

MATH 8030 Homework 7

Directions: Complete all problems. Upload solutions to Problems 3 and 4 by 11:59 pm Thursday, March 13.

When not otherwise stated, $\{X_n : n \geq 0\}$ is a Markov chain on discrete state space S with transition matrix $P = \{p_{ij}\}_{i,j \in S}$.

1. $\{X_n\}$ is said to be *reversible* with respect to a distribution π on S if

$$\pi_i p_{ij} = \pi_j p_{ji} \quad \text{for all } i, j \in S.$$

(Sometimes the above equalities are called the *detailed balance equations*.)

- (a) Show that if $\{X_n\}$ is reversible with respect to π then π is a stationary distribution for the chain. Give an example showing that the converse is false in general, i.e., that when a stationary distribution π exists the chain is not necessarily reversible with respect to π .
- (b) Show that if $\{X_n\}$ is reversible with respect to π then

$$P_\pi(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = P_\pi(X_0 = i_n, X_1 = i_{n-1}, \dots, X_n = i_0),$$

for any $i_0, \dots, i_n \in S$ (hence the term “reversible”).

2. For the following transition matrices, find any stationary distributions and determine if there is a limiting distribution.

$$(a) \ S = \{0, 1, 2, 3, 4, 5\} \text{ and } P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix}.$$

$$(b) \ S = \{0, 1, 2, 3, \dots\} \text{ and } P = \begin{pmatrix} 1-\theta & \theta & 0 & 0 & \dots \\ 1-\theta & 0 & \theta & 0 & \dots \\ 1-\theta & 0 & 0 & \theta & \dots \\ 1-\theta & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ where } \theta \in (0, 1).$$

3. Define the process $\{Y_n : n \geq 0\}$ by $Y_n = (X_n, X_{n+1})$ for all $n \geq 0$.

- (a) Show that $\{Y_n\}$ is a Markov chain, and determine its state space and transition probabilities.
- (b) Suppose that $\{X_n\}$ has stationary distribution π . Find a stationary distribution for $\{Y_n\}$ in terms of π and P .

4. Consider the reflected random walk defined on the previous homework assignment. Recall that the chain is transient when $p > q$ and recurrent when $p \leq q$.
- (a) Compute the stationary distribution when $p < q$ (thereby showing the chain is positive recurrent in that case), and show that the chain is null recurrent when $p = q = 1/2$.
 - (b) When $p < q$, what is the long run fraction of time the walk spends at odd states?