HW5

Friday, March 21, 2025 8:41 PM

1. Consider the following equivalent linear programs:

$$\begin{array}{ll} \text{minimize} & x_1 - x_2 \\ \text{subject to} & x_1 + 2x_2 \geq 4 \\ & -x_2 \geq -4 \\ & 3x_1 - 2x_2 \leq 0 \\ & x_1 \text{free}, x_2 \geq 0 \end{array}$$

and

$$\begin{array}{ll} \text{minimize} & x_1^+ - x_1^- - x_2 \\ \text{subject to} & x_1^+ - x_1^- + 2x_2 \geq 4 \\ & -x_2 \geq -4 \\ & -x_1^+ + 3x_1^- + 2x_2 \geq 0 \\ & x_1^+, x_1^-, x_2 \geq 0. \end{array}$$

(I assume there is a typo here

or else the problems are not equivalent

Write the dual problems and analyze the equivalence between them.

a) $\mathcal{L}(x_1, x_2; y) = (x_1 - x_2) + y_1(4 - x_1 - 2x_2) + y_2(x_2 - 4) + y_3(-3x_1 + 2x_2)$

$\mathcal{L}_{x_1} = 1 - y_1 - 3y_3 \leq 0 \Rightarrow y_1 - 3y_3 \leq 1$

$\mathcal{L}_{x_2} = -1 - 2y_1 + y_2 + 2y_3 \leq 0 \Rightarrow -2y_1 + y_2 + 2y_3 \leq 1$

$\min_{x_1, x_2} \mathcal{L}(x_1, x_2; y) = 4y_1 - 4y_2$

$$\begin{array}{ll} \max_{y_1, y_2, y_3} & 4y_1 - 4y_2 \\ \text{st} & y_1 - 3y_3 \leq 1 \\ & -2y_1 + y_2 + 2y_3 \leq 1 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

$\mathcal{L}(x_1^+, x_1^-, x_2; y) = (x_1^+ - x_2^- - x_2) + y_1(4 - x_1^+ + x_1^- - 2x_2) + y_2(-4 + x_2) + y_3(x_1^+ - 3x_1^- - 2x_2)$

$\mathcal{L}_{x_1^+} = 1 - y_1 - 3y_3 \leq 0 \Rightarrow -y_1 - 3y_3 \leq 1$

$\mathcal{L}_{x_1^-} = -1 + y_1 + 3y_3 \leq 0 \Rightarrow y_1 + 3y_3 \leq 1$

$\mathcal{L}_{x_2} = -1 - 2y_1 + y_2 - 2y_3 \leq 0 \Rightarrow -2y_1 + y_2 - 2y_3 \leq 1$

$\min_{x_1^+, x_1^-, x_2} \mathcal{L}(x_1^+, x_1^-, x_2; y) = 4y_1 - 4y_2$

$$\begin{array}{ll} \max & 4y_1 - 4y_2 \\ \text{st} & -y_1 - 3y_3 \leq 1 \\ & y_1 + 3y_3 \leq 1 \\ & -2y_1 + y_2 - 2y_3 \leq 1 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

Assuming the typo, these duals are nearly the same

Assuming the typo, these duals are nearly the same just the signs on the LHS of second problem are flipped which came from splitting $x_1 = x_1^+ - x_1^-$

2. Consider the following problem:

$$\begin{array}{llcllcl} \min & 6x_1 & + & 4x_2 & & & \\ s.t. & x_1 & + & x_2 & = & 1 \\ & 3x_1 & + & x_2 & \geq & 0 \\ & x_1 & & & \leq & 0 \\ & & & x_2 & \geq & 0. \end{array}$$

(a) Write its dual;

(b) Solve the dual with the graphical method;

(c) Find an optimal solution of the primal by applying complementary slackness.

(d) Verify the primal's optimal solution by solving the primal through the graphical method.

