HW11

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No.1. (8 pts.) Let H_1, H_2, $B(\cdot,\cdot)$ and F be as described in the Brezzi-Nečas-Babuška Theorem. Additionally, let $H_{1,h}$ and $H_{2,h}$ be closed subspaces of H_1 and H_2, respectively.
(i) Does it immediately follow from the Brezzi-Nečas-Babuška Theorem that there exists a unique $u_h \in H_{1,h}$ satisfying $B(u_h, v_h) = F(v_h)$, $\forall v_h \in H_{2,h}$?
If not, what additional assumptions are needed??

(ii) Follow the proof of Céa's Lemma to establish an error bound for $\|u - u_h\|_{H_1}$.

i) Conditions need for B

- $|B(w,v)| \leq C_1 \|w\|_{H_1} \|v\|_{H_2} \quad \forall w \in H_1\ \forall v \in H_2$
- $\displaystyle\sup_{v \neq 0 \in H_2} \frac{|B(w,v)|}{\|v\|_{H_2}} \geq C_2 \|w\|_{H_1} \quad \forall w \in H_1$
- $\displaystyle\sup_{w \in H_1} |B(w,z)| > 0 \quad \forall z \neq 0 \in H_2$

- If H_1 and H_2 are closed spanning subsets the bddedness would not change.
- If we require the last two conditions to hold for $v \in H_{2h}$ and $w \in H_{1h}$ then we can still apply the theorem.

ii) Let H_{1h} H_{2h} be closed subspaces
Let $v_h = 0 \in H_{2h}$ & $u_h \in H_{1h}$ and let u be the true solution from BNB such that

$B(u, v_h) = F(v_h)$ and $B(u_h, v_h) = F(v_h) \quad \forall v_h \in H_{2h}$

$B(u - u_h, v_h) = F(v_h) - F(v_h) = 0$

Let $w_h \in H_{1h}$ so that $\chi = u - w_h \in H_1 \setminus H_{1h}$ and $\eta = w_h - u_h \in H_{1h}$

$u - u_h = (u - w_h) + (w_h - u_h) = \chi + \eta$

Hence $B(\chi + \eta, v_h) = 0 \Rightarrow |B(\chi, v_h)| = |B(\eta, v_h)|$

Let $\varepsilon > 0$ $\quad (C_2 - \varepsilon)\|\eta\| \leq \displaystyle\sup_{v_h \neq 0 \in H_{2h}} \frac{|B(\eta, v_h)|}{\|v_h\|_{H_2}} = \sup_{v_h \neq 0 \in H_{2h}} \frac{|B(\chi, v_h)|}{\|v_h\|_{H_2}} \leq \frac{C_c \|\chi\| \cancel{\|v_h\|}}{\cancel{\|v_h\|}}$

as $\varepsilon \to 0$ we have $\|\eta\| \leq \frac{C_c}{C_2}\|\chi\|$

$\|u - u_h\| = \|\chi + \eta\| \leq \|\chi\| + \|\eta\| \leq \left(1 + \frac{C_c}{C_2}\right)\|\chi\|$

Now taking inf over w_h we get

$\|u - u_h\| \leq \left(1 + \frac{C_c}{C_2}\right) \displaystyle\inf_{w_h \in H_{2h}} \|u_h - w_h\|$

No.2. (8 pts.) Consider the problem of determining u satisfying

$$\mathcal{L}(u)(x) := -\nabla \cdot k(\mathbf{x})\nabla u(\mathbf{x}) + \mathbf{b}(\mathbf{x}) \cdot \nabla u(\mathbf{x}) + e(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x}), \ \mathbf{x} \in \Omega, \quad (0.1)$$

$$\text{subject to } \begin{cases} u(\mathbf{x}) &= 0, \ \mathbf{x} \in \Gamma \subset \partial\Omega, \\ -k(x)\frac{\partial u(\mathbf{x})}{\partial \mathbf{n}} &= g_2(\mathbf{x}), \ \mathbf{x} \in \partial\Omega \backslash \Gamma \end{cases} . \quad (0.2)$$

Give the weak formulation for (0.1), (0.2).

$\int_\Omega -\nabla \cdot k(x) \nabla u(x) v(x) + b(x) \cdot \nabla u(x) v(x) + e(x) u(x) v(x) \, d\Omega = \int_\Omega f(x) v(x) \, d\Omega$

$\int_\Omega \nabla \cdot k \nabla u v \, d\Omega = \int_{\partial\Omega} -k \nabla u \cdot \hat{n} \, v \, ds + \int_\Omega k \nabla u \cdot \nabla v \, d\Omega$

$\int_\Omega b \cdot \nabla u v \, d\Omega = \int_{\partial\Omega} u v b \cdot \hat{n} ds - \int_\Omega u (b \cdot \nabla v + \nabla \cdot b v) d\Omega$

$\int_\Omega k \nabla u \cdot \nabla v - u(b \cdot \nabla v + \nabla \cdot b v) + e u v \, d\Omega + \int_{\partial\Omega} u v b \cdot \hat{n} - k \nabla u \cdot \hat{n} v ds = \int_\Omega f v \, d\Omega$

$\int_{\partial\Omega} u v b \cdot \hat{n} - k v \nabla u \cdot \hat{n} \, ds = \int_\Gamma 0 - k v \nabla u \cdot \hat{n} ds + \int_{\partial\Omega \backslash \Gamma} u v b \cdot \hat{n} + g_2 v ds$

