HW03

Thursday, January 30, 2025 11:14 AM

2. Consider the record process from Homework 1: $X_0 = -\infty$, and $\{X_i : i \geq 1\}$ are i.i.d. continuous random variables. We say a record happens at time n if the event $A_n = \{X_n > \max_{k<n} X_k\}$ occurs. Recall that $N_n = \sum_{i=1}^{n} 1_{A_i}$ denotes the number of records that have occurs up to (and including) time n. For fixed positive integer k, define the sequence of random variables $R_1^{(k)}, R_2^{(k)}, \ldots$ by

$$R_n^{(k)} = N_{(k+1)n} - N_{kn},$$

which denotes the number of records that occur during the length-n time interval $\{kn+1, \ldots, (k+1)n\}$. Show that $R_n^{(k)} \Rightarrow \text{Poisson}(\lambda_k)$ as $n \to \infty$, and find λ_k.

$$R_n^{(k)} = \sum_{i=1}^{(k+1)n} 1_{A_i} - \sum_{i=1}^{kn} 1_{A_i} = \sum_{i=kn+1}^{(k+1)n} 1_{A_i}$$

$\lim E[R_n^{(k)}] = \lim(H_{(k+1)n} - H_{kn})$ From HW 1

$= \lim(H_{(k+1)n} - \log((k+1)n) + \log((k+1)n) - (H_{kn} - \log kn + \log kn))$

$= \gamma - \gamma + \lim(\lim((k+1)n) - \lim kn)$ By hint

$= \lim \log \frac{(k+1)n}{kn}$

$= \log(1 + \frac{1}{k})$

Define $\lambda_k := \log(1+\frac{1}{k})$

Consider

$$\sum_{i=kn+1}^{(k+1)n} \frac{1}{i^2} \leq \sum_{i=kn+1}^{(k+1)n} \frac{1}{(kn)^2} = \frac{n}{(kn)^2} = \frac{1}{k^2 n} \to 0 \text{ as } n \to \infty$$

Thus by the Law of Rare events $R_n^{(k)} \Rightarrow \text{Poisson}(\log(1+\frac{1}{k}))$

3. For each $n \geq 1$, suppose that $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$ are i.i.d. random variables with cumulative distribution function F_n. Let $M_n = \max\{X_{1,n}, \ldots, X_{n,n}\}$, and for $x \in \mathbb{R}$, let $N_n^{(x)}$ denote the number of values among $\{X_{1,n}, \ldots, X_{n,n}\}$ that are larger than x. Finally, suppose M is a random variable with cumulative distribution function $0 < F < 1$. Show that $M_n \Rightarrow M$ as $n \to \infty$ if and only if $N_n^{(x)} \Rightarrow \text{Poisson}(-\log F(x))$ for all $x \in \mathbb{R}$ at which F is continuous.

$P(M_n \leq x) = P(X_{1,n} \leq x)^n = F_n(x)^n$

$N_n^{(x)} \sim Bin(n, P(X_{1,n} \geq x)) = Bin(n, 1 - F_n(x))$

$(\Rightarrow)$ $F_n(x)^n \to F(x)$

$n \log F_n(x) \to \log F(x)$

$$n \log F_n(x) \to \log F(x)$$

$$\log(F_n(x)) = \log(1-(1-F_n(x)))$$

$$= -(1-F_n(x)) + O((1-F_n(x))^2)$$

$$E[N_n^{(x)}] = n(1-F_n(x)) \approx -n\log F_n(x) \to -\log F(x)$$

$\therefore N_n^{(x)} \Rightarrow \text{Poisson}(-\log F(x))$ By corollary of LRE

$(\Leftarrow)$ $n(1-F_n(x)) \to -\log F(x)$

$$n \log F_n(x) \approx -n(1-F_n(x)) \to \log F(x)$$

$$\Rightarrow (F_n(x))^n \to F(x)$$

4. Let $X_0 = 0$, and let $X_1, X_2, \ldots$ be i.i.d. $\{0,1,2,\ldots\}$-valued random variables with

$$P(X_1 = j) = \alpha_j, \qquad j = 0,1,2,\ldots.$$

We say a record occurs at time k if $X_k > \max\{X_1, \ldots, X_{k-1}\}$, and in this case we call X_k a record value. For each $n \geq 1$, let R_n denote the nth record value.

(a) Argue that $\{R_n : n \geq 1\}$ is a Markov chain, and compute its transition probabilities.

(b) Let T_n denote the time between the nth and $(n+1)$th records. Is $\{T_n : n \geq 1\}$ a Markov chain? What about $\{(T_n, R_n) : n \geq 1\}$? Compute transition probabilities for each in the case it is a Markov chain.

a) Since a new record only is dependent on the current record and not on any other previous records, R_n has the Markov property.
There can also only be a countable number of records which is unrelated to which record it is.

$$P(R_{n+1}=j \mid R_n=i) = \frac{\alpha_j}{\sum_{i=j+1}^{\infty} \alpha_i}$$

b) $\{T_n \mid n \geq 1\}$ is not a Markov chain since T_n depends on R_n which depends on R_{n-1}

$\{(T_n, R_n) \mid n \geq 1\}$ is a Markov chain since T_n depends only on R_n and R_n only depends on R_{n-1}.
Similarly to part a there is only a countable amount of records independent to which one it is.

$$P((T_{n+1}=t, R_{n+1}=j) \mid (T_n=s, R_n=i)) = P(T_{n+1}=t \mid T_n=s)\,P(R_{n+1}=j \mid R_n=i)$$

Since T and R are independent

Note $T_j \sim \text{Geom}(\sum_{i=j+1}^{\infty} \alpha_i)$ since we are looking for first success time.

Thus

$$P((T_{n+1}=t,\ R_{n+1}=j)\mid(T_n=s,\ R_n=i))=\left(1-\sum_{k=j+1}^{\infty}\alpha_k\right)^{t-1}\sum_{k=j+1}^{\infty}\alpha_k\,\frac{\alpha_j}{\sum_{i=j+1}^{\infty}\alpha_i}$$