MATH 8100 Mathematical Programming Spring 2025
First Midterm Exam

February 17–18, 2025

- This take-home test is due in Gradescope at noon on February 18, 2025. Be sure to mark your answers for each of the questions in Gradescope when you upload.
- There are a total of 70 points. Point value is listed next to each question.
- Mark your answers clearly. *Show your work.* Unsupported correct answers receive partial credit.
- Good luck!

Name: Jacob Manning

Student ID #: C43549874

I certify that I have not received any unauthorized assistance in completing this examination.

Signature: Jacob Manning

Date: 2/17/25

1. (8 points) Recall that a cone is a set $S \subseteq \mathbb{R}^n$ such that $\boldsymbol{x} \in S$ implies $\alpha\boldsymbol{x} \in S$ for all $\alpha \geq 0$. Prove that a set $S \subseteq \mathbb{R}^n$ is a convex cone if and only if it is closed under vector addition and nonnegative scalar multiplication.

($\Rightarrow$) Since $S \subseteq \mathbb{R}^n$ is a cone, by the first sentence, S is closed under non-negative scalar multiplication.

Since S is convex, for $x, y \in S$, $\frac{1}{2}x + \frac{1}{2}y \in S$. We also know S is closed under positive multiplication. Thus $2(\frac{1}{2}x + \frac{1}{2}y) = x + y \in S$. Therefore S is closed under vector addition.

($\Leftarrow$) Let $\alpha \geq 0$ and $x \in S$. Since S is closed under non-negative scalar multiplication, $\alpha x \in S$. Thus S is a cone.

Let $\lambda \in [0,1]$. It follows $1-\lambda \geq 0$. Thus for $x, y \in S$ λx and $(1-\lambda)y$ are in S. Since S is closed under vector addition, $\lambda x + (1-\lambda)y \in S$. Therefore, S is also convex.

2. (7 points) Let $g_1, g_2, \ldots, g_m$ be concave functions on $\mathbb{R}^n$, f be a convex function on $\mathbb{R}^n$, and μ be a positive constant. Show that the function

$$\beta(\boldsymbol{x}) = f(\boldsymbol{x}) - \mu \sum_{i=1}^{m} \ln g_i(\boldsymbol{x})$$

is convex on the set $S = \{\boldsymbol{x} : g_i(\boldsymbol{x}) > 0,\ i = 1, 2, \ldots, m\}$.

Since g_i and $\ln$ are concave functions and $\ln$ is continuous and monotonic, for $\lambda \in [0,1]$ and $x, y \in S$

and ln is continuous and monotonic,
for $\lambda\in[0,1]$ and $x,y\in S$

$$\ln(g_i(\lambda x+(1-\lambda)y))\geq \ln(\lambda g_i(x)+(1-\lambda)g_i(y))$$
$$\geq \lambda \ln g_i(x)+(1-\lambda)\ln g_i(y)$$

Thus $-\ln(g_i(\lambda x+(1-\lambda)y))\leq -\lambda \ln g_i(x)-(1-\lambda)\ln g_i(y)$.

It follows,

$$\beta(\lambda x+(1-\lambda)y)=f(\lambda x+(1-\lambda)y)+\mu\sum_{i=1}^{m}-\ln(g_i(\lambda x+(1-\lambda)y))$$
$$\leq \lambda f(x)+(1-\lambda)f(y)+\mu\sum_{i=1}^{m}-\lambda\ln g_i(x)-(1-\lambda)\ln g_i(y)$$
$$=\lambda\left(f(x)-\mu\sum_{i=1}^{m}\ln g_i(x)\right)+(1-\lambda)\left(f(y)-\mu\sum_{i=1}^{m}\ln g_i(y)\right)$$
$$=\lambda\beta(x)+(1-\lambda)\beta(y)$$

3. (6 points each part) Consider the function

$$f(x_1,x_2)=\frac{1}{3}x_1^3+\frac{1}{2}x_1^2+2x_1x_2+\frac{1}{2}x_2^2-x_2+9.$$

For each of the following points,

- determine if the point is stationary;
- if stationary, determine if the point is a local minimum, maximum, or neither;
- if not stationary, identify a descent direction.