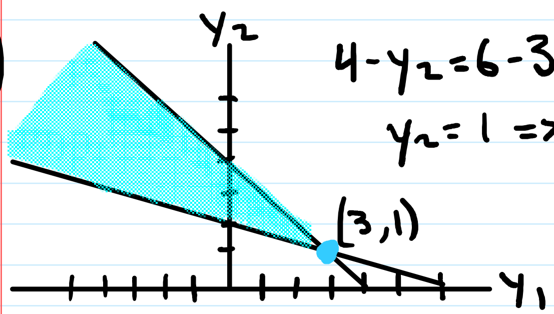
a) $\mathcal{L}(x_1,x_2;y) = (6x_1+4x_2)+y_1(1-x_1-x_2)+y_2(-3x_1-x_2)$

$\mathcal{L}_{x_1} = 6+y_1+3y_2 \leq 0 \Rightarrow y_1+3y_2 \leq 6$

$\mathcal{L}_{x_2} = 4-y_1-y_2 \leq 0 \Rightarrow y_1+y_2 \geq 4$

$\min_{x_1,x_2} \mathcal{L}(x_1,x_2;y) = y_1$

$$\begin{array}{ll} \max & y_1 \\ \text{st} & y_1+3y_2 \leq 6 \\ & y_1+y_2 \geq 4 \\ & y_1 \text{ free} \\ & y_2 \geq 0 \end{array}$$

b) 

$4-y_2 = 6-3y_2$

$y_2 = 1 \Rightarrow y_1 = 3$

So $\max y_1 = 3$

c) $x_1+x_2=1$ is always active because equality

$y_2 = 1 \neq 0$ so $3x_1+x_2=0$

Thus $\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow x_2 = -3x_1 = \frac{3}{2}$

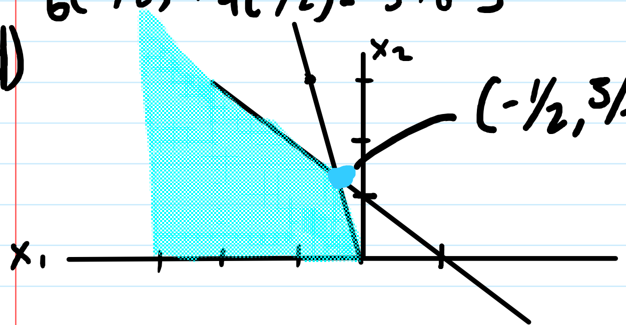
$x_1 - 3x_1 = 1$

$x_1 = -1/2$

$6(-1/2)+4(3/2) = -3+6=3$

$x_1 = -1/2$

d) $6(-1/2) + 4(3/2) = -3 + 6 = 3$



$(-1/2, 3/2) \Rightarrow \min\ 6x_1 + 4x_2 = 3$

3. What does the Lagrangian function $\mathcal{L}(\boldsymbol{\lambda})$ look like for an optimization problem? Apply it to the Knapsack problem (for simplicity, we'll use its minimization variant):

$$\begin{array}{ll} \min & \boldsymbol{c}^T\boldsymbol{x} \\ s.t. & \boldsymbol{a}^T\boldsymbol{x} \geq b \\ & \boldsymbol{x} \in \{0,1\}^n. \end{array}$$

(a) Write the LP relaxation of this problem by relaxing integrality on the variables (but not their bounds!). Call this problem (LP).

(b) Apply Lagrangian relaxation to (LP) by relaxing $\boldsymbol{a}^T\boldsymbol{x} \geq b$ only.

(c) The newly constructed problem should only depend on one (scalar) parameter λ. Write the value of the optimal solution for any λ. Note: relaxing the linear constraint makes it a very trivial optimization problem.

(d) Depict the function $y = \mathcal{L}(\lambda)$ on the Cartesian plane (λ, y) for the following values of $\boldsymbol{a}$, $\boldsymbol{c}$, and b (note that $\boldsymbol{a}$ and $\boldsymbol{c}$ are sorted in increasing order of $\frac{c_i}{a_i}$):

$n = 4$; $b = 20$;

i	1	2	3	4
a_i	9	7	4	3
c_i	2	3	6	7

(e) What is the maximum value of $\mathcal{L}(\lambda)$?

(f) What value of λ does it correspond to?

(g) What is the value of the optimal solution of (LP)? Find out with AMPL and CPLEX or Gurobi—no need to submit code, it will just save you some time—or solve it by hand.

a) LP $\min\ c^T x$
$\quad$ st $a^T x \geq b$
$\quad 0 \leq x \leq 1$

b) $\mathcal{L}(\lambda; x) = c^T x + \lambda(b - a^T x)$ $\qquad \lambda > 0$

$\min_x \mathcal{L}(\lambda; x) = \lambda b + \min_x (c^T - \lambda a^T)x$

Since $\lambda > 0$ it follows

$$\min_x (c^T - \lambda a^T)x = \min_{0 \leq x_i \leq 1} (c_i - \lambda a_i)x_i = \begin{cases} 0 & c_i - \lambda a_i > 0 \\ c_i - \lambda a_i & c_i - \lambda a_i \leq 0 \end{cases}$$

