Solution Homework No 11

No 2. $$L(u)(x) := -\nabla\cdot k(x)\nabla u(x) + \underset{\sim}{b}(x)\cdot\nabla u(\underset{\sim}{x}) + e(x)u(x) = f(x), \quad x\in\Omega \quad (0.1)$$

subject to
$$\begin{cases} u(x) = 0, & x\in\Gamma\subset\partial\Omega \\ -k(x)\dfrac{\partial u(x)}{\partial n} = g_2(x), & x\in\partial\Omega\setminus\Gamma \end{cases} \quad (0.2)$$

[This is a combination of §1.1 and §1.2 in Essential and Natural Boundary Conditions].

Multiplying (0.1) thru by a suitably nice $v(x)$ and integrating over Ω,

$$\underbrace{-\int_\Omega \nabla\cdot k(x)\nabla u(x)\,v(x)\,d\Omega}_{} + \int_\Omega \underset{\sim}{b}(x)\cdot\nabla u(\underset{\sim}{x})v(x)\,d\Omega + \int_\Omega e(x)u(x)v(x)\,d\Omega = \int_\Omega f(x)v(x)\,d\Omega \quad (0.3)$$

$$= -\int_{\partial\Omega} k(x)\frac{\partial u(x)}{\partial n}v(x)\,ds + \int_\Omega k(x)\nabla u(x)\cdot\nabla v(x)\,dx$$

$$= -\int_\Gamma k(x)\frac{\partial u}{\partial n}(x)v(x)\,ds - \int_{\partial\Omega\setminus\Gamma} k(x)\frac{\partial u}{\partial n}(x)v(x)\,ds + \int_\Omega k(x)\nabla u(x)\cdot\nabla v(x)\,d\Omega$$

$$= \underbrace{-\int_\Gamma k(x)\frac{\partial u}{\partial n}(x)v(x)\,ds}_{} + \int_{\partial\Omega\setminus\Gamma} g_2(x)v(x)\,ds + \int_\Omega k(x)\nabla u(x)\cdot\nabla v(x)\,d\Omega$$

As we have no information on $\frac{\partial u}{\partial n}(x)$ along Γ, we can make this term go away by imposing on v (i.e. the test space) $v|_\Gamma = 0$. Note that this matches (0.2)(a).

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(0.3) then becomes

$$\int_\Omega k(x)\nabla u(x)\cdot\nabla v(x)\,d\Omega + \int_\Omega \underset{\sim}{b}(x)\cdot\nabla u(x)\,v(x)\,d\Omega + \int_\Omega e(x)\,u(x)\,v(x)\,d\Omega$$
$$= \int_\Omega f(x)\,v(x)\,d\Omega - \int_{\partial\Omega\setminus\Gamma} g_2(x)\,v(x)\,ds. \qquad (0.4)$$

Now, for $k(x)$, $\underset{\sim}{b}(x)$, $e(x)$ sufficiently nice, all of the terms on the LHS of (0.4) are well defined for $u,v\in H^1(\Omega)$. In view of (0.2)(a) and the condition $v|_\Gamma=0$, in this case we have the same test and trial space:

$$H^1_\Gamma(\Omega)=\{w\in H^1(\Omega):\ w|_\Gamma=0\}$$

In terms of the "weak formulation", nothing is gained by integrating by parts the second integral on the L.H.S of (0.4). The first integral requires $u,v\in H^1$, so integrating by parts on the second integral will not "weaken" this requirement. For the first integral on the RHS of (0.4) we require $f\in(H^1_\Gamma(\Omega))'$ and the second integral $g_2\in(H^{1/2}_\Gamma(\partial\Omega))'$.

<u>Weak Formulation</u>: Given $f\in(H^1(\Omega))'$ and $g_2\in(H^{1/2}_\Gamma(\partial\Omega\setminus\Gamma))'$ determine $u\in H^1_\Gamma(\Omega)$ such that for all $v\in H^1_\Gamma(\Omega)$

$$B(u,v)=F(v)$$

where

$$B(w,v):=\int_\Omega k(x)\nabla w(x)\cdot\nabla v(x)\,d\Omega + \int_\Omega \underset{\sim}{b}(x)\cdot\nabla w(x)\,v(x)\,d\Omega + \int_\Omega e(x)\,w(x)\,v(x)\,d\Omega$$

and $F(v):=\langle f,v\rangle_{(H^1_\Gamma(\Omega))',\,H^1_\Gamma(\Omega)} - \langle\langle g_2,v\rangle\rangle_{(H^{\frac12}_\Gamma(\partial\Omega\setminus\Gamma))',\,H^{\frac12}_\Gamma(\partial\Omega\setminus\Gamma)}$

No3.
$$-\nabla\cdot k(x)\nabla u(x) + \underset{\sim}{b}(x)\cdot\nabla u(x) + c(x)u(x) = f \text{ in } \Omega \quad (0.3)$$
$$\text{s.t. } \frac{\partial u(x)}{\partial n} = g(x), \quad x\in\partial\Omega. \quad (0.4)$$

[This is similar to Example 2 in The Aubin-Nitsche Trick and Negative Norm Estimates]

