3. Prove Wald's Second Moment Identity: If $X_1, X_2, \ldots$ are i.i.d. random variables with $\mu = E[X_1]$ and $\sigma^2 = \mathrm{Var}(X_1) < \infty$ and N is a $\{0, 1, 2, \ldots\}$-valued random variable independent of the $\{X_i\}$ with $E[N] < \infty$, then

$$E[(S_N - N\mu)^2] = \sigma^2 E[N].$$

(As stated, all assumptions needed are given, but you may also assume that all generating functions exist.)

$$E[(S_N - N\mu)^2] = E[(\textstyle\sum_{i=1}^N X_i - \sum_{i=1}^N \mu)^2]$$
$$= E[(\textstyle\sum_{i=1}^N (X_i - \mu))^2]$$
$$= E[\textstyle\sum_{i=1}^N (X_i - \mu)^2] + E[\sum_{i\neq j} (X_i - \mu)(X_j - \mu)]$$
$$= E[\textstyle\sum_{i=1}^N (X_1 - \mu)^2] + \sum_{i\neq j} E[X_i - \mu]E[X_j - \mu]$$
$$= E[N]E[(X_1 - \mu)^2] + 0$$
$$= \sigma^2 E[N]$$

4. Let $\{S_n : n \in \mathbb{N}\}$ be a simple random walk with $S_0 = 0$, so that $S_n = \sum_{i=1}^n X_i$ where $\{X_i\}$ are i.i.d. with $P(X_1 = 1) = P(X_1 = -1) = 1/2$.

   (a) For fixed n, compute the moment generating function $M_{S_n}(t)$ of S_n.

   (b) Show that $n^{-1/2} S_n \Rightarrow \mathcal{N}(0, 1)$ as $n \to \infty$.

a) $M_{S_n}(t) = E[e^{S_n t}] = E[e^{t\sum_{i=1}^n X_i}] = E[e^{tX_1}]^n = M_{X_1}(t)^n$

$M_{X_1}(t) = E[e^{tX_1}] = \dfrac{e^t + e^{-t}}{2}$

$M_{X_1}(t)^n = \left(\dfrac{e^t + e^{-t}}{2}\right)^n$

b) $K_{\frac{S_n}{\sqrt{n}}}(t) = \sqrt{n}\mu t + \dfrac{(\sigma^2 + \mu^2)t^2}{2} + O\left(\dfrac{t^3}{\sqrt{n}}\right)$

$= \dfrac{\sigma^2 t^2}{2} + O\left(\dfrac{t^3}{\sqrt{n}}\right)$ since $\mu = 0$

Thus

$\lim K_{\frac{S_n}{\sqrt{n}}}(t) = \dfrac{\sigma^2 t^2}{2} \Rightarrow M_{\frac{S_n}{\sqrt{n}}}(t) = e^{\frac{\sigma^2 t^2}{2}}$ as $n \to \infty$

$$V[n^{-1/2}S_n] = \frac{1}{n}V[S_n] = \frac{1}{n}V[\textstyle\sum_{i=1}^{n} X_i]$$

$$= \frac{1}{n}\sum_{i=1}^{n} V[X_1]$$

$$= V[X_1]$$

$$= E[X_1^2] - E[X_1]^2$$

$$= \tfrac{1}{2}(1)^2 + \tfrac{1}{2}(-1)^2 - \cancel{\tfrac{1}{2}(1) - \tfrac{1}{2}(-1)}^{0}$$

$$= 1$$

$\therefore\ n^{-1/2}S_n \Rightarrow N[0,1]$ as $n \to \infty$