HW7

Sunday, November 24, 2024 3:14 PM

1. **Reverse-engineering a Boolean model.** Consider a Boolean model $f = (f_1, f_2, f_3)$ whose (synchronous) phase spaces consists of the following:

$$(0,0,1) \xrightarrow{f} (1,0,1) \xrightarrow{f} (1,1,1) \xrightarrow{f} (1,1,0) \xrightarrow{f} (0,1,0) \xrightarrow{f} (0,0,0).$$

In this problem, you will reverse-engineer the wiring diagram and then the models space.

(a) Do the following steps for each $k = 1, 2, 3$.

i. Write down the corresponding set of *data*

$$\mathcal{D}_k := \{(s_1, t_{1k}), \ldots, (s_5, t_{5k})\}$$

that arises from the k^{th} coordinate of this time-series.

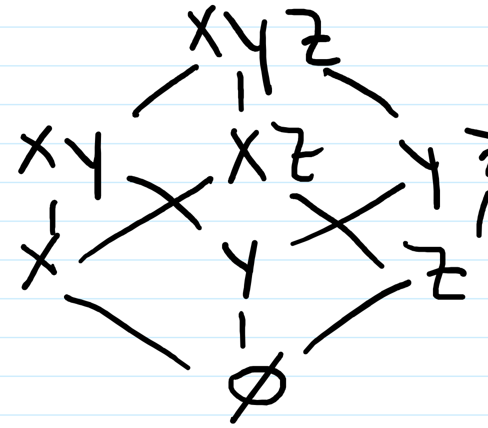
ii. Find the monomial ideal $I_{\Delta_k^c}$ of non-disposable sets, and compute its primary decomposition to find the min-sets.

iii. Find the pseudomonomial ideal $J_{\Delta_k^c}$, and compute its primary decomposition to find the signed min-sets.

(b) Find all min-sets and signed min-sets of the original Boolean model.

(c) Use Macaulay2 to compute the vanishing ideal $I(\mathcal{D})$. How big is it?

(d) Find the model space $\text{Mod}(\mathcal{D}) = \prod_{i=1}^{3} [f + I(\mathcal{D})]$. Use Lagrange interpolation to find $f = (f_1, f_2, f_3)$, and Macaulay2 to reduce them modulo $I(\mathcal{D})$.



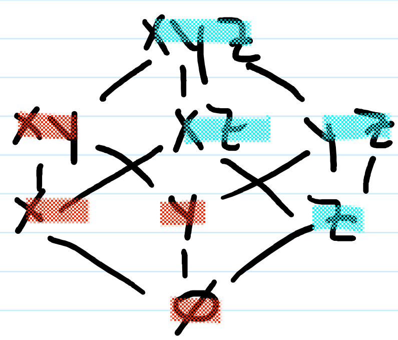
i)

x	0	1	1	1	0
y	0	0	1	1	1
z	1	1	1	0	0
f_1	1	1	1	0	0
f_2	0	1	1	1	0
f_3	1	1	0	0	0

ii) f_1: 001 → 1, 101 → 1, 111 → 1, 110 → 0, 010 → 0

yz / xyz, yz / xyz, z / xz

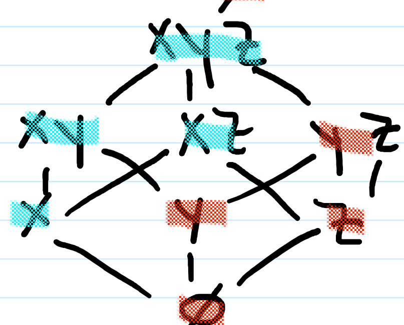
$I_{\Delta^c} = \langle xyz, yz, xz, z \rangle = \langle z \rangle$



f_2: 001 → 0, 101 → 1, 111 → 1, 110 → 1, 010 → 0

x / xyz, xy / xz, xyz / x

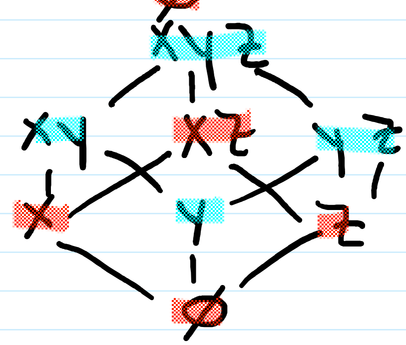
$I_{\Delta^c} = \langle xyz, xy, xz, x \rangle = \langle x \rangle$



