit starts in state $|0\rangle$
3. Operator X acts on the state.

Funded by
the European Union

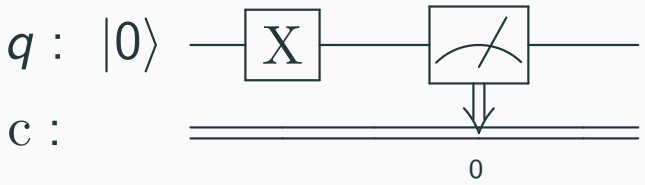
Diagrams with measurements



1. q for **q**ubit and c for **c**lassical bit.
2. qubit starts in state $|0\rangle$
3. Operator X acts on the state.
4. The new state is measured (acted on by the measuring operator)

Funded by
the European Union

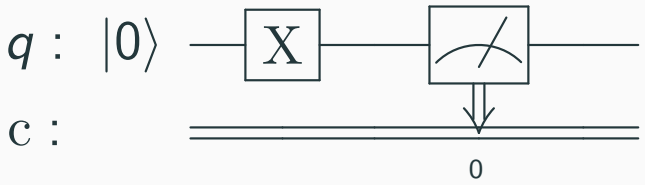
Diagrams with measurements



1. q for **q**ubit and c for **c**lassical bit.
2. qubit starts in state $|0\rangle$
3. Operator X acts on the state.
4. The new state is measured (acted on by the measuring operator)
5. The outcome of the measurement is stored in classical bit number 0 (counting from 0 instead of 1).

Funded by the European Union

Diagrams with measurements

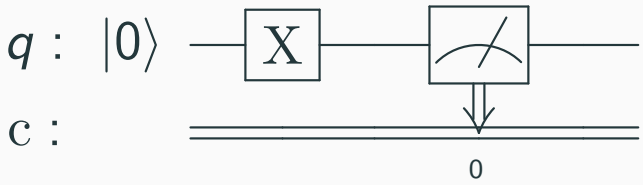


1. q for **q**ubit and c for **c**lassical bit.
2. qubit starts in state $|0\rangle$
3. Operator X acts on the state.
4. The new state is measured (acted on by the measuring operator)
5. The outcome of the measurement is stored in classical bit number 0 (counting from 0 instead of 1).
6. The diagram before the measurement translates to the following calculation:

$$X|0\rangle = (|1\rangle\langle 0| + |0\rangle\langle 1|)|0\rangle$$

Funded by the European Union

Diagrams with measurements



1. q for **q**ubit and c for **c**lassical bit.
2. qubit starts in state $|0\rangle$
3. Operator X acts on the state.
4. The new state is measured (acted on by the measuring operator)
5. The outcome of the measurement is stored in classical bit number 0 (counting from 0 instead of 1).
6. The diagram before the measurement translates to the following calculation:

$$X|0\rangle = (|1\rangle\langle 0| + |0\rangle\langle 1|)|0\rangle$$

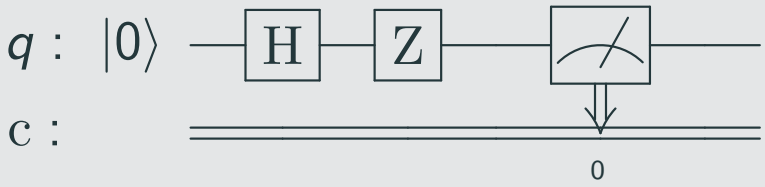
7. Predict the outcome of the measurement.

Funded by the European Union

Example and exercise

Read through the example concerning the Hadamard operator on page 17 and 18 in the notes. Then do exercise 21 on page 18 in the notes.

Exercise 21



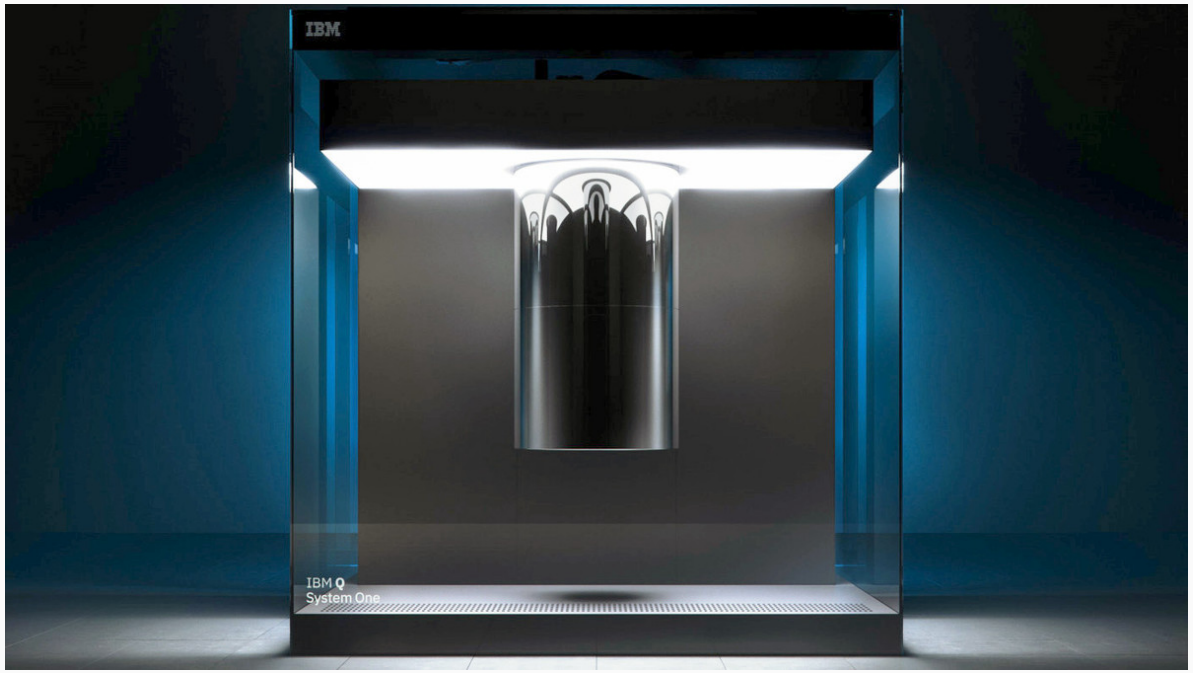
Our qubit start out in the state $|0\rangle$. Explain what the state of our qubit is after H and then Z have acted on it, i.e. calculate $ZH|0\rangle$. Then determine what the probability is of measurement giving 0 and 1, respectively.

