MATH 8610 (SPRING 2024) HOMEWORK 4

Assigned 02/16/24, due 02/26/24 by 11:59pm (Monday).
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1. **[Q1] [Q2] (10 points each)** Trefethen's book, Problems 11.1 and 11.3. You are encouraged to give a serious consideration of 11.2 (not required to work on it, though).

2. **[Q3] (15 points)** Implement Householder reduced QR factorization with *column pivoting*. At step k, consider columns k through n of the current A (has been updated in previous steps), find the column j ($k \le j \le n$) such that $\|A(k:m,j)\|_2 = \max_{k\le \ell \le n} \|A(k:m,\ell)\|_2$, and switch columns k and j. Similar to GEPP, we need a permutation matrix P to record column swapping, such that $AP = QR$ numerically.

   Generate a new test matrix as follows.

   ```
   U = randn(1024,10);
   A4 = U*randn(10,15);
   ```

   (a) Test your code on A_2, A_3 (see HW3 [Q2]) and A_4, compare your upper triangular matrices with those generated by MATLAB's command `[Q,R,P] = qr(A,0);`
   (b) Show that the diagonal elements of R are monotonically decreasing in modulus.
   (c) Comment on the use of this algorithm to extract a set of numerically linearly *independent* columns from a matrix with numerically linearly *dependent* columns.

3. **[Q4*] (10 extra points)** Consider the Householder reduced QR applied to the $(m+n)\times n$ matrix $B = \begin{bmatrix} 0_n \\ A \end{bmatrix} = Q_1^{(B)} R_1^{(B)}$. Show that in exact arithmetic, up to a sign change column-wise, $Q_1^{(B)} = \begin{bmatrix} 0_n \\ Q_1^{(A)} \end{bmatrix}$, where $Q_1^{(A)} \in \mathbb{R}^{m\times n}$ is the reduced Q factor obtained by applying MGS to A. In computer arithmetic, explore what you *actually* get for $\hat{Q}_1^{(B)}$, for matrices A_2 and A_3 in [Q2]. Explain why we necessarily have a nonzero top block in $\hat{Q}_1^{(B)}$ (Hint: consider the level of orthogonality of the columns of $Q_1^{(B)}$ and of the columns of $Q_1^{(A)}$ obtained by the two algorithms).