f_3: 001 → 1, 101 → 1, 111 → 0, 110 → 0, 010 → 0

xy / y, xyz / yz, yz / xyz

$I_{\Delta^c} = \langle xyz, xy, yz, y \rangle = \langle y \rangle$



iii f_1 001 101 111 110 010
$\quad$ 1 1 1 0 0

$xy\bar{z}$ $\quad y\bar{z}$ $\quad \bar{z}$
$y\bar{z}$ $\quad \bar{x}y\bar{z}$ $\quad \bar{x}\bar{z}$

$J_{\Delta^c} = \langle \bar{z} \rangle$

f_2 001 101 111 110 010
$\quad$ 0 1 1 1 0

$\bar{x}$ $\quad \bar{x}\bar{y}$ $\quad \bar{x}\bar{y}z$
$\bar{x}y\bar{z}$ $\quad \bar{x}\bar{z}$ $\quad \bar{x}$

$J_{\Delta^c} = \langle \bar{x} \rangle$

f_3 001 101 111 110 010
$\quad$ 1 1 0 0 0

$\bar{x}\bar{y}$ $\quad \bar{x}\bar{y}z$ $\quad \bar{y}z$
$\bar{y}$ $\quad \bar{y}z$ $\quad x\bar{y}z$

$J_{\Delta^c} = \langle \bar{y} \rangle$

b) $\langle x \rangle \cup \langle y \rangle \cup \langle z \rangle$ min
$\langle \bar{x} \rangle \cup \langle \bar{y} \rangle \cup \langle \bar{z} \rangle$ signed min

c) $I(D) = \langle yz+y+z+1,\ xy+xz+x+y+z+1 \rangle$
$|I(D)| = 2^{6-3} = 2^3 = 8$

d) f_1 001 101 111 110 010
$\quad$ 1 1 1 0 0

$f_1 = (x+1)(y+1) + x(y+1) + xyz$

f_2 001 101 111 110 010
$\quad$ 0 1 1 1 0

$f_2 = x(y+1) + yz + y(z+1)x$

f_3 001 101 111 110 010
$\quad$ 1 1 0 0 0

$f_3 = (y+1)(x+1) + (x+1)(y+1)(z+1)$

$f = \{f_1, f_2, f_3\}$

$f = \{f_1, f_2, f_3\}$

$f \setminus G = \{z, x, xz + z\}$

2. **Reverse-engineering a ternary model**. Repeat the previous problem but for the following *time series* of a 3-node algebraic model over $\mathbb{F}_3$:

$$(1,1,1) \xrightarrow{f} (2,0,1) \xrightarrow{f} (2,0,0) \xrightarrow{f} (0,2,2) \xrightarrow{f} (0,2,2).$$

(a) Do the following steps for each $k = 1, 2, 3$.

i. Write down the corresponding set of *data*

$$\mathcal{D}_k := \{(s_1, t_{1k}), \ldots, (s_5, t_{5k})\}$$

that arises from the k^{th} coordinate of this time-series.

ii. Find the monomial ideal $I_{\Delta_k^c}$ of non-disposable sets, and compute its primary decomposition to find the min-sets.

iii. Find the pseudomonomial ideal $J_{\Delta_k^c}$, and compute its primary decomposition to find the signed min-sets.

(b) Find all min-sets and signed min-sets of the original Boolean model.

(c) Use Macaulay2 to compute the vanishing ideal $I(\mathcal{D})$. How big is it?

