Test 2

Wednesday, April 9, 2025 11:46 AM

MATH 8100 Mathematical Programming Spring 2025
Second Midterm Exam

April 9, 2025

- This take-home exam is due in Gradescope by 12:00 noon Friday, April 11, 2025.
- You may use only your textbook, course Canvas materials, your notes, the online pivot tool, and your instructor as resources.
- There are a total of 65 points. Point value is listed next to each question.
- Mark your answers clearly in the space provided. Gradescope is set to recognize the page formats, so you don't need to select your answers when you submit.
- *Show your work.* Unsupported correct answers receive partial credit.
- Be sure to write your name and ID number on *each* page.
- Good luck!

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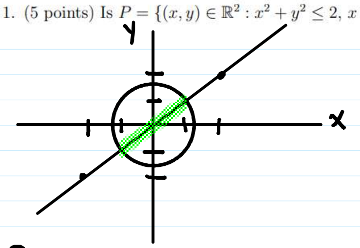
I certify that I have not received any unauthorized assistance in completing this examination.

Signature: Jacob Manning

Date: 04/09/2025

The word model is used as a noun, adjective, and verb, and in each instance it has a slightly different connotation. As a noun "model" is a representation in the sense in which an architect constructs a small-scale model of a building or a physicist a large-scale model of an atom. As an adjective "model" implies a degree of perfection or idealization, as in reference to a model home, a model student, or a model husband. As a verb "to model" means to demonstrate, to reveal, to show what a thing is like. —Russell Ackoff

1. (5 points) Is $P = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2,\ x - 2y = 0\}$ a polyhedron? Explain.



P is the green line

$x - 2y = 0$

$\Rightarrow x - 2y \geq 0$

$\quad x - 2y \leq 0$

$x = 2y \quad (2y)^2 + y^2 \leq 2$

$y^2 \leq \frac{2}{5}$

$-\sqrt{\frac{2}{5}} \leq y \leq \sqrt{\frac{2}{5}}$

It follows P can be expressed as $P = \{x \in \mathbb{R}^2 \mid Ax \geq b\}$ where

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ 0 \\ \sqrt{2/5} \\ -\sqrt{2/5} \end{bmatrix}$$

So P is a polyhedron

2. Consider the following linear program in standard form:

$$\begin{array}{llllllll} \text{minimize} & 4x_1 & + & 3x_2 & + & 2x_3 & + & 4x_4 \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & & & = & 4 \\ & x_1 & + & 2x_2 & & & + & x_4 & = & 12 \\ & & & & & & & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

(a) (3 points) Show that $(0, 0, 4, 12)$ is a BFS.

(b) (7 points) Solve the LP with the simplex method, using Bland's rule to select entering and leaving variables, starting with the BFS $(0, 0, 4, 12)$. Report your pivot entering and leaving variables for each iteration, your final dictionary, the final primal and dual solutions, and the optimal objective value.

(c) (3 points) Find the basic solution with x_2 and x_3 basic. Is it feasible for the primal? Is it feasible for the dual? Explain.

(d) (5 points) Solve the LP starting from the basis in part 2c using the dual simplex method. Select pivots using Bland's rule.

(e) (5 points) Consider the problem above, with greater-or-equal constraints replacing the equalities. Write the representation of the feasible set as convex combinations of extreme points plus a conic combination of extreme directions.

(f) (7 points) For each point you found in part 2e, compute the objective value. Formulate an LP equivalent to that of part 2e where the variables are the weights of a convex combination of the extreme points and a conic combination of the extreme directions that you found.

a) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$ Thus $(0\ 0\ 4\ 12)$ is a BFS

b) $x_3 = 4 - x_1 - x_2$
$x_4 = 12 - x_1 - 2x_2$
$z = 4x_1 + 3x_2 + 2(4 - x_1 - x_2) + 4(12 - x_1 - 2x_2)$
$z = -2x_1 - 7x_2 + 56$

$$\boxed{\begin{aligned} x_3 &= 4 - x_1 - x_2 \\ x_4 &= 12 - x_1 - 2x_2 \\ z &= -2x_1 - 7x_2 + 56 \end{aligned}}$$

x_1 in x_3 out
$(4 < 12)$

$x_1 = 4 - x_2 - x_3$
$x_4 = 12 - (4 - x_2 - x_3) - 2x_2$
$= 8 - x_2 - x_3$
$z = -2(4 - x_2 - x_3) - 7x_2 + 56$
$= -5x_2 + 2x_3 + 48$

$$\boxed{\begin{aligned} x_1 &= 4 - x_2 - x_3 \\ x_4 &= 8 - x_2 - x_3 \\ z &= -5x_2 + 2x_3 + 48 \end{aligned}}$$

x_2 in x_1 out
$(4 < 8)$

$x_2 = 4 - x_1 - x_3$
$x_4 = 8 - (4 - x_1 - x_3) + x_3$
$= 4 + x_1$
$z = -5(4 - x_1 - x_3) + 2x_3 + 48$
$= 5x_1 + 7x_3 + 28$

$$\boxed{\begin{aligned} x_2 &= 4 - x_1 - x_3 \\ x_4 &= 4 + x_1 \\ z &= 5x_1 + 7x_3 + 28 \end{aligned}}$$

optimal $x = (0\ 4\ 0\ 4)$
$z = 28$

Dual
max $4y_1 + 12y_2$
st $y_1 + y_2 \leq 4$
$y_1 + 2y_2 \leq 3$
$y_1 \leq 2$
$y_2 \leq 4$

By complementary slackness
$y_1 + 2y_2 = 3 \quad y_2 = 4$
$\Rightarrow y_1 = -5$

$4(-5)+12(4)=28=z$

Is $(-5\ \ 4)$ feasible?

$-5+4\le 4$ ✓

$-5+8=3\le 3$ ✓

$-5\le 2$ ✓

$y_2=4\le 4$ ✓

c) $x_1+x_2+x_3=4$

$x_1+2x_2+x_4=12$ $\qquad x_2\ x_3$ basic

$x_2+x_3=4$

$2x_2=12$

$x_2=6$

$x_3=-2 \implies x=\begin{bmatrix}0\\6\\-2\\0\end{bmatrix}$

d) $x_2=6-\frac{x_1}{2}-\frac{x_4}{2}$

$x_3=4-(6-\frac{x_1}{2}-\frac{x_4}{2})-x_1$

$=-2-\frac{x_1}{2}+\frac{x_4}{2}$

$z=\cancel{4}^{3}x_1+3(6-\frac{x_1}{2}-\frac{x_4}{2})+2(-2-\frac{x_1}{2}+\frac{x_4}{2})+\cancel{4}^{5}x_4$

$=\frac{3}{2}x_1+\frac{7}{2}x_4+14$

$$\boxed{\begin{aligned}x_2&=6-\tfrac{x_1}{2}-\tfrac{x_4}{2}\\ x_3&=-2-\tfrac{x_1}{2}+\tfrac{x_4}{2}\\ z&=\tfrac{3}{2}x_1+\tfrac{7}{2}x_4+14\end{aligned}}$$

x_3 out $\qquad x_4$ in

$\frac{1}{2}>0$

$x_4=4+x_1+2x_3$

$x_2=6-\frac{x_1}{2}-\frac{1}{2}(4+x_1+2x_3)$

$=4-x_1-x_3$

$z=\frac{3}{2}x_1+\frac{7}{2}(4+x_1+2x_3)+14$

$=5x_1+7x_3+28$

$$\boxed{\begin{aligned}x_2&=4-x_1-x_3\\ x_4&=4+x_1+2x_3\\ z&=28+5x_1+7x_3\end{aligned}}$$

optimal $(0\ 4\ 0\ 4)$

$z=28$

e) $x_1+x_2+x_3\ge 4$

$x_1+2x_2+x_4\ge 12$

We want to check basic solutions and use complementary slackness

$x_1,x_2=0$

$x_3=4\quad x_4=12$ ✓

$x_1,x_3=0$

$x_2=4\quad x_4=12-8=4$ ✓

$x_1,x_4=0$

$x_2+x_3=4\quad x_2=6$ ✗

$x_3=-2$

$x_2,x_3=0$

$x_1=4\quad x_4=8$ ✓

$x_2,x_4=0$

Thus there are 3 basic solutions that are feasible. Label them $\{v_i\}_{i=1}^{3}$ respectively

$(0\ 0\ 4\ 12),(0\ 4\ 0\ 4),(4\ 0\ 0\ 8)$

Looking at unbdd directions

$d_1+d_2+d_3\ge 0$

$d_1+2d_2+d_4\ge 0$

we can see if $d_4\to\infty$ this system will always be satisfied.

