Komplekse tal - Besvarelse af opgaver

Supplerende stof

Matematik A

Vibenshus Gymnasium

Opgave 1

Betragt de to komplekse tal $z = 3 + 4i$ og $w = 2 - i$. Beregn og indtegn følgende sammenhænge i et Argand-diagram

1. $z + w$

Komponentform:

$$
\begin{aligned}
z + w &= 3 + 4i + 2 - i \\
z + w &= 3 + 2 + (4 - 1)i \\
z + w &= 5 + 3i
\end{aligned}
$$

Eksponentialform:

$$
\begin{aligned}
\sqrt{5^2 + 3^2} &= 5.83095189485 \\
\tan^{-1}\left(\frac{3}{5}\right) &= 0.540419500271
\end{aligned}
$$

$$z + w = 5.83e^{i0.54}$$

2. $w - z$

Komponentform:

$$
\begin{aligned}
w - z &= 2 - i - (3 + 4i) \\
w - z &= -1 - i5
\end{aligned}
$$

Eksponentialform:

$$
\begin{aligned}
\sqrt{(-1)^2 + (-5)^2} &= 5.09901951359 \\
\tan^{-1}\left(\frac{-5}{-1}\right) &= 1.373400766945016
\end{aligned}
$$

Modulus skal omskrives da det komplekse tal ligger i 3. kvadrant: $\theta = -(\pi - 1.373) = -1.768$

$$w - z = 5.099e^{-1.768i}$$

3. $w \cdot z$

Komponentform:

$$w \cdot z = (3 + 4i) \cdot (2 - i) = 6 - i3 + 8i + 4 = 10 + 5i$$

Eksponentialform:

$$
\begin{aligned}
\sqrt{10^2 + 5^2} &= 11.180339887498950 \\
\tan^{-1}\left(\frac{5}{10}\right) &= 0.463647609000806
\end{aligned}
$$

$$w \cdot z = 11.180e^{i0.464}$$

4. $\frac{z}{w}$

Komponentform:

$$
\begin{aligned}
\frac{z}{w} &= \frac{3+4i}{2-i} \\
\frac{z}{w} &= \frac{3+4i}{2-i} \cdot \frac{2+i}{2+i} \\
\frac{z}{w} &= \frac{(3+4i)\cdot(2+i)}{2^2-i^2} \\
\frac{z}{w} &= \frac{6+11i-4}{5} \\
\frac{z}{w} &= \frac{2+11i}{5} \\
\frac{z}{w} &= \frac{2}{5} + \frac{11}{5}i
\end{aligned}
$$

Eksponentialform:

$$
\sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{11}{5}\right)^2} = 2.23606797749979
$$

$$
\tan^{-1}\left(\frac{11}{2}\right) = 1.390942827002418
$$

$$
\frac{z}{w} = 2.236e^{i1.391}
$$

5. $z^* \cdot w + w^* \cdot z$

Komponentform:

$$
\begin{aligned}
z^* \cdot w + w^* \cdot z &= (3-4i)\cdot(2-i) + (2+i)\cdot(3+4i) \\
z^* \cdot w + w^* \cdot z &= 6-3i-8i-4+6+8i+3i-4 \\
z^* \cdot w + w^* \cdot z &= 4
\end{aligned}
$$

Eksponentialform:

$$
z^* \cdot w + w^* \cdot z = 4 \cdot e^{i\cdot 0} = 4
$$

6. w^2

Komponentform:

$$
\begin{aligned}
w^2 &= (2-i)^2 \\
w^2 &= 2^2 + i^2 - 2 \cdot 2 \cdot i \\
w^2 &= 4 - 1 - 4 \cdot i \\
w^2 &= 3 - 4 \cdot i
\end{aligned}
$$

