# Egyptian Fractions

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## 1 Introduction

Ancient Egyptians could only represent fractions as **unit fractions** of the form $\frac{1}{n}$, where $n$ is a positive integer. Therefore, any fraction had to be expressed as a sum of **distinct** unit fractions.

For example:

$$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}.$$

## 2 Greedy Algorithm for Egyptian Fractions

One method to represent a fraction $\frac{x}{y}$ as a sum of distinct unit fractions is the **greedy algorithm**:

1. Choose the largest unit fraction $\frac{1}{n}$ such that $\frac{1}{n} \leq \frac{x}{y}$.
2. Subtract it:
$$\frac{x}{y} - \frac{1}{n} = \frac{xn - y}{yn}.$$
3. Repeat the process with the remaining fraction until the remainder is zero.

## 3 Example

Consider $\frac{5}{6}$:

1. The largest unit fraction $\leq \frac{5}{6}$ is $\frac{1}{2}$.
2. Subtract: $\frac{5}{6} - \frac{1}{2} = \frac{2}{6} = \frac{1}{3}$.
3. Resulting sum:
$$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}.$$

## 4 Proof of Termination

Let $\frac{x}{y}$ be a positive fraction in lowest terms. At each step of the greedy algorithm, we select

$$n = \left\lceil \frac{y}{x} \right\rceil,$$

the smallest integer such that $\frac{1}{n} \leq \frac{x}{y}$. Then the remainder is

$$\frac{x}{y} - \frac{1}{n} = \frac{xn - y}{yn}.$$

We now show that the numerator of the remainder, $xn - y$, satisfies

$$0 < xn - y < x.$$

- Since $n = \lceil y/x \rceil$, we have $n - 1 < y/x \leq n$, which implies

$$(n-1)x < y \leq nx.$$

- Subtracting $y$ from $nx$ gives

$$0 < nx - y < x.$$

Thus, the numerator of the remaining fraction is strictly positive and strictly smaller than the previous numerator. Consequently, the numerators form a strictly decreasing sequence of positive integers, which must eventually reach 1. At that point, the remainder is a unit fraction, and the algorithm terminates.

Since each selected unit fraction is strictly less than the remaining fraction, all terms are **distinct**. Therefore, the greedy algorithm always produces a finite sum of distinct unit fractions for any positive rational number.

## 5 Conclusion

The greedy algorithm provides a systematic and finite method to represent any positive fraction as a sum of distinct unit fractions, reflecting the ancient Egyptian approach to fractions.