HW06

Wednesday, March 5, 2025 11:48 AM

3. Consider the "reflected simple random" walk on $S = \{0, 1, 2, \ldots\}$: for $p \in (0,1)$ and $q = 1-p$, $\{W_n : n \geq 0\}$ is the chain with transition matrix

$$P = \begin{pmatrix} q & p & 0 & 0 & 0 & \cdots \\ q & 0 & p & 0 & 0 & \cdots \\ 0 & q & 0 & p & 0 & \cdots \\ 0 & 0 & q & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

That is, $P(W_n = i+1 | W_{n-1} = i) = 1 - P(W_n = i-1 | W_{n-1} = i) = p$ for all $i \geq 1$ and $P(W_n = 1 | W_{n-1} = 0) = 1 - P(W_n = 0 | W_{n-1} = 0) = p$.

(a) Show that for any $n \geq 2$,

$$f_{20}^{(n)} = \sum_{k=1}^{n} f_{21}^{(k)} f_{10}^{(n-k)},$$

and therefore that $f_{20} = f_{21} f_{10}$.

(b) Argue that

$$f_{21} = f_{10}, \qquad f_{00} = q + p f_{10}, \qquad \text{and} \qquad f_{10} = q + p(f_{10})^2.$$

(c) Compute f_{00}, and determine recurrence or transience of the chain in the cases $p > q$ and $p \leq q$.

a) $f_{20}^{(n)} = P_2(\eta_0 = n)$ $\quad f_{21}^{(k)} = P_2(\eta_1 = k)$ $\quad f_{10}^{(n-k)} = P_1(\eta_0 = n-k)$

$$P_2(\eta_0 = n) = \sum_1^n P_2(\eta_0 = n, \eta_1 = k)$$
$$= \sum_1^n P_2(\eta_0 = n | \eta_1 = k) P_2(\eta_1 = k)$$
$$= \sum_1^n f_{21}^{(k)} P_1(\eta_0 = n-k)$$
$$= \sum_1^n f_{21}^{(k)} f_{10}^{(n-k)}$$

$$f_{20} = \sum_0^\infty f_{20}^{(n)} = \sum_{n=0}^\infty \sum_{k=1}^n f_{21}^{(k)} f_{10}^{(n-k)}$$

$1 \leq k \leq n$ $\quad k \leq n \leq \infty$

$1 \leq n \leq \infty$ $\quad 1 \leq k \leq \infty$

$$= \sum_{k=1}^\infty f_{21}^{(k)} \sum_{n=k}^\infty f_{10}^{(n-k)}$$
$$= f_{21} \sum_{l=0}^\infty f_{10}^{(l)}$$

$l = n-k$ $\quad f_{10}^{(0)} = 0$

$$= f_{21} f_{10}$$

b) $f_{ij} = \sum_0^\infty f_{ij}^{(n)}$ $\quad f_{21} = \sum_0^\infty P_2(\eta_1 = n)$ $\quad f_{10} = \sum_0^\infty P_1(\eta_0 = n)$

WTS $f_{21} = f_{10}$

Note $P_2(\eta_1 = 2n) = 0 = P_1(\eta_0 = 2n)$ because you have to take odd steps to go to an adjacent spot.

$P_2(\eta_1 = 1) = q$ $\quad P_2(\eta_1 = 3) = pqq$ $\quad P_2(\eta_1 = 2n+1) = \overbrace{p \ldots p}^{n} \overbrace{q \ldots q}^{n} q$

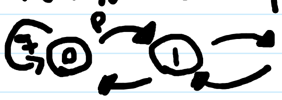
$P_2(\eta_1=1)=q \qquad P_2(\eta_1=3)=pqq \qquad \cdots \qquad P_2(\eta_1=2n+1)=\overbrace{p\ldots p}^{n}\overbrace{q\ldots q}^{n}q$

$P_1(\eta_0=1)=q \qquad P_1(\eta_0=3)=pqq \qquad\qquad P_1(\eta_0=2n+1)=\underbrace{p\ldots p}_{n}\underbrace{q\ldots q}_{n}q$

It follows $\quad P_2(\eta_1=n)-P_1(\eta_0=n)=0 \quad \forall n\in\mathbb{N}$

$$\Rightarrow \sum_0^\infty P_2(\eta_1=n)-P_1(\eta_0=n)=0$$
$$\sum_0^\infty P_2(\eta_1=n)=\sum_0^\infty P_1(\eta_0=n)$$
$$\therefore\ f_{21}=f_{10}$$

$P_0(\eta_0=1)=q \qquad P_0(\eta_0=k) \quad k>1$



$\Rightarrow \quad P_0(\eta_0=k)=p\,P_1(\eta_0=k)$

You can go $0\to0$ w.p. q and $0\to1$ w.p. p then starting at state 1 we are still interested in returning to 0.