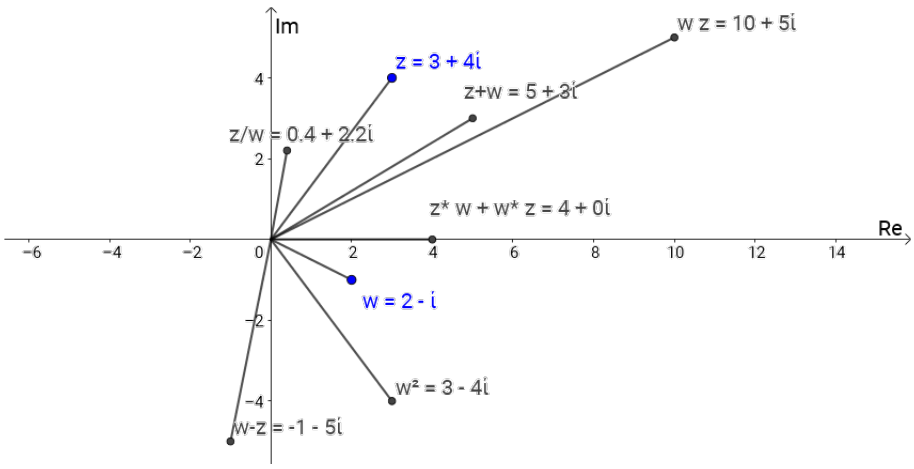
Eksponentialform:

$$
\begin{aligned}
\sqrt{3^2 + 4^2} &= 5.0 \\
\tan^{-1}\left(\frac{-4}{3}\right) &= -0.9272952180016121
\end{aligned}
$$

w^2 befinder sig i 4. kvadrant så vinklen stemmer overens.

$$w^2 = 5e^{-i0.927}$$

Alle kombinationerne for z og w kan ses på den følgende figur.



Opgave 2

Evaluering eller simplificering af komplekse udtryk.

1. $Re\left(e^{2iz}\right)$

Husk at $z = x + iy$

$$
\begin{aligned}
e^{2iz} &= e^{2i(x+iy)} = e^{2ix-2y} \\
&= e^{2ix} \cdot e^{-2y} \\
&= (\cos(2x) + i\sin(2x)) \cdot e^{-2y} \quad \text{Benytter de Moivres formel} \rightarrow \\
Re\left(e^{2iz}\right) &= Re\left((\cos(2x) + i\sin(2x)) \cdot e^{-2y}\right) \\
Re\left(e^{2iz}\right) &= \cos(2x) \cdot e^{-2y} \quad \triangle
\end{aligned}
$$

2. $\left(-1+\sqrt{3}\cdot i\right)^{\frac{1}{2}}$

Først defineres $w = -1 + \sqrt{3} \cdot i$ således at udtrykket kan skrives som

$$w^{\frac{1}{2}}.$$

w skrives på polær form:

$$|w| = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$$

$$arg(w) = tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = -\frac{\pi}{3} + \pi \cdot n, \text{ hvor } n \text{ er et heltal}$$

I dette tilfælde vælges $n = 1$ sa w vil ligge i 2. kvadrant. Dermed er $arg(w) = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$.

Det oprindelige udtryk kan nu skrives som

$$
\begin{aligned}
\left(-1+\sqrt{3}\cdot i\right)^{\frac{1}{2}} &= \left(2 \cdot e^{i\cdot\left(\frac{2\pi}{3}\right)}\right)^{\frac{1}{2}} \\
\left(-1+\sqrt{3}\cdot i\right)^{\frac{1}{2}} &= \sqrt{2} \cdot e^{i\cdot\frac{\pi}{3}} \quad \triangle
\end{aligned}
$$

3. $\left|e^{\left(i^{\frac{1}{2}}\right)}\right|$

i omskrives til $i = e^{i\frac{\pi}{2}}$

$$e^{\left(i^{\frac{1}{2}}\right)} = e^{\left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}}} = e^{\left(e^{i\frac{\pi}{4}}\right)}$$

Udnytter nu, at $|z|^2 = z \cdot z^*$

$$
\begin{aligned}
\left|e^{\left(i^{\frac{1}{2}}\right)}\right|^2 &= e^{\left(e^{i\frac{\pi}{4}}\right)} \cdot e^{\left(e^{-i\frac{\pi}{4}}\right)} \\
\left|e^{\left(i^{\frac{1}{2}}\right)}\right|^2 &= e^{e^{i\frac{\pi}{4}}+e^{-i\frac{\pi}{4}}} \\
\left|e^{\left(i^{\frac{1}{2}}\right)}\right|^2 &= e^{2\cos\left(\frac{\pi}{4}\right)} \quad \text{Her benyttes ligning (3.11) i kompendiet.} \\
\left|e^{\left(i^{\frac{1}{2}}\right)}\right| &= e^{\cos\left(\frac{\pi}{4}\right)} \\
\left|e^{\left(i^{\frac{1}{\sqrt{2}}}\right)}\right| &= e^{\frac{1}{\sqrt{2}}} \vee e^{-\frac{1}{\sqrt{2}}} \quad \triangle
\end{aligned}
$$