(a) $\boldsymbol{x}=(1,-1)$
(b) $\boldsymbol{x}=(2,-3)$
(c) $\boldsymbol{x}=(0,0)$

$$\nabla f(x)=\langle x_1^2+x_1+2x_2,\ 2x_1+x_2-1\rangle$$
$$\nabla^2 f(x)=\begin{bmatrix}2x_1+1 & 2\\ 2 & 1\end{bmatrix}$$

a) $\nabla f(1,-1)=\langle 1+1-2,\ 2-1-1\rangle=0$

$$|\nabla^2 f(1,-1)|=\begin{vmatrix}3 & 2\\ 2 & 1\end{vmatrix}=-1<0$$

$(1,-1)$ is a saddle point

b) $\nabla f(2,-3)=\langle 4+2-2(3),\ 4-3-1\rangle=0$

$$|\nabla^2 f(2,-3)|=\begin{vmatrix}5 & 2\\ 2 & 1\end{vmatrix}=4>0 \qquad 5>0$$

$(2,-3)$ is a relative min

c) $\nabla f(0,0)=\langle 0,-1\rangle\neq 0$

Let $p = \langle 0, 1 \rangle$

$p^T \nabla f(0,0) = -1 < 0$

Thus p is a search direction

4. Consider the primal-dual pair

$$\begin{array}{ll} \text{minimize} & p = f(\boldsymbol{x}) \\ \text{subject to} & g_i(\boldsymbol{x}) \leq 0 \quad i = 1, 2, \ldots, m \\ & \boldsymbol{x} \in X \end{array} \tag{1}$$

and

$$\begin{array}{ll} \text{maximize} & d = L(\boldsymbol{\lambda}) = \min_{\boldsymbol{x} \in X} \left(f(\boldsymbol{x}) + \sum_{i=1}^m \lambda_i g_i(\boldsymbol{x}) \right) \\ \text{subject to} & \boldsymbol{\lambda} \geq \mathbf{0}. \end{array} \tag{2}$$

Suppose $\hat{\boldsymbol{x}} \in X$ and $\hat{\boldsymbol{\lambda}} \geq \mathbf{0}$ satisfy the following saddle point condition for all $\boldsymbol{x} \in X$ and all $\boldsymbol{\lambda} \geq \mathbf{0}$:

$$f(\hat{\boldsymbol{x}}) + \sum_{i=1}^m \lambda_i g_i(\hat{\boldsymbol{x}}) \leq f(\hat{\boldsymbol{x}}) + \sum_{i=1}^m \hat{\lambda}_i g_i(\hat{\boldsymbol{x}}) \leq f(\boldsymbol{x}) + \sum_{i=1}^m \hat{\lambda}_i g_i(\boldsymbol{x}).$$

(a) (8 points) Show that $\hat{\boldsymbol{x}}$ solves the primal nonlinear program (1). (Hint: Use the saddle point condition to show that complementary slackness holds at $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\lambda}})$.)

(b) (7 points) Show that the saddle point condition implies strong duality, i.e., $p^* = d^*$ where p^* is the optimal value of (1) and d^* is the optimal value of (2).

a) Consider $g_i(x) < 0 \quad \forall x \in X$

$\hat{\lambda}_i g_i(x) \leq 0 \quad \forall x \in X$

It follows for $\lambda = 0$ since $\sum_{i=1}^m \hat{\lambda}_i g_i(\hat{x}) \leq 0$

$f(\hat{x}) = f(\hat{x}) + \sum_{i=1}^m \lambda_i g_i(\hat{x}) \leq f(\hat{x}) + \sum_{i=1}^m \hat{\lambda}_i g_i(\hat{x}) \leq f(\hat{x})$

Thus $f(\hat{x}) = f(\hat{x}) + \sum_{i=1}^m \hat{\lambda}_i g_i(\hat{x})$ whenever $g_i(\hat{x}) < 0$

$\Rightarrow \hat{\lambda}_i g_i(\hat{x}) = 0$

If $g_i(\hat{x}) = 0$ then $\hat{\lambda}_i g_i(\hat{x}) = 0$.

