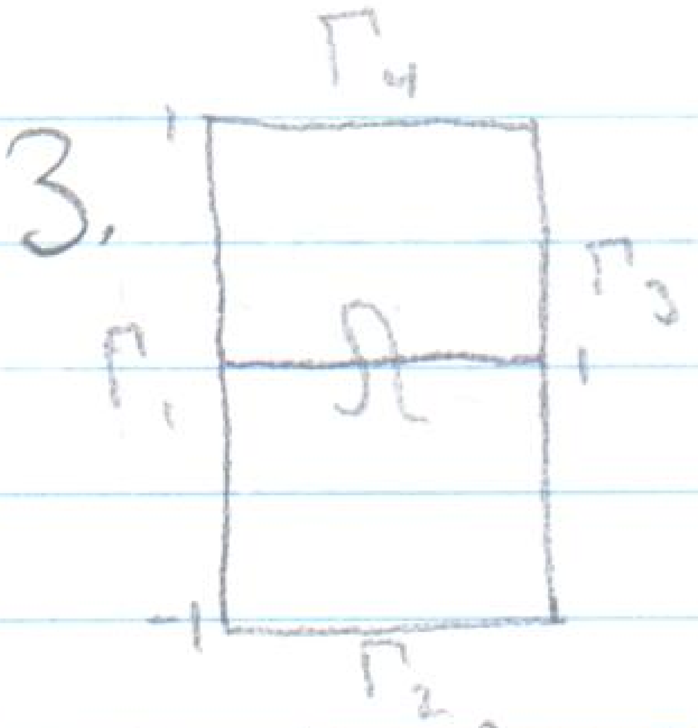
1. We will let P be the partition of the cross section between $[x, x+\delta x]$. A each point in the partition, the physics suggests $\Delta E = u(\cancel{\xi_i}, t)\, C\, M \Delta\xi$
where dE is the energy at x_i, $u(x_i, t)$ is the tempature, C is the specific heat constant and m is the mass.
Thus to get the energy in the whole section we need to take a fine enough partition to integrate and get
$E = \int_x^{x+\delta x} dE = \int_x^{x+\delta x} u(\xi, t)\, cm\, d\xi.$
Lastly we notice that $\rho = \frac{M}{V} \Rightarrow M = \rho A$ since we are adding up slices of the bar.
$\therefore\ E = \int_x^{x+\delta x} u(\xi, t)\, c\rho A\, d\xi$

2. $\frac{dE}{dt} = q(x,t)A - q(x+\delta x, t)A + \int_x^{x+\delta x} \tilde{f}(\xi, t)\rho A\, d\xi$
$= \int_x^{x+\delta x} \left(-\frac{\partial q(\xi, t)}{\partial \xi} A + \tilde{f}(\xi, t)\rho A\right) d\xi$
It follows $dE/dt = \frac{d}{dt}\int_x^{x+\delta x} u(\xi, t)\, c\rho A\, d\xi$
$= \int_x^{x+\delta x} \frac{\partial}{\partial t} u(\xi, t)\, c\rho A\, d\xi$
We also know $q(x,t) = -k\frac{\partial u(x,t)}{\partial x}$
where k is a proportionality constant
Note $q(x,t)$ is rate of energy flow and $\tilde{f}$ is a forcing function.
Thus $\int_x^{x+\delta x} \frac{\partial u(\xi,t)}{\partial t} c\rho A\, d\xi = \int_x^{x+\delta x} \left(\frac{k\, \partial^2 u(\xi,t) A}{\partial \xi^2} + \tilde{f}(\xi, t)\rho A\right) d\xi$
$\Rightarrow \int_x^{x+\delta x} \frac{\partial u(\xi,t)}{\partial t} c\rho A - \frac{kA\, \partial^2 u(\xi,t)}{\partial \xi^2}\, d\xi = \int_x^{x+\delta x} \tilde{f}(\xi, t)\rho A\, d\xi$
Since this is true for all $0 < x < x+\delta x < L$,
Let $\kappa = \frac{k}{c\rho}$ and $f = \frac{1}{c}\tilde{f}$
It follows $\frac{\partial u(x,t)}{\partial t} - \kappa \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t)$ for $0 < x < L$ $t > 0$

3.



