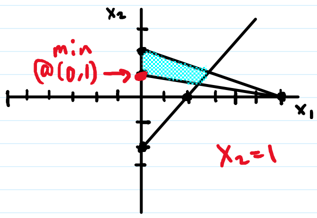
HW3

Friday, February 28, 2025 8:05 AM

1. Solve the following problem using the graphical method:

$$
\begin{array}{llcrcl}
\min & 5x_1 & + & x_2 & & \\
s.t. & x_1 & + & 3x_2 & \leq & 6 \\
 & x_1 & + & 6x_2 & \geq & 6 \\
 & x_1 & - & x_2 & \leq & 2 \\
 & x_1 & & & \geq & 0.
\end{array}
$$

Then write the problem in standard form. Do not attempt to solve the problem in standard form with the graphical method.



$x_1 - x_2 = 2 \qquad x_1 + 6x_2 = 6$

$6 - 6x_2 = x_1 = 2 + x_2$

$4/7 = x_2 \qquad x_1 = 18/7$

$5(18/7) + 4/7 > 1$

$$\min \begin{bmatrix} 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 1 & 6 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 2 \end{bmatrix}$$

2. In class, we discussed the following "tracks on the ceiling" problems:

> An engineering project needs to place three sensors in the ceiling, each connected to a central reporting system. A map looking like a coordinate system describes where the sensors are to be placed. If the "origin" is in the southwest corner of the room, then the three sensors are located at (10, 3), (5, 15), and (20, 25). Our job is to place the central unit. Due to the design of the ceiling, th e wiring between the sensors can run only along horizontal or vertical tracks in th e ceiling. Design two convex models to complete the following tasks:
>
> - Minimize the total amount of wire used;
> - Minimize the maximum wire length from a sensor to the central unit.

Convex models were introduced in class with respect to the Manhattan distance. Solve the models using AMPL and CPLEX or Gurobi. Your AMPL formulation should use descriptive variable names, informative comments, legible layout, and appropriate index sets and data tables so that your model can be used with different data without editing the model file.

$$X = \{\|(x,y)^T - (10,3)^T\|_1, \|(x,y)^T - (5,15)^T\|_1, \|(x,y)^T - (20,25)^T\|_1\}$$

Model

$$\min_{x,y} \sum_{i=1}^{3} X_i$$

$$\min_{x,y} \max X$$

Results

```
ampl: include Test.run;
CPLEX 22.1.1:               CPLEX 22.1.1: optimal solution; objective 37
6 simplex iterations
min_tot_wire = 37
x = 10
y = 15
```

```
ampl: include Test.run;
CPLEX 22.1.1:               CPLEX 22.1.1: optimal solution; objective 16
6 simplex iterations
min_max_wire = 16
x = 14
y = 15
```

.mod

```
#import from data
param x_num;
param y_num;

#make indexing sets
set x_set = 1..x_num;
set y_set = 1..y_num;

#import from data
param x_pos {x_set};
param y_pos {y_set};

#restrict x,y to first quadrant
var x >= 0;
var y >= 0;

#solve the minimizations
#minimize min_max_wire: max(abs(x-x_pos[1])+abs(y-y_pos[1]),abs(x-x_pos[2])+abs(y-y_pos[2]),abs(x-x_pos[3])+abs(y-y_pos[3]));

minimize min_tot_wire: abs(x-x_pos[1])+abs(y-y_pos[1])+abs(x-x_pos[2])+abs(y-y_pos[2])+abs(x-x_pos[3])+abs(y-y_pos[3]);
```

```
#import from data
param x_num;
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#make indexing sets
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param x_pos {x_set};
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#restrict x,y to first quadrant
var x >= 0;
var y >= 0;

#solve the minimizations
minimize min_max_wire: max(abs(x-x_pos[1])+abs(y-y_pos[1]),abs(x-x_pos[2])+abs(y-y_pos[2]),abs(x-x_pos[3])+abs(y-y_pos[3]));

#minimize min_tot_wire: abs(x-x_pos[1])+abs(y-y_pos[1])+abs(x-x_pos[2])+abs(y-y_pos[2])+abs(x-x_pos[3])+abs(y-y_pos[3]);
```

.run

```
reset; #clean slate

model Test.mod; #choose model type
data Test.dat; #choose data for model

option solver cplex; #choose solver

solve; #solve command

#display the minimum calculated as well as x and y value assosicated with the minimum
#display min_max_wire, x, y;

#display the minimum calculated as well as x and y value assosicated with the minimum
display min_tot_wire, x, y;
```

```
reset; #clean slate

model Test.mod; #choose model type
data Test.dat; #choose data for model

option solver cplex; #choose solver

solve; #solve command

#display the minimum calculated as well as x and y value assosicated with the minimum
display min_max_wire, x, y;

#display the minimum calculated as well as x and y value assosicated with the minimum
#display min_tot_wire, x, y;
```

.dat

```
#how many x coordinantes there are
param x_num = 3;

#how many y coordinantes there are
param y_num = 3;

#x position values
param x_pos:= 1 10 2 5 3 20;

#y position values
param y_pos:= 1 3 2 15 3 25;
```

```
#how many x coordinantes there are
param x_num = 3;

#how many y coordinantes there are
param y_num = 3;

#x position values
param x_pos:= 1 10 2 5 3 20;

#y position values
param y_pos:= 1 3 2 15 3 25;
```

Exercise 2.4 We know that every linear programming problem can be converted to an equivalent problem in standard form. We also know that nonempty polyhedra in standard form have at least one extreme point. We are then tempted to conclude that every nonempty polyhedron has at least one extreme point. Explain what is wrong with this argument.

