MATH 8100 Mathematical Programming Spring 2025
Second Midterm Exam

April 9, 2025

- This take-home exam is due in Gradescope by 12:00 noon Friday, April 11, 2025.
- You may use only your textbook, course Canvas materials, your notes, the online pivot tool, and your instructor as resources.
- There are a total of 65 points. Point value is listed next to each question.
- Mark your answers clearly in the space provided. Gradescope is set to recognize the page formats, so you don't need to select your answers when you submit.
- *Show your work.* Unsupported correct answers receive partial credit.
- Be sure to write your name and ID number on *each* page.
- Good luck!

Name:____

Student ID #:____________________

I certify that I have not received any unauthorized assistance in completing this examination.

Signature:____

Date:____________________

The word model is used as a noun, adjective, and verb, and in each instance it has a slightly different connotation. As a noun "model" is a representation in the sense in which an architect constructs a small-scale model of a building or a physicist a large-scale model of an atom. As an adjective "model" implies a degree of perfection or idealization, as in reference to a model home, a model student, or a model husband. As a verb "to model" means to demonstrate, to reveal, to show what a thing is like. —Russell Ackoff

1. (5 points) Is $P = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2,\ x - 2y = 0\}$ a polyhedron? Explain.

2. Consider the following linear program in standard form:

$$\begin{array}{rrcrcrcrcl} \text{minimize} & 4x_1 & + & 3x_2 & + & 2x_3 & + & 4x_4 & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & & & = & 4 \\ & x_1 & + & 2x_2 & & & + & x_4 & = & 12 \\ & & & & & \multicolumn{3}{r}{x_1, x_2, x_3, x_4} & \geq & 0 \end{array}$$

   (a) (3 points) Show that $(0, 0, 4, 12)$ is a BFS.

   (b) (7 points) Solve the LP with the simplex method, using Bland's rule to select entering and leaving variables, starting with the BFS $(0, 0, 4, 12)$. Report your pivot entering and leaving variables for each iteration, your final dictionary, the final primal and dual solutions, and the optimal objective value.

   (c) (3 points) Find the basic solution with x_2 and x_3 basic. Is it feasible for the primal? Is it feasible for the dual? Explain.

   (d) (5 points) Solve the LP starting from the basis in part 2c using the dual simplex method. Select pivots using Bland's rule.

   (e) (5 points) Consider the problem above, with greater-or-equal constraints replacing the equalities. Write the representation of the feasible set as convex combinations of extreme points plus a conic combination of extreme directions.

   (f) (7 points) For each point you found in part 2e, compute the objective value. Formulate an LP equivalent to that of part 2e where the variables are the weights of a convex combination of the extreme points and a conic combination of the extreme directions that you found.

3. (10 points) Prove the following theorem of alternative:

   Exactly one of the following systems has a solution:

   - There exists $\boldsymbol{x} \neq \boldsymbol{0}$ such that $A\boldsymbol{x} = \boldsymbol{0}$, $\boldsymbol{x} \geq \boldsymbol{0}$.
   - There exists $\boldsymbol{u}$ such that $\boldsymbol{u}^T A > \boldsymbol{0}^T$.

4. (10 points) Consider the LP in standard form and its dual:

$$\begin{array}{rrcl} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} & & \\ \text{subject to} & A\boldsymbol{x} & = & \boldsymbol{b} \\ & \boldsymbol{x} & \geq & \boldsymbol{0} \end{array} \qquad \begin{array}{rrcl} \text{maximize} & \boldsymbol{b}^T\boldsymbol{y} & & \\ \text{subject to} & A^T\boldsymbol{y} & \leq & \boldsymbol{c} \end{array}$$

   Suppose that $x_j^* = 0$ for any optimal solution $\boldsymbol{x}^*$. Show that there exists a dual solution $\boldsymbol{y}^*$ such that $A_{\cdot j}^T\boldsymbol{y}^* < c_j$.

5. (10 points) Consider the following two linear programs:

$$\begin{array}{rl} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \tag{1}$$

and

$$\begin{array}{rc} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{d} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \tag{2}$$

Suppose that (1) has a finite solution. Prove that if (2) has a feasible solution, it has a finite optimal solution.