$B(u,v) = \int_\Omega k \nabla u \cdot \nabla v - u(b \cdot \nabla v + \nabla \cdot b v) + e u v d\Omega - \int_\Gamma k v \nabla u \cdot \hat{n} ds + \int_{\partial\Omega \backslash \Gamma} u v b \cdot \hat{n} ds$

$F(v) = \int f v d\Omega - \int_{\partial\Omega \backslash \Gamma} g_2 v ds$

Given $k, b, \& e$ are nice enough ($L^\infty(\Omega)$)

$B(u,v)$ is well defined if $\nabla u, u, \nabla u|_{\partial\Omega}$, and $u|_{\partial\Omega}$ are in $L^2(\Omega)$

and v and ∇v in $L^2(\Omega)$

Thus $v \in H^1(\Omega)$ and by the trace thrm $u \in H^{3/2}$ so that $\nabla u|_{\partial\Omega} \in L^2(\Omega)$ and $u|_{\partial\Omega} \in L^2(\Omega)$

$g_2 \in (H^{1/2}(\partial\Omega))'$ since by the trace thrm $v|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$

$\therefore \ B(u,v) = F(v) \quad \forall v \in H^1(\Omega)$

subject to $u = 0$ on Γ

$-k\frac{\partial u}{\partial \hat{n}} = g_2$ on $\partial\Omega \backslash \Gamma$

For $g_2 \in (H^{1/2}(\partial\Omega))'$ and $u \in H^{3/2}(\Omega)$ $f \in (H^1(\Omega))'$

No.3. (8 pts.) Consider the boundary value problem: *Given f and g determine u satisfying*

$$-\nabla \cdot (k(\mathbf{x}) \nabla u(\mathbf{x})) + \mathbf{b}(\mathbf{x}) \cdot \nabla u(\mathbf{x}) + c(\mathbf{x}) u(\mathbf{x}) = f \ \text{ in } \Omega, \quad (0.3)$$

$$\text{subject to } \frac{\partial u(\mathbf{x})}{\partial \mathbf{n}} = g(\mathbf{x}), \ \mathbf{x} \in \partial\Omega. \quad (0.4)$$

(i) Derive the weak formulation for the solution of (0.3), (0.4), $B(u,v) = F(v)$.
(ii) Identify $\mathcal{L} : X \longrightarrow Y$ such that $\mathcal{L}(w)(\cdot) = B(w, \cdot)$.
(iii) Determine the adjoint operator, $\mathcal{L}^*$, of $\mathcal{L}$, and give an associated adjoint problem to (0.3), (0.4).

i) This problem is similar to #2
Thus

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Thus

$$\int_\Omega k\nabla u\cdot\nabla v - u(b\cdot\nabla v+\nabla\cdot bv)+euv\,d\Omega+\int_{\partial\Omega}uvb\cdot\hat{n}-k\nabla u\cdot\hat{n}v\,ds=\int_\Omega fv\,d\Omega$$

$$\int_{\partial\Omega}uvb\cdot\hat{n}-kv\nabla u\cdot\hat{n}\,ds=\int_{\partial\Omega}uvb\cdot\hat{n}-kvg\,ds$$

$$B(u,v)=\int_\Omega k\nabla u\cdot\nabla v-u(b\cdot\nabla v+\nabla\cdot bv)+euv\,d\Omega+\int_{\partial\Omega}uvb\cdot\hat{n}\,ds$$

$$F(v)=\int_\Omega fv\,d\Omega+\int_{\partial\Omega}kvg\,ds$$

$$B(u,v)=F(v)\quad\forall v\in H^1(\Omega)\quad g\in(H^{1/2}(\partial\Omega))'\quad u\in H^1(\Omega)\quad f\in(H^1(\Omega))'$$

ii) $\mathcal{L}(w)(\cdot)=B(w,\cdot)$ where $B(\cdot,\cdot)$ is given above

iii) $$\int_\Omega k\nabla u\cdot\nabla v-u(b\cdot\nabla v+\nabla\cdot bv)+euv\,d\Omega+\int_{\partial\Omega}uvb\cdot\hat{n}\,ds$$

$$\int_\Omega\nabla v\cdot k\nabla u-u\nabla\cdot bv+euv\,d\Omega+\int_{\partial\Omega}uvb\cdot\hat{n}\,ds$$

$$\int_\Omega\nabla v\cdot k\nabla u\,d\Omega=\int_{\partial\Omega}uk\nabla v\cdot\hat{n}\,ds-\int_\Omega u\nabla\cdot k\nabla v\,d\Omega$$

$$\mathcal{L}^*(u)(v)=B(v,u)=F(u)\quad\Rightarrow$$

$$\int_\Omega -u\nabla\cdot k\nabla v-u\nabla\cdot bv+veu-fu\,d\Omega-\int_{\partial\Omega}uvb\cdot\hat{n}+uk\nabla v\cdot\hat{n}-kug\,ds=0$$

$$\int_\Omega u(-\nabla\cdot k\nabla v-\nabla\cdot bv+ev-f)\,d\Omega-\int_{\partial\Omega}u((vb+k\nabla v)\cdot\hat{n}-kg)\,ds=0$$

$\therefore$

$$-\nabla\cdot k\nabla v-\nabla\cdot bv+ev=f\quad\text{in }\Omega$$

subject to $vb+k\nabla v=kg$ on $\partial\Omega$