MATH 8610 (SPRING 2024) HOMEWORK 2

Assigned 01/26/24, due 02/04/24 (Sunday) 11:59pm.

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1. **[Q1] (10 pts)** Let $a_0, a_1, \ldots, a_n$ be $n+1$ equispaced points on $[-1, 1]$, where $a_0 = -1$ and $a_n = 1$. Assemble these $n+1$ values into a column vector u, and use MATLAB's `vander` to generate Vandermonde matrices A from vector u for $n = 9, 19, 29, 39$.

   Let $x = [1\,1\,\ldots\,1]^T$ and $b = Ax$. Pretend that we do not know x and use numerical algorithms to solve this linear system for x. Let $\hat{x}$ be the computed solution. Compute the relative forward errors $\frac{\|\hat{x}-x\|}{\|x\|}$ and the smallest relative backward errors $\frac{\|b-A\hat{x}\|_2}{\|A\|_2\|\hat{x}\|_2} = \min\left\{\frac{\|\Delta A\|_2}{\|A\|_2} : (A+\Delta A)\hat{x} = b\right\}$ for (a) GEPP (MATLAB's backslash), (b) QR factorization of A, (c) Cramer's rule, (d) A^{-1} multiplied with b, and (e) GE without pivoting. Comment on the forward/backward stability of these methods.

2. **[Q2] (10 pts)** Consider the eigenvalue problem $Av = \lambda v$. Let $(\widehat{\lambda}, \widehat{v})$ be a computed eigenpair, which is assumed to be the exact eigenpair of a perturbed matrix $A + \Delta A$. Show that the minimum 2-norm of all such ΔA is $\frac{\|A\widehat{v}-\widehat{\lambda}\widehat{v}\|_2}{\|\widehat{v}\|_2}$, and find a particular ΔA whose 2-norm is the minimum. (Note that this result can help us experimentally determine if an eigenvalue algorithm is backward stable)

3. **[Q3] (10 pts)** Give a proof that the worst-case growth factor $\rho_n = 2^{n-1}$ for GEPP. Compared to $\rho_n \le C n^{\frac{1}{2}+\frac{1}{4}\ln n}$ with complete pivoting and $\rho_n \le 1.5 n^{\frac{3}{4}\ln n}$ with *rook pivoting*, this is much larger. However, we construct matrices with random elements, each are independent samples from the normal distribution of mean 0 and standard deviation $\frac{1}{\sqrt{n}}$ (run `A = randn(n,n)/sqrt(n);`). Let $n = 32, 64, \ldots, 512$, and for each n, repeat the experiment 1000 times with partial pivoting. Find the percentage of experiments when $\rho_n > \sqrt{n}$. Make brief comments on the chance of having a large ρ_n.

4. **[Q4] (20 pts)** Though pivoting is needed for factorizing general matrices, it is not needed for symmetric positive definite and diagonally dominant matrices.

   (a) For a symmetric positive definite A, with the one-step Cholesky factorization $A = \begin{bmatrix} a_{11} & w^T \\ w & K \end{bmatrix} = \begin{bmatrix} \sqrt{a_{11}} & 0 \\ \frac{w}{\sqrt{a_{11}}} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & K - \frac{ww^T}{a_{11}} \end{bmatrix} \begin{bmatrix} \sqrt{a_{11}} & \frac{w^T}{\sqrt{a_{11}}} \\ 0 & I \end{bmatrix} = R_1^T A_1 R_1$, show that the submatrix $K - \frac{ww^T}{a_{11}}$ is symmetric positive definite. Therefore, the factorization $A = R^T R$ can be done without break-down. Show that $\|R\|_2 = \|A\|_2^{\frac{1}{2}}$, which means the elements in R are uniformly bounded by those of $\|A\|$. Explain why this observation leads to the backward stability of Cholesky factorization.

   (b) Suppose that $A = \begin{bmatrix} \alpha & w^T \\ v & C \end{bmatrix}$ is column diagonally dominant, with one-step LU factorization $A = \begin{bmatrix} 1 & 0 \\ \frac{v}{\alpha} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & C - \frac{1}{\alpha}vw^T \end{bmatrix} \begin{bmatrix} \alpha & w^T \\ 0 & I \end{bmatrix}$. Show that the submatirx $C - \frac{1}{\alpha}vw^T$ is also column diagonally dominant, and no pivoting is needed.