$\mathcal{L}(\lambda; x) = \lambda b + \sum_{i=1}^{n} \min(0, c_i - \lambda a_i)$

c) $g(\lambda) = \max_{\lambda \geq 0} \lambda b + \sum_{i=1}^{n} \min(0, c_i - \lambda a_i)$

d) $\mathcal{L}(\lambda; x) = 20\lambda + \sum_{i=1}^{n} \min(0, c_i - \lambda a_i)$ This will be piecewise

Note $\lambda \geq 0$

For $\lambda \in [0, \frac{2}{9}]$ $\sum_{i=1}^{n} \min(0, c_i - \lambda a_i) = 0 \Rightarrow \mathcal{L}(\lambda; x) = 20\lambda$

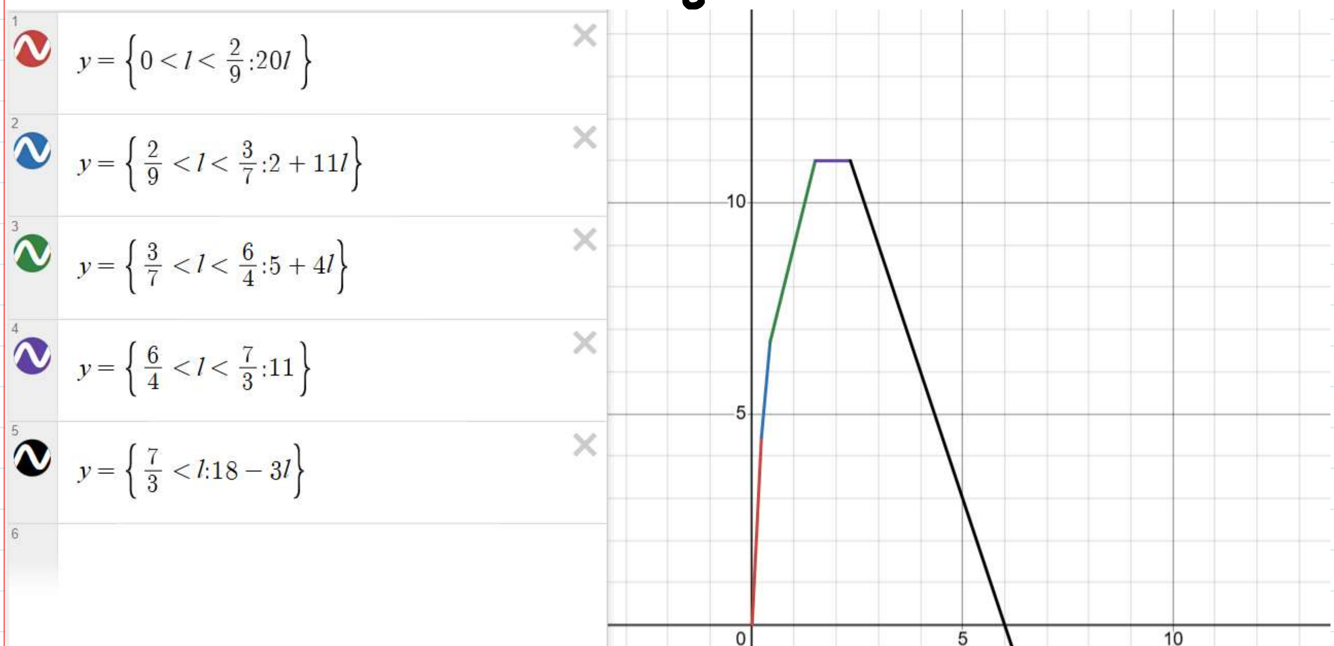
For $\lambda \in [\frac{2}{9}, \frac{3}{7}]$, $c_1 - \lambda a_1 \leq 0 \Rightarrow \mathcal{L}(\lambda; x) = 20\lambda + 2 - 9\lambda = 2 + 11\lambda$

For $\lambda \in [\frac{3}{7}, \frac{6}{4}]$ $c_{1,2} - \lambda a_{1,2} \leq 0 \Rightarrow \mathcal{L}(\lambda; x) = 2 + 11\lambda + 3 - 7\lambda = 5 + 4\lambda$

For $\lambda \in [\frac{6}{4}, \frac{7}{3}]$ $c_{1,2,3} - \lambda a_{1,2,3} \leq 0 \Rightarrow \mathcal{L}(\lambda; x) = 5 + 4\lambda + 6 - 4\lambda = 11$

