MATH 8030 Homework 7 Hints

2. (a) You can always write down the system of equations $\pi = \pi P$ and solve for all solutions π. However, the computation can be simplified by using the structure of the chain and the different representations of π when it exists. (b) The chain is irreducible and positive recurrent (see Midterm solutions).

3. (a) For use in part (b), it is important that the state space is specified carefully. (b) Once you come up with a guess for a stationary distribution ρ for $\{Y_n\}$, you just need to check that it satisfies $\rho_x = \sum_{y \in R} \rho_y r_{yx}$, where R and $\{r_{xy}\}$ are the state space and transition probabilities for $\{Y_n\}$. The following may help for coming up with a guess: Recall that when $\{Y_n\}$ has stationary distribution ρ, then $\{Y_n\}$ is a stationary process with respect to P_ρ. This means that

$$(Y_0, \ldots, Y_k) \stackrel{d}{=}_\rho (Y_n, \ldots, Y_{k+n}), \quad \text{any} \quad k, n \geq 0, \tag{1}$$

where "$\stackrel{d}{=}_\rho$" indicates equality in distribution with respect to P_ρ. Since $Y_n = (X_n, X_{n+1})$ for each n, we can write the process $\{X_n\}$ as $X_n = f(Y_n)$, where $f(i,j) = i$. (So $\{X_n\}$ can be thought of as a kind of projection of $\{Y_n\}$.) Then the stationarity (1) implies that

$$(X_0, \ldots, X_k) = (f(Y_0), \ldots, f(Y_k)) \stackrel{d}{=}_\rho (f(Y_n), \ldots, f(Y_{k+n})) = (X_n, \ldots, X_{k+n}),$$

for any k and n. This suggests that the projection $\tilde{\rho}_i = \rho_{(i,j)}$, $(i,j) \in R$, should be a stationary distribution for $\{X_n\}$, i.e., that $\tilde{\rho} = \pi$. To come up with a candidate ρ from this intuition, pick ρ so that for each n the distribution of Y_n under ρ is the same as the distribution of (X_n, X_{n+1}) under π, namely

$$P_\rho(Y_0 = (i,j)) = P_\pi(X_0 = i, X_1 = j).$$

4. (a) Attempt the computation of the stationary distribution for $p \leq q$, and observe that it can't be done when $p = q$. Along with the recurrence shown on the previous homework, this shows null recurrence when $p = q = 1/2$. (b) Use the law of large numbers for Markov chains.