HW12

Monday, November 25, 2024 5:33 PM

No.1. (8 pts.) (6 pts.) Consider the mathematical model for the steady-state temperature distribution across a (sufficiently) smooth domain Ω:

$$-\nabla \cdot \nabla u = f(\mathbf{x}), \ \mathbf{x} \text{ in } \Omega, \quad (0.1)$$

$$u = 0, \text{ on } \partial\Omega, \quad (0.2)$$

where $f \in H^{5/4}(\Omega)$. Suppose a continuous piecewise quadratic Finite Element approximation is computed for (0.1), (0.2). Of interest is the average temperature:
(i) over the domain, i.e., $\frac{1}{|\Omega|}\int_\Omega u\,1\,d\Omega$, and
(ii) over a portion of the domain $\Omega_0 \subset \Omega$, i.e., $\frac{1}{|\Omega_0|}\int_\Omega u\,1_{\Omega_0}\,d\Omega$, where 1_{Ω_0} is the characteristic function on Ω_0.
Determine the expected rate of convergence for the FEM estimates to (i) and (ii).
(Recall for $h(x) = \begin{cases} 0, & 0 < x < 1/2, \\ 1, & 1/2 \le x < 1 \end{cases}$, that $h \in H^{1/2-\epsilon}(0,1)$ for any $\epsilon > 0$.)

i) Let $z \in D(\mathcal{L}^*) \ni \mathcal{L}^* z = 1$

$|\frac{1}{\Omega}\int_\Omega u-u_h\,d\Omega| = \frac{1}{|\Omega|}|\langle u-u_h, \mathcal{L}^* z\rangle| = \frac{1}{|\Omega|}|B(u-u_h, z)|$

Galerkin $\perp$ $= \frac{1}{|\Omega|}|B(u-u_h, z-z_h)|$

Bddness $\le \frac{C_t}{|\Omega|}\|u-u_h\|\|z-z_h\|$

Cea's Lemma $\le \frac{C_t}{|\Omega|}\inf\|u-v_h\|\inf\|z-z_h\|$

Choose z_h to be inf

Since $f \in H^{5/4}$, $u \in H^{13/4}$ by elliptic irregularity

$z \in H^s$ $\forall s$ since $\mathcal{L}^* z = 1$ and $1 \in C^\infty$

Since we are looking at quadratics in H^1,

$err \sim h^2|u|_{H^3}$ since $3 < \frac{13}{4}$

$err \sim h^2|z|_{H^3}$

Returning to the previous bound,

$|\frac{1}{\Omega}\int_\Omega u-u_h\,d\Omega| \le Ch^4\|u\|_{H^3}$

ii) Let $z \in D(\mathcal{L}^*) \ni \mathcal{L}^* z = 1_{\Omega_0}$ and $\varepsilon > 0$.

$z \in H^{1/2-\varepsilon}$ since $\mathcal{L}^* z = 1_{\Omega_0}$ and the discontinuities.

Similarly to i), we are looking at quadratics in H^1

$err \sim h^2|z|_{H^{1/2-\varepsilon}}$

It follows $\frac{C_t}{\quad}\inf\|u-v_h\|\inf\|z-z_h\| \le Ch^2\|u\|_{H^3}\,h^{-1/2}|z|_{H^{1/2}}$

It follows $\frac{C_t}{|\Omega|} \inf\|u-v_h\| \inf\|z-z_h\| \le C h^2 \|u\|_{H^3} h^{-1/2} |z|_{H^{1/2}}$

$$= C h^{7/2} \|u\|_{H^3}$$

Since $|z|_{H^{1/2}} \le \|u-u_h\|$

No.2. (5 pts.) Let $\Omega \subset \mathbb{R}^2$, $\mathbf{w}(\mathbf{x}) = [w_1(\mathbf{x}), w_2(\mathbf{x})]^T \in \left(H^1(\Omega)\right)^2$, and $\mathbf{v}(\mathbf{x}) = [v_1(\mathbf{x}), v_2(\mathbf{x})]^T \in \left(H_0^1(\Omega)\right)^2$.

$$\text{Notation: } \nabla \mathbf{w}(\mathbf{x}) = \begin{bmatrix} \frac{\partial w_1(\mathbf{x})}{\partial x_1} & \frac{\partial w_1(\mathbf{x})}{\partial x_2} \\ \frac{\partial w_2(\mathbf{x})}{\partial x_1} & \frac{\partial w_2(\mathbf{x})}{\partial x_2} \end{bmatrix},$$

$$\nabla \cdot \nabla \mathbf{w}(\mathbf{x}) = \begin{bmatrix} \nabla \cdot [\frac{\partial w_1(\mathbf{x})}{\partial x_1}, \frac{\partial w_1(\mathbf{x})}{\partial x_2}]^T \\ \nabla \cdot [\frac{\partial w_2(\mathbf{x})}{\partial x_1}, \frac{\partial w_2(\mathbf{x})}{\partial x_2}]^T \end{bmatrix},$$

scalar tensor product $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$.

