MATH 8030 Homework 12

Directions: Complete all problems. Upload solutions to Problems 1 and 2 by 11:59 pm Thursday, April 24.

1. Recall the context of time-dependent thinning of a rate-λ Poisson process $\{N(t)\}$: For $\{p_1(t), \ldots, p_m(t) : t \geq 0\})$ where $\sum_{j=1}^m p_j(t) = 1$ for every t, the labels $\{L_n : n \geq 0\}$ are $\{1, \ldots, m\}$-valued random variables with distribution defined by

$$P(L_1 = j_1, \ldots, L_n = j_n | T_1, \ldots, T_n) = \prod_{k=1}^{n} p_{j_k}(T_k),$$

for any $j_1, \ldots, j_n \in \{1, \ldots, m\}$ and $n \geq 0$. For each $1 \leq j \leq m$, the process $N_j(t)$ counts the number of j-labeled events that have occurred by time t. For $s < t$, show that the increments $N_j(s)$ and $N_j(s, t]$ are independent. (This can be generalized to any number of increments. Along with results from class, this shows that $N_j(t)$ is a nonhomogeneous Poisson process with intensity $\lambda p_j(t)$.)

2. Let $\{N(t) : t \geq 0\}$ be a homogeneous Poisson process with rate λ and points $\{T_n : n \geq 0\}$. It can be shown that $P\big(\lim_{t\to\infty} N(t) = \infty\big) = 1$, and you should assume this to do this problem.

   (a) Find constants $a, b \in \mathbb{R}$ so that

$$P\Big(\lim_{n\to\infty} \frac{T_n}{n} = a\Big) = 1 \qquad \text{and} \qquad P\Big(\lim_{t\to\infty} \frac{N(t)}{t} = b\Big) = 1.$$

   (b) Find a constant $c \in \mathbb{R}$ and a distribution $\mathcal{D}$ so that for the sequence $\{N(k) : k \in \mathbb{N}\}$,

$$\frac{N(k) - ck}{\sqrt{k}} \Rightarrow \mathcal{D}$$

   as $k \to \infty$.

   (c) Do limits analogous to part (a) hold for a nonhomogeneous Poisson process $\{N_\alpha(t) : t \geq 0\}$ with (local) intensity $\alpha(t)$?

3. Let $\{N(t) : t \geq 0\}$ be a conditional Poisson process driven by a positive random variable Λ with c.d.f. F. That is, conditional on $\Lambda = \lambda$, $N(t)$ is a homogeneous Poisson process with rate λ.

   (a) If $\{T_n : n \geq 1\}$ are the points of $N(t)$, what is the conditional distribution of $T_1, \ldots, T_n$ given $N(t) = n$?

   (b) Suppose $\Lambda \sim \text{Exp}(\mu)$ and fix $t \geq 0$. Compute (i) the distribution of $N(t)$ for each t, and (ii) the conditional distribution of Λ given $N(t) = n$.