1 Introduction

In this section a few background topics and terminology are introduced to the reader. This information is largely similar to topics that we have covered in the slides presented in class. The main question of the paper is posed as, "Consider P a finite subset of $[0,1]^n$ such that $f|_P$ is known. How can we use knowledge of $f|_P$ to reconstruct the network?" where f is the time discretized, space continuous, Boolean functions to be studied.
As the amount of time allotted for the presentation was limited and these proofs are multiple paragraphs long, the proofs in this section were omitted, though a few examples from the paper are included.

2 Preliminaries

Here is a section that is largely similar to what was covered in class for discrete space discrete time models. However, since the problem is fundamentally different as it is a continuous space model, we need to make sure that the same things we did in class still apply to the new problems that will be looked at.
Let $h : [0,1]^n \to [0,1]$. define $g : [0,1] \to [0,1]$ $x_i \mapsto h(c_1, ..., c_{i-1}, x_i, c_{i+1}, ..., c_n)$ where $c = \{c_j\}_{j \neq i} \subseteq [0,1]^{n-1}$ are different inputs for the other f_j's.

Definition 1 (2.1)

- h *is called independent of* x_i *if* g *is constant for all* c
- h *is called monotone increasing with respect to* x_i *if for* $a \leqslant b$ $g(a) \leqslant g(b)$ *for all* c
- h *is called monotone decreasing with respect to* x_i *if for* $a \leqslant b$ $g(a) \geqslant g(b)$ *for all* c
- h *is called monotone if any of the previous apply*

Remark
If x_i is a monotone increasing variable it is called an activator.
If x_i is a monotone decreasing variable it is called a repressor or inhibitor.

Definition 2 (2.3) *Given a monotone function* $h : [0,1]^n \to [0,1]$*, the local wiring diagram of* h *is defined as*

$$w(h) := \{(x_k, s) | h \textit{ is monotone, but not independent of } x_k\}_{k=1}^{n}$$

Where s *is the* ± 1 *if* h *is increasing or decreasing with respect to* x_i*.*

Ex
Consider $h : [0,1]^5 \to [0,1]$ $x \mapsto \frac{x_1}{1+x_1} \frac{x_3^2}{1+x_3^2} \frac{1}{1+x_5}$. The local wiring diagram would be $w(h) = \{(x_,1), (x_3, 1), (x_5, -1)\}$

Definition 3 (2.6) *Given* D *as the data and a Boolean function* h, h *is consistent if* $w(h)$ *agrees with* D. *If* h *is consistent and* $w(h)$ *does not include any "fluff" it is called a minimal local wiring diagram.*

Definition 4 (2.9) *Let* $p, p' \in P$ *such that* $h(p) < h(p')$. *We say that the local wiring diagram* w *is consistent with the pair* $(p, p') \in P^2$ *if for some* i, $p_i < p'_i$ *and* $(x_i, 1) \in w$, *or* $p_i > p'_i$ *and* $(x_i, -1) \in w$. *Equivalently, for some* i, $(x_i, \mathit{sign}(p'_i - p_i)) \in w$. *We denote* $W_{(p,p')}$ *the set of all local wiring diagrams that are consistent with the pair* (p, p')

Theorem 1 (2.11) *Let* $D = (P, h|_p)$ *be the observed data. Then the set of local wiring diagrams consistent with the data* D *is exactly the set of wiring diagrams which are consistent with each pair of points* $p, p' \in P$ *satisfying* $h(p) < h(p')$, *that is*

$$W_D = \bigcap_{\substack{(p,p') \in P^2 \\ h(p) < h(p')}} W_{(p,p')}$$

3 Algebraic Approach for Network Reconstruction

Now with the background established, the authors start developing the tools needed for their proofs.

Definition 5 (3.1-2) *Let* w *be a local wiring diagram and* $(p, p') \in P^2$ *such that* $h(p) < h(p')$. *define the ideals*

$$\mathfrak{I}_w := \langle \{x_i - s_i | (x_i, s_i) \in w\} \rangle$$

$$\mathfrak{I}_{(p,p')} := \left\langle \prod_{p_i \neq p'_i} (x_i - \mathrm{sign}(p'_i - p_i)) \right\rangle$$

Definition 6 (3.3) *Let* $D = (P, h|_P)$ *be observed data. define the ideal*

$$\mathfrak{I}_D := \sum_{\substack{(p,p') \in P^2 \\ h(p) < h(p')}} \mathfrak{I}_{(p,p')}$$

