HW6

Sunday, April 20, 2025 1:49 PM

1. Consider the LP problem

$$\begin{array}{lrcl} \min & x_1 + 2x_2 & & \\ \text{s.t.} & 3x_1 + x_2 & \leq & 9 \\ & x_1 + x_2 & \geq & 5 \\ & x_1, x_2 & \geq & 0. \end{array}$$

(a) Put it in standard form.

(b) Verify that the optimal basis is $\mathcal{B} = \{1, 2\}$ by computing $B^{-1}\boldsymbol{b}$ and the reduced costs of the nonbasic variables.

(c) Suppose a variable x_5, unrestricted in sign, is added to the original problem, with objective coefficient α and column $A_5 = (1, 1)^\top$. How does the standard form change?

(d) Determine the values of α such that the optimal solution of the new problem has the same entries for x_1 and x_2.

(e) Determine the values of α such that the problem is bounded.

a) $\min \begin{bmatrix} 1 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix}$

st $\begin{bmatrix} -3 & -1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \end{bmatrix}$

$x_1, x_2, s_1, s_2 \geq 0$

b) $\begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -9 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$6+3=9$ ✓ $2+3=5$ ✓ $2,3 \geq 0$ ✓

$\underline{s_1}$) $0 - \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = 1/2 > 0$

$\underline{s_2}$) $0 - \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 5/2 > 0$

Thus $\{1, 2\}$ is optimal and the optimal solution is 8

c) $\min \begin{bmatrix} 1 & 2 & \alpha & -\alpha & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_5^+ \\ x_5^- \\ s_1 \\ s_2 \end{bmatrix}$

st $\begin{bmatrix} -3 & -1 & 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_5^+ \\ x_5^- \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \end{bmatrix}$

$x_1, x_2, x_5^+, x_5^-, s_1, s_2 \geq 0$

d) $-6 - 3 + x_5^+ - x_5^- + s_1 = 9$ $\qquad 2 + 3 + x_5^+ - x_5^- + s_2 = 5$

$x_5^+ - x_5^- + s_1 = 0$ $\qquad x_5^+ - x_5^- + s_2 = 0$

since $x_5^+, x_5^-, s_1, s_2 \geq 0$, $x_5 = 0$ $s_1 = 0$ $s_2 = 0$

We need the reduced cost of x_5 to be non-negative

$\underline{x_5^+}$) $\alpha - \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \alpha - 2 \geq 0 \Rightarrow \alpha \geq 2$

$\underline{x_5}|\ -\alpha - [1\ \ 2]\begin{bmatrix}-3 & -1\\ 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}-1\\-1\end{bmatrix} = -\alpha + 2 \geq 0 \Rightarrow 2 \geq \alpha$

Thus $\alpha = 2$

e) If $\alpha > 0$ then if $x_5 \to -\infty$ we can adjust x_1, x_2 to remain feasible.
Similarly if $\alpha < 0$ then we can choose x_1, x_2 to keep feasibility as $x_5 \to \infty$
Thus $\alpha = 0$ to be bold.

2. Consider the LP problem:

$$\begin{array}{lrcl} \min & 2x_1 + x_2 & & \\ \text{s.t.} & 3x_1 + x_2 & \geq & 6 \\ & x_1 + x_2 & \geq & 4 \\ & x_1, x_2 & \geq & 0. \end{array}$$

(a) After putting it in standard form, verify that the basis $\mathcal{B} = \{1, 2\}$ is optimal.

(b) Add the constraint $x_1 + 3x_2 = \alpha$. For what value of α is the problem feasible?