4. e^{i^3}

Skrives let op som

$$
\begin{aligned}
e^{i^3} &= e^{i\cdot i\cdot i} = e^{-i} \\
e^{i^3} &= \cos(-1) + i\sin(-1) \quad \text{Benytter Eulers ligning (2.18)} \\
e^{i^3} &= 0.54 - i0.84 \quad \triangle
\end{aligned}
$$

5. $Im\left(2^{i+3}\right)$

I første omgang omskrives 2^{i+3} ved hjælp af ligning (4.3)

$$2^{i+3} = e^{(i+3)\cdot\ln(2)} = e^{i\ln(2)} \cdot e^{3\ln(2)} = e^{i\ln(2)} \cdot \left(e^{\ln(2)}\right)^3 = e^{i\ln(2)} \cdot 2^3 = e^{i\ln(2)} \cdot 8\,.$$

Første faktor i sidste ligning omskrives ved hjælp af Eulers ligning

$$2^{i+3} = e^{i\ln(2)} \cdot 8 = (\cos(\ln(2)) + i\sin(\ln(2))) \cdot 8\,.$$

Nu kan den imaginære del findes:

$$Im\left(2^{i+3}\right) = Im\left((\cos(\ln(2)) + i\sin(\ln(2))) \cdot 8\right) = \sin(\ln(2)) \cdot 8 = 5.11 \quad \triangle$$

6. $z = 1^i$

Denne opgave løses let ved hjælp af ligning (4.3)

$$z = 1^i = e^{i\cdot\ln(1)} = e^{i\cdot 0} = e^0 = 1 \quad \triangle$$

7. $z = i^i$

Udnytter at i selv kan skrives som $i = e^{i\left(\frac{\pi}{2}+2\pi n\right)}$

$$z = i^i = \left(e^{i\left(\frac{\pi}{2}+2\pi n\right)}\right)^i = e^{i\cdot i\cdot\left(\frac{\pi}{2}+2\pi n\right)} = e^{-1\cdot\left(\frac{\pi}{2}+2\pi n\right)} = e^{-\frac{\pi}{2}-2\pi n} \quad \triangle$$

Opgave 3

Skitsér de dele af Argand-diagrammet hvor følgende udsagn gælder

1. $|z| = 2$
2. $|z| < 1$
3. $1 < |z| < 2$

De tre udsagn kan ses på den figur 1.

Opgave 4

1. Benyt de Moivres formel med $n = 4$ til at bevise at

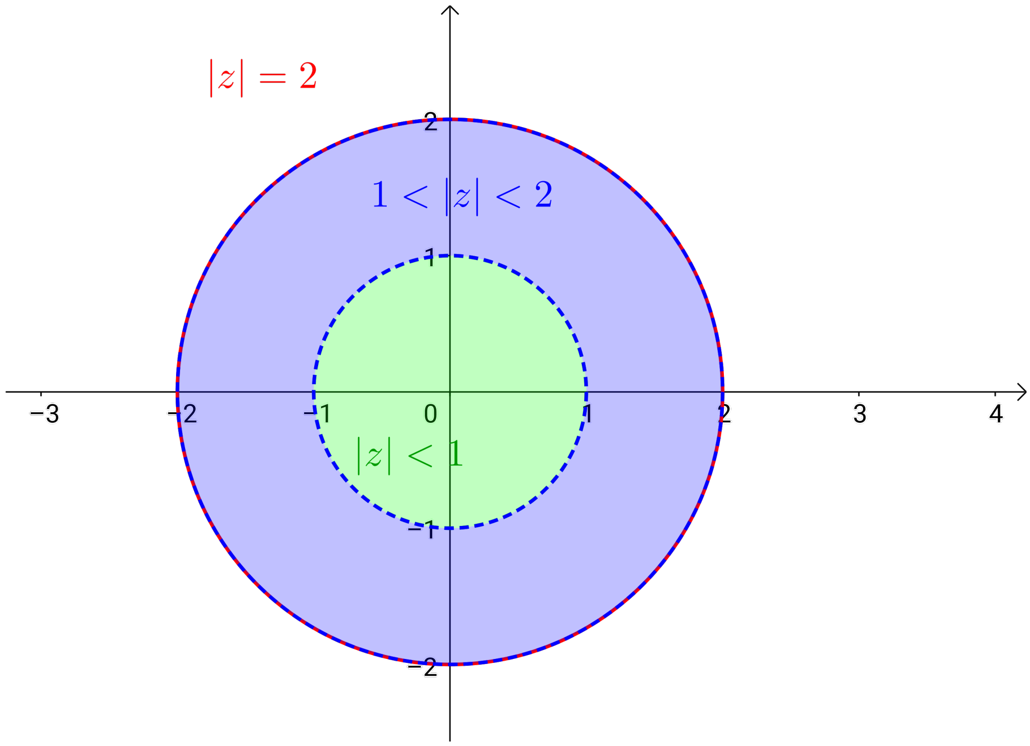
$$\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$$

