MATH 8030 Homework 5

Directions: Complete all problems. Upload solutions to Problems 3 and 4 by 11:59 pm Thursday, February 20.

When not otherwise stated, $\{X_n : n \geq 0\}$ is a Markov chain on discrete state space S with transition matrix $P = \{p_{ij}\}_{i,j \in S}$.

1. Consider the following two transition matrices for a Markov chain on $S = \{0, 1, 2, 3, 4, 5\}$:

$$P_1 = \begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & \frac{4}{5} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{7}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{6} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{3}{8} & \frac{1}{8} \end{pmatrix}.$$

   For each, (a) give the communicating classes, (b) determine which classes are closed, and (c) determine which states are transient and which are recurrent.

2. Show that the different characterizations of closure we gave in class are equivalent. Namely, for a communicating class $C \subset S$, show that the following statements are equivalent:

   (i) $P_i(\tau_{C^c} = \infty) = 1$ for all $i \in C$,

   (ii) $P_i(X_n \in C) = 1$ for all $i \in C$ and all $n \geq 0$.

   (iii) $p_{ij} = 0$ for all $i \in C$ and $j \in C^c$.

3. Given an i.i.d. sequence $\{V_n : n \geq 0\}$ of random variables taking values in some space E, and functions $g, h : S \times E \to S$, define the process $\{X_n : n \geq 0\}$ as follows:

$$\begin{aligned} X_0 &= g(j, V_0) \text{ for some } j \in S, \\ X_n &= h(X_{n-1}, V_n) \text{ for all } n \geq 1. \end{aligned}$$

   (a) Show that $\{X_n : n \geq 0\}$ is a Markov chain.

   (b) Consider the following model of a single-server queue: Customers arrive to a server and are served on a first come, first served basis. Between times $n-1$ and n, the number of customer arrivals is a random variable A_n taking values in $\{0, 1, \ldots\}$ with mass function

$$\alpha_k = P(A_n = k), \quad k = 0, 1, \ldots.$$

   The service time of each customer is deterministically 1, i.e., between times $n-1$ and n exactly one customer (if any) is served and leaves the system. Let X_n be the number of customers present in the system at time n. Use part (a) to conclude that $\{X_n : n \geq 0\}$ is a Markov chain, and find its transition probabilities.

4. Suppose C is a closed and finite communicating class. Let $\eta_i = \inf\{k \geq 1 : X_k = i\}$ for $i \in C$.

   (a) Show that there exists an integer $n > 0$ and an $\varepsilon \in (0, 1)$ with the following property: For any states $i, j \in C$, there exists $m \leq n$ such that $p_{ij}^{(m)} \geq \varepsilon$.

   (b) Suppose $k \geq 0$ is an integer and $i, j \in C$. Show that $P_i(\eta_j > kn) \leq (1 - \varepsilon)^k$, where n and ε are as in part (a).

   (c) Use part (b) to conclude that every state in C is positive recurrent.