Finance Essentials

Proof that Market Portfolio is Value Weighted

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1. Setup:
   (a) There are N securities in the market. The market value of security i is V_i.
   (b) There are J investors with identical beliefs but different wealths and risk aversions. Since investors have identical beliefs, they choose the same mean-variance efficient portfolio of risky securities.
   (c) Investor j invests W_j amount of money in the risky securities.
   (d) w_i is the weight of risky security i in the portfolio.

2. Total amount invested by investor j in the risky securities must equal his/her investment in risky securities:

$$W_j = \sum_i W_j w_i = W_j \sum_i w_i \qquad \Rightarrow \qquad \sum w_i = 1 \tag{1}$$

3. The market value of the security i must equal the total amount invested by all the investors in risky security i:

$$V_i = \sum_j W_j w_i = w_i \sum_j W_j \tag{2}$$

$$\Rightarrow \qquad w_i = \frac{V_i}{\sum_j W_j} \tag{3}$$

4. The market value of all the securities must equal the total amount invested by all the investors in all the risky securities. To get this relationship, sum equation (3) across all securities:

$$\sum_i V_i = \sum_i \left(w_i \sum_j W_j \right) = \left(\sum_j W_j \right) \left(\sum_i w_i \right) = \sum_j W_j \tag{4}$$

   This equation should make intuitive sense: the total value of all risky securities equals the total amount invested in the risky securities.

5. Substitute $\sum_j W_j$ from (4) into (3) to get:

$$w_i = \frac{V_i}{\sum_i V_i} \tag{5}$$

   This equation shows that the weights of risky securities in the mean-variance efficient portoflio are the fractions of the securities relative to the values of all the securities in the market. In other words, market portfolio weights are equal to market value weights, or that the market portfolio is value weighted.

6. Example: The economy has 4 risky securities with market values as shown below:

$$V_1 = \$10{,}000, \qquad V_2 = \$20{,}000, \qquad V_3 = \$30{,}000, \qquad V_4 = \$40{,}000$$

The total market value of all risky securities is $\sum_i V_i = \$100{,}000$ The risky portfolio weights are 10%, 20%, 30% 40%. A sample calculation is shown below:

$$w_3 = \frac{V_3}{\sum_i V_i} = \frac{\$30{,}000}{\$100{,}000} = 30\%$$

So, all investors will hold the risky securities in the same proportion: (10%, 20%, 30%, 40%).

7. Why doesn't this happen in real life? Note the assumption about identical beliefs in item 1 above.

8. What does happen in real life? Investors have different beliefs and create portfolios with different wights in risky securities. So, the weight of risky security i in the portfolio would be different for different investors. Let's denote it by w_{ij}. With differing beliefs, wealth weighted average of the portfolio weights are equal to market value weights. The proof follows.

9. From the market clearing condition for security i (corresponding to equation (3)):

$$V_i = \sum_j W_j w_{ij} \tag{6}$$

Sum equation (6) across all securities to get the market clearing condition for all securities:

$$\sum_i V_i = \sum_i \sum_j W_j w_{ij} = \sum_j W_j$$

combining the above two equation, we see that the wealth weighted average weight of security i across investors is

$$\frac{\sum_j W_j w_{ij}}{\sum_j W_j} = \frac{V_i}{\sum_i V_i}$$

10. Going back to the numerical example above: Suppose there are 3 investors A, B and C with wealths in risky securities of \$20,000, \$30,000, and \$50,000, respectively. Tht total wealth invested in risky securities is \$100,000, same as the market value of securities.[1] The investors guided by their beliefs, create the following portfolios of risky securities:

$$\begin{aligned} A &= (\$2{,}000, \$6{,}000, \$4{,}000, \$8{,}000) = (10\%, 30\%, 20\%, 40\%) \\ B &= (\$3{,}000, \$4{,}000, \$9{,}000, \$14{,}000) = (10\%, 13.33\%, 30\%, 46.67\%) \\ C &= (\$5{,}000, \$10{,}000, \$17{,}000, \$18{,}000) = (10\%, 20\%, 34\%, 36\%) \end{aligned}$$

Note that

- For each investor, the total amount invested in the risky securities is equal to his/her desired investment in the risky securities. For example, the total amount invested by investor A is $\$2{,}000 + \$6{,}000 + \$4{,}000 + \$8{,}000 = \$20{,}000$ which matches his desired investment in the risky securities.
- For each security, the total amount invested is equal to its market value. For example the total amount invested in security 2 is equal to $\$4{,}000 + \$9{,}000 + \$17{,}000 = \$30{,}000$ which is equal to the market value of the security.

[1] If the investible wealth were less (more), the market value of securities would fall (rise) accordingly.

Let us now calculate the wealth weighted average of portfolio weights:

$$\text{Security 1} = \frac{\$20{,}000 \times 10\% + \$30{,}000 \times 10\% + \$50{,}000 \times 10\%}{\$100{,}000} = 10\%$$
$$\text{Security 2} = \frac{\$20{,}000 \times 30\% + \$30{,}000 \times 13.33\% + \$50{,}000 \times 20\%}{\$100{,}000} = 20\%$$
$$\text{Security 3} = \frac{\$20{,}000 \times 20\% + \$30{,}000 \times 30\% + \$50{,}000 \times 34\%}{\$100{,}000} = 30\%$$
$$\text{Security 4} = \frac{\$20{,}000 \times 40\% + \$30{,}000 \times 46.67\% + \$50{,}000 \times 36\%}{\$100{,}000} = 40\%$$

Note that these wealth weighted average portfolio weights are equal to the market value weights of the securities shown in item 6 above.