You could have an unbounded polyhedron. For example $\{(x,y) \mid x \geq 0\}$ or $\{(x,y) \mid y=0\}$. These do not have extrema.

4. Find an extreme point of each of the following polyhedra, or a line contained in it. If you find an extreme point $\boldsymbol{x}^\star$, write the active linearly independent constraints at $\boldsymbol{x}^\star$.

(a) $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \geq \boldsymbol{b}\}$, where A is a square matrix with n linearly independent columns;

(b) $P = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{e}^\top \boldsymbol{x} \geq 0\}$ with $\boldsymbol{e}^\top = (1\ 1\ \cdots\ 1)$.

a) An extreme point of P has all active constraints. Since the cols of A are L.I. it is invertible. It follows $x^* = A^{-1}b$.

The active L.I. constraints are the rows of A.

b) $e^T x = \sum x_i \geq 0$. This problem is unbounded and thus will have not have extrema

Exercise 2.10 Consider the standard form polyhedron $P = \{\mathbf{x} \mid \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$. Suppose that the matrix $\mathbf{A}$ has dimensions $m \times n$ and that its rows are linearly independent. For each one of the following statements, state whether it is true or false. If true, provide a proof, else, provide a counterexample.

(a) If $n = m + 1$, then P has at most two basic feasible solutions.

(b) The set of all optimal solutions is bounded.

(c) At every optimal solution, no more than m variables can be positive.

(d) If there is more than one optimal solution, then there are uncountably many optimal solutions.

(e) If there are several optimal solutions, then there exist at least two basic feasible solutions that are optimal.

(f) Consider the problem of minimizing $\max\{\mathbf{c}'\mathbf{x}, \mathbf{d}'\mathbf{x}\}$ over the set P. If this problem has an optimal solution, it must have an optimal solution which is an extreme point of P.

a) Consider $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

It follows the $BFS = \{x \in \mathbb{R}^3 \mid x_1 + x_2 = 1,\ x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\}$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in BFS$ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in BFS$ $\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in BFS$

Which is greater than 2.

b) Similarly, $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\max \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \max x_1 + x_3$$

$$\begin{aligned} Ax &\geq b \\ x &\geq 0 \end{aligned} \qquad \begin{aligned} x_1 + x_2 &= 1 \\ x_2 + x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Pick $t \in \mathbb{R}$. Set $x_1 = t = x_3$ and $x_2 = 1 - t$.

It follows the set of BFS is unbounded.

c) Since $\operatorname{rank}(A) = m$, a BFS will have $n-m$ variables set to zero to use the m L.I. eqs to find an extrema. Thus we can only have m positive values because any more would require more L.I. eqs.

d) Let x_1, x_2 be optimal solutions to

$$\begin{aligned} \max\ & c^T x = p^* \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Let $\lambda \in [0,1]$. Consider $x_\lambda = \lambda x_1 + (1-\lambda) x_2$

It follows

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It follows

$$c^T x_\lambda = c^T(\lambda x_1 + (1-\lambda)x_2) = \lambda p^* + (1-\lambda)p^* = p^*$$

$$A x_\lambda = A(\lambda x_1 + (1-\lambda)x_2) \leq \lambda b + (1-\lambda) b = b$$

$$x_\lambda = \lambda x_1 + (1-\lambda)x_2 \geq 0$$

Thus x_λ is an optimal solution as well. As $[0,1]$ is uncountable, I have finished.

e) The previous part in reverse works here.

f) $\max\{c'x, d'x\}$ s.t. $x \in P$ has an optimal solution by hypothesis

Consider rewritting this problem as

min t

s.t. $c'x \leq t$

$d'x \leq t$

$x \in P$

This problem is convex and since $\max\{c'x, d'x\}$ has an optimal solution, $t = \max\{c'x, d'x\}$ and will be a vertex of P since the new problem is convex.

Exercise 2.13 Consider the standard form polyhedron $P = \{\mathbf{x} \mid \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$. Suppose that the matrix $\mathbf{A}$, of dimensions $m \times n$, has linearly independent rows, and that all basic feasible solutions are nondegenerate. Let $\mathbf{x}$ be an element of P that has exactly m positive components.

(a) Show that $\mathbf{x}$ is a basic feasible solution.

(b) Show that the result of part (a) is false if the nondegeneracy assumption is removed.

a) Let $B = [a_{ij}]_{x_i > 0}$. Since $x \in P$, x is a feasible solution.
$\text{rank} B = m = \text{rank} A$ and $B \in \mathbb{R}^{m \times m}$ since x has m positive components.

It follows since $x \in P$, $Ax = b = Bx = b$

$$\Rightarrow x^* = B^{-1} b$$

where $x^* = \{x\}_{x_i > 0}$. Note this step depends on the nondegeneracy assumption since we can use the fact B is invertible and will be a bijection.

b) Without the nondegeneracy assumption, we cannot use the bijection property to get that x is a BFS.

7. Suppose that a free variable x_i is replaced by $x_i^+ - x_i^-$ ($x_i^+, x_i^- \geq 0$) when a linear program is converted to standard form. Show that, in a basic s olution of the resulting standard form, at least one of the variables x_i^+ and x_i^- has value 0.

In standard form we need to have L.I. eqs, however, we can write $x_i^+ = x_i + x_i^-$ and $x_i^- = x_i^+ - x_i$.
Thus these are not L.I. and atleast one will have to be zero.