Algebraic Network Reconstruction of Discrete Dynamical Systems

Jacob Manning

Clemson University

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Section 1 Introduction

Consider P a finite subset of $[0,1]^n$ such that $f|_P$ is known. How can we use knowledge of $f|_P$ to reconstruct the network? Where f is the time discretized, space continuous, Boolean functions to be studied.

Section 2 Preliminaries

Let $h : [0,1]^n \to [0,1]$. define $g : [0,1] \to [0,1]$
$x_i \mapsto h(c_1, ..., c_{i-1}, x_i, c_{i+1}, ..., c_n)$ where $c = \{c_j\}_{j \neq i} \subseteq [0,1]^{n-1}$ are different inputs for the other f_j's.

Definition (2.1)

- *h is called independent of x_i if g is constant for all c*
- *h is called monotone increasing with respect to x_i if for $a \leqslant b$ $g(a) \leqslant g(b)$ for all c*
- *h is called monotone decreasing with respect to x_i if for $a \leqslant b$ $g(a) \geqslant g(b)$ for all c*
- *h is called monotone if any of the previous apply*

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- *h is called monotone if any of the previous apply*

Remark

If x_i is a monotone increasing variable it is called an activator.
If x_i is a monotone decreasing variable it is called a repressor or inhibitor.

Definition (2.3)

Given a monotone function $h : [0,1]^n \to [0,1]$*, the local wiring diagram of h is defined as*

$$w(h) := \{(x_k, s) | h \textit{ is monotone, but not independent of } x_k\}_{k=1}^{n}$$

Where s is the ± 1 *if h is increasing or decreasing with respect to* x_i.

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Ex

Consider $h : [0,1]^5 \to [0,1]$ $x \mapsto \frac{x_1}{1+x_1} \frac{x_3^2}{1+x_3^2} \frac{1}{1+x_5}$. The local wiring diagram would be $w(h) = \{(x_, 1), (x_3, 1), (x_5, -1)\}$

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Definition (2.6)

Given D as the data and a Boolean function h, h is consistent if $w(h)$ *agrees with D. If h is consistent and* $w(h)$ *does not include any "fluff" it is called a minimal local wiring diagram.*

Preliminaries

Definition (2.9)

Let $p, p' \in P$ such that $h(p) < h(p')$. We say that the local wiring diagram w is consistent with the pair $(p, p') \in P^2$ if for some i, $p_i < p'_i$ and $(x_i, 1) \in w$, or $p_i > p'_i$ and $(x_i, -1) \in w$. Equivalently, for some i, $(x_i, sign(p'_i - p_i)) \in w$. We denote $W_{(p,p')}$ the set of all local wiring diagrams that are consistent with the pair (p, p')

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Theorem (2.11)

Let $D = (P, h|_p)$ be the observed data. Then the set of local wiring diagrams consistent with the data D is exactly the set of wiring diagrams which are consistent with each pair of points $p, p' \in P$ satisfying $h(p) < h(p')$, that is

$$W_D = \bigcap_{\substack{(p,p') \in P^2 \\ h(p) < h(p')}} W_{(p,p')}$$

Section 3 Algebraic Approach for Network Reconstruction

Definition (3.1-2)

Let w be a local wiring diagram and $(p, p') \in P^2$ such that $h(p) < h(p')$. define the ideals

$$\mathfrak{I}_w := \langle \{x_i - s_i | (x_i, s_i) \in w\} \rangle$$

$$\mathfrak{I}_{(p,p')} := \left\langle \prod_{p_i \neq p_i'} (x_i - sign(p_i' - p_i)) \right\rangle$$

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Let $D = (P, h|_P)$ be observed data. define the ideal

$$\mathfrak{I}_D := \sum_{\substack{(p,p') \in P^2 \\ h(p) < h(p')}} \mathfrak{I}_{(p,p')}$$

Algebraic Approach for Network Reconstruction

Proposition (3.5)

Let $D = (P, h|_P)$ be observed data and consider w to be a local wiring diagram. Then, w is a minimal local wiring diagram of W_D if and only if $\mathfrak{I}_w$ is a minimal prime of $\mathfrak{I}_D$. Furthermore, every minimal prime of $\mathfrak{I}_D$ is of this form.

