MATH 8610 (SPRING 2024) HOMEWORK 7

Assigned 03/25/24, due 04/03/23 (Wednesday) by 11:59pm.
Instructor: Dr. Fei Xue, Martin O-203, fxue@clemson.edu.

1. **[Q1]** (10 pts) **(a)** A is Hermitian if $A^H = A$, and skew-Hermitian if $A^H = -A$. Show that the eigenvalues of a Hermitian matrix (e.g., real symmetric) are real, and those of a skew-Hermitian (e.g., real skew-symmetric) are purely imaginary. In both cases, show that the eigenvectors associated with distinct eigenvalues are orthogonal.

   (b) For a block upper triangular $F = \begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1n} \\ & F_{22} & \cdots & F_{2n} \\ & & \ddots & \vdots \\ & & & F_{nn} \end{bmatrix}$, show that $\Lambda(F) = \bigcup_{k=1}^{n} \Lambda(F_{kk})$, where $\Lambda(\cdot)$ denotes the spectrum (all eigenvalues) of a matrix.

2. **[Q2]** (10 pts) **(a)** Given a complex Schur form $U^H AU = T$, where T is upper triangular, show that the first k columns of U, $u_1, u_2, \ldots, u_k$, span an invariant subspace of A. That is, $A\,\mathrm{span}\{u_1, \ldots, u_k\} \equiv \mathrm{span}\{Au_1, \ldots, Au_k\} \subset \mathrm{span}\{u_1, \ldots, u_k\}$.

   (b) Let $U \in \mathbb{R}^{n\times p}$ $(n > p)$ contains basis vectors of an invariant subspace of A, such that $AU = UM$ for some $M \in \mathbb{R}^{p\times p}$. Show that the eigenvalues of M are also eigenvalues of A. If, in addition, $W \in \mathbb{R}^{n\times m}$ $(n > m > p)$ has orthonormal columns, and $\mathrm{col}(U) \subset \mathrm{col}(W)$, show that the eigenvalues of M are eigenvalues of $W^T AW$.

3. **[Q3]** (15 pts) **(a)** Describe a procedure to post-process the Q and R factors of Givens or Householder QR, such that the R factor has all non-negative diagonal entries.

   (b) Verify numerically that the Simultaneous Iteration is equivalent to the unshifted QR iteration. To this end, first construct an upper Hessenberg H_0 as follows

   ```
   rng('default'); H0 = triu(randn(7,7),-1);
   ```

   Implement the Simultaneous Iteration and the QR iteration, described in Trefethen's book, Chapter 28. Feel free to use MATLAB's `qr`, followed by the post-processing in part (a), and the `'*'` operation directly to form $H^{(k)} = R^{(k)}Q^{(k)}$ (that is, no need to use the Givens rotations to perform the QR iteration as usually supposed to).

   Compare the projection matrices $H_{SI}^{(k)}$ in (28.10) for simultaneous iteration and $H_{QR}^{(k)}$ in (28.13) in the QR iteration. Find the relative difference $\frac{\|H_{SI}^{(k)} - H_{QR}^{(k)}\|_F}{\|H_{SI}^{(k)}\|_F}$ for $k = 3$, 30, 300 and 3000, and the relative difference in the eigenvalues of $H_{SI}^{(k)}$ and $H_{QR}^{(k)}$ at these steps? What if the post-processing is not used, and in this case, do $H_{SI}^{(k)}$ and $H_{QR}^{(k)}$ have numerically the same eigenvalues?

   (c) Find the eigenvalues of H_0, then use the theory we learned from class to estimate the rate of convergence of $H_{QR}^{(k)}$ toward the quasi-upper triangular T of the real Schur form. About how many iterations are needed to achieve $\|H_{QR}^{(k)} - T\|_F / \|T\|_F \approx \epsilon_{mach}$?

4. **[Q4]** (10 pts) (Trefethen's book Prob. 28.2, but for the nonsymmetric case).
   (a) Explore the nonzero structure of the Q factor of the QR factorization of an upper Hessenberg matrix, and verify that RQ is also upper Hessenberg. For clarity, you may give an illustration for a 5×5 upper Hessenberg.

(b) The computation of $H^{(k)} = R^{(k)}Q^{(k)}$, if done naively (by direct evaluation of the matrix-matrix multiplication), would need $\mathcal{O}(n^3)$ operations. Fortunately, $H^{(k)}$ can be computed only in $\mathcal{O}(n^2)$ operations. Explain, by Givens rotations, how this is achieved. Make sure that you do see the difference in cost.