$x_2, x_3 = 0$
$x_1 = 4 \quad x_4 = 8$ ✓
$x_2, x_4 = 0$
$x_1 + x_3 = 4 \quad x_1 = 12$ ✗
$x_3 = -8$
$x_3, x_4 = 0$
$x_1 + x_2 = 4 \quad x_1 + 2x_2 = 12$ ✗
$x_1 = 4 - x_2 \quad x_2 = 8$
$x_1 = -4$
$x_1 = 0$
$x_2 + x_3 = 4 \quad 2x_2 + x_4 = 12$
∞ many solutions
$x_2 = 0$
$x_1 + x_3 = 4 \quad x_1 + x_4 = 12$
∞ many solutions
$x_3 = 0$
$x_1 + x_2 = 4 \quad x_1 + 2x_2 + x_4 = 12$
∞ many solutions
$x_4 = 0$
$x_1 + x_2 + x_3 = 4 \quad x_1 + 2x_2 = 12$
∞ many solutions

$x_1, + 2x_2 + x_4 \geq 0$
we can see if $d_4 \to \infty$ this system will always be satisfied.
Thus
$S = \{x \in \mathbb{R}^4 \mid x = \sum_{i=1}^{3} \lambda_i v_i + \mu e_4, \sum_{i=1}^{3} \lambda_i = 1, \mu \geq 0\}$

f) $c^T v_1 = 16 + 32 = 48$
$c^T v_2 = 12 + 16 = 28$ so v_2 is the optimum where $c^T v_2 = 28$
$c^T v_3 = 8 + 48 = 56$

$x = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \mu e_4$

$$= \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 12 & 4 & 8 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \mu \end{bmatrix}$$

$\min 4x_1 + 3x_2 + 2x_3 + 4x_4 \Rightarrow \min 16\lambda_3 + 12\lambda_2 + 8\lambda_1 + 4(12\lambda_1 + 4\lambda_2 + 8\lambda_3 + \mu)$
$\min 56\lambda_1 + 28\lambda_2 + 48\lambda_3 + 4\mu$

$\min 56\lambda_1 + 28\lambda_2 + 48\lambda_3 + 4\mu$
st $\lambda_1 + \lambda_2 + \lambda_3 = 1$
$\lambda_1, \lambda_2, \lambda_3, \mu \geq 0$

We know the min occurs at $(0,1,0,0)$ with objective value of 28

3. (10 points) Prove the following theorem of alternative:

Exactly one of the following systems has a solution:

- There exists $\boldsymbol{x} \neq \mathbf{0}$ such that $A\boldsymbol{x} = \mathbf{0}$, $\boldsymbol{x} \geq \mathbf{0}$.
- There exists $\boldsymbol{u}$ such that $\boldsymbol{u}^T A > \mathbf{0}^T$.

$\underline{1 \Rightarrow \neg 2}$

$\exists x \neq 0 \ni Ax = 0 \quad x \geq 0 \Rightarrow \nexists u \ni u^T A > 0^T$

ATC $\exists u \ni 0^T < u^T A$
It follows $0^T x < (u^T A) x = u^T (Ax) = u^T 0 = 0$
Since $0^T x = 0$
$0 < u^T A x = 0$ ✗
Thus $\nexists u \ni u^T A > 0^T$

$0 < u^T A x = 0$ $\times$

Thus $\exists u \ni u^T A > 0^T$

2 => ¬1)

$\exists u \ni u^T A > 0 \Rightarrow \nexists x \neq 0 \ni Ax = 0 \quad x \geq 0$

ATC $\exists x \neq 0 \quad Ax = 0 \quad x \geq 0$

It follows

$\mu^T (Ax) = \mu^T 0 = 0$

$(\mu^T A) x = 0$

$\sum_i (\mu^T A)_i x_i = 0$

However since $\mu^T A > 0$, $(u^T A)_i > 0 \ \forall i$ and since $x \neq 0$ and $x \geq 0$ $\exists$ at least one index, lets call k, such that $x_k > 0$.