Funded by the European Union

Funded by
the European Union

Funded by
the European Union

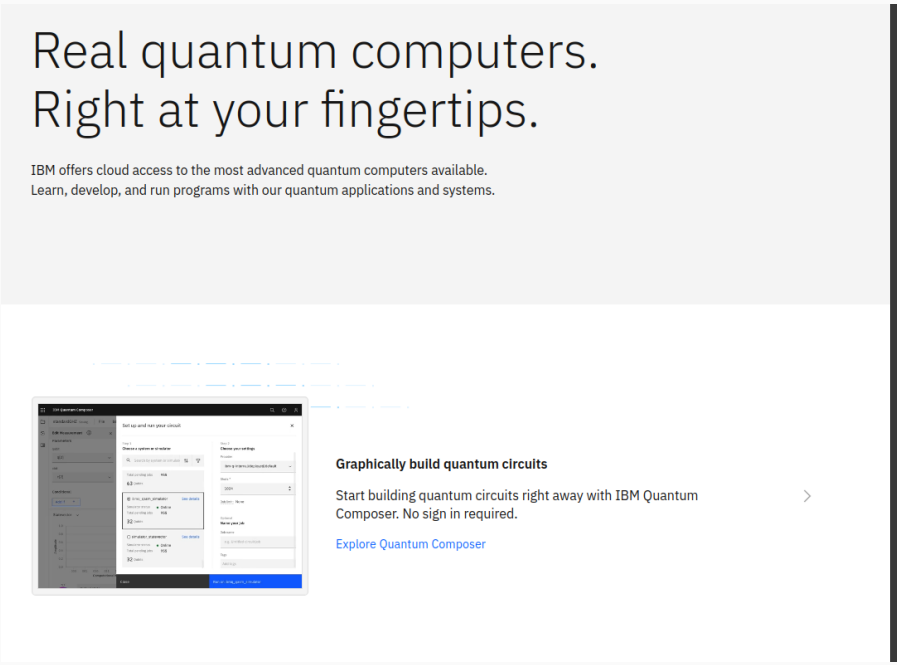
IBM Q

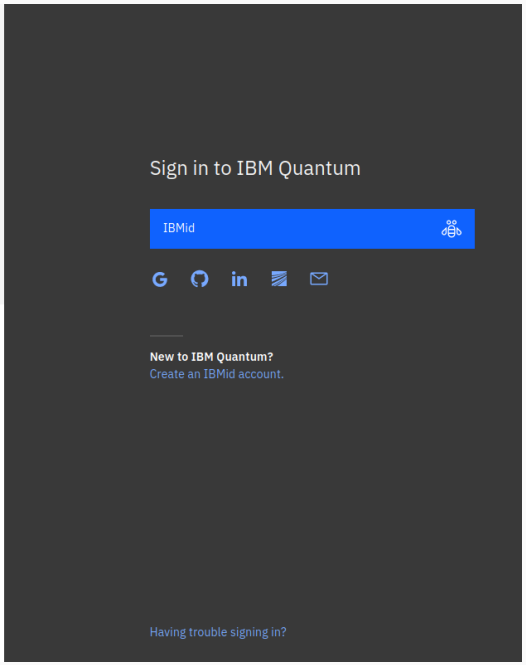


Funded by
the European Union

Access IBM Q

- Visit https://quantum-computing.ibm.com/
- Create an IBM account or sign in with one of the other options.
- Go to “Lab”.



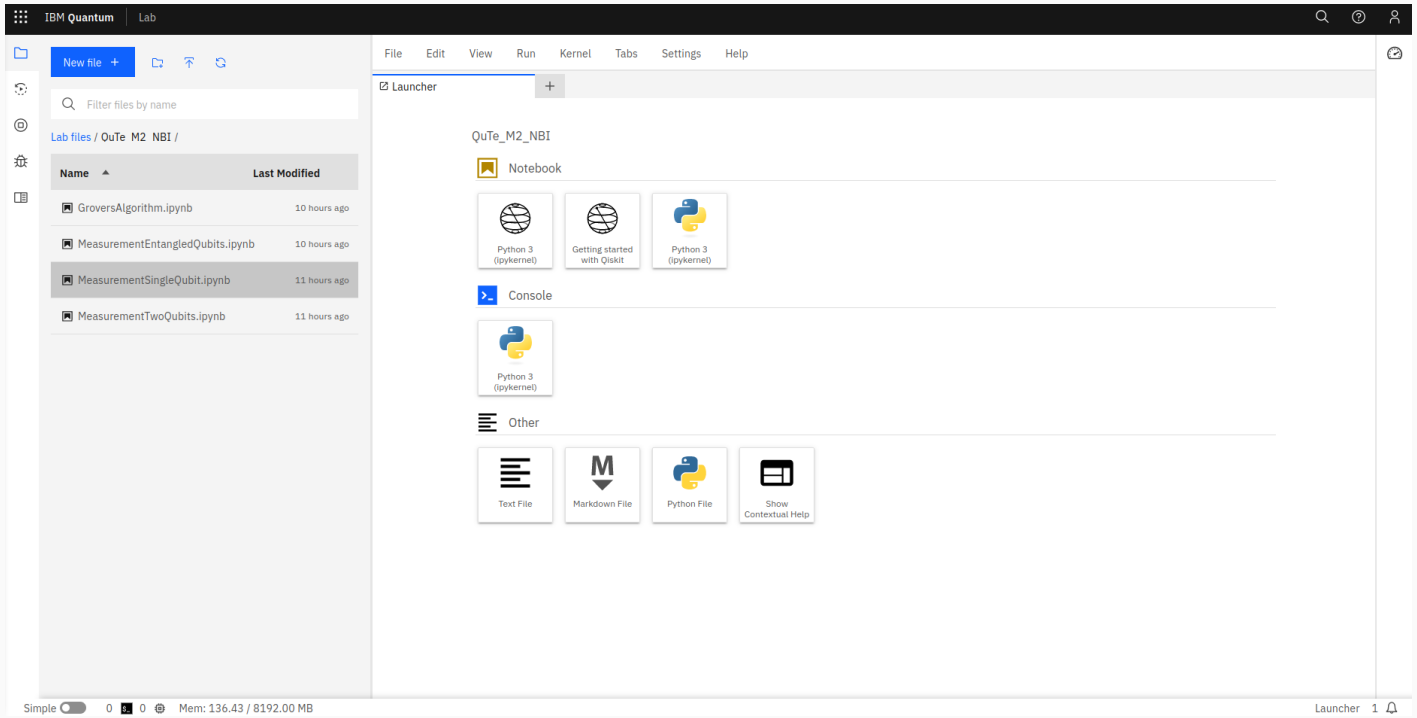


Funded by the European Union

Upload the IBM Q exercises

Upload the following exercises to the lab file browser. (You can use drag and drop).

