HW09

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*When not otherwise stated, $\{N(t) : t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ and event times $\{T_n : n \geq 1\}$.*

3. Let $0 < s < t$.

   (a) Find the conditional distribution of $N(s)$ given $N(t)$.

   (b) Compute $E[T_k|N(t)]$ for all $k \geq 1$.

a) $P(N(s)=k \mid N(t)=n) = \dfrac{P(N(s)=k, N(t)=n)}{P(N(t)=n)}$

$$= \frac{P(N(s)=k,\ N(t)-N(s)=n-k)}{P(N(t)=n)}$$

$$= \frac{P(N(s)=k)P(N(t)-N(s)=n-k)}{P(N(t)=n)}$$

$$= \frac{\cancel{e^{-\lambda s}}(\lambda s)^k}{k!}\frac{\cancel{e^{-\lambda(t-s)}}(\lambda(t-s))^{n-k}}{(n-k)!}\frac{n!}{e^{-\lambda t}(\lambda t)^n}$$

$$= \frac{\binom{n}{k} s^k (t-s)^{n-k}}{t^n}$$

$$= \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}$$

So $\sum_{k=0}^{n} P(N(s)=k \mid N(t)=n) = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}$

$\therefore\ N(s) \mid N(t)=n \sim Bin(n, \frac{s}{t})$

b) Note $\{T_n \leq t\} \Leftrightarrow \{N(t) \geq n\}$

$P(T_k > s \mid N(t)=n) = P(N(s) < k \mid N(t)=n)$

$$= \sum_{j=0}^{k-1}\binom{n}{j}\left(\frac{s}{t}\right)^j\left(1-\frac{s}{t}\right)^{n-j}$$

If $n<k$, $P(T_k > s \mid N(t)=n) = P(N(s) < k \mid N(t)=n) = 0$.
This is because $N(u) = \sum_{n=1}^{\infty} 1\{T_n \leq u\}$ so $N(t)$ cannot equal $n$ if $N(s) > k$ and $s<t$.

$E[T_k \mid N(t)=n] = \int_0^\infty P(T_k > s \mid N(t)=n)\,ds$

$$= \int_0^\infty P(N(s) < k \mid N(t)=n)\,ds$$

$$= \int_0^t \sum_{j=0}^{k-1}\binom{n}{j}\left(\frac{s}{t}\right)^j\left(1-\frac{s}{t}\right)^{n-j} ds \quad \text{because } s<t$$

$$= \sum_{j=0}^{k-1}\binom{n}{j}\int_0^t\left(\frac{s}{t}\right)^j\left(1-\frac{s}{t}\right)^{n-j} ds$$

$$= \sum_{j=0}^{k-1}\binom{n}{j}\int_0^1 u^j(1-u)^{n-j}\, t\, du \qquad u=\frac{s}{t}\quad du=\frac{1}{t}ds$$

$$= t\sum_{j=0}^{k-1}\binom{n}{j}\frac{\Gamma(n-j+1)\Gamma(j+1)}{\Gamma(n-j+j+2)}$$ Beta normalization constant

$$= t\sum_{j=0}^{k-1}\frac{n!}{j!(n-j)!}\frac{(n-j)!\,j!}{(n+1)!}$$

$$= t\sum_{j=0}^{k-1}\frac{1}{n+1}$$

$$= t\,\frac{k}{n+1}$$

4. Let $\{M(t) : t \geq 0\}$ be a Poisson process with rate $\mu$ and event times $\{S_n : n \geq 1\}$ independent of $\{N(t)\}$ and $\{T_n\}$. Determine if the process $\{L(t) : t \geq 0\}$ is a Poisson process, and if so compute its rate.

   (a) $L(t) = N(t) + M(t)$.

   (b) $L(t)$ is the counting process on $[0, \infty)$ with event times $R_n = \min\{S_n, T_n\}$, $n \geq 1$.

