MATH 8600 Scientific Computing

Homework No.6 (Due: Friday, Dec. 8)

1. Consider the ODE $y' = -y^2$ with the initial condition $y(0) = 1$. We will use the backward Euler's method with mesh size $h = 0.1$. Since the backward Euler method is implicit, and the ODE is nonlinear, we will need to solve a nonlinear algebraic equation for y_1.
   a. (3pts.) Write out that nonlinear algebraic equation for y_1.
   b. (3pts.) Write out the Newton iteration for solving the nonlinear algebraic equation.
   c. (3pts.) Obtain the starting guess for the Newton iteration by using one step of Euler's method for the ODE.
   d. (3pts.) Finally, compute an approximate value for the solution y_1 by using one iteration of Newton's method for the nonlinear algebraic equation.

2. Given the differential equation and initial conditions

$$\frac{dy}{dx} = \frac{2}{x}\,y + x^2\,e^x, \qquad y(1) = 0,$$

   with the exact solution $y(x) = x^2(e^x - e)$,
   a. (3pts.) use Euler method with $h = 0.1$ to approximate the solution at $x = 1.1$ and compute the numerical error.
   b. (3pts.) repeat (a) using the Backward Euler method.
   c. (3pts.) repeat (a) using Heun's method.
   d. (3pts.) repeat (a) using the second-order Taylor method.

3. Consider the ODE $y' = -5y$ with initial condition $y(0) = 1$. We will solve this ODE numerically using a mesh size of $h = 0.5$.
   a. (3pts.) Is Euler's method stable for this ODE using this mesh size?
   b. (3pts.) Is backward Euler's method stable for this ODE using this mesh size?

4. (Extra credit problem 15pts.) The parts (b) and (c) are computing problems. Consider the third-order differential equation

$$x^3 y''' - x^2 y'' + 3xy' - 4y = 5x^3\,ln\,x + 9x^3$$

   for $1 \leq x \leq 2$ with $y(1) = 0$, $y'(1) = 1$, $y''(1) = 3$. The equation has the actual solution $y(x) = -x^2 + x\,cos(ln\,x) + x\,sin(ln\,x) + x^3 ln\,x$.
   a. Write the third-order equation as a first-order system of differential equations.
   b. Solve the system obtained in (a) by the Runge-Kutta method of order 4 using $h = 0.1$.
   c. Plot your numerical solution and the actual solution in a common window for comparison