Jacob Manning HW03

1. $\mathcal{H}$ w/ $\langle \cdot, \cdot \rangle$ real I.P. w/ $\|\cdot\|$
Let $f, g \in \mathcal{H}$

a) i) $g_{\parallel} = \frac{\langle g, f \rangle f}{\langle f, f \rangle}$ WTS $g_{\perp} = g - g_{\parallel} \perp f$

$$\langle g_{\perp}, f \rangle = \langle g, f \rangle - \langle g_{\parallel}, f \rangle$$
$$= \langle g, f \rangle - \frac{\langle g, f \rangle}{\cancel{\langle f, f \rangle}} \cancel{\langle f, f \rangle}$$
$$= 0$$

Thus $g_{\perp} \perp f$

ii) WTS C.S. given $\|g_{\perp}\| \geq 0$

By part i) since $g_{\perp} \perp f$ we can write $g = g_{\perp} + g_{\parallel}$ and

$$\|g\|^2 = \|g_{\perp}\|^2 + \|g_{\parallel}\|^2 \geq \|g_{\parallel}\|^2 = \left(\frac{\langle g, f \rangle}{\langle f, f \rangle}\right)^2 \langle f, f \rangle$$

since $\|g\| \geq 0$

It follows $\|g\|^2 \|f\|^2 \geq (\langle g, f \rangle)^2$

$$\Rightarrow |\langle g, f \rangle| \leq \|g\| \, \|f\|$$

b) WTS $\langle f, g \rangle = \frac{1}{2}(\|f\|^2 + \|g\|^2 - \|f - g\|^2)$

$$\frac{1}{2}(\|f\|^2 + \|g\|^2 - \langle f - g, f - g \rangle) = \frac{1}{2}(\|f\|^2 + \|g\|^2 - (\|f\|^2 + \|g\|^2 - 2\langle f, g \rangle))$$
$$= \frac{1}{2}(2\langle f, g \rangle) = \langle f, g \rangle$$

c) WTS Parallelogram

$\|f-g\|^2+\|f+g\|^2=\langle f-g,f-g\rangle+\langle f+g,f+g\rangle$

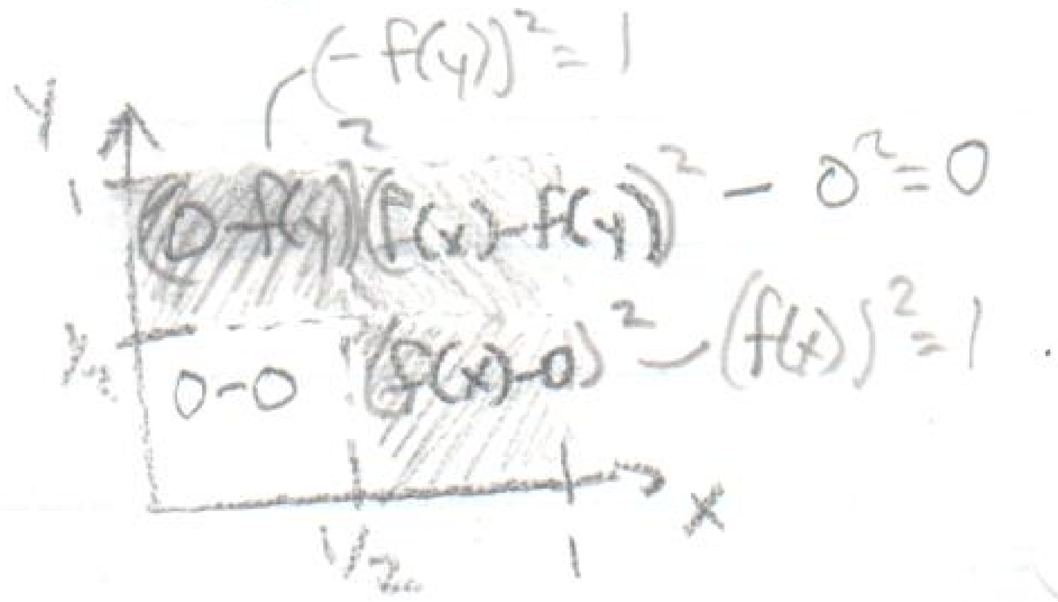
$=\|f\|^2+\|g\|^2-2\langle f,g\rangle+\|f\|^2+\|g\|^2+2\langle f,g\rangle$

$=2(\|f\|^2+\|g\|^2)$

2) $f(x)=\begin{cases}0 & 0<x<\frac{1}{2}\\ 1 & \frac{1}{2}\le x<1\end{cases}$

Use Sobolev-Slobedetskii norm to determine $s\ge 0 \ni f\in H^s(0,1)$

$|f|^2_{H^s(0,1)}=\int_0^1\int_0^1\frac{(f(x)-f(y))^2}{|x-y|^{1+2s}}\,dxdy$



The density is the same over both regions so, by semmetry

$|f|^2_{H^s(0,1)}=2\int_{1/2}^1\int_0^{1/2}\frac{1}{|x-y|^{1+2s}}\,dxdy$

We also know $y>x$ on $\Omega \Rightarrow |x-y|=|y-x|=y-x$

$\Rightarrow 2\int_{1/2}^1\int_0^{1/2}\frac{1}{(y-x)^{1+2s}}\,dxdy$ $\qquad y-x=u \Rightarrow du=-dx$