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Theorem (3.8)

Consider a monotone function $h : [0,1]^n \to [0,1]$ and suppose we obtain data, D by sampling points in $[0,1]^n$ using a uniform distribution. Then, with probability 1, W_D will eventually have $w(h)$ as its unique minimal wiring diagram. Equivalently, with probability 1, $\mathfrak{I}_D$ will eventually be equal to $\mathfrak{I}_{w(h)}$.

Ex

Let $f : [0,1]^5 \to [0,1]^5$ be given by the following equations

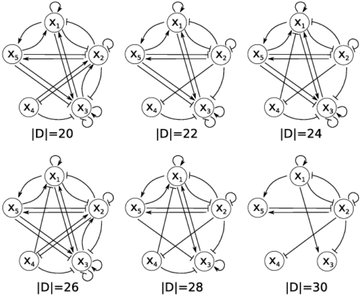
$$f_1 = \frac{x_1}{1+x_1}\frac{1}{1+x_2^2}$$
$$f_2 = \frac{1}{1+x_1x_2}\frac{1}{1+x_5}$$
$$f_3 = \frac{x_1^2}{1+x_1^2}\frac{1}{1+x_2}$$
$$f_4 = \frac{1}{1+x_2^2}$$
$$f_5 = \frac{x_1}{1+x_1}\frac{x_2}{1+x_2}$$

By randomly sampling points in $[0,1]^5$ and applying Proposition 3.5 the minimal wiring diagrams for each coordinate function were constructed.

Algebraic Approach for Network Reconstruction

Table 3.1
Example of a data set.

P	x					$f(x)$				
1	.75	.30	.17	.90	.70	.39	.48	.28	.77	.10
2	.69	.98	.71	.20	.31	.21	.46	.16	.51	.20
3	.99	.50	.31	.98	.97	.40	.34	.33	.67	.17
4	.96	.75	.04	.94	.25	.31	.47	.27	.57	.21
5	.30	.16	.26	.18	.66	.23	.57	.07	.86	.03
6	.18	.53	.15	.22	.28	.12	.71	.02	.65	.05
7	.58	.05	.62	.27	.88	.37	.52	.24	.95	.02
8	.25	.65	.06	.09	.75	.14	.49	.04	.61	.08
9	.43	.10	.73	.90	.61	.30	.60	.14	.91	.03
10	.08	.79	.79	.89	.52	.05	.62	.00	.56	.03
11	.17	.99	.04	.28	.73	.07	.49	.01	.50	.07
12	.70	.54	.52	.63	.62	.32	.45	.21	.65	.14
13	.57	.20	.75	.22	.05	.35	.85	.20	.83	.06
14	.73	.51	.25	.48	.93	.33	.38	.23	.66	.14
15	.20	.10	.77	.05	.61	.17	.61	.03	.91	.02
16	.65	.79	.40	.85	.48	.24	.45	.17	.56	.17
17	.34	.99	.50	.58	.64	.13	.46	.05	.50	.13
18	.92	.64	.65	.71	.39	.34	.45	.28	.61	.19
19	.53	.53	.43	.54	.79	.27	.44	.14	.65	.12
20	.47	.33	.78	.58	.07	.29	.81	.14	.75	.08
21	.14	.42	.61	.65	.96	.10	.48	.01	.70	.04
22	.49	.32	.66	.48	.74	.30	.50	.15	.76	.08
23	.78	.83	.18	.50	.66	.26	.37	.21	.55	.20
24	.85	.55	1.0	.97	.93	.35	.35	.27	.65	.16
25	.36	.84	.78	.43	.66	.16	.46	.06	.54	.12
26	.09	.23	.24	.27	.39	.08	.70	.01	.81	.02
27	.95	.74	.70	.12	.18	.31	.50	.27	.57	.21
28	.67	.84	.50	.06	.47	.24	.44	.17	.54	.18
29	.28	.76	.38	.14	.03	.14	.80	.04	.57	.09
30	.57	.97	.87	.28	.64	.19	.39	.12	.51	.18



