1. $-\varepsilon y'' + y' = 0 \quad x \in (0,1) := I \quad y_0 = 1 \quad y_1 = 0$

$-\varepsilon r^2 + r = 0$

$r(r - \frac{1}{\varepsilon}) = 0$

$r = 0 \quad r = \frac{1}{\varepsilon}$

Thus $y = c_1 + c_2 e^{x/\varepsilon}$

$1 = c_1 + c_2$

$0 = c_1 + c_2 e^{1/\varepsilon}$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & e^{1/\varepsilon} & 0 \end{array}\right] \Rightarrow \begin{array}{l} c_1 = \frac{e^{1/\varepsilon}}{e^{1/\varepsilon}-1} \\ c_2 = \frac{-1}{e^{1/\varepsilon}-1} \end{array}$$

Thus $y(x) = \frac{e^{1/\varepsilon}}{e^{1/\varepsilon}-1} - \frac{e^{x/\varepsilon}}{e^{1/\varepsilon}-1}$

2a. $-\varepsilon \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} + \frac{y(x+h) - y(x-h)}{2h} = 0$

$\left(\frac{-\varepsilon}{h^2} - \frac{1}{2h}\right) y_{i-1} + \left(\frac{2\varepsilon}{h^2}\right) y_i + \left(\frac{-\varepsilon}{h^2} + \frac{1}{2h}\right) y_{i+1} \quad 2 \le i \le N$

$y_1 = 1 \quad y_{N+1} = 0$

b. $\varepsilon = 1/20 \quad N_1 = 20 \quad N_2 = 40 \quad \Delta x_i = \frac{1-0}{N}$

See code section

3. $N \in \mathbb{N} \quad 0 = x_1 < x_2 < \ldots < x_{N+1} = 1$ partition of I. S_N continuous & piecewise linear subject to I.

WTS $\{\phi\}_{i=1}^{N+1}$ is a basis for S_N

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} \le x \le x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & x_i \le x \le x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Consider $\sum c_i \phi_i(x_j) = c_j$

Thus $\sum c_i \phi_i(x_j) = 0$ only if $c_j = 0$.

Therefore since we picked an arbitrary x_j,

$\sum c_i \phi_i = 0 \Rightarrow \{c_i\} = 0 \Rightarrow \{\phi_i\}$ is linearly independent

3 cont Let $f \in S_N$. Thus we can denote $f_i = f(x_i)$ and f will be uniquely denoted by $\{f_i\}$.

Consider $f(x) = \sum c_i \phi_i(x)$. To determine $\{c_i\}$ evaluate at x_j

$$f(x_j) = \sum c_i \phi_i(x_j) = c_j$$

Thus $c_j = f(x_j)$ and since we chose an arbitiary x_j, $\{c_i\} = \{f_i\}$.

Since, as noted before, $f(x)$ is uniquely defined by $\{f_i\}$, we can express f as the sum of $\{\phi_i\}$.

As the sum of piecewise linear functions, f is also piecewise linear.

Let $\varepsilon > 0$ $\delta > 0$ and $|x - x'| < \delta$ $\quad \varepsilon = \frac{f_i \delta}{h(N+1)}$

(I) $x, x' \in [x_{i-1}, x_i]$

$$\begin{aligned} |f(x) - f(x')| &= |\sum f_i \phi_i(x) - \sum f_i \phi_i(x')| \\ &= |\sum \tfrac{f_i}{h}(x - \cancel{x_{i-1}} - x' + \cancel{x_{i-1}})| \\ &< |\sum \tfrac{f_i}{h} \delta| \\ &= |\sum \varepsilon/(N+1)| \\ &= \varepsilon \end{aligned}$$

(C II)

$$\begin{aligned} |f(x) - f(x')| &= |\sum f_i \phi_i(x) - f_i \phi_i(x')| \\ &= |\sum \tfrac{f_i}{h}(\cancel{x_{i+1}} - x - \cancel{x_{i+1}} + x')| \\ &< |\sum \tfrac{f_i \delta}{h}| \\ &= \varepsilon \end{aligned}$$

Thus f is continuous

4. $y_N(x) = \phi_1(x) + \sum_{j=2}^{N} c_j \phi_j(x) + 0\phi_{N+1}(x)$

$y'_N(x) = \phi'_1(x) + \sum_{j=2}^{N} c_j \phi'_j(x)$

For fixed i

$$\varepsilon \int_0^1 y'_N(x)\phi'_i(x)dx + \int_0^1 y'_N(x)\phi_i(x)dx = 0$$

$$\varepsilon \int_0^1 \sum_{j=1}^{N} c_j \phi'_j(x)\,\phi'_i(x)dx = \sum_{j=1}^{N} c_j \int_0^1 \varepsilon \phi'_j(x)\,\phi'_i(x)dx$$

As discussed in class $\int_0^1 \phi'_j(x)\phi'_i(x)dx = \begin{cases} \frac{2}{h} & j=i \\ -\frac{1}{h} & j=i-1 \\ -\frac{1}{h} & j=i+1 \\ 0 & \text{otherwise} \end{cases}$

$$\int_0^1 \sum_{j=1}^{N} c_j \phi'_j(x)\phi_i(x)dx = \sum_{j=1}^{N} c_j \int_0^1 \phi'_j(x)\phi_i(x)dx$$