Thus $\quad P_0(\eta_0=n)=q+pP_1(\eta_0=n)$

$$\sum_1^\infty P_0(\eta_0=n)=q+\sum_1^\infty pP_1(\eta_0=n)$$
$$f_{00}=q+pf_{10}$$

$P_1(\eta_0=1)=q \qquad P_1(\eta_0=k) \quad k>1$

$$P_1(\eta_0=k)=p\,P_2(\eta_0=k)$$

Then $\quad \sum_1^\infty P_1(\eta_0=n)=q+p\sum_1^\infty P_2(\eta_0=n)$

$$f_{10}=q+pf_{20}$$
$$=q+pf_{21}f_{10} \quad \leftarrow \text{part a}$$
$$=q+pf_{10}^2 \quad \leftarrow \text{part b}$$

c) $pf_{10}^2-f_{10}+q=0$

$f_{10}=\dfrac{1\pm\sqrt{1-4pq}}{2p} \qquad f_{00}=q+pf_{10}$

$1-4p(1-p)=1-4p+4p^2=(1-2p)^2$

$f_{10}=\dfrac{1\pm(1-2p)}{2p} \qquad$ so $\quad f_{10}=\dfrac{1+1-2p}{2p}=\dfrac{2(1-p)}{2p}=\dfrac{q}{p}$

$f_{10} = \frac{1 \pm (1-2p)}{2p}$ so $f_{10} = \frac{1+1-2p}{2p} = \frac{2(1-p)}{2p} = \frac{q}{p}$

or $f_{10} = \frac{1-1+2p}{2p} = 1$

Thus $f_{00} = q + p$ or $f_{00} = q + q$

$= 1$ $\quad\quad$ $= 2q$

If $q \geq p$, $q \geq \frac{1}{2}$ which means $2q \geq 1$ which is impossible since $f_{00} \in [0,1]$. So when $q \geq p$, 0 is recurrent because we have to choose the case that results in $f_{00} = 1$.

If $q < p$, $q < \frac{1}{2}$ which means $2q < 1$ and thus f_{00} can be less than 0. Therefore $q < p$ is transient.

4. (Resnick 2.16) Without benefit of dirty tricks, Happy Harry's restaurant business fluctuates in successive years between three states: 0 (bankruptcy), 1 (verge of bankruptcy), and 2 (solvency). The transition matrix giving the probabilities of evolving from state to state is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ .5 & .25 & .25 \\ .5 & .25 & .25 \end{pmatrix}.$$

(a) What is the expected number of years until Happy Harry's restaurant goes bankrupt, assuming that he starts from the state of solvency?

(b) Harry's rich uncle Zeke decides it is bad for the family name if his nephew Harry is allowed to go bankrupt. Thus when state 0 is entered, Zeke infuses Harry's business with cash, returning him to solvency with probability 1. Thus the transition matrix for this new Markov chain is

$$P' = \begin{pmatrix} 0 & 0 & 1 \\ .5 & .25 & .25 \\ .5 & .25 & .25 \end{pmatrix}.$$

What is the expected number of years between cash infusions from Zeke?

a) Let $\hat{P} = \begin{array}{c|ccc} & VB & S & B \\ \hline VB & 0.25 & 0.25 & 0.5 \\ S & 0.25 & 0.25 & 0.5 \\ B & 0 & 0 & 1 \end{array} = \begin{bmatrix} Q & R \\ 0 & \tilde{P} \end{bmatrix}$

$W_2 = [(I - Q)^{-1} \mathbb{1}]_2 = 2$

Q:matrix([0.25, 0.25],[0.25, 0.25]);

$$\begin{pmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{pmatrix}$$

I:matrix([1,0],[0,1]);

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

e:matrix([1],[1]);

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

invert(I−Q).e;

$$\begin{pmatrix} 2.0 \\ 2.0 \end{pmatrix}$$

b) $$P' = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0.25 & 0.25 \\ 0.5 & 0.25 & 0.25 \end{bmatrix}$$

Since $|S| < \infty$ $\exists!\ \pi \ni \pi = \pi P$

$v = [1\ \ 0.5\ \ 1.5]$

$\pi = \frac{v}{|v|} = \frac{1}{3}\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$

$\pi_0 = \frac{1}{3} \Rightarrow E_0[\eta_0] = 3$

P:matrix([0,0,1],[0.5,0.25, 0.25],[0.5,0.25, 0.25]);

$$\begin{pmatrix} 0 & 0 & 1 \\ 0.5 & 0.25 & 0.25 \\ 0.5 & 0.25 & 0.25 \end{pmatrix}$$

eigenvectors(transpose(P));

$$\left[\left[\left[1, -\left(\frac{1}{2}\right), 0\right], [1,1,1]\right], \left[\left[\left[1, \frac{1}{2}, \frac{3}{2}\right]\right], \left[\left[1, \frac{1}{2}, -\left(\frac{3}{2}\right)\right]\right], [[0,1,-1]]\right]\right]$$

Note this chain is irreduceable since P^2 is fully dense.

P:matrix([0,0,1],[0.5,0.25, 0.25],[0.5,0.25, 0.25]);

$$\begin{pmatrix} 0 & 0 & 1 \\ 0.5 & 0.25 & 0.25 \\ 0.5 & 0.25 & 0.25 \end{pmatrix}$$

P.P;

$$\begin{pmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.125 & 0.625 \\ 0.25 & 0.125 & 0.625 \end{pmatrix}$$