

MATH 8100 Mathematical Programming Spring 2025  
First Midterm Exam

February 17–18, 2025

- This take-home test is due in Gradescope at noon on February 18, 2025. Be sure to mark your answers for each of the questions in Gradescope when you upload.
- There are a total of 70 points. Point value is listed next to each question.
- Mark your answers clearly. *Show your work.* Unsupported correct answers receive partial credit.
- Good luck!

Name:\_\_\_\_\_

Student ID #:\_\_\_\_\_

I certify that I have not received any unauthorized assistance in completing this examination.

Signature:\_\_\_\_\_

Date:\_\_\_\_\_

“Without data, you’re just another person with an opinion.”

—Anon.

- (8 points) Recall that a cone is a set  $S \subseteq \mathbb{R}^n$  such that  $\mathbf{x} \in S$  implies  $\alpha\mathbf{x} \in S$  for all  $\alpha \geq 0$ . Prove that a set  $S \subseteq \mathbb{R}^n$  is a convex cone if and only if it is closed under vector addition and nonnegative scalar multiplication.
- (7 points) Let  $g_1, g_2, \dots, g_m$  be concave functions on  $\mathbb{R}^n$ ,  $f$  be a convex function on  $\mathbb{R}^n$ , and  $\mu$  be a positive constant. Show that the function

$$\beta(\mathbf{x}) = f(\mathbf{x}) - \mu \sum_{i=1}^m \ln g_i(\mathbf{x})$$

is convex on the set  $S = \{\mathbf{x} : g_i(\mathbf{x}) > 0, i = 1, 2, \dots, m\}$ .

- (6 points each part) Consider the function

$$f(x_1, x_2) = \frac{1}{3}x_1^3 + \frac{1}{2}x_1^2 + 2x_1x_2 + \frac{1}{2}x_2^2 - x_2 + 9.$$

For each of the following points,

- determine if the point is stationary;
  - if stationary, determine if the point is a local minimum, maximum, or neither;
  - if not stationary, identify a descent direction.
- (a)  $\mathbf{x} = (1, -1)$   
 (b)  $\mathbf{x} = (2, -3)$   
 (c)  $\mathbf{x} = (0, 0)$

- Consider the primal-dual pair

$$\begin{aligned} & \text{minimize} && p = f(\mathbf{x}) \\ & \text{subject to} && g_i(\mathbf{x}) \leq 0 \quad i = 1, 2, \dots, m \\ & && \mathbf{x} \in X \end{aligned} \tag{1}$$

and

$$\begin{aligned} & \text{maximize} && d = L(\boldsymbol{\lambda}) = \min_{\mathbf{x} \in X} (f(\mathbf{x}) + \sum_{i=1}^m \lambda_i g_i(\mathbf{x})) \\ & \text{subject to} && \boldsymbol{\lambda} \geq \mathbf{0}. \end{aligned} \tag{2}$$

Suppose  $\hat{\mathbf{x}} \in X$  and  $\hat{\boldsymbol{\lambda}} \geq \mathbf{0}$  satisfy the following saddle point condition for all  $\mathbf{x} \in X$  and all  $\boldsymbol{\lambda} \geq \mathbf{0}$ :

$$f(\hat{\mathbf{x}}) + \sum_{i=1}^m \lambda_i g_i(\hat{\mathbf{x}}) \leq f(\hat{\mathbf{x}}) + \sum_{i=1}^m \hat{\lambda}_i g_i(\hat{\mathbf{x}}) \leq f(\mathbf{x}) + \sum_{i=1}^m \hat{\lambda}_i g_i(\mathbf{x}).$$

- (8 points) Show that  $\hat{\mathbf{x}}$  solves the primal nonlinear program (1). (Hint: Use the saddle point condition to show that complementary slackness holds at  $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}})$ .)

(b) (7 points) Show that the saddle point condition implies strong duality, i.e.,  $p^* = d^*$  where  $p^*$  is the optimal value of (1) and  $d^*$  is the optimal value of (2).

5. (8 points) Solve the following problem.

$$\begin{aligned} & \text{minimize}_{\boldsymbol{x} \in X} \quad \sum_{i=1}^n x_i \ln x_i \\ & \text{subject to} \quad \sum_{i=1}^n x_i = 1 \end{aligned}$$

where  $X = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} > \mathbf{0}\}$ .

6. (a) (7 points) Consider the problem

$$\begin{aligned} & \text{minimize} \quad x_1^3 + x_2 \\ & \text{subject to} \quad x_2 \geq 1. \end{aligned}$$

holds. Can we conclude that the KKT point is an optimal solution? Why or why not?

(b) (7 points) Consider the problem

$$\begin{aligned} & \text{minimize} \quad x_1^2 + x_2 \\ & \text{subject to} \quad x_2 \geq 1. \end{aligned}$$

holds. Can we conclude that the KKT point is an optimal solution? Why or why not?