(c) Choose an α such that the problem is feasible and perform the necessary operations to find the new optimal solution.

a) $\min\ [2\ \ 1\ \ 0\ \ 0]\begin{bmatrix}x_1\\x_2\\s_1\\s_2\end{bmatrix}$

st $\begin{bmatrix}3 & 1 & -1 & 0\\ 1 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\s_1\\s_2\end{bmatrix} = \begin{bmatrix}6\\4\end{bmatrix}$

$x_1, x_2, s_1, s_2 \geq 0$

$\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}6\\4\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$

$3+3=6$ ✓ $\quad 1+3=4$ ✓ $\quad 1,3 \geq 0$

$\underline{s_1}|\ 0 - [2\ \ 1]\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}-1\\0\end{bmatrix} = 1/2 > 0$

$\underline{s_2}|\ 0 - [2\ \ 1]\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}0\\-1\end{bmatrix} = 1/2 > 0$

Thus $\{1,2\}$ is optimal and 5 is the optimal value

b) $\min\ [2\ \ 1\ \ 0\ \ 0]\begin{bmatrix}x_1\\x_2\\s_1\\s_2\end{bmatrix}$

st $\begin{bmatrix}3 & 1 & -1 & 0\\ 1 & 1 & 0 & -1\\ 1 & 3 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\s_1\\s_2\end{bmatrix} = \begin{bmatrix}6\\4\\\alpha\end{bmatrix}$

$x_1 = \alpha - 3x_2$

$3(\alpha - 3x_2) + x_2 - s_1 = 6 \qquad \alpha - 3x_2 + x_2 - s_2 = 4$

$3\alpha - 8x_2 - s_1 = 6 \qquad \alpha - 2x_2 - s_2 = 4$

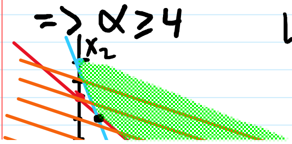
$3\alpha - 8x_2 - 6 = s_1 \qquad \alpha - 2x_2 - 4 = s_2$

$x_1 \geq 0 \Rightarrow x_2 \leq \frac{\alpha}{3} \qquad s_2 \geq 0 \Rightarrow x_2 \leq \frac{\alpha - 4}{2}$

$s_1 \geq 0 \Rightarrow x_2 \leq \frac{3\alpha - 6}{8}$

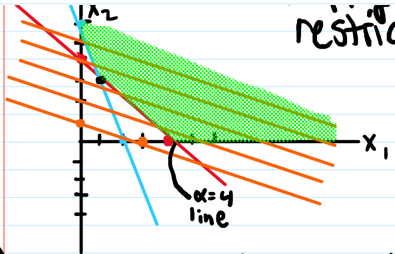
$0 \leq x_2 \leq \min\left\{\frac{\alpha}{3}, \frac{3\alpha - 6}{8}, \frac{\alpha - 4}{2}\right\}$

$\Rightarrow \alpha \geq 4$



Looking at this graph, the only restriction is $\alpha \geq 4$

restriction is $\alpha \geq 4$



c) Choose $\alpha = 4 \Rightarrow x_1 = 4 - 3x_2$

$3(4-3x_2)+x_2 \geq 6$ $\qquad$ $4-3x_2+x_2 \geq 4$

$12-9x_2+x_2 \geq 6$ $\qquad$ $x_2 \leq 0 \Rightarrow x_2 = 0$

$x_2 \leq 3/4$

min $2(4-3x_2)+x_2 = 8$ since $x_2 = 0$

$x_1 = 4 \quad x_2 = 0$

3. Consider the LP problem:

$$\begin{aligned} \min \quad & (1+\alpha)x_1 + (2-\alpha)x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \geq 9 \\ & x_1 + x_2 \geq 5 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Compute the values of α such that the problem is infeasible, unbounded, or has a finite optimal solution.