For $\lambda \geq 7/3$ $c_i - \lambda a_i \leq 0 \ \forall i \Rightarrow \mathcal{L}(\lambda; x) = 11 + 7 - 3\lambda = 18 - 3\lambda$

These values are hard to graph so here is desmos



e/f) $\lambda \in [6/4, 7/3]$ gives max value of $y = 11$

g) Assuming we are talking about the same n, b, a, c from d)

The max of $\mathcal{L}(\lambda; x) = 11$ when $x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

$a^T x = 9 + 7 + 4 + 0 = 20 \geq 20$ (feasible)

$c^T x = 2 + 3 + 6 + 0 = 11$ (no duality gap)

Thus this is optimal since $\max \mathcal{L}(\lambda; x) = \min c^T x$

4. Solve the following problem using dictionaries and starting from basis $\mathcal{B} = \{3, 4, 5\}$, after reducing the problem to standard form:

$$\begin{array}{lrcrcr} \text{minimize} & 5x_1 & + & 3x_2 & & \\ \text{subject to} & -x_1 & + & x_2 & \leq & -2 \\ & x_1 & - & x_2 & \leq & 2 \\ & -2x_1 & + & x_2 & \leq & -5 \\ & x_1, & & x_2 & \geq & 0. \end{array}$$

The dictionary will be infeasible but with non-negative reduced cost. Apply dual-simplex pivoting operations until feasibility is reached. Represent the value of x_1 and x_2 on the Cartesian plane for every basis visited.

$$\begin{array}{ll} \min & 5x_1 + 3x_2 \\ \text{st} & -x_1 + x_2 + x_3 = -2 \\ & x_1 - x_2 + x_4 = 2 \\ & -2x_1 + x_2 + x_5 = -5 \end{array}$$

$$\min \begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -5 \end{bmatrix}$$

$x \geq 0$

$$\boxed{\begin{aligned} x_3 &= -2 + x_1 - x_2 \\ x_4 &= 2 - x_1 + x_2 \\ x_5 &= -5 + 2x_1 - x_2 \\ z &= 5x_1 + 3x_2 \end{aligned}}$$

-5 is smallest

$\left|\frac{5}{2}\right| < \left|\frac{3}{-1}\right|$ so x_5 in x_1 out

$x_1 = \frac{5}{2} + \frac{1}{2}x_5 + \frac{1}{2}x_2$

$x_3 = -2 + \frac{5}{2} + \frac{1}{2}x_5 + \frac{1}{2}x_2 - x_2 = \frac{1}{2} + \frac{1}{2}x_5 - \frac{1}{2}x_2$

$x_4 = 2 - (\frac{5}{2} + \frac{1}{2}x_5 + \frac{1}{2}x_2) + x_2 = -\frac{1}{2} - \frac{1}{2}x_5 + \frac{1}{2}x_2$

$z = 5(\frac{5}{2} + \frac{1}{2}x_5 + \frac{1}{2}x_2) + 3x_2 = \frac{25}{2} + \frac{5}{2}x_5 + \frac{11}{2}x_2$

$-\frac{1}{2} < 0$

$\left|\frac{-1}{2} / \frac{5}{2}\right| = \frac{2}{10} = \frac{1}{5}$

$\left|\frac{1}{2} / \frac{11}{2}\right| = \frac{1}{11}$

x_2 out x_4 in

$$\boxed{\begin{aligned} x_1 &= \frac{5}{2} + \frac{1}{2}x_5 + \frac{1}{2}x_2 \\ x_3 &= \frac{1}{2} + \frac{1}{2}x_5 - \frac{1}{2}x_2 \\ x_4 &= -\frac{1}{2} - \frac{1}{2}x_5 + \frac{1}{2}x_2 \\ z &= \frac{25}{2} + \frac{5}{2}x_5 + \frac{11}{2}x_2 \end{aligned}}$$

$x_2 = 1 + 2x_4 + x_5$

$x_1 = \frac{5}{2} + \frac{1}{2}x_5 + \frac{1}{2}(1 + 2x_4 + x_5) = 3 + x_4 + x_5$

$x_3 = \frac{1}{2} + \frac{1}{2}x_5 - \frac{1}{2}(1 + 2x_4 + x_5) = -x_4$

$z = \frac{25}{2} + \frac{5}{2}x_5 + \frac{11}{2}(1 + 2x_4 + x_5) = 18 + 11x_4 + 8x_5$

$$\boxed{\begin{aligned} x_2 &= 1 + 2x_4 + x_5 \\ x_1 &= 3 + x_4 + x_5 \\ x_3 &= -x_4 \\ z &= 18 + 11x_4 + 8x_5 \end{aligned}}$$