Similarly we know

$$\int_0^1 \phi'_j(x)\phi_i(x)dx = \begin{cases} 0 & i=j \\ -\frac{1}{2} & j=i-1 \\ \frac{1}{2} & j=i+1 \\ 0 & \text{otherwise} \end{cases}$$

It follows (fixing $c_1=1$)

$$c_{j-1}\left(-\frac{\varepsilon}{h}-\frac{1}{2}\right) + c_j\left(\frac{2\varepsilon}{h}\right) + c_{j+1}\left(-\frac{\varepsilon}{h}+\frac{1}{2}\right)$$

Which is only off by a factor of $\frac{1}{h}$ over all c_i's which will not change the set of solutions for the resulting matrices

$\therefore$ the linear systems are equivalent

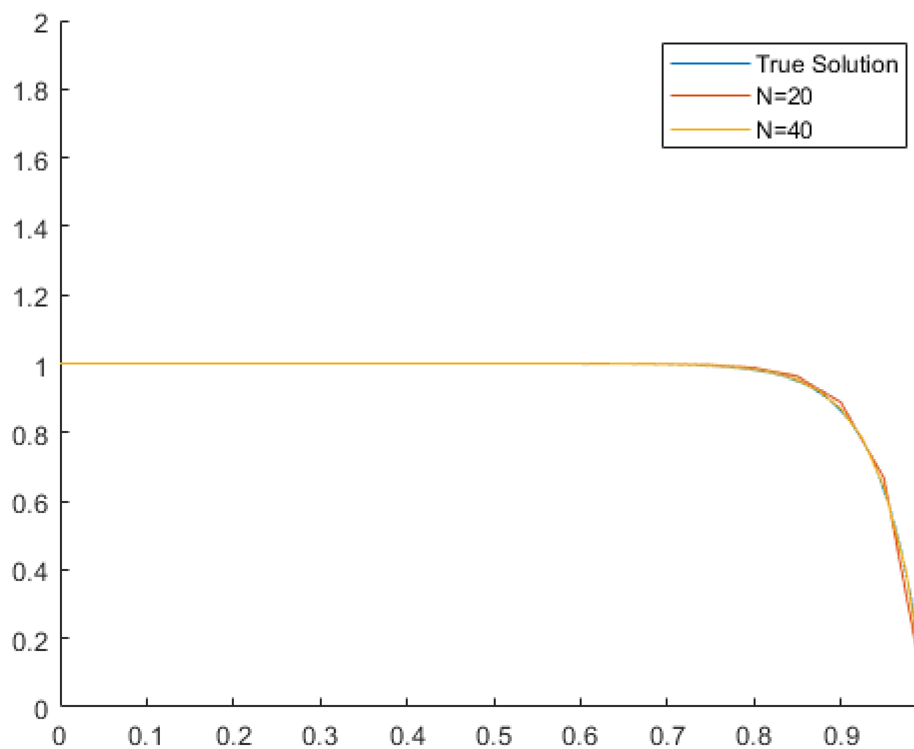
5. See code

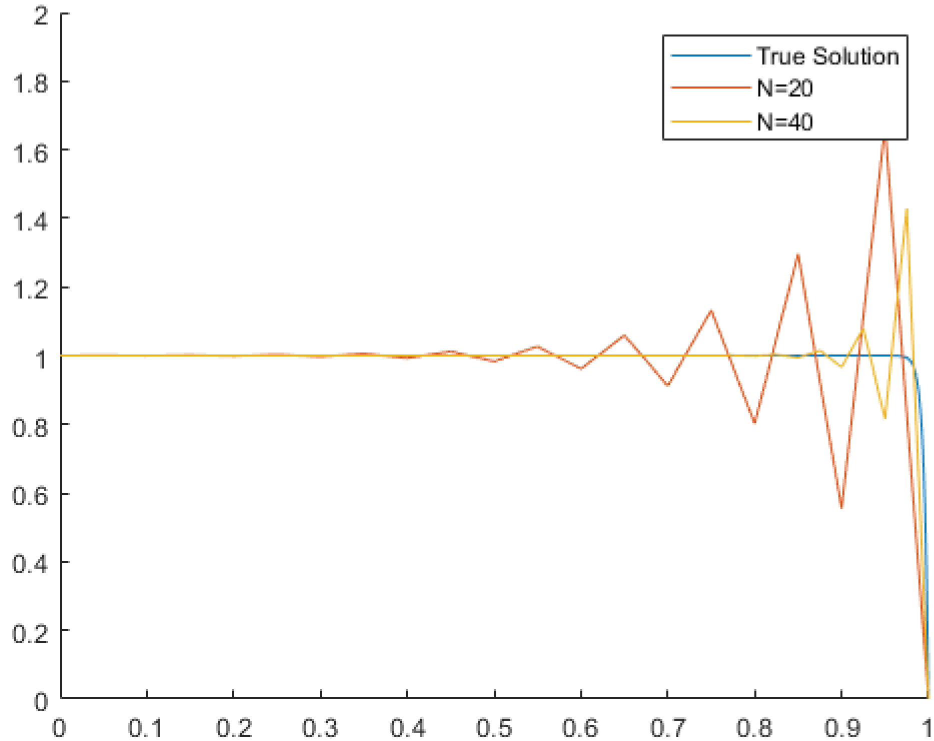
Both solutions blow up. The $N=20$ solution blows up first and $N=40$ blows up near the end