Directions of unbddness satisfy

$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} d \geq 0$, $d \geq 0$, and $(1+\alpha)d_1 + (2-\alpha)d_2 < 0$

$d_1 \geq 0,\ d_2 = 0)$

$\begin{bmatrix} 3 \\ 1 \end{bmatrix} d_1 \geq 0$, $d \geq 0$, and $(1+\alpha)d_1 < 0$

$(1+\alpha)d_1 < 0 \Rightarrow \alpha < -1$

$d_1 = 0,\ d_2 \geq 0)$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} d_2 \geq 0$, $d \geq 0$, and $(2-\alpha)d_2 < 0$

$(2-\alpha)d_2 < 0 \Rightarrow \alpha > 2$

Consider the region $[-1, 2]$

$\alpha = -1)$

min $3x_2 \Rightarrow x_2 = 0$

$\Rightarrow x_1 \geq 3,\ x_1 \geq 5$ $\quad$ so $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ is finite

optimal solution

$\alpha = 2)$

min $3x_1 \Rightarrow x_1 = 0$

$\Rightarrow x_2 \geq 5,\ x_2 \geq 9$ $\quad$ so $\begin{bmatrix} 0 \\ 9 \end{bmatrix}$ is finite

optimal solution

So $\alpha \in (-\infty, -1) \cup (2, \infty)$ the problem is unbounded and $\alpha \in [-1, 2]$ the problem has finite optimal solution

The constraints don't depend on α so α won't change feasibility

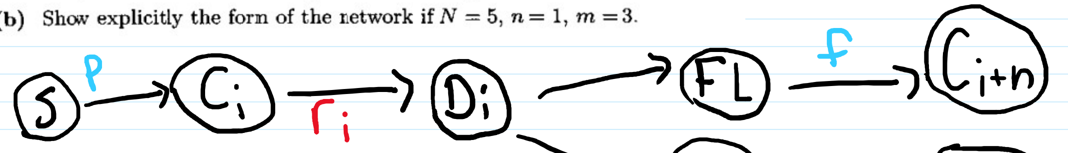
Exercise 7.1 (The caterer problem) A catering company must provide to a client r_i tablecloths on each of N consecutive days. The catering company can buy new tablecloths at a price of p dollars each, or launder the used ones. Laundering can be done at a fast service facility that makes the tablecloths unavailable for

Exercise 7.1 (The caterer problem) A catering company must provide to a client r_i tablecloths on each of N consecutive days. The catering company can buy new tablecloths at a price of p dollars each, or launder the used ones. Laundering can be done at a fast service facility that makes the tablecloths unavailable for the next n days and costs f dollars per tablecloth, or at a slower facility that makes tablecloths unavailable for the next m days (with $m > n$) at a cost of g dollars per tablecloth ($g < f$). The caterer's problem is to decide how to meet the client's demand at minimum cost, starting with no tablecloths and under the assumption that any leftover tablecloths have no value.

(a) Show that the problem can be formulated as a network flow problem. *Hint:* Use a node corresponding to clean tablecloths and a node corresponding to dirty tablecloths for each day; more nodes may also be needed.

(b) Show explicitly the form of the network if $N = 5$, $n = 1$, $m = 3$.

a)



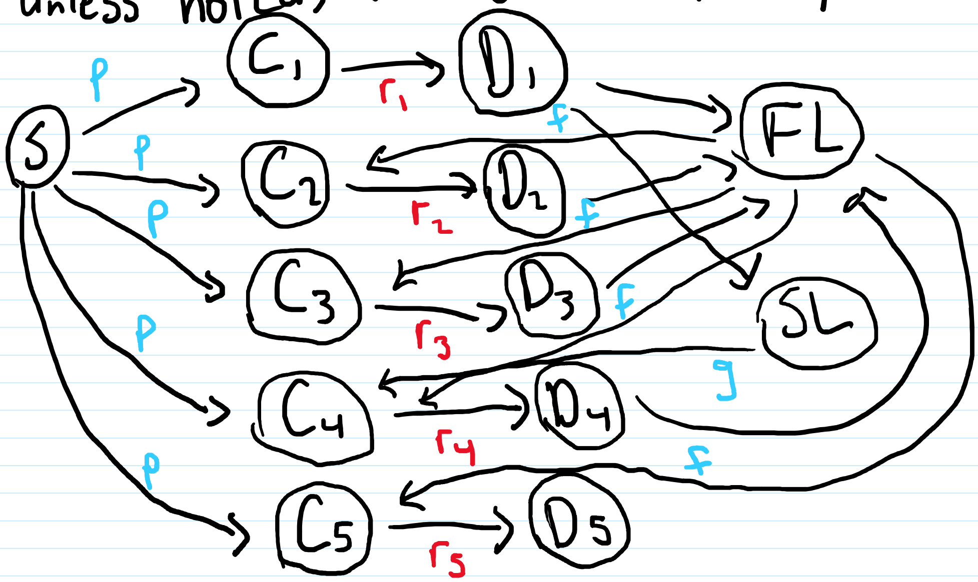
These nodes can purchase from source and lead to dirty nodes