Multiplying through by a suitably nice function $v(x)$ and integrating over Ω:

$$\underbrace{-\int_\Omega \nabla\cdot k(x)\nabla u(x)\, v(x)\, d\Omega}_{} + \int_\Omega b(x)\cdot\nabla u(x)\, v(x)\, d\Omega + \int_\Omega c(x)u(x)v(x)\, d\Omega = \int_\Omega f(x)v(x)\, d\Omega \quad (0.6)$$

$$= -\int_{\partial\Omega} k(x)\frac{\partial u(x)}{\partial n} v(x)\, ds + \int_\Omega k(x)\nabla u(x)\cdot\nabla v(x)\, d\Omega$$
$$= -\int_{\partial\Omega} k(x)\, g(x)\, v(x)\, ds + \int_\Omega k(x)\nabla u(x)\cdot\nabla v(x)\, d\Omega$$

Thus, (0.6) becomes

$$(0.7) \quad \int_\Omega k(x)\nabla u(x)\cdot\nabla v(x)\, d\Omega + \int_\Omega b(x)\cdot\nabla u(x)\, v(x)\, d\Omega + \int_\Omega c(x)u(x)v(x)\, d\Omega = \int_\Omega f(x)\, v(x)\, d\Omega + \int_{\partial\Omega} k(x)\, g(x)\, v(x)\, d\Omega.$$

We obtain the weak formulation: (See discussion above regarding the spaces) Given $f\in(H^1(\Omega))'$, $g\in(H^{1/2}(\partial\Omega))'$ determine $u\in H^1(\Omega)$ such that for all $v\in H^1(\Omega)$

$$B(u,v) = F(v) \longrightarrow$$

where $B(w,v) = \int_\Omega k(x)\,\nabla w(x)\cdot\nabla v(x)\,d\Omega + \int_\Omega \underset{\sim}{b}(x)\cdot\nabla w(x) v(x)\,d\Omega + \int_\Omega c(x)\,w(x)\,v(x)\,d\Omega$

$$F(v) = \langle f, v\rangle_{(H^1(\Omega))',\, H^1(\Omega)} + \langle\langle kg, v\rangle\rangle_{(H^{1/2}(\partial\Omega))',\, H^{1/2}(\partial\Omega)}.$$

(ii) $\mathcal{L} : H^1(\Omega) \to (H^1(\Omega))'$ defined by

$$\mathcal{L}(w)(\cdot) := -\nabla\cdot k(x)\nabla w(x) + \underset{\sim}{b}(x)\cdot\nabla w(x) + c(x)\,w(x).$$

(iii). We need to throw the derivatives off u in (0.7)

$$\int_\Omega k(x)\,\nabla u(x)\cdot\nabla v(x)\,d\Omega$$

$$= \int_{\partial\Omega} k(x)\frac{\partial v(x)}{\partial n}\,u(x)\,ds - \int_\Omega \nabla\cdot k(x)\nabla v(x)\,d\Omega.$$

$$\int_\Omega \underset{\sim}{b}(x)\circ\nabla u(x)\,v(x)\,d\Omega$$

$$= \int_{\partial\Omega} v(x)\,\underset{\sim}{b}(x)\cdot\underset{\sim}{n}(x).\,u(x)\,ds - \int_\Omega \nabla\cdot(\underset{\sim}{b}(x).v(x))\,u(x)\,d\Omega$$

$$= \int_{\partial\Omega} v(x)\,\underset{\sim}{b}(x)\cdot\underset{\sim}{n}(x)\,u(x)\,ds - \int_\Omega \left(\underset{\sim}{b}(x)\cdot\nabla v(x) + \nabla\cdot\underset{\sim}{b}(x)\,v(x)\right)u(x)\,d\Omega.$$

Thus the LHS of (0.7) becomes

$$\int_\Omega \underbrace{\left(-\nabla\cdot k(x)\nabla v(x) - \underset{\sim}{b}(x)\cdot\nabla v(x) + (c(x) - \nabla\cdot\underset{\sim}{b}(x))v(x)\right)}_{:=\mathcal{L}^*(v)} u(x)\,d\Omega + \underbrace{\int_{\partial\Omega}\left(k(x)\nabla v(x) + \underset{\sim}{b}(x)v(x)\right)\cdot\underset{\sim}{n}\,u(x)\,ds}_{\text{goes in B.C.}}$$

Thus $\mathcal{L}^* : H^1(\Omega) \to (H^1(\Omega))'$ is defined by

$$\mathcal{L}^*(v) = -\nabla\cdot k(x)\nabla v(x) - \underset{\sim}{b}(x)\cdot\nabla v(x) + (c(x) - \nabla\cdot\underset{\sim}{b}(x))\,v(x)$$

$\longrightarrow$

and an associated adjoint problem

$$-\nabla \cdot k(x) \nabla v(x) - \underset{\sim}{b}(x) \cdot \nabla v(x) + \left(c(x) - \nabla \cdot \underset{\sim}{b}(x)\right) v(x) = h(x), \text{ in } \Omega$$

subject to $\quad k(x) \dfrac{\partial v}{\partial n} + \underset{\sim}{b}(x) \cdot \underset{\sim}{n}(x)\, v(x) = 0 \quad$ on $\partial\Omega$.