Benytter som sagt de Moivres formel til at skrive

$$(\cos(\theta) + i\sin(\theta))^4 = \cos(4\theta) + i\sin(4\theta)\,.$$

Parentesen ophæves ved at multiplicere ud,

$$\begin{aligned}(\cos(\theta) + i\sin(\theta))^4 =& \cos^4(\theta) + 4\cos^3(\theta)i\sin(\theta) + 6\cos^2(\theta)i^2\sin^2(\theta)\\ &+ 4\cos(\theta)i^3\sin^3(\theta) + i^4\sin^4(\theta)\\ (\cos(\theta) + i\sin(\theta))^4 =& \cos^4(\theta) + 4\cos^3(\theta)i\sin(\theta) - 6\cos^2(\theta)\sin^2(\theta)\\ &- 4\cos(\theta)i\sin^3(\theta) + \sin^4(\theta)\end{aligned}$$



Figur 1: Opgave 3

Det vides nu at

$$\cos(4\theta) + i\sin(4\theta) = \cos^4(\theta) + 4\cos^3(\theta)i\sin(\theta) - 6\cos^2(\theta)\sin^2(\theta) \\ - 4\cos(\theta)i\sin^3(\theta) + \sin^4(\theta)$$

Ser nu kun på den reelle del

$$\cos(4\theta) = \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)$$

Udnytter at $\sin^2(\theta) + \cos^2(\theta) = 1 \to \sin^2(\theta) = 1 - \cos^2(\theta)$.

$$\begin{aligned}
\cos(4\theta) &= \cos^4(\theta) - 6\cos^2(\theta)\left(1 - \cos^2(\theta)\right) + \left(1 - \cos^2(\theta)\right)^2 \\
\cos(4\theta) &= \cos^4(\theta) - 6\cos^2(\theta) + 6\cos^4(\theta) + 1 + \cos^4(\theta) - 2\cos^2(\theta) \\
\cos(4\theta) &= 8\cos^4(\theta) - 8\cos^2(\theta) + 1 \quad \triangle
\end{aligned}$$

2. og udled at

$$\cos\left(\frac{\pi}{8}\right) = \left(\frac{2+\sqrt{2}}{4}\right)^{\frac{1}{2}}.$$

$\frac{\pi}{8}$ indsættes i første omgang på θ's plads.

$$\begin{aligned}
\cos(4\theta) &= 8\cos^4(\theta) - 8\cos^2(\theta) + 1 \\
\cos\left(4 \cdot \frac{\pi}{8}\right) &= 8\cos^4\left(\frac{\pi}{8}\right) - 8\cos^2\left(\frac{\pi}{8}\right) + 1 \\
0 &= 8\cos^4\left(\frac{\pi}{8}\right) - 8\cos^2\left(\frac{\pi}{8}\right) + 1 \quad , \cos\left(\frac{\pi}{2}\right) = 0
\end{aligned}$$

Nu indføres der en midlertidlig variabel $w = \cos^2\left(\frac{\pi}{8}\right)$, så ligningen bliver til

$$0 = 8w^2 - 8w + 1\,.$$

Nu er der tale om en andengradsligning, som let løses:

$$\begin{aligned}
a &= 8 \\
b &= -8 \\
c &= 1 \\
d &= b^2 - 4ac \\
d &= (-8)^2 - 4 \cdot 8 \cdot 1 = 32 \\
w &= \frac{-b \pm \sqrt{d}}{2a} \\
w &= \frac{8 \pm \sqrt{32}}{2 \cdot 8} \\
w &= \frac{8 \pm \sqrt{32}}{16} \\
w &= \frac{2 \pm \frac{\sqrt{32}}{4}}{4} \\
w &= \frac{2 \pm \frac{\sqrt{32}}{\sqrt{16}}}{4} \\
w &= \frac{2 \pm \sqrt{\frac{32}{16}}}{4} \\
w &= \frac{2 \pm \sqrt{2}}{4}
\end{aligned}$$

