MATH 8030 Homework 3 Hints

2. $R_n^{(k)}$ is a sum of n independent Bernoulli random variables:

$$R_n^{(k)} = \sum_{i=kn+1}^{(k+1)n} 1_{A_i}.$$

From Homework 1, the mean is

$$E\big[R_n^{(k)}\big] = H_{(k+1)n} - H_{kn},$$

where H_m denotes the mth harmonic number $H_m = \sum_{i=1}^m 1/i$. You may use without proof that

$$\lim_{m\to\infty} (H_m - \log m) =: \gamma$$

exists in $(0,\infty)$. ($\gamma \approx 0.5772156649$ is sometimes called Euler's constant.)

3. There's more than one way to argue this. Note the following:

   (a) For fixed x, $N_n^{(x)} \sim \mathrm{Bin}(n, p_n)$ for $p_n = P(X_{1,n} > x)$. The law of rare events gives necessary and sufficient conditions for $\mathrm{Bin}(n, p_n) \Rightarrow \mathrm{Pois}(\lambda)$.

   (b) $M_n \le x$ if and only if all of $X_{1,n}, \dots, X_{n,n}$ are less than or equal to x, or equivalently none of $X_{1,n}, \dots, X_{n,n}$ are larger than x. This allows you to relate $P(M_n \le x)$ to both $N_n^{(x)}$ and $F_n(x)$.

   (c) By the same inequality we used to prove the law of rare events in class,

$$x \le -\log(1-x) \le x + 2x^2 \quad \text{when} \quad |x| < 1/2.$$

   (d) Assuming $M_n \Rightarrow M$, since $P(M_n \le x) = F_n(x)^n$, we have $F_n(x)^n \to F(x)$ for all x at which F is continuous. Argue that this means $F_n(x) \to 1$ as $n \to \infty$. The place you may want to use the above inequality is to then say that $n(1 - F_n(x)) \to -\log F(x)$ as $n \to \infty$. Alternatively, L'Hôpital's rule can be used:

$$\lim_{t\to 0} \frac{-\log(1-t)}{t} = \lim_{t\to 0} \frac{1}{1-t} = 1.$$

   Then if $F_n(x) \to 1$ as $n \to \infty$, so that $1 - F_n(x) \to 0$,

$$\lim_{n\to\infty} \frac{-n \log F_n(x)}{n(1 - F_n(x))} = \lim_{n\to\infty} \frac{-\log(1 - (1 - F_n(x)))}{1 - F_n(x)} = 1.$$

   So,

$$\lim_{n\to\infty} n(1 - F_n(x)) = \lim_{n\to\infty} -n \log F_n(x),$$

   when the limits exist.