   (c) $L(t)$ counts the number of intervals $(k-1, k]$, $k = 1, 2, \ldots, N(t)$ for which $M(k-1, k] = 0$, i.e., on which the process $\{M(t)\}$ sees no events.

a) • $L(0) = N(0) + M(0) = 0$

• $L(t) - L(s) = N(t) + M(t) - (N(s) + M(s))$
$= N(t) - N(s) + M(t) - M(s)$

So $L$ has stationary increments

• $G_{N(t)}(s) = \sum_{k=0}^{\infty} s^k \frac{e^{-\lambda}}{k!} \lambda^k = e^{-\lambda} e^{s\lambda} = e^{\lambda(s-1)}$

$G_{M(t)}(s) = e^{\mu(s-1)}$

$G_{N(t)}(s) G_{M(t)}(s) = e^{\lambda(s-1)} e^{\mu(s-1)} = e^{(\lambda+\mu)(s-1)} = G_{N(t)+M(t)}(s) = G_{L(t)}(s)$

Thus the rate is $\lambda + \mu$

b) • $L(t) = \sum_{n=1}^{\infty} I\{\min(T_n, S_n) \leq t\}$

$L(0) = \sum_{n=1}^{\infty} I\{\min(T_n, S_n) \leq 0\} = 0$ Since $T_n, S_n \geq 0$

• WLOG assume $T_n = \min(T_n, S_n)$

$L(t) - L(s) = \sum_{n=1}^{\infty} I\{T_n \leq t\} - I\{T_n \leq s\}$
$= N(t) - N(s)$

Thus $L$ has stationary increments

• $R_n = \min\{T_n, S_n\}$

$P(R_n \leq x) = 1 - P(R_n > x)$
$= 1 - P(T_n > x) P(S_n > x)$
$= 1 - e^{-(\lambda+\mu)tx}$

Thus the rate is $\lambda + \mu$

c) • From class $L(t) = \sum_{k=1}^{N(t)} I\{M(k-1, k] = 0\}$

$I\{M(k-1,k] = 0\} \sim Ber(e^{-\mu})$

It follows $L(t) | N(t) = n \sim Bin(n, e^{-\mu})$

$P(L(t) = m) = \sum_{n=0}^{\infty} P(L(t) = m | N(t) = n) P(N(t) = n)$

$= \sum_{n=m}^{\infty} \binom{n}{m} e^{-\mu m} (1 - e^{-\mu})^{n-m} \frac{e^{-\lambda t} (\lambda t)^m}{n!}$ Because $Bin(n, e^{-\mu}) = 0$ for $n < m$

$= \frac{e^{-\lambda t - \mu m}}{m!} \sum_{n=m}^{\infty} \frac{(\lambda t)^n}{(n-m)!} (1 - e^{-\mu})^{n-m} \frac{(\lambda t)^m}{(\lambda t)^m}$

$= \frac{e^{-\lambda t} (e^{-\mu} \lambda t)^m}{m!} \sum_{k=0}^{\infty} \frac{(\lambda t (1 - e^{-\mu}))^k}{k!}$ $k = n - m$

$= e^{-\lambda t} e^{\lambda t (1 - e^{-\mu})} \frac{(e^{-\mu} \lambda t)^m}{m!}$

$(e^{-\mu} \lambda t)^m \quad e^{-e^{-\mu} \lambda t}$

$$= e \; e \; \frac{(e \quad \lambda t)}{m!}$$

$$= \frac{(e^{-\mu}\lambda t)^m}{m!} e^{-e^{-\mu}\lambda t}$$

Thus $L(t) \sim \text{Poisson}(e^{-\mu}\lambda t) \Rightarrow$ rate of $L$ is $e^{-\mu}\lambda$

- $L(0) = 0$
- Since $L(t) \sim \text{Poisson}(e^{-\mu}\lambda t)$

  $L(t) - L(s) = L(t-s)$