Nu kan udtrykket for w sættes tilbage ind

$$\begin{aligned}
\cos^2\left(\frac{\pi}{8}\right) &= \frac{2 \pm \sqrt{2}}{4} \rightarrow \\
\cos\left(\frac{\pi}{8}\right) &= \pm\left(\frac{2 \pm \sqrt{2}}{4}\right)^{\frac{1}{2}}
\end{aligned}$$

Af dette kan det ses at

$$\cos\left(\frac{\pi}{8}\right) = \left(\frac{2+\sqrt{2}}{4}\right)^{\frac{1}{2}}.$$

er indeholdt i løsningen. △

Opgave 5

1. Udtryk $\sin^4(\theta)$ kun ved hjælp af trigonometriske funktioner med multiplum af vinkler (læs $\sin(n\theta)$ eller $\cos(n\theta)$.

Benytter i første omgang ligning (3.11):

$$\begin{aligned}
z - \frac{1}{z} &= 2i\sin(\theta) \rightarrow \\
\left(z - \frac{1}{z}\right)^4 &= (2i\sin(\theta))^4 \\
\left(z - \frac{1}{z}\right)^4 &= 2^4 i^4 \sin^4(\theta) \\
\left(z - \frac{1}{z}\right)^4 &= 16\sin^4(\theta) \\
z^4 + \frac{1}{z^4} - 4\cdot z^2 - 4\cdot\frac{1}{z^2} + 6 &= 16\sin^4(\theta) \\
z^4 + \frac{1}{z^4} - 4\cdot\left(z^2 + \frac{1}{z^2}\right) + 6 &= 16\sin^4(\theta)
\end{aligned}$$

Benytter nu ligning (3.6) til omskrivning

$$\begin{aligned}
2\cos(4\theta) - 4\cdot 2\cos(2\theta) + 6 &= 16\sin^4(\theta) \rightarrow \\
\frac{2\cos(4\theta) - 4\cdot 2\cos(2\theta) + 6}{16} &= \sin^4(\theta) \\
\frac{1}{8}\cos(4\theta) - \frac{1}{2}\cos(2\theta) + \frac{3}{8} &= \sin^4(\theta) \quad \triangle
\end{aligned}$$

2. Eftervis at den gennemsnitslige værdi over en periode er $\frac{3}{8}$.

Den gennemsnitslige værdi findes ved at udføre følgende integrale

$$\frac{\int_0^{2\pi} \sin^4(\theta) d\theta}{2\pi} = \frac{\int_0^{2\pi} \frac{1}{8}\cos(4\theta) - \frac{1}{2}\cos(2\theta) + \frac{3}{8}\, d\theta}{2\pi}$$
$$\frac{\int_0^{2\pi} \sin^4(\theta) d\theta}{2\pi} = \frac{\frac{1}{8}\int_0^{2\pi} \cos(4\theta)\, d\theta - \frac{1}{2}\int_0^{2\pi} \cos(2\theta)\, d\theta + \int_0^{2\pi} \frac{3}{8}\, d\theta}{2\pi}$$
$$\frac{\int_0^{2\pi} \sin^4(\theta) d\theta}{2\pi} = \frac{\frac{1}{8}\left[\frac{\sin(4\theta)}{4}\right]_0^{2\pi} - \frac{1}{2}\left[\frac{\sin(2\theta)}{2}\right]_0^{2\pi} + \left[\frac{3}{8}\cdot\theta\right]_0^{2\pi}}{2\pi}$$
$$\frac{\int_0^{2\pi} \sin^4(\theta) d\theta}{2\pi} = \frac{\frac{1}{8}\left(\frac{\sin(8\cdot\pi)}{4} - \frac{\sin(0)}{4}\right) - \frac{1}{2}\left(\frac{\sin(4\pi)}{2} - \frac{\sin(0)}{2}\right) + \left(\frac{3}{8}\cdot 2\pi - \frac{3}{8}\cdot 0\right)}{2\pi}$$
$$\frac{\int_0^{2\pi} \sin^4(\theta) d\theta}{2\pi} = \frac{\frac{1}{8}(0-0) - \frac{1}{2}(0-0) + \left(\frac{3}{8}\cdot 2\pi - 0\right)}{2\pi}$$
$$\frac{\int_0^{2\pi} \sin^4(\theta) d\theta}{2\pi} = \frac{\frac{3}{8}\cdot 2\pi}{2\pi}$$
$$\frac{\int_0^{2\pi} \sin^4(\theta) d\theta}{2\pi} = \frac{3}{8}$$

