MATH 8030 Homework 9

Directions: Complete all problems. Upload solutions to Problems 3 and 4 by 11:59 pm Thursday, April 3.

When not otherwise stated, $\{N(t) : t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ and event times $\{T_n : n \geq 1\}$.

1. For $0 < s < t$, compute the following.
   (a) $E[N(t)^2 | N(s)]$,
   (b) $\text{Cov}\,(N(s), N(t))$,
   (c) The joint distribution of $T_{N(s)}$ and $T_{N(t)}$.

2. Recall the defining properties of $\{N(t)\}$: (i) $N(0) = 0$ with probability 1, (ii) $N(t)$ has independent increments, and (iii) $N(t) \sim \text{Poisson}(\lambda t)$ for all $t \geq 0$.
   (a) Verify that $N(t)$ has stationary increments. That is, show that (i), (ii), and (iii) imply that $N(t) - N(s) \overset{d}{=} N(t - s)$ for all $s < t$.
   (b) Verify that (i)–(iii) imply
   $$P(N(t, t + h] = 1) = \lambda h + o(h), \qquad h \downarrow 0,$$
   and
   $$P(N(t, t + h] \geq 2) = o(h), \qquad h \downarrow 0,$$
   for any $t \geq 0$.

3. Let $0 < s < t$.
   (a) Find the conditional distribution of $N(s)$ given $N(t)$.
   (b) Compute $E[T_k | N(t)]$ for all $k \geq 1$.

4. Let $\{M(t) : t \geq 0\}$ be a Poisson process with rate μ and event times $\{S_n : n \geq 1\}$ independent of $\{N(t)\}$ and $\{T_n\}$. Determine if the process $\{L(t) : t \geq 0\}$ is a Poisson process, and if so compute its rate.
   (a) $L(t) = N(t) + M(t)$.
   (b) $L(t)$ is the counting process on $[0, \infty)$ with event times $R_n = \min\{S_n, T_n\}$, $n \geq 1$.
   (c) $L(t)$ counts the number of intervals $(k - 1, k]$, $k = 1, 2, \ldots, N(t)$ for which $M(k - 1, k] = 0$, i.e., on which the process $\{M(t)\}$ sees no events.