(d) Find the model space $\operatorname{Mod}(\mathcal{D}) = \prod_{i=1}^{3} [f + I(\mathcal{D})]$. Use Lagrange interpolation to find $f = (f_1, f_2, f_3)$, and Macaulay2 to reduce them modulo $I(\mathcal{D})$.

i)

	1	2	2	0
	1	0	0	2
	1	1	0	2
f_1	2	2	0	0
f_2	0	0	2	2
f_3	1	0	2	2

111	201	200	022
2	2	0	0
0	0	2	2
1	0	2	2

ii) f_1

111	201	200	022
2	2	0	0

xyz z

xyz xyz

$I_{\Delta_1^c} = \langle xyz, z \rangle = \langle x, z \rangle \cap \langle y, z \rangle$

f_2

111	201	200	022
0	0	2	2

xyz xyz

z xyz

$I_{\Delta_2^c} = \langle x, z \rangle \cap \langle y, z \rangle$

f_3

111	201	200	022
1	0	2	2

xy xy

xyz z

xyz xyz

$I_{\Delta_3^c} = \langle xy, z \rangle$

iii) f_1 $\underset{2}{111}$ $\underset{2}{201}$ $\underset{0}{200}$ $\underset{0}{022}$

$x\bar{y}\bar{z}$ z

$\bar{x}yz$ $\bar{x}yz$

$J_{\Delta^c} = \langle z, \bar{y}\rangle \cap \langle z, x\rangle$

f_2 $\underset{0}{111}$ $\underset{0}{201}$ $\underset{2}{200}$ $\underset{2}{022}$

$\bar{x}yz$ $x\bar{y}\bar{z}$

z $xy\bar{z}$

$J_{\Delta^c} = \langle z, \bar{y}\rangle \cap \langle z, x\rangle$

f_3 $\underset{1}{111}$ $\underset{0}{201}$ $\underset{2}{200}$ $\underset{2}{022}$

$x\bar{y}$ z

$x\bar{y}\bar{z}$ $\bar{x}yz$

$\bar{x}yz$ $\bar{x}y$

$J_{\Delta^c} = \langle x, y, z\rangle \cap \langle z, \bar{y}, \bar{x}\rangle$

b) $(\langle x,z\rangle \cap \langle y,z\rangle) \cup (\langle x,z\rangle \cap \langle y,z\rangle) \cup (\langle xy, z\rangle)$ min

$(\langle z, \bar{y}\rangle \cap \langle z,x\rangle) \cup (\langle x,y,z\rangle \cap \langle z, \bar{y}, \bar{x}\rangle)$ signed min

c) $\mathcal{I}(D) = \langle \bar{x}, \bar{y}, \bar{z}\rangle$

$|\mathcal{I}(D)| = 3^{5-3} = 9$

d) f_1 $\underset{2}{111}$ $\underset{2}{201}$ $\underset{0}{200}$ $\underset{0}{022}$

$0 = 3$

$-1 = 2$

$-2 = 1$

$f_1 = 2r_1 + 2r_2$

$r_1 = (1-2)(x-2)(1-2)(x-2)x$

$r_2 = (2-1)(x-1)(2-2)(x-2)2x$

$f_1 = 2(-1)^2(x-2)^2 x = 2(x^2-4x-1)x = 2x^3+2x^2+2x$

f_2 $\underset{0}{111}$ $\underset{0}{201}$ $\underset{2}{200}$ $\underset{2}{022}$