$x_3 = 0 = x_4 = x_5$
$x_1 = 3$ $\quad z = 18$
$x_2 = 1$

5. Consider the LP in Problem 4. Apply Phase I of the simplex method to find a feasible initial basis, then solve the problem to optimality. How many pivot operations were performed in Phase I and in Phase II?

$$\min \begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$-x_1 + x_2 \leq -2$ and $x_1 - x_2 \leq 2$
$\Rightarrow x_1 - x_2 = 2$

$$\text{st} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$x \geq 0$

$x_1 - x_2 + s_1 = 2$

$2x_1 - x_2 - x_3 + s_2 = 5$

$w = 5 - 2x_1 + x_2 + x_3 + 2 - x_1 + x_2$
$w = 7 - 3x_1 + 2x_2 + x_3$

$$\boxed{\begin{aligned} s_1 &= 2 - x_1 + x_2 \\ s_2 &= 5 - 2x_1 + x_2 + x_3 \\ w &= 7 - 3x_1 + 2x_2 + x_3 \end{aligned}}$$

x_1 enters
s_1 leaves

$x_1 = 2 + x_2 - s_1$
$s_2 = 5 - 2(2 + x_2 - s_1) + x_2 + x_3 = 1 - x_2 + x_3 - 2s_1$
$W = 7 - 3(2 + x_2 - s_1) + 2x_2 + x_3 = 1 - x_2 + x_3 + 3s_1$

$$\boxed{\begin{aligned} x_1 &= 2 + x_2 - s_1 \\ s_2 &= 1 - x_2 + x_3 - 2s_1 \\ W &= 1 - x_2 + x_3 + 3s_1 \end{aligned}}$$

x_2 enters
s_2 leaves

$x_2 = 1 + x_3 - 2s_1 - s_2$
$x_1 = 2 + 1 + x_3 - 2s_1 - s_2 - s_1 = 3 + x_3 - 3s_1 - s_2$
$W = 1 - (1 + x_3 - 2s_1 - s_2) + x_3 + 3s_1 = s_1 + s_2$

$$\boxed{\begin{aligned} x_2 &= 1+x_3-2s_1-s_2 \\ x_1 &= 3+x_3-3s_1-s_2 \\ w &= s_1+s_2 \end{aligned}}$$

$x_1 = 3 \quad x_2 = 1$

$w = 0$

$0 = s_1 + s_2 \Rightarrow s_1 = -s_2$

$\Rightarrow s_1 = 0 = s_2$

since $s \geq 0$

2 steps

$x_1 = 3 + x_3$

$x_2 = 1 + x_3$

However since $x_1 - x_2 = 2$, $x_3 = 0$ so $x_1 = 3$ $x_2 = 1$ is optimal

min $5(3) + 3(1) = 18$

0 steps

Exercise 4.4 Let $\mathbf{A}$ be a symmetric square matrix. Consider the linear programming problem

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}'\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{c} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Prove that if $\mathbf{x}^*$ satisfies $\mathbf{A}\mathbf{x}^* = \mathbf{c}$ and $\mathbf{x}^* \geq \mathbf{0}$, then $\mathbf{x}^*$ is an optimal solution.

Consider the dual problem

max $p^T c$

st $p^T A \leq c^T$

$p \geq 0$

$Ax^* = c$

$x^{*\prime} A^T = x^{*\prime} A = c^T$

$\Rightarrow x^{*T} A \leq c^T$ so x^{*T} is feasible for the dual problem.

x^* is also feasible for the primal problem.

Since $\mathbb{R}^n$ is over the reals, $x^{*T} c = c^T x^*$.

Thus by weak duality corollary we see x^*

pg 148 is optimal for the primal and dual problem

Exercise 4.5 Consider a linear programming problem in standard form and assume that the rows of **A** are linearly independent. For each one of the following statements, provide either a proof or a counterexample.

(a) Let $\mathbf{x}^*$ be a basic feasible solution. Suppose that for every basis corresponding to $\mathbf{x}^*$, the associated basic solution to the dual is infeasible. Then, the optimal cost must be strictly less that $\mathbf{c}'\mathbf{x}^*$.

(b) The dual of the auxiliary primal problem considered in Phase I of the simplex method is always feasible.