Proposition 1 (3.5) *Let* $D = (P, h|_P)$ *be observed data and consider* w *to be a local wiring diagram. Then,* w *is a minimal local wiring diagram of* W_D *if and only if* $\mathfrak{I}_w$ *is a minimal prime of* $\mathfrak{I}_D$. *Furthermore, every minimal prime of* $\mathfrak{I}_D$ *is of this form.*

Theorem 2 (3.8) *Consider a monotone function* $h : [0, 1]^n \to [0, 1]$ *and suppose we obtain data,* D *by sampling points in* $[0, 1]^n$ *using a uniform distribution. Then, with probability 1,* W_D *will eventually have* $w(h)$ *as its unique minimal wiring diagram. Equivalently, with probability 1,* $\mathfrak{I}_D$ *will eventually be equal to* $\mathfrak{I}_{w(h)}$.

<u>Ex</u>

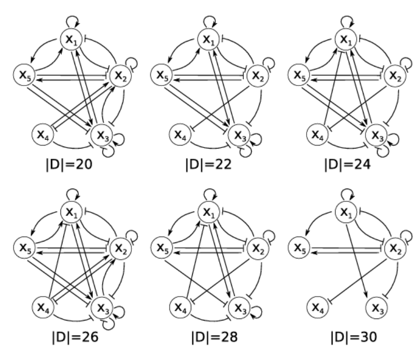
Let $f : [0,1]^5 \to [0,1]^5$ be given by the following equations

$$
\begin{aligned}
f_1 &= \frac{x_1}{1+x_1}\frac{1}{1+x_2^2} \\
f_2 &= \frac{1}{1+x_1x_2}\frac{1}{1+x_5} \\
f_3 &= \frac{x_1^2}{1+x_1^2}\frac{1}{1+x_2} \\
f_4 &= \frac{1}{1+x_2^2} \\
f_5 &= \frac{x_1}{1+x_1}\frac{x_2}{1+x_2}
\end{aligned}
$$

By randomly sampling points in $[0,1]^5$ and applying Proposition 3.5 the minimal wiring diagrams for each coordinate function were constructed.

Table 3.1
Example of a data set.

P	x					$f(x)$				
1	.75	.30	.17	.90	.70	.39	.48	.28	.77	.10
2	.69	.98	.71	.20	.31	.21	.46	.16	.51	.20
3	.99	.50	.31	.98	.97	.40	.34	.33	.67	.17
4	.96	.75	.04	.94	.25	.31	.47	.27	.57	.21
5	.30	.16	.26	.18	.66	.23	.57	.07	.86	.03
6	.18	.53	.15	.22	.28	.12	.71	.02	.65	.05
7	.58	.05	.62	.27	.88	.37	.52	.24	.95	.02
8	.25	.65	.06	.09	.75	.14	.49	.04	.61	.08
9	.43	.10	.73	.90	.61	.30	.60	.14	.91	.03
10	.08	.79	.79	.89	.52	.05	.62	.00	.56	.03
11	.17	.99	.04	.28	.73	.07	.49	.01	.50	.07
12	.70	.54	.52	.63	.62	.32	.45	.21	.65	.14
13	.57	.20	.75	.22	.05	.35	.85	.20	.83	.06
14	.73	.51	.25	.48	.93	.33	.38	.23	.66	.14
15	.20	.10	.77	.05	.61	.17	.61	.03	.91	.02
16	.65	.79	.40	.85	.48	.24	.45	.17	.56	.17
17	.34	.99	.50	.58	.64	.13	.46	.05	.50	.13
18	.92	.64	.65	.71	.39	.34	.45	.28	.61	.19
19	.53	.53	.43	.54	.79	.27	.44	.14	.65	.12
20	.47	.33	.78	.58	.07	.29	.81	.14	.75	.08
21	.14	.42	.61	.65	.96	.10	.48	.01	.70	.04
22	.49	.32	.66	.48	.74	.30	.50	.15	.76	.08
23	.78	.83	.18	.50	.66	.26	.37	.21	.55	.20
24	.85	.55	1.0	.97	.93	.35	.35	.27	.65	.16
25	.36	.84	.78	.43	.66	.16	.46	.06	.54	.12
26	.09	.23	.24	.27	.39	.08	.70	.01	.81	.02
27	.95	.74	.70	.12	.18	.31	.50	.27	.57	.21
28	.67	.84	.50	.06	.47	.24	.44	.17	.54	.18
29	.28	.76	.38	.14	.03	.14	.80	.04	.57	.09
30	.57	.97	.87	.28	.64	.19	.39	.12	.51	.18



4 Network Reconstruction with Noise

In this section, there is noise introduced into the input and output of our calculations. I skipped over the section about just error from the input/output and went straight the combined definitions.

