HW 4 Jacob Manning

1. $Q = \int_\Omega u^2 u' v \, dx \quad u, v \in H^1(\Omega)$

$\leq (\int_\Omega u'^2 dx)^{1/2} (\int_\Omega u^4 v^2 dx)^{1/2}$

$p=4 \quad q = \frac{4}{3}$

$(\int_\Omega u^4 v^2 dx)^{1/2} \leq ((\int_\Omega (v^2)^4 dx)^{1/4})^{1/2} ((\int_\Omega (u^4)^{3/4} dx)^{4/3})^{1/2}$

$= (\int_\Omega v^8 dx)^{1/8} (\int_\Omega u^3 dx)^{2/3}$

$= \|v\|_{L^8} \|u\|^2_{L^3}$

i) $\Omega \subseteq \mathbb{R}^2 \Rightarrow n=2 \qquad u, v \in H^1(\Omega) \Rightarrow 2(1) = 2 = n$

$u, v \in L^q(\Omega)$ for $q \in [2, \infty)$

Thus $u \in L^3$ and $v \in L^8$ so $Q < \infty$ for $\Omega \subseteq \mathbb{R}^2$

ii) $2(1) = 2 < 3 \quad \Omega \subseteq \mathbb{R}^3$

Thus $u, v \in L^q(\Omega)$ for $q \in [2, \frac{2(3)}{3-2(1)}]$

$[2, \frac{6}{1}]$

Therefore Q is not guarented to be finite for $\Omega \subseteq \mathbb{R}^3$

2. $V = \mathbb{R}^3 \quad f = [-2, 5, 1]^T \quad F(v) = f \cdot v$

i) WTS F is a linear functional

The codomain of F is the reals so F is a functional.

Let $\alpha \in \mathbb{R}$ $v_1, v_2 \in V$ were $v_1 = \begin{bmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \end{bmatrix}$ $v_2 = \begin{bmatrix} v_2^1 \\ v_2^2 \\ v_2^3 \end{bmatrix}$

$F(\alpha v_1 + v_2) = f \cdot (\alpha v_1 + v_2) = f \cdot \begin{bmatrix} \alpha v_1^1 + v_2^1 \\ \alpha v_1^2 + v_2^2 \\ \alpha v_1^3 + v_2^3 \end{bmatrix}$

$= -2(\alpha v_1^1 + v_2^1) + 5(\alpha v_1^2 + v_2^2) + 1(\alpha v_1^3 + v_2^3)$

$= \alpha[(-2v_1^1) + 5v_1^2 + 1v_1^3] + (-2)v_2^1 + 5v_2^2 + 1v_2^3$

$= \alpha f \cdot v_1 + f \cdot v_2$

$= \alpha F(v_1) + F(v_2)$

Thus F is a linear functional.

2iii) WTS F is bdd and find $\|F\|$

$|F(v)| = |\langle v, f\rangle| \leq \|f\|\,\|v\|$

Thus F is bdd

$\|F\|^2 = \langle f, f\rangle = (-2)^2 + 5^2 + 1^2 = 30 \Rightarrow \|F\| = \sqrt{30}$

3. X nls $F: X \to \mathbb{R}$ WTS F is continuous iff F is bdd

($\Leftarrow$) Let $\varepsilon > 0$ $\delta = \frac{\varepsilon}{\|F\|}$ $\|x - x'\| < \delta$ It follows

$\|F(x) - F(x')\| = \|F(x - x')\| \leq \|F\|\,\|x - x'\|$
$< \|F\|\delta = \varepsilon$

Thus F is continuous

($\Rightarrow$) WTS F is bounded given F is continuous.
We will show F is bounded given F is continuous at the origion.

Since F is continuous at 0 $\exists$ $x \in B_\delta(0)$ that implies $\|F(x) - \cancel{F(0)}\| < 1$

Thus

$\|F(x)\| = \|F(\frac{\|x\|}{\delta}\frac{\delta}{\|x\|}x)\| = \frac{\|x\|}{\delta}\|F(\frac{x}{\|x\|}\delta)\| \leq \frac{1}{\delta}\|x\|$. Therefore F is bdd

Note $\|\frac{x}{\|x\|}\delta\| = \delta$ thus $\frac{x\delta}{\|x\|} \in B_\delta(0)$

4i) $B(f,g) = \int_0^1\int_0^y \frac{\partial^2 f}{\partial x^2}\frac{\partial g}{\partial y}\,dx\,dy + f(\frac{1}{2},\frac{1}{2})\,g(\frac{1}{4},\frac{1}{2})$

$f, g \in H^2(\Omega)$

By embedding theorems $\Omega \subseteq \mathbb{R}^2 \Rightarrow$ pointwise evaluation of all $H^s(\Omega)$ functions make sense for $2s > 2 = n$ or $s > 1$. Since $f,g \in H^s(\Omega)$, we are guarenteed pointwise evaluation.

