MATH 8030 Homework 4 Hints

2. (a) If f is one-to-one on S, then $\{f(X_n) : n \geq 0\}$ is a Markov chain, since then

$$P(f(X_{n+1}) = j | f(X_0) = i_0, \ldots, f(X_{n-1}) = i_{n-1}, f(X_n) = i)$$
$$= P(X_{n+1} = f^{-1}(j) | X_0 = f^{-1}(i_0), \ldots, X_{n-1} = f^{-1}(i_{n-1}), X_n = f^{-1}(i))$$
$$= p_{f^{-1}(i), f^{-1}(j)},$$

where $\{p_{ij}\}$ are transition probabilities for the chain $\{X_n : n \geq 0\}$. (b) The independence assumption is important.

3. Follow the analysis of the general two-state chain we did in class. You may use a computer to find eigenvalues and eigenvectors or invert a matrix. However, finding the eigenvalues of P by hand by factoring the degree 3 polynomial $\det(P - \lambda I)$ becomes easier upon recognizing that part (a) means 1 is a root. It is also not necessary to diagonalize P if you compute P^n another way.

4. Start with $E_i[S_T]$ and recall we found the mass function of S_T with respect to P_i in class. $E_i[T]$ can be computed by deriving a difference equation with boundary values, however it is quicker in the $p \neq 1/2$ case to recognize that S_T is a random sum and use Wald's identity. When $p = 1/2$, use the method of first step analysis.

5. Model the chain with 3 as an absorbing state. As an example, we know that if instead we considered the number of tosses until one H, the expected value is $1/p$, since this takes a Geom(p) number of tosses. If $\{Y_n\}$ is the chain where we stop after seeing a single H, then $S = \{0, 1\}$ and the transition matrix is

$$P = \begin{pmatrix} 1-p & p \\ 0 & 1 \end{pmatrix}.$$

If $\eta = \inf\{n \geq 0 : Y_n = 1\}$, then a first step analysis with $g(i) = E_i[\eta]$ gives

$$g(0) = 1 + (1-p)g(0) + pg(1), \qquad g(1) = 0.$$

Solving yields $g(0) = 1/p$, as expected.