Let h be a monotone function with $\tilde{h} = h + \Delta h$. Where Δh is introduced due to error.

Definition 7 (4.4-5) *Suppose we have noisy data* D *such that* $(\tilde{p_i}, \tilde{p_i}') \in P^2$

and the noise bound is given by $\epsilon = (\epsilon_{in}, \epsilon_{out})$. *definee the ideals*

$$\mathfrak{I}_{\epsilon,(\tilde{p},\tilde{p}')} := \left\langle \prod_{\substack{|\tilde{p_i}'-\tilde{p_i}|>2\epsilon_{in} \\ \tilde{p_i}\neq\tilde{p}'_i}} (x_i - \mathrm{sign}(\tilde{p_i}' - \tilde{p_i})) \prod_{|\tilde{p_i}'-\tilde{p_i}|<2\epsilon_{in}} (x_i^2 - 1) \right\rangle$$

$$\mathfrak{I}_{\epsilon,D} := \sum_{\substack{(p,p')\in P^2 \\ \tilde{h}(p')-\tilde{h}(p)>2\epsilon_{out}}} \mathfrak{I}_{\epsilon,(\tilde{p},\tilde{p}')}$$

Theorem 3 (4.7) *Consider a monotone function* $h : [0,1]^n \to [0,1]$ *and suppose we obtain noisy data,* D *by sampling points in* $[0,1]^n$ *using a uniform distribution. If* ϵ *is small enough,* W_D *will eventually have* $w(h)$ *as its unique minimal wiring diagram with probability 1. Equivalently, with probability 1,* $\mathfrak{I}_{\epsilon,D}$ *will eventually be equal to* $\mathfrak{I}_{w(h)}$.

Theorem 4 (4.8) *Suppose we obtain noisy data* D *by sampling points in* $[0,1]^n$ *using a uniform distribution. For small enough noise, with probability 1 a wiring diagram* w *is consistent with* D *if and only if* $\mathfrak{I}_{\epsilon,D} \subseteq \mathfrak{I}_{w(h)}$

5 Selection of Wiring Diagrams

As we have seen in the example at the end of section 3, there were many different wiring diagrams. The authors thus decided to create a score function that balances time and complexity of the minimal wiring diagram. They assumed that the in-degree follows a power law distribution which implies that $P(|W_{true}| = k) = \frac{c}{k^\gamma}$ for all $k > k_0$ and zero for $k < k_0$ where W_{true} is the true local wiring diagram, γ is a parameter, and $c = \left(\sum_{k=k_0}^{n} k^{-\gamma}\right)^{-1}$ is a normalization constant. In order to look at W_D, from the previous work, it is enough to find its minimal local wiring diagrams.

Proposition 2 (5.1)
Let $\{W_i\}_{i=1}^{l}$ *be the elements in* W_D.
Define

$$\begin{aligned} N_k &= |\{W_j : |W_j| = k\}| \\ N_{ki}^{+} &= |\{W_j : |W_j| = k \mathrm{and} (x_i, 1) \in W_j\}| \\ N_{ki}^{-} &= |\{W_j : |W_j| = k \mathrm{and} (x_i, -1) \in W_j\}| \\ N_{ki} &= N_{ki}^{+} + N_{ki}^{-} \end{aligned}$$

Note that $n \geqslant k \geqslant k_0$ *iff* $N_k \neq 0$
Then up to a rescaling factor, $P((x_i, \pm 1) \in W_{true}) = \sum_{k=1}^{n} \frac{N_{ki}^{+}}{k^\gamma N_k}$

Definition 8 *The score of* x_i *is*

$$S(i) = s^{+}(i) + s^{-}(i) = c \sum_{k=1}^{n} \frac{N_{ki}}{k^\gamma N_k}$$

Similarly, the score of the local wiring diagram is

$$S(W) = \prod_{(x_i,1)\in W} S^+(i) \prod_{(x_i,-1)\in W} S^-(i) \prod_{\substack{(x_i,1)\notin W \\ \text{and} \\ (x_i,-1)\notin W}} (1 - S^+(i) - S^-(i))$$

6 Conclusions

This paper covered algorithms for finding signed minimal wiring diagrams with space continuous data. One pro to this method is that you do not need to worry about fitting parameters which can become fairly arbitrary. They are looking at continuing with these algebraic approaches to reverse engineering and maximizing information in the biological math models.

7 Citation

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