$f_2 = 2r_3 + 2r_4$

$r_3 = (0-1)(y-1)(0-0)y(0-2)(y-2)$

$r_4 = (2-1)(y-1)(2-0)y(2-0)y$

$f_2 = 2^3(y-1)y^2 = 2y^3 + y^2$

f_3 111 201 200 022
1 0 2 2

$f_3 = 1r_1 + 2r_3 + 2r_4$

$r_1 = (1-1)(z-1)z(1-2)(z-2)$

$r_3 = (0-1)(z-1)(0-0)z(0-2)(z-2)$

$r_4 = (2-1)(z-1)(2-0)z(2-z)$

$f_3 = 8(z-1)z^2 = 2z3 + z^2$

$f = (f_1, f_2, f_3)$

$f \setminus G = \{1 + z - y - z^2, z^2 - z, z^2 - z\}$

This is all the output for Macauley2

```
i1 : R = ZZ/2[x1,x2,x3];
i2 :
-- Shortcut for AND and OR functions
RingElement | RingElement :=(x,y)->x+y+x*y;
i3 : RingElement & RingElement :=(x,y)->x*y;
i4 :
-- Define a quotient ring, where each x_i^2 = x_i
J = ideal(x1^2-x1, x2^2-x2, x3^2-x3);
o4 : Ideal of R
i5 : Q = R / J;
i6 : ideal(x1*x2*x3,x2*x3,x1*x3,x3)
o6 = ideal(x1x2x3,x2x3,x1x3,x3)
o6 : Ideal of Q
i7 : Q1=ideal(x1*x2*x3,x2*x3,x1*x3,x3);
o7 : Ideal of Q
i8 : Q1
o8 = ideal(x1x2x3,x2x3,x1x3,x3)
o8 : Ideal of Q
i9 : intersect(ideal(x1,x3),ideal(x3))
o9 = ideal x3
o9 : Ideal of Q
i10 :
~RingElement:=x->x+1;
stdio:17:0:(3): error: syntax error at '~'
i10 :
-- Define a quotient ring, where each x_i^2 = x_i
R = ZZ/2[x,y,z];
i11 : J_nonDisp = ideal(x*y*~z, ~y*x,~z,y*~z,~x*y*~z,~x*~z);
stdio:22:22:(3): error: syntax error at '~'
i11 : primaryDecomposition J_nonDisp
stdio:23:22:(3): error: no method for binary operator _ applied to objects:
```

- ideal (x12+x1,x22+x2,x32+x3) (of class Ideal)
- nonDisp (of class Symbol)

```
stdio:23:23-23:30: here is the first use of nonDisp
i12 : -- Define a quotient ring, where each x_i^2 = x_i
R = ZZ/2[x,y,z];
i13 : J_nonDisp = ideal(x*y*(z+1), (1+y)*x,(z+1),y*(1+z),(1+x)*y*(1+z),(1+x)*(1+z));
stdio:27:10:(3): error: no method for assignment to binary operator _ applied to objects:
```

- ideal (x12+x1,x22+x2,x32+x3) (of class Ideal)
- nonDisp (of class Symbol)

```
stdio:23:23-23:30: here is the first use of nonDisp
i14 : primaryDecomposition J_nonDisp
stdio:28:22:(3): error: no method for binary operator _ applied to objects:
```

- ideal (x12+x1,x22+x2,x32+x3) (of class Ideal)
- nonDisp (of class Symbol)

```
stdio:23:23-23:30: here is the first use of nonDisp
i15 :
```

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Macaulay2, version 1.24.11-1522-g1dadf80809 (vanilla)
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, Isomorphism,
LLLBases, MinimalPrimes, OnlineLookup, PackageCitations, Polyhedra, PrimaryDecomposition,
ReesAlgebra, Saturation, TangentCone, Truncations, Varieties
i1 : -- Define a quotient ring, where each x_i^2 = x_i
R = ZZ/2[x,y,z];
i2 :
J_nonDisp = ideal(x*y*(z+1), (1+y)*x,(z+1),y*(1+z),(1+x)*y*(1+z),(1+x)*(1+z));
o2 : Ideal of R
i3 : primaryDecomposition J_nonDisp
o3 = {ideal(z+1,y+1),ideal(z+1,x)}
o3 : List
i4 : -- Define a quotient ring, where each x_i^2 = x_i
R = ZZ/2[x,y,z];
i5 :
J_nonDisp = ideal((x+1), (1+x)*y*(z+1),(x+1)*(y+1),(x+1)*(z+1),(x+1)*(1+y)*z,(1+x));
o5 : Ideal of R
i6 : primaryDecomposition J_nonDisp
o6 = {ideal(x+1)}
o6 : List
i7 : -- Define a quotient ring, where each x_i^2 = x_i
R = ZZ/2[x,y,z];
i8 :
J_nonDisp = ideal((x+1)*(y+1),(y+1),(x+1)*(y+1)*z,(y+1)*z,(y+1)*z,x*(y+1)*z);
o8 : Ideal of R
i9 : primaryDecomposition J_nonDisp
o9 = {ideal(y+1)}
o9 : List
i10 : -- Define a quotient ring, where each x_i^2 = x_i
R = ZZ/2[x,y,z];
i11 :
J_nonDisp = ideal(x*y*(z+1),y*(z+1),y*(z+1),(x+1)*y*(z+1),(z+1),(x+1)*(z+1));
o11 : Ideal of R
i12 : primaryDecomposition J_nonDisp
o12 = {ideal(z+1)}
o12 : List
i13 :

