# MATH 8660 Sec.1: Homework No.3

**Due: Tuesday, September 10**

### Hilbert Spaces and $H^s(\Omega)$

**No.1**. (8 pts.) Let $H$ denote a Hilbert space, $\langle \cdot, \cdot \rangle$ its associated inner product, and $\| \cdot \|$ its associated norm. Let $f, g \in H$.

(a) (i) Let $g_{\|} = \frac{\langle g,f \rangle}{\langle f,f \rangle} f$. Show that $g_{\perp} = g - g_{\|}$ is orthogonal to f.

(ii) Use the fact that $\|g_{\perp}\| \geq 0$ to establish the Cauchy-Schwartz inequality, $|\langle g, f \rangle| \leq \|g\| \, \|f\|$.

(b) Show that: $\langle f, g \rangle = \frac{1}{2}\|f\|^2 + \frac{1}{2}\|g\|^2 - \frac{1}{2}\|f - g\|^2$ .

(c) Show that (Parallelogram equality): $\|f - g\|^2 + \|f + g\|^2 = 2\|f\|^2 + 2\|g\|^2$ .

**No.2**. (4 pts.)

$$\text{Let} \quad f(x) = \begin{cases} 0, & 0 < x < \frac{1}{2} \\ 1, & \frac{1}{2} \leq x < 1 \end{cases} .$$

Using the definition of the Sobolev-Slobedetskii norm, determine the values of $s \geq 0$ such that $f \in H^s(0,1)$.

Hint: For $0 < s < 1$, begin by showing

$$|f|^2_{H^s(0,1)} = \int_{y=0}^{1} \int_{x=0}^{1} \frac{(f(x) - f(y))^2}{|x - y|^{1+2s}} \, dx \, dy = 2 \int_{y=1/2}^{1} \int_{x=0}^{1/2} \frac{1}{|x - y|^{1+2s}} \, dx \, dy \, .$$

**No.3**. (8 pts.) Point singularities in $\mathbb{R}^2$.

(Recall that in $\mathbb{R}^1$ we have that $f(x) = x^r \in H^s(0,1)$ for $r > s - \frac{1}{2}$. )

Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$, and $f(\mathbf{x}) = (x^2 + y^2)^{r/2}$.

(a) Determine $r$ such that $f(\mathbf{x}) \in L^2(\Omega)$.

(b) Determine $r$ such that $f(\mathbf{x}) \in H^1(\Omega)$.

(c) Comment on the connection between point singularities on a function, the Hilbert space it lies in, and how it depends on the space dimension $\mathbb{R}^n$.

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