MATH 8600 Scientific Computing

HW #1 (Due on Sep. 12)

1. (5pts.) Consider the normalized system $\mathcal{F}(\beta, p, L, U)$ with $\beta = 10$.

   a. what are the smallest values of p and U, and the largest values of L, such that both 365.27 and 0.000512 can be represented exactly in a normalized floating-point system?

   b. How would your answer change if the system is not normalized?

2. (5pts.) Let x be a given nonzero floating-point number in a normalized system $\mathcal{F}(\beta, p, L, U)$, and let y be an adjacent floating-point number, also nonzero.

   a. What is the minimum possible spacing between x and y?

   b. What is the maximum possible spacing between x and y?

3. (6pts.) Prove that the induced norm

$$\|A\|_\alpha = \max_{x \neq 0} \frac{\|Ax\|_\alpha}{\|x\|_\alpha}, \quad \alpha = 1, 2, \infty$$

   is a matrix norm.

4. (5pts.) For $A_{m \times n}$ prove that

$$\|A\|_1 = \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|$$

5. (6pts.) Show that for all $x \in R^n$

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_\infty.$$

6. (6pts.) Make use of the inequalities from problem 5 to prove that for all nonsingular $A \in R^{n \times n}$

$$\frac{cond_2(A)}{n} \leq cond_1(A) \leq n\, cond_2(A),$$

$$\frac{cond_\infty(A)}{n} \leq cond_2(A) \leq n\, cond_\infty(A),$$

   and

$$\frac{cond_1(A)}{n^2} \leq cond_\infty(A) \leq n^2 cond_1(A).$$

7. (8pts.) Prove the following properties of the condition number for $\alpha = 1, 2, \infty$.

   a. $cond_\alpha(I) = 1$

   b. $cond_\alpha(A) \geq 1$ for any $A \in R^{n \times n}$.

   c. $cond_\alpha(P) = 1$ for any permutation matrix P.

   d. $cond_\alpha(D) = \frac{max_i |d_i|}{min_i |d_i|}$, where $D = (d_i)$ is a diagonal matrix.