HW04

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3. Consider a Markov chain $\{X_n : n \geq 0\}$ with state space $S = \{0, 1, 2\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

(a) Find all probability distributions π on S such that $\pi = \pi P$.

(b) Given an initial distribution $X_0 \sim \mu_0$ on S, find a formula for $\mu_n = (P(X_n = 0), P(X_n = 1), P(X_n = 2))$ in terms of n.

(c) Does μ_n converge to a distribution on S as $n \to \infty$?

a) π is a left eigenvector with eigen value of 1

Since $(I-P)^T = I-P$ by inspection,

$(\pi(I-P))^T = (I-P)^T\pi^T = (I-P)\pi^T \Rightarrow \pi^T$ is a right eigenvector of P with eigenvalue of 1

$$\begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}\pi^T = 0 \Rightarrow \pi = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \text{ by inspection}$$

b) $\mu_n = \mu_{n-1}P = \mu_{n-2}P^2 = \ldots = \mu_0 P^n$

c)

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(%i16) P:matrix([1/2, 1/2,0],[1/2, 0, 1/2],[0, 1/2, 1/2]);
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$$P \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

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(%i39) eigenvectors(P);
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(%o39) $\left[\left[\left[1,-\left(\frac{1}{2}\right),\frac{1}{2}\right],[1,1,1]\right],[[[1,1,1]],[[1,-2,1]],[[1,0,-1]]]\right]$

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(%i47) V:transpose(matrix([1,1,1],[1,-2,1],[1,0,-1]));
V1:invert(V);
L:matrix([1,0,0],[0,-1/2,0],[0,0,1/2]);
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$$V \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$V1 \quad \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & -\left(\frac{1}{3}\right) & \frac{1}{6} \\ \frac{1}{2} & 0 & -\left(\frac{1}{2}\right) \end{pmatrix}$$

$$L \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\left(\frac{1}{2}\right) & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$P^n = V\Lambda^n V^{-1}$$

$$= V\begin{bmatrix} 1 & & \\ & (-1/2)^n & \\ & & (1/2)^n \end{bmatrix}V^{-1}$$

$$\lim P^n = V\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}V^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/6 & -1/3 & 1/6 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{vmatrix} & \binom{2}{} & \\ 0 & 0 & \frac{1}{2}\end{vmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\lim \mu_n = \lim \mu_0 P^n = \mu_0 \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\left[|\mu_0|_1 \;\; |\mu_0|_1 \;\; |\mu_0|_1 \right]$$

$$= \frac{1}{3}\left[1 \;\; 1 \;\; 1 \right]$$

$|\mu_0|_1 = 1$ since it is stochastic

This is a uniform distribution

4. Consider the setup of the "Gambler's ruin" chain: $\{S_n : n \geq 0\}$ is a simple random walk on $\{0, 1, \ldots, N\}$ with transition probabilities $p_{i,i+1} = p$, $p_{i,i-1} = q = 1 - p$ for $i \in \{1, \ldots, N-1\}$ and absorbing states $\{0, N\}$. Recall that $P_i(\cdot) = P(\cdot | S_0 = i)$ and

$$T = \inf\{n \geq 0 : S_n \in \{0, N\}\}.$$

Compute $E_i[S_T]$ and $E_i[T]$ for each $i \in \{0, \ldots, N\}$.

$$E_i[S_T] = 0P_i(S_T = 0) + NP_i(S_T = N)$$
$$= N(1 - P_i(S_T = 0))$$

From class
$$P_i(S_T = 0) = \begin{cases} \dfrac{1 - (p/q)^{N-i}}{1 - (p/q)^N} & p \notin \{0, 1/2, 1\} \\ 1_{i \neq N} & p = 0 \\ 1_{i=0} & p = 1 \\ 1 - \frac{i}{N} & p = 1/2 \end{cases}$$

$$E_i[S_T] = \begin{cases} N - N\dfrac{1 - (p/q)^{N-i}}{1 - (p/q)^N} & p \notin \{0, 1/2, 1\} \\ 0 & p = 0 \\ 0 & p = 1 \\ i & p = 1/2 \end{cases}$$

$\boxed{p \neq 1/2}$

$$S_T = S_0 + \sum_{i=1}^{T} X_i \qquad E[X_i] = 1p + (-1)(1-p)$$
$$= 2p - 1$$

$$E_i[S_T] = E_i[S_0 + \sum_{i=1}^{T} X_i]$$
$$= E_i[S_0] + E_i[\sum_{i=1}^{T} X_i]$$

$$= E_i[S_0] + E_i\left[\sum_{i=1}^{T} X_i\right]$$

$$= i + E_i[T]E_i[X_i] \quad \text{By Wald's lemma given}$$

$$= i + E_i[T](2p-1) \quad \text{By iid}$$

$$\frac{E_i[S_T]-i}{2p-1} = E_i[T]$$

$\boxed{p = 1/2}$ $\quad f(i) = \frac{1}{2}(f(i+1)+f(i-1))+1 \qquad f(i) = E_i[T]$

