# Notes on Utility Theory

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## 1 Utility Functions

Utility is derived from consumption of goods. Utility functions are used to provide an *ordering* of the utilities provided by different consumption packets. For example, if an individual would rather have a house than a new BMW 735i, he has a higher utility for the house than that for the BMW. Another individual (or even the same individual at a different time!), may have an opposite belief. This concept of ordering is based on quite a few sound logical axioms. Some of them are listed below:

1. Individuals derive a higher utility from more quantity of a (consumption) good that less.
2. If an individual prefers packet (of consumption goods) A over packet B and he prefers packet B over packet C, then he (logically) prefers packet A over packet C.
3. Given two packets A and B, the individual is able to decide whether he prefers A to B, B to A or he is indifferent between A and B.

Many a times, for convenience, we define the utility to be a direct function of a commodity other than a consumption good. Utility functions that serve the purpose of providing ordering between different packets, without assigning numerical values to the utilities are called *ordinal* functions. As you can well imagine, this is not very useful for analytical purposes. This led to development of *cardinal* utility functions by von Neumann and Morgenstern based on some additional axioms. These utility functions assign a number to each packet and a packet with a higher utility number is said to be preferred over the one with a lower utility number.

Different individuals have different utility functions. Since we prefer more over less, the utility functions have a positive slope. Also, due to the fact that (normally) consuming the first unit of any good give us a lot more utility than consuming one unit after we have consumed (say) 100 of these, the slope is high in the beginning and then goes on declining. In mathematical notation, if the utility function of an individual is denoted as $U(W)$, then $U' > 0$ and $U'' < 0$.

Cardinal utility functions are estimated by comparing the utility of a gamble with 2 outcomes with that of a sure outcome. The probabilities of the different outcomes in the gamble are adjusted till the individual is *indifferent* between the gamble and the sure situation. This gives us an equation that can be used to get numbers for utility of different packets.

To assign numeric values to cardinal utility functions we need to fix some standards against which the utilities of all other packets will be measured. This is just like we assign (arbitrary) values of 32 and 212 to the temperature at which the water freezes and boils respectively and then we assign values to all other temperatures with respect to these standard. We know that these assignments are arbitrary because values of 0 and 100 to these temperatures do equally well (or at least the Canadians think so!). Ultimately the objective of the temperature scale is to provide us with some notion of relative warmth (or coolnees). Both the Fahrenheit and the Celsius do a good job of it, once we get used to them. Coming back to utility functions, there are, as yet no universal standards as for the temperature scale and therefore one is free to assign his or her own.

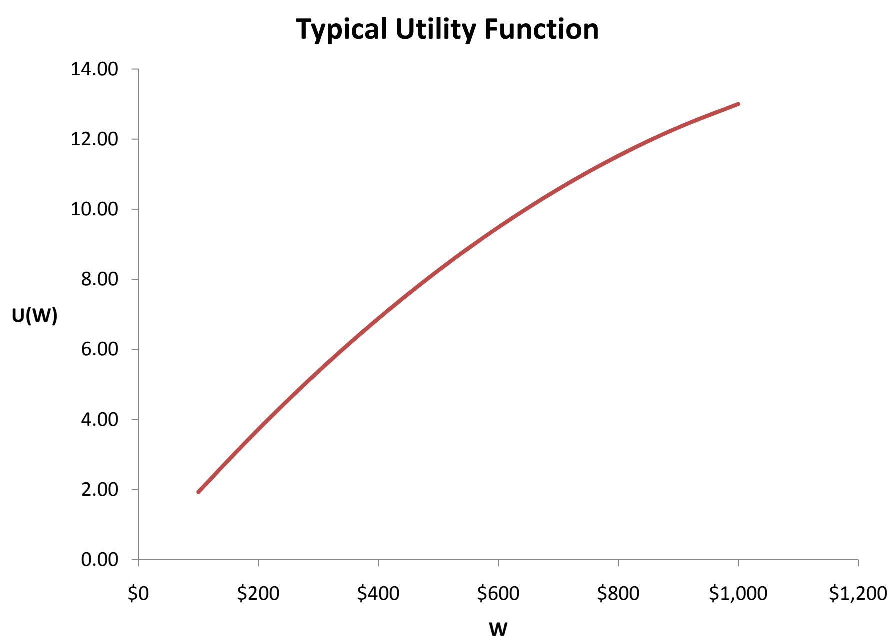
*Example*: We will derive utility function for dollars for a person. We assume that to start with she has no wealth of her own. We will arbitrarily assign $U(0) = 0$ and $U(100) = 10$. Now suppose we want to find the value of utility for \$50. We will present her with a situation in which she has the option to take either \$50 or play a lottery in which she can get \$0 or \$100. Then we will keep on adjusting the probabilities for the two outcomes in the lottery till she is *indifferent* between the gamble and the sure amount of \$50. Clearly in this situation the wealth after the gamble is providing her as much utility as \$50. We use this information to form an equation. Let us assume that the probabilities were 0.3 and 0.7 for \$0 and \$100, respectively at the point of indifference. Then we can write as follows:

$$\begin{aligned}
E(U(\text{Sure situation})) &= E(U(\text{Gamble})) \\
U(50) &= 0.3 \times U(0) + 0.7 \times U(100) \\
U(50) &= 0.3 \times 0 + 0.7 \times 10 \\
U(50) &= 7
\end{aligned}$$

What if we wanted to find the utility of \$150? Construct the following situation: \$100 for sure or a gamble with outcomes of \$0 or \$150. Suppose she became indifferent at the probabilities of .2 and .8, respectively, then:

$$\begin{aligned}
E(U(\text{Gamble}) &= E(U(\text{Sure situation})) \\
0.2 \times U(0) + 0.8 \times U(150) &= U(100) \\
0.2 \times 0 + 0.8 \times U(150) &= 10 \\
U(150) &= 12.5
\end{aligned}$$

So, this way we can get utilities for various amounts. Now we can plot such a utility function on a graph and attempt to fit an algebraic equation through the plotted points. Figure 1 shows a typical utility function. Algebraically, commonly fitted equations are $U(W) = a\ ln(bW)$, $U(W) = -ae^{-bW}$ and $U(W) = aW - bW^2$. $a$ and $b$ are positive numbers for risk averse persons.




## 2 Decision Making

Without going into details, we will intuitively believe that a person would want to maximize the utility (expected utility in a situation involving uncertainty). If we have utility functions that depend on only one good, we can see very easily that the person's objective is identical to that of maximizing the total quantity of the good consumed.

The situation becomes tricky when we have more than one goods (or 'bads'!). For example, deciding between 2 apples & 3 oranges and 3 apples & 2 oranges; \$100 today & \$200 a year from now and \$200 today & \$125 a year from now; an expected rate of return of 0.10 & a standard deviation of 0.07 and an expected rate of return of 0.12 & a standard deviation of 8%. We will limit ourselves to considering two objects only.
To answer such questions we have to define what are known as the *indifference curves*. These curves are the contours of combinations of the two commodities that have the same amount of utility for a person. Two examples are given below:
Convince yourself that the shapes do indeed look like as drawn above. The individual would like to be on a contour with the highest value of utility.

# 3 Risk Premium

Let us assume an investor with initial wealth of $W$ is faced with a gamble (an investment with a random outcome) $\tilde{Z}$. For illustration purposes we will consider the following gamble:
The individual has two choices: take the gamble or take a sure amount and avoid the gamble. If the individual takes the gamble, the end of period wealth will be $W + \tilde{Z}$. Carrying on with our example, we can show it as:
The expected value of wealth after the gamble is $W + E(\tilde{Z})$. Instead if the individual takes a *sure* amount $CE$, the end of period wealth will be $W + CE$.

*Definition*: CE is the *certainty equivalent* of the gamble for the individual at the wealth level $W$ if and only if:

$$U(W + CE) = E(U(W + \tilde{Z})) \tag{1}$$

In words, the gamble will make the individual as happy (will provide as much utility) as receiving $CE$ for sure.