Each other nonzero entry of x is also positive.

Thus we can bound the sum below as follows

$\sum_i (u^T A)_i x_i \geq (u^T A)_k x_k > 0$ $\times$

Thus $\nexists x \neq 0 \ni Ax = 0 \quad x \geq 0$

4. (10 points) Consider the LP in standard form and its dual:

$$\begin{array}{lrcl} \text{minimize} & \boldsymbol{c}^T \boldsymbol{x} & & \\ \text{subject to} & A\boldsymbol{x} & = & \boldsymbol{b} \\ & \boldsymbol{x} & \geq & \boldsymbol{0} \end{array} \qquad \begin{array}{lrcl} \text{maximize} & \boldsymbol{b}^T \boldsymbol{y} & & \\ \text{subject to} & A^T \boldsymbol{y} & \leq & \boldsymbol{c} \end{array}$$

Suppose that $x_j^* = 0$ for any optimal solution $\boldsymbol{x}^*$. Show that there exists a dual solution $\boldsymbol{y}^*$ such that $A_{\cdot j}^T \boldsymbol{y}^* < c_j$.

Let the optimal solution be p

We know $p \leq C^T x^* \leq p$, $x^* \geq 0$, $Ax^* = b$, and in particular $x_j = 0$

Appealing to complementary slackness

Consider

$$\begin{array}{ll} \max & x_j \\ \text{st} & c^T x^* \leq p \\ & x^* \geq 0 \\ & Ax^* = b \end{array}$$

So that it is still optimal for the original

In standard form with the respective dual

$$\begin{array}{ll} \min & -e_j^T x^* \\ \text{st} & -c^T x^* \geq -p \\ & x^* \geq 0 \\ & Ax^* = b \end{array} \qquad \begin{array}{ll} \max & b^T y - \lambda p \\ \text{st} & A^T y - \lambda c + e_j \leq 0 \\ & y \geq 0 \quad \lambda \geq 0 \end{array}$$

The optimal value is 0 by the original problem and by strong duality the optimal value of the new dual is also 0

The optimal value is 0 by the original problem and by strong duality the optimal value of the new dual is also 0.

Let y^* and λ^* be such that $b^Ty^* - \lambda^* p = 0$ and $b^Ty^* \leq p$

$\Rightarrow \lambda^* \leq 1$

Thus $(A^Ty^*)_j \leq \lambda^* c_j - 1 < c_j$

It follows $b^Ty^* \leq p$ and $A^Ty^* \leq \lambda^* c - e_j \leq c$

Thus y^* is feasible for the original dual.

If $\lambda^* < 1$ then there would be a duality gap, which we know does not exist by assumption.

$\therefore b^Ty^* = d$ and thus y^* is optimal for the original $\blacksquare$

5. (10 points) Consider the following two linear programs:

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \tag{1}$$

and

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{d} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \tag{2}$$

Suppose that (1) has a finite solution. Prove that if (2) has a feasible solution, it has a finite optimal solution.

Consider the dual of 1

$$\begin{array}{ll} \max & c^Tb \\ \text{st} & y^TA \leq c^T \\ & y \geq 0 \end{array}$$

by strong duality $\exists$ a finite optimal solution to the dual since $\exists$ one for the primal problem. Let y^* be the optimum of the dual.

Let $\bar{x}$ be the feasible solution to 2.

It follows

$$y^{*T}A \leq c^T$$

$$y^{*T}A\bar{x} \leq c^T\bar{x} \qquad \text{since } A\bar{x} = d$$

$$y^{*T}d \leq c^T\bar{x}$$

Thus $c^T\bar{x}$ is bdd below for any feasible solution to 2.

Since 2 is bdd and $\exists$ a feasible solution, $\exists$ an optimal solution which is finite.