MATH 8660 Sec.1: Homework No.5
Due: Tuesday, September 24

No.1. (10 pts.) Consider

$$-(k(x)u')' + b(x)u' + c(x)u = f(x), \ x \in \Omega := (\alpha, \beta), \tag{0.1}$$
$$u(\alpha) = 0, u(\beta) = 0. \tag{0.2}$$

(a) Show how to reformulate (0.1),(0.2) as $B(u, v) = F(v)$, where

$$B(w, v) := \int_\Omega \big(k(x)w'(x)v'(x) + b(x)w'(x)v(x) + c(x)w(x)v(x)\big)dx,$$
$$F(v) := \int_\Omega f(x)v(x)\,dx.$$

(b) Assume $0 < k_m \leq k(x) \leq k_M < \infty$ for all $x \in \Omega$. Give conditions on b and c in order that for all w, $v \in H_0^1(\Omega)$ there exists C_1, $C_2 > 0$ such that
(i) $|B(w, v)| \leq C_1 \|w\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$,
(ii) $|B(w, w)| \geq C_2 \|w\|^2_{H^1(\Omega)}$.
(c) Show that for $f \in L^2(\Omega)$, there exists $C_3 > 0$ such that $|F(v)| \leq C_3 \|v\|_{H^1(\Omega)}$. Hence conclude that as

$$\|F\| = \sup_{v \in H_0^1(\Omega)} \frac{|F(v)|}{\|v\|_{H^1(\Omega)}} \leq C_3 < \infty,$$

F is a bounded linear functional.
(d) Apply the Lax-Miligram theorem to conclude that there exists a unique solution $u \in H_0^1(\Omega)$ to $B(u, v) = F(v)$, for all $v \in H_0^1(\Omega)$.

No.1 cont. (4 pts.) Let S_N denote a partition of (α, β) into N equally spaced subintervals of length $h = (\beta - \alpha)/N$. Additionally, let $x_i = \alpha + (i - 1)h, \ i = 1, 2, \ldots, N + 1$, and $\phi_i(x)$ denote the continuous, piecewise affine function such that

$$\phi_i(x) = \begin{cases} 1, & \text{for } x = x_i, \\ 0, & \text{for } x = x_j, \ j \neq i. \end{cases}$$

Note that $S_N = \text{span}\{\phi_2(x), \phi_3(x), \ldots, \phi_N(x)\} \subset H_0^1(\alpha, \beta)$.

Discuss the Galerkin approximation $u_h \in S_N$ to u.