HW 2 Jacob Manning

1a. Prove the product of two upper triangular matrices are upper triangular.

Let U_1 & U_2 be upper triangular & size $n \times n$. It follows $U_{1ij} = 0 = U_{2ij}$ $\forall\ i > j$. Consider $U_0 = U_1 U_2$. $U_{0ij} = \sum_{k=1}^{n} U_{1ik} U_{2kj}$. Since $U_{1ij} = U_{2ij} = 0\ \forall\ i > j$, $U_{0ij} = 0\ \forall\ i > j$. $\therefore$ U_0 is also upper triangular.

b. Prove the inverse of a nonsingular upper triangular matrix is upper triangular.

Let U be upper triangular $n \times n$ and nonsingular. Consider $U^{-1} U = I$. It follows $\delta_{ij} = \sum_{k=1}^{n} u^{-1}_{ik} u_{kj}$. Since $u_{ij} = 0\ \forall\ i > j$, u^{-1}_{ij} will also need to be zero $\forall\ i > j$ so that the sum of the products equals δ_{ij}. $\therefore$ U^{-1} is also upper triangular.

2. $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix}$ $b = [-2\ 10\ 14]^T$ $[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ 1 & 2 & 3 & 10 \\ 2 & -1 & 4 & 14 \end{array}\right]$ $P_{31}[A|b] = \left[\begin{array}{ccc|c} 2 & -1 & 4 & 14 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & -1 & -2 \end{array}\right]$

$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{bmatrix}$ $M P_{31}[A|b] = \left[\begin{array}{ccc|c} 2 & -1 & 4 & 14 \\ 0 & 5/2 & 1 & 3 \\ 0 & 5/2 & -3 & -9 \end{array}\right]$ $\hat{M} P_{31}[A|b] = \left[\begin{array}{ccc|c} 2 & -1 & 4 & 14 \\ 0 & 5/2 & 1 & 3 \\ 0 & 0 & -4 & -12 \end{array}\right]$

$U = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 5/2 & 1 \\ 0 & 0 & -4 \end{bmatrix}$ $P_{31} A = LU$

$A = P_{31} L U$ $P_{31} L U x = b$

back sub $LUx = P_{31} b$ $\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{bmatrix} Ux = \begin{bmatrix} 14 \\ 10 \\ -2 \end{bmatrix}$

Front sub

$Ux = [14\ \ 3\ \ -12]^T$ $\begin{bmatrix} 2 & -1 & 4 \\ 0 & 5/2 & 1 \\ 0 & 0 & -4 \end{bmatrix} x = \begin{bmatrix} 14 \\ 3 \\ -12 \end{bmatrix}$ $x = [1\ 0\ 3]^T$

3a. $n \times n$ upper Hessenberg, nonsingular, no pivoting

Let A_{i+1} be the matrix A_i but with gaussian elemination in $[a_i]$

$A_{i+1} = L_i A_i$ $L = \prod_{i=1}^{n} L_i$ and $U = A_{n+1}$ $LA = \prod_{i=1}^{n} L_i A_0$

b. $L_{ii} = 1$ $1 \le i \le n$ & $L_{i+1,i} = \alpha$ where α is a scalar

else $L_{ij} = 0$

c. We would still need $\sim O(n^2)$ operations because we need back substitution

4. This algorithm is numerically stable. The stability comes the facts that all permutations of size $n \times n$ are normal in $\mathbb{R}^{n \times n}$ ($P \in \mathbb{R}^{n \times n}$). You can always find a permutation to commute with any change made to A. It follows permutations are the same done before or after LU factorization. By the numerical stability of $A = LU$, it follows $PA = LU$ is also stable.

5. $$\begin{bmatrix} L_1 & 0 \\ B & L_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}$$

First solve $L_1 x = b$ for $x \Rightarrow x = L_1^{-1} b$

Next $Bx + L_2 y = c$ since $x = L_1^{-1} b$ it follows

$$B L_1^{-1} b + L_2 y = c$$
$$L_2 y = c - B L_1^{-1} b$$

Then solve this vector equation for y.

It follows $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} L_1^{-1} b \\ L_2^{-1}(c - B L_1^{-1} b) \end{bmatrix}$