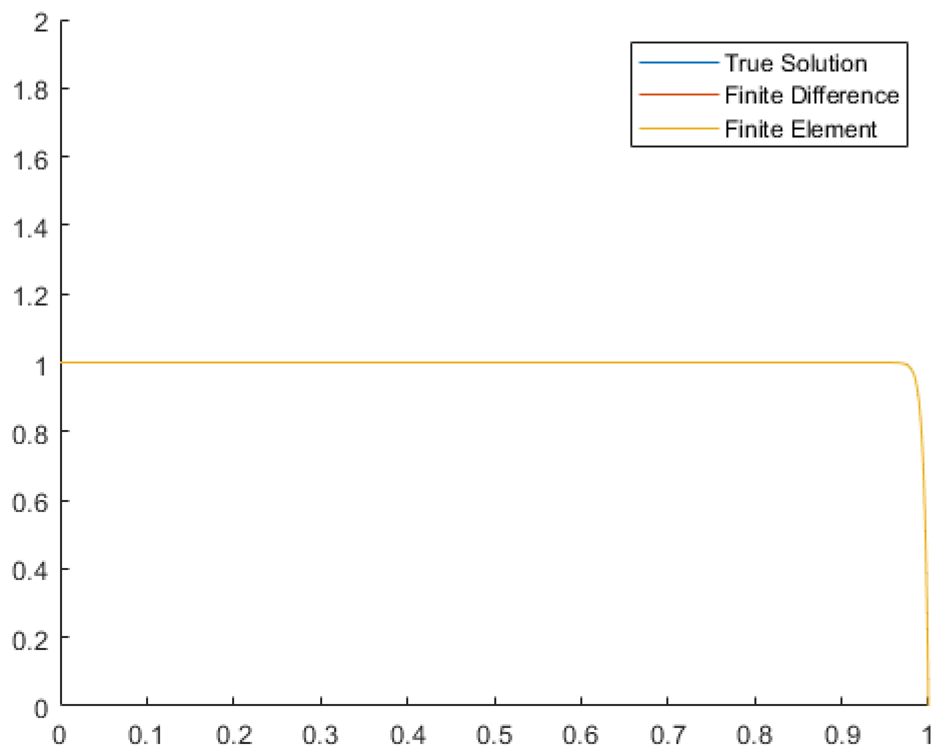
6. The approximation is very good, I had to double check to make sure I was graphing every plot.

```
%Problem 2b
e2=1/20;

x21=fd(e2,20);
x22=fd(e2,40);

figure
hold on
fplot(@(x) exp(1/e2)/(exp(1/e2)-1)-exp(x/e2)/(exp(1/e2)-1))
plot(linspace(0,1,21), x21)
plot(linspace(0,1,41), x22)
axis([0 1 0 2])
hold off

legend('True Solution','N=20','N=40')

%Problem 5
e5=1/200;

x51=fd(e5,20);
x52=fd(e5,40);

figure
hold on
fplot(@(x) exp(1/e5)/(exp(1/e5)-1)-exp(x/e5)/(exp(1/e5)-1))
plot(linspace(0,1,21), x51)
plot(linspace(0,1,41), x52)
axis([0 1 0 2])
hold off

legend('True Solution','N=20','N=40')

%Problem 6

x61=fd(e5,1000);
x62=fe(e5,1000);

figure
hold on
fplot(@(x) exp(1/e5)/(exp(1/e5)-1)-exp(x/e5)/(exp(1/e5)-1))
plot(linspace(0,1,1001), x61)
plot(linspace(0,1,1001), x62)
axis([0 1 0 2])
hold off

legend('True Solution','Finite Difference','Finite Element')

function xs=fd(ep,N)
    h=1/N;
    B=diag([1 2*ep/h^2*ones(1,N-1) 1])+diag((-ep/h^2+1/
(2*h))*ones(1,N),1)+diag((-ep/h^2-1/(2*h))*ones(1,N),-1);
    B(1,2)=0;
```

```
    B(N+1,N)=0;
    s=zeros(N+1,1);
    s(1)=1;

    xs=B\s;
end

function xs=fe(ep,N)
    h=1/N;
    B=diag([1 2*ep/h*ones(1,N-1) 1])+diag((-ep/h+1/2)*ones(1,N),1)+diag((-
ep/h-1/2)*ones(1,N),-1);
    B(1,2)=0;
    B(N+1,N)=0;
    s=zeros(N+1,1);
    s(1)=1;

    xs=B\s;
end
```




True Solution
N=20
N=40
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


True Solution
Finite Difference
Finite Element
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

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