Final

Monday, April 28, 2025 12:25 PM

MATH 8100 Mathematical Programming Spring 2025

Final Exam

April 30, 2025

- There are a total of ?? points. Point value is listed next to each question.
- Mark your answers clearly. *Show your work.* Unsupported correct answers receive partial credit.
- Be sure to write your name and ID number on *each* page.
- Good luck!

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I certify that I have not received any unauthorized assistance in completing this examination.

Signature: Jacob Manning

Date: 04/29/2025

The conclusions of most good operations research studies are obvious. —Robert E. Machol

Cited in: Paul Dickson (1999), *The Official Rules and Explanations*, p. 14.

Machol named this the "Billings Phenomenon." Dickson explains: "The name refers to a well-known [Josh] Billings story in which a farmer becomes concerned that his black horses are eating a lot more than his white horses. He does a detailed study of the situation and finds that he has more black horses than white horses. Machol points out that the obvious conclusions are not likely to be obvious *a priori* but obvious after the results are in. In other words, good research does not have to yield dramatic findings."

1. (10 points) Let S be a nonempty set in $\mathbb{R}^n$. The *polar set* of S, denoted by S_P, is given by

$$S_P = \{\boldsymbol{y} : \boldsymbol{y}^T\boldsymbol{x} \leq 1, \forall \boldsymbol{x} \in S\}.$$

Prove that S_P is convex.

Let $y_1, y_2 \in S_p$, $x \in S$, and $\lambda \in [0,1]$

WTS $\lambda y_1 + (1-\lambda) y_2 \in S_p$

$$(\lambda y_1 + (1-\lambda) y_2)^T x = \lambda y_1^T x + (1-\lambda) y_2^T x$$
$$\leq \lambda + 1 - \lambda$$
$$= 1$$

Thus $\lambda y_1 + (1-\lambda) y_2 \in S_p$

$\therefore S_p$ is convex

2. (10 points) Use the KKT conditions to solve the following constrained optimization problem:

$$\begin{array}{ll} \text{minimize} & x_1x_2 \\ \text{subject to} & x_1^2 + x_2^2 \leq 1 \end{array}$$

We need primal and dual feasibility as well as complementary slackness.

Let $f_1(x_1,x_2) = x_1x_2$ and $f_2(x_1,x_2) = x_1^2 + x_2^2 - 1$

$$\nabla f_1 + \lambda \nabla f_2 = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + 2\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for } \lambda \geq 0$$

So $x_1 = -2\lambda x_2 \Rightarrow x_2 - 4\lambda^2 x_2 = 0$

$x_2(1-4\lambda^2) = 0$

so $x_2 = 0$ or $1 - 4\lambda^2 = 0 \Rightarrow \lambda = 1/2$ (because $\lambda \geq 0$)

Either way $x_1 = -x_2$

Complementary Slackness says $\lambda(x_1^2 + x_2^2 - 1) = 0$

So if $x = 0$ then $\lambda = 0$. If $x_1^2 + x_2^2 - 1 = 0$ then $\lambda = 1/2$.

So possibilites are $x = 0$ and

So if $\lambda=0$ then $x=0$. If $x_1+x_2-1=0$ then $\lambda=1/2$.

So possibilites are $x=0$ and

$2x_i^2-1=0 \Rightarrow x_1=\frac{\pm 1}{\sqrt{2}}$, $x_2=\frac{\mp 1}{\sqrt{2}}$ (since $x_1=-x_2$)

Checking the possible minimums

$f_1(0,0)=0$ and $f_1\left(\frac{\pm 1}{\sqrt{2}}, \frac{\mp 1}{\sqrt{2}}\right)=-\frac{1}{2}$

$-\frac{1}{2}<0$ so $-\frac{1}{2}$ is the minimum at $x=\begin{bmatrix} \pm\frac{1}{\sqrt{2}} \\ \mp\frac{1}{\sqrt{2}} \end{bmatrix}$

3. Consider the following linear program and the accompanying optimal dictionary:

$$\begin{array}{lrcrcrcl} \text{maximize} & 10x_1 & + & 7x_2 & + & 6x_3 & & \\ \text{subject to} & 3x_1 & + & 2x_2 & + & x_3 & \leq & 36 \\ & x_1 & + & x_2 & + & 2x_3 & \leq & 32 \\ & 2x_1 & + & x_2 & + & x_3 & \leq & 22 \\ & & & x_1, x_2, x_3 \geq 0 & & & & \end{array}$$

$$\begin{array}{rcrcrcrcr} \zeta & = & 146 & - & 3x_1 & - & x_4 & - & 5x_6 \\ x_2 & = & 14 & - & x_1 & - & x_4 & + & x_6 \\ x_5 & = & 2 & + & 2x_1 & - & x_4 & + & 3x_6 \\ x_3 & = & 8 & - & x_1 & + & x_4 & - & 2x_6 \end{array}$$

(a) (5 points) Write the dual problem.

