HW2

Monday, February 10, 2025 2:04 PM

1. Consider the nonlinear program:

$$\begin{array}{ll} \min & 4(x_1-1)^2+(x_2-1)^2 \\ \text{s.t..} & 16x_1+6x_2=63 \end{array}$$

   (a) Form the Lagrangian function for this model.

   (b) Write the stationary conditions for the Lagrangian.

   (c) Solve your stationary condition for x_1 and x_2. Explain why your answers are optimal for the original model.

   (d) Explain why the constraint $x_1 \le 2$ would be active if added to the original model.

   (e) Use the Lagrangian approach to compute an optimal solution to the model with the added constraint.

a) $\mathcal{L}(x;\lambda,\mu) = f_0(x) + \sum \lambda_i f_i(x) + \sum \mu_i h_i(x)$

$= 4(x_1^2-2x_1+1)+x_2^2-2x_2+1+\mu(16x_1+6x_2-63)$

b) PF) $16x_1+6x_2-63=0$

DF) $\langle 8x_1-8+16\mu,\ 2x_2-2+6\mu\rangle = 0$

c) $\begin{bmatrix} 8 & 0 & 16 \\ 0 & 2 & 6 \\ 16 & 6 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \mu \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 63 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ \mu \end{bmatrix} = \begin{bmatrix} 66/25 \\ 173/50 \\ -41/50 \end{bmatrix}$

```
(%i11) m:matrix([8,0,16],[0,2,6],[16,6,0]);
       v:matrix([8],[2],[63]);
       m1:invert(m);
       m1.v;
```

m $\begin{pmatrix} 8 & 0 & 16 \\ 0 & 2 & 6 \\ 16 & 6 & 0 \end{pmatrix}$

v $\begin{pmatrix} 8 \\ 2 \\ 63 \end{pmatrix}$

m1 $\begin{pmatrix} \frac{9}{200} & -\left(\frac{3}{25}\right) & \frac{1}{25} \\ -\left(\frac{3}{25}\right) & \frac{8}{25} & \frac{3}{50} \\ \frac{1}{25} & \frac{3}{50} & -\left(\frac{1}{50}\right) \end{pmatrix}$

(%o11) $\begin{pmatrix} \frac{66}{25} \\ \frac{173}{50} \\ -\left(\frac{41}{50}\right) \end{pmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 66/25 \\ 173/50 \end{bmatrix}$ f_0 is convex as it is parabolas with positive coefficients

$h(x)$ is affine

$f(x^*) = 4(66/25-1)^2+(173/50-1)^2 = \frac{1681}{100}$

Thus the problem is convex and the local min from the KKT solution is the global min

d) $66/25 > 2 \Rightarrow x_1^* = 2$ So this would be an active constraint.

e) PF) $x_1-2\le 0$

$16x_1+6x_2-63=0$

DF) $\lambda \ge 0$

$\langle 8x_1-8+\lambda+16\mu,\ 2x_2-2+6\mu\rangle = 0$

CS) $\lambda(x_1-2)=0$

Set $x_1=2$

$32+6x_2=63$

$x_2 = 31/6$

$16-8+\lambda+16\mu=0$

$\lambda = -16\mu-8$

$-16\mu-8\ge 0$

$-\frac{1}{2}\ge\mu$

$\frac{31}{3}-2+6\mu=0$

$= \frac{1}{3}-\frac{31}{18} \approx -1.389 < -1/2$

Thus $(2, 31/6)$ is the new optimal solution as it fits the KKT conditions

$f(x^*) = 4(2-1)^2+(31/6-1)^2 = \frac{769}{36}$

2. Consider the nonlinear program:

$$\begin{array}{ll} \min & x_1 \\ \text{s.t.} & x_1^2 - x_2 = 0 \\ & 2x_1 - x_2 = 1 \end{array}$$

(a) Form the Lagrangian function for this model.

(b) Write the stationary conditions for the Lagrangian.

(c) Solve your stationary condition for x_1 and x_2.

(d) Identify an optimal solution to this problem. Explain your reasoning.

