MATH 8030 Homework 6

Directions: Complete all problems. Upload solutions to Problems 3 and 4 by 11:59 pm Thursday, March 6.

When not otherwise stated, $\{X_n : n \geq 0\}$ is a Markov chain on discrete state space S with transition matrix $P = \{p_{ij}\}_{i,j \in S}$.

1. (a) Suppose $0 < n < m$. For $i_0, i_1, \ldots, i_m \in S$, Show that

$$P(X_n = i_n | X_k = i_k \text{ for all } k \neq n) = P(X_n = i_n | X_{n-1} = i_{n-1}, X_{n+1} = i_{n+1}),$$

provided that all events conditioned on have positive probability.

   (b) Compute $P(X_n = j | X_{n-1} = i, X_{n+1} = k)$ in terms of the transition probabilities $\{p_{ij} : i, j \in S\}$.

2. Consider the chain on S $= \{0, 1, 2, 3, 4, 5, 6\}$ with transition matrix

$$P = \begin{pmatrix} .2 & .2 & 0 & .1 & 0 & .3 & .1 & .1 \\ 0 & .3 & .2 & .1 & 0 & .1 & .1 & .2 \\ 0 & 0 & .1 & 0 & .6 & .3 & 0 & 0 \\ .25 & .1 & .2 & .13 & .12 & .1 & 0 & .1 \\ 0 & 0 & .4 & 0 & .6 & 0 & 0 & 0 \\ 0 & 0 & .14 & 0 & .36 & .5 & 0 & 0 \\ .11 & .11 & .28 & .1 & .2 & .1 & .1 & 0 \\ .1 & .4 & .05 & .04 & .06 & 0 & .1 & .25 \end{pmatrix}$$

Let $T \subset S$ denote the transient states of the chain, and let T^c denote the recurrent states. Let $\tau = \inf\{n \geq 0 : X_n \in T^c\}$. For initial distribution $\mu = (.2, .01, .32, .1, .17, .05, .05, .1)$, compute $\{P_\mu(X_\tau = k) : k \in T^c\}$ and $E_\mu[\tau]$.

3. Consider the "reflected simple random" walk on $S = \{0, 1, 2, \ldots\}$: for $p \in (0, 1)$ and $q = 1 - p$, $\{W_n : n \geq 0\}$ is the chain with transition matrix

$$P = \begin{pmatrix} q & p & 0 & 0 & 0 & \cdots \\ q & 0 & p & 0 & 0 & \cdots \\ 0 & q & 0 & p & 0 & \cdots \\ 0 & 0 & q & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

That is, $P(W_n = i + 1 | W_{n-1} = i) = 1 - P(W_n = i - 1 | W_{n-1} = i) = p$ for all $i \geq 1$ and $P(W_n = 1 | W_{n-1} = 0) = 1 - P(W_n = 0 | W_{n-1} = 0) = p$.

   (a) Show that for any $n \geq 2$,

$$f_{20}^{(n)} = \sum_{k=1}^{n} f_{21}^{(k)} f_{10}^{(n-k)},$$

and therefore that $f_{20} = f_{21} f_{10}$.

(b) Argue that

$$f_{21} = f_{10}, \qquad f_{00} = q + pf_{10}, \qquad \text{and} \qquad f_{10} = q + p(f_{10})^2.$$

(c) Compute f_{00}, and determine recurrence or transience of the chain in the cases $p > q$ and $p \leq q$.

4. (Resnick 2.16) Without benefit of dirty tricks, Happy Harry's restaurant business fluctuates in successive years between three states: 0 (bankruptcy), 1 (verge of bankruptcy), and 2 (solvency). The transition matrix giving the probabilities of evolving from state to state is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ .5 & .25 & .25 \\ .5 & .25 & .25 \end{pmatrix}.$$

(a) What is the expected number of years until Happy Harry's restaurant goes bankrupt, assuming that he starts from the state of solvency?

(b) Harry's rich uncle Zeke decides it is bad for the family name if his nephew Harry is allowed to go bankrupt. Thus when state 0 is entered, Zeke infuses Harry's business with cash, returning him to solvency with probability 1. Thus the transition matrix for this new Markov chain is

$$P' = \begin{pmatrix} 0 & 0 & 1 \\ .5 & .25 & .25 \\ .5 & .25 & .25 \end{pmatrix}.$$

What is the expected number of years between cash infusions from Zeke?