MATH 8100, Spring 2025
Homework 1

Due: January 29, 2025

1. (5 points) B&T 1.15.

2. The function $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if for any $\boldsymbol{x}$, $\boldsymbol{y}$ in f's domain and $\lambda \in [0, 1]$,
$$f(\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) \leq \lambda f(\boldsymbol{x}) + (1 - \lambda) f(\boldsymbol{y}).$$
   (a) (5 points) Prove that any local minimizer of f is a global minimizer.
   (b) (5 points) f is *strictly convex* if the above inequality is strict. Show that if f has a minimizer on its domain, then that minimizer is unique.

3. The *Minkowski sum* of sets $S, T \subseteq \mathbb{R}^n$ is
$$V = \{\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y} : \boldsymbol{x} \in S \text{ and } \boldsymbol{y} \in T\}.$$
   (a) (5 points) Show that if S and T are convex, V is convex.
   (b) (5 points) Does the converse hold? Prove or give a counterexample.

4. Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and $c \in \mathbb{R}$, the *level set* of f with respect to c is $L^{=}(c) = \{\boldsymbol{x} : f(\boldsymbol{x}) = c\}$. The *lower level set* of f with respect to c is $L^{\leq}(c) = \{\boldsymbol{x} : f(\boldsymbol{x}) \leq c\}$.
   (a) (5 points) Prove that if f is convex, then $L^{\leq}(c)$ is convex.
   (b) (5 points) Prove that $L^{=}(c)$ is convex if c is the minimum value of f.

5. (5 points) Prove the well known inequality between the arithmetic and geometric means of a set of positive numbers:

$$(x_1 + x_2 + \cdots + x_k)/k \geq (x_1 x_2 \cdots x_k)^{1/k}.$$

(Hint: Apply Jensen's inequality to $f(x) = -\log(x)$.)

6. (5 points) Let f be a twice continuously differentiable function. Recall the second-order Taylor expansion at $\boldsymbol{x}$ is

$$f(\boldsymbol{x} + \boldsymbol{d}) = f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^T \boldsymbol{d} + \frac{1}{2} \boldsymbol{d}^T \nabla^2 f(\boldsymbol{x}) \boldsymbol{d} + o(\|\boldsymbol{d}\|^2).$$

Prove that f is convex if and only if the Hessian of f is positive semidefinite at any $\boldsymbol{x}$ in the domain of f.

7. (5 points) Find the minimum of $f(\boldsymbol{x}) = \frac{1}{3}(x_1^2 + x_2^2)$ using the steepest descent algorithm with initial vector $\boldsymbol{x} = (1, 2)$ and constant $\lambda = 1$.

8. Consider the following unconstrained nonlinear program:

$$f(\boldsymbol{x}) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

   (a) (5 points) What are the gradient and Hessian of f?

   (b) (5 points) Starting from the initial point $\begin{bmatrix} 1 & 2 \end{bmatrix}^T$, find a stationary point $\boldsymbol{x}^*$ using the steepest descent method with constant step size $\alpha = 1/4$.

   (c) (5 points) Is $\boldsymbol{x}^*$ an optimal solution? if so, is it unique? Explain.

9. (5 points) Perform one Newton iteration for finding the minimum of $f(x, y) = x^4 y^2 + x^2 y^4 - 3x^2 y^2 + 1$ using initial vector $\boldsymbol{x} = (2, 2)$.

10. (5 points) Let $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x} + c$ for some positive definite A. Show that Newton's method to find a minimum of f starting from an arbitrary initial vector $\boldsymbol{x}^0$ in a single iteration. Give an intuitive explanation for this finding.