MATH 8030 Homework 3

Directions: Complete all problems. Upload solutions to Problems 2, 3, and 4 by 11:59 pm Thursday, January 30.

1. Suppose that for each $n \in \mathbb{N}$, $X_n \sim \text{Geom}(p_n)$, so that

$$P(X_n = k) = (1 - p_n)^{k-1} p_n, \qquad k = 1, 2, 3, \ldots.$$

Also suppose that $np_n \to \lambda > 0$ as $n \to \infty$. Find a sequence $\{a_n\}_{n=1}^{\infty}$ so that X_n/a_n converges in distribution to a (non-constant) random variable X as $n \to \infty$, and identify the distribution of X. (This limit can be seen as a complement to the law of rare events for the binomial distribution. While the $\text{Bin}(n, p_n)$ distribution counts the number of occurrences of (rare) events in n trials, the $\text{Geom}(p_n)$ distribution counts the number of trials until the first occurrence of one of these events. Thus we have an analogy: geometric is to binomial as the limit distribution of X_n/a_n is to Poisson. In other words, the limit distribution of X_n/a_n is a good model for the amount of time needed to wait until the first occurrence of an event that happens at rate λ. This construction becomes important later in the course when we consider stochastic processes in *continuous* time, rather than discrete time steps indexed by n.)

2. Consider the record process from Homework 1: $X_0 = -\infty$, and $\{X_i : i \geq 1\}$ are i.i.d. continuous random variables. We say a record happens at time n if the event $A_n = \{X_n > \max_{k<n} X_k\}$ occurs. Recall that $N_n = \sum_{i=1}^{n} 1_{A_i}$ denotes the number of records that have occurs up to (and including) time n. For fixed positive integer k, define the sequence of random variables $R_1^{(k)}, R_2^{(k)}, \ldots$ by

$$R_n^{(k)} = N_{(k+1)n} - N_{kn},$$

which denotes the number of records that occur during the length-n time interval $\{kn + 1, \ldots, (k+1)n\}$. Show that $R_n^{(k)} \Rightarrow \text{Poisson}(\lambda_k)$ as $n \to \infty$, and find λ_k.

3. For each $n \geq 1$, suppose that $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$ are i.i.d. random variables with cumulative distribution function F_n. Let $M_n = \max\{X_{1,n}, \ldots, X_{n,n}\}$, and for $x \in \mathbb{R}$, let $N_n^{(x)}$ denote the number of values among $\{X_{1,n}, \ldots, X_{n,n}\}$ that are larger than x. Finally, suppose M is a random variable with cumulative distribution function $0 < F < 1$. Show that $M_n \Rightarrow M$ as $n \to \infty$ if and only if $N_n^{(x)} \Rightarrow \text{Poisson}(-\log F(x))$ for all $x \in \mathbb{R}$ at which F is continuous.

4. Let $X_0 = 0$, and let $X_1, X_2, \ldots$ be i.i.d. $\{0, 1, 2, \ldots\}$-valued random variables with

$$P(X_1 = j) = \alpha_j, \qquad j = 0, 1, 2, \ldots.$$

We say a record occurs at time k if $X_k > \max\{X_1, \ldots, X_{k-1}\}$, and in this case we call X_k a record value. For each $n \geq 1$, let R_n denote the nth record value.

   (a) Argue that $\{R_n : n \geq 1\}$ is a Markov chain, and compute its transition probabilities.

   (b) Let T_n denote the time between the nth and $(n+1)$th records. Is $\{T_n : n \geq 1\}$ a Markov chain? What about $\{(T_n, R_n) : n \geq 1\}$? Compute transition probabilities for each in the case it is a Markov chain.