MATH 8610 (SPRING 2024) HOMEWORK 5

Assigned 02/27/24, due 03/08/24 at 11:59pm.

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1. **[Q1] (10 points)** Trefethen's book, Problem 5.3 (page 37).

2. **[Q2] (10 points)** (a) If $A \in \mathbb{R}^{m\times n}$ and $E \in \mathbb{R}^{m\times n}$, show that

$$\sigma_{\max}(A+E) \le \sigma_{\max}(A) + \|E\|_2 \quad \text{and} \quad \sigma_{\max}(A+E) \ge \sigma_{\max}(A) - \|E\|_2.$$

Comment on the (absolute) condition number of $\|A\|_2$ as a function of A.

(b) If $A \in \mathbb{R}^{m\times n}$, $m > n$ and $z \in \mathbb{R}^m$, show that

$$\sigma_{\max}([A\ \ z]) \ge \sigma_{\max}(A) \quad \text{and} \quad \sigma_{\min}([A\ \ z]) \le \sigma_{\min}(A).$$

3. **[Q3] (10 points)** (a) Show that if $A \in \mathbb{R}^{m\times n}$, then $\|A\|_F \le \sqrt{\operatorname{rank}(A)}\|A\|_2$.

(b) Show that if $A \in \mathbb{R}^{m\times n}$ has rank n, then $\|A(A^TA)^{-1}A^T\|_2 = 1$.

4. **[Q4] (a) (10 points)** Given $A \in \mathbb{R}^{n\times n}$, let $A = U\Sigma V^T$ be an SVD of A, where $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_n > 0$. Let $B = [U\operatorname{diag}(1,\ldots,1,-1)]\,\Sigma V^T$ such that $\det(B) = -\det(A)$ and $\|A-B\|_F = 2\sigma_n$. Show that for *any singular values* $\sigma_1, \sigma_2, \ldots \sigma_{n-1}$ $(\ge \sigma_n)$, there exists $C \in \mathbb{R}^{n\times n}$ such that $\det(C) = \det(B) = -\det(A)$, and $\|A - C\|_F < \|A - B\|_F = 2\sigma_n$.

(Hint: to construct C, modify σ_n and σ_{n-1} of A only. Change the sign of one and keep the other, change the values of the two, and make sure that the product of the two modified scalars does not change in absolute value.)

(b) (5 points) Trefethen's book, Problem 6.1.

5. **[Q5] (10 points)** (a) Implement the Golub-Kahan (GK) bidiagonalization of a matrix. Test it on $F \in \mathbb{R}^{10\times 10}$ obtained as follows

```
rgn('default');
F = randn(10,10);
```

Make sure that your bidiagonal matrix has the same singular values as F.

(b) Generate a matrix $A \in \mathbb{R}^{(1024^2+1)\times 32}$ as follows

```
col = linspace(-1,1,1024*1024+1)';
A = col.^(0:31);
```

Apply Householder QR to A and get $R \in \mathbb{R}^{32\times 32}$, then apply GK to R and get bidiagonal $B \in \mathbb{R}^{32\times 32}$ (no need to retrieve Q for this problem). Compute the 5 largest and 5 smallest singular values of A from the eigenvalues of $\begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix}$. Compare these singular values with those computed by taking the square root of the 5 largest and 5 smallest eigenvalues of A^TA. What conclusion do you draw? Is it a good idea to compute the eigenvalues of $\begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}$ directly, and why?