MATH 8030 Homework 12 Hints

1. Independence can be shown by computing the joint m.g.f. $E[e^{\alpha N_j(s)+\beta N_j(s,t]}]$ and seeing that it factors. From class we know $N_j(s) \sim \text{Poisson}\,(\lambda \int_0^s p_j(r)\,dr)$, and if $\{N_j(t) : t \geq 0\}$ really is a nonhomogeneous Poisson process with intensity $\lambda p_j(t)$, then $E[e^{\beta N_j(s,t]}]$ should be the m.g.f. of $\text{Poisson}(\lambda \int_s^t p_j(r)\,dr)$. To compute the joint m.g.f., you want to use the conditional independence of $\{L_n\}$ given the points $\{T_n : n \geq 0\}$ of the process $\{N(t) : t \geq 0\}$, and then use the order statistic property of $\{N(t) : t \geq 0\}$. This means finding some symmetric function of points $\{T_n\}$. It's not obvious at first how to do this, since $e^{\alpha N_j(s)+\beta N_j(s,t]}$ does not look symmetric in $\{T_n\}$. It helps to write

$$N_j(s) = \sum_{n=1}^{\infty} 1(T_n \leq s, L_n = j) = \sum_{n=1}^{N(t)} 1(T_n \leq s, L_n = j),$$

and

$$N_j(s,t] = \sum_{n=1}^{\infty} 1(s < T_n \leq t, L_n = j) = \sum_{n=1}^{N(t)} 1(s < T_n, L_n = j).$$

Then,

$$\alpha N_j(s) + \beta N_j(s,t] = \sum_{n=1}^{N(t)} \big\{\alpha 1(T_n \leq s, L_n = j) + \beta 1(s < T_n, L_n = j)\big\}$$

is symmetric in $T_1, \ldots, T_{N(t)}$. When computing the m.g.f., it may be useful to recall that for an indicator random variable 1_A and a constant $c \in \mathbb{R}$,

$$e^{c1_A} = e^c 1_A + 1_{A^c} = 1 + (e^c - 1)1_A.$$

2. (a) Since $T_n = T_1 + \sum_{k=2}^n (T_k - T_{k-1})$ is an i.i.d. sum, with $T_k - T_{k-1} \sim \text{Exp}\,(\lambda)$, the limit of T_n/n is the subject of the strong law of large numbers. If we think about taking the limit of $N(t)/t$ along the positive integers, then the independent and stationary increment property means that $N(k) = \sum_{j=1}^k N(j-1, j]$ is also an i.i.d. sum. This allows you to guess that constant b (as does the interpretation of λ as a rate). Technically the limit along the integers does not imply the limit as $t \to \infty$. However, recall that

$$T_{N(t)} \leq t \leq T_{N(t)+1},$$

and since $N(t) \to \infty$ as $t \to \infty$ w.p. 1,

$$\lim_{t\to\infty} \frac{T_{N(t)}}{N(t)} = \lim_{n\to\infty} \frac{T_n}{n} \qquad \text{w.p. } 1.$$

To finish, it may help to look back at how we proved the law of large numbers for Markov chains. (b) Use $N(k) = \sum_{j=1}^k N(j-1, j]$ and the central limit theorem. (c) This is open ended, but think about it in terms of a transformation/time change that turns the nonhomogeneous process into a homogeneous one. Using the result from part (a) for the homogeneous process, what can you say about the nonhomogeneous one? In particular, suppose that $\int_0^\infty \alpha(s)\,ds = \infty$.