HW08

Monday, March 24, 2025 1:39 PM

2. The distribution of a random variable X is said to be *memoryless* if

$$P(X > s + t | X > s) = P(X > t)$$

holds for all $s, t \in \mathbb{R}$. For this problem, the Geom(p) distribution is the one with values $k \in \{1, 2, 3, \ldots\}$ and mass function $P(X = k) = (1 - p)^{k-1} p$.

(a) Verify that the Geom(p) and Exp(λ) distributions are memoryless.

(b) Consider a random variable $X \geq 0$ with a memoryless distribution. Show that if X is continuous then it has an exponential distribution, and if X is $\{1, 2, \ldots\}$-valued then it has a geometric distribution.

a) • $P(X=x) = \lambda e^{-\lambda x}$ $\quad$ CDF $= 1 - e^{-\lambda x}$

$$P(X>s+t|X>s) = \frac{P(X>s+t)}{P(X>s)} = \frac{1-1+e^{-(s+t)\lambda}}{1-1+e^{-s\lambda}} = e^{-t\lambda} = P(X>t)$$

• $P(X=k) = (1-p)^{k-1}p$ $\quad$ CDF $= 1-(1-p)^x$

$$P(X>s+t|X>s) = \frac{P(X>s+t)}{P(X>s)} = \frac{1-1+(1-p)^{s+t}}{1-1+(1-p)^s} = (1-p)^t = P(X>t)$$

b) • Let X be continuous

We know $P(X>s+t|X>s) = P(X>t)$

$$\frac{P(X>s+t)}{P(X>s)} = P(X>t) \Rightarrow P(X>s+t) = P(X>t)P(X>s)$$

Note $P(X>0)=1$. Let $P'(X>0) = -\lambda$.

It follows The hint said to assume P is differentiable

$$P'(X>t) = \lim_{s\to 0} \frac{P(X>s+t) - P(X>t)}{s}$$

$$= \lim_{s\to 0} \frac{P(X>t)(P(X>s)-1)}{s}$$

$$= P(X>t)P'(X>0) \quad \text{(l'Hopital)}$$

$$\frac{P'(X>t)}{P(X>t)} = -\lambda$$

Also suggested in the hint

$$P(X>t) = Ce^{-\lambda t} \Rightarrow P(X<t) = 1 - e^{-\lambda t}$$ (Set $C=1$ so dist)

Thus if X is continuous it is exponential by uniqueness of CDFs

- Let X be a $\mathbb{N}$ valued rv

We know $P(X>s+t)=P(X>s)P(X>t)$.

Let $P(X>1)=1-p$ $\quad p\in[0,1]$

It follows

$$P(X>s+1)=P(X>s)P(X>1)$$
$$=(1-p)P(X>s)$$

$$\Rightarrow P(X>n)=(1-p)^n$$

$$\Rightarrow P(X\leq n)=1-(1-p)^n$$

$\therefore$ X is a geometric dist by uniqueness of CDFs