MATH 8100, Spring 2025
Homework 2

Due: February 12, 2025

1. Consider the nonlinear program:

$$\begin{array}{ll} \min & 4(x_1-1)^2+(x_2-1)^2 \\ \text{s.t..} & 16x_1+6x_2=63 \end{array}$$

   (a) Form the Lagrangian function for this model.

   (b) Write the stationary conditions for the Lagrangian.

   (c) Solve your stationary condition for x_1 and x_2. Explain why your answers are optimal for the original model.

   (d) Explain why the constraint $x_1 \leq 2$ would be active if added to the original model.

   (e) Use the Lagrangian approach to compute an optimal solution to the model with the added constraint.

2. Consider the nonlinear program:

$$\begin{array}{ll} \min & x_1 \\ \text{s.t.} & x_1^2-x_2=0 \\ & 2x_1-x_2=1 \end{array}$$

   (a) Form the Lagrangian function for this model.

   (b) Write the stationary conditions for the Lagrangian.

   (c) Solve your stationary condition for x_1 and x_2.

   (d) Identify an optimal solution to this problem. Explain your reasoning.

   (e) Explain the relationship between your answers to the previous two parts.

3. For the following mathematical programs, state the KKT conditions and determine whether a KKT point is a global optimum

   (a)
$$\begin{array}{ll} \max & 6x_1 + 40x_2 + 5x_3 \\ \text{s.t.} & x_1 \sin x_2 + 9x_3 \le 2 \\ & e^{18x_1+3x_2} + 14x_3 \ge 50 \\ & x_1, x_2 \ge 0 \end{array}$$

   (b)
$$\begin{array}{ll} \max & 7\ln x_1 + 4\ln x_2 + 11\ln x_3 \\ \text{s.t.} & (x_1+2)^2 - x_1x_2 + (x_2-7)^2 \le 80 \\ & 5x_1 + 7x_3 = 22 \\ & x_1, x_2, x_3 \ge 1 \end{array}$$

4. We will show in this problem that redundant constraints in constraint optimization can be useful. Consider the non-convex optimization problem

$$\begin{aligned} &\min_{x_1,x_2\in\mathbb{R}} x_1x_2 \\ &\text{s.t.} x_1^2 + x_2^2 \le 1, x_1 \ge 0, \text{ and } x_2 \ge 0. \end{aligned} \tag{1}$$

and

$$\begin{aligned} &\min_{x_1,x_2\in\mathbb{R}} x_1x_2 \\ &\text{s.t.} x_1^2 + x_2^2 \le 1, x_1x_2 \ge 0, x_1 \ge 0, \text{ and } x_2 \ge 0. \end{aligned} \tag{2}$$

The only difference between (1) and (2) is that there is a redundant constraint $x_1x_2 \ge 0$ in (2). Indeed, both problems have exactly the same feasible set and objective function, with optimal primal objective value 0.

   (a) List the KKT conditions of both problems.
   (b) Show that the optimal value of the dual problem to (1) is $-1/2$.
   (c) Show that the optimal value of the dual problem to (2) is 0.