HW05

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3. Given an i.i.d. sequence $\{V_n : n \geq 0\}$ of random variables taking values in some space E, and functions $g, h : S \times E \to S$, define the process $\{X_n : n \geq 0\}$ as follows:

$$X_0 = g(j, V_0) \text{ for some } j \in S,$$
$$X_n = h(X_{n-1}, V_n) \text{ for all } n \geq 1.$$

(a) Show that $\{X_n : n \geq 0\}$ is a Markov chain.

(b) Consider the following model of a single-server queue: Customers arrive to a server and are served on a first come, first served basis. Between times $n-1$ and n, the number of customer arrivals is a random variable A_n taking values in $\{0, 1, \dots\}$ with mass function

$$\alpha_k = P(A_n = k), \quad k = 0, 1, \dots.$$

The service time of each customer is deterministically 1, i.e., between times $n-1$ and n exactly one customer (if any) is served and leaves the system. Let X_n be the number of customers present in the system at time n. Use part (a) to conclude that $\{X_n : n \geq 0\}$ is a Markov chain, and find its transition probabilities.

a) WTS $P(X_n | X_0, \dots, X_{n-1}) = P(X_n | X_{n-1})$

Since $\{V_i\}$ are iid, given $m+1 > n$ V_m is independent of X_n because X_n depends on previous V_i's and V_m has not been used to construct a X_i.

When $m = n+1$ $X_m = h(X_n, V_m)$ so it is dependent on V_m.

$$P(X_n | X_0, \dots, X_{n-1}) = P(h(X_{n-1}, V_n) | g(j, V_0), h(X_0, V_1), \dots, h(X_{n-2}, V_{n-1}))$$
$$= P(h(X_{n-1}, V_n) | h(X_{n-2}, V_{n-1}))$$
$$= P(X_n | X_{n-1})$$

Thus $\{X_n | n \geq 0\}$ is a Markov Chain

b) Let $X_0 = k$ $k \in \mathbb{Z}^+$ (nonnegative)

Let $X_n = \max(0, X_{n-1} - 1) + \alpha_n$

This lets us start at some number of people waiting and each point in the future is determined by the previous amount and arrivals. The max tells us if everyone at time n was serviced between n-1 and n.

These are two functions as in part a for X_n and X_0. It follows $\{X_n | n \geq 0\}$ is a MC.

$$P_{ij} = P(X_n = j \mid X_{n-1} = i) = P(\max(0, i-1) + \alpha_n = j)$$

4. Suppose C is a closed and finite communicating class. Let $\eta_i = \inf\{k \geq 1 : X_k = i\}$ for $i \in C$.

   (a) Show that there exists an integer $n > 0$ and an $\varepsilon \in (0,1)$ with the following property: For any states $i, j \in C$, there exists $m \leq n$ such that $p_{ij}^{(m)} \geq \varepsilon$.

   (b) Suppose $k \geq 0$ is an integer and $i, j \in C$. Show that $P_i(\eta_j > kn) \leq (1-\varepsilon)^k$, where n and ε are as in part (a).

   (c) Use part (b) to conclude that every state in C is positive recurrent.

a) Let $i,j \in C$.

Since C is a closed and finite communicating class $i \leftrightarrow j$ and $\exists\, m_{ij} \in \mathbb{Z}^+ \ni P_{ij}^{(m_{ij})} > 0$.

Let $n = \max_{i,j} m_{ij}$ the length of the longest path back.

Let $\Sigma = \min_{i,j,\, m \leq n} P_{ij}^{(m)} > 0$.

Thus $\forall i,j \in C \;\exists\, n > 0 \ni \exists\, m \leq n \ni P_{ij}^{(m)} > \Sigma$.

b) From the hint given

$P_i(\eta_j > kn) \leq (1-\Sigma) P_i(\eta_j > (k-1)n)$ since in part a it was shown $P_i(X_1 \neq j, \ldots, X_n \neq j) = 1 - \Sigma$.

It follows

$$P_i(\eta_j > kn) \leq (1-\Sigma)^2 P_i(\eta_j > (k-2)n) \leq \ldots \leq (1-\Sigma)^k P_i(\eta_j > n)$$
$$\leq (1-\Sigma)^k$$

c) WTS $E_i[\eta_i] < \infty$

$$E_i[\eta_i] = \sum_{n=0}^{\infty} P_i(\eta_i > n) = \sum_{n=0}^{km-1} P_i(\eta_j > n) + \sum_{n=km}^{\infty} P_i(\eta_j > n)$$
$$\leq \sum_{n=0}^{km-1} P_i(\eta_j > n) + \sum_{n=km}^{\infty} (1-\Sigma)^n$$
$$= \sum_{n=0}^{km-1} P_i(\eta_j > n) + \frac{(1-\Sigma)^{km}}{\Sigma}$$

$$= \sum_{n=0}^{km-1} P_i(\eta_j > n) + \frac{(1-\varepsilon)^{km}}{\varepsilon}$$

$$< \infty$$

Thus $\forall i \in C$ i is positive recurrent