Jacob Manning HW1

large and big is the vocab we used in class

1a. $\kappa_{abs} \approx \|J_F(x)\|$ $f(x)=e^{-2x}$ $\kappa_{abs} = \|-2e^{-2x}\|$
$\kappa_{rel} = \kappa_{abs} \frac{\|x\|}{\|f(x)\|}$ $= 2|e^{-2x}|$

$\kappa_{rel} = \frac{2|e^{-2x}||x|}{|e^{-2x}|} = 2|x|$

For $f(x)$, κ_{abs} is big for $x \in (-\infty, 0)$.
κ_{rel} is big when $|x|$ is large.

$g(x) = (\ln x)^3$ $\kappa_{abs} = 3(\ln x)^2 |\frac{1}{x}|$
$\kappa_{rel} = \frac{3(\ln x)^2 |\frac{1}{x}||x|}{(\ln x)^3} = \frac{3}{\ln x}$ $\ln x = 0$ when $x=1$

For $g(x)$ κ_{abs} is large for $x \in (0,1)$.
κ_{rel} is large when $x \in (0,1) \cup (1,2)$.

b. $x_1, x_2 \in \mathbb{R}^+$ $f(x_1,x_2) = x_1^{x_2}$
$\kappa_{abs} = \|\nabla x_1^{x_2}\| = \sqrt{(x_2 x_1^{x_2-1})^2 + (x_1^{x_2} \ln x_1)^2}$
$\kappa_{rel} = \frac{\kappa_{abs}\|x\|}{\|f(x)\|} = \frac{\sqrt{(x_2 x_1^{x_2-1})^2 + (x_1^{x_2} \ln x_1)^2}\sqrt{x_1^2 + x_2^2}}{|x_1^{x_2}|}$

κ_{rel} is large when x_1 and x_2 are large. It also increases for $x_1 \in (0,1)$ and x_2 big enough as the denominator gets small.

2a. $x_k = \frac{6^{k+1} + 5^{k+1}}{6^k + 5^k}$. Firstly we see $x_0 = \frac{11}{2} < \frac{61}{11} = x_1$.

Next consider $6^{k+1} + 5^{k+1} \geq 6^k + 5^k$ $\forall k \in \mathbb{N}$
$\therefore \{x_k\}_{k=1}^{\infty}$ is monotonically increasing by the definition.

2a cont. Let $\varepsilon > 0$. Consider

$$\lim_{k\to\infty}\left\|\frac{6^{k+1}+5^{k+1}}{6^k+5^k}-6\right\| = \lim_{k\to\infty}\left\|\frac{6^{k+1}+5^{k+1}-6^{k+1}-6\cdot 5^k}{6^k+5^k}\right\|$$

$$= \lim_{k\to\infty}\left\|\frac{5^k(5-6)}{6^k+5^k}\right\| = \lim_{k\to\infty}\left\|\frac{5^k}{6^k+5^k}\right\|$$

Since $6^k > 0 \ \forall k \in \mathbb{N}$, $6^k + 5^k > 5^k \ \forall k \in \mathbb{N}$.

It follows $\left\|\frac{5^k}{6^k+5^k}\right\| \to 0 < \varepsilon$ as $k \to \infty$.

b. Code attached

$\frac{61}{11}$ cannot be represented in binary exactly so the numeric solution diverges to 100 as $a \neq 0$ in the numeric recursion.

The condition number is large if x_0 and x_1 cannot be represented exactly in binary because then 100^{k+1} becomes the dominating term as a will not be exactly equal to 0.

3. $P(x) = \prod_{i=1}^{24}(x-i)$ Let a_i be the coefficient of the x^i term. Let x_j be the j^{th} root.

Consider $a_i \to a_i + \Delta a_i$

$$P(a_0, a_1, \dots, a_i+\Delta a_i, \dots, a_{24}, x_j) = 0$$
$$- P(a_0, a_1, \dots, a_{23}, a_{24}, x_j) = 0$$

$$\approx \left.\frac{\partial P}{\partial a_i}\right|_{(a_0,\dots,a_{24},x_j)} \Delta a_i + \left.\frac{\partial P}{\partial x_j}\right|_{(a_0,\dots,a_{24},x_j)} \Delta x_j = 0$$

Thus, $|\Delta x_j| = \left|\frac{\partial P}{\partial a_i}\Delta a_i \Big/ \frac{\partial P}{\partial x_j}\right| \Rightarrow K_{abs} = \frac{x_j^i}{P'(x_j)}$ for the a_i^{th} x_j.

It follows $K_{rel} = \frac{x_j^i}{P'(x_j)}\left|\frac{a_i}{x_j}\right| = \frac{x_j^{i-1}a_i}{P'(x_j)}$.

$$\left.\frac{\partial P(x_j)}{\partial x_j}\right|_{x=x_j} = \left.\frac{\partial \prod_{i=1}^{24}(x-i)}{\partial x_j}\right|_{x=x_j} = (j-1)!\,(24-j)!$$ Since it is the product of exponents.