price

capacity

Unless noted, assume ∞ capacity and 0 cost

b)



Exercise 7.9 Consider the uncapacitated network flow problem shown in Fi-ure 7.37. The label next to each arc is its cost.

(a) What is the matrix $\mathbf{A}$ corresponding to this problem?

(b) Solve the problem using the network simplex algorithm. Start with t tree indicated by the dashed arcs in the figure.

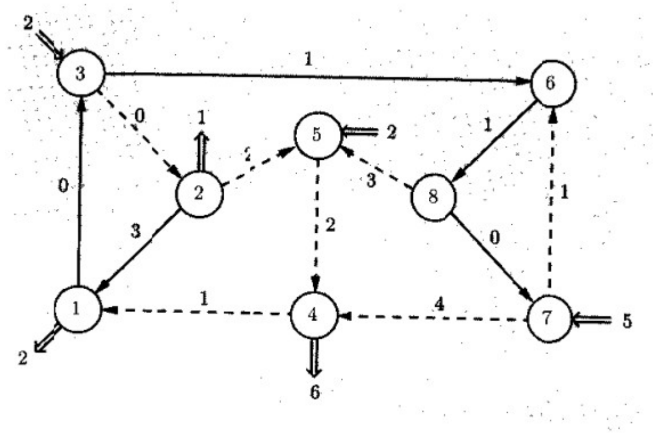


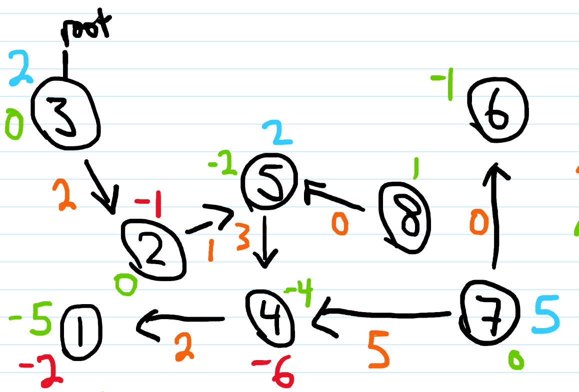
Figure 7.37: The network flow problem in Exercise 7.9.

a)

supply/demanel ... doted line arcs

$$A = \begin{array}{c|cccccccccccccccccc} & & & & & & & & * & & * & * & * & & & * & & * & * \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 5 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 \\ 7 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 \\ \hline \text{Costs} & 2 & 2 & 5 & 2 & 1 & 6 & 0 & 0 & 3 & 2 & 2 & 1 & 1 & 1 & 3 & 0 & 1 & 4 \end{array}$$

b)



