

# MATH 8030 Homework 7

*Directions: Complete all problems. Upload solutions to Problems 3 and 4 by 11:59 pm Thursday, March 13.*

*When not otherwise stated,  $\{X_n : n \geq 0\}$  is a Markov chain on discrete state space  $S$  with transition matrix  $P = \{p_{ij}\}_{i,j \in S}$ .*

1.  $\{X_n\}$  is said to be *reversible* with respect to a distribution  $\pi$  on  $S$  if

$$\pi_i p_{ij} = \pi_j p_{ji} \quad \text{for all } i, j \in S.$$

(Sometimes the above equalities are called the *detailed balance equations*.)

- Show that if  $\{X_n\}$  is reversible with respect to  $\pi$  then  $\pi$  is a stationary distribution for the chain. Give an example showing that the converse is false in general, i.e., that when a stationary distribution  $\pi$  exists the chain is not necessarily reversible with respect to  $\pi$ .
- Show that if  $\{X_n\}$  is reversible with respect to  $\pi$  then

$$P_\pi(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = P_\pi(X_0 = i_n, X_1 = i_{n-1}, \dots, X_n = i_0),$$

for any  $i_0, \dots, i_n \in S$  (hence the term “reversible”).

2. For the following transition matrices, find any stationary distributions and determine if there is a limiting distribution.

$$(a) S = \{0, 1, 2, 3, 4, 5\} \text{ and } P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix}.$$

$$(b) S = \{0, 1, 2, 3, \dots\} \text{ and } P = \begin{pmatrix} 1 - \theta & \theta & 0 & 0 & \dots \\ 1 - \theta & 0 & \theta & 0 & \dots \\ 1 - \theta & 0 & 0 & \theta & \dots \\ 1 - \theta & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ where } \theta \in (0, 1).$$

3. Define the process  $\{Y_n : n \geq 0\}$  by  $Y_n = (X_n, X_{n+1})$  for all  $n \geq 0$ .

- Show that  $\{Y_n\}$  is a Markov chain, and determine its state space and transition probabilities.
- Suppose that  $\{X_n\}$  has stationary distribution  $\pi$ . Find a stationary distribution for  $\{Y_n\}$  in terms of  $\pi$  and  $P$ .

4. Consider the reflected random walk defined on the previous homework assignment. Recall that the chain is transient when  $p > q$  and recurrent when  $p \leq q$ .
- (a) Compute the stationary distribution when  $p < q$  (thereby showing the chain is positive recurrent in that case), and show that the chain is null recurrent when  $p = q = 1/2$ .
  - (b) When  $p < q$ , what is the long run fraction of time the walk spends at odd states?