HW13

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The Darcy fluid flow equations (for modeling fluid flow through a porous medium) are: Given $\mathbf{f} \in L^2(\Omega)$, $\beta \in \mathbb{R}^+$, determine $\mathbf{u}$, p satisfying

$$\begin{aligned} \beta\mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega, & (1)\\ \nabla\cdot\mathbf{u} &= 0, \quad \text{in } \Omega, & (2)\\ \mathbf{u}\cdot\mathbf{n} &= 0, \quad \text{on } \partial\Omega, & (3)\end{aligned}$$

where $\mathbf{n}$ denotes the unit outer normal to $\partial\Omega$.

The function spaces of interest are:

$$\text{for the velocity}: \quad X = \mathbf{H}_0^{\mathbf{div}}(\Omega) := \{\mathbf{v} \in L^2(\Omega) : \nabla\cdot\mathbf{v} \in L^2(\Omega),\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\},$$

$$\text{for the pressure}: \quad Q = L_0^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q\, d\Omega = 0\},$$

$$\text{with associated norms } \|\mathbf{v}\|_X := \left(\|\mathbf{v}\|^2_{L^2(\Omega)} + \|\nabla\cdot\mathbf{v}\|^2_{L^2(\Omega)}\right)^{1/2}, \quad \|q\|_Q := \|q\|_{L^2(\Omega)}.$$

The weak formulation of (1) - (3): *Given* $\mathbf{f} \in L^2(\Omega)$, *determine* $(\mathbf{u}, p) \in X \times Q$ *satisfying for all* $(\mathbf{v}, q) \in X \times Q$

$$\begin{aligned} A(\mathbf{u},\mathbf{v}) + B(\mathbf{v},p) &= (\mathbf{f},\mathbf{v}),\\ B(\mathbf{u},q) &= 0, \end{aligned} \qquad (4)$$

where $A(\cdot,\cdot) : X \times X \longrightarrow \mathbb{R}$, and $B(\cdot,\cdot) : X \times Q \longrightarrow \mathbb{R}$ are defined by

$$A(\mathbf{w},\mathbf{v}) := \int_\Omega \mathbf{w}\cdot\mathbf{v}\, d\Omega, \quad B(\mathbf{v},q) := -\int_\Omega q\,\nabla\cdot\mathbf{v}\, d\Omega.$$

Let $V := \{\mathbf{z} \in X : \int_\Omega q\,\nabla\cdot\mathbf{z}\, d\Omega = 0,\ \forall q \in Q\}$.

No.1 (12 pts.).

(a) (4 pts.). Show that

(i) $|A(\mathbf{w},\mathbf{v})| \le C_1 \|\mathbf{w}\|_X \|\mathbf{v}\|_X, \quad \forall\, \mathbf{w}, \mathbf{v} \in X$.

(ii) $|A(\mathbf{w},\mathbf{w})| \ge C_2 \|\mathbf{w}\|^2_X, \quad \forall\, \mathbf{w} \in V$.

NOTE: (ii) ONLY holds for $\mathbf{w} \in V$.

(b) (4 pts.). Show that

(iii) $|B(\mathbf{v},q)| \le C_3 \|\mathbf{v}\|_X \|q\|_Q, \quad \forall\, \mathbf{v} \in X,\ q \in Q$.

(iv) $\sup_{q\in Q} B(\mathbf{v},q) > 0, \quad \forall \mathbf{v} \in V^\perp$.

(c) (4 pts.). Assume

(v) $\sup_{\mathbf{v}\in V^\perp} \frac{|B(\mathbf{v},q)|}{\|\mathbf{v}\|_X} \ge C_5 \|q\|_Q, \quad \forall\, q \in Q$.

Show how (i) – (v) guarantee the existence and uniqueness of solution to (4).

(a i) $|\int w\cdot v\, d\Omega| \le \|w\|_2 \|v\|_2 \le \|w\|_X \|v\|_X \qquad \forall\, w, v \in X$

ii) $|A(w,w)| = |\int w\cdot w\, d\Omega| = \|w\|_2^2$

From the lemma in class, $\|\nabla\cdot w\|_2^2 = 0$

Thus, $\|w\|_2^2 = \|w\|_X^2 \qquad \forall\, w \in V$

$\therefore\ |A(w,w)| = \|w\|_X^2$

b iii) $|B(v,q)| = |\int q\,\nabla\cdot v\, d\Omega| \le \|\nabla\cdot v\|_2 \|q\|_Q \le \|v\|_X \|q\|_Q \qquad \forall\, v \in X \quad q \in Q$

iv) Let $q = \nabla\cdot v - \frac{1}{|\Omega|}\int \nabla\cdot v\, d\Omega$

By the definition of B, $v \in H_1 = V \oplus V^\perp$

So we can write $v = v_1 + v_2$ and $\int q\,\nabla\cdot v\, d\Omega = \int q\,\nabla\cdot v_1\, d\Omega + \int q\,\nabla\cdot v_2\, d\Omega$

$= \int q\,\nabla\cdot v_2\, d\Omega$

We can thus assume WLOG $v \in V^\perp$ in this context.

It follows, $|\int q\, d\Omega| = |\int \nabla\cdot v - \frac{1}{|\Omega|}\int \nabla\cdot v\, d\Omega\, d\Omega|$

$= |\int \nabla\cdot v\, d\Omega - \frac{1}{|\Omega|}\int\int \nabla\cdot v\, d\Omega\, d\Omega|$

$= 0$

$\therefore\ q \in Q$

$\sup_{\tilde{q}\in Q} |B(v,\tilde{q})| \ge |\int q\,\nabla\cdot v\, d\Omega| > 0$

$\sup_{\tilde q \in Q} |B(v,\tilde q)| \geq |\int q \nabla \cdot v \, d\Omega| > 0$

since $v \in V^\perp$

(v) by assuming $\sup_{v \in V^\perp} \frac{|B(v,q)|}{\|v\|_X} \geq c_5 \|q\|_Q \quad \forall q \in Q$

Let $\langle f, v \rangle = F(v)$

It follows since V is a closed subspace of X, V is a Hilbert space.