(e) Explain the relationship between your answers to the previous two parts.

a) $\mathcal{L}(x;\lambda,\mu) = x_1 + 0 + \mu_1(x_1^2 - x_2) + \mu_2(2x_1 - x_2 - 1)$

b) PF) $x_1^2 - x_2 = 0$ DF) $\langle 1 + 2x_1\mu_1 + 2\mu_2, -\mu_1 - \mu_2 \rangle = 0$

$2x_1 - x_2 - 1 = 0$

CS)

c) $x_2 = x_1^2$

$x_1^2 - 2x_1 + 1 = 0$ $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$(x_1 - 1)^2 = 0$

d) This is the only feasible solution so it is optimal

e) This has only one feasible result so by default it is optimal. The only one has many feasible points so we have to choose a number

3. For the following mathematical programs, state the KKT conditions and determine whether a KKT point is a global optimum

(a)

$$\begin{array}{ll} \max & 6x_1 + 40x_2 + 5x_3 \\ \text{s.t.} & x_1 \sin x_2 + 9x_3 \le 2 \\ & e^{18x_1 + 3x_2} + 14x_3 \ge 50 \\ & x_1, x_2 \ge 0 \end{array}$$

(b)

$$\begin{array}{ll} \max & 7\ln x_1 + 4\ln x_2 + 11\ln x_3 \\ \text{s.t.} & (x_1+2)^2 - x_1x_2 + (x_2-7)^2 \le 80 \\ & 5x_1 + 7x_3 = 22 \\ & x_1, x_2, x_3 \ge 1 \end{array}$$

a) Pf) $x_1 \sin x_2 + 9x_3 - 2 \le 0$ Df) $\lambda_i \ge 0$

$50 - \exp(18x_1 + 3x_2) - 14x_3 \le 0$

$-x_1 \le 0$

$-x_2 \le 0$

$6 - \lambda_1 \sin x_2 - 18\lambda_2 e^{3x_2 + 18x_1} - \lambda_3 = 0$

$40 - \lambda_1 x_1 \cos x_2 - 3\lambda_2 e^{3x_2 + 18x_1} - \lambda_4 = 0$

$5 - 9\lambda_1 - 14\lambda_2 = 0$

CS) $\lambda_1(x_1 \sin x_2 + 9x_3 - 2) = 0$

$\lambda_2(50 - \exp(18x_1 + 3x_2) - 14x_3) = 0$

$\lambda_3 x_1 = 0$

$\lambda_4 x_2 = 0$

```
grad(var, e) := makelist(diff(e, var1), var1, var);
grad(var,e):=makelist(d/dvar1 e,var1,var)
f0(x,y,z):=6·x+40·y+5·z;
f1(x,y,z):=x·sin(y)+9·z−2;
f2(x,y,z):=50−e^(18·x+3·y)−14·z;
f3(x,y,z):=−x;
f4(x,y,z):=−y;
f0(x,y,z):=6x+40y+5z
f1(x,y,z):=x sin(y)+9z−2
f2(x,y,z):=50−%e^(18x+3y)+−14z
f3(x,y,z):=−x
f4(x,y,z):=−y
grad([x, y, z], f0(x,y,z));
grad([x, y, z], f1(x,y,z));
grad([x, y, z], f2(x,y,z));
grad([x, y, z], f3(x,y,z));
grad([x, y, z], f4(x,y,z));
[6,40,5]
[sin(y),x cos(y),9]
[−(18 %e^(3y+18x)),−(3 %e^(3y+18x)),−14]
[−1,0,0]
[0,−1,0]
```

$x_1 \sin x_2 + 9x_3 - 2 \le 0$ is not convex/concave so the KKT points are not neccessarily global optimum

$x_1 \sin x_2 + 9x_3 - 2 \le 0$ is not convex/concave so The KKT points are not neccessarily global optimum

1) $[0,-1,0]$

b) Pf) $(x_1+2)^2 - x_1 x_2 + (x_2-7)^2 - 80 \le 0$

$1 - x_1 \le 0$

$1 - x_2 \le 0$

$1 - x_3 \le 0$

$5x_1 + 7x_3 - 22 = 0$

DF) $\frac{7}{x_1} + 2\lambda_1(x_1+2) - \lambda_1 x_2 - \lambda_2 + 5\mu = 0$

$\frac{4}{x_2} + 2\lambda_1(x_2-7) - \lambda_1 x_1 - \lambda_3 = 0$

$\frac{11}{x_3} - \lambda_4 + 7\mu = 0$

$\lambda_i \ge 0$

CS) $\lambda_1((x_1+2)^2 - x_1x_2 + (x_2-7)^2 - 80) = 0$

$\lambda_2(1-x_1) = 0 \quad \lambda_4(1-x_3) = 0$

$\lambda_3(1-x_2) = 0$

```
grad(var, e) := makelist(diff(e, var1), var1, var);
```
grad(var,e):= makelist$\left(\frac{d}{d\,var1} e, var1, var\right)$

```
f0(x,y,z):=7·log(x)+4·log(y)+11·log(z);
f1(x,y,z):=(x+2)^2−x·y+(y−7)^2−80;
f2(x,y,z):=1−x;
f3(x,y,z):=1−y;
f4(x,y,z):=1−z;
h(x,y,z):=5·x+7·z−22;
```
f0(x,y,z):= 7 log(x) + 4 log(y) + 11 log(z)