(i) Show that

$$\nabla \cdot \nabla \mathbf{w}(\mathbf{x}) = \begin{bmatrix} \Delta w_1(\mathbf{x}) \\ \Delta w_2(\mathbf{x}) \end{bmatrix}.$$

(ii) Verify, by considering each component,

$$\int_\Omega -\nabla \cdot \nabla \mathbf{w}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\Omega = -\int_{\partial\Omega} \mathbf{v}(\mathbf{x}) \cdot \nabla \mathbf{w}(\mathbf{x})\mathbf{n}(\mathbf{x})\, ds + \int_\Omega \nabla \mathbf{w}(\mathbf{x}) : \nabla \cdot \mathbf{v}(\mathbf{x})\, d\Omega.$$

i) $\nabla \cdot \nabla W(x) = \nabla \cdot \begin{bmatrix} \frac{\partial w_1}{\partial x_1} & \frac{\partial w_1}{\partial x_2} \\ \frac{\partial w_2}{\partial x_1} & \frac{\partial w_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \nabla \cdot \left[\frac{\partial w_1}{\partial x_1} \quad \frac{\partial w_1}{\partial x_2}\right]^T \\ \nabla \cdot \left[\frac{\partial w_2}{\partial x_1} \quad \frac{\partial w_2}{\partial x_2}\right]^T \end{bmatrix}$

$$= \begin{bmatrix} \frac{\partial^2 w_1}{\partial x_1^2} & \frac{\partial^2 w_1}{\partial x_2^2} \\ \frac{\partial^2 w_2}{\partial x_1^2} & \frac{\partial^2 w_2}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} \Delta w_1 \\ \Delta w_2 \end{bmatrix}$$

ii) $\int_\Omega -\nabla \cdot \nabla w \cdot v \, d\Omega \stackrel{?}{=} \int_{\partial\Omega} v \cdot \nabla w \hat{n} \, ds + \int_\Omega \nabla w \cdot \nabla v \, d\Omega$

$$= \int_\Omega \begin{bmatrix} \nabla w_1 \\ \nabla w_2 \end{bmatrix} \cdot \begin{bmatrix} \nabla v_1 \\ \nabla v_2 \end{bmatrix} d\Omega - \int_{\partial\Omega} v \cdot \nabla w \hat{n} \, ds$$

$$= \int_\Omega \frac{\partial w_1}{\partial x_1}\frac{\partial v_1}{\partial x_1} + \frac{\partial w_1}{\partial x_2}\frac{\partial v_1}{\partial x_2} + \frac{\partial w_2}{\partial x_1}\frac{\partial v_2}{\partial x_1} + \frac{\partial w_2}{\partial x_2}\frac{\partial v_2}{\partial x_2} \, d\Omega - \int_{\partial\Omega} v \cdot \nabla w \hat{n} \, ds$$

$$= \int_\Omega \nabla w : \nabla v \, d\Omega - \int_{\partial\Omega} v \cdot \nabla w \hat{n} \, ds$$

No.3. (4 pts.) Let

$$X := \left(H^1(\Omega)\right)^n, \ Q := \{q \in L^2(\Omega) : \int_\Omega q \, d\Omega = 0\},$$
$$\text{and } V := \{\mathbf{z} \in X : \int_\Omega q \, \nabla \cdot \mathbf{z} \, d\Omega = 0, \ \forall q \in Q\}.$$

Show that V is a closed subspace of X.

Let $\{z_n\} \subseteq V \ni z_n \to z \in X$

$\left| \int q \nabla \cdot z \, d\Omega \right| \leq \overbrace{\|q\|}^{0} \|\nabla \cdot z\| = 0$ since $\|\nabla \cdot z\| < \infty$ because $z \in X$

Thus $z \in V$ and $\therefore$ V is a closed subspace of X.