HW9

Saturday, April 27, 2024 1:59 PM

[Q1] (10 pts) (a) For a generic Krylov subspace method that takes the initial approximation x_0, gets the initial residual $r_0 = b - Ax_0$, develops the sequence of Krylov subspaces $\mathcal{K}_k(A, r_0)$ and constructs the approximate solution $x_k = x_0 + z_k$ where $z_k \in \mathcal{K}_k(A, r_0)$, the residual $r_k = b - Ax_k$ can be written as $r_k = p_k(A)r_0$, where p_k is a polynomial of degree no greater than k with $p_k(0) = 1$.

Consider $r_k = r_0 - AV_k y_k$

Since $\text{col}(V_k) = \mathcal{K}_k(A, r_0)$, we can write $V_k y_k$ as

$q_{k-1}(A) r_0$

Thus, $r_k = (I - A q_{k-1}(A)) r_0$

$= p_k(A) r_0$

Note $p_k(0) = 1$ since $I \to 1$ $\blacksquare$

(b) Let A be SPD, and x_0 and $r_0 = b - Ax_0$ be the initial approximation and residual, respectively. Consider the Lanczos relation $AU_k = U_k T_k + \beta_k u_{k+1} e_k^T$ (Arnoldi's method applied to a symmetric A), where $u_1 = \frac{r_0}{\|r_0\|_2}$. Show that the k-th iterate of CG can be written as $x_k = x_0 + U_k y_k$, where y_k satisfies $T_k y_k = \|r_0\|_2 e_1$.
(Hint: use the fact that $r_k = b - Ax_k = r_0 - AU_k y_k \perp \mathcal{K}_k(A, r_0) = \text{col}(U_k)$)

$r_k = b - Ax_k = r_0 - AU_k y_k$

$r_k \perp \mathcal{K}_k(A, r_0) = \text{col}(U_k)$

$r_0 = b - Ax_0$

$x_k = x_0 + U_k y_k$

$r_k = b - Ax_k$

$= b - A(x_0 + U_k y_k)$

$= b - Ax_0 - AU_k y_k$

$= r_0 - AU_k y_k$

$r_k = r_0 - (U_k T_k + \beta_k u_{k+1} e_k^T) y_k$

$U_k^T r_k = U_k^T r_0 - T_k y_k$

$U_k^T (b - Ax_k) = U_k^T (b - Ax_0) - T_k y_k$

$U_k^T A x_0 = U_k^T A x_k - T_k y_k$

$U_k^T A x_0 + T_k y_k = U_k^T A x_k \qquad U_k^T A U_k = T_k$

$\cancel{U_k^T A}(x_0 + U_k y_k) = \cancel{U_k^T A} x_k$

$x_k = x_0 + U_k y_k$

(c) Show that the k-th residual of GMRES $r_k = b - Ax_k$ satisfies $r_k \in \mathcal{K}_{k+1}(A, r_0)$, $r_k \perp A\mathcal{K}_k(A, r_0)$, $(r_k, r_k) = (r_j, r_k)$ for all $0 \le j \le k-1$, and therefore $\|r_k\|_2 \le \|r_j\|_2$.

$\text{span}(\{A^n r_0\}_{n=1}^{k}) \qquad r_1 = b - Ax_0$

$r_k = b - AQ_k y \qquad r_1 \in K_1(A, r_0)$

Assume

$r_k \in K_k(A, r_0) \qquad r_k = b - Ax_k \in \text{span}\{A^n r_0\}_{n=0}^{k-1}$

$r_{k+1} = b - Ax_{k+1} \qquad x_{k+1} = Q_k y$

$x_{k+1} \in \text{Col}(Q_k) = K_k(A, r_0)$

$\Rightarrow b - Ax_{k+1} \in K_{k+1}(A, r_0)$

$\therefore r_{k+1} \in K_{k+1}(A, r_0)$

$\langle r_k, v \rangle = \langle r_0 - Ax_k, v \rangle$ Let $v \in AK_k(A, r_0)$

$= v^T r_0 - v^T A x_k$ since by Arnoldi process

$= 0 + 0$ there are orthogonal

$\therefore \|r_k\| \le \|r_j\| \quad \forall \ k \ge j$

35.2. (a) Let $S \subseteq \mathbb{C}$ be a set whose convex hull contains 0 in its interior. That is, S is contained in no half-plane disjoint from the origin. Show that there is no $p \in P_1$ (i.e., no polynomial p of degree 1 with $p(0) = 1$) such that $\|p\|_S < 1$.

(b) Let A be a matrix, not necessarily normal, whose spectrum $\Lambda(A)$ has the property (a). Show that there is no $p \in P_1$ such that $\|p(A)\| < 1$.