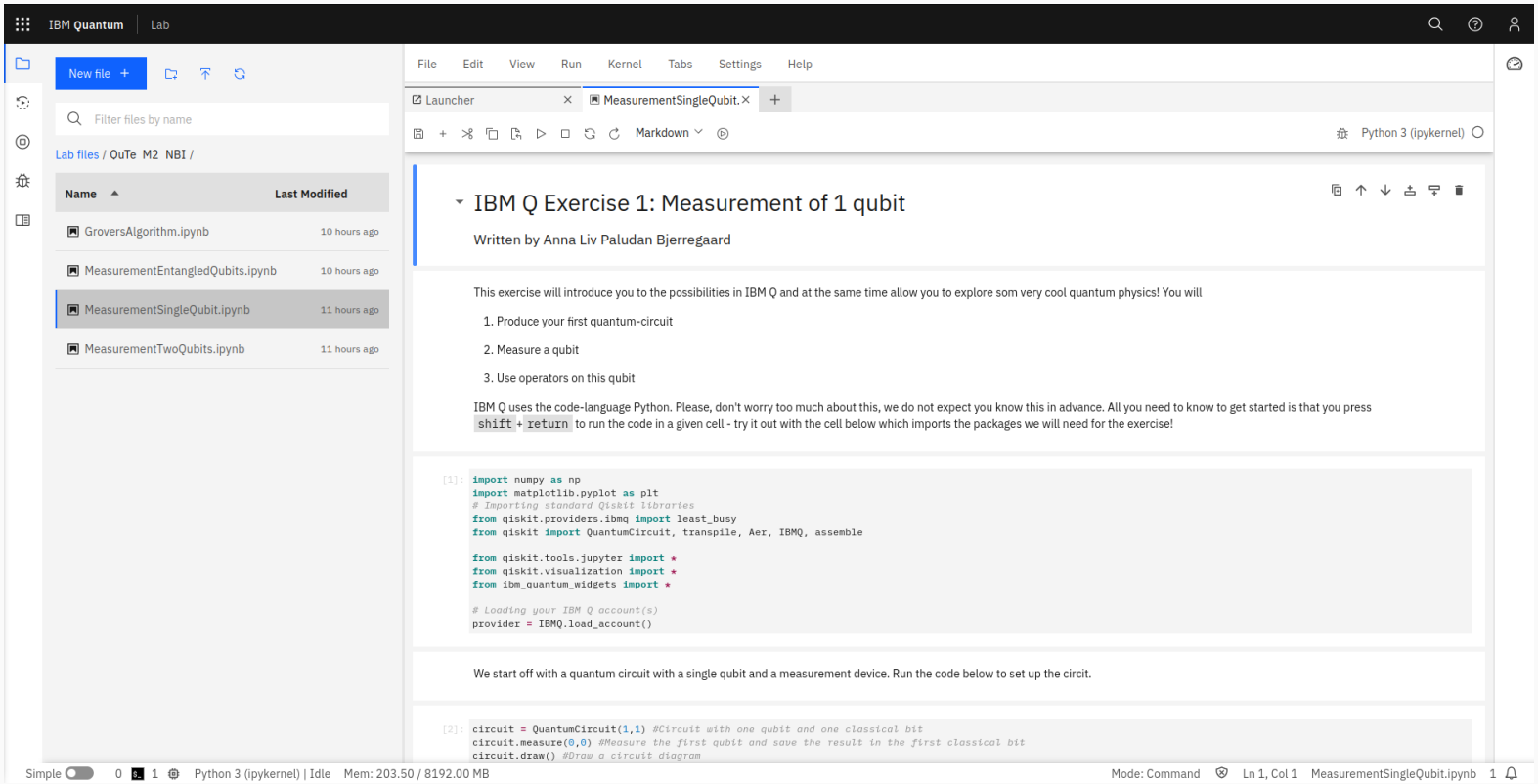
- MeasurementSingleQubit.ipynb
- MeasurementTwoQubits.ipynb
- MeasurementEntangledQubits.ipynb
- GroversAlgorithm.ipynb



Funded by the European Union

Single Qubit Measurements

Run the *Measurement SingleQubit.ipynb* notebook and follow the instructions. You are now able to perform **real** work on a real **quantum computer**! (If you set the variable sym=False)



Funded by the European Union

Pun intended!

Did you get the pun?

Funded by
the European Union

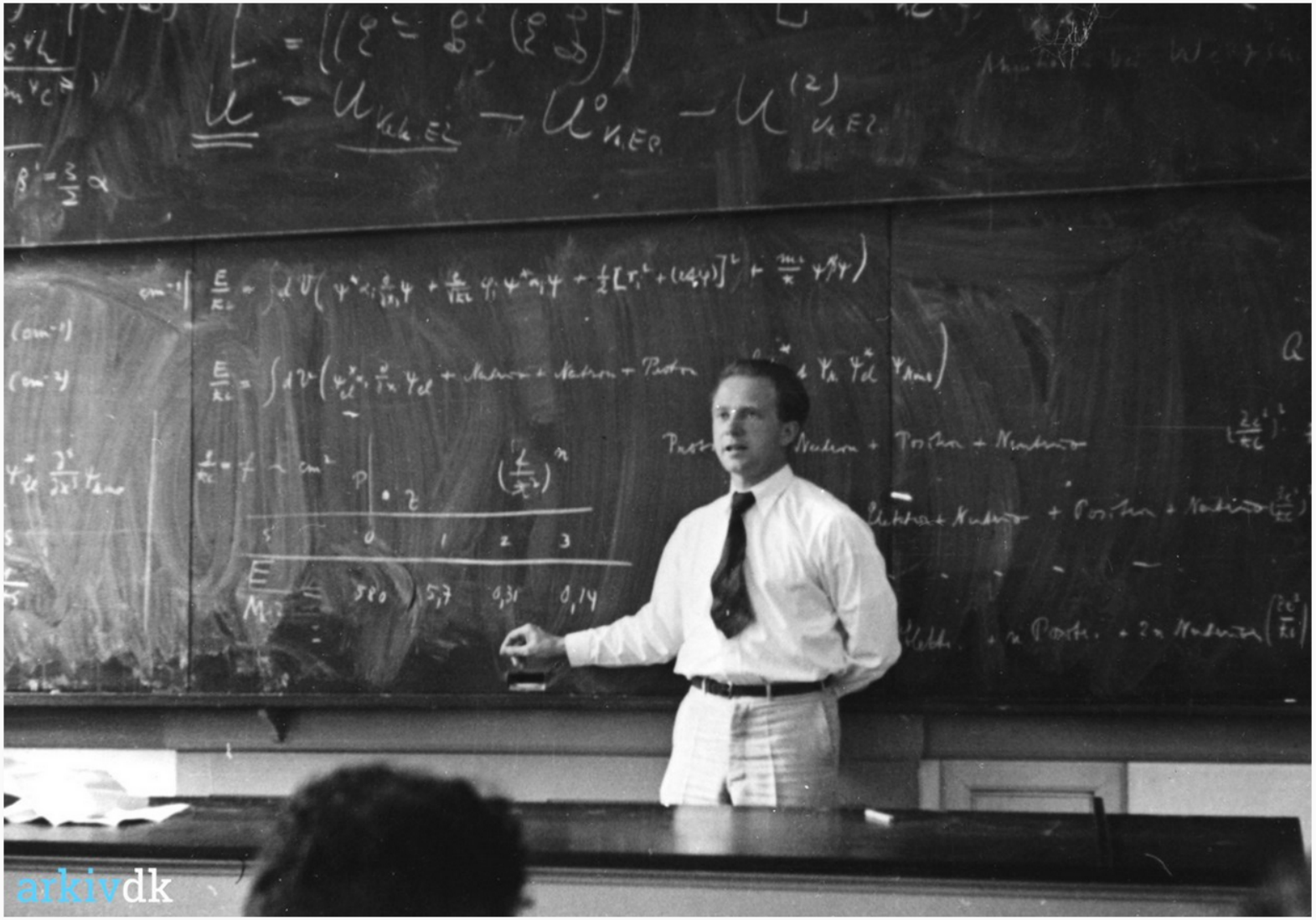
Pun intended!