By Lax-Milgram $\exists!\, u \in V \ni$

$$A(u,v) = F(v) \quad \forall\, v \in V \quad \text{since } A \text{ fits } i \,\&\, ii$$

Let $\tilde F(v) = F(v) - A(u,v)$.

by iii - v and BNB $\exists!\, p \in Q \ni$

$$B(v,p) = \tilde F(v) \qquad \forall\, v \in V^\perp$$

Thus $\exists!$ u and p satisfying (4)

No.2 (16 pts.). Discrete approximation.

(a) (4 pts.). Assume $X_h \subset X$, and $Q_h \subset Q$. Discuss the existence and uniqueness of $(\mathbf{u}_h, p_h) \in X_h \times Q_h$ satisfying

$$A(\mathbf{u}_h, \mathbf{v}_h) + B(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in X_h,$$
$$B(\mathbf{u}_h, q_h) = 0, \quad \forall q_h \in Q_h.$$

In particular, are the conditions (i)–(v) sufficient to guarantee the existence and uniqueness of $(\mathbf{u}_h, p_h) \in X_h \times Q_h$?

Let $V_h := \{\mathbf{z}_h \in X_h : \int_\Omega q_h \nabla \cdot \mathbf{z}_h \, d\Omega = 0, \ \forall q_h \in Q_h\}$.

(b) (6 pts.). Derive the error estimate for the velocity

$$\|\mathbf{u} - \mathbf{u}_h\|_X \leq C \left(\inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_X + \inf_{q_h \in Q_h} \|p - q_h\| \right).$$

(c) (6 pts.). Derive the error estimate for the pressure

$$\|p - p_h\|_Q \leq C \left(\inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_X + \inf_{q_h \in Q_h} \|p - q_h\| \right).$$

2a) We need to assume that the conditions hold for V_h and $V_h^\perp$ since these are not subsets of V and $V^\perp$ as well as X_h and Q_h.

This will guarentee the existance and uniqueness in $X_h \times Q_h$

b) We know $A(u,v) + B(v,p) = F(v) \ \forall\, v \in X$ and $A(u_h, v_h) = F(v_h) \ \forall v_h \in V_h \subseteq X$

$$A(u - u_h, v_h) = F(v_h) - F(v_h) - B(v_h, p)$$

By Galerkin Orthogonality,

$$A(u - u_h, v_h) = B(v_h, q_h - p) \quad \forall q_h \in Q_h$$

Let $\eta_h \in V_h$ and $\chi \in X \setminus V_h$

where $\chi = u - w_h$ and $\eta_h = w_h - u_h$ for some $w_h \in V_h$

It follows $u - u_h = \chi + \eta_h$ and

$$C_2 \|\eta_h\|^2 \leq |A(\eta_h, \eta_h)| = |-A(\chi, \eta_h) - B(\eta_h, p - q_h)| \leq C_1 \|\chi\|_X \|\eta_h\|_X + C_3 \|\eta_h\|_X \|p - q_h\|_Q$$

Finally $\|u-u_h\|_X \le \|\chi\|_X + \|\eta_h\|_X$
$$\le \|\chi\|_X + \frac{c_1}{c_2}\|\chi\|_X + \frac{c_3}{c_2}\|p-q_h\|_Q$$

Let $C = \max\{1+\frac{c_1}{c_2}, \frac{c_3}{c_2}\}$

Since w_h and q_h were arbitrary, taking the infinum we get
$$\|u-u_h\|_X \le C\left(\inf_{v_h\in V_h}\|u-v_h\|_X + \inf_{q_h\in Q_h}\|p-q_h\|_Q\right)$$

c) Note $B(v_h,p_h) = F(v_h) - A(u_h,v_h)$ and $F(v_h) = A(u,v_h) + B(v_h,p)$

Let $q_h \in Q_h$

By v) $C_5\|q_h-p_h\|_Q \le \sup_{v_h\in V_h^\perp} \frac{|B(v_h,q_h-p_h)|}{\|v_h\|_X}$
$$= \sup_{v_h\in V_h^\perp} \frac{|B(v_h,q_h)+A(u_h,v_h)-F(v_h)|}{\|v_h\|_X}$$
$$= \sup_{v_h\in V_h^\perp} \frac{|B(v_h,q_h)+A(u_h,v_h)-A(u,v_h)-B(v_h,p)|}{\|v_h\|_X}$$
$$= \sup_{v_h\in V_h^\perp} \frac{|B(v_h,q_h-p)+A(u_h-u,v_h)|}{\|v_h\|_X}$$
$$\le C_1\|u_h-u\|_X + C_3\|q_h-p\|_Q$$

It follows
$$\|p-p_h\|_Q \le \|p-q_h\|_Q + \|q_h-p_h\|_Q$$
$$\le \|p-q_h\|_Q + \frac{C_1}{C_5}\|u-u_h\|_X + \frac{C_3}{C_5}\|p-q_h\|_Q$$

Let $C = \max\{1+\frac{C_3}{C_5}, \frac{C_1}{C_5}\}$

Since u_h and q_h were arbitrary, taking the infinum we get
$$\|p-p_h\|_Q \le C\left(\inf_{v_h\in V_h}\|u-v_h\|_X + \inf_{q_h\in Q_h}\|p-q_h\|_Q\right)$$