Thus we have complementary slackness.

From the given inequality and complementary slackness

$f(\hat{x}) = f(\hat{x}) + \sum_{i=1}^m \hat{\lambda}_i g_i(\hat{x}) \leq f(x) + \sum_{i=1}^m \hat{\lambda}_i g_i(x) \quad \forall x \in X$

$\leq f(x) \quad \forall x \in X \quad$ since $\hat{\lambda}_i g_i(x) < 0$

$\therefore f(\hat{x})$ is the min.

b) We know $\mathcal{L}(\hat{\lambda}) = f(\hat{x}) = p^*$ by complementary slackness

$\therefore d^* = \max_{\lambda \geq 0} \mathcal{L}(\lambda) \leq p^* = \mathcal{L}(\hat{\lambda})$ by weak duality

Thus $d^* = \mathcal{L}(\hat{\lambda})$ by definition of the maximum
$= p^*$

5. (8 points) Solve the following problem.

$$\begin{aligned} \text{minimize}_{\boldsymbol{x} \in X} \quad & \sum_{i=1}^{n} x_i \ln x_i \\ \text{subject to} \quad & \sum_{i=1}^{n} x_i = 1 \end{aligned}$$

where $X = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} > \boldsymbol{0}\}$.

Consider $\mathcal{L}(x, \lambda) = \sum_{i=1}^n x_i \ln x_i + \lambda(\sum_{i=1}^n x_i - 1)$

$\frac{\partial \mathcal{L}}{\partial x_i} = \ln x_i + 1 + \lambda = 0$

$\ln x_i = -\lambda - 1$

$x_i = e^{-\lambda - 1} \Rightarrow x_i = x_j \ \forall i, j$

Dual Feasibility

$\sum_{i=1}^n x_i = \sum_{i=1}^n x_1 = 1$

$\Rightarrow x_i = \frac{1}{n}$

Primal Feasibility

$f(x) = x \ln x$

The problem is convex

$\frac{\partial^2 f}{\partial x_i^2} = \frac{1}{x_i} \qquad \frac{\partial^2 f}{\partial x_i \partial x_j} = 0$ for $i \neq j$

Thus $\nabla^2 f(x)$ is diagonal and p.d. since

$|\nabla^2 f(x)| = \frac{1}{\prod_{i=1} x_i} > 0$ since $x_i > 0$

We also know the constraint is affine and $x_i = \frac{1}{n}$ is Feasible so we have LCQ

$\lambda(1 - \sum_{i=1}^n x_i) = 0$ complementary slackness

Thus $\{\frac{1}{n}\}_{i=1}^n$ is the global min by KKT thrm with LCQ and

$p^* = \sum_{i=1}^n \frac{1}{n} \ln \frac{1}{n} = \sum_{i=1}^n -\frac{\ln n}{n} = -\ln n$

6. (a) (7 points) Consider the problem

$$\begin{array}{ll} \text{minimize} & x_1^3 + x_2 \\ \text{subject to} & x_2 \geq 1. \end{array}$$

holds. Can we conclude that the KKT point is an optimal solution? Why or why not?

(b) (7 points) Consider the problem

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2 \\ \text{subject to} & x_2 \geq 1. \end{array}$$

holds. Can we conclude that the KKT point is an optimal solution? Why or why not?

a) No because x_1^3 is not convex and in particular this min is $-\infty$ as it is unbdd below

b) Yes because x_1^2 is convex which makes the objective function convex (x_2 is convex as it is linear)
The constraint also produces a convex feasibility region.
$(1,1) \in D^\circ$ so by Slater's and the convexity of the problem, KKT would give an optimal solution.