MATH 8100
Mathematical Programming
Spring 2025
Homework 5

Due March 28, 2025

1. Consider the following equivalent linear programs:

$$\begin{array}{rc}
\text{minimize} & x_1 - x_2 \\
\text{subject to} & x_1 + 2x_2 \geq 4 \\
 & -x_2 \geq -4 \\
 & 3x_1 - 2x_2 \leq 0 \\
 & x_1 \text{free}, x_2 \geq 0
\end{array}$$

and

$$\begin{array}{rc}
\text{minimize} & x_1^+ - x_1^- - x_2 \\
\text{subject to} & x_1^+ - x_1^- + 2x_2 \geq 4 \\
 & -x_2 \geq -4 \\
 & -x_1^+ + 3x_1^- + 2x_2 \geq 0 \\
 & x_1^+, x_1^-, x_2 \geq 0.
\end{array}$$

Write the dual problems and analyze the equivalence between them.

2. Consider the following problem:

$$\begin{array}{rrcrcl}
\min & 6x_1 & + & 4x_2 & & \\
s.t. & x_1 & + & x_2 & = & 1 \\
 & 3x_1 & + & x_2 & \geq & 0 \\
 & x_1 & & & \leq & 0 \\
 & & & x_2 & \geq & 0.
\end{array}$$

   (a) Write its dual;
   (b) Solve the dual with the graphical method;
   (c) Find an optimal solution of the primal by applying complementary slackness.
   (d) Verify the primal's optimal solution by solving the primal through the graphical method.

3. What does the Lagrangian function $\mathcal{L}(\boldsymbol{\lambda})$ look like for an optimization problem? Apply it to the Knapsack problem (for simplicity, we'll use its minimization variant):

$$\begin{array}{ll}
\min & \boldsymbol{c}^T\boldsymbol{x} \\
s.t. & \boldsymbol{a}^T\boldsymbol{x} \geq b \\
 & \boldsymbol{x} \in \{0,1\}^n.
\end{array}$$

(a) Write the LP relaxation of this problem by relaxing integrality on the variables (but not their bounds!). Call this problem (LP).

(b) Apply Lagrangian relaxation to (LP) by relaxing $\boldsymbol{a}^T\boldsymbol{x} \geq b$ only.

(c) The newly constructed problem should only depend on one (scalar) parameter λ. Write the value of the optimal solution for any λ. Note: relaxing the linear constraint makes it a very trivial optimization problem.

(d) Depict the function $y = \mathcal{L}(\lambda)$ on the Cartesian plane (λ, y) for the following values of $\boldsymbol{a}$, $\boldsymbol{c}$, and b (note that $\boldsymbol{a}$ and $\boldsymbol{c}$ are sorted in increasing order of $\frac{c_i}{a_i}$):

$n = 4$; $b = 20$;

i	1	2	3	4
a_i	9	7	4	3
c_i	2	3	6	7

(e) What is the maximum value of $\mathcal{L}(\lambda)$?

(f) What value of λ does it correspond to?

(g) What is the value of the optimal solution of (LP)? Find out with AMPL and CPLEX or Gurobi—no need to submit code, it will just save you some time—or solve it by hand.

4. Solve the following problem using dictionaries and starting from basis $\mathcal{B} = \{3, 4, 5\}$, after reducing the problem to standard form:

$$\begin{array}{llcll} \text{minimize} & 5x_1 & + & 3x_2 & \\ \text{subject to} & -x_1 & + & x_2 & \leq -2 \\ & x_1 & - & x_2 & \leq 2 \\ & -2x_1 & + & x_2 & \leq -5 \\ & x_1, & & x_2 & \geq 0. \end{array}$$

The dictionary will be infeasible but with non-negative reduced cost. Apply dual-simplex pivoting operations until feasibility is reached. Represent the value of x_1 and x_2 on the Cartesian plane for every basis visited.

5. Consider the LP in Problem 4. Apply Phase I of the simplex method to find a feasible initial basis, then solve the problem to optimality. How many pivot operations were performed in Phase I and in Phase II?

6. B&T, Problem 4.4

7. B&T, Problem 4.5

8. Consider the system $A\boldsymbol{x} \geq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$. Find a system with the property that either this one or yours has a feasible solution, but not both. Prove your result.

9. Verify that your theorem holds for the system

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & -1 & -3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

by checking that exactly one of the systems has a solution.

10. B&T, Problem 4.45. Prove or provide a counterexample.