HW12

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1. Recall the context of time-dependent thinning of a rate-λ Poisson process $\{N(t)\}$: For $\{p_1(t), \ldots, p_m(t) : t \geq 0\})$ where $\sum_{j=1}^m p_j(t) = 1$ for every t, the labels $\{L_n : n \geq 0\}$ are $\{1, \ldots, m\}$-valued random variables with distribution defined by

$$P(L_1 = j_1, \ldots, L_n = j_n | T_1, \ldots, T_n) = \prod_{k=1}^n p_{j_k}(T_k),$$

for any $j_1, \ldots, j_n \in \{1, \ldots, m\}$ and $n \geq 0$. For each $1 \leq j \leq m$, the process $N_j(t)$ counts the number of j-labeled events that have occurred by time t. For $s < t$, show that the increments $N_j(s)$ and $N_j(s,t]$ are independent. (This can be generalized to any number of increments. Along with results from class, this shows that $N_j(t)$ is a nonhomogeneous Poisson process with intensity $\lambda p_j(t)$.)

$N_j(s) = \sum_{k=1}^{N(s)} 1\{L_k = j\}$ $\quad N_j(s,t] = \sum_{k=N(s)+1}^{N(t)} 1\{L_k = j\}$

$P(N_j(s)=a, N_j(s,t]=b) = \sum_{n,m} P(N_j(s)=a, N_j(s,t]=b \mid N(s)=n, N(s,t]=m) \times P(N(s)=n)P(N(s,t]=m)$

Since the labels and T_n related to $N(t)$ are independent

$= \sum_{n,m} P(N_j(s)=a \mid N(s)=n) P(N_j(s,t]=b \mid N(s,t]=m) \times P(N(s)=n)P(N(s,t]=m)$

$= \sum_n P(N_j(s)=a \mid N(s)=n)P(N(s)=n) \sum_m P(N_j(s,t]=b \mid N(s,t]=m) \times P(N(s,t]=m)$

$= P(N_j(s)=a)P(N_j(s,t]=b)$

2. Let $\{N(t) : t \geq 0\}$ be a homogeneous Poisson process with rate λ and points $\{T_n : n \geq 0\}$. It can be shown that $P\big(\lim_{t\to\infty} N(t) = \infty\big) = 1$, and you should assume this to do this problem.

   (a) Find constants $a, b \in \mathbb{R}$ so that

$$P\Big(\lim_{n\to\infty} \frac{T_n}{n} = a\Big) = 1 \qquad \text{and} \qquad P\Big(\lim_{t\to\infty} \frac{N(t)}{t} = b\Big) = 1.$$

   (b) Find a constant $c \in \mathbb{R}$ and a distribution $\mathcal{D}$ so that for the sequence $\{N(k) : k \in \mathbb{N}\}$,

$$\frac{N(k) - ck}{\sqrt{k}} \Rightarrow \mathcal{D}$$

   as $k \to \infty$.

   (c) Do limits analogous to part (a) hold for a nonhomogeneous Poisson process $\{N_\alpha(t) : t \geq 0\}$ with (local) intensity $\alpha(t)$?

a) We can define each T_n as the telescoping sum of interarrival times. We also know that the interarrival times are distributed $Exp(\lambda)$

Thus by the SLLN $\frac{T_n}{n} = \sum_{i=1}^n \frac{T_i - T_{i-1}}{n} \to E[T_2 - T_1]$
$= \frac{1}{\lambda}$

For $t \in [T_{N(t)}, T_{N(t)+1}]$

$$\frac{T_{N(t)}}{N(t)} \le \frac{t}{N(t)} \le \frac{T_{N(t)+1}}{N(t)+1}$$

Since $N(t) \to \infty$ a.s. by the SLLN and uniquess of limits

$$\frac{1}{\lambda} \le \lim \frac{t}{N(t)} \le \frac{1}{\lambda} \Rightarrow \lim \frac{N(t)}{t} = \lambda$$

$\therefore a = \frac{1}{\lambda}$ and $b = \lambda$

b) $$M_{\frac{N(k)-ck}{\sqrt{k}}}(s) = E\left[e^{s\frac{N(k)-ck}{\sqrt{k}}}\right] = 1 + sE\left[\frac{N(k)-ck}{\sqrt{k}}\right] + \frac{s^2}{2}E\left[\left(\frac{N(k)-ck}{\sqrt{k}}\right)^2\right] + o\left(\frac{1}{k}\right)$$

$$\log M_{\frac{N(k)-ck}{\sqrt{k}}}(s) = \frac{s}{\sqrt{k}}(\lambda k - ck) + \frac{s^2}{2k} E\left[(N(k)-ck)^2\right] + o\left(\frac{1}{k}\right)$$

If $c = \lambda$ $\quad = 0 + \frac{s^2}{2k} Var(N(k)) + o\left(\frac{1}{k}\right)$

$$\to \frac{s^2\lambda}{2}$$

Thus $\frac{N(k)-ck}{\sqrt{k}} \Rightarrow N(0,\lambda)$ and $c = \lambda$

c) We know we can write any nonhomogeneous P.P. as a homogenous P.P. with rate 1 composed with $\mu(t) = \int_0^t \alpha(s)ds$. The mean of the P.P. will be $\mu(t)$. Looking at the results for part a, and knowing that the mean is $\mu(t)$

$a = \lim \frac{1}{\mu(t)}$ $\qquad b = \frac{N(\mu(t))}{t} \to \lim \frac{\mu(t)t}{t} = \lim \mu(t)$

If $\lim \mu(t) = \infty$, $a = 0$ $\quad b = \infty$.