Hermed er det vist, at gennemsnittet over en periode er $\frac{3}{8}$ $\triangle$

Opgave 6

Find samtlige løsninger til følgende ligninger

1. $x^3 + 8 = 0$

Benytter samme strategi som i eksemplerne i kapitel 3.3

$$
\begin{aligned}
z^3 + 8 = 0 \rightarrow \\
z^3 &= -8 \\
z^3 &= -8 \cdot 1 \\
z^3 &= -8 \cdot e^{2\pi k i} \rightarrow \\
z &= \sqrt[3]{-8} \cdot e^{\frac{2\pi k i}{3}} \qquad \text{hvor } k = 0, 1, 2 \\
z &= -2 \cdot e^{\frac{2\pi k i}{3}} \qquad \text{hvor } k = 0, 1, 2 \\
z_1 &= -2 \cdot e^{\frac{2\pi \cdot 0 \cdot i}{3}} \qquad \text{for } k = 0 \\
z_1 &= -2 \\
z_2 &= -2 \cdot e^{\frac{2\pi \cdot 1 \cdot i}{3}} \qquad \text{for } k = 1 \\
z_2 &= -2 \cdot \left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right) \\
z_2 &= -2 \cdot \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\
z_2 &= 1 + \sqrt{3}i \\
z_3 &= -2 \cdot e^{\frac{4\pi \cdot 1 \cdot i}{3}} \qquad \text{for } k = 2 \\
z_3 &= -2 \cdot \left(\cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right)\right) \\
z_3 &= -2 \cdot \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \\
z_3 &= 1 - \sqrt{3}i
\end{aligned}
$$

Ergo er der tre løsninger: $z_1 = -2$, $z_2 = 1 + \sqrt{3}i$ og $z_3 = 1 - \sqrt{3}i$.

2. $z^4 = 16$

Løses på tilsvarende vis som i forrige opgave:

$$
\begin{aligned}
z^4 &= 16 \\
z^4 &= 16 \cdot 1 \\
z^4 &= 16 \cdot e^{2\pi k i} \rightarrow \\
z &= \sqrt[4]{16} \cdot e^{\frac{2\pi k i}{4}} \qquad \text{for } k = 0, 1, 2, 3 \\
z &= 2 \cdot e^{\frac{2\pi k i}{4}} \qquad \text{for } k = 0, 1, 2, 3 \\
z_1 &= 2 \cdot e^{\frac{2\pi \cdot 0 \cdot i}{4}} \qquad \text{for } k = 0 \\
z_1 &= 2 \\
z_2 &= 2 \cdot e^{\frac{2\pi \cdot 1 \cdot i}{4}} \qquad \text{for } k = 1 \\
z_2 &= 2 \cdot \left(\cos\left(\frac{2\pi \cdot 1 \cdot i}{4} \right) + i \sin\left(\frac{2\pi \cdot 1 \cdot i}{4} \right) \right) \\
z_2 &= 2 \cdot (0 + i1) \\
z_2 &= 2i \\
z_3 &= 2 \cdot e^{\frac{2\pi \cdot 2 \cdot i}{4}} \qquad \text{for } k = 2 \\
z_3 &= 2 \cdot \left(\cos\left(\frac{2\pi \cdot 2 \cdot i}{4} \right) + i \sin\left(\frac{2\pi \cdot 2 \cdot i}{4} \right) \right) \\
z_3 &= 2 \cdot (-1 + i0) \\
z_3 &= -2 \\
z_4 &= 2 \cdot e^{\frac{2\pi \cdot 3 \cdot i}{4}} \qquad \text{for } k = 3 \\
z_4 &= 2 \cdot \left(\cos\left(\frac{2\pi \cdot 3 \cdot i}{4} \right) + i \sin\left(\frac{2\pi \cdot 3 \cdot i}{4} \right) \right) \\
z_4 &= 2 \cdot (0 - i1) \\
z_4 &= -2i
\end{aligned}
$$

