MATH 8610 (SPRING 2024) HOMEWORK 0 (LINEAR ALGEBRA WARMUP EXERCISE)

Assigned 01/15/18, due 01/21/18 (Sunday) 11:59pm.

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1. **[Q1]** Let matrix S be skew-Hermitian, i.e., $S^* = -S$.
(a) Show that the eigenvalues of S are pure imaginary (a similar well-known result is that the eigenvalues of a Hermitian matrix are all real)
(b) Show that $I - S$ is nonsingular.
(c) Show that the matrix $Q = (I-S)^{-1}(I+S)$ is unitary, i.e., $Q^{-1} = Q^*$, or equivalently, $Q^*Q = QQ^* = I$.

2. **[Q2]** Let $u, v \in \mathbb{C}^n$ and matrix $A = I + uv^* \in \mathbb{C}^{n\times n}$ is a rank-one update of the identity matrix. If A is nonsingular, show that $A^{-1} = I + \alpha uv^*$ for some scalar α, and find the expression of α. Under what conditions on u and v is A singular? If A is singular, what is null(A)?

3. **[Q3]** Prove the Cauchy-Schwarz inequality $|\langle x, y\rangle| \le \langle x, x\rangle^{\frac{1}{2}} \langle y, y\rangle^{\frac{1}{2}}$.
There are multiple ways to show, but please follow the hint: let $x = sy + z := \frac{\langle x,y\rangle}{\langle y,y\rangle} y + z$. Show that $\langle y, z\rangle = 0$. Pythagorean theorem gives $\langle x, x\rangle = \langle sy, sy\rangle + \langle z, z\rangle \ge \langle sy, sy\rangle$.

4. **[Q4]** Consider m-by-n matrices A, B with elements a_{ij} and b_{ij} ($1 \le i \le m$, $1 \le j \le n$). Show that $\langle A, B\rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij}\overline{b_{ij}}$ (the bar denotes the conjugate) is an inner product, and that the Frobenius norm is induced by this inner product.

5. **[Q5]** Let $\|\cdot\|$ denote any vector norm on $\mathbb{C}^n$ and the induced matrix norm on $\mathbb{C}^{n\times n}$. Let $\rho(A) = \max_{1\le i\le n} |\lambda_i(A)|$ be the largest absolute value of all eigenvalues of A (called the *spectral radius* of A). Show that $\rho(A) \le \|A\|$. Under what condition does $\lim_{k\to\infty} A^k = 0$? You may use the Jordan canonical form to gain an insight.

6. **[Q6]** Let $\|\cdot\|$ be any vector norm on $\mathbb{C}^n$, and let $\|\cdot\|_D$ be the dual vector norm defined as $\|x\|_D = \sup_{\|y\|=1} |y^*x|$.
(a) Show that $\|\cdot\|_D$ is a norm.
(b) Let $x, y \in \mathbb{C}^n$ with $\|x\| = \|y\| = 1$ be given. Show that there exists a rank-one matrix $B = yz^*$ (where vector z depends on x and y) such that $Bx = y$ and $\|B\| = 1$. Here $\|B\|$ is the induced matrix norm based on the vector norm $\|\cdot\|$. You may use the following result: given $x \in \mathbb{C}^n$, there exists a $z \in \mathbb{C}^n$, such that $|z^*x| = \|x\|\|z\|_D$.

7. **[Q7]** Show that $\|A\|_{\infty,1} = \max_{i,j} |a_{ij}|$.

8. **[Q8]** Let A be an m-by-n matrix, and B an p-by-q submatrix of A ($1 \le p \le m$ and $1 \le q \le n$). Find matrices $P \in \mathbb{R}^{p\times m}$ and $Q \in \mathbb{R}^{n\times q}$ such that $B = PAQ$. Show that $\|B\|_\ell \le \|A\|_\ell$ for any $1 \le \ell \le \infty$.

9. **[Q9]** Look up and self-study the following two results. Find a proof for each result you feel comfortable with and read it carefully.

   (a) **Sylvester's law of inertia.** Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix, i.e., $A = A^*$.

Let the number of negative, zero, and positive eigenvalues of A be n_-, n_0, and n_+, respectively. states that if matrix $B = SAS^*$ with a nonsingular matrix S (B is *congruent* to A), then B has the same triplet (n_-, n_0, n_+) as A.

(b) **Bauer-Fike theorem.** Let $A \in \mathbb{C}^{n\times n}$ be diagonalizable, such that $A = V\Lambda V^{-1}$ with diagonal $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ of eigenvalues, and $V = [v_1, \ldots, v_n]$ is the corresponding eigenvector matrix. Let ΔA be a matrix of perturbation, and $\{\mu_i\}$ are eigenvalues of $A+\Delta A$. Then for a given eigenvalue λ of A, there exists an eigenvalue μ of $A+\Delta A$, such that $|\lambda - \mu| \leq \kappa_p(V)\|\Delta A\|_p$, where $\kappa_p(\cdot)$ is the condition number based on the matrix norm $\|\cdot\|_p$ induced by the vector norm $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$.