$b(x) = \begin{bmatrix} 1 \\ x_1 x_2 \end{bmatrix}$ $\quad u(x) = \sin(2x_1)$

WTS

$$\int_\Omega b(x) \cdot \nabla u(x)\, d\Omega = \int_{\partial\Omega} b(x) \cdot n(x)\, u(x)\, ds - \int_\Omega \nabla \cdot b(x)\, u(x)\, d\Omega$$

(Γ_4) $\int_{\Gamma_4} \begin{bmatrix} 1 \\ x_1 x_2 \end{bmatrix} \cdot \hat{j} \sin(2x_1)\, ds$

$x_1 = 1 - t \quad x_2 = 1 \quad ds = \sqrt{(-1)^2}\, dt \qquad u = 1-t \quad dv = \sin 2(1-t)$

$\int_{\Gamma_4} (1-t)(1) \sin(2(1-t))\, dt \qquad du = -1 \quad v = \frac{-1}{2}\cos 2(1-t)$

$= -\frac{1-t}{2}\cos 2(1-t) + \frac{1}{4}\sin 2(1-t) \Big|_0^1 = \frac{-1}{2}\cos 2 + \frac{1}{4}\sin 2 \qquad V = \frac{1}{4}\sin 2(1-t)$

(Γ_3) $\int_{\Gamma_3} \begin{bmatrix} 1 \\ x_1 x_2 \end{bmatrix} \cdot \hat{i} \sin(2x_1)\, ds \qquad x_2 = -1 \quad t = 0$

$x_1 = 1 \quad x_2 = t - 1 \quad ds = \sqrt{1^2}\, dt \quad x_2 = 1 \quad t = 2$

$\int_{\Gamma_3} 1 \sin 2\, dt = t \sin 2 \Big|_0^2 = 2\sin 2$

(Γ_2) $\int_{\Gamma_2} \begin{bmatrix} 1 \\ x_1 x_2 \end{bmatrix} \cdot (-\hat{j}) \sin(2x_1)\, ds$

$\int_{\Gamma_2} -x_1 x_2 \sin 2x_1\, ds \qquad ds = \sqrt{1^2}\, dt \quad u = +t \quad dv = \sin 2t$

$x_2 = -1 \qquad x_1 = t \qquad du = +1 \quad v = \frac{-1}{2}\cos 2t$

$\int_{\Gamma_2} +t \sin 2t\, dt \qquad V = \frac{-1}{4}\sin 2t$

$= \frac{-t}{2}\cos 2t + \frac{1}{4}\sin 2t \Big|_0^1$

$= \frac{-1}{2}\cos 2 + \frac{1}{4}\sin 2 - 0 + \frac{1}{4}(0)$

(Γ_1) $\int_{\Gamma_1} \begin{bmatrix} 1 \\ x_1 x_2 \end{bmatrix} \cdot (-\hat{i}) \sin(2x_1)\, ds$

$= \int_{\Gamma_1} \sin 2x_1\, ds$

$x_1 = 0 \qquad x_2 = t + 1 \qquad ds = \sqrt{1^2}\, dt$

$\int_{\Gamma_1} \cancel{-\sin 2}\, dt = \cancel{-t \sin 2 \Big|_0^2} = \cancel{-2\sin 2} = 0$

(Ω_1) $\int_\Omega \nabla \cdot b(x)\, u(x)\, d\Omega = \int_\Omega x_1 \sin 2x_1\, d\Omega$

$= \int_{-1}^1 \int_0^1 x_1 \sin 2x_1\, dx_1\, dx_2$

$= \int_{-1}^1 -\frac{x_1}{2}\cos 2x_1 + \frac{1}{4}\sin 2x_1 \Big|_0^1 dx_2$

$= \int_{-1}^1 \frac{-1}{2}\cos 2 + \frac{1}{4}\sin 2\, dx_2$

$= \left(\frac{-1}{2}\cos 2 + \frac{1}{4}\sin 2\right) x_2 \Big|_{-1}^1$

$= -\cos 2 + \frac{1}{2}\sin 2$

(Ω_2) $\int_\Omega b(x) \cdot \nabla u(x)\, d\Omega = \int_{-1}^1 \int_0^1 \begin{bmatrix} 1 \\ x_1 x_2 \end{bmatrix} \cdot \begin{bmatrix} 2\cos 2x_1 \\ 0 \end{bmatrix} dx_1\, dx_2$

$= \int_{-1}^1 \int_0^1 2\cos 2x_1\, dx_1\, dx_2$

$= \int_{-1}^1 \sin 2x_1 \Big|_0^1 dx_2$

$= 2\sin 2$

3cont, Thus,

$$2\sin 2 = 2\left(-\frac{1}{2}\cos 2 + \frac{1}{4}\sin 2\right) + 2\sin 2 + 0 - \left(-\cos 2 + \frac{1}{2}\sin 2\right)$$

$$= -\cancel{\cos 2} + \cancel{\tfrac{1}{2}\sin 2} + 2\sin 2 + \cancel{\cos 2} - \cancel{\tfrac{1}{2}\sin 2}$$