HW07

Tuesday, March 11, 2025 3:38 PM

3. Define the process $\{Y_n : n \geq 0\}$ by $Y_n = (X_n, X_{n+1})$ for all $n \geq 0$.

   (a) Show that $\{Y_n\}$ is a Markov chain, and determine its state space and transition probabilities.

   (b) Suppose that $\{X_n\}$ has stationary distribution π. Find a stationary distribution for $\{Y_n\}$ in terms of π and P.

a) • $P(Y_n = (i_n, j_n) \mid \{Y_k = (i_k, j_k)\}_{k=0}^{n-1})$

$= P((X_n, X_{n+1}) = (i_n, j_n) \mid \{(X_k, X_{k+1}) = (i_k, j_k)\}_{k=0}^{n-1})$

$= P((X_n, X_{n+1}) = (i_n, j_n) \mid (X_{n-1}, X_n) = (i_{n-1}, j_n))$

$= P(Y_n = (i_n, j_n) \mid Y_{n-1} = (i_{n-1}, j_n))$

• $S_Y = S_X \times S_X$

• $P(Y_{n+1} = (k, l) \mid Y_n = (i, j)) = \begin{cases} p_{jl} & j = k \\ 0 & \text{o.w.} \end{cases}$

b) $\mu_{ij} = P(X_n = i, X_{n+1} = j)$

$= P(X_{n+1} = j \mid X_n = i) P(X_n = i)$

$= \pi_i p_{ij}$

4. Consider the reflected random walk defined on the previous homework assignment. Recall that the chain is transient when $p > q$ and recurrent when $p \leq q$.

   (a) Compute the stationary distribution when $p < q$ (thereby showing the chain is positive recurrent in that case), and show that the chain is null recurrent when $p = q = 1/2$.

   (b) When $p < q$, what is the long run fraction of time the walk spends at odd states?

Consider the "reflected simple random" walk on $S = \{0, 1, 2, \ldots\}$: for $p \in (0,1)$ and $q = 1-p$, $\{W_n : n \geq 0\}$ is the chain with transition matrix

$$P = \begin{pmatrix} q & p & 0 & 0 & 0 & \cdots \\ q & 0 & p & 0 & 0 & \cdots \\ 0 & q & 0 & p & 0 & \cdots \\ 0 & 0 & q & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

That is, $P(W_n = i+1 | W_{n-1} = i) = 1 - P(W_n = i-1 | W_{n-1} = i) = p$ for all $i \geq 1$ and $P(W_n = 1 | W_{n-1} = 0) = 1 - P(W_n = 0 | W_{n-1} = 0) = p$.

a) $\pi_j = \sum_k \pi_k P_{kj}$

Note Z^+ is nonnegative integers

$p<q$

$j=0$ $\quad \pi_0 = \sum_k \pi_k P_{k0}$

$$\pi_0 = \pi_0 q + \pi_1 q$$

$$p\pi_0 = \pi_1 q$$

$$\pi_1 = \frac{p}{q}\pi_0$$

$j \geq 1$ $\quad \pi_j = \sum \pi_k P_{kj}$

$$\pi_j = p\pi_{j-1} + q\pi_{j+1}$$

Assume $\pi_j = \frac{p}{q}\pi_{j-1}$

$$\pi_{j+1} = \frac{\pi_j - p\pi_{j-1}}{q}$$

$$= \frac{\left(\frac{p}{q} - p\right)\pi_{j-1}}{q}$$

$$= \frac{p(1-q)}{q^2}\pi_{j-1}$$

$$= \frac{p^2}{q^2}\pi_{j-1} = \frac{p}{q}\pi_j$$

Thus by induction $\pi_{j+1} = \frac{p}{q}\pi_j \ \forall j \in \mathbb{Z}^+$

Thus by induction $\pi_{j+1} = \frac{p}{q}\pi_j \ \forall j \in \mathbb{Z}^+$

$\Rightarrow \pi_j = \left(\frac{p}{q}\right)^j \pi_0.$

We know $\sum \pi_i = 1$

$\sum \left(\frac{p}{q}\right)^i \pi_0 = 1$

For $p<q$, $\frac{p}{q}<1 \Rightarrow \frac{\pi_0}{1-p/q} = 1$

Thus $\pi_0 = 1 - p/q$

$= \frac{q-p}{q}$

And $\pi_j = \frac{q-p}{q}\left(\frac{p}{q}\right)^j$

$\underline{p=q=\frac{1}{2}}$

ATC $\exists\ \mu = \mu P \quad \mu_i = \sum_k \mu_k P_{ki}$

(μ is an stationary measure)

$\underline{i=0}$ $\mu_0 = \frac{1}{2}\mu_0 + \frac{1}{2}\mu_1 \Rightarrow \mu_0 = \mu_1$

$\underline{i\geq 1}$ Assume $\mu_i = \mu_{i-1}$

$\mu_i = \frac{1}{2}\mu_{i-1} + \frac{1}{2}\mu_{i+1} \Rightarrow \mu_{i+1} = \mu_i$

By induction $\mu_i = \mu_{i-1}$

and thus $\mu_i = \mu_0 \ \forall i \in \mathbb{Z}^+$.

However $\sum_i \mu_i = \sum_i \mu_0 = \infty \neq 1$ $\times$ From last HW

This chain is irreduceable and recurrant but $\nexists!$ stationary measure so by a Prop in class we know the chain is null recurrant.

b) $\sum_n \pi_{2n+1} = \frac{q-p}{q} \sum_n \left(\frac{p}{q}\right)^{2n+1}$ for $n \in \mathbb{Z}^+$

$$= \frac{q-p}{q} \frac{p}{q} \frac{1}{1-\frac{p^2}{q^2}}$$

$$= \frac{(q-p)p}{q^2} \frac{q^2}{q^2-p^2}$$

$$= \frac{p}{p+q}$$

$$= p$$