source

sink

flow

dual

x_{32} $0-0+0=0$

x_{25} $2-0+{}^{-}2=0$

x_{85} $3-1+{}^{-}2=0$

x_{54} $2-{}^{-}2+{}^{-}4=0$

x_{47} $4-0+-4=0$

x_{76} $1-0+-1=0$

x_{41} $1-{}^{-}4+{}^{-}5=0$

Reduced Cost

$C-T+H$

x_{13} $0-{}^{-}5+0=5$

x_{36} $1-0+-1=0$

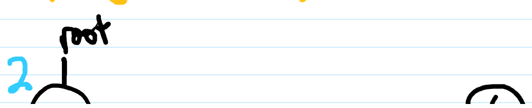
x_{68} $1-{}^{-}1+1=3$

x_{87} $0-1+0=-1$

x_{21} $3-0+-5=-2$

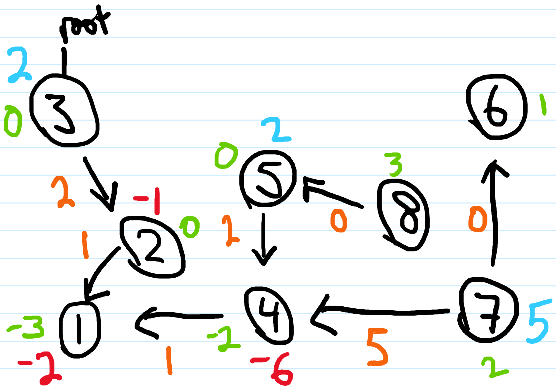
So x_{21} enters

x_{25} leaves



$C-T+H=0$

x_{32} $0-0+0=0$



$C - T + H = 0$

x_{32} $0 - 0 + 0 = 0$

x_{21} $3 - 0 + {}^{-}3 = 0$

x_{41} $1 - {}^{-}2 + {}^{-}3 = 0$

x_{54} $2 - 0 + {}^{-}2 = 0$

x_{85} $3 - 3 + 0 = 0$

x_{74} $4 - 2 + {}^{-}2 = 0$

x_{76} $1 - 2 + 1 = 0$

x_{13} $0 - {}^{-}3 + 0 = 3$

x_{36} $1 - 0 + 1 = 2$

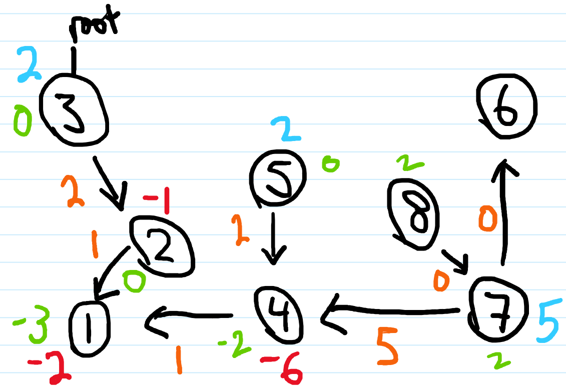
x_{68} $1 - 1 + 3 = 3$

x_{25} $2 - 0 + 0 = 2$

x_{87} $0 - 3 + 2 = -1$

x_{87} enters

x_{85} leaves



x_{87} $0 - 2 + 2 = 0$

$$\left.\begin{array}{ll} x_{13} & 0 - {}^{-}3 + 0 = 3 \\ x_{36} & 1 - 0 + 1 = 2 \\ x_{68} & 1 - 1 + 3 = 3 \\ x_{25} & 2 - 0 + 0 = 2 \\ x_{85} & 3 - 2 + 0 = 1 \end{array}\right\} \text{all positive}$$

So optimum is $\cancel{2 \cdot 0} + 1\cdot 3 + 1 \cdot 1 + 2 \cdot 2 + 5 \cdot 4 + \cancel{0 \cdot 0} + \cancel{0} \cdot 1 = 4 + 24 = 28$

Exercise 7.10 Consider the uncapacitated network flow problem shown in Figure 7.38. The label next to each arc is its cost. Consider the spanning tree indicated by the dashed arcs in the figure and the associated basic solution.

(a) What are the values of the arc flows corresponding to this basic solution? Is this a basic feasible solution?

(b) For this basic solution, find the reduced cost of each arc in the network.

(c) Is this basic solution optimal?

(d) Does there exist a nondegenerate basic feasible solution?

(e) Find an optimal dual solution.

(f) By how much can we increase c_{56} [the cost of arc (5,6)] and still have the same optimal basic feasible solution?