(c) Though the convergence in Figure 35.5 is slow, it is clear that $\|r_1\| < \|r_0\|$. Explain why this does not contradict the result of (b). Describe what kind of polynomial $p_1 \in P_1$ GMRES has probably found to achieve $\|r_1\| < \|r_0\|$.

a) $\|p\|_S = \sup_{z\in S} |p(z)|$

Assume $\exists\, p \in P_1 \ni$

$p(0) = 1$

$p(z) = a + bz$

$p(0) = a = 1$

$\Rightarrow p(z) = 1 + bz$

WLOG assume S is a circle, if not take $S \subseteq S'$ that is a circle and what is shown for S' will hold for S

Let $S = B(0, r)$ so $\|p\|_S = \sup_{z\in S} |1 + bz|$

$= 1 + br > 1$ $\blacksquare$

(b) Let A be a matrix, not necessarily normal, whose spectrum $\Lambda(A)$ has the property (a). Show that there is no $p \in P_1$ such that $\|p(A)\| < 1$.

b) $\|p(0)\| = 1 \Rightarrow p(A) = I + \beta A$

Consider an eigenpair (λ, v) of A.

Let v be normalized. Choose direction of $v \ni \beta\lambda \geq 0$

$$\begin{aligned}
\|I + \beta A\| &= \sup_{\|x\|=1} \|(I + \beta A)x\| \\
&\geq \|v + \beta A v\| \\
&= \|v + \beta\lambda v\| \\
&= \|(1 + \beta\lambda)v\| \\
&= |1 + \beta\lambda| \\
&> 1
\end{aligned}$$

(c) Though the convergence in Figure 35.5 is slow, it is clear that $\|r_1\| < \|r_0\|$. Explain why this does not contradict the result of (b). Describe what kind of polynomial $p_1 \in P_1$ GMRES has probably found to achieve $\|r_1\| < \|r_0\|$.

$r_0 = b \qquad r_1 = b - Ax_0$

It does not contradict since the residual decreases by looking at the minized residual. So $\|r_{n+1}\| \le \|r_n\| \ \forall n \in \mathbb{N}$ (problem 1)

$p(A) = I + bA$

$r_1 = p(A) r_0$

$\| r_1 \| = \|p(A) r_0\| \le \|p(A)\| \, \|r_0\|$

If $\|p(A)\| = 1$ then $\|r_1\| \le \|r_0\|$

(b) Let $A \in \mathbb{R}^{n\times n}$ be nonsymmetric and diagonalizable. Assume that all eigenvalues of A lie in the disk centered at $c \in \mathbb{C} \setminus \{0\}$ with radius $r < |c|$. Consider using GMRES to solve the linear system $Ax = b$ iteratively. Show that the k-th relative residual satisfies $\frac{\|r_k\|_2}{\|r_0\|_2} \le C\left(\frac{r}{|c|}\right)^k$ for some constant C independent of k. What if A has a small number, say, $m \ll n$ eigenvalues outside such a disk?

$\|r_k\| = \min_p \|p_k(A) r_0\|$

$\frac{\|r_k\|}{\|r_0\|} \le \min_p \|p_k(A)\|$

$A = VDV^{-1}$

$\frac{\|r_k\|}{\|r_0\|} \le \min \kappa(V) \|p_k(A)\|$

$= \max \kappa(V) |H_k - \lambda I|$ from class min is characteristic polynomial of H

$= \kappa(V) \lambda_1^k$

$\le \kappa(V) \left(\frac{r}{|c|}\right)$ max from eigval

Since $\Lambda(A) \subseteq B(|c|, r)$

It would depend on the magnatude of the eigenvalues outside of the disk.

c) (c) If A is an SPD matrix with the smallest eigenvalue λ_1 and the largest eigenvalue λ_n, what is the convergence factor obtained in part (b)? Compare this factor with that of CG we learned in class. Which one is better?

$\frac{\lambda_1}{\lambda_n}$ Which would be larger

$\frac{\lambda_1}{\lambda_n}$ which would be larger

than CG's $\sqrt{\frac{\lambda_1}{\lambda_n}}$ rate

Note $\Lambda(A) = B(|c|, \lambda_n) \setminus B(|c|, \lambda_1)$ in this case

3. **[Q3]** (10 pts) Let x^* be the true solution of $Ax = b$ with SPD A, x_k be the k-th iterate of CG, and $\varphi(x) = \frac{1}{2}x^T Ax - b^T x$ for CG minimization.

   (a) Note that $r_k \perp r_j$ for $0 \le j \le k-1$, and hence $r_k \perp U_k = \text{span}\{p_0, p_1, \ldots, p_{k-1}\}$. Also note that $r_k = -\nabla\varphi(x_k)$, and any vector $x \in W_k = x_0 + U_k$. Explain from the optimization point of view, why $x_k = \text{argmin}_{x \in W_k} \varphi(x)$.

