1. Let H be a Hilbert space and M be a closed subspace of H. Suppose φ is a bounded linear functional defined on M. Use Hilbert space methods to show that φ can be uniquely extended to H, i.e., show that there exists a unique bounded linear functional $\tilde{\varphi}$ on H such that $\tilde{\varphi}(m) = \varphi(m)$, $\forall m \in M$ and $\|\tilde{\varphi}\| = \|\varphi\|$.

2. Let X be a normed linear space and $\mathbf{0} \neq x_0 \in X$. Show that there exists $\varphi \in X'$ such that $\|\varphi\| = \|x_0\|$ and $|\varphi(x_0)| = \|x_0\|^2$.

3. Let X be a normed linear space and Y be a closed proper subspace of X. For any $x_0 \in X \backslash Y$, let $d(x_0, Y) := \inf\limits_{y \in Y} \|x_0 - y\|$ be the distance from x_0 to Y. Show that there exists $\varphi \in X'$ such that $\|\varphi\| = 1$, $\varphi(x_0) = d(x_0, Y)$ and $\varphi(y) = 0$ for all $y \in Y$.

4. Let X be a normed linear space and $x_1, x_2, \cdots, x_n$, $n \in \mathbb{N}$, be linearly independent vectors in X. Show that there exist $\varphi_1, \varphi_2, \cdots, \varphi_n \in X'$ such that
$$\varphi_i(x_j) = \delta_{ij} := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad 1 \leqslant i, j \leqslant n.$$

5. Let X be a normed linear space and $\{y_n\} \subset X$. Show that $z \in Y = \overline{span\{y_n\}}$ if and only if $\forall \varphi \in X'$, $\varphi(y_n) = 0$, $\forall n \in \mathbb{N}$ implies $\varphi(z) = 0$.

6. Let X be a normed linear space and $A, B \subset X$ be two nonempty convex subsets such that $A \cap B = \varnothing$. Assume that A is closed and B is compact, show that A and B can be strictly separated by a hyperplane, i.e., there exists $c \in \mathbb{R}$ and $\varphi \in X'$ such that $\varphi(a) < c < \varphi(b)$, $\forall a \in A$, $b \in B$.

7. Let X be a normed linear space and $Y \leqslant X$ be a subspace such that $\overline{Y} \neq X$. Show that there exists $\varphi \in X'$, $\varphi \neq \mathbf{0}$, such that $\varphi(y) = 0$, $\forall y \in Y$.