(g) If we increase the supply at node 1 and the demand at node 9 by a small positive amount δ, what is the change in the value of the optimal cost?

(h) Does this problem have a special structure that makes it simpler than the general uncapacitated network flow problem?

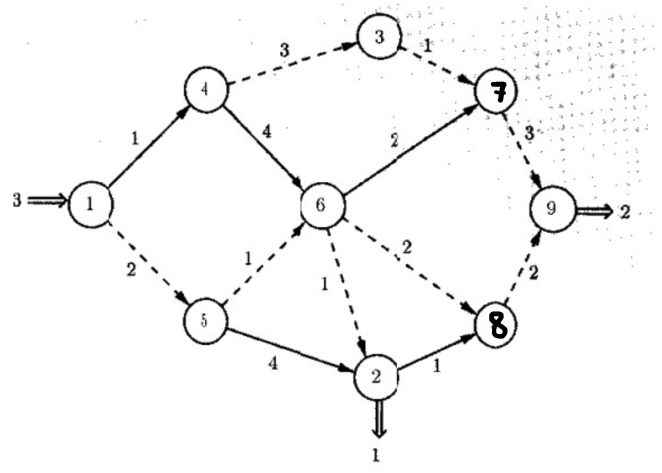
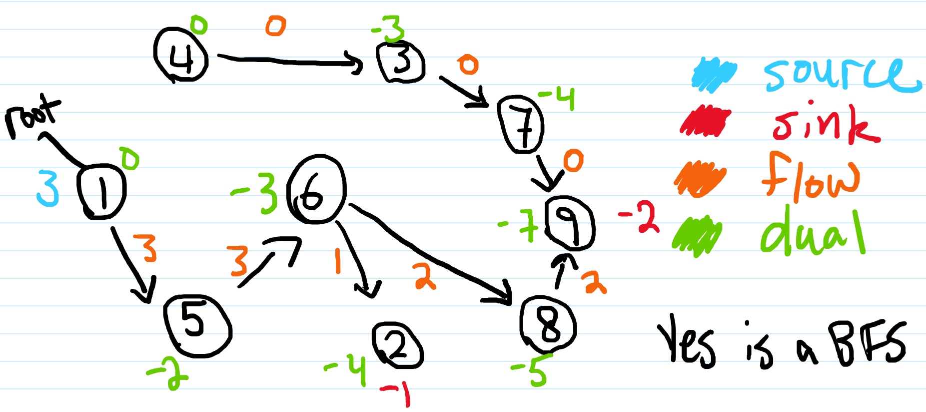


Figure 7.38: The network flow problem in Exercise 7.10.

a)



Yes is a BFS

$C - T + H = 0$

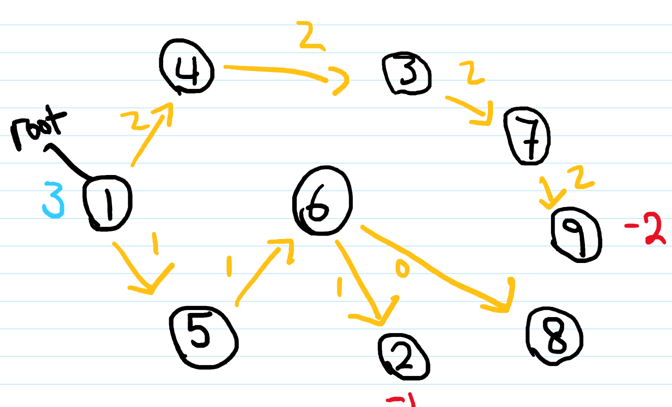
x_{15}	$2 - 0 + {-2} = 0$
x_{56}	$1 - {-2} + {-3} = 0$
x_{62}	$1 - {-3} + {-4} = 0$
x_{68}	$2 - {-3} + {-5} = 0$
x_{89}	$2 - {-5} + {-7} = 0$
x_{79}	$3 - {-4} + {-7} = 0$
x_{37}	$1 - {-3} + {-4} = 0$
x_{43}	$3 - 0 + {-3} = 0$

b)

x_{14}	$1 - 0 + 0 = 1$
x_{46}	$4 - 0 + {-3} = 1$
x_{67}	$2 - {-3} + {-4} = 1$
x_{52}	$4 - {-2} + {-4} = 2$
x_{28}	$1 - {-4} + {-5} = 0$

c) Yes it is optimal with value $3 \cdot 2 + 3 \cdot 1 + 1 \cdot 1 + 2 \cdot 2 + 2 \cdot 2 = 18$