Did you get the pun?

Monty Python → IBM Q *is* related → IBM Q uses *Python* → Python is named after Monty Python, because programming should be fun!

Funded by the European Union

Back to school!



Funded by
the European Union

One, Many

How many is *many?*

Funded by
the European Union

One, Many

How many is *many?*

- One

Funded by
the European Union

One, Many

How many is *many?*

- One, two

Funded by
the European Union

One, Many

How many is *many?*

- One, two, many

Funded by
the European Union

One, Many

How many is *many?*

- ~~One, two, many~~

Funded by
the European Union

One, Many

How many is *many?*

- ~~One, two, many~~

This time it is only

Funded by
the European Union

One, Many

How many is *many?*

- ~~One, two, many~~

This time it is only

- One

Funded by
the European Union

One, Many

How many is *many?*

- ~~One, two, many~~

This time it is only

- One, many

Funded by
the European Union

One, Many

How many is *many?*

- ~~One, two, many~~

This time it is only

- One, many

This time around we are going to work with **two** qubit systems.

Funded by
the European Union

One, Many

How many is *many?*

- ~~One, two, many~~

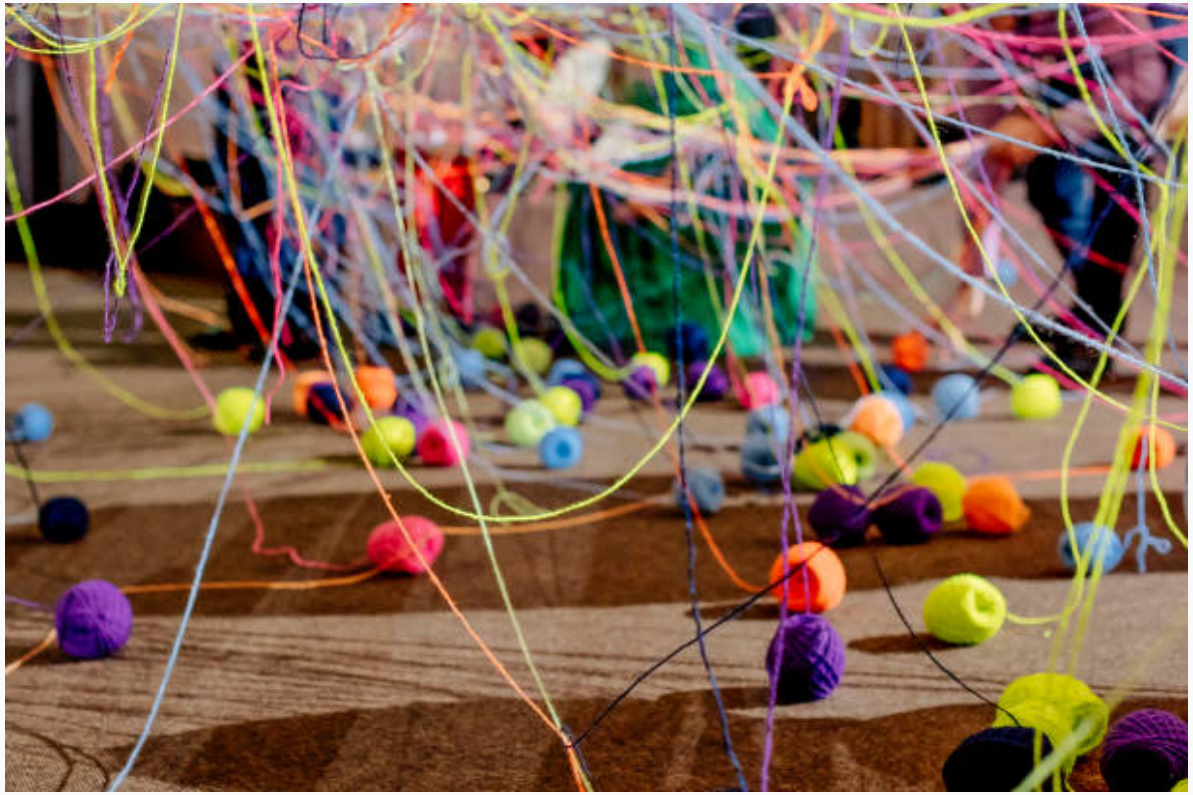
This time it is only

- One, many

This time around we are going to work with **two** qubit systems. With two qubits we can do...

Funded by
the European Union

Entanglement



Funded by
the European Union

Two-qubit entanglement

Definition: Entanglement

Two qubits are entangled if the state of one qubit depends on the other. When two qubits are entangled it is *impossible* to write the total state as a product of a state of the first qubits and a state of the second.

Funded by
the European Union

Two-qubit entanglement

Definition: Entanglement

Two qubits are entangled if the state of one qubit depends on the other. When two qubits are entangled it is *impossible* to write the total state as a product of a state of the first qubits and a state of the second.