(b) (5 points) Identify the current basis matrix and its inverse.

(c) (5 points) Write the optimal solution to the dual.

(d) (5 points) At what rate does the objective change from the optimal solution value if the right-hand side of the first constraint changes by a small amount?

(e) (5 points) Over what range could the right-hand side of the first constraint change without changing the optimal basis?

(f) (5 points) Write an expression showing how the optimal basic variables change with respect to a change of β in the first constraint's right-hand side.

(g) (5 points) If the right-hand side of the first constraint falls below the bound you found above, what pivot must be performed to maintain feasibility? (You don't need to perform the pivot, just identify the leaving and entering variables.)

(h) (5 points) Suppose a new column is added with coefficients of 2 and 2 in the first three constraints and 0 in the last one. What is the least profit on this new column to make it attractive to increase the level of that variable?

(i) (5 points) If the profit is increased beyond the value you found above, what pivot would have to be performed to maintain optimality? (You don't need to perform the pivot, just identify the entering and leaving variables.)

a) $A=\begin{bmatrix} 3 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ $c=\begin{bmatrix} 10 \\ 7 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $b=\begin{bmatrix} 36 \\ 32 \\ 22 \end{bmatrix}$ $x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$

Then the problem in standard form is

$$\begin{aligned} \max\ & c^T x \\ \text{st}\ & Ax=b \\ & x \geq 0 \end{aligned}$$

Thus the dual is

$$\begin{aligned} \min\ & b^T y \\ \text{st}\ & A^T y \geq c \end{aligned}$$

where y is a vector of Lagrange multipliers

or $\min \begin{bmatrix} 36 & 32 & 22 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

st $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 1 \\ \ldots \end{bmatrix} \begin{bmatrix} y_1 \\ \ldots \end{bmatrix} \geq \begin{bmatrix} 10 \\ 7 \\ \ldots \end{bmatrix}$

$$\text{st} \quad \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 10 \\ 7 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

b) The current basis is $\{2,5,3\}$

so $B = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ $\quad B^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ -1 & 0 & 2 \end{bmatrix}$

```
B:matrix([2,0,1],[1,1,2],[1,0,1])$;
invert(B);
```

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ -1 & 0 & 2 \end{pmatrix}$$

c) Using the basis B we can solve the following

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 7 \\ 0 \\ 6 \end{bmatrix}$$

for y. (this comes from the dual problem above) Since our dual is a minimization problem, we would like a tight bound on the above system because all coeffients in B^T are positive.

Thus $y = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$.

```
v:matrix([7],[0],[6])$;
invert(transpose(B)).v;
```

$$\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$$

Checking

$$\begin{bmatrix} 36 & 32 & 22 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} = 36 + 5 \cdot 22 = 146$$

Same as primal optimal value from optimal dictionary

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \geq \begin{bmatrix} 10 \\ 7 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

```
A:matrix([3,1,2],[2,1,1],[1,2,1],[1,0,0],[0,1,0],[0,0,1])$;
u:matrix([1],[0],[5])$;
A.u;
```

$$\begin{pmatrix} 13 \\ 7 \\ 6 \\ 1 \\ 0 \\ 5 \end{pmatrix} \geq \begin{bmatrix} 10 \\ 7 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

d) The slack variable x_4 is related to the first constraint. It follows $\left|\frac{\partial \zeta}{\partial x_4}\right| = 1$ so as the first constraint changes ζ changes by the same amount.

e) Using the current basis $\{2,5,3\}$

ζ changes by the same amount.

