

MATH 8100 Mathematical Programming Spring 2025
First Midterm Exam

February 17–18, 2025

- This take-home test is due in Gradescope at noon on February 18, 2025. Be sure to mark your answers for each of the questions in Gradescope when you upload.
- There are a total of 70 points. Point value is listed next to each question.
- Mark your answers clearly. *Show your work.* Unsupported correct answers receive partial credit.
- Good luck!

Name:_____

Student ID #:_____

I certify that I have not received any unauthorized assistance in completing this examination.

Signature:_____

Date:_____

“Without data, you’re just another person with an opinion.”

—Anon.

- (8 points) Recall that a cone is a set $S \subseteq \mathbb{R}^n$ such that $\mathbf{x} \in S$ implies $\alpha \mathbf{x} \in S$ for all $\alpha \geq 0$. Prove that a set $S \subseteq \mathbb{R}^n$ is a convex cone if and only if it is closed under vector addition and nonnegative scalar multiplication.
- (7 points) Let g_1, g_2, \dots, g_m be concave functions on \mathbb{R}^n , f be a convex function on \mathbb{R}^n , and μ be a positive constant. Show that the function

$$\beta(\mathbf{x}) = f(\mathbf{x}) - \mu \sum_{i=1}^m \ln g_i(\mathbf{x})$$

is convex on the set $S = \{\mathbf{x} : g_i(\mathbf{x}) > 0, i = 1, 2, \dots, m\}$.

- (6 points each part) Consider the function

$$f(x_1, x_2) = \frac{1}{3}x_1^3 + \frac{1}{2}x_1^2 + 2x_1x_2 + \frac{1}{2}x_2^2 - x_2 + 9.$$

For each of the following points,

- determine if the point is stationary;
- if stationary, determine if the point is a local minimum, maximum, or neither;
- if not stationary, identify a descent direction.

(a) $\mathbf{x} = (1, -1)$

(b) $\mathbf{x} = (2, -3)$

(c) $\mathbf{x} = (0, 0)$

- Consider the primal-dual pair

$$\begin{aligned} & \text{minimize} && p = f(\mathbf{x}) \\ & \text{subject to} && g_i(\mathbf{x}) \leq 0 \quad i = 1, 2, \dots, m \\ & && \mathbf{x} \in X \end{aligned} \tag{1}$$

and

$$\begin{aligned} & \text{maximize} && d = L(\boldsymbol{\lambda}) = \min_{\mathbf{x} \in X} (f(\mathbf{x}) + \sum_{i=1}^m \lambda_i g_i(\mathbf{x})) \\ & \text{subject to} && \boldsymbol{\lambda} \geq \mathbf{0}. \end{aligned} \tag{2}$$

Suppose $\hat{\mathbf{x}} \in X$ and $\hat{\boldsymbol{\lambda}} \geq \mathbf{0}$ satisfy the following saddle point condition for all $\mathbf{x} \in X$ and all $\boldsymbol{\lambda} \geq \mathbf{0}$:

$$f(\hat{\mathbf{x}}) + \sum_{i=1}^m \lambda_i g_i(\hat{\mathbf{x}}) \leq f(\hat{\mathbf{x}}) + \sum_{i=1}^m \hat{\lambda}_i g_i(\hat{\mathbf{x}}) \leq f(\mathbf{x}) + \sum_{i=1}^m \hat{\lambda}_i g_i(\mathbf{x}).$$

- (8 points) Show that $\hat{\mathbf{x}}$ solves the primal nonlinear program (1). (Hint: Use the saddle point condition to show that complementary slackness holds at $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}})$.)

- (b) (7 points) Show that the saddle point condition implies strong duality, i.e., $p^* = d^*$ where p^* is the optimal value of (1) and d^* is the optimal value of (2).

5. (8 points) Solve the following problem.

$$\begin{array}{ll}\text{minimize}_{\mathbf{x} \in X} & \sum_{i=1}^n x_i \ln x_i \\ \text{subject to} & \sum_{i=1}^n x_i = 1\end{array}$$

where $X = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} > \mathbf{0}\}$.

6. (a) (7 points) Consider the problem

$$\begin{array}{ll}\text{minimize} & x_1^3 + x_2 \\ \text{subject to} & x_2 \geq 1.\end{array}$$

holds. Can we conclude that the KKT point is an optimal solution? Why or why not?

(b) (7 points) Consider the problem

$$\begin{array}{ll}\text{minimize} & x_1^2 + x_2 \\ \text{subject to} & x_2 \geq 1.\end{array}$$

holds. Can we conclude that the KKT point is an optimal solution? Why or why not?