f1(x,y,z):= $(x+2)^2 - x\,y + (y-7)^2 - 80$

f2(x,y,z):= 1 − x

f3(x,y,z):= 1 − y

f4(x,y,z):= 1 − z

h(x,y,z):= 5 x + 7 z − 22

```
grad([x, y, z], f0(x,y,z));
grad([x, y, z], f1(x,y,z));
grad([x, y, z], f2(x,y,z));
grad([x, y, z], f3(x,y,z));
grad([x, y, z], f4(x,y,z));
grad([x, y, z], h(x,y,z));
```
$\left[\frac{7}{x}, \frac{4}{y}, \frac{11}{z}\right]$

[2 (x+2) − y, 2 (y − 7) − x, 0]

[−1, 0, 0]

[0, −1, 0]

[0, 0, −1]

[5, 0, 7]

The objective Function is concave

h is affine and $x_i \ge 1$ is convex

so we just need to check if f_1 is convex

$\nabla^2 f_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad tr(\nabla^2 f_1) = 4 \quad det(\nabla^2 f) = 4 - 1 = 3$

$\therefore \sigma_p(\nabla^2 f_1) = \{3, 1\} \Rightarrow \nabla^2 f_1$ is p.d. $\Rightarrow f_1$ is convex

Thus the KKT points would give global optimum

4. We will show in this problem that redundant constraints in constraint optimization can be useful. Consider the non-convex optimization problem

$$\begin{aligned} &\min_{x_1, x_2 \in \mathbb{R}} x_1 x_2 \\ &\text{s.t.} x_1^2 + x_2^2 \le 1, x_1 \ge 0, \text{ and } x_2 \ge 0. \end{aligned} \tag{1}$$

and

$$\begin{aligned} &\min_{x_1, x_2 \in \mathbb{R}} x_1 x_2 \\ &\text{s.t.} x_1^2 + x_2^2 \le 1, x_1 x_2 \ge 0, x_1 \ge 0, \text{ and } x_2 \ge 0. \end{aligned} \tag{2}$$

The only difference between (1) and (2) is that there is a redundant constraint $x_1 x_2 \ge 0$ in (2). Indeed, both problems have exactly the same feasible set and objective function, with optimal primal objective value 0.

(a) List the KKT conditions of both problems.

(b) Show that the optimal value of the dual problem to (1) is $-1/2$.

(c) Show that the optimal value of the dual problem to (2) is 0.

a) (1)

Pf) $x_1^2 + x_2^2 - 1 \le 0$

$-x_1 \le 0$

$-x_2 \le 0$

(2)

$x_1^2 + x_2^2 - 1 \le 0$

$-x_1 \le 0$

```
grad(var, e) := makelist(diff(e, var1), var1, var
```
grad(var,e):= makelist$\left(\frac{d}{d\,var1} e, var1, var\right)$

```
f0(x,y,z):=x·y;
f1(x,y,z):=x^2+y^2−1;
```

