Jacob Manning HW0

1a) Let $S^* = -S$ and eigenvalues of S be σ

$\sigma\|x\|^2 = x^*\sigma x = x^*Sx = \langle Sx, x\rangle = \langle x, S^*x\rangle = x^*S^*x = -\sigma\|x\|^2$

$\therefore \sigma = -\sigma \Rightarrow \sigma$ is prue imaginary.

b) Show $I-S$ is nonsingular.

$(I-S)x = x - Sx = x - \sigma x = (1-\sigma)x$

Since σ is prue imaginary, $1-\sigma \neq 0$.

It follows $I-S$ does not have any 0 eigenvalues.

Hence, $I-S$ is nonsingular.

c) Show $Q = (I-S)^{-1}(I+S)$ is unitary.

Note $(I-S)^* = I+S$. Also note $I^2 - S^2 = (I-S)^*(I-S)$

$= (I-S)(I-S)^*$

So $(I-S)^*$ comuetes with $I-S$,

It follows $QQ^* = (I-S)^{-1}(I-S)^*[(I-S)^{-1}(I-S)^*]^*$

$= (I-S)^{-1}(I-S)^*(I-S)(I-S)^{*-1}$

$= (I-S)^{-1}(I-S)(I-S)^*(I-S)^{*-1}$

$= I$

$u, v \in \mathbb{C}^n$ $A = I + uv^*$

2a) A nonsingular Show $A^{-1} = I + \alpha uv^*$ find α.

Let $v^*u = \beta$,

Consider $AA^{-1} = I$

$(I+uv^*)(I+\alpha uv^*) = I$

$uv^* + \alpha uv^* + \alpha uv^*uv^* = 0$

$(1+\alpha+\alpha\beta)uv^* = 0$

Since A is nonsingular, $uv^* \neq 0$

It follows $\alpha = \frac{-1}{1+\beta}$

b) If A is singular what is u and v?

$\det A = 0$

$\det(I+uv^*) = 0$ By Sylvester's Thrm

$1 + v^*u = 0$

$v^*u = -1$

2c) If A is singular What is null A?

Since A is singular, $\exists\, x \neq 0 \ni Ax = 0$.

$(I + uv^*)x = 0$ Let $v^*x = \gamma$

$x + \gamma u = 0$

$x = -\gamma u$

It follows null A = span u

3. Prove $|\langle x,y\rangle| \leq \sqrt{\langle x,x\rangle\langle y,y\rangle}$

Let $x = \frac{\langle x,y\rangle}{\langle y,y\rangle} y + z$, It follows $z = x - Sy$.

Consider $\langle y,z\rangle = \langle y, x - Sy\rangle$

$= \langle y,x\rangle - \underline{\langle y, Sy\rangle}$

$= \langle y,x\rangle - \frac{\langle x,y\rangle}{\cancel{\langle y,y\rangle}}\cancel{\langle y,y\rangle}$

$= 0$

Hence $\langle x,x\rangle = \langle Sy+z, Sy+z\rangle = \langle Sy,Sy\rangle + \langle z,z\rangle$.

Consider $\langle Sy,Sy\rangle \leq \langle Sy,Sy\rangle + \langle z,z\rangle = \langle x,x\rangle$

$\left|\frac{\langle x,y\rangle}{\langle y,y\rangle}\right|^2 \langle y,y\rangle \leq \langle x,x\rangle$

$\therefore$ $|\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle \Rightarrow |\langle x,y\rangle| \leq \sqrt{\langle x,x\rangle\langle y,y\rangle}$

4. Show $\langle A,B\rangle$ is an I.P. Let $C \in \mathbb{C}^{m\times n}$

Using Einstein sumation notation.

- $\langle A,A\rangle = a_{ij}\overline{a_{ij}} = |a_{ij}|^2 = 0$ iff $a_{ij} = 0\ \forall\, i,j$
- $\langle \alpha A, B\rangle = (\alpha a_{ij})\overline{b_{ij}} = \alpha(a_{ij}\overline{b_{ij}}) = \alpha\langle A,B\rangle$
- $\langle A,B\rangle = a_{ij}\overline{b_{ij}} = \overline{\overline{a_{ij}}\, b_{ij}} = \overline{b_{ij}\overline{a_{ij}}} = \overline{\langle B,A\rangle}$
- $\langle A+C, B\rangle = (a_{ij}+c_{ij})\overline{b_{ij}} = a_{ij}\overline{b_{ij}} + c_{ij}\overline{b_{ij}} = \langle A,B\rangle + \langle C,B\rangle$.

