MATH 8660 Sec.1: Homework No.12

Due: Tuesday, November 26 at 8:05am.

No.1. (8 pts.) (6 pts.) Consider the mathematical model for the steady-state temperature distribution across a (sufficiently) smooth domain Ω:

$$-\nabla \cdot \nabla u = f(\mathbf{x}), \ \mathbf{x} \text{ in } \Omega, \tag{0.1}$$

$$u = 0, \text{ on } \partial\Omega, \tag{0.2}$$

where $f \in H^{5/4}(\Omega)$. Suppose a continuous piecewise quadratic Finite Element approximation is computed for (0.1), (0.2). Of interest is the average temperature:
(i) over the domain, i.e., $\frac{1}{|\Omega|}\int_\Omega u\, 1\, d\Omega$, and
(ii) over a portion of the domain $\Omega_0 \subset \Omega$, i.e., $\frac{1}{|\Omega_0|}\int_\Omega u\, 1_{\Omega_0}\, d\Omega$, where 1_{Ω_0} is the characteristic function on Ω_0.
Determine the expected rate of convergence for the FEM estimates to (i) and (ii).
(Recall for $h(x) = \begin{cases} 0, & 0 < x < 1/2, \\ 1, & 1/2 \le x < 1 \end{cases}$, that $h \in H^{1/2-\epsilon}(0,1)$ for any $\epsilon > 0$.)

No.2. (5 pts.) Let $\Omega \subset \mathbb{R}^2$, $\mathbf{w}(\mathbf{x}) = [w_1(\mathbf{x}), w_2(\mathbf{x})]^T \in \left(H^1(\Omega)\right)^2$, and $\mathbf{v}(\mathbf{x}) = [v_1(\mathbf{x}), v_2(\mathbf{x})]^T \in \left(H_0^1(\Omega)\right)^2$.

$$\text{Notation: } \nabla\mathbf{w}(\mathbf{x}) = \begin{bmatrix} \frac{\partial w_1(\mathbf{x})}{\partial x_1} & \frac{\partial w_1(\mathbf{x})}{\partial x_2} \\ \frac{\partial w_2(\mathbf{x})}{\partial x_1} & \frac{\partial w_2(\mathbf{x})}{\partial x_2} \end{bmatrix},$$

$$\nabla \cdot \nabla\mathbf{w}(\mathbf{x}) = \begin{bmatrix} \nabla \cdot [\frac{\partial w_1(\mathbf{x})}{\partial x_1}, \frac{\partial w_1(\mathbf{x})}{\partial x_2}]^T \\ \nabla \cdot [\frac{\partial w_2(\mathbf{x})}{\partial x_1}, \frac{\partial w_2(\mathbf{x})}{\partial x_2}]^T \end{bmatrix},$$

$$\text{scalar tensor product } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}.$$

(i) Show that

$$\nabla \cdot \nabla\mathbf{w}(\mathbf{x}) = \begin{bmatrix} \Delta w_1(\mathbf{x}) \\ \Delta w_2(\mathbf{x}) \end{bmatrix}.$$

(ii) Verify, by considering each component,

$$\int_\Omega -\nabla \cdot \nabla\mathbf{w}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\Omega = -\int_{\partial\Omega} \mathbf{v}(\mathbf{x}) \cdot \nabla\mathbf{w}(\mathbf{x})\mathbf{n}(\mathbf{x})\, ds + \int_\Omega \nabla\mathbf{w}(\mathbf{x}) : \nabla \cdot \mathbf{v}(\mathbf{x})\, d\Omega.$$

No.3. (4 pts.) Let

$$X := \left(H^1(\Omega)\right)^n, \ Q := \{q \in L^2(\Omega) : \int_\Omega q\, d\Omega = 0\},$$

$$\text{and } V := \{\mathbf{z} \in X : \int_\Omega q\, \nabla \cdot \mathbf{z}\, d\Omega = 0, \ \forall q \in Q\}.$$

Show that V is a closed subspace of X.