MATH 8100
Mathematical Programming
Spring 2025
Homework 4

Due March 14, 2025

1. Consider the following problem:

$$\begin{array}{lrcrcl} \min & 2x_1 & + & x_2 & & \\ s.t. & x_1 & + & 2x_2 & \geq & 4 \\ & x_1 & - & x_2 & \leq & 1 \\ & x_1, & & x_2 & \geq & 0. \end{array}$$

   (a) Rewrite it in standard form.
   (b) Consider the basic solution obtained using x_1 and x_4 as basic variables. What is the corresponding value of $\boldsymbol{x}_B = B^{-1}\boldsymbol{b}$ and $\boldsymbol{x}_N$? Is this a feasible solution?
   (c) Consider another basic solution that has x_1 and x_2 as basic variables. What is the corresponding value of $\boldsymbol{x}_B$ and $\boldsymbol{x}_N$? Is this a feasible solution?
   (d) Consider the basic feasible solution (BFS) you found in either step 2 or 3. What is its objective function value?
   (e) Compute a feasible direction $\boldsymbol{d}$ by choosing a suitable non-basic variable x_j.
   (f) What is the reduced cost $\bar{c}_j$?
   (g) Can you obtain a new BFS if you move along direction $\boldsymbol{d}$? If so, what is that BFS and what is its objective function value?

2. B&T, Problem 3.4.

3. B&T, Problem 3.5. Remember that the set of feasible solutions is infinite.

4. Solve the following problem (after an appropriate transformation) using the simplex method. Use dictionaries as seen in class or tableaux. Feel free to write B, B^{-1}, N, $\mathbf{d}_B$ if that helps you, but write all dictionaries/tableaux until the optimal one.

$$\begin{array}{lrcl} \min & 2x_1 - x_2 & & \\ s.t. & 3x_1 + x_2 & \leq & 9 \\ & x_1 + 2x_2 & \leq & 8 \\ & x_1, x_2 & \geq & 0. \end{array}$$

Use $\mathcal{B} = \{3, 4\}$ as the first basis. For this and the following problems, you might find the following URL useful, especially if you hate arithmetic: `http://www.princeton.edu/~rvdb/JAVA/pivot/simple.html`. There is a link to a tool for minimization problems at the bottom of that page.

5. B&T, Problem 3.12.

6. Solve the following problem, after reducing it to standard form, using the simplex method (with dictionaries):

$$\begin{array}{rrcrcrcl} \min & -2x_1 & + & x_2 & + & 2x_3 & & \\ s.t. & x_1 & + & 2x_2 & + & x_3 & \leq & 4 \\ & x_1 & - & x_2 & + & x_3 & \leq & 1 \\ & x_1, & & x_2, & & x_3 & \geq & 0. \end{array}$$

   Start with the slack basis $\mathcal{B} = \{4, 5\}$.

7. B&T, Problem 3.19.

8. Let

$$B = \begin{bmatrix} 4 & 1 & 4 & 4 \\ 3 & 4 & 4 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 4 & 1 & 2 \end{bmatrix} \text{ and } B^{-1} = \frac{1}{125} \begin{bmatrix} 26 & 24 & -57 & -12 \\ -19 & 4 & 8 & 28 \\ -4 & 14 & 28 & -27 \\ 14 & -49 & 27 & 32 \end{bmatrix}$$

   Use the product-form update from the revised simplex method to compute the inverse of the new matrix

$$\bar{B} = \begin{bmatrix} 4 & 1 & 2 & 4 \\ 3 & 4 & 4 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 4 & 2 & 2 \end{bmatrix}$$

9. One application of linear programming is to "blending problems." Here, several ingredients are to be blended to produce a final product. The ingredients have different levels of component features, and the constraints may require certain levels of features as percentages of the total amount of final product. Such a constraint can be modeled as a ratio. For example:

$$\frac{\sum_j a_j x_j}{\sum_j x_j} \geq p$$

where a_j is the amount of feature per unit in ingredient j, x_j is the total amount of ingredient j used, and p is the fraction of feature j required in the final product. The constraint can be converted to a linear inequality by multiplying both sides by the denominator, which represents the total amount produced:

$$\sum_j a_j x_j \geq p \left(\sum_j x_j \right).$$

One needs to be careful that the denominator does not end up being zero.

Use this idea to formulate the following LP. Solve your LP using AMPL. A .CSV file is provided with the data table.

The *risk index* of an investment can be obtained from return on investment (ROI) by taking the percentage change in the value of the investment (in absolute terms) for each year and averaging them.

Suppose you are trying to determine what percentage of your money should be invested in T-bills, gold, and stocks. In the table below and the file `riskindex.csv` you are given the annual returns (change in value) for these investments for the years 2004–2024. Let the risk index of a portfolio be the weighted (according to the fraction of your money assigned to each investment) average of the risk index of each individual investment. Suppose that the amount of each investment must be between 20% and 50% of the total invested. You would like the risk index of your portfolio to equal 0.15 and your goal is to maximize the expected return on your portfolio. Formulate an LP whose solution will maximize the expected return on your portfolio, subject to the given constraints. Use the average return earned by each investment during the years 2004–2024 as your estimate of expected return.

Year	Stocks	Gold	T-bills
2004	11	11	5
2005	−9	8	7
2006	4	−14	7
2007	14	14	4
2008	19	44	4
2009	−15	66	7
2010	−27	64	8
2011	37	0	6
2012	24	−22	5
2013	−7	18	5
2014	7	31	7
2015	19	59	10
2016	33	99	11
2017	−5	−25	15
2018	22	4	11
2019	23	−11	9
2020	6	−15	10
2021	32	−12	8
2022	19	16	6
2023	5	22	5
2024	17	−2	6

Write clear and readable AMPL models with descriptive identifiers, well structured layout and white space, and helpful comments. Submit your `.mod` and `.dat` files as a zip file to the Homework 4 Code assignment on Canvas. Name your zip file `math-8100-<userid>-hw<hwnum>.zip`, where `<userid>` is your Clemson userid and `<hwnum>` is the homework number. For example, `math-8100-mjs-hw5.zip`. Also include your model and data listings in the PDF you submit to Gradescope.