$\|A\|_F^2$ is defined as $|a_{ij}|^2 = \langle A,A\rangle$.

It follows $\|\cdot\|_F = \sqrt{\langle\cdot,\cdot\rangle}$ and $\therefore$ $\|\cdot\|_F$ is induced by $\langle\cdot,\cdot\rangle$

5. Show $\rho(A) \le \|A\|$. When does $\lim_{k\to\infty} A^k = 0$?
Let $|\lambda| = \rho(A)$.
Consider $|\lambda|\|x\| = \|\lambda x\| = \|Ax\| \le \|A\|\|x\|$.
Since x is an eigenvector, $x \ne 0$. It follows
$|\lambda| = \rho(A) \le \|A\|$.
$\lim_{k\to\infty} A^k x = \lim_{k\to\infty} \lambda^k x = 0$ only if $|\lambda| < 1$

6. $\|x\|_D := \sup_{\|y\|=1} |y^* x|$ Show $\|\cdot\|_D$ is a norm.
- $\|x\|_D = \sup_{\|y\|=1} |y^* x| \le \sup_{\|y\|=1} \|y^*\|\|x\| = 0$ iff $x = 0$
- $\|\alpha x\|_D = \sup_{\|y\|=1} |\alpha y^* x| = |\alpha| \sup_{\|y\|=1} |y^* x| = |\alpha|\|x\|_D$
- $\|x+z\|_D = \sup_{\|y\|=1} |y^*(x+z)| \le \sup_{\|y\|=1} |y^* x| + \sup_{\|y\|=1} |y^* z| = \|x\|_D + \|z\|_D$

Let $\|x\| = \|y\| = 1$ Show $\exists\ B = yz^* \ni$
$Bx = y$ & $\|B\| = 1$
Hint $x \in \mathbb{C}^n\ \exists\ z \in \mathbb{C}^n \ni |z^* x| = \|x\|\|z\|_D = \|z\|_D$
Choose $z^* \ni z^* x \ge 0$ and $\|z\|_D = 1$.
B is a rank 1 matrix.
Consider $Bx = yz^* x = y|z^* x| = \|z\|_D y = y$.
Now, $\|B\| = \sup_{\|x\|=1} \|Bx\| = \|y\| = 1$.

7. Show $\|A\|_{\infty,1} = \max_{i,j} |a_{ij}|$
$$\|A\|_{\infty,1} = \sup_{\|x\|_1=1} = \|\textstyle\sum_{j=1}^n a_j x_j\|_\infty$$
$$\le \sup_{\|x\|_1=1} \textstyle\sum_{j=1}^n |x_j|\|a_j\|_\infty$$
$$\le \max_{1\le j\le n} \|a_j\|_\infty$$
$$\le \max_{i,j} |a_{ij}|.$$
Next consider $x = e_k \ni k$ maximizes $|a_{ik} e_k|$
$$\|A\|_{\infty,1} \ge \|Ax\|_\infty$$
$$= \max_{1\le i\le m} |a_{ij} x_j|$$
$$= \max_{i,j} |a_{ij}|$$
It follows $\|A\|_{\infty,1} = \max_{i,j} |a_{ij}|$

8. $A_{m\times n}$ $B_{p\times q}$ submatrix $1 \le p \le m$ & $1 \le q \le n$
Find $P \in \mathbb{R}^{p\times m}$ and $Q \in \mathbb{R}^{n\times q}$ $\ni$ $B = PAQ$.
Show $\|B\|_\ell \le \|A\|_\ell$ $\forall 1 \le \ell \le \infty$.
WLOG, by pivoting, transform A so that
B is the first p rows and q columns
of A. It follows $P = I_p$ and $Q = I_q$.
Consider $\|B\|_\ell = \|PAQ\|_\ell \le \|P\|_\ell \|A\|_\ell \|Q\|_\ell$
$= \|I_p\|_\ell \|A\|_\ell \|I_q\|_\ell$
$= \|A\|_\ell$.
It follows $\|B\|_\ell \le \|A\|_\ell$.

9. I read the proofs