# MATH 8660 Sec.1: Homework No.2

**Due: Tuesday, September 3**

### Finite Difference and Finite Element Approximations

Consider the differential equation

$$-\epsilon\, y''(x) \ + y'(x) = \ 0\,,\ x \in (0,1) := \mathrm{I}\,, \tag{0.1}$$

$$\text{subject to} \ \ y(0) = 1\,, \quad y(1) = 0\,. \tag{0.2}$$

**No.1**. (3 pts.) Determine the exact solution to (0.1),(0.2).

**No.2**. (8 pts.) Consider

$$y''(x) \approx \frac{y(x+h) \ - \ 2y(x) \ + \ y(x-h)}{h^2} \tag{0.3}$$

$$y'(x) \approx \frac{y(x+h) \ - \ y(x-h)}{2h} \tag{0.4}$$

For $N \in \mathbb{N}$, let $0 = x_1 < x_2 < \ldots < x_{N+1} = 1$ represent a uniform partition of I, i.e., $x_{i+1} - x_i = h := 1/N$, for $i = 1, 2, \ldots, N$.

(a) Let $y_i \approx y(x_i)$. Using (0.3) and (0.4), give the finite difference equation approximation to (0.1) at $x_i$, for $2 \le i \le N$, in terms of the $y_{i-1}$, $y_i$ and $y_{i+1}$.

(b) Using MATLAB, compute a finite difference approximation to (0.1),(0.2) for $\epsilon = \frac{1}{20}$, using $N = 20$ and $N = 40$ equally spaced subintervals. Plot your approximations and the true solution on the same axes. (Use *fplot* to plot the true solution.) Turn in your code along with the plots.

**No.3**. (4 pts.) For $N \in \mathbb{N}$, let $0 = x_1 < x_2 < \ldots < x_{N+1} = 1$ represent a partition of I, and $S_N$ denote the collection of continuous, piecewise linear functions, subject to the partition $\{x_1, x_2, \ldots, x_{N+1}\}$. That is, for $\phi \in S_N$, $\phi$ is continuous, and $\phi|_{[x_i, x_{i+1}]}$ is a linear (actually affine) function. Let

$$\phi_i(x) \ = \ \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & , \quad x_{i-1} \le x \le x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & , \quad x_i \le x \le x_{i+1} \\ 0 & , \qquad \text{otherwise}\,. \end{cases}$$

Show that $\{\phi\}_{i=1}^{N+1}$ is a basis for $S_N$.

**No.4**. (6 pts.) Let $\{x_1, x_2, \ldots, x_{N+1}\}$ denote a uniform partition of I. For $\phi_j(x)$ defined in **No.3**, let

$$y_N(x) \ = \ 1\,\phi_1(x) \ + \ \sum_{j=2}^{N} c_j\,\phi_j(x) \ + \ 0\,\phi_{N+1}(x)\,.$$

Note that $y_N(0) = 1$, and $y_N(1) = 0$.

Multiplying (0.1) through by $v(x)$, satisfying $v(0) = v(1) = 0$, integrating by parts, and then replacing $y(x)$ by $y_N(x)$ we obtain

$$\epsilon \int_0^1 y_N'(x)\, v'(x)\, dx \ + \ \int_0^1 y_N'(x)\, v(x)\, dx \ = \ 0\,. \tag{0.5}$$

For $v(x) = \phi_i(x)$ show that (0.5) gives an equivalent linear system of equations as in **No.2**(a) (for $y_i$ replaced by $c_i$).

**No.5**. (2 pts.) For $\epsilon = 1/200$, repeat **No.2**(b). **Comment on the approximation**.

**No.6**. (2 pts.) For $\epsilon = 1/200$ and $N = 1000$, compute the finite difference (and finite element) approximation to (0.1),(0.2). Plot your approximation and the true solution on the same axes. **Comment on the approximation**.

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