MATH 8030 Homework 1

Directions: Complete all problems. Upload solutions to Problems 4 and 5 by 11:59 pm Thursday, January 16.

1. Prove that it is not possible to choose a positive integer uniformly at random. I.e., show that there does not exist a probability measure on $\Omega = \{1, 2, \ldots\}$ that assigns equal likelihood to the occurance of each $\omega \in \Omega$.

2. X is a random variable with $E[X^2] < \infty$.
   (a) Show that $\mathrm{Var}\,(X) = \min_{t \in \mathbb{R}} E[(X - t)^2]$.
   (b) (Ross 1.12) Show that if $P(0 \le X \le a) = 1$, then $\mathrm{Var}\,(X) \le a^2/4$. *(Hint: Use part (a) and the right value of t.)*

3. Suppose $a_1, a_2, \ldots, a_n$ are nonnegative real numbers. Use Jensen's inequality to verify the following.
   (a) $(a_1 + a_2 + \cdots + a_n)^p \le n^{p-1}(a_1^p + a_2^p + \cdots + a_n^p)$ when $p \ge 1$.
   (b) The AM-GM inequality, $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \ge (a_1 a_2 \cdots a_n)^{1/n}$.

4. (a) Suppose X and Y are random variables such that
   $$E[X|Y] = 18 - \frac{3}{5}Y \qquad \text{and} \qquad E[Y|X] = 10 - \frac{1}{3}X.$$
   Find $E[X]$ and $E[Y]$.
   (b) Suppose $X_1, X_2, X_3, \ldots$ be i.i.d. random variables where $\mu = E[X_1]$ exists, and let $S_n = X_1 + \cdots + X_n$ for each $n \ge 1$. For $n, m \ge 1$, find an explicit formula (in terms of n, m, S_n, and μ) for $E[S_m|S_n]$. *(Hint: The $m \le n$ and $m > n$ cases are different. Exploit the "symmetry" that $E[X_i|S_n]$ does not depend on the value of i, but only whether or not $i \le n$ or $i > n$.)*

5. (Ross 1.6) Let $X_1, X_2, \ldots$ be i.i.d. continuous random variables and let $X_0 = -\infty$. We say that a record occurs at time $n \ge 1$ and has value X_n when $X_n > \max\{X_1, \ldots, X_{n-1}\}$.
   (a) Let $A_n = \{\text{a record occurs at time } n\}$, $n \ge 1$. Compute $P(A_n)$ and explain why $A_1, A_2, \ldots A_n$ are independent for each n.
   (b) Let N_n denote the total number of records that have occurred up to (and including) time n. Find $E[N_n]$ and $\mathrm{Var}\,(N_n)$. *(Hint: Use indicator variables.)*
   (c) Let $T = \min\{n \ge 2 : A_n \text{ occurs}\}$ be the time of the first record (other than X_1). Compute $P(T > n)$ and show that $P(T < \infty) = 1$ and $E[T] = \infty$. *(Hint: by continuity of probability, $P(T = \infty) = \lim_{n\to\infty} P(T > n)$.)*
   (d) Let T_y denote the time of the first record value greater than y, i.e., $T_y = \min\{n \ge 1 : X_n > y\}$. Show that T_y and X_{T_y} are independent, i.e., the time of the first value greater than y is independent of that value.