HW11

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1. Suppose cars enter a one-way infinite highway, represented by $[0, \infty)$, according to $\{N(t) : t \geq 0\}$, a Poisson process with rate λ (i.e., one car enters at location 0 at each event time T_i). The ith car to enter chooses a velocity V_i and travels at this velocity. Assume that $\{V_i\}$ are i.i.d. positive random variables with distribution function F.

   (a) For $(a, b) \subset [0, \infty)$ and $t > 0$, find the distribution of the number of cars located in (a, b) at time t.

   (b) Suppose the speed limit on this highway is $v > 0$. Find the joint distribution of the number of cars going above the speed limit and the number going below the speed limit located in a given interval $(a, b) \subset [0, \infty)$ at time t.

Let the cars in (a,b) be $\xi(t)$

a) Let $X_i(s)$ be the distance traveled by each car.

$X_i(s) = V_i(s - T_i) = V_i \Delta t_i$

where T_i is the time the car entered the highway.

For $X_i(s) \in (a,b)$, $a < V_i \Delta t_i < b$

$$\frac{a}{\Delta t_i} < V_i < \frac{b}{\Delta t_i}$$

Using F, $P\left(\frac{a}{\Delta t_i} < V_i < \frac{b}{\Delta t_i}\right) = F\left(\frac{b}{\Delta t_i}\right) - F\left(\frac{a}{\Delta t_i}\right)$

$\xi(t)$ can also be viewed as a thinning process of $N(t)$. Thus

$$\xi(t) \sim \text{Poisson}\left(\lambda \int_0^t F\left(\frac{b}{\Delta t_i}\right) - F\left(\frac{a}{\Delta t_i}\right) ds\right)$$

b) Cars going above and below v as well as in (a,b) can be viewed as a thinning of $N(t)$. Since the thinnings are independent, the joint distribution is the product of the marginal distributions.

Let $\xi_g(t)$ be those in (a,b) and $V_i \geq v$

Continuing from part a, for $V_i \geq v$ we want to look at different cases.

Consider $P_g(s) = P(a < X_i(s) < b, V_i \geq v)$

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If $v\Delta t_i < a$, $v \leq V_i$ automatically and

$$P_g(s) = F(\tfrac{b}{\Delta t_i}) - F(\tfrac{a}{\Delta t_i})$$

If $a \leq v\Delta t_i < b$ we have to look at cars where $V_i \geq v$.

$$\Rightarrow P_g(s) = F(\tfrac{b}{\Delta t_i}) - F(v)$$

If $v\Delta t_i \geq b$, $V_i < v$ so

$$P_g(s) = 0$$

Thus $\xi_g(t) \sim \text{Poisson}(\lambda \int_0^t F(\tfrac{b}{\Delta t_i}) - \max(v, F(\tfrac{a}{\Delta t_i}))ds)$

Similarly for $\xi_\ell(t)$ those in (a,b) and $V_i < v$

$$P_\ell(s) = P(a < X_i(s) < b, V_i < v)$$

If $v\Delta t_i \leq a$, $V_i > v$ so

$$P_\ell(s) = 0$$

If $a < v\Delta t_i < b$, we have to choose V_i where $V_i < v$
so

$$P_\ell(s) = F(v) - F(\tfrac{a}{\Delta t_i})$$

If $v\Delta t_i \geq b$, $V_i < v$ so

$$P_\ell(s) = F(\tfrac{b}{\Delta t_i}) - F(\tfrac{a}{\Delta t_i})$$

Thus $\xi_\ell(t) \sim \text{Poisson}(\lambda \int_0^t \min(v, F(\tfrac{b}{\Delta t_i})) - F(\tfrac{a}{\Delta t_i})ds)$

Let $\mu_g = \lambda \int_0^t F(\tfrac{b}{\Delta t_i}) - \max(v, F(\tfrac{a}{\Delta t_i}))ds$

and $\mu_\ell = \lambda \int_0^t \min(v, F(\tfrac{b}{\Delta t_i})) - F(\tfrac{a}{\Delta t_i})ds$

It follows the joint distribution is distributed as Poissoin$(\mu_g \mu_\ell)$