MATH 8660 Sec.1: Homework No.11

Due: Tuesday, November 19 at 8:05am.

No.1. (8 pts.) Let H_1, H_2, $B(\cdot,\cdot)$ and F be as described in the Brezzi-Nečas-Babuška Theorem. Additionally, let $H_{1,h}$ and $H_{2,h}$ be closed subspaces of H_1 and H_2, respectively.
(i) Does it immediately follow from the Brezzi-Nečas-Babuška Theorem that there exists a unique $u_h \in H_{1,h}$ satisfying $B(u_h, v_h) = F(v_h)\,,\ \ \forall v_h \in H_{2,h}$?
If not, what additional assumptions are needed??

(ii) Follow the proof of Céa's Lemma to establish an error bound for $\|u - u_h\|_{H_1}$.

No.2. (8 pts.) Consider the problem of determining u satisfying

$$\mathcal{L}(u)(x) := -\nabla \cdot k(\mathbf{x})\nabla u(\mathbf{x}) + \mathbf{b}(\mathbf{x}) \cdot \nabla u(\mathbf{x}) + e(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x})\,,\ \mathbf{x} \in \Omega\,, \tag{0.1}$$

$$\text{subject to} \left\{ \begin{array}{rcl} u(\mathbf{x}) & = & 0\,,\ \ \mathbf{x} \in \Gamma \subset \partial\Omega\,, \\ -k(x)\,\frac{\partial u(\mathbf{x})}{\partial \mathbf{n}} & = & g_2(\mathbf{x})\,,\ \ \mathbf{x} \in \partial\Omega\backslash\Gamma \end{array} \right. . \tag{0.2}$$

Give the weak formulation for (0.1), (0.2).

No.3. (8 pts.) Consider the boundary value problem: *Given f and g determine u satisfying*

$$-\nabla \cdot (k(\mathbf{x})\,\nabla u(\mathbf{x})) + \mathbf{b}(\mathbf{x}) \cdot \nabla u(\mathbf{x}) + c(\mathbf{x})\,u(\mathbf{x}) = f \ \ \text{in } \Omega\,, \tag{0.3}$$

$$\text{subject to } \frac{\partial u(\mathbf{x})}{\partial \mathbf{n}} = g(\mathbf{x})\,,\ \ \mathbf{x} \in \partial\Omega\,. \tag{0.4}$$

(i) Derive the weak formulation for the solution of (0.3), (0.4), $B(u, v) = F(v)$.
(ii) Identify $\mathcal{L} : X \longrightarrow Y$ such that $\mathcal{L}(w)(\cdot) = B(w, \cdot)$.
(iii) Determine the adjoint operator, $\mathcal{L}^*$, of $\mathcal{L}$, and give an associated adjoint problem to (0.3), (0.4).