e) Using the current basis $\{2,5,3\}$

$$B\begin{bmatrix}x_2\\x_5\\x_3\end{bmatrix}=\begin{bmatrix}36\\32\\22\end{bmatrix}+\begin{bmatrix}\Delta\\0\\0\end{bmatrix}$$

```
w:matrix([36],[32],[22])$;
invert(B).w;
```

$$\begin{pmatrix}14\\2\\8\end{pmatrix}$$

$$\begin{bmatrix}x_2\\x_5\\x_3\end{bmatrix}=B^{-1}\begin{bmatrix}36\\32\\22\end{bmatrix}+B^{-1}\begin{bmatrix}\Delta\\0\\0\end{bmatrix}$$

$$=\begin{bmatrix}14+\Delta\\2+\Delta\\8-\Delta\end{bmatrix}$$

Since $x\geq 0$ it follows

$\Delta \geq -14$
$\Delta \geq -2$
$\Delta \leq 8$

Thus $\Delta\in[-2,8] \Rightarrow [34,44]$ is acceptable without changing the optimal basis

f) $x(\beta)=\begin{bmatrix}14+\beta\\2+\beta\\8-\beta\end{bmatrix}$

g) As we saw in e), x_5 would be the exiting variable

	x_1	x_4	x_6
Ratios	$-3/2$	1	$-5/3$

Since we have a maximum problem, the dual is a minimum so we choose x_4 to enter as $1>0$.

h) We want to look at the reduced cost of the new x_7.

$$\bar{C}_7 = C_7 - C_B^T B^{-1} A_7$$

$$= C_7 - \begin{bmatrix}7 & 0 & 6\end{bmatrix}\begin{bmatrix}1 & 0 & -1\\1 & 1 & -3\\-1 & 0 & 2\end{bmatrix}\begin{bmatrix}2\\2\\0\end{bmatrix}$$

```
cb:matrix([7],[0],[6])$;
x7:matrix([2],[2],[0])$;
B:matrix([2,0,1],[1,1,2],[1,0,1])$;
-transpose(cb).invert(B).x7;
```

-2

$$= C_7 - 2$$

For x_7 to enter, $C_7 - 2 \geq 0 \Rightarrow C_7 \geq 2$

i) We need to update the optimal dictionary. Since we know B and B^{-1}, it shouldn't be too hard

$$\begin{aligned}\zeta &= 146 - 3x_1 - x_4 - 5x_6\\ x_2 &= 14 - x_1 - x_4 + x_6\\ x_5 &= 2 + 2x_1 - x_4 + 3x_6\\ x_3 &= 8 - x_1 + x_4 - 2x_6\end{aligned} \Rightarrow \begin{aligned}\zeta &= 146 - 3x_1 - x_4 - 5x_6 + (2+\delta)x_7\\ x_2 &= 14 - x_1 - x_4 + x_6 - 2x_7\\ x_5 &= 2 + 2x_1 - x_4 + 3x_6 - 4x_7\\ x_3 &= 8 - x_1 + x_4 - 2x_6 + 2x_7\end{aligned}$$

```
x7:matrix([2],[2],[0])$;
B:matrix([2,0,1],[1,1,2],[1,0,1])$;
-invert(B).x7;
```

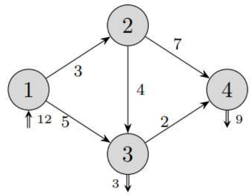
$$\begin{pmatrix}-2\\-4\\2\end{pmatrix}$$

Since $x_3 \propto x_7$, it will not enter.

x_2 $\frac{14}{2}=7$ $\quad x_5$ $\frac{2}{4}=1/2$

$1/2 < 7$ so x_5 exits basis

4. Consider the following uncapacitated minimum-cost flow problem. The cost of each arc and the supply at each node is marked on the graph.



(a) (3 points) Write the linear program corresponding to this minimum-cost flow problem.

(b) (7 points) Starting from the feasible tree solution corresponding to $\{(1,2),(1,3),(2,4)\}$, solve the problem by the network simplex method.

(c) (3 points) Instead of costs, interpret the arc labels as capacities. Ignore the supply labels on the nodes. Formulate the problem of finding the maximum flow from node 1 to node 4 as a linear program.