*Definition*: Risk premium is defined as the difference between the expected wealth from the gamble and the certainty equivalent wealth, i.e. premium denoted by $\pi$ is defined as below:

$$\begin{aligned} \pi &= [W + E(\tilde{Z})] - [W + CE] \\ &= E(\tilde{Z}) - CE \end{aligned} \tag{2}$$

The Markowitz risk premium is calculated by combining equations (**??**) and (**??**). Substitute $CE$ from (**??**) into (**??**) to get:

$$U(W + E(\tilde{Z}) - \pi) = E(U(W + \tilde{Z}))$$

or,

$$W + E(\tilde{Z}) - \pi = U^{-1}[E(U(W + \tilde{Z}))]$$

and therefore,

$$\pi = W + E(\tilde{Z}) - U^{-1}[E(U(W + \tilde{Z}))] \tag{3}$$

where $U^{-1}$ is the inverse of the utility function. A person is classified as risk loving, risk neutral or risk averse if the risk premium is negative, zero or positive, respectively. The risk premium can be considered to be the *additional* amount that our investor needs before he is lured into taking the gamble. A more detailed explanation is provided in the example below. Naturally, if the calculated premium turns out to be negative, our investor will be willing to pay a fee to be allowed to take the gamble. For computational purposes use equation (**??**) to calculate $CE$ and then use equation (**??**) to calculate the premium.

*Example*: Consider an individual with $W = \$10$ and $U(W) = ln(W)$. The individual is faced with the following gamble:

$$\begin{aligned} E(U(W + \tilde{Z})) &= 0.1 \times U(10 + 10) + 0.9 \times U(10 + 100) \\ &= 0.1 \times ln(20) + 0.9 \times ln(110) \\ &= 0.1 \times (3.00) + 0.9 \times (4.70) \\ &= 4.53 \end{aligned}$$

By applying equation(**??**)

$$U(W + CE) = 4.53$$

or,

$$ln(W + CE) = 4.53$$

therefore,

$$W + CE = e^{4.53} = 92.76$$

so that,

$$CE = 82.76$$

also,

$$E(\tilde{Z}) = 0.1 \times 10 + 0.9 \times 100 = 91$$

text,

$$\pi = (10 + 91) - (10 + 82.76) = 8.24$$

This means that the individual will be indifferent between the gamble and a sure amount of \$91-8.24=\$82.76. The gamble, *on an average* pays \$91. Due to the risk averse nature of our investor, he likes this uncertain situation (that pays \$91 on an average) as much as he likes the sure amount of \$82.76. If the gamble was paying an amount even slightly less than \$91 on an average, our investor would not take it. This means that the gamble is paying the investor an excess of \$8.24 (on an average) to lure him into taking the gamble.
The certainty equivalent amount for the same gamble will change if the individual started out with a different amount of wealth. Go through the above example with the initial wealth of \$100.
Given the same initial level of wealth and the same gamble, different individuals will have different certainty equivalents and will, therefore, demand different amounts of risk premia. This happens because of the difference in the utility functions. Go through the above example with the utility function $U(W) = -e^{-W}$. Everything else being the same, the higher the risk premium demanded, the more risk averse the individual.
Markovitz calculation of risk premium does not allow us to categorize individuals into various degrees of risk aversion, because the answer depends on the particular gamble on hand. Pratt and Arrow overcame this difficulty by making an approximation. They combined equations (**??**) and (**??**) to get:

$$U(W + E(\tilde{Z}) - \pi) = E(U(W + \tilde{Z})) \tag{4}$$

From Taylor expansion (See the mathematical appendix),

$$U(W + E(\tilde{Z}) - \pi) = U(W) + (E(\tilde{Z}) - \pi)U'(W) + \cdots$$

and,

$$U(W + \tilde{Z}) = U(W) + \tilde{Z}U'(W) + \frac{1}{2}\tilde{Z}^2U''(W) + \cdots$$

so that,

$$E(U(W + \tilde{Z})) = U(W) + E(\tilde{Z})U'(W) + \frac{1}{2}E(\tilde{Z}^2)U''(W) + \ldots \cdots$$