Let $f_1, f_2, g_1, g_2 \in H^2(\Omega)$ $\alpha \in \mathbb{R}$

$B(\alpha f_1 + f_2, g_1) = \int_0^1\int_0^y \frac{\partial^2(\alpha f_1 + f_2)}{\partial x^2}\frac{\partial g_1}{\partial y}\,dx\,dy + (\alpha f_1 + f_2)|_{(1/2,1/2)}\,g_1(\frac{1}{4},\frac{1}{2})$

$= \alpha\left[\int_0^1\int_0^y \frac{\partial^2 f_1}{\partial x^2}\frac{\partial g_1}{\partial y}\,dx\,dy + f_1(\frac{1}{2},\frac{1}{2})g_1(\frac{1}{4},\frac{1}{2})\right] + \int_0^1\int_0^y \frac{\partial^2 f_2}{\partial x^2}\frac{\partial g_1}{\partial y}\,dx\,dy + f_2(\frac{1}{2},\frac{1}{2})\,g_1(\frac{1}{4},\frac{1}{2})$

$= \alpha B(f_1, g_1) + B(f, g_1)$

4i cont. $B(f_1, \alpha g_1 + g_2) = \int_0^1\int_0^y \frac{\partial^2 f_1}{\partial x^2}\frac{\partial(\alpha g_1+g_2)}{\partial y}\,dx\,dy + f_1(\tfrac{1}{2},\tfrac{1}{2})(\alpha g_1+g_2)\big|_{(\frac{1}{4},\frac{1}{2})}$

$= \alpha\left[\int_0^1\int_0^y \frac{\partial^2 f_1}{\partial x^2}\frac{\partial g_1}{\partial y}\,dx\,dy + f_1(\tfrac{1}{2},\tfrac{1}{2})\,g_1(\tfrac{1}{4},\tfrac{1}{2})\right]$

$+ \int_0^1\int_0^y \frac{\partial^2 f_1}{\partial x^2}\frac{\partial g_2}{\partial y}\,dx\,dy + f_1(\tfrac{1}{2},\tfrac{1}{2})\,g_2(\tfrac{1}{4},\tfrac{1}{2})$

$= \alpha B(f_1, g_1) + B(f_1, g_2)$

Note integrals and the evaluation function are linear functionals and the partial derivative is linear.

Thus B is a bilinear form

ii) As noted in part i) there is a continuous embedding such that $f(\tfrac{1}{2},\tfrac{1}{2})\,g(\tfrac{1}{4},\tfrac{1}{2}) \le C_f C_g \|f\|_{H^2}\|g\|_{H^2}$

Now consider

$$\left|\int_0^1\int_0^y \frac{\partial^2 f}{\partial x^2}\frac{\partial g}{\partial y}\,dx\,dy\right| \le \int_0^1\int_0^y \left|\frac{\partial^2 f}{\partial x^2}\frac{\partial g}{\partial y}\right| dx\,dy$$

$$\le \int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial x^2}\frac{\partial g}{\partial y}\right| dx\,dy$$

$$= \int_\Omega \left|\frac{\partial^2 f}{\partial x^2}\frac{\partial g}{\partial y}\right| d\Omega$$

$$\le \left(\int_\Omega \left(\frac{\partial^2 f}{\partial x^2}\right)^2 d\Omega\right)^{1/2}\left(\int_\Omega \left(\frac{\partial g}{\partial y}\right)^2 d\Omega\right)^{1/2}$$

$$\le \|f\|_{H^2}\|g\|_{H^2} < \infty$$

Thus $|B(f,g)| \le (1 + C_f C_g)\|f\|_{H^2}\|g\|_{H^2} \Rightarrow B$ is bdd

5. X nls $B: X\times X \to \mathbb{R}$ bilinear form

$\|(a,b)\|_{X\times X} = (\|a\|_X^2 + \|b\|_X^2)^{1/2}$ WTS B is continuous iff B is bdd

($\Rightarrow$) WTS B is bdd if B is continuous

Since B is continuous $\exists\ (a,b) \in B_\delta(0,0)$

That implies $\|B(a,b) - \cancel{B(0,0)}\| \le 1$

Thus $\|B(a,b)\| = \|B(\frac{\|a\|}{\sqrt{\delta}}\frac{\sqrt{\delta}}{\|a\|}a, \frac{\|b\|}{\sqrt{\delta}}\frac{\sqrt{\delta}}{\|b\|}b)\| = \frac{\|a\|\,\|b\|}{\delta}\|B(\frac{\sqrt{\delta}}{\|a\|}a, \frac{\sqrt{\delta}}{\|b\|}b)\|$

$\le \frac{1}{\delta}\|a\|\,\|b\|$

Since $\|B(\cdot,\cdot)\| \le 1\ \forall\ (x,y) \in B_\delta(0,0)$

Thus B is bdd

5 cont. $(\Leftarrow)$ Let $\Sigma > 0$ and $(a,b) \in X \times X$

For $\delta = \min\{1, \frac{\Sigma}{C+(1+\|b\|+\|a\|)}\}$

It follows

$$\|B(x,y) - B(a,b)\| \leq \|B(x-a,y)\| + \|B(a,y-b)\|$$
$$\leq C+[\|x-a\|\|y\| + \|a\|\|y-b\|]$$

Note $\|(x,y)-(a,b)\|^2 = \|x-a\|^2 + \|y-b\|^2$

Thus

$$C+[\|x-a\|\|y\| + \|a\|\|y-b\|] \leq C+[(\|y\|+\|a\|)\|(x,y)-(a,b)\|]$$

Similarly by our bound on δ,

$$\|y-b\| \leq \|(x,y)-(a,b)\| < \delta = 1$$

and $\|y\| \leq \|y-b\| + \|b\|$
$\leq 1 + \|b\|$

Thus
$$C+(\|y\|+\|a\|)\|(x,y)-(a,b)\|$$
$$\leq C+(1+\|b\|+\|a\|)\|(x,y)-(a,b)\|$$
$$< C+(1+\|b\|+\|a\|)\delta$$
$$< \Sigma$$

Therefore B is continuous