Finance Essentials

Meaning of Expected Value

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There are several measures of central tendency. Some are very easy to describe: Mode is the most likely outcome; median is the midpoint of all the possible outcomes. Expected value does not lend itself to such an easy explanation. Try providing an intuitive explanation, and you will end up describing the formula for calculating the expected value; something like sum of all the possible outcomes divided by the number of outcomes or worse yet, the probability weighted sum of all the possible outcomes. Obviously, these definitions are nowhere near as appealing as mode or median. This document makes an attempt to put some intuition behind expected value.

Let's take the simple case of tossing a coin. Designate heads as 0 and tails as 1. Let's also assume that the coin is fair so that each outcome is equally likely. Suppose before tossing the coin I ask you to make a guess. If you are wrong, you pay a penalty. The bigger the mistake in your guess, the higher the penalty. In fact, the penalty is the square of the error in your guess. Let's enumerate the possibilities:

- Suppose you guess heads (0).
  - Suppose the result of the toss is tails (1). You are penalized $(1-0)^2 = 1$ or \$1.
  - Suppose the result of the toss is heads (0). You are penalized $(0-0)^2 = 0$ or \$0.
- Suppose you guess tails (1).
  - Suppose the result of the toss is tails (1). You are penalized $(1-1)^2 = 0$ or \$0.
  - Suppose the result of the toss is heads (0). You are penalized $(0-1)^2 = 1$ or \$1.

You are facing a maximum penalty of \$1 regardless of what you guess. Your average penalty is $(0+1)/2 = 0.5$ or \$0.50.

You try to find a way out of this situation by guessing an amount that would minimize your penalty. You could either look to minimize your maximum penalty[1] or average penalty. Here let's talk about the strategy for minimizing the average penalty. Suppose you choose to guess x as the outcome. Then your penalty would be $(0-x)^2$ if the result is heads and $(1-x)^2$ if the toss produces heads. The average penalty is:

$$\frac{(0-x)^2 + (1-x)^2}{2}$$

[1]This strategy is known as MiniMax. Look it up!

You want to minimize the average penalty by choosing the value x.

$$\min_x \frac{(0-x)^2 + (1-x)^2}{2}$$

The answer turns out to be $x = 0.5$. The average penalty with this choice is

$$\frac{(0-0.5)^2 + (1-0.5)^2}{2} = 0.25$$

or \$0.25. What does 0.5 mean in the context of coin toss. If heads is 0 and tails is 1, then 0.5 means that the coin will come up standing on its side! We know that this will never (well, almost never) happen. Regardless, that would be your best guess to minimize the average penalty.

You can verify that the average penalty (where the penalty is proportional to squared error in your guess) is minimized when guessing the outcome of rolling a die if the guess is 3.5.

Suppose you are about to walk into a typical American family home and are asked to guess the number of children in the family. If there is a penalty for you being wrong, and the penalty is proportional to the square of the error, your best guess would be 2.1 or 2.3 or something like that.

The examples above seemed nonsensical, especially the one involving the number of children in the family. That's because the probability distributions were all *discrete*. Let's consider a continuous probability distribution: cubic feet of propane an energy company will sell to its customers during the upcoming year. If the company guesses too high, it will have to pay storage cost. If the company guesses too low, it will have lost sale. Let's assume that the cost incurred by the company—storage or lost sale—will be square of the amount of mistake in its estimate. The company will try to come up with an estimate that minimizes the average cost taking all possible scenarios into account. That estimate will be called the expected value.

So, we can define expected value as

> Expected value is the estimated outcome that minimizes the average penalty due to a bad estimate, assuming that the penalty is proportional to the square of the estimation error.

I know it is still not anywhere as easy to describe as median or mode but, in my opinion, better than having to describe the mathematical formula to calculate the expected value.[2]

[2]Those of you who notice the circularity problem in the definition are very astute. I am not sure how I can get around that problem.