HW9

Sunday, October 27, 2024 3:24 PM

No.1. (3 pts.) Let G denote a closed subspace of a Hilbert space H. Show that $G^{\perp}$ is a closed subspace of H.

Let $\{\varphi_n\} \subseteq G^{\perp} \ni \varphi_n \to \varphi \in H$ Let $z \in G$

WTS $\langle \varphi, z \rangle = 0$

$0 = \langle 0, z \rangle = \lim \langle \varphi - \varphi_n, z \rangle = \langle \varphi, z \rangle - \underbrace{\lim \langle \varphi_n, z \rangle}^{0} = \langle \varphi, z \rangle$

Thus $\varphi \in G^{\perp} \Rightarrow G^{\perp}$ is closed

No.2. (3 pts.) Suppose $H = \mathbb{R}^3$, endowed with the usual vector dot product, and $G = span\{[1,2,3]^T\}$. Give an explicit representation for $G^{\perp}$.

Since $\dim H = 3$ and $\dim G = 1$, $\dim G^{\perp} = 2$

Let $[1,2,3]^T = v^T$ and $u^1, u^2 \in G^{\perp}$

$v^T u = 0$

$u_1 + 2u_2 + 3u_3 = 0$

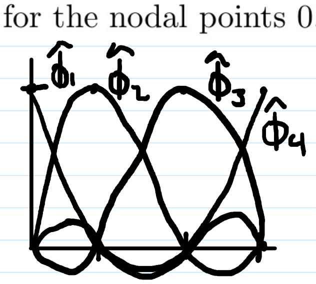
$u_1 = -2u_2 - 3u_3$

By inspection we see two possibilities are as follows

$u^1 = \begin{bmatrix} 1 \\ -1 \\ 1/3 \end{bmatrix}$ $\quad u^2 = \begin{bmatrix} 0 \\ -1 \\ 2/3 \end{bmatrix}$

Since $\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & -1 \\ 3 & 1/3 & 2/3 \end{bmatrix} \sim I_3$ and $v^T u^1 = 0 = v^T u^2$, $G^{\perp} = span(\{u^1, u^2\})$

No.3. (3 pts.) Give the Lagrangian, cubic basis functions on the reference interval, i.e. $[0,1] \subset \mathbb{R}$, for the nodal points 0, 1/3, 2/3, 1.



$\hat\phi_1 = -\frac{9}{2}(\xi - \frac{1}{3})(\xi - \frac{2}{3})(\xi - 1)$

$\hat\phi_2 = \frac{27}{2}\xi(\xi - \frac{2}{3})(\xi - 1)$

$\hat\phi_3 = -\frac{27}{2}\xi(\xi - \frac{1}{3})(\xi - 1)$

$\hat\phi_4 = \frac{9}{2}\xi(\xi - 1/3)(\xi - 2/3)$

I used a computer algebra system for this problem

No.4. (5 pts.) Continuous piecewise cubic basis functions on $\widehat{\mathcal{T}}$.

There are ten piecewise cubic basis functions on $\widehat{\mathcal{T}}$: three nodal basis functions, six edge basis function (two per side), and one *bubble function*.

The *bubble function* is given by: $\widehat{\phi}_{10}(\xi, \eta) = 27\,\xi\,\eta\,(1 - \xi - \eta)$.

Note that the bubble function is zero on all the edges of $\widehat{\mathcal{T}}$ and is equal to 1 at $(\xi, \eta) = (1/3\,,\,1/3)$.

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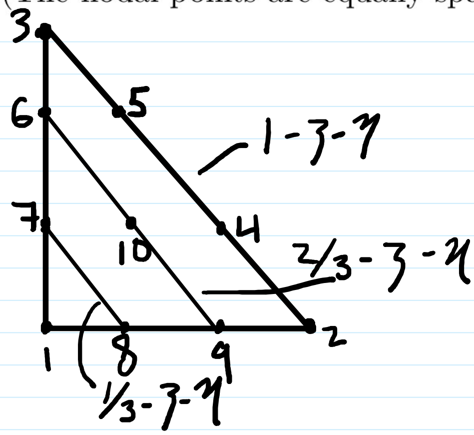
Determine the other nine basis functions, which are Lagrangian.
(The nodal points are equally spaced along each edge of $\widehat{\mathcal{T}}$.



$\hat{\phi}_1 = \frac{9}{2}(1-\xi-\eta)(2/3-\xi-\eta)(1/3-\xi-\eta)$
$\hat{\phi}_2 = \frac{9}{2}\xi(\xi-2/3)(\xi-1/3)$
$\hat{\phi}_3 = \frac{9}{2}\eta(\eta-2/3)(\eta-1/3)$
$\hat{\phi}_4 = -\frac{27}{2}\xi\eta(\eta-2/3)$
$\hat{\phi}_5 = \frac{27}{4}\xi\eta(\eta-1/3)$
$\hat{\phi}_6 = \frac{27}{2}(1-\xi-\eta)(\xi-2/3)(1/3-\xi-\eta)$
$\hat{\phi}_7 = -\frac{27}{2}(1-\xi-\eta)(\xi-1/3)(2/3-\xi-\eta)$
$\hat{\phi}_8 = -\frac{27}{2}(1-\xi-\eta)(\eta-1/3)(2/3-\xi-\eta)$
$\hat{\phi}_9 = \frac{27}{2}(1-\xi-\eta)(\eta-2/3)(1/3-\xi-\eta)$

I used a computer algebra system for this problem

No.5. (4 pts.) Let H denote a Hilbert space, $\mathcal{L} : H \longrightarrow H'$ a linear operator, and $\{G_n\}_{n=1}^{\infty}$ a Cauchy sequence in $Range(\mathcal{L}) \subset H'$. Show that there exists a Cauchy sequence $\{g_n\} \in H$ such that $G_n(v) \;=\; \langle g_n, v\rangle$, for all $v \in H$.

By the RRT for each $G_n \in R(\mathcal{L})$ $\exists!\, g_n \in H \ni G_n(v) = \langle g_n, v\rangle\ \forall v \in H$.
Let $\varepsilon > 0$ & $h > 0$ WTS $\|g_{n+h} - g_n\| < \varepsilon$
Since $\{G_n\}$ is Cauchy, $\exists N \in \mathbb{N} \ni \forall n \geq N$ $|(G_{n+h} - G_n)(v)| < \varepsilon^2\ \forall v \in H$
It follows

$$\begin{aligned} |(G_{n+h} - G_n)(v)| &= |G_{n+h}(v) - G_n(v)| = |\langle g_{n+h}, v\rangle - \langle g_n, v\rangle| \quad \forall v \in H \\ &= |\langle g_{n+h} - g_n, v\rangle| \\ &= |\langle g_{n+h} - g_n, g_{n+h} - g_n\rangle| \quad \text{Let } v = g_{n+h} - g_n \\ &= \|g_{n+h} - g_n\|^2 \end{aligned}$$

$\therefore \|g_{n+h} - g_n\|^2 < \varepsilon^2$
$\Rightarrow \{g_n\} \subseteq H$ is Cauchy and $G_n(v) = \langle g_n, v\rangle\ \forall v \in H$