Perturbation of Semigroups

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December 5, 2024

Section 3.1 Perturbations by Bounded Linear Operators

3.1 Perturbations by Bounded Linear Operators

Definition

- If A is the infinitesimal generator of $T(t)$ (a semigroup of linear operators) then we say A generates $\{T(t) : t \geq 0\}$.
- C_0 semigroup $T(t)$ is the semigroup of strongly continuous operators that are bounded on X.

Theorem (3.1.1)

Let X be a Banach space and let A be the infinitesimal generator of a C_0 semigroup $T(t)$ on X, satisfying $||T(t)|| \leq Me^{\omega t}$. If B is a bounded linear operator on X then $A + B$ is the infinitesimal generator of a C_0 semigroup $S(t)$ on X, satisfying $||S(t)|| \leq Me^{(\omega + M||B||)t}$.

Remark

- An equivalent definition of infinitesimal generator of $\{T(t) : t \geq 0$ is, let $A : D(A) \subset X \to X$ is defined by $D(A) = \{x \in X : \lim_{t \downarrow 0} \frac{1}{t}(T(t)x - x)$ exists$\}$ and $Ax = \lim_{t \downarrow 0} \frac{1}{t}(T(t)x - x)$
- The initial bound of $||T(t)|| \leq Me^{\omega t}$ is given to the true generator A. However, there will exist some error when computing this generating numerically. The numerically computed generator could be $\tilde{A} = A + \Delta A$ where ΔA is a bounder linear operator.

3.1 Perturbations by Bounded Linear Operators

We are interested in relations between the semigroup $T(t)$ generated by A and the semigroup $S(t)$ generated by $A+B$. Consider the following operator,

$$H(s) = T(t-s)S(s)$$

for $x \in D(A) = D(A+B)$ such that $s \to H(s)x$. Notice that $H(s)$ is differentiable.

$$H'(s)x = T(t-s)BS(s)x \text{ for } x \in D(A)$$

Integrating this from 0 to t, we can find that

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)xds$$

Both sides are bounded, meaning $S(t)$ is a solution.

Theorem (3.1.2)

Let $T(t)$ be a C_0 semigroup satisfying $||T(t)|| \leq Me^{\omega t}$. Let B be a bounded operator on X. Then there exists a unique family $V(t)$, $t \geq 0$ of bounded operators on X such that $t \to V(t)x$ is continuous on $[0, \infty)$ for every $x \in X$ and

$$V(t)x = T(t)x + \int_0^t T(t-s)BV(s)xds \qquad x \in X.$$

Remark

We can immediately gain an explicit representation of $S(t)$ in terms of $T(t)$ such that

$$S(t) = \sum_{n \in \mathbb{N}} S_{n-1}(t)$$

where $S_0(t) = T(t)$ and $S_n(t) = \int_0^t T(t-s)BS_{n-1}(s)xds$ with $x \in X$. The series converges uniformly.

Corollary (3.1.3)

Let A be the infinitesimal generator of a C_0 semigroup $T(t)$ satisfying $||T(t)|| \leq Me^{\omega t}$. Let B be a bounded operator and let $S(t)$ be the C_0 semigroup generated by $A + B$. Then

$$||S(t) - T(t)|| \leq Me^{\omega t}(e^{M||B||t} - 1).$$

Proof.

From our theorems 3.1.1 and 3.1.2 we gain the following

$$||S(t)x - T(t)x|| \leq \int_0^t ||T(t-s)||\ ||B||\ ||S(s)||\ ||x||ds$$

$$\leq M^2 e^{\omega t}||B|| \int_0^t e^{M||B||s}||x||ds$$

$$= Me^{\omega t}(e^{M||B||t} - 1)||x||$$

Remark

The big takeaway of this section is theorem 3.1.1, which shows that we can add a bounded linear operator to an infinitesimal operator and it does not destroy the property.

Follow up question

What other properties can a semigroup $T(t)$ generated by an infinitesimal operator A carry over?

It turns our perturbation $(A + B)$ will be compact or analytic if our infinitesimal operator A is compact or analytic.

3.1 Perturbations by Bounded Linear Operators

Proposition (3.1.4)

Let A be the infinitesimal generator of a compact C_0 semigroup $T(t)$. Let B be a bounded operator, then $A + B$ is the infinitesimal generator of a compact C_0 semigroup $S(t)$.

Proposition (3.2.1)

Let A be the infinitesimal generator of an analytic semigroup. Let B be a closed linear operator satisfying $D(B) \supset D(A)$ and

$$||Bx|| \leq a||Ax|| + b||x||$$

for $x \in D(A)$ There exists a positive number δ such that if $0 \leq a \leq \delta$ then $A + B$ is the infinitesimal generator of an analytic semigroup

Remark

Not all of the properties of the semigroup of $T(t)$ are preserved by a bounded perturbation of its infinitesimal generator.

