HW01

Monday, January 13, 2025 10:39 AM

4. (a) Suppose X and Y are random variables such that

$$E[X|Y] = 18 - \frac{3}{5}Y \qquad \text{and} \qquad E[Y|X] = 10 - \frac{1}{3}X.$$

Find $E[X]$ and $E[Y]$.

(b) Suppose $X_1, X_2, X_3, \ldots$ be i.i.d. random variables where $\mu = E[X_1]$ exists, and let $S_n = X_1 + \cdots + X_n$ for each $n \geq 1$. For $n, m \geq 1$, find an explicit formula (in terms of n, m, S_n, and μ) for $E[S_m|S_n]$. *(Hint: The $m \leq n$ and $m > n$ cases are different. Exploit the "symmetry" that $E[X_i|S_n]$ does not depend on the value of i, but only whether or not $i \leq n$ or $i > n$.)*

a) $E[X] = E[E[X|Y]] = E[18 - \frac{3}{5}Y] = 18 - \frac{3}{5}E[Y]$

$E[Y] = E[E[Y|X]] = E[10 - \frac{1}{3}X] = 10 - \frac{1}{3}E[X]$

$$\left[\begin{array}{cc|c} 1 & 3/5 & 18 \\ 1/3 & 1 & 10 \end{array}\right] \Rightarrow \begin{array}{l} E[X] = 15 \\ E[Y] = 5 \end{array}$$

b) $\{X_i\} \sim$ iid $\quad \mu = E[X_1]$ exists $\quad S_n = \sum_{i=1}^{n} X_i \quad n \geq 1$

Find $E[S_m|S_n]$

Case 1 $\quad m \leq n$

$$E[S_m|S_n] = \sum_{i=1}^{m} E[X_i|S_n]$$
$$= mE[X_1|S_n]$$
$$= \frac{m}{n}E[nX_1|S_n]$$
$$= \frac{m}{n}S_n$$

Since $\{X_i\}$ are idd

Case 2 $\quad m > n$

$$E[S_m|S_n] = \sum_{i=1}^{m} E[X_i|S_n]$$
$$= \sum_{i=1}^{n} E[X_i|S_n] + \sum_{i=n+1}^{m} E[X_i|S_n]$$
$$= \frac{n}{n}S_n + (m-n+1)E[X_1]$$

$\{X_i\}$ iid

$= \frac{n}{n} S_n + (m-n+1) E[X_i]$

$= S_n + (m-n+1)\mu$

5. (Ross 1.6) Let $X_1, X_2, \ldots$ be i.i.d. continuous random variables and let $X_0 = -\infty$. We say that a record occurs at time $n \geq 1$ and has value X_n when $X_n > \max\{X_1, \ldots, X_{n-1}\}$.

   (a) Let $A_n = \{\text{a record occurs at time } n\}$, $n \geq 1$. Compute $P(A_n)$ and explain why $A_1, A_2, \ldots A_n$ are independent for each n.

   (b) Let N_n denote the total number of records that have occurred up to (and including) time n. Find $E[N_n]$ and $\mathrm{Var}(N_n)$. *(Hint: Use indicator variables.)*

   (c) Let $T = \min\{n \geq 2 : A_n \text{ occurs}\}$ be the time of the first record (other than X_1). Compute $P(T > n)$ and show that $P(T < \infty) = 1$ and $E[T] = \infty$. *(Hint: by continuity of probability, $P(T = \infty) = \lim_{n\to\infty} P(T > n)$.)*

   (d) Let T_y denote the time of the first record value greater than y, i.e., $T_y = \min\{n \geq 1 : X_n > y\}$. Show that T_y and X_{T_y} are independent, i.e., the time of the first value greater than y is independent of that value.

$\{X_i\}_{i\in\mathbb{N}}$ cont iid $X_0 = -\infty$

$n \geq 1$ $X_n > \max\{X_i\}_{i=0}^{n-1}$

a) $\{A_i\}$ is independent since only 1 event can be a record at a time

$\{A_i\}$ will be distributed uniformly in t. Thus $P(A_n) = \frac{1}{n}$

b) $N_n = \sum_{k=1}^{n} 1_{A_k}$

$E[N_n] = \sum_{k=1}^{n} P(A_k)$

$= \sum_{k=1}^{n} \frac{1}{k}$

$V[N_n] = E[(N_n)^2] - (E[N_n])^2$

Let $i<j$

$E[N_n^2] = E[(\sum_{k=1}^{n} 1_{A_k})^2]$

$= E[\sum_{l=1}^{n} 1_{A_l}^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} 1_{A_i \cap A_j}]$

$= \sum_{l=1}^{n} \frac{1}{l} + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} P(A_i \cap A_j)$

$$= \sum_{l=1}^{n} \frac{1}{l} + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} P(A_i \cap A_j)$$

$$= \sum_{l=1}^{n} \frac{1}{l} + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} P(A_i) P(A_j)$$

$$V[N_n] = \sum_{k=1}^{n} \frac{1}{k} + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{1}{ij} - \left(\sum_{k=1}^{n} \frac{1}{k}\right)^2$$

c) $P(T > n) = P(2 < X_k < X_n \mid X_k \text{ is not a record})$

$$= \prod_{k=2}^{n} P(X_k \text{ not a record})$$

$$= \prod_{k=2}^{n} 1 - P(X_k \text{ is a record})$$

$$= \prod_{k=2}^{n} 1 - \frac{1}{k}$$

$$= \prod_{k=2}^{n} \frac{k-1}{k}$$

$$= \frac{1}{\cancel{2}} \cancel{\frac{2}{3}} \cancel{\frac{3}{4}} \cdots \cancel{\frac{n-1}{n}}$$

$$= \frac{1}{n}$$

$$P(T = \infty) = \lim P(T > n) = \lim \frac{1}{n} = 0$$

$$P(T < \infty) = 1 - P(T = \infty) = 1$$

$$E[T] = \sum_{k=0}^{\infty} P(T > k) = \sum_{k=0}^{\infty} \frac{1}{k} = \infty$$

d) $P(X_n > \{X_i\}_{i=1}^{n-1} \mid n = T_y) = P(X_1 < X_{T_y})^{n-1}$

$$= P(X_1 < X_n)^{n-1}$$

$$= P(X_n > \{X_i\}_{i=1}^{n-1})$$

$$P(T_y = n \mid X_n > \{X_i\}_{i=1}^{n-1}) = P(\{X_{T_y} \geq y\} \cap \{X_{T_y} > \{X_i\}_{i=1}^{n-1}\})$$

$$= P(X_1 < X_{T_y})^{n-1} P(X_{T_y} > y)$$

$$= P(X_1 < X_n)^{n-1} P(X_n > y)$$

$$= P(T_y = n)$$

Thus T_y and X_{T_y} are independent

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