MATH 8610 (SPRING 2024) MIDTERM EXAM

Assigned 03/12/24 at 10am, due 03/15/24 by 12:00pm (noon).

1. **[Q1]** (a) Let $f(x_1, x_2) = x_1^2 \ln x_2$ where $x_2 > 0$. Find the relative condition number of f. If $x_1 \approx 1$, for what values of x_2 is this evaluation ill-conditioned?

   (b) Let $A \in \mathbb{R}^{n\times n}$ and (λ, v) be an eigenpair such that $Av = \lambda v$. Let $(\hat{\lambda}, \hat{v})$ be a numerically computed approximation to (λ, v). Assume that $\|v\|_2 = \|\hat{v}\|_2 = 1$.
   Find the vector x, such that $(\hat{\lambda}, \hat{v})$ is an eigenpair of $A + \Delta A$, where $\Delta A = x\hat{v}^H$. Give the expression of $\|\Delta A\|_2$ (should not contain x), and explain why $\|A\hat{v} - \hat{\lambda}\hat{v}\|_2 \leq \mathcal{O}(\|A\|_2 \epsilon_{mach})$ means $(\hat{\lambda}, \hat{v})$ is computed by a backward stable algorithm.

   (c) Consider $I_n := \int_0^1 \frac{x^n}{x+10} dx$, which satisfies $I_n + 10I_{n-1} = \int_0^1 x^{n-1} dx = \frac{1}{n}$. Given $I_0 = \ln \frac{11}{10}$ (with $n = 1$), we can evaluate $I_1 = -10I_0 + \frac{1}{1}$, $I_2 = -10I_1 + \frac{1}{2}$, etc. Use this method to evaluate I_{20}, then call MATLAB's `integral` function with absolute and relative tolerance set to ϵ_{mach} to evaluate I_{20} numerically. Discuss your findings.

2. **[Q2]** (a) Given a 6×4 matrix A with all nonzero entries, illustrate the procedure of Golub-Kahan bidiagonalization, and explain how to compute all singular values of A.

   (b) Let $x \in \mathbb{R}^n$, and consider the vector $z = \begin{bmatrix} 0_{n-1} \\ \|x\|_2 \\ x \end{bmatrix} \in \mathbb{R}^{2n}$. Find the Householder reflector $H = I - 2vv^T$ that reduces z such that Hz is a multiple of e_1 (sufficient to find the expression of v). For $y = \begin{bmatrix} 0_n \\ x \end{bmatrix} \in \mathbb{R}^{2n}$, give the simplfied expression of Hy.

   (c) Let $U \in \mathbb{R}^{n\times m}$ with $m \ll n$, and $S \in \mathbb{R}^{s\times n}$ with s being a small multiple of m (e.g., $s = 4m$) and $s \ll n$. Suppose that S is well-conditioned (therefore has full rank s), and U is of full rank m. Assume that $SU \in \mathbb{R}^{s\times m}$ has orthonormal columns. Show that $P = I - U(SU)^T S$ is a projector and find its null space and the range. (hint: consider the reduced QR factorization of S^T)

3. **[Q3]** (a) For $A \in \mathbb{R}^{m\times n}$ $(m \geq n)$, let $A^\dagger = (A^T A)^{-1} A^T$. Show that $\|A^\dagger\|_2 = \frac{1}{\sigma_n(A)}$. (assume that A has full column rank)

   (b) Let $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, where $A_1 \in \mathbb{R}^{n\times n}$ is nonsingular. Show that $\sigma_n(A) \geq \sigma_n(A_1)$ (explore the relation between $\frac{\|Ax\|_2}{\|x\|_2}$ and $\frac{\|A_1x\|_2}{\|x\|_2}$), and $\|A^\dagger\|_2 \leq \|A_1^{-1}\|_2$.

   (c) Define the numerical rank of $A \in \mathbb{R}^{m\times n}$ $(m \geq n)$ as $\text{rank}(A, \epsilon) = \max\{k : \sigma_k \geq \epsilon\}$ $(\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n)$. If A has numerical rank $k < n$ for a given ϵ, find a numerically full rank B satisfying $\inf_{\text{rank}(B,\epsilon)=n} \|A - B\|_F$ and show that $\|B - A\|_F \leq \sqrt{n-k}\,\epsilon$.