Ex

Let A be the infinitesimal generator of a semigroup $T(t)$ which is continuous in the uniform operator topology for $t \geq t_0 \geq 0$, or is differentiable for $t \geq t_0 \geq 0$, or is compact for $t \geq t_0 \geq 0$. Then $S(t)$, the semigroup generated by $A + B$ where B is a bounded operator need not have the corresponding property.

<u>3.3 Perturbations of Infinitesimal Generators of Contraction Semigroups</u>

3.3 Perturbations of Infinitesimal Generators of Contraction Semigroups

Recall

Theorem (1.4.3 Lumer-Phillips)

- *If A is dissipative and there is a $\lambda_0 > 0$ such that $R(\lambda_0 I - A) = X$, then A is the infinitesimal generator of a C_0 semigroup of contractions on X.*
- *If A is the infinitesimal generator of a C_0 semigroup of contractions on X then $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative.*

3.3 Perturbations of Infinitesimal Generators of Contraction Semigroups

Definition (3.3.1)

- A linear operator A is called dissipative if $Re\langle Ax, x\rangle < 0 \quad \forall x \in D(A)$
- A dissipative operator A such that $R(I - A) = X$ is called a maximal dissipative operator.
- An operator A is called closable if there exists a closed operator $\bar{A}$ such that $G(A) \subset G(\bar{A})$.

Remark

If A is dissipative, for any $c > 0$, cA is also dissipative.

Theorem (Lumer-Phillips)

A densely defined operator A is the infinitesimal generator a C_0 semigroup of contractions iff A is maximal dissipative.

3.3 Perturbations of Infinitesimal Generators of Contraction Semigroups

Theorem (3.3.2)

Let A and B be linear operators in X such that $D(B) \supset D(A)$ and $A + tB$ is dissipative for $0 \leq t \leq 1$. If

$$||Bx|| \leq \alpha ||Ax|| + \beta ||x|| \qquad x \in D(A)$$

where $0 \leq \alpha < 1$, $\beta \geq 0$ and for some $0 \leq t_0 \leq 1$, $A + t_0 B$ is maximal dissipative then $A + tB$ is maximal dissipative for all $0 \leq t \leq 1$.

This theorem is often used with the following corollary.

Corollary (3.3.3)

Let A be the infinitesimal generator of a C_0 semigroup of contractions. Let B be dissipative and satisfy $D(B) \supset D(A)$ and

$$||Bx|| \leq \alpha ||Ax|| + \beta ||x|| \qquad x \in D(A)$$

where $0 \leq \alpha < 1$, $\beta \geq 0$. Then $A + B$ is the infinitesimal generator of a C_0 semigroup of contractions.

Remark

If $A + tB$ is dissipative for $0 \leq t \leq 1$, $D(B) \supset D(A)$, $\overline{D(A)} = X$, and

$$||Bx|| \leq \alpha ||Ax|| + \beta ||x|| \qquad x \in D(A)$$

where $0 \leq \alpha < 1$, $\beta \geq 0$. Then either both A and $A + B$ are maximal dissipative or neither.

3.3 Perturbations of Infinitesimal Generators of Contraction Semigroups

$\alpha = 1$ can be an issue since $A + B$ may not be closed and thus cannot be the infinitesimal generator of a C_0 semigroup.

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Ex

Let iA be a self adjoint operator in a Hilbert space. It follows that A and $-A$ are infinitesimal generators of C_0 semigroups of contractions. Taking $\alpha = 1$ and $\beta = 0$. It follows

$$||-Ax|| \leq 1 * ||Ax|| + 0 * ||x|| = ||Ax||$$

However $\overline{A + (-A)}\Big|_{D(A)}$ is not closed.

What conditions can we set $\alpha = 1$?

What conditions can we set $\alpha = 1$?

Theorem (3.3.4)

Let A be the infinitesimal generator of a C_0 semigroup of contractions. Let B be dissipative and satisfy $D(B) \supset D(A)$ and

$$||Bx|| \leq ||Ax|| + \beta ||x|| \qquad x \in D(A)$$

where $\beta \geq 0$. If B^* is densely defined, then the closure $\overline{A + B}$ of $A + B$ is the infinitesimal generator of a C_0 semigroup of contractions.

If X is a reflexive Banach space and T is closable and densely defined, we know that T^* is closed and $D(T^*)$ is dense in X^*. It follows

Corollary (3.3.5)

Let X be a reflexive Banach space and let A be the infinitesimal generator of a C_0 semigroup of contractions in X. Let B be dissipative such that $D(B) \supset D(A)$ and

$$||Bx|| \leq ||Ax|| + \beta ||x|| \qquad x \in D(A)$$

where $\beta \geq 0$. Then $\overline{A+B}$, the closure of $A+B$, is the infinitesimal generator of a C_0 semigroup of contractions in X.

Citations

Amnon Pazy. "Perturbations and Approximations." Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer Science & Business Media, 6 Dec. 2012.