$f(i+1) - 2f(i) + f(i-1) = -2 \qquad f(0) = 0 = f(N)$

Characteristic eq for homogeneous solution is

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0 \Rightarrow f_h(i) = A + Bi$$

-2 is a constant so $f_p(i) = Ci^2$

$$C(i+1)^2 - 2Ci^2 + C(i-1)^2 = -2$$

$$C(\cancel{i^2} + \cancel{2i} + 1) - \cancel{2Ci^2} + C(\cancel{i^2} - \cancel{2i} + 1) = -2$$

$$2C = -2 \Rightarrow C = -1 \Rightarrow f_p(i) = -i^2$$

$$f(i) = f_h(i) + f_p(i) = A + Bi - i^2$$

$$f(0) = 0 = A + 0 - 0^2 \Rightarrow A = 0$$

$$f(N) = 0 = BN - N^2 \Rightarrow B = N$$

$$f(i) = Ni - i^2$$

$$\therefore E_i[T] = \begin{cases} \dfrac{N - N\dfrac{1-(p/q)^{N-i}}{1-(p/q)^N} - i}{2p-1} & p \notin \{0, \frac{1}{2}, 1\} \\ Ni - i^2 & p = 1/2 \\ 0 & p \in \{0, 1\} \end{cases}$$

$$E_i[\tau] = \begin{cases} \dfrac{1-(p/q)}{2p-1} & p \notin \{0, \tfrac{1}{2}, 1\} \\ Ni - i^2 & p = 1/2 \\ 0 & p \in \{0,1\} \end{cases}$$

5. You have a coin for which the chance of heads is $p \in (0,1)$. Find the expected number of tosses of this coin needed to see three consecutive heads using the following steps.

   (a) You flip the coin repeatedly. Let $\{X_n : n \geq 0\}$ be the Markov chain on state space $S = \{0,1,2,3\}$ where X_n gives the number of consecutive heads you've seen immediately after the nth toss, and where you stop after you see three in a row. For example, $X_0 = 0$, and if the sequence of flips is $HTHHH$ then $X_1 = 1$, $X_2 = 0$, $X_3 = 1$, $X_4 = 2$, and $X_n = 3$, $n \geq 5$. Find the transition matrix of this chain.

   (b) Let $\tau = \inf\{n \geq 0 : X_n = 3\}$. Use a "first-step analysis" to write down a difference equation with boundary value for $f(i) = E_i[\tau]$ and solve it to find $E_0[\tau]$.

a) Note, 3 is an absorbing state
so last row is [0 0 0 1]
For $0 \leq i \leq 2$
$P_{i0} = 1-p$ since we can always start over with a tails
$P_{01} = P_{12} = P_{23} = p$ because we need a heads to move forward with chance of p
Thus

$$P = \begin{bmatrix} 1-p & p & 0 & 0 \\ 1-p & 0 & p & 0 \\ 1-p & 0 & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) $f(i) = E_i[\tau] = 1 + Pf(i)$ $\quad f(3) = 0$

$f(0) = 1 + (1-p)f(0) + pf(1)$

$f(0) = \dfrac{1+pf(1)}{p}$

$f(2) = 1 + (1-p)f(0) + p\cancel{f(3)}^{0}$

$f(2) = 1 + (1-p)\dfrac{1+pf(1)}{p}$

$= 1 + \dfrac{1+pf(1)}{p} - 1 - pf(1)$

$f(1) = 1 + (1-p)f(0) + pf(2)$

$= 1 + (1-p)\dfrac{1+pf(1)}{p} + 1 + pf(1) - p^2f(1)$

$= \cancel{1} + \dfrac{1+pf(1)}{p} - \cancel{1} - \cancel{pf(1)} + 1 + \cancel{pf(1)} - p^2f(1)$

$= \dfrac{1+pf(1)}{p} + 1 - p^2f(1)$

$p\cancel{f(1)} = 1 + p\cancel{f(1)} + p - p^3f(1)$

$$= 1 + \frac{1 + pf(1)}{p} - 1 - pf(1)$$

$$= \frac{1 + pf(1) - p^2 f(1)}{p}$$

$$\cancel{pf(1)} = 1 + \cancel{pf(1)} + p - p^3 f(1)$$

$$f(1) = \frac{1+p}{p^3}$$

$$f(0) = \frac{1}{p} + \frac{1+p}{p^3} = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3}$$

$$f(1) = \frac{1+p}{p^3} = \frac{1}{p^3} + \frac{1}{p^2}$$

$$f(2) = \frac{1}{p} + \frac{1+p}{p^3} - \frac{1+p}{p^2} = \cancel{\frac{1}{p}} + \frac{1}{p^3} + \cancel{\frac{1}{p^2}} - \cancel{\frac{1}{p^2}} - \cancel{\frac{1}{p}} = \frac{1}{p^3}$$

$$E_0[\tau] = f(0) = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3}$$