# MATH 8030 Homework 4

*Directions: Complete all problems. Upload solutions to Problems 3, 4, and 5 by 11:59 pm Thursday, February 6.*

1. $\{X_n : n \geq 0\}$ is a Markov chain on discrete state space $S$ and transition probabilities $\{p_{ij} : i, j \in S\}$. Suppose $i \in S$ is such that $p_{ii} > 0$, and let $\eta_i$ be the exit time from state $i$:
$$\eta_i = \inf\{n \geq 1 : X_n \neq i\}.$$
Find the distribution of $\eta_i$ with respect to $P_i(\cdot) = P(\cdot | X_0 = i)$.

2. $\{X_n : n \geq 0\}$ and $\{Y_n : n \geq 0\}$ are two independent Markov chains on discrete state space $S$. For each part, either show the given process is a Markov chain or provide a counterexample.
   (a) $\{f(X_n) : n \geq 0\}$ on discrete state space $S'$, where $f : S \to S'$ is a function.
   (b) $\{(X_n, Y_n) : n \geq 0\}$ on state space $S \times S$.
   (c) When $S \subset \mathbb{Z}$, $\{X_n + Y_n : n \geq 0\}$ on state space $S' = \{i + j : i, j \in S\}$.

3. Consider a Markov chain $\{X_n : n \geq 0\}$ with state space $S = \{0, 1, 2\}$ and transition matrix
$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$
   (a) Find all probability distributions $\pi$ on $S$ such that $\pi = \pi P$.
   (b) Given an initial distribution $X_0 \sim \mu_0$ on $S$, find a formula for $\mu_n = (P(X_n = 0), P(X_n = 1), P(X_n = 2))$ in terms of $n$.
   (c) Does $\mu_n$ converge to a distribution on $S$ as $n \to \infty$?

4. Consider the setup of the "Gambler's ruin" chain: $\{S_n : n \geq 0\}$ is a simple random walk on $\{0, 1, \ldots, N\}$ with transition probabilities $p_{i,i+1} = p$, $p_{i,i-1} = q = 1 - p$ for $i \in \{1, \ldots, N-1\}$ and absorbing states $\{0, N\}$. Recall that $P_i(\cdot) = P(\cdot | S_0 = i)$ and
$$T = \inf\{n \geq 0 : S_n \in \{0, N\}\}.$$
Compute $E_i[S_T]$ and $E_i[T]$ for each $i \in \{0, \ldots, N\}$.

5. You have a coin for which the chance of heads is $p \in (0, 1)$. Find the expected number of tosses of this coin needed to see three consecutive heads using the following steps.
   (a) You flip the coin repeatedly. Let $\{X_n : n \geq 0\}$ be the Markov chain on state space $S = \{0, 1, 2, 3\}$ where $X_n$ gives the number of consecutive heads you've seen immediately after the $n$th toss, and where you stop after you see three in a row. For example, $X_0 = 0$, and if the sequence of flips is $HTHHH$ then $X_1 = 1$, $X_2 = 0$, $X_3 = 1$, $X_4 = 2$, and $X_n = 3$, $n \geq 5$. Find the transition matrix of this chain.
   (b) Let $\tau = \inf\{n \geq 0 : X_n = 3\}$. Use a "first-step analysis" to write down a difference equation with boundary value for $f(i) = E_i[\tau]$ and solve it to find $E_0[\tau]$.