Let us see some examples. (Read the examples in Chapter V section 10 in the notes, page 10 and 11)

Funded by the European Union

None entangled two qubit systems

Two qubits not entangled

If qubit 1 is in the state $|0\rangle$ and qubit 2 is in the state $|1\rangle$, then the total state of the qubits is $|1\rangle|0\rangle = |10\rangle$

Another two qubits not entangled

If the state of qubit 1 is $|0\rangle$ and the state of qubit 2 is $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, then the total state of the two qubits is
$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$.

A third example

Two *not entangled* qubits: The two qubits in the state
$\frac{1}{5}(3|01\rangle + 4|11\rangle)$ are not entangled, since we can write the state as
$\frac{1}{5}(3|01\rangle + 4|11\rangle) = \frac{1}{5}(3|0\rangle + 4|1\rangle)|1\rangle$.

Funded by
the European Union

Entangled two qubit systems

Example: Entangled qubits

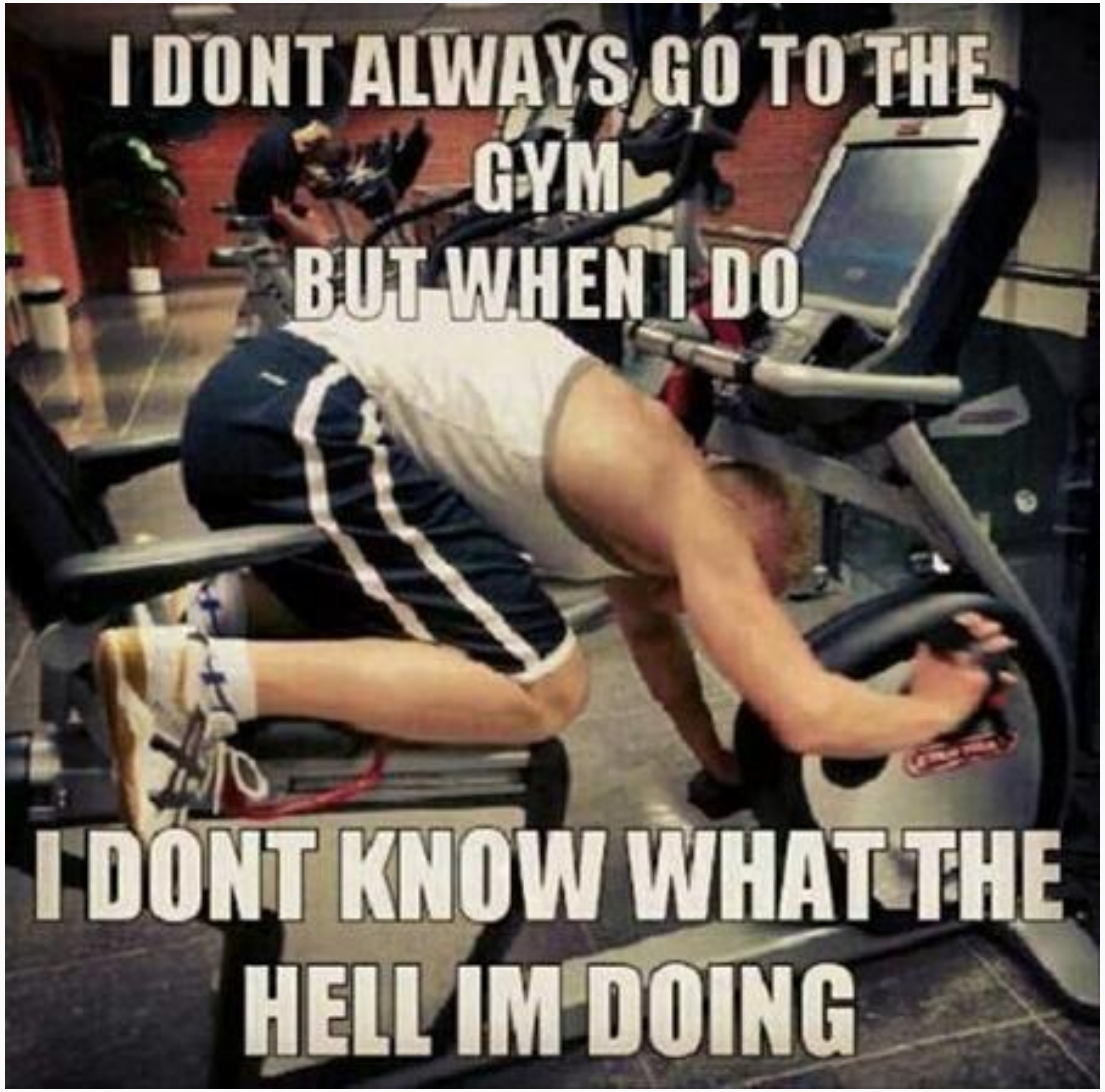
Two qubits are in the state $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$. They are *entangled* because the state of the first qubit depends on the state of the second: If the first qubit is 1 then the other is 0, and vice versa (if the first qubit is 0 then the second i 1). To put it differently, you can no twrite the total state $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ as a product of two states $|\ldots\rangle\,|\ldots\rangle$.

Another example

The two qubits in the state $\frac{1}{5}(3\,|00\rangle + 4\,|11\rangle)$ are entangled.

Funded by
the European Union

Time for some exercise



Funded by the European Union

Exercises

Exercise 11

Two qubits are in the state $|01\rangle$. What is the state of qubit 1? what about qubit 2?

Exercise 12

Determine which of the following two-qubit states are entangled:

1. $|11\rangle$
2. $\frac{1}{\sqrt{2}}(|11\rangle + |00\rangle$
3. $\frac{1}{\sqrt{2}}(|10\rangle + |11\rangle$

Funded by
the European Union

But what is the difference between entangled and none entangled states?

Funded by
the European Union

But what is the difference between entangled and none entangled states?

It's all about the measurements.

Funded by
the European Union

None entangled qubits

Two qubits are in the state $|01\rangle$(read from the right).

- Qubit 2 is in state $|0\rangle$.

- Qubit 1 is in state $|1\rangle$.

Funded by
the European Union

None entangled qubits

Two qubits are in the state $|01\rangle$(read from the right).

- Qubit 2 is in state $|0\rangle$.
- Qubit 1 is in state $|1\rangle$.

Measuring on any of the qubits.

Funded by
the European Union

None entangled qubits

Two qubits are in the state $|01\rangle$(read from the right).

- Qubit 2 is in state $|0\rangle$.
- Qubit 1 is in state $|1\rangle$.

