MATH 8030 Homework 2

Directions: Complete all problems. Upload solutions to Problems 3 and 4 by 11:59 pm Thursday, January 23.

1. Suppose X is a $\{0, 1, 2, \ldots\}$-valued random variable with generating function $G_X(s)$. Find formulas for $\mathrm{Var}\,(X)$ and $E[X^3]$ in terms of derivatives of G_X.

2. You have a biased coin where the probability of heads is $p \in (0, 1)$. You flip this coin a random number $N \sim \mathrm{Poisson}(\lambda)$ times, and record the number of heads X and the number of tails Y. Determine the joint distribution of (X, Y).

3. Prove Wald's Second Moment Identity: If $X_1, X_2, \ldots$ are i.i.d. random variables with $\mu = E[X_1]$ and $\sigma^2 = \mathrm{Var}\,(X_1) < \infty$ and N is a $\{0, 1, 2, \ldots\}$-valued random variable independent of the $\{X_i\}$ with $E[N] < \infty$, then

$$E[(S_N - N\mu)^2] = \sigma^2 E[N].$$

   (As stated, all assumptions needed are given, but you may also assume that all generating functions exist.)

4. Let $\{S_n : n \in \mathbb{N}\}$ be a simple random walk with $S_0 = 0$, so that $S_n = \sum_{i=1}^n X_i$ where $\{X_i\}$ are i.i.d. with $P(X_1 = 1) = P(X_1 = -1) = 1/2$.

   (a) For fixed n, compute the moment generating function $M_{S_n}(t)$ of S_n.

   (b) Show that $n^{-1/2} S_n \Rightarrow \mathcal{N}(0, 1)$ as $n \to \infty$.