HW10

Wednesday, April 9, 2025 10:05 AM

2. Find the conditional distribution $T_1, \ldots, T_n | T_{n+1} = t$.

$$f_{T_1,\ldots T_n|T_{n+1}=t}(t_1,\ldots,t_n,t) = \frac{f_{T_1,\ldots,T_{n+1}}(t_1,\ldots,t_{n+1})}{f_{T_{n+1}}(t)}$$

Note the interarrival times are distributed $Exp(\lambda)$ and the Jacobian of the following transformation is

$$J = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix} \quad \text{So} \quad |J| = 1$$

Thus

$$f_{T_1,\ldots,T_{n+1}}(t_1,\ldots,t_{n+1}) = f_{T_1,\ldots,T_{n+1}}(t_1, t_2-t_1, \ldots t_n - t_{n-1}, t_{n+1}-t_n)|J|$$
$$= \prod_{k=1}^{n+1} \lambda e^{-\lambda(t_{k+1}-t_k)}$$
$$= \lambda^{n+1} e^{-\lambda \sum t_{k+1}-t_k}$$
$$= \lambda^{n+1} e^{-\lambda t_{n+1}}$$

Using the transformation again and writing $T_{n+1} = \sum_{k=1}^{n} T_{k+1} - T_k$ it follows T_{n+1} is a sum of exponential variables and thus distributed as $T_{n+1} \sim \Gamma(n+1, \lambda)$.

Therefore

$$f_{T_1,\ldots T_n|T_{n+1}=t}(t_1,\ldots,t_n,t) = \frac{\cancel{\lambda^{n+1}} \cancel{e^{-\lambda t_{n+1}}}}{\frac{\cancel{\lambda^{n+1}} t^{n+1-1} \cancel{e^{-\lambda t_{n+1}}}}{\Gamma(n+1)}}$$
$$= \frac{n!}{t^n}$$

Thus $T_1, \ldots, T_n | T_{n+1} = t$ is distributed as the order statistics of $Uniform[0,t]$

3. Busloads of customers arrive at an infinite server queue according to a Poisson process with rate λ. Each customer is served independently with common service time distribution having distribution function G. A bus contains j customers with probability α_j, $j = 1, 2, 3, \ldots$. Let $X(t)$ denote the number of customers that have has service completed by time t.

   (a) Compute $E[X(t)]$.

   (b) Does $X(t)$ have a Poisson distribution?

a) Let μ be the average number of costumers per bus.

Let a bus arrive at time $s<t$. Then

$$E[\text{customers from } s \text{ served by } t] = \mu G(t-s)$$

It follows

$$E[X(t)] = \lambda \int_0^\infty \mu G(t-s) \mathbb{1}\{s \leq t\} ds$$
$$= \lambda \mu \int_0^t G(t-s) ds$$

b) We can view $X(t)$ as a time dependent thinning of the original Poisson Process because being served can be used as a time dependent labeling. Thus $X(t) \sim \text{Pois}(\lambda \mu \int_0^t G(t-s) ds)$ by thrm in class.