Does not alter the other.

Funded by
the European Union

None entangled qubits

Two qubits are in the state $|01\rangle$(read from the right).

- Qubit 2 is in state $|0\rangle$.
- Qubit 1 is in state $|1\rangle$.

The state remains unaltered.

Funded by
the European Union

Entangled qubits

Two qubits are in the entangled state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

Qubit 2

Qubit 1

Funded by
the European Union

Entangled qubits

Two qubits are in the entangled state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

Qubit 2

Qubit 1

We measure qubit 1

Funded by
the European Union

Entangled qubits

Two qubits are in the entangled state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

Qubit 2

Qubit 1

and get 0.

Funded by
the European Union

Entangled qubits

Two qubits are in the entangled state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

Qubit 2

Qubit 1

Qubit 1 is now in state $|0\rangle$ forcing the total state to be $|10\rangle$

Funded by
the European Union

Entangled qubits

Two qubits are in the entangled state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

Qubit 2

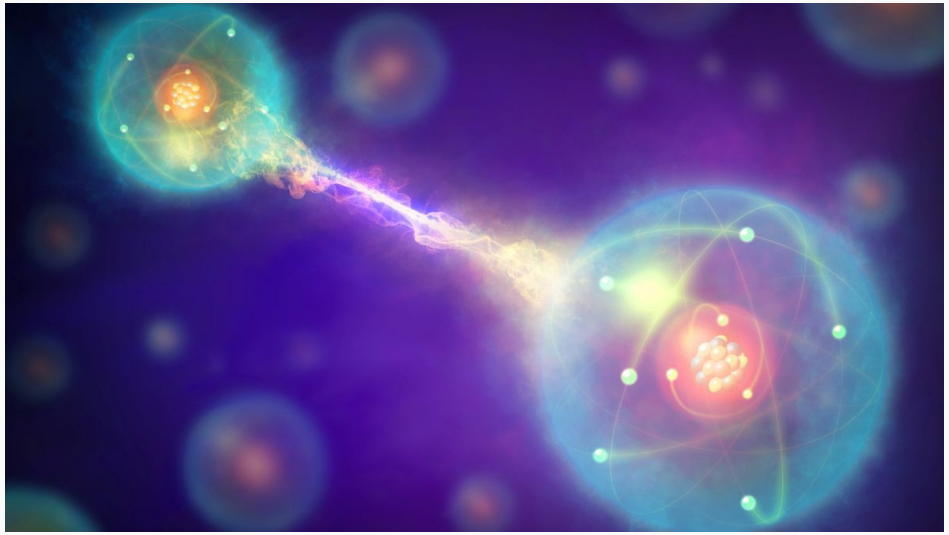
Qubit 1

The measurement on qubit 1 affects qubit 2 instantaneously.

Funded by
the European Union

Albert Einstein

I cannot seriously believe in it because the theory cannot be reconciled with the idea that physics should represent a reality in time and space, free from **spooky action at a distance**.



Funded by
the European Union

Untangle these exercises

Exercise 13

Two qubits are in the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. What is the probability of a measurement on the first qubit giving 0 and 1, respectively. If we measure the first qubit, and the outcome is 1, what is the total state of the two qubits immediately after the measurement?

Exercise 14

Two qubits are in the state $\frac{1}{5}(3\,|01\rangle - 4\,|10\rangle$. Will a measurement on the second qubit affect the state of the first qubit?

Funded by
the European Union

Operating on two qubits

The Controlled NOT gate

Funded by
the European Union

The CNOT gate

The CNOT operator

First qubit is the control bit

$CNOT = |00\rangle\langle 00| + |11\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 11|$

Second qubit is the control bit

$CNOT = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$

Funded by
the European Union

The CNOT gate

The CNOT operator

First qubit is the control bit

$CNOT = |00\rangle\langle 00| + |11\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 11|$

Second qubit is the control bit

$CNOT = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$

But what does it do?

Funded by
the European Union

The CNOT gate

The CNOT operator

First qubit is the control bit

$CNOT = |00\rangle\langle 00| + |11\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 11|$

Second qubit is the control bit

$CNOT = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$

But what does it do?

If the control qubit is 0, it does nothing.

Funded by
the European Union

The CNOT gate

The CNOT operator

First qubit is the control bit

$CNOT = |00\rangle\langle 00| + |11\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 11|$

Second qubit is the control bit

$CNOT = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$

But what does it do?

If the control qubit is 0, it does nothing.

If the control qubit is 1, it changes the state of the other qubit (by applying the X operator in the other qubit.)

Funded by the European Union

Let us check

$CNOT\,|01\rangle$ with first qubit as control

$$
\begin{aligned}
CNOT\,|01\rangle &= (|00\rangle\langle 00| + |11\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 11|)\,|01\rangle \\
&= |00\rangle\langle 00|01\rangle + |11\rangle\langle 01|01\rangle + |10\rangle\langle 10|01\rangle + |01\rangle\langle 11|01\rangle \\
&= |00\rangle\,0 + |11\rangle\,1 + |10\rangle\,0 + |10\rangle\,0 \\
&= |11\rangle
\end{aligned}
$$

As you can see the first qubit remains unchanged while the second qubit changes from state 0 to 1.

Funded by
the European Union

Let us check

$CNOT\,|01\rangle$ with first qubit as control

$$
\begin{aligned}
CNOT\,|01\rangle &= (|00\rangle\langle 00| + |11\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 11|)\,|01\rangle \\
&= |00\rangle\langle 00|01\rangle + |11\rangle\langle 01|01\rangle + |10\rangle\langle 10|01\rangle + |01\rangle\langle 11|01\rangle \\
&= |00\rangle\,0 + |11\rangle\,1 + |10\rangle\,0 + |10\rangle\,0 \\
&= |11\rangle
\end{aligned}
$$

As you can see the first qubit remains unchanged while the second qubit changes from state 0 to 1.

