1. Let X be a Banach space, $S, T \in \mathcal{B}(X, X)$ and $ST = TS$.
   (a) Show that $r_\sigma(ST) \leqslant r_\sigma(S)r_\sigma(T)$.
   (b) Show (with a counterexample) that the commutativity $ST = TS$ can not be dropped in part (a).

2. Let H be a Hilbert space. Recall $T \in \mathcal{B}(H, H)$ is called a *normal* operator if $TT^* = T^*T$, where T^* is the (Hilbert) adjoint of T. Show that if T is normal, then
$$r_\sigma(T) = \|T\|.$$

3. Let $T : C[0,1] \to C[0,1]$ be defined by
$$(Tf)(x) = \int_0^x f(y)\, dy, \ \forall f \in C[0,1].$$
Find the spectrum of T.

4. Let $\{a_n\} \subset \mathbb{C}$ be a sequence of scalars such that $a_n \to 0$, $n \to \infty$. Define $T : l^2 \to l^2$ by $T(\{x_1, x_2, x_3, \cdots\}) = \{a_1x_1, a_2x_2, a_3x_3, \cdots\}$. Show that T is compact.

5. Let H be a Hilbert space and $T \in \mathcal{K}(H, H)$. Suppose $\{e_n\}$ is an orthonormal basis of H, show that $\langle Te_n, e_n\rangle \to 0$, $n \to \infty$.

6. Let X be a normed linear space and $T \in \mathcal{K}(X, X)$. Suppose $S \in \mathcal{B}(X, X)$, show that both TS and ST are compact linear operators on X.

7. Let H be a Hilbert space and $T \in \mathcal{B}(H, H)$. Let T^* denote the Hilbert adjoint operator of T. Show that T is compact if and only if T^*T is compact.

8. Let $T : l^2 \to l^2$ be defined by
$$Tx = \left\{\frac{x_2}{1}, \frac{x_3}{2}, \frac{x_4}{3}, \cdots\right\}, \ \forall x = \{x_1, x_2, x_3, \cdots\} \in l^2.$$
Show that T is compact and the point spectrum $\sigma_p(T) = \{0\}$.

9. (*Extra credits*) Let $T : L^2[0,1] \to L^2[0,1]$ be defined by $(Tf)(x) = \displaystyle\int_0^1 K(x,y)f(y)dy$ where $K(x,y)$ satisfies $\displaystyle\int_0^1\int_0^1 |K(x,y)|^2 dxdy < \infty$. Show that T is compact.