MATH 8030 Homework 10

Directions: Complete all problems. Upload solutions to Problems 1 and 4 by 11:59 pm Thursday, April 17.

1. Suppose cars enter a one-way infinite highway, represented by $[0, \infty)$, according to $\{N(t) : t \geq 0\}$, a Poisson process with rate λ (i.e., one car enters at location 0 at each event time T_i). The ith car to enter chooses a velocity V_i and travels at this velocity. Assume that $\{V_i\}$ are i.i.d. positive random variables with distribution function F.

   (a) For $(a, b) \subset [0, \infty)$ and $t > 0$, find the distribution of the number of cars located in (a, b) at time t.

   (b) Suppose the speed limit on this highway is $v > 0$. Find the joint distribution of the number of cars going above the speed limit and the number going below the speed limit located in a given interval $(a, b) \subset [0, \infty)$ at time t.

2. Let $\{N(t) : t \geq 0\}$ be a nonhomogeneous Poisson process with intensity $\alpha(t)$, which is a continuous function of t. Define the event times $\{T_n : n \geq 0\}$ by $T_0 = 0$ and $T_n = \inf\{t \geq 0 : N(t) = n\}$, $n \geq 1$.

   (a) Compute the density function of T_n for $n \geq 1$.

   (b) Are the interarrival times $\{T_n - T_{n-1} : n \geq 1\}$ independent? Are they identically distributed?

3. Let $\{N(t) : t \geq 0\}$ be a nonhomogeneous Poisson process with intensity $\alpha(t)$. Let $\mu(t) = \int_0^t \alpha(s)\, ds$, and define its "inverse"

$$\mu^{-1}(t) = \inf\{s \geq 0 : \mu(s) \geq t\}.$$

   Verify that $\{N(\mu^{-1}(t)) : t \geq 0\}$ is a homogeneous Poisson process with rate 1.

4. Recall the context of time-dependent thinning of a rate-λ Poisson process $\{N(t)\}$: For $\{p_1(t), \dots, p_m(t) : t \geq 0\})$ where $\sum_{j=1}^m p_j(t) = 1$ for every t, the labels $\{L_n : n \geq 0\}$ are $\{1, \dots, m\}$-valued random variables with distribution defined by

$$P(L_1 = j_1, \dots, L_n = j_n | T_1, \dots, T_n) = \prod_{k=1}^n p_{j_k}(T_k),$$

   for any $j_1, \dots, j_n \in \{1, \dots, m\}$ and $n \geq 0$. For each $1 \leq j \leq m$, the process $N_j(t)$ counts the number of j-labeled events that have occurred by time t. Show that $N_j(t)$ is a nonhomogeneous Poisson process with intensity $\lambda p_j(t)$.