MATH 8660 Sec.1: Homework No.4

Due: Tuesday, September 17

No.1. (4 pts.) Let Ω denote a bounded domain and $Q = \int_\Omega u^2(\mathbf{x})\, u'(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x}$. For $u,\, v \in H^1(\Omega)$, is Q guaranteed to be finite if
(i) $\Omega \subset \mathbb{R}^2$,
(ii) $\Omega \subset \mathbb{R}^3$?

Recall:
Hölder's inequaity : $\|fg\|_1 \ \le \ \|f\|_p\, \|g\|_q$, where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$,
Young's inequality: For $a, b \ge 0$, $\quad a\, b \ \le \ \frac{a^p}{p} + \frac{b^q}{q}$, where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

No.2. (5 pts.) Let $V = \mathbb{R}^3$, $\mathbf{f} := [-2, 5, 1]^T$, and $F(\mathbf{v}) := \mathbf{f} \cdot \mathbf{v}$.
(i) Show that $F : V \longrightarrow \mathbb{R}$ is a linear functional.
(ii) With $\langle \cdot, \cdot \rangle$ denoting the vector dot product, and $\| \cdot \|_V := (\langle \cdot, \cdot \rangle)^{1/2}$, show that F is bounded and determine $\|F\|$.

No.3. (5 pts.) For X a normed space, let $F : X \longrightarrow \mathbb{R}$ denote a linear functional. Show that F is continuous if and only if it is bounded.
(Hint: To establish that continuity $\Rightarrow$ boundedness, consider continuity about $x = 0$ for a specific value of ϵ, and then use scaling.)

No.4. (5 pts.) Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$, $B(\cdot, \cdot) : H^2(\Omega) \times H^2(\Omega) \longrightarrow \mathbb{R}$ be defined by

$$B(f, g) \ = \ \int_0^1 \int_0^y \frac{\partial^2 f(x,y)}{\partial^2 x}\, \frac{\partial g(x,y)}{\partial y}\, dx\, dy \ + \ f(1/2\,,\, 1/2)\, g(1/4\,, 1/2)\,.$$

(i) Determine if $B(\cdot, \cdot)$ is a bilinear form.
(ii) Determine if $B(\cdot, \cdot)$ is a bounded bilinear form. If so, give an explicit value for C (may involve an embedding constant).

No.5. (8 pts.) For X a normed space, let $B(\cdot, \cdot) : X \times X \longrightarrow \mathbb{R}$ denote a bilinear form. Assume that for $(a, b) \in X \times X$, $\|(a,b)\|_{X \times X} \ = \ (\|a\|_X^2 + \|b\|_X^2)^{1/2}$. Show that B is continuous if and only if it is bounded.