$=-2\int_{1/2}^1\int_y^{y-\frac{1}{2}} u^{-2s-1}\,dudy$ $\qquad y-\frac{1}{2}=v$

$=\frac{1}{s}\int_{1/2}^1 \frac{1}{u^{2s}}\Big|_y^{y-\frac{1}{2}}\,dy$

$=\frac{1}{s}\int_{1/2}^1 \frac{1}{(y-\frac{1}{2})^{2s}}-\frac{1}{y^{2s}}\,dy$

$=\frac{1}{s}\left(\frac{1}{-2s+1}\right)\left(v^{1-2s}\Big|_0^{1/2}-y^{1-2s}\Big|_{1/2}^1\right)$

$=\frac{1}{-2s^2+s}\left(\left(\frac{1}{2}\right)^{1-2s}-1+\left(\frac{1}{2}\right)^{1-2s}\right)=\frac{2(\frac{1}{2})^{1-2s}-1}{-2s^2+s}$

$-2s^2+s\neq 0$

$s(1-2s)\neq 0 \qquad s\neq 0,\frac{1}{2}$

$\int v^{1-2s}\,dv<\infty$ iff $1-2s>0 \Rightarrow \frac{1}{2}>s \Rightarrow 0<s<\frac{1}{2}$

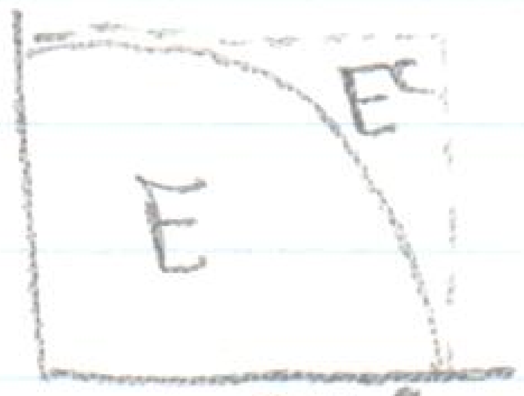
$f(x)=\delta(x-\frac{1}{2}) \Rightarrow f'(x)\in L^2$ because $f'(x)=\delta(x-\frac{1}{2})$

Similary $f^{n+1}\in L^2$ $\forall n\in\mathbb{N}$ because $\delta(x-\frac{1}{2})\in H^k$ $\forall k\in\mathbb{N}$

2cont. Consider $|f'|^2_{H^s(0,1)} = \int_\Omega \frac{(0-0)^2}{|x-y|^{1+2s}} d\Omega = 0 < \infty$

Thus we only need to restrict s on $0 < s < 1/2$

3a.



$\Omega = (0,1) \times (0,1)$ $\quad f(\bar{x}) = (x^2+y^2)^{r/2}$

Let E be the semi circle

$\|f\|_2^2 = \int_\Omega f(\bar{x})^2 d\Omega = \int_E f(\bar{x})^2 d\mu + \int_{E^c} f(\bar{x})^2 d\mu$

$\int_E f(\bar{x})^2 d\mu = \int_0^{\pi/4} \int_0^1 \rho^{2r+1} d\rho\, d\theta$

$= \frac{\pi}{4} \frac{1}{2r+2} < \infty$ if $2r+1 \neq -1 \Rightarrow r \neq -1$

Since if $r = -1$, $\frac{\pi}{4} \int_0^1 \frac{d\rho}{\rho} = \frac{\pi}{4}[\ln 1 - \ln 0] = \infty$

$\int_{E^c} f(\bar{x})^2 d\mu = \int_{E^c} (x^2+y^2)^r d\mu$

If $r \geq 0$ $\int_{E^c} (x^2+y^2)^r d\mu \leq \int_{E^c} 2^r d\mu \leq \int_\Omega 2^r d\Omega < \infty$

Since $(x^2+y^2)^r \leq 2^r$ on Ω $\quad$ if $r < \infty$

If $r < 0$ $\int_{E^c} (x^2+y^2)^r d\mu \leq m(E^c) < \infty$ $\forall r < 0$

since $\overline{(x^2+y^2)^r} < 1$ $\forall (x,y) \in \Omega$

b. $\nabla f = \{2\frac{r}{2} x (x^2+y^2)^{r/2 - 1}, 2\frac{r}{2} y (x^2+y^2)^{r/2-1}\}$

Thus $2(r/2 - 1) \neq -1 \Rightarrow r - 2 \neq -1 \Rightarrow r \neq 1$

c. It seems the restriction on r increases as s ($H^s(\Omega)$) increases. It also seems as the spacial dimension increases, the gap inbetween the "r's" increase as well