Ergo er der altså de fire løsninger $z_1 = 2$, $z_2 = 2i$, $z_3 = -2$ og $z_4 = -2i$. $\triangle$

3. $z^3 = 27i$

I denne opgave udnyttes det at $i = e^{i\left(\frac{\pi}{2} + 2\pi k\right)}$

$$z^3 = 27i$$
$$z^3 = 27e^{i\left(\frac{\pi}{2}+2\pi k\right)} \rightarrow$$
$$z = \sqrt[3]{27}e^{i\left(\frac{\frac{\pi}{2}+2\pi k}{3}\right)} \quad \text{for } k = 0,1,2$$
$$z = 3e^{i\left(\frac{\pi+4\pi k}{6}\right)} \text{ for } k = 0,1,2$$
$$z_1 = 3e^{i\left(\frac{\pi+4\pi\cdot 0}{6}\right)} \text{ for } k = 0$$
$$z_1 = 3e^{i\frac{\pi}{6}}$$
$$z_1 = 3\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$
$$z_1 = 3\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$
$$z_1 = \frac{3}{2}\left(\sqrt{3} + i\right)$$
$$z_2 = 3e^{i\left(\frac{\pi+4\pi\cdot 1}{6}\right)} \text{ for } k = 1$$
$$z_2 = 3\left(\cos\left(\frac{\pi+4\pi\cdot 1}{6}\right) + i\sin\left(\frac{\pi+4\pi\cdot 1}{6}\right)\right)$$
$$z_2 = 3\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$
$$z_2 = \frac{3}{2}\left(-\sqrt{3} + i\right)$$
$$z_3 = 3e^{i\left(\frac{\pi+4\pi\cdot 1}{6}\right)} \text{ for } k = 2$$
$$z_3 = 3\left(\cos\left(\frac{\pi+4\pi\cdot 2}{6}\right) + i\sin\left(\frac{\pi+4\pi\cdot 2}{6}\right)\right)$$
$$z_3 = 3\left(0 - i\right)$$
$$z_3 = -3i$$

Ergo er der altså de tre løsninger $z_1 = \frac{3}{2}\left(\sqrt{3}+i\right)$,$z_2 = \frac{3}{2}\left(-\sqrt{3}+i\right)$ og $z_3 = -3i$.
$\triangle$

4. $z^3 + z^2 - 2z = 0$

I første omgang kan den trivielle løsning $z_1 = 0$ let ses. Ligningen kan nu reduceres til:

$$z^2 + z - 2 = 0$$

Denne kan løses som en almindelig andengradsligning.

$$
\begin{aligned}
a &= 1 \\
b &= 1 \\
c &= -2 \\
d &= 1^2 - 4 \cdot 1 \cdot (-2) = 9 \\
z &= \frac{-1 \pm \sqrt{9}}{2 \cdot 1} \\
z &= \frac{-1 \pm 3}{2 \cdot 1} \\
z_2 &= 1 \\
z_3 &= -2
\end{aligned}
$$

Altså er der de tre løsninger $z_1 = 0$, $z_2 = 1$ og $z_3 = -2$.$\triangle$

5. $z^3 - 2z^2 + 2z = 0$

Igen ses den trivielle løsning $z_1 = 0$ let, og ligningen kan reduceres til

$$
z^2 - 2z + 2 = 0\,.
$$

Løses igen som en almindelig andengradsligning:

$$
\begin{aligned}
a &= 1 \\
b &= -2 \\
c &= 2 \\
d &= (-2)^2 - 4 \cdot 1 \cdot 2 = 4 - 8 = -4 \\
z &= \frac{2 \pm \sqrt{-4}}{2 \cdot 1} \\
z &= \frac{2 \pm 2i}{2} \\
z &= 1 \pm i \rightarrow \\
z_2 &= 1 + i \\
z_3 &= 1 - i
\end{aligned}
$$

Altså er der de tre løsninger $z_1 = 0$, $z_2 = 1 + i$ og $z_3 = 1 - i$.$\triangle$