(c) Let p_i be the dual variable associated with the ith equality constraint in the primal. Eliminating the ith primal equaity constraint is equivalent to introducing the additional constraint $p_i = 0$ in the dual problem.

(d) If the unboundedness criterion in the primal simplex algorithm is satisfied, then the dual problem is infeasible

a) Assume c^Tx^* is optimal. Let B be a basis of x^*. Then $c^TB^{-1}b$ is optimal for the dual problem by the Strong duality thrm. Thus by contraposition we have shown a)

pg 148

b) As the auxitery objective is the sum of the nonnegative auxitery variables, it is a bdd minimization problem. This auxitery problem always has a solution as well which implies there always exists an optimum. By the strong duality thrm ~~pg 148~~ the dual also has an optimum which is feasible.

c) AWLOG The mth Constraint is eliminated.

It follows $\sum_{j=1}^{m} p_j a_{ij} \to \sum_{j=1}^{m-1} p_j a_{ij} \Rightarrow p_m = 0$

Similarly for the objective function

$\sum_{j=1}^{m} p_j b_j \to \sum_{j=1}^{m-1} p_j b_j \Rightarrow p_m = 0$

d) By weak duality thrm pg 146 the optimum of the dual $p^* \le d^*$ which is the optimum of the primal problem. If the primal problem is unbdd then $p^* \le d^* = -\infty$. By the corollary pg 147, the dual problem is infeasible

8. Consider the system $Ax \geq b$, $x \geq 0$. Find a system with the property that either this one or yours has a feasible solution, but not both. Prove your result.

Let $y \geq 0 \ni y^T A \leq 0$ & $y^T b > 0$.

Consider If $\exists\ x \ni Ax \geq b$ and $x \geq 0$

$Ax \geq b$

$y^T Ax \geq y^T b > 0$ Since $y^T A \leq 0$ and $x \geq 0$ $y^T Ax \leq 0$

$0 \geq y^T b > 0$ so there cannot exist a $y \ni y^T b > 0$

Next consider if $\exists y \ni y^T A \leq 0$ & $y^T b > 0$

$y^T Ax < y^T b$ Since $y^T A \leq 0$ and $x \geq 0$

$Ax < b \Rightarrow Ax \ngeq b$

9. Verify that your theorem holds for the system

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & -1 & -3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

by checking that exactly one of the systems has a solution.

$Ax \geq b$

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \qquad x \geq 0$$

$-x_2 - 3x_3 \geq 0$

$\Rightarrow x_2 + 3x_3 \leq 0$ $\ast$ since $x_2, x_3 \geq 0$

$y^T A \leq 0 \quad y^T b > 0$

$$\begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & -1 & -3 \end{bmatrix} \leq 0$$

$$\begin{bmatrix} y_1 - y_2 + y_3 & y_1 + y_2 - y_3 - y_4 & 2y_1 + y_2 + y_3 - 3y_4 \end{bmatrix} \leq 0$$

$$\begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} > 0$$

$y_1 + 2y_2 + y_3 > 0$

$y \stackrel{?}{=} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $1+2+1 > 0$ ✓

$1+1 \leq 1$ ×

$y_1 + y_2 \leq y_3$

$y_1+y_3 \le y_2$

$y_1+y_2 \le y_3+y_4$

$2y_1+y_2+y_3 \le 3y_4$

$y \overset{?}{=} \begin{bmatrix}1\\1\\0\\2\end{bmatrix}$
$1+2>0$ ✓
$1+0\le 1$ ✓
$1+1\le 0+2$ ✓
$2+1+0\le 6$ ✓

$\boxed{y=\begin{bmatrix}1\\1\\0\\2\end{bmatrix}}$

$y = \begin{bmatrix}\\1\\1\\1\end{bmatrix}$
$1+1\le 1$ ×

$y \overset{?}{=} \begin{bmatrix}1\\1\\0\\1\end{bmatrix}$
$1+2>0$ ✓
$1+0\le 1$ ✓
$1+1\le 0+1$ ×

Exercise 4.45 Let P be a polyhedron with at least one extreme point. Is it possible to express an arbitrary element of P as a convex combination of its extreme points plus a nonnegative multiple of a single extreme ray?

Counter example

$P=\{(x,y)\mid x\ge 0,\ y\ge 0\}$

Extreme point at $(0,0)$

Extreme rays $(1,0),(0,1)$

Consider $(1,1)=(0,0)+(1,0)+(0,1)$.

This point uses both extreme rays and cannot be expressed by a single extreme ray.