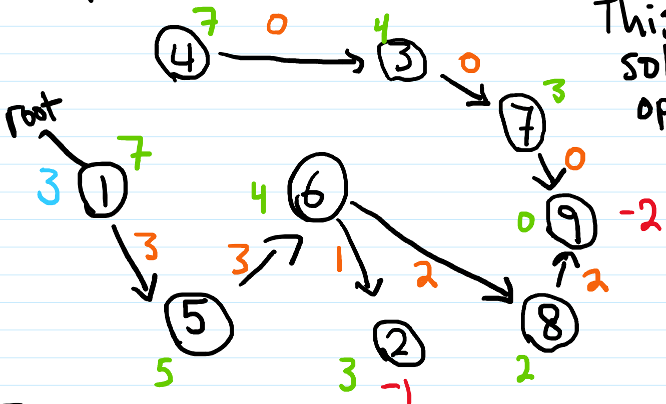
d)



Since the only other option to avoid degeneracy introduces degeneracy in arc 68 so there does not exist a nondegenerate solution without a cycle.

e) Since the dual values we calculated are potentials, we can shift all of them to make sure all dual variables

e) Since the dual values we calculated are potentials, we can shift all of them to make sure all dual variables are positive



This gives optimal dual solution since it is primally optimal and satisfies complementary slackness by construction.
With optimal value of 18
$(21-3+0=18)$

f) I chose to do the problem with excel to visual the changes easier

$C_{56}=1$

	C	T	H	=
x15	2	0	-2	0
x56	1	-2	-3	0
x62	1	-3	-4	0
x68	2	-3	-5	0
x89	2	-5	-7	0
x79	3	-4	-7	0
x37	1	-3	-4	0
x43	3	0	-3	0
x14	1	0	0	1
x46	4	0	-3	1
x67	2	3	-4	1
x52	4	2	-4	2
x28	1	4	-5	0

$C_{56}=2$

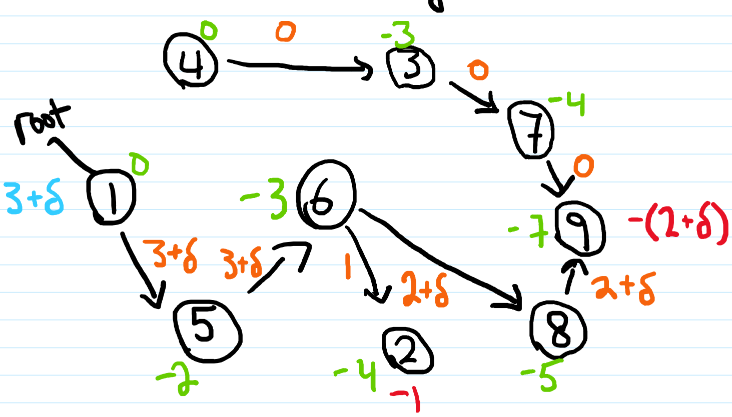
	C	T	H	=
x15	2	0	-2	0
x56	2	-2	-4	0
x62	1	-4	-5	0
x68	2	-4	-6	0
x89	2	-6	-8	0
x79	3	-5	-8	0
x37	1	-4	-5	0
x43	3	-1	-4	0
x14	1	0	-1	0
x46	4	1	-4	1
x67	2	4	-5	1
x52	4	2	-5	1
x28	1	5	-6	0

