MATH 8610 (SPRING 2024) HOMEWORK 3

Assigned 02/06/24, due 02/15/24 by 11:59pm (Thursday).
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1. **[Q1] (10 pts)** (a) Determine the eigenvalues, determinant, and singular values of a Householder reflector $H = I - 2\frac{vv^T}{v^Tv}$. For the eigenvalues, give a geometric argument as well as an algebraic proof.

   (b) Consider the *Givens rotation* $G = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Give a geometric interpretation of the action of G on a vector in $\mathbb{R}^2$. Do the same analysis as part (a) for G, but no geometric interpretation is needed for the eigenvalues.

2. **[Q2] (10 pts)** Implement QR factorizations in MATLAB based on
   (i) classical Gram-Schmidt (CGS)
   (ii) modified Gram-Schmidt (MGS)
   (iii) MGS with double orthogonalization, and
   (iv) Householder reflectors (for Householder $H = I - \frac{2vv^T}{v^Tv}$, let $v = x + \text{sign}(x_1)\|x\|_2 e_1$, with $\text{sign}(z) = 1$ for $z = 0$ and $e^{i\theta}$ for $z = \rho e^{i\theta} \neq 0$).
   Then we construct three matrices as follows.

```
A1 = randn(2^20,15);   % (large but well-conditioned)
u = (-1:2/40:1)';
A2 = u.^(0:23);  % (partial Vandermonde)
A3 = u.^(0:40);  % (full Vandermonde)
```

   For each matrix, run the algorithms, then compute $\frac{\|A-\hat{Q}\hat{R}\|_F}{\|A\|_F}$ and $\|\hat{Q}^T\hat{Q} - I_n\|_2$. Draw conclusions about the backward stability of these algorithms, and the orthogonality of the computed Q factors, probably related to the condition numbers of the matrices.

3. **[Q3] (10 pts)** Evaluate the arithmetic work needed to retrieve the reduced factor $Q_L \in \mathbb{R}^{m\times n}$ from the Householder and Givens reduction of A to R, respectively (second phase of QR). Compare the cost with that for the first phase.

4. **[Q4] (10 pts)** Implement the algorithm for solving linear system $Ax = b$ or linear least squares problem $\min \|b - Ax\|_2$ based on Householder QR. Make sure that the reduced Q factor is NOT formed explicitly to save the cost of the second phase.

   Then solve the linear least squares problem $\min \|b - Ax\|_2$ where $A = A_2$, and the linear system $Ax = b$, where $A = A_3$ in [Q2], and $b = [1, -1, 1, -1, \ldots]^T$. Report your $\frac{\|b-A\hat{x}\|_2}{\|A\|_2\|\hat{x}\|_2}$ for both solves, and compare with this quantity associated with the solutions obtained by MATLAB's backslash.