(d) (7 points) Solve the maximum flow problem using the shortest augmenting path method. Identify the maximum flow and the minimum-capacity cut.

a) $\min\ 3x_{12}+4x_{23}+7x_{24}+5x_{13}+2x_{34}$

st $x_{12}+x_{13}=12$

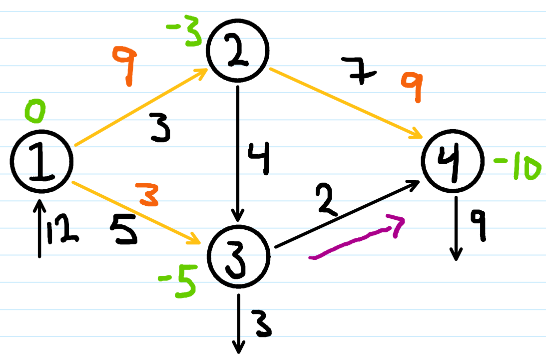
$-x_{13}-x_{23}+x_{34}=-3$

$-x_{24}-x_{34}=-9$

$-x_{12}+x_{23}+x_{24}=0$

$x_{12},x_{13},x_{23},x_{24},x_{34}\geq 0$

b) I chose node 1 as the root



flow

dual

x_{34} enters

x_{12} $3-0+{}^-3=0$

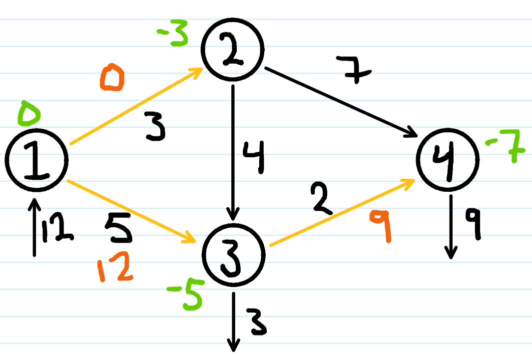
x_{13} $5-0+{}^-5=0$

x_{24} $7-{}^-3+{}^-10=0$

x_{23} $4-{}^-3+{}^-5=2>0$

x_{34} $2-{}^-5+{}^-10=-3<0$

There is a tie for exiting arcs, since $7>3$ we will choose x_{24} to leave



x_{34} $2-{}^-5+{}^-7=0$

x_{24} $7-{}^-3+{}^-7=3$

x_{23} $4-{}^-3+{}^-5=2$

both positive

Thus the min is $5(12)+2(9)=78$

c) max F

c) $\max F$

st $f_{12}+f_{13}=F$

$f_{24}+f_{34}=F$

$f_{24}+f_{23}=f_{12}$

$f_{13}+f_{23}=f_{34}$

$0\le f_{12}\le 3$

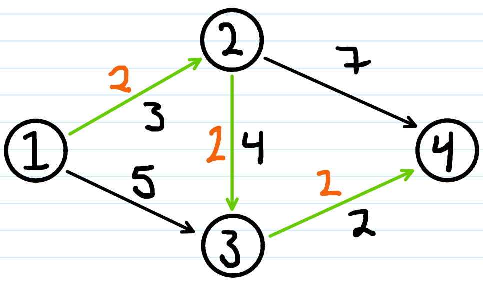
$0\le f_{13}\le 5$

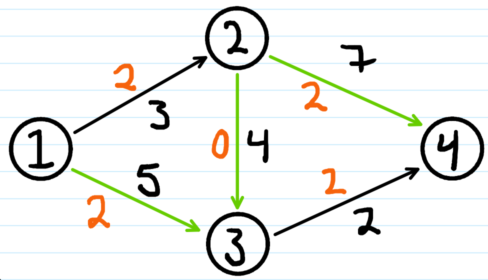
$0\le f_{23}\le 4$

$0\le f_{24}\le 7$

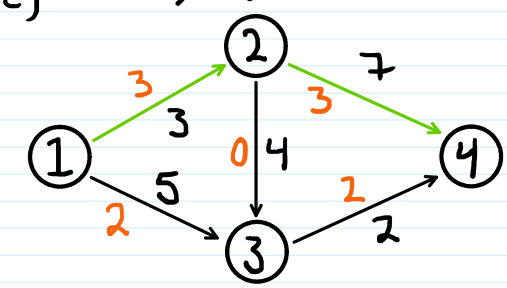
$0\le f_{34}\le 2$

d)





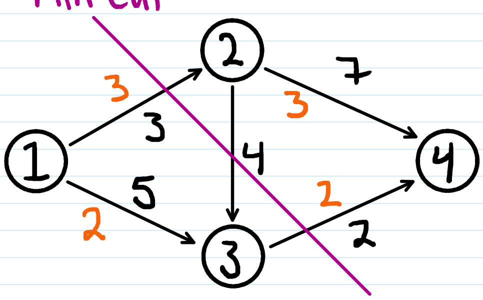
(Iterations not an arc)



This is saturated

Similarly $|F|=2+3=5$ which is also expected by the max flow min cut theorem.

min cut



$3+2=5$