Macaulay2, version 1.24.11-1522-g1dadf80809 (vanilla)
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, Isomorphism,
LLLBases, MinimalPrimes, OnlineLookup, PackageCitations, Polyhedra, PrimaryDecomposition,
ReesAlgebra, Saturation, TangentCone, Truncations, Varieties
i1 : R=ZZ/2[x,y,z] / ideal(x^2-x, y^2-y, z^2-z);
i2 :
I1=ideal(x,y,z+1)
o2 = ideal(x,y,z+1)
o2 : Ideal of R
i3 : I2=ideal(x+1,y,z+1)
o3 = ideal(x+1,y,z+1)
o3 : Ideal of R
i4 : I3=ideal(x+1,y+1,z+1)
o4 = ideal(x+1,y+1,z+1)
o4 : Ideal of R
i5 : I4=ideal(x+1,y+1,z)
o5 = ideal(x+1,y+1,z)
o5 : Ideal of R
i6 : I5=ideal(x,y+1,z)
o6 = ideal(x,y+1,z)
o6 : Ideal of R
i7 :
I=intersect{I1,I2,I3,I4,I5}
o7 = ideal(yz+y+z+1,xy+xz+x+y+z+1)
o7 : Ideal of R
i8 : G=gens gb I
o8 = (yz+y+z+1xy+xz+x+y+z+1)
o8 : Matrix R1←R2
i9 : R=ZZ/2[x,y,z] / ideal(x^2-x, y^2-y, z^2-z);
i10 :
I1=ideal(x,y,z+1)
o10 = ideal(x,y,z+1)
o10 : Ideal of R
i11 : I2=ideal(x+1,y,z+1)
o11 = ideal(x+1,y,z+1)
o11 : Ideal of R
i12 : I3=ideal(x+1,y+1,z+1)
o12 = ideal(x+1,y+1,z+1)
o12 : Ideal of R
i13 : I4=ideal(x+1,y+1,z)
o13 = ideal(x+1,y+1,z)
o13 : Ideal of R
i14 : I5=ideal(x,y+1,z)
o14 = ideal(x,y+1,z)
o14 : Ideal of R
i15 :
I=intersect{I1,I2,I3,I4,I5}
o15 = ideal(yz+y+z+1,xy+xz+x+y+z+1)
o15 : Ideal of R
i16 : G=gens gb I
o16 = (yz+y+z+1xy+xz+x+y+z+1)
o16 : Matrix R1←R2
i17 :
f1=(x+1)*(y+1)+x*(y+1)+x*y*z;
i18 : f2=x*(y+1)+y*z+y*x*(z+1);
i19 : f3=(y+1)*(x+1)+(x+1)*(y+1);
i20 :
f1=f1%G;
i21 : f2=f2%G;
i22 : f3=f3%G;
i23 :
(f1,f2,f3)
```

```
o23 = (z,x,0)
o23 : Sequence
i24 : R=ZZ/2[x,y,z] / ideal(x^2-x, y^2-y, z^2-z);
i25 :
I1=ideal(x,y,z+1)
o25 = ideal(x,y,z+1)
o25 : Ideal of R
i26 : I2=ideal(x+1,y,z+1)
o26 = ideal(x+1,y,z+1)
o26 : Ideal of R
i27 : I3=ideal(x+1,y+1,z+1)
o27 = ideal(x+1,y+1,z+1)
o27 : Ideal of R
i28 : I4=ideal(x+1,y+1,z)
o28 = ideal(x+1,y+1,z)
o28 : Ideal of R
i29 : I5=ideal(x,y+1,z)
o29 = ideal(x,y+1,z)
o29 : Ideal of R
i30 :
I=intersect{I1,I2,I3,I4,I5}
o30 = ideal(yz+y+z+1,xy+xz+x+y+z+1)
o30 : Ideal of R
i31 : G=gens gb I
o31 = (yz+y+z+1xy+xz+x+y+z+1)
o31 : Matrix R1←R2
i32 :
f1=(x+1)*(y+1)+x*(y+1)+x*y*z;
i33 : f2=x*(y+1)+y*z+y*x*(z+1);
i34 : f3=(y+1)*(x+1)+(x+1)*(y+1)*(z+1);
i35 :
f1=f1%G;
i36 : f2=f2%G;
i37 : f3=f3%G;
i38 :
(f1,f2,f3)
o38 = (z,x,xz+z)
o38 : Sequence
i39 :

