MATH 8100 Mathematical Programming Spring 2025
Final Exam

April 30, 2025

- There are a total of ?? points. Point value is listed next to each question.

- Mark your answers clearly. *Show your work.* Unsupported correct answers receive partial credit.

- Be sure to write your name and ID number on *each* page.

- Good luck!

Name:⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

Student ID #:⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

I certify that I have not received any unauthorized assistance in completing this examination.

Signature:⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

Date:⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

The conclusions of most good operations research studies are obvious. —Robert E. Machol

Cited in: Paul Dickson (1999), *The Official Rules and Explanations*, p. 14.

Machol named this the "Billings Phenomenon." Dickson explains: "The name refers to a well-known [Josh] Billings story in which a farmer becomes concerned that his black horses are eating a lot more than his white horses. He does a detailed study of the situation and finds that he has more black horses than white horses. Machol points out that the obvious conclusions are not likely to be obvious *a priori* but obvious after the results are in. In other words, good research does not have to yield dramatic findings."

1. (10 points) Let S be a nonempty set in $\mathbb{R}^n$. The *polar set* of S, denoted by S_P, is given by
$$S_P = \{\boldsymbol{y} : \boldsymbol{y}^T\boldsymbol{x} \leq 1, \forall \boldsymbol{x} \in S\}.$$
   Prove that S_P is convex.

2. (10 points) Use the KKT conditions to solve the following constrained optimization problem:
$$\begin{aligned} \text{minimize} \quad & x_1 x_2 \\ \text{subject to} \quad & x_1^2 + x_2^2 \leq 1 \end{aligned}$$

3. Consider the following linear program and the accompanying optimal dictionary:

$$\begin{array}{rlllll}
\text{maximize} & 10x_1 & + & 7x_2 & + & 6x_3 \\
\text{subject to} & 3x_1 & + & 2x_2 & + & x_3 & \leq & 36 \\
& x_1 & + & x_2 & + & 2x_3 & \leq & 32 \\
& 2x_1 & + & x_2 & + & x_3 & \leq & 22 \\
& & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$\begin{array}{rlllll}
\zeta & = & 146 & - & 3x_1 & - & x_4 & - & 5x_6 \\
x_2 & = & 14 & - & x_1 & - & x_4 & + & x_6 \\
x_5 & = & 2 & + & 2x_1 & - & x_4 & + & 3x_6 \\
x_3 & = & 8 & - & x_1 & + & x_4 & - & 2x_6
\end{array}$$

   (a) (5 points) Write the dual problem.

   (b) (5 points) Identify the current basis matrix and its inverse.

   (c) (5 points) Write the optimal solution to the dual.

   (d) (5 points) At what rate does the objective change from the optimal solution value if the right-hand side of the first constraint changes by a small amount?

   (e) (5 points) Over what range could the right-hand side of the first constraint change without changing the optimal basis?

   (f) (5 points) Write an expression showing how the optimal basic variables change with respect to a change of β in the first constraint's right-hand side.

   (g) (5 points) If the right-hand side of the first constraint falls below the bound you found above, what pivot must be performed to maintain feasibility? (You don't need to perform the pivot, just identify the leaving and entering variables.)

   (h) (5 points) Suppose a new column is added with coefficients of 2 and 2 in the first three constraints and 0 in the last one. What is the least profit on this new column to make it attractive to increase the level of that variable?

   (i) (5 points) If the profit is increased beyond the value you found above, what pivot would have to be performed to maintain optimality? (You don't need to perform the pivot, just identify the entering and leaving variables.)

4. Consider the following uncapacitated minimum-cost flow problem. The cost of each arc and the supply at each node is marked on the graph.
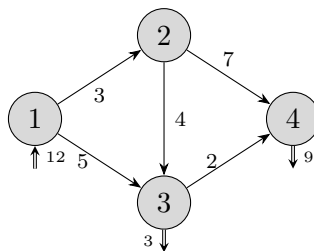
(a) (3 points) Write the linear program corresponding to this minimum-cost flow problem.

(b) (7 points) Starting from the feasible tree solution corresponding to $\{(1, 2), (1, 3), (2, 4)\}$, solve the problem by the network simplex method.

(c) (3 points) Instead of costs, interpret the arc labels as capacities. Ignore the supply labels on the nodes. Formulate the problem of finding the maximum flow from node 1 to node 4 as a linear program.

(d) (7 points) Solve the maximum flow problem using the shortest augmenting path method. Identify the maximum flow and the minimum-capacity cut.