3cont. See code for various K_{rel}.
The most sensitive was a_{17} with x_{17}. $K_{rel} \approx 9.1024 \cdot 10^{15}$. I explored other a_i with x_j combinations and this was still the largest. The roots function finds imaginary roots about half way between the real solutions. $a_0, \ldots, a_9$ are pretty close with $a_0, \ldots, a_5$ being displayed exactly.

```matlab
%Problem 2
% Initial conditions
x(1) = 11/2;
x(2) = 61/11;

% Number of iterations
N = 100;

% Recurrence relation
for k = 2:N
    x(k+1) = 111 - (1130 - 3000/x(k-1))/x(k);
end

% Exact solution
a = 0; b = 1; c = 1;
for k = 1:N
    exact(k) = (100^(k+1)*a + 6^(k+1)*b + 5^(k+1)*c) / (100^k*a + 6^k*b +
5^k*c);
end

% Plotting
figure
plot(1:N+1, x, 'r', 1:N, exact, 'b')
legend('Recurrence', 'Exact solution')
xlabel('k')
ylabel('x_k')
title('Comparison of Recurrence and Exact Solution')

%Problem 3
r = roots(wilkinson24_monomial_coeffs)

ls = [];
for j= 1 : 5
    for i = 1:5
        a = wilkinson24_monomial_coeffs(6+i);
        k = (19-j)^(18-i)*a/(factorial(18-j)*factorial(5+j));
        ls = [ls,abs(k)];
    end
end

max(ls)
```

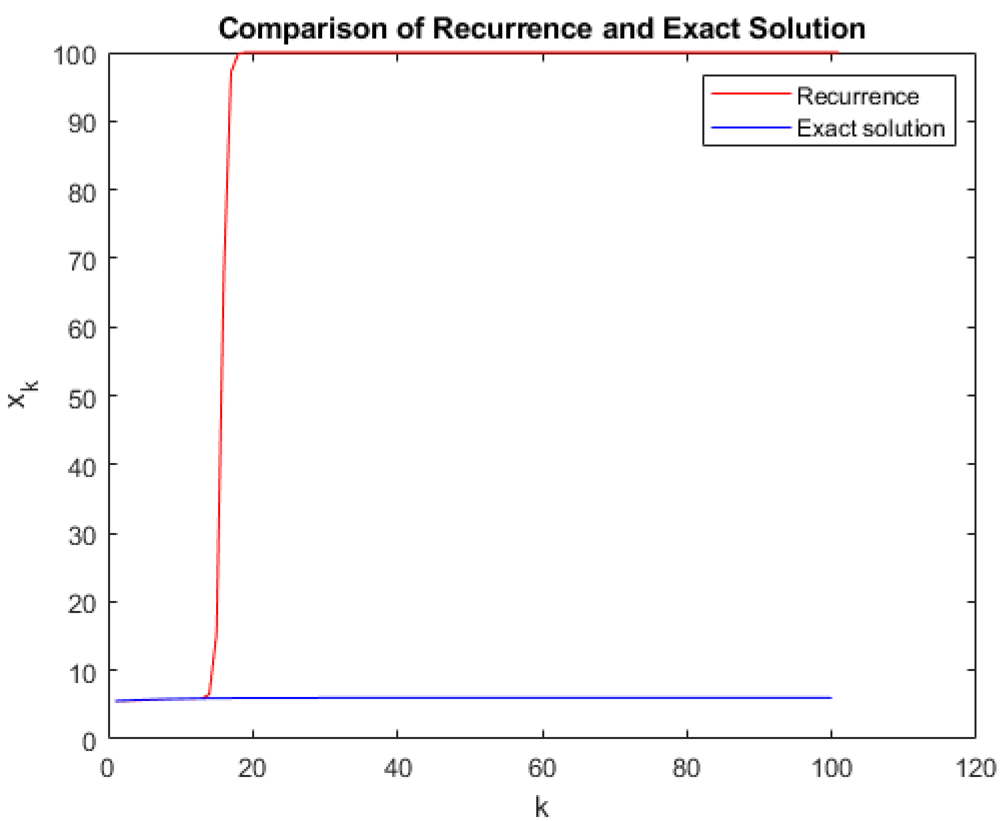
```
r =

  24.1395 + 0.0000i
  23.0556 + 0.9091i
  23.0556 - 0.9091i
  21.0637 + 1.9217i
  21.0637 - 1.9217i
  18.6551 + 2.3831i
  18.6551 - 2.3831i
  16.2606 + 2.2741i
```

```
  16.2606 - 2.2741i
  14.0989 + 1.8491i
  14.0989 - 1.8491i
  12.0993 + 1.2880i
  12.0993 - 1.2880i
  10.2453 + 0.5708i
  10.2453 - 0.5708i
   8.8822 + 0.0000i
   8.0239 + 0.0000i
   6.9974 + 0.0000i
   6.0002 + 0.0000i
   5.0000 + 0.0000i
   4.0000 + 0.0000i
   3.0000 + 0.0000i
   2.0000 + 0.0000i
   1.0000 + 0.0000i


ans =

   9.1024e+15
```



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