Do a quick check of the other combinations of $CNOT$ and the states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

Funded by the European Union

Creating an entangled state

$CNOT$ on $\frac{1}{\sqrt{2}}(|01\rangle + |00\rangle)$ with qubit 1 as control

$$
\begin{aligned}
CNOT \frac{1}{\sqrt{2}}(|01\rangle + |00\rangle) &= (|00\rangle\langle 00| + |11\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 11|)\frac{1}{\sqrt{2}}(|01\rangle + |00\rangle) \\
&= \frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)
\end{aligned}
$$

Funded by
the European Union

Creating an entangled state

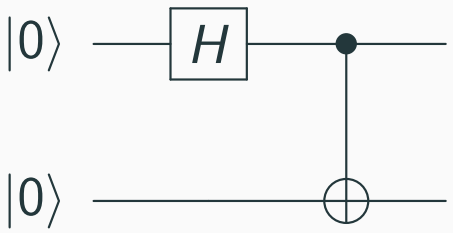
$CNOT$ on $\frac{1}{\sqrt{2}}(|01\rangle + |00\rangle)$ with qubit 1 as control

$$
\begin{aligned}
CNOT \frac{1}{\sqrt{2}}(|01\rangle + |00\rangle) &= (|00\rangle\langle 00| + |11\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 11|)\frac{1}{\sqrt{2}}(|01\rangle + |00\rangle) \\
&= \frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)
\end{aligned}
$$

We went from an none entangled state to an entangled state!

Funded by the European Union

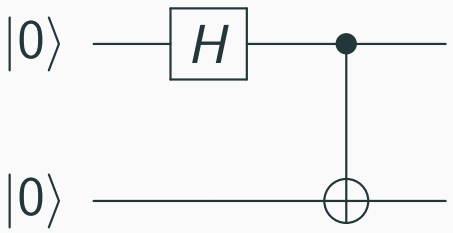
Creating an entangled state from scratch



The diagram shows how we, using a Hadamard and a *CNOT*-operator, can produce the entangled state $\frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)$. The dot in the *CNOT*-operator indicates the control qubit and the other end, which looks like a sight, points to the qubit which the CNOT operators does something to (the target qubit).

Funded by
the European Union

Creating an entangled state from scratch



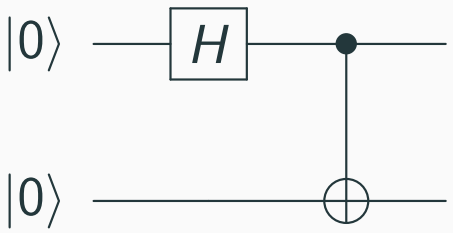
The diagram shows how we, using a Hadamard and a *CNOT*-operator, can produce the entangled state $\frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)$. The dot in the *CNOT*-operator indicates the control qubit and the other end, which looks like a sight, points to the qubit which the CNOT operators does something to (the target qubit).

The state went

from $|0\rangle\,|0\rangle$

Funded by
the European Union

Creating an entangled state from scratch



The diagram shows how we, using a Hadamard and a *CNOT*-operator, can produce the entangled state $\frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)$. The dot in the *CNOT*-operator indicates the control qubit and the other end, which looks like a sight, points to the qubit which the CNOT operators does something to (the target qubit).

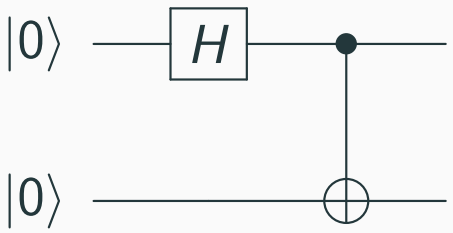
The state went

from $|0\rangle\,|0\rangle$

to $|0\rangle\,\frac{1}{\sqrt{1}}\,(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\,(|00\rangle + |01\rangle)$

Funded by
the European Union

Creating an entangled state from scratch



The diagram shows how we, using a Hadamard and a *CNOT*-operator, can produce the entangled state $\frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)$. The dot in the *CNOT*-operator indicates the control qubit and the other end, which looks like a sight, points to the qubit which the CNOT operators does something to (the target qubit).

The state went

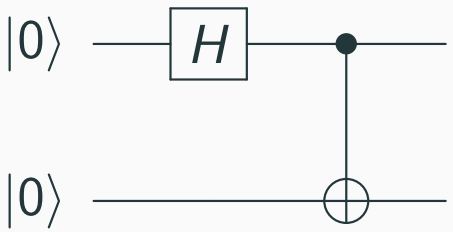
from $|0\rangle\,|0\rangle$

to $|0\rangle\,\frac{1}{\sqrt{1}}\,(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\,(|00\rangle + |01\rangle)$

to $CNOT\,\frac{1}{\sqrt{2}}\,(|00\rangle + |01\rangle)$

Funded by
the European Union

Creating an entangled state from scratch



The diagram shows how we, using a Hadamard and a *CNOT*-operator, can produce the entangled state $\frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)$. The dot in the *CNOT*-operator indicates the control qubit and the other end, which looks like a sight, points to the qubit which the CNOT operators does something to (the target qubit).

The state went

from $|0\rangle\,|0\rangle$

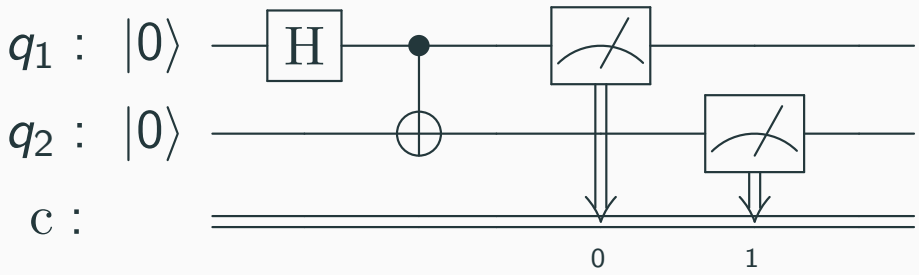
to $|0\rangle\,\frac{1}{\sqrt{1}}\,(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\,(|00\rangle + |01\rangle)$

to $CNOT\,\frac{1}{\sqrt{2}}\,(|00\rangle + |01\rangle)$

to the final *entangled* state $\frac{1}{\sqrt{2}}\,(|11\rangle + |00\rangle)$

Funded by
the European Union

Measuring entangled states



In the figure we perform measurements on each of the two qubits which are in the state $\frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)$ before the measurement. The measurement on the first qubit will give 0 with 50% probability. If the measurement gives 0, the state of the first qubit is $|0\rangle$ after the measurement. Therefore, the total state of the two qubits is $|00\rangle$ after we have measured 0 for the first qubit. We now measure the second qubit and get 0 with 100% probability, since the second qubit is in the state $|0\rangle$.

