b. $-(k(x)u')' + b(x)u' + c(x)u = f(x) \quad x\in\Omega=(\alpha,\beta)$

$u(\alpha)=u(\beta)=0$

Let $v\in H^1_0(\Omega)$

$$\int_\Omega -(k(x)u')'v(x) + b(x)u'v(x) + c(x)u\,v(x)\,d\Omega = \int_\Omega f(x)v(x)\,d\Omega$$

$$\int_\Omega -(k(x)u')'v(x)\,d\Omega = \overbrace{\int_{\partial\Omega}(-k(x)u')\cdot\hat{n}\,v\,ds}^{0} + \int_\Omega (k(x)u')v'(x)\,d\Omega$$

Let $w(x)$ be the representation of $u$ in $\Omega$

$$\int_\Omega k(x)w'(x)v'(x) + b(x)w'(x)v(x) + c(x)w(x)v(x)\,d\Omega = \int_\Omega f(x)v(x)\,d\Omega$$

$B(w,v)=F(v)$

bi. Since $0<k_m\le k(x)\le k_M<\infty \;\forall x\in\Omega$, we know $k$ is bdd and positive.

$$|B(w,v)| = \left|\int_\Omega k(x)w'(x)v'(x) + b(x)w'(x)v(x) + c(x)w(x)v(x)\,d\Omega\right|$$

$$\le \left|\int_\Omega k(x)w'(x)v'(x)\,d\Omega\right| + \left|\int_\Omega b(x)w'(x)v(x)\,d\Omega\right| + \left|\int_\Omega c(x)w(x)v(x)\,d\Omega\right|$$

$$\le k_M\left(\int_\Omega |w'|^2\,d\Omega\right)^{1/2}\left(\int_\Omega |v'|^2\,d\Omega\right)^{1/2} + \|b\|_\infty\left(\int_\Omega|w'|^2\,d\Omega\right)^{1/2}\left(\int_\Omega|v|^2\,d\Omega\right)^{1/2}$$
$$+ \|c\|_\infty\left(\int_\Omega|w|^2\,d\Omega\right)^{1/2}\left(\int_\Omega|v|^2\,d\Omega\right)^{1/2}$$

$$\le k_M\|w\|_{H^1}\|v\|_{H^1} + \|b\|_\infty\|w\|_{H^1}\|v\|_{H^1} + \|c\|_\infty\|w\|_{H^1}\|v\|_{H^1}$$

$$=(k_M+\|b\|_\infty+\|c\|_\infty)\|w\|_{H^1}\|v\|_{H^1}$$

Thus $b$ & $c$ need to be in $L^\infty$

bii. $|B(w,w)| = \left|\int_\Omega k(x)(w')^2 + b\,w'w + c(w)^2\,d\Omega\right|$

$$= \left|\int_\Omega k(x)(w')^2\,d\Omega + \int_\Omega \tfrac{1}{2}b(w^2)'\,d\Omega + \int_\Omega c\,(w)^2\,d\Omega\right|$$

$$\tfrac{1}{2}\int_\Omega b(w^2)'\,d\Omega = \underbrace{\cancel{\tfrac{1}{2}\int_{\partial\Omega} b\cdot n(w^2)\,ds}}_{0} - \tfrac{1}{2}\int_\Omega b'w^2\,d\Omega$$

$$\ge \left|k_m\|w'\|_2^2 - \tfrac{1}{2}\int_\Omega b'w^2\,d\Omega + \int_\Omega c\,w^2\,d\Omega\right|$$

$$= \left|k_m\|w'\|_2^2 + \int_\Omega\left(c-\tfrac{1}{2}b'\right)w^2\,d\Omega\right|$$

$\ge (k_m + C_{PF})\|w\|_V^2$ given $c-\frac{1}{2}b'\ge 0 \;\forall x\in\Omega$ since $w(\alpha)=w(\beta)=0$

$\ge C_E(k_m+C_{PF})\|w\|_{H^1}^2$ since equivalent norms

Thus $c-\frac{1}{2}b'\ge 0$ is the restriction so that $B$ is coercive

1c. $f \in L^2$ $\quad |F(v)| = |\int_\Omega f\, v\, d\Omega|$
$$\leq \left(\int |f|^2 d\Omega\right)^{1/2} \left(\int_\Omega |v|^2 d\Omega\right)^{1/2}$$
$$= \|f\|_2 \|v\|_2$$
$$\leq \|f\|_2 \|v\|_{H^1}$$

Since $f \in L^2$, $\|f\|_2 < \infty$ Thus $F(v)$ is bdd

$$\|F\| = \sup_{\|v\|_{H^1} = 1} |F(v)| \leq \sup_{\|v\|_{H^1} = 1} \|f\|_2 \|v\|_{H^1} = \|f\|_2 < \infty$$

$\therefore$ F is a bdd linear functional

1d. Let $v_1, v_2, w_1, w_2 \in H_0^1(\Omega)$ and $\delta \in \mathbb{R}$

$$F(\delta v_1 + v_2) = \int_\Omega f(\delta v_1 + v_2)\, d\Omega = \delta \int_\Omega f v_1\, d\Omega + \int_\Omega f v_2\, d\Omega$$
$$= \delta F(v_1) + F(v_2)$$

$$B(\delta w_1 + w_2, v_1) = \int_\Omega k(\delta w_1 + w_2)' v_1' + b(\delta w_1 + w_2)' v_1 + c(\delta w_1 + w_2) v_1\, d\Omega$$
$$= \delta \int_\Omega k w_1' v_1' + b w_1' v_1 + c w_1 v_1\, d\Omega + \int_\Omega k w_2' v_1' + b w_2' v_1 + c w_2 v_1\, d\Omega$$
$$= \delta B(w_1, v_1) + B(w_2, v_1)$$

$$B(w_1, \delta v_1 + v_2) = \int_\Omega k w_1' (\delta v_1 + v_2)' + b w_1' (\delta v_1 + v_2) + c w_1 (\delta v_1 + v_2)\, d\Omega$$
$$= \delta \int_\Omega k w_1' v_1' + b w_1' v_1 + c w_1 v_1\, d\Omega + \int_\Omega k w_1' v_2' + b w_1' v_2 + c w_1 v_2\, d\Omega$$
$$= \delta B(w_1, v_1) + B(w_1, v_2)$$

Thus $B(\cdot,\cdot)$ is coercive, bilinear, and bdd and $F(\cdot)$ is a bdd linear functional, therefore $\exists!\ u \in H_0^1 \ni B(u,v) = F(v)\ \forall v \in H_0^1$

1cont. As the span of L.I. elements of $H_0^1(\Omega)$, $S_N \subseteq H_0^1(\Omega)$.
Note $B|_{S_N}$ and $F|_{S_N}$ still fit the requirements for the Lax-Milgram theorem since the domain does not change bddness, coercivity, or (bi)linearity of B or F

lcont. cont $\therefore$ by Lax-Milgram $\exists!\ u_n \in S_N \ni$

$$B(u_n, v_n) = F(v_n) \quad \forall\ v_n \in S_N$$

It follows by Cea's lemma that:

$$\|e\|_{H_0^1} = \|u - u_n\|_{H^1} \le \left(1 + \frac{c_1}{c_2}\right) \inf_{w_n \in S_N} \|u - w_n\|_{H_0^1}$$