Some Comments on the Utility Function

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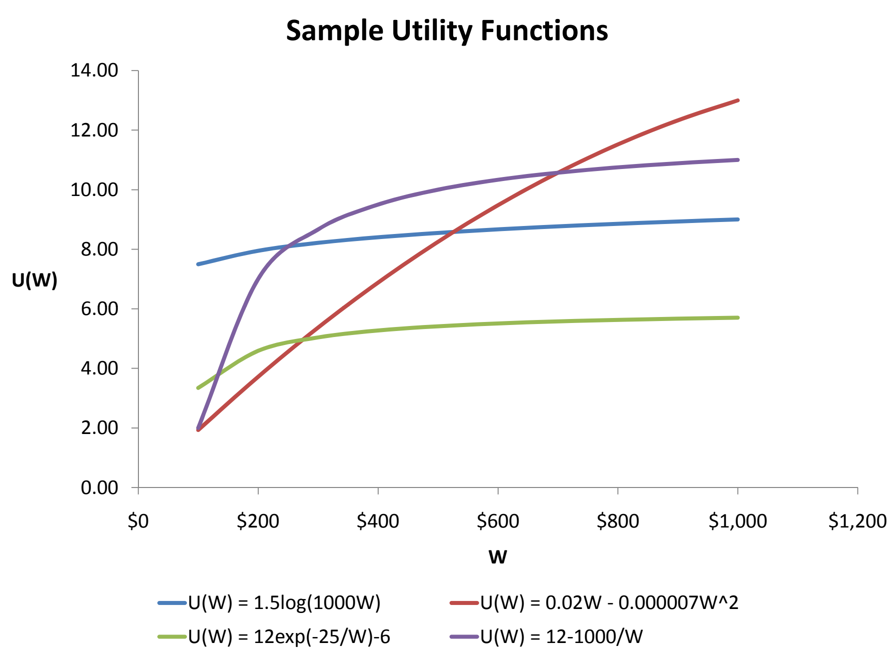
- Utility is the pleasure or satisfaction derived from consumption of goods and services. We assume that wealth and consumption are directly related, and define utility as a function of wealth.
- The functional form of the utility function is based on two premises:
  - The higher the wealth, the higher the utility. So, the slope of the utility function, i.e., the first derivative of the utility function, should be positive.
  - The marginal utility of incremental dollar diminishes as wealth increases. For example, the incremental utility of an additional \$100 for a person who has \$1,000 is higher than that for someone who has \$100,000. So, the slope should decrease with wealth, i.e., the second derivative of the utility function should be negative.
- Some common functional forms that satisfy the conditions listed above are:

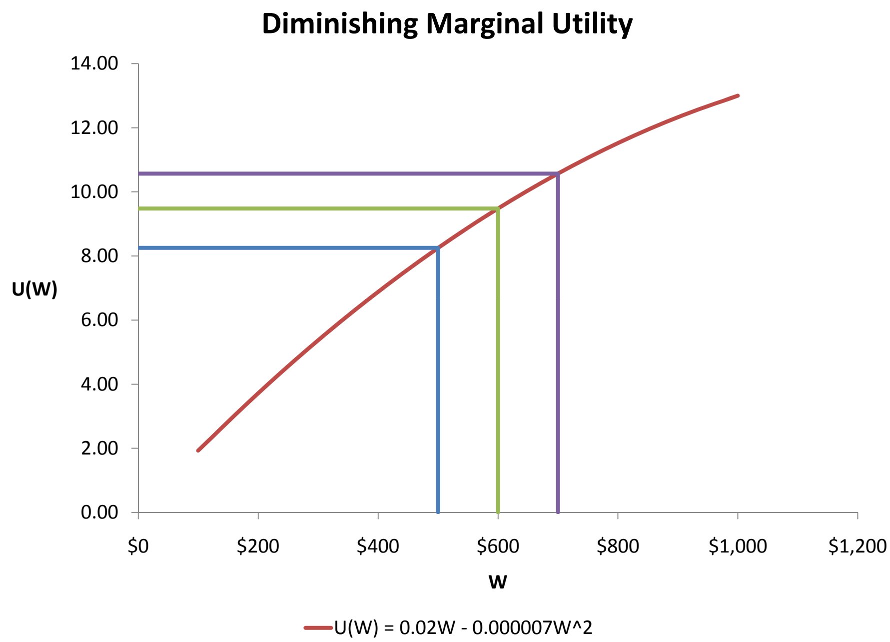
$$U(W) = 1.5\log(1000W) \tag{1}$$
$$U(W) = 0.02W - 0.000007W^2 \tag{2}$$
$$U(W) = 12e^{-\frac{25}{W}} - 6 \tag{3}$$
$$U(W) = 12 - \frac{1000}{W} \tag{4}$$

- The chart below shows the functions graphically:[1]



[1] Some of these utility functions obey the premises listed above only in a narrow range of wealth.

- Because of diminishing marginal utility, the impact of a loss of \$100 on utility is bigger than that of a gain of \$100. The figure below uses the quadratic utility function above ($U(W) = 0.02W - 0.000007W^2$) to illustrate the point. Using this function you can see that $U(\$500) = 8.25$, $U(\$600) = 9.48$ and $U(\$700) = 10.57$. Suppose a person with this utility function has the wealth of \$600, which would lead to a utility of 9.48. If the person were to lose \$100 to bring the wealth down to \$500, the utility would drop to 8.25, a decrease of $9.48 - 8.25 = 1.23$. On the other hand, if this person were to gain \$100 to bring the wealth up to \$700, the utility would rise to 10.57, an increase of $10.57 - 9.48 = 1.09$. So, the loss of utility from the loss of \$100 is larger than the gain of utility from a gain of \$100. Therefore, a prospect of losing and gaining \$100 with equal probabilities will be shunned by people. In order for a 50:50 prospect to be desirable, the value of the possible gain must be larger than the value of the possible loss (say \$120:\$100). This behavior is known as *risk aversion*.