$C_{56}=3$

	C	T	H	=
x15	2	0	-2	0
x56	3	-2	-5	0
x62	1	-5	-6	0
x68	2	-5	-7	0
x89	2	-7	-9	0
x79	3	-6	-9	0
x37	1	-5	-6	0
x43	3	-2	-5	0
x14	1	0	-2	-1
x46	4	2	-5	1
x67	2	5	-6	1
x52	4	2	-6	0
x28	1	6	-7	0

Since we are only using integer values, the reduced costs will also be integer values.
Thus 2 is the largest C_{56} can be to keep optimality

g)



$$2(3+\delta)+(3+\delta)+1+2(2+\delta)+2(2+\delta)=\boxed{18+7\delta}$$

h) There are no cycles and the flow generally moves forward toward the sink. It does not backtrack which also lets us order the nodes.

Exercise 7.22 **(Connectivity and vulnerability)** Consider a directed graph, and let us fix an origin node s and a destination node t. We define the *connectivity* of the graph as the maximum number of directed paths from s to t that do not share any nodes. We define the *vulnerability* of the graph as the minimum number of nodes (besides s and t) that need to be removed so that there exists no directed path from s to t. Prove that connectivity is equal to vulnerability. *Hint:* Convert the connectivity problem to a maximum flow problem.

For each node that is not the source or sink, split the node into e_{in} and e_{out} with an arc between them that has capicity of 1. For all original arcs, set capacity to ∞.
From flow decomposition, the flow can be decomposed into paths that each contribute 1 unit. Let there be n paths from $s \to t$. Thus the max flow is n.
A min cut of this network would go between all of the e_{in}/e_{out} arcs which would be n cuts and would remove all arcs between s and t.
By max flow min cut theorem, the connectivity and vulnerability are both equal to n.

Exercise 7.31 **(Dual simplex method for network flow problems)** Consider the uncapacitated network flow problem.

(a) Show that every spanning tree determines a basic solution to the dual probem.

(b) Given a basic feasible solution to the dual problem, associated with a certain tree, show that it is optimal if and only if the corresponding tree solution to the primal is feasible.

(c) If the tree solution in part (b) is infeasible, remove an arc that carries negative flow. Given that we wish to maintain dual feasibility, how should an arc be chosen to enter the tree?

(d) Note that the entering arc divides the tree into two parts. Consider the dual variables following a dual simplex update. Show that the dual variables in one part of the tree remain unchanged and in the other part of the tree, they are all changed by the same amount.

a) For a spanning tree T with $|V|-1$ arcs by looking at dual constraints $\pi_i - \pi_j \le c_{ij}$
By fixing a π_r for the root of the tree, we can determine the other $|V|-1$ π_i variables using a system of equations and binding constraints.
Thus we have $|V|-1$ system of linearly independent equations that implies it is a basic solution.

b) ($\Rightarrow$) The primal solution x is determined by the flow balance constraints and setting each $x_{ij}=0$ $\forall (i,j) \notin T$.
By complementary slackness for each $x_{ij}>0$, $\pi_i - \pi_j = c_{ij}$
Thus X is primally feasible.
($\Leftarrow$) Similarly $\forall (i,j) \in T$ $\pi_i - \pi_j = c_{ij}$ and $\forall (i,j) \notin T$ $\pi_i - \pi_j \le c_{ij}$ by complementary slackness. Thus the tree is a BFS for the dual problem.

c) We want to keep all reduced costs positive. We first cut the tree at the negative flow arc, splitting the tree into

c) We want to keep all reduced costs positive. We first cut the tree at the negative flow arc, splitting the tree into two disjoint sets R (for root) and NR (for no root). We will choose the arc with the minimum reduced cost going in the opposite direction of the leaving arc and keeps the two sets connected.

d) Assuming the potentials are based off of the root potential set at 0. Since all potentials in R are based on the root potential nothing will change in R.
For NR, we have to update all of the potentials such that the entering arc has a reduced cost of 0. This propagates to every node in NR and they all change by this amount.