Section 4 Network Reconstruction with Noise

There is noise introduced into the input and output of our calculations.

Let h be a monotone function with $\tilde{h} = h + \Delta h$. Where Δh is introduced due to error.

Definition (4.4-5)

Suppose we have noisy data D such that $(\tilde{p}_i, \tilde{p}_i{}') \in P^2$ and the noise bound is given by $\epsilon = (\epsilon_{in}, \epsilon_{out})$. define the ideals

$$\mathfrak{I}_{\epsilon,(\tilde{p},\tilde{p}')} := \left\langle \prod_{\substack{|\tilde{p}_i{}' - \tilde{p}_i| > 2\epsilon_{in} \\ \tilde{p}_i \neq \tilde{p}_i'}} (x_i - sign(\tilde{p}_i{}' - \tilde{p}_i)) \prod_{|\tilde{p}_i{}' - \tilde{p}_i| < 2\epsilon_{in}} (x_i^2 - 1) \right\rangle$$

$$\mathfrak{I}_{\epsilon,D} := \sum_{\substack{(p,p')\in P^2 \\ \tilde{h}(p') - \tilde{h}(p) > 2\epsilon_{out}}} \mathfrak{I}_{\epsilon,(\tilde{p},\tilde{p}')}$$

Theorem (4.7)

Consider a monotone function $h : [0,1]^n \to [0,1]$ and suppose we obtain noisy data, D by sampling points in $[0,1]^n$ using a uniform distribution. If ϵ is small enough, W_D will eventually have $w(h)$ as its unique minimal wiring diagram with probability 1. Equivalently, with probability 1, $\mathfrak{I}_{\epsilon,D}$ will eventually be equal to $\mathfrak{I}_{w(h)}$.

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Consider a monotone function $h : [0,1]^n \to [0,1]$ *and suppose we obtain noisy data,* D *by sampling points in* $[0,1]^n$ *using a uniform distribution. If* ϵ *is small enough,* W_D *will eventually have* $w(h)$ *as its unique minimal wiring diagram with probability 1. Equivalently, with probability 1,* $\mathfrak{I}_{\epsilon,D}$ *will eventually be equal to* $\mathfrak{I}_{w(h)}$.

Theorem (4.8)

Suppose we obtain noisy data D *by sampling points in* $[0,1]^n$ *using a uniform distribution. For small enough noise, with probability 1 a wiring diagram* w *is consistent with* D *if and only if* $\mathfrak{I}_{\epsilon,D} \subseteq \mathfrak{I}_{w(h)}$

Section 5 Selection of Wiring Diagrams

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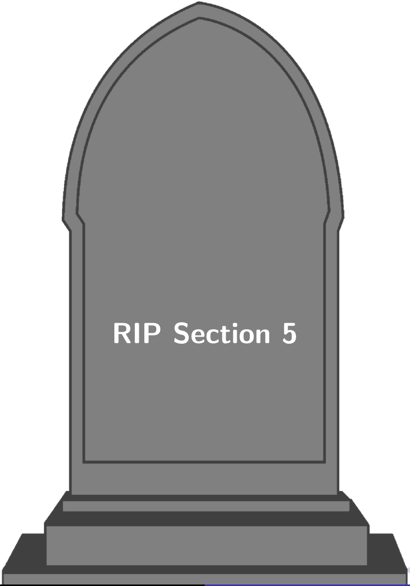

Citations

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