- Let's consider a game involving a series of coin tosses. In the first toss, if the coin shows "head," you get \$1, and the game is over. If the coin shows "tail", the coin is tossed a second time. In the second toss, if the coin shows "head," you get \$2 and the game is over. If the coin shows "tail," the coin is tossed a third time. If the coin shows "head" in this toss, you get \$4, and the game is over. If the coin shows tail, it is tossed again. In summary, the coin is tossed until it shows head, and the payoff is doubled in successive tosses. The question is, how much would you be willing to pay to play this game. The *expected payoff* of this game is infinity: There is a 1/2 chance of getting \$1 (H), 1/4 chance of getting \$2 (TH), 1/8 chance of getting \$4 (TTH), etc.[2] So, the expected payoff is:

$$\frac{1}{2} \times \$1 + \frac{1}{4} \times \$2 + \frac{1}{8} \times \$4 + \frac{1}{16} \times \$8 + \cdots = \infty$$

  Obviously nobody would pay an infinite sum of money to play this game even though the expected payoff is infinity. That divergence of price from expected payoff is known as St. Petersburg Paradox. How do we explain this reluctance to pay for the game despite the fact that the expected value of the game is infinity?

- Eighteenth century mathematician Daniel Bernoulli resolved the paradox by suggesting that the proper metric to use in decisions involving uncertainty is expected utility of wealth ($E(U(W))$), instead of

[2]The letters in the parentheses indicate the results of coin tosses, with T for tail and H for head. Remember that the game ends when a head shows.

expected wealth ($E(W)$) or utility of expected wealth ($U(E(W))$). The expected utility of the game described above is finite, and the person will pay a finite sum to play the game depending on his/her utility function.

- It can be shown that if either the investor has a quadratic utility function (of the form $U(W) = aW + bW^2$) or the ending wealth $W = W_0(1+r)$ has a normal distribution because return r has a normal distribution, the expected utility of wealth is a function of expected return ($E(r)$) and standard deviation (σ).[3] Specifically, the functional form of expected utility is,

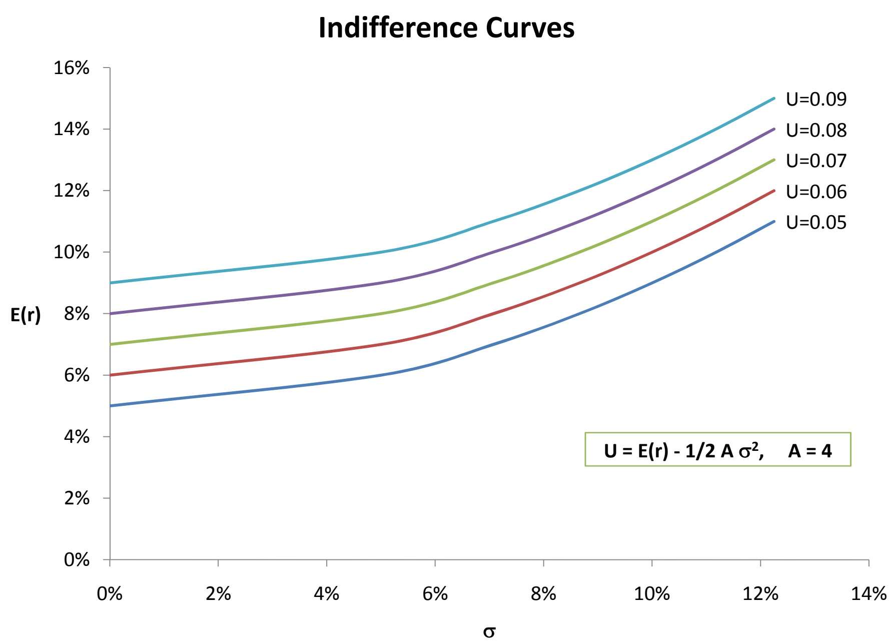
$$\text{Expected utility} = E(U) = aE(r) + b\sigma^2$$

Since expected utility is positively related to the expected return and negatively related to the standard deviation for risk averse investors, a reasonable form for the expected utility is

$$E(U) = E(r) - {}^1\!/_2 A\sigma^2$$

where A is a measure of the investor's risk aversion.

- Combinations of $E(r)$ and σ that provide the same level of utility are known as indifference sets, Plotting them on a chart with σ $E(r)$ as the axes gives us the *indifference curves*. The figure below shows the indifference curves for various values of expected utility. The investor would like to be on the highest possible indifference curve given the opportunities available.



Here is the process for creating the indifference curves for, say $E(U) = 0.06$ and $A = 4$.

$$E(U) = E(r) - {}^1\!/_2 A\sigma^2$$
$$0.06 = E(r) - {}^1\!/_2 \times 4\sigma^2$$
$$\sigma = \sqrt{\frac{0.06 - E(r)}{2}}$$

Now use some value of $E(r)$ (e.g., 6%) and calculate σ using the equation.[4] Repeat this process for various values of σ and make a table of sets of ($E(r)$, σ). Plot these points with σ on the x-axis and $E(r)$ on the y-axis.

[3]See my document titled *Notes on Utility Theory* for full mathematical details.
[4]Don't try a value of $E(r)$ less than $E(U)$.