Substitute in equation (**??**) and simplify to get the following approximation:

$$\pi \simeq -\frac{1}{2}E(\tilde{Z}^2)\frac{U''(W)}{U'(W)}$$

but,

$$E(\tilde{Z}^2) = \sigma^2 + \mu^2$$

where $\mu$ and $\sigma$ are the mean and standard deviation of the gamble outcomes (See the mathematical appendix). Using this we get

$$\pi \simeq -\frac{1}{2}(\sigma^2 + \mu^2)\frac{U''(W)}{U'(W)} \tag{5}$$

For an *actuarially neutral* gamble $\mu = 0$, so that

$$\pi \simeq -\frac{1}{2}\sigma^2 \frac{U''(W)}{U'(W)} \tag{6}$$

Now we can have a measure of the risk aversion of individuals that does not depend on the gamble on hand. For any gamble an individual will demand a higher risk premium if the value of $-\frac{U''(W)}{U'(W)}$ is higher for him. $-\frac{U''(W)}{U'(W)}$, therefore, is a measure of risk aversion. It is called the *absolute risk aversion.* A related measure $-\frac{WU''(W)}{U'(W)}$ is called *relative risk aversion.*

Note that the Pratt-Arrow measures of risk aversion are based on approximations. The risk premia given by them are a good approximation only if the gambles being considered are much small compared to the starting level of wealth.

Consider the example we had above:

$$\begin{aligned}
\mu &= E(\tilde{Z}) = 91 \\
\sigma^2 &= Var(\tilde{Z}) = 0.1(10-91)^2 + 0.9(100-91)^2 = 729 \\
U(W) &= ln(W) \\
U'(W) &= \frac{d(ln(W))}{dW} = \frac{1}{W} = \frac{1}{10}
\end{aligned}$$

and,

$$U''(W) = \frac{dU'}{dW} = \frac{-1}{W^2} = \frac{-1}{100}$$

so,

$$\begin{aligned}
\pi &= -\frac{1}{2}(91^2 + 729)\frac{(-1/100)}{(1/10)} \\
&= 450.50
\end{aligned}$$

Pratt-Arrow measure of risk premium is absurdly off because the essential approximation that the gamble should be small compared to the initial wealth is violated ($W = 10, E(\tilde{Z}) = 91$).

In general, Pratt-Arrow measures of risk aversion are quite good because in reality we consider only those gambles that are small compared to the initial wealth. Pratt-Arrow measure are useful because they can characterize individuals into more or less risk averse without concentrating on a specific gamble or a risky investment.

In this section we considered only those gambles that had two possible outcomes. The definitions and results are quite general and do not depend on the nature of distribution of the random outcome.

# 4 Mean-Variance Sufficiency

In this part we will examine the conditions under which an investor may just look at the mean and variance of the outcome of the risky investment and ignore all other aspects of the probability distribution.

The investor's goal is to maximize the wealth after the investment $\tilde{Z}$, i.e.,

$$Max \qquad E(U(W + \tilde{Z}))$$

We will show that there are two situations under which the investor can solve this problem just by concentrating on the mean and variance of the outcome of the investment.

*Case I*: Assume $\tilde{Z}$ is distributed normally with mean $\mu$ and variance $\sigma^2$. Then we can transform $\tilde{Z}$ to a standard unit normal by:

$$\tilde{z} = \frac{\tilde{Z} - \mu}{\sigma}$$

or,

$$\tilde{Z} = \mu + \sigma\tilde{z}$$

where $\tilde{z}$ is distributed normally with mean 0 and standard deviation 1. A special property of the standard normal distribution is that:

$$E(\tilde{z}^{2k-1}) = 0 \qquad \text{and} \qquad E(\tilde{z}^{2k}) = \frac{2k!}{2^k k!} \qquad k = 1, 2, \ldots, \infty$$

Realizing this, the investor's problem is

$$Max \qquad E(U(W + \mu + \sigma\tilde{z}))$$

Expand $U(W + \mu + \sigma\tilde{z})$ using the Taylor expansion:

$$U(W + \mu + \sigma\tilde{z}) = U(W + \mu) + \sum_{k=1}^{\infty}\Big[\frac{\sigma^{2k-1}}{(2k-1)!}\tilde{z}^{2k-1}U^{2k-1}(W + \mu) + \frac{\sigma^{2k}}{2k!}\tilde{z}^{2k}U^{2k}(W + \mu)\Big]$$

where $U^n$ denotes the $n^{th}$ derivative of $U$.
Taking expectation and substituting for $E(\tilde{z}^{2k-1}), E(\tilde{z}^{2k})$ from above and simplifying we get the following expression:

$$E(U(W + \mu + \sigma\tilde{z})) = U(W + \mu) + \sum_{k=1}^{\infty}\Big[\frac{\sigma^{2k}}{2^k\ k!}U^{2k}(W + \mu)\Big]$$

which can be written as:

$$E(U(W + \mu + \sigma\tilde{z})) = U(W + \mu) + F(\sigma, U)$$

where $F$ is the function as defined by the equations above. Now to maximize $E(U(W+\mu+\sigma\tilde{z}))$, the investor just needs to concentrate on $\mu$ and $\sigma$ i.e., mean and variance only.

*Case II*: Assume that the investor has the following utility function:

$$U(W) = aW + bW^2$$

where $a$ and $b$ are some constants. This is called the quadratic utility function. For risk averse investors $b < 0$. Now,

$$\begin{aligned} U(W + \tilde{Z}) &= a(W + \tilde{Z}) + b(W + \tilde{Z})^2 \\ &= aW + a\tilde{Z} + b(W^2 + \tilde{Z}^2 + 2W\tilde{Z}) \\ &= aW + a\tilde{Z} + bW^2 + b\tilde{Z}^2 + 2bW\tilde{Z} \end{aligned}$$

Taking expectation,

$$\begin{aligned} E(U(W + \tilde{Z})) &= (aW + bW^2) + (a + 2bW)E(\tilde{Z}) + bE(\tilde{Z}^2) \\ &= aW + bW^2 + (a + 2bW)\mu + b(\mu^2 + \sigma^2) \\ &= a(W + \mu) + b(W^2 + 2W\mu + \mu^2) + b\sigma^2 \\ &= a(W + \mu) + b(W + \mu)^2 + b\sigma^2 \end{aligned}$$

Once again, to maximize $E(U(W + \tilde{Z}))$ the investor needs to consider $\mu$ and $\sigma^2$ only (a, b and W are fixed). Since $b < 0$ for a risk averse investor, we have

$$E(U + (W + \tilde{Z})) = a(W + \mu) - |b|(W + \mu)^2 - |b|\sigma^2$$

Although, it is not obvious from the derivations above, it can be shown that for a risk averse person the tradeoff between risk and return is positive ($\frac{d\mu}{d\sigma}$ is positive). This means that a risk averse person will ask for a higher expected (or mean) return if the risk (as measured by standard deviation) is increased.

Therefore, the mean and the variance (or the standard deviation) are sufficient statistics about the return on an investment if *either* of the two conditions hold:

(a) The returns are distributed normally.

(b) The investors have a quadratic utility function.

Neither of these assumptions are satisfied exactly in the real world. The second assumption is more difficult to justify. There is some justification for the first assumption. So finance goes on.

# A Mathematical Appendix

## A.1 Taylor Expansion

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \cdots$$

where,

$$\begin{aligned} f(x) &= \text{a function of } x \\ f(x_0) &= \text{value of } f(x) \quad \text{at } x = x_0 \\ f'(x_0) &= \frac{df}{dx} \quad \text{at } x = x_0 \\ f''(x_0) &= \frac{d^2 f}{dx^2} \quad \text{at } x = x_0 \qquad \text{etc.} \end{aligned}$$

and,

$$\begin{aligned} 2! &= 2 \times 1 \\ 3! &= 3 \times 2! \\ 4! &= 4 \times 3! \qquad \text{etc.} \end{aligned}$$

Note: The higher order terms can be neglected if $h << x_0$.

## A.2 Valuation of $E(\tilde{Z}^2)$

$$\begin{aligned} Var(\tilde{Z}) &= E(\tilde{Z} - \mu)^2 \qquad \mu = E(\tilde{Z}) \\ &= E(\tilde{Z}^2 + \mu^2 - 2\mu\tilde{Z}) \\ &= E(\tilde{Z}^2) + \mu^2 - 2\mu E(\tilde{Z}) \\ &= E(\tilde{Z}^2) + \mu^2 - 2\mu^2 \\ &= E(\tilde{Z}^2) - \mu^2 \end{aligned}$$

or,

$$E(\tilde{Z}^2) = \mu^2 + \sigma^2$$