Macaulay2, version 1.24.11-1522-g1dadf80809 (vanilla)
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, Isomorphism,
LLLBases, MinimalPrimes, OnlineLookup, PackageCitations, Polyhedra, PrimaryDecomposition,
ReesAlgebra, Saturation, TangentCone, Truncations, Varieties
i1 :
R = QQ[x,y,z];
i2 : J_nonDisp = ideal(x*(y+1)*(z+1),(x+1)*y*z,z,(x+1)*y*z);
o2 : Ideal of R
i3 : primaryDecomposition J_nonDisp
o3 = {ideal(z,y+1),ideal(z,x)}
o3 : List
i4 :
R = QQ[x,y,z];
i5 : J_nonDisp = ideal((x+1)*x*z,z,x*(y+1)*(z+1));
o5 : Ideal of R
i6 : primaryDecomposition J_nonDisp
o6 = {ideal(z,y+1),ideal(z,x)}
o6 : List
i7 :
R = QQ[x,y,z];
i8 : J_nonDisp = ideal(x*(y+1),x*(y+1)*(z+1),(x+1)*y*z,z,(x+1)*y*z,(x+1)*y);
o8 : Ideal of R
i9 : primaryDecomposition J_nonDisp
o9 = {ideal(z,y,x),ideal(z,y+1,x+1)}
o9 : List
i10 : R=ZZ/3[x,y,z] / ideal(x^2-x, y^2-y, z^2-z);
i11 :
I1=ideal(x-1,y-1,z-1);
o11 : Ideal of R
i12 : I2=ideal(x-2,y,z-1);
o12 : Ideal of R
i13 : I3=ideal(x-2,y,z);
o13 : Ideal of R
i14 : I4=ideal(x,y-2,z-2)
o14 = ideal(x,y+1,z+1)
o14 : Ideal of R
i15 :
I=intersect{I1,I2,I3,I4};
o15 : Ideal of R
i16 : G=gens gb I;
o16 : Matrix R1←R3
i17 : G
o17 = (z−1y−1x−1)
o17 : Matrix R1←R3
i18 : 2*(x-2)^2*x
o18 = −x
o18 : R
i19 : 2^3*(y-1)*y^2
o19 = 0
o19 : R
i20 : 8
o20 = 8
i21 :

Macaulay2, version 1.24.11-1522-g1dadf80809 (vanilla)
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LLLBases, MinimalPrimes, OnlineLookup, PackageCitations, Polyhedra, PrimaryDecomposition,
ReesAlgebra, Saturation, TangentCone, Truncations, Varieties
i1 : R=ZZ/3[x,y,z]
o1 = R
```

o1 : PolynomialRing
i2 :
I1=ideal(x-1,y-1,z-1);
o2 : Ideal of *R*
i3 : I2=ideal(x-2,y,z-1);
o3 : Ideal of *R*
i4 : I3=ideal(x-2,y,z);
o4 : Ideal of *R*
i5 : I4=ideal(x,y-2,z-2)
o5 = ideal(*x*,*y*+1,*z*+1)
o5 : Ideal of *R*
i6 :
f1=2*x^3+2*x^2+2*x;
i7 : f2=2*y^3+y^2;
i8 : f3=2*z^3+z^2;
i9 :
I=intersect{I1,I2,I3,I4};
o9 : Ideal of *R*
i10 : G=gens gb I;
o10 : Matrix $R^1 \leftarrow R^4$
i11 :
f1=f1%G;
i12 : f2=f2%G;
i13 : f3=f3%G;
i14 : (f1,f2,f3)
o14 = $(-z^2-y+z+1, z^2-z, z^2-z)$
o14 : Sequence
i15 :

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