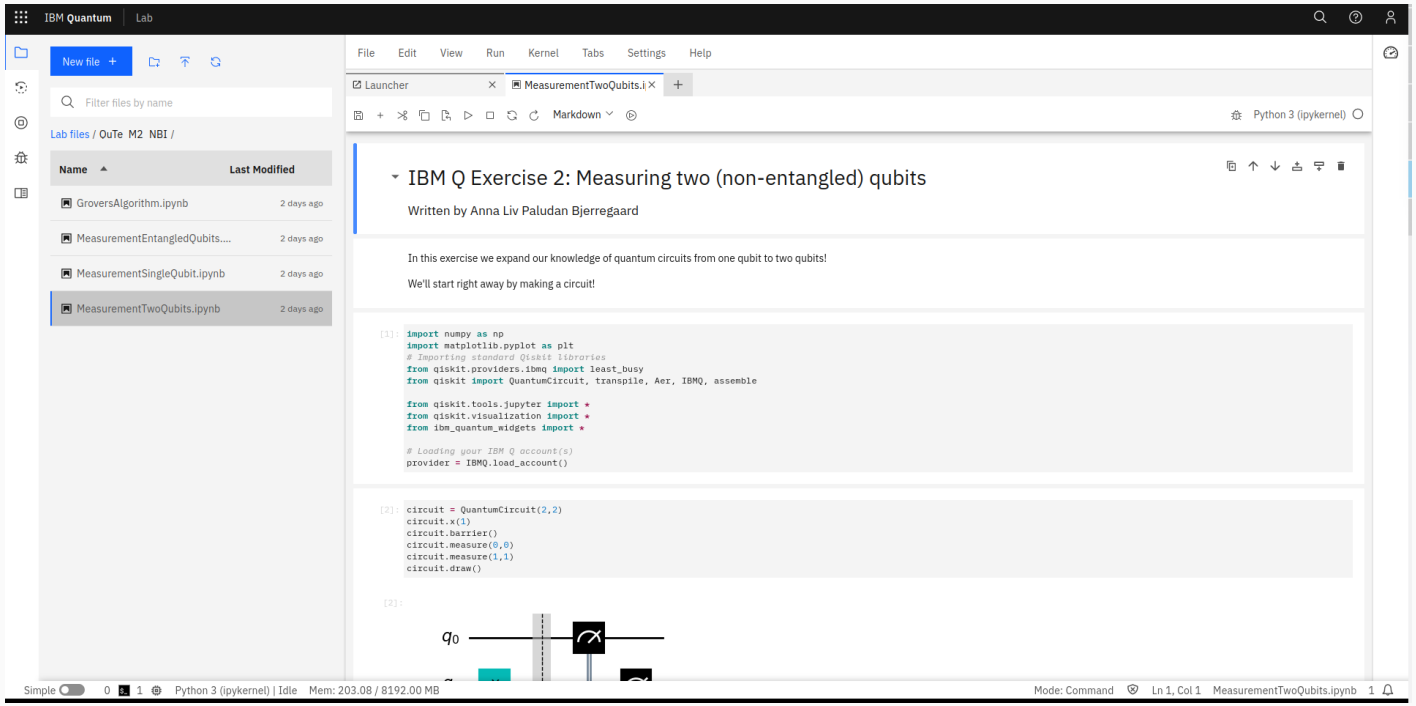
Funded by the European Union

Back to IBM Q

Funded by
the European Union

Measuring two (non-entangled) qubits

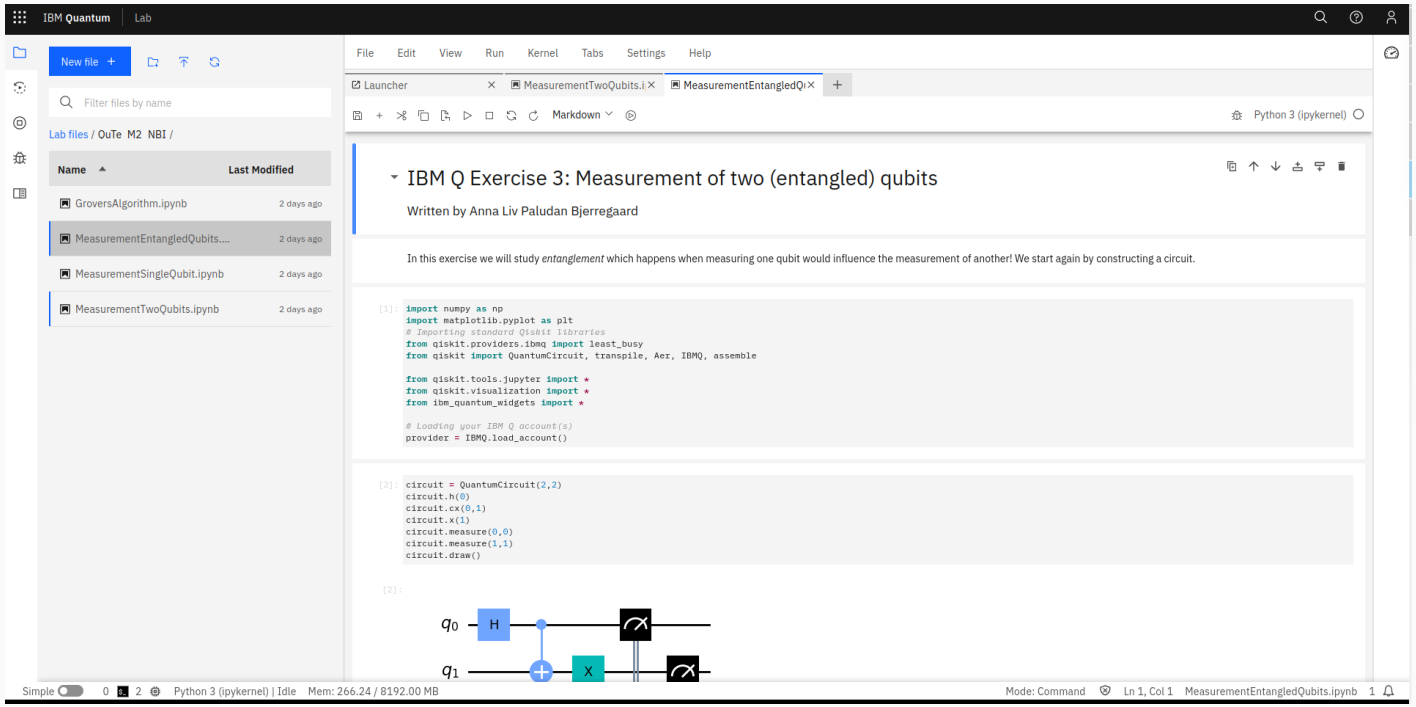
Log in to https://lab.quantum-computing.ibm.com and fire up exercise *MeasurementTwoQubits.pynb*



Funded by the European Union

Measurement of two (entangled) qubits

While still at https://lab.quantum-computing.ibm.com work on exercise *MeasurementEntangledQubits.pynb*



Funded by the European Union