Pf) $x_1^2+x_2^2-1\le 0$
$-x_1\le 0$
$-x_2\le 0$

Df) $\lambda_i\ge 0$

$x_2+2\lambda_1x_1-\lambda_2=0$

$x_1+2\lambda_1x_2-\lambda_3=0$

CS) $\lambda_1(x_1^2+x_2^2-1)=0$

$-\lambda_2x_1=0$

$-\lambda_3x_2=0$

$x_1^2+x_2^2-1\le 0$
$-x_1\le 0$
$-x_2\le 0$
$-x_1x_2\le 0$

$\lambda_i\ge 0$

$x_2+2\lambda_1x_1-\lambda_2-\lambda_4x_2=0$

$x_1+2\lambda_1x_2-\lambda_3-\lambda_4x_1=0$

$\lambda_1(x_1^2+x_2^2-1)=0$

$-\lambda_2x_1=0$

$-\lambda_3x_2=0$

$-\lambda_4x_1x_2=0$

```
grad(var,e):=makelist(d/d var1 e,var1,var)
f0(x,y,z):=x·y;
f1(x,y,z):=x^2+y^2−1;
f2(x,y,z):=−x;
f3(x,y,z):=−y;
f4(x,y,z):=−x·y;
f0(x,y,z):=x y
f1(x,y,z):=x^2+y^2 −1
f2(x,y,z):=−x
f3(x,y,z):=−y
f4(x,y,z):=−x y

grad([x, y, z], f0(x,y,z));
grad([x, y, z], f1(x,y,z));
grad([x, y, z], f2(x,y,z));
grad([x, y, z], f3(x,y,z));
grad([x, y, z], f4(x,y,z));
[y,x,0]
[2 x,2 y,0]
[−1,0,0]
[0,−1,0]
[−y,−x,0]
```

b) $g(\lambda)=\inf \mathcal{L}(x;\lambda)=\inf(x_1x_2+\lambda_1(x_1^2+x_2^2-1)-\lambda_2x_1-\lambda_3x_2)$

$=-\lambda_1$

$x_2=\lambda_2-2\lambda_1x_1$

$x_1+2\lambda_1\lambda_2-4\lambda_1^2x_1-\lambda_3=0$

$x_1=\dfrac{\lambda_3-2\lambda_1\lambda_2}{1-4\lambda_1^2}$

$x_2=\lambda_2-2\lambda_1\left(\dfrac{\lambda_3-2\lambda_1\lambda_2}{1-4\lambda_1^2}\right)=\dfrac{\lambda_2(1-\cancel{4\lambda_1^2})-2\lambda_1\lambda_3+\cancel{4\lambda_1^2\lambda_2}}{1-4\lambda_1^2}=\dfrac{\lambda_2-2\lambda_1\lambda_3}{1-4\lambda_1^2}$

$\left(\dfrac{\lambda_2-2\lambda_1\lambda_3}{1-4\lambda_1^2}\right)^2+\left(\dfrac{\lambda_3-2\lambda_1\lambda_2}{1-4\lambda_1^2}\right)^2\le 1$

$(\lambda_2-2\lambda_1\lambda_3)^2+(\lambda_3-2\lambda_1\lambda_2)^2\le(1-4\lambda_1^2)^2$

$\min(\lambda_2-2\lambda_1\lambda_3)^2+(\lambda_3-2\lambda_1\lambda_2)^2\le(1-4\lambda_1^2)^2$

$0\le 1-4\lambda_1^2$

$\Rightarrow \lambda_1\le\frac{1}{2}$

$\therefore \max g(\lambda)=-\frac{1}{2}$

c) $g(\lambda)=\inf \mathcal{L}(x;\lambda)=\inf(x_1x_2+\lambda_1(x_1^2+x_2^2-1)-\lambda_2x_1-\lambda_3x_2-\lambda_4x_1x_2)$

$=-\lambda_1$

$\max g(\lambda)=\max -\lambda_1=-\min\lambda_1$

$\left(\dfrac{\lambda_2-2\lambda_1\lambda_3}{1-4\lambda_1^2}\right)\left(\dfrac{\lambda_3-2\lambda_1\lambda_2}{1-4\lambda_1^2}\right)\ge 0$

$\lambda_2\lambda_3-2\lambda_1\lambda_2^2-2\lambda_1\lambda_3^2-4\lambda_1^2\lambda_2\lambda_3\ge 0$

$$4\lambda_1^2\lambda_2\lambda_3 + 2\lambda_1(\lambda_2^2+\lambda_3^2) - \lambda_2\lambda_3 \leq 0$$

$$\frac{-2(\lambda_2^2+\lambda_3^2) \pm \sqrt{4(\lambda_2^2+\lambda_3^2)^2+16\lambda_2^2\lambda_3^2}}{8\lambda_2\lambda_3} \geq \lambda_1$$

$$\min \frac{-2(\lambda_2^2+\lambda_3^2) \pm \sqrt{4(\lambda_2^2+\lambda_3^2)^2+16\lambda_2^2\lambda_3^2}}{8\lambda_2\lambda_3} \geq \min \lambda_1$$

$$0 \geq \min \lambda_1 \geq 0$$

$$\Rightarrow \min \lambda_1 = 0$$

$$\max g(\lambda) = -\min \lambda_1 = -0 = 0$$