MATH 8660 Sec. 1: Homework No.13

Due: Monday, December 9, 1:00pm

The Darcy fluid flow equations (for modeling fluid flow through a porous medium) are: Given $\mathbf{f} \in L^2(\Omega)$, $\beta \in \mathbb{R}^+$, determine $\mathbf{u}$, p satisfying

$$\begin{aligned} \beta\mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega, && (1)\\ \nabla \cdot \mathbf{u} &= 0, \quad \text{in } \Omega, && (2)\\ \mathbf{u}\cdot\mathbf{n} &= 0, \quad \text{on } \partial\Omega, && (3) \end{aligned}$$

where $\mathbf{n}$ denotes the unit outer normal to $\partial\Omega$.

The function spaces of interest are:

$$\text{for the velocity}: \quad X = \mathbf{H_0^{div}(\Omega)} := \{\mathbf{v} \in L^2(\Omega) : \nabla\cdot\mathbf{v} \in L^2(\Omega),\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\},$$

$$\text{for the pressure}: \quad Q = L_0^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q\, d\Omega = 0\},$$

$$\text{with associated norms} \quad \|\mathbf{v}\|_X := \left(\|\mathbf{v}\|^2_{L^2(\Omega)} + \|\nabla\cdot\mathbf{v}\|^2_{L^2(\Omega)}\right)^{1/2}, \quad \|q\|_Q := \|q\|_{L^2(\Omega)}.$$

The weak formulation of (1) - (3): *Given* $\mathbf{f} \in L^2(\Omega)$, *determine* $(\mathbf{u}, p) \in X \times Q$ *satisfying for all* $(\mathbf{v}, q) \in X \times Q$

$$\begin{aligned} A(\mathbf{u},\mathbf{v}) + B(\mathbf{v},p) &= (\mathbf{f},\mathbf{v}),\\ B(\mathbf{u},q) &= 0, \end{aligned} \qquad (4)$$

where $A(\cdot,\cdot) : X \times X \longrightarrow \mathbb{R}$, and $B(\cdot,\cdot) : X \times Q \longrightarrow \mathbb{R}$ are defined by

$$A(\mathbf{w},\mathbf{v}) := \int_\Omega \mathbf{w}\cdot\mathbf{v}\, d\Omega, \qquad B(\mathbf{v},q) := -\int_\Omega q\,\nabla\cdot\mathbf{v}\, d\Omega.$$

Let $V := \{\mathbf{z} \in X : \int_\Omega q\,\nabla\cdot\mathbf{z}\, d\Omega = 0,\ \forall q \in Q\}$.

No.1 (12 pts.).

(a) (4 pts.). Show that

(i) $|A(\mathbf{w},\mathbf{v})| \le C_1 \|\mathbf{w}\|_X \|\mathbf{v}\|_X, \quad \forall\, \mathbf{w}, \mathbf{v} \in X$.

(ii) $|A(\mathbf{w},\mathbf{w})| \ge C_2 \|\mathbf{w}\|_X^2, \quad \forall\, \mathbf{w} \in V$.

NOTE: (ii) ONLY holds for $\mathbf{w} \in V$.

(b) (4 pts.). Show that

(iii) $|B(\mathbf{v},q)| \le C_3 \|\mathbf{v}\|_X \|q\|_Q, \quad \forall\, \mathbf{v} \in X,\ q \in Q$.

(iv) $\sup_{q\in Q} B(\mathbf{v},q) > 0, \quad \forall \mathbf{v} \in V^\perp$.

(c) (4 pts.). Assume

(v) $\sup_{\mathbf{v}\in V^\perp} \frac{|B(\mathbf{v},q)|}{\|\mathbf{v}\|_X} \ge C_5 \|q\|_Q, \quad \forall\, q \in Q$.

Show how (i) – (v) guarantee the existence and uniqueness of solution to (4).

No.2 (16 pts.). Discrete approximation.
(a) (4 pts.). Assume $X_h \subset X$, and $Q_h \subset Q$. Discuss the existence and uniqueness of $(\mathbf{u}_h, p_h) \in X_h \times Q_h$ satisfying

$$\begin{aligned} A(\mathbf{u}_h, \mathbf{v}_h) \,+\, B(\mathbf{v}_h, p_h) &= \,(\mathbf{f}, \mathbf{v}_h)\,, \quad \forall \mathbf{v}_h \in X_h\,, \\ B(\mathbf{u}_h, q_h) &= 0\,, \quad \forall q_h \in Q_h\,. \end{aligned}$$

In particular, are the conditions (i)–(v) sufficient to guarantee the existence and uniqueness of $(\mathbf{u}_h, p_h) \in X_h \times Q_h$?

Let $V_h := \{\mathbf{z}_h \in X_h \,:\, \int_\Omega q_h \nabla \cdot \mathbf{z}_h \, d\Omega \;=\; 0\,,\ \forall q_h \in Q_h\}$.
(b) (6 pts.). Derive the error estimate for the velocity

$$\|\mathbf{u} \,-\, \mathbf{u}_h\|_X \;\leq\; C \left(\inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} \,-\, \mathbf{v}_h\|_X \;+\; \inf_{q_h \in Q_h} \|p \,-\, q_h\| \right).$$

(c) (6 pts.). Derive the error estimate for the pressure

$$\|p \,-\, p_h\|_Q \;\leq\; C \left(\inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} \,-\, \mathbf{v}_h\|_X \;+\; \inf_{q_h \in Q_h} \|p \,-\, q_h\| \right).$$