   Hint: one possible (and easier) solution is to show that W_k is a convex set, and $\varphi(x)$ is a convex function defined on W_k; then local minimizer of $\varphi(x)$ is necessarily a global minimizer. Please do a little search on convex set/functions yourselves. The condition $r_k \perp U_k$ is crucial to show the optimality here.

   (b) Show directly that $x_k = \text{argmin}_{x \in W_k} \|x - x^*\|_A$, without referring to the connection between $\varphi(x)$ and $\|e_k\|_A$. (Hint: consider a different $\tilde{x}_k \in W_k$, with $d_k = \tilde{x}_k - x_k \neq 0$. Show that $\|\tilde{x}_k - x^*\|_A = \|d_k + x_k - x^*\|_A \ge \|x_k - x^*\|_A$)

a) Since $W_k = x_0 + U_k$ where U_k is a vector space, it follows U_k is convex $\Rightarrow$ W_k is convex.

Let $x_1 = \sum_{l=1}^{k} \alpha_l p_l$ $\quad x_2 = \sum_{l=1}^{k} \beta_l p_l$

WTS $\varphi(\lambda x_1 + (1-\lambda)x_2) \le \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2)$

$$\tfrac{1}{2}(\lambda x_1 + (1-\lambda)x_2)^T A(\lambda x_1 + (1-\lambda)x_2) - b^T(\lambda x_1 + (1-\lambda)x_2)$$

A is spd $\in \mathbb{R}^{n\times n}$

$$= \tfrac{1}{2}\lambda^2 x_1^T A x_1 + \tfrac{1}{2}(1-\lambda)^2 x_2^T A x_2 + \lambda(1-\lambda)x_1^T A x_2 - \lambda b^T x_1 + (1-\lambda)b^T x_2$$

$$= \lambda^2\varphi(x_1) + (1-\lambda)^2\varphi(x_2) + \lambda(1-\lambda)x_1^T A x_2$$

$$\le \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2) + \lambda(1-\lambda)x_1^T A x_2$$

Bc A is spd $\lambda(1-\lambda)x_1^T A x_2$ is convex

Thus by the sum of two convex functions being convex, we see $\varphi(x)$ is convex

Therefore the local minimizer $x_k \in W_k$ for $\varphi(x)$ is the global minimizer.

(b) Show directly that $x_k = \text{argmin}_{x \in W_k} \|x - x^*\|_A$, without referring to the connection

(b) Show directly that $x_k = \text{argmin}_{x \in W_k} \|x - x^*\|_A$, without referring to the connection between $\varphi(x)$ and $\|e_k\|_A$. (Hint: consider a different $\tilde{x}_k \in W_k$, with $d_k = \tilde{x}_k - x_k \neq 0$. Show that $\|\tilde{x}_k - x^*\|_A = \|d_k + x_k - x^*\|_A \geq \|x_k - x^*\|_A$)

$d_k = \tilde{x}_k - x_k \neq 0 \implies \tilde{x}_k = d_k + x_k$

$\|\tilde{x}_k - x^*\|_A = \|d_k + x_k - x^*\| \geq | \|x_k - x^*\| - \|d_k\| | \geq \|x_k - x^*\|$

since $\|d_k\| \geq 0$

4a) B has pos & neg eig vals ✓

$\hat{B}$ has pos eig vals ✓

Both symmetric

A & $\hat{A}$ are not symmetric

The eigvals are close to the origin

There does not seem to be an obvious correlation to condition number as $\hat{B}$ was largest

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>> HW9_linsolvecomp
Loading problem data completed.
SPD matrix A: preconditioner setup time 4.762 secs.
P-CG converges in 70 iters (12.203 secs) to relative tol 1.000e-08.
MATLAB's sparse Cholesky direct solver takes 369.842 secs.
nnz of A, incomplete and complete Cholesky factors: 6141493 27725520
   1140363654.

Symmetric indefinite C: indefinite preconditioner constructed already.
P-SQMR converges in 67 iters (2.059 secs) to relative tol 1.000e-08.

Transforming the indefinite preconditioner to an SPD preconditioner.
Transformation to SPD preconditioner takes 0.186 secs.
P-MINRES converges in 210 iters (2.701 secs) to relative tol 1.000e-08.
MATLAB's sparse LDL direct solver takes 156.538 secs.
nnz of C, incomplete and complete LDL factors: 738598 617410 82593444.

Unsymmetric matrix A: preconditioner setup time 2.858 secs.
P-BICGSTABL(2) solve: 93 mvps (10.342 secs) to tolerance 1.000e-08.
P-IDR(4) solve: 86 mvps (10.190 secs) to tolerance 1.000e-08.
P-GMRES(60) solve: 90 mvps (24.967 secs) to tolerance 1.000e-08.
MATLAB's sparse LU direct solution takes 1129.186 secs.
nnz of B, incomplete and complete LU factors: 3604989 5659826 856629224.
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It was more manageable and faster