1. Let X and Y be Banach spaces and $\{T_n\} \subset \mathcal{B}(X,Y)$. Show that the following are equivalent:

   (a) $\{\|T_n\|\}$ is bounded.

   (b) $\{\|T_nx\|\}$ is bounded for all $x \in X$.

   (c) $\{|\varphi(T_nx)|\}$ is bounded for all $x \in X$ and all $\varphi \in Y'$.

$a \to b$

$\{\|T_n\|\}$ is bdd WTS $\{\|T_nx\|\}$ is bdd $\forall x \in X$

Let $x \in X$ Let $C > 0 \ni \|T_n\| < C \ \forall n \in \mathbb{N}$

$\|T_nx\| \leq \|T_n\| \|x\| \leq C\|x\|$

Thus $\{\|T_nx\|\}$ is bdd $\forall n \in \mathbb{N}$ $x \in X$

$b \to c$

Let $\{\|T_nx\|\}$ be bdd $\forall x \in X$

WTS $\{|\varphi(T_nx)|\}$ is bdd $\forall x \in X$ $\varphi \in Y'$

Let $\varphi \in Y'$ and $C > 0 \ni \|T_nx\| < C$ $\forall n \in \mathbb{N}$ $x \in X$

$|\varphi(T_nx)| \leq \|\varphi\| \|T_nx\| \leq C\|\varphi\|$

Thus $\{|\varphi(T_nx)|\}$ is bdd $\forall$ $x \in X$ $\varphi \in Y'$

$c \to a$

Let $\{|\varphi(T_nx)|\}$ be bdd $\ni \exists C > 0$ $|\varphi(T_nx)| < C$ $\forall \varphi \in Y'$ $x \in X$

WTS $\{\|T_n\|\}$ is bdd

$$\|y\|_Y = \sup_{\varphi \neq 0} \frac{|\varphi(y)|}{\|\varphi\|}$$

$$\Rightarrow \|T_nx\|_Y = \sup_{\varphi \neq 0} \frac{|\varphi(T_nx)|}{\|\varphi\|} \leq \sup_{\varphi \neq 0} \frac{C}{\|\varphi\|} \leq C$$

Thus $\|T_nx\|_Y$ is bdd $\forall n \in \mathbb{N}$ $x \in X$

$\therefore$ By UBP $\{\|T_n\|\}$ is bdd $\forall n \in \mathbb{N}$

2. Let $y = \{y_n\} \subset \mathbb{C}$ be such that $\sum \overline{\overline{x_n} y_n}$ converges for every $x = \{x_n\} \in c_0$, where

2. Let $y = \{y_n\} \subset \mathbb{C}$ be such that $\sum_{n=1} \overline{x_n y_n}$ converges for every $x = \{x_n\} \in c_0$, where $c_0 \subset l^\infty$ denotes the space of all (complex) sequences that converge to zero. Show that

$$\sum_{n=1}^{\infty} |y_n| < \infty.$$

WTS $\{x_n y_n\} \subseteq l^1 \ \forall \{x_n\} \subseteq c_0 \Rightarrow y \in l^1$

Let $\{x^{(N)}\} \subseteq c_0 \ni x^{(N)}: x_n^{(N)} := \begin{cases} \frac{y_n}{|y_n|} & n \leq N \\ 0 & n > N \end{cases}$

Since $x^{(N)} \in c_0 \ \forall N \in \mathbb{N}$

$\infty > \sum_{n=1}^{\infty} x_n^{(N)} \overline{y_n} = \sum_{n=1}^{N} \frac{y_n \overline{y_n}}{|y_n|} = \sum_{n=1}^{N} |y_n| \quad \forall N \in \mathbb{N}$

as $N \to \infty$ $\sum_{n=1}^{N} |y_n| \to \sum_{n=1}^{\infty} |y_n| < \infty$ from previous bound

3. Let X be a Banach space, Y be a normed linear space and $\{T_n\} \subset \mathcal{B}(X, Y)$. Suppose $\{T_n\}$ satisfies the property that for any $\{x_n\} \subset X$ with $\|x_n\| \to 0$, then $\|T_n(x_n)\| \to 0$. Show that

$$\sup_{n \in \mathbb{N}} \|T_n\| < \infty.$$

ATC $\sup \|T_n\| = \infty$ WTS $\sup_{n \in \mathbb{N}} \|T_n x\| < \infty \ \forall x \in X$

Let $x \in X$ choose $y \in c_0 \ni \|x y_n\| \to 0$

y exists since $\|x_n\| < \infty \ \forall n \in \mathbb{N}$ since $x \in X$

Define $S_n : c_0 \to X \quad z \mapsto \sum_{k=1}^{n} T_k z_k \quad z_n = x y_n$

$\|S_n z\| = \|\sum_{k=1}^{n} T_k x y_k\| \leq \sum_{k=1}^{n} \|T_k x y_k\|$

Since $\|x y_n\| \to 0$, $\|T_n x y_n\| \to 0 \Rightarrow \sum_{k=1}^{\infty} \|T_k x y_k\| < \infty$

Since X is complete $S_n z$ converges to a limiting Sz

By construction of S_n and bdd linearity of T_n,

S_n is bdd and linear $\forall n \in \mathbb{N}$. Thus S is also bdd linear.

$\|S\| = \sup_{\|y\| \leq 1} |\sum_{k=1}^{\infty} T_k x y_k| \geq |\sum_{k=1}^{N} T_k x| \Rightarrow \sup_{n \in \mathbb{N}} \|T_n x\| < \infty$

$\therefore$ By corollary to Banach-Steinhaus X is not complete $\ast$

4. Let H be a Hilbert space and sequences $\{x_n\}, \{y_n\} \subset H$ such that $x_n \xrightarrow{w} x \in H$ and $y_n \to y \in H$, as $n \to \infty$. Show that

$$\langle x_n, y_n \rangle \to \langle x, y \rangle,\ n \to \infty.$$

By RRT we can represent $\{y_n\}$ as a sequence $\{\varphi_n\} \subseteq H' \cong H$

Since $x_n \xrightarrow{w} x$ $\quad \varphi(x_n) \to \varphi(x)$ $\forall \varphi \in H' \cong H$

Thus $\langle x_n, y_m \rangle = \varphi_m(x_n) \underset{\text{as } n\to\infty}{\to} \varphi_m(x) = \langle x, y_m \rangle$

$|\langle x, y_m - y\rangle| \leq \|x\| \|y_m - y\| \to 0$ as $m \to \infty$

Thus $\langle x, y_m - y \rangle \to 0 \Rightarrow \langle x, y_m \rangle \to \langle x, y \rangle$

$\therefore$ as $m, n \to \infty$ $\langle x_n, y_m \rangle \to \langle x, y \rangle$

5. A *weak Cauchy sequence* in a normed linear space X is a sequence $\{x_n\} \subset X$ such that for every $\varphi \in X'$ the sequence $\{\varphi(x_n)\}$ is Cauchy in $\mathbb{C}$. Show that a weak Cauchy sequence is bounded.

$|\varphi(x_n) - \varphi(x_m)| < \varepsilon$ $\quad$ WTS $\exists C > 0 \ni \|x_n\| < C$ $\forall n \in \mathbb{N}$

Since $\mathbb{C}$ is complete $\exists$ $\varphi(x) = \lim \varphi(x_n)$ $\forall \varphi \in X'$

Consider the canonical mapping $\ell_{x_n}(\varphi) = \varphi(x_n) \to \varphi(x)$

Since $\ell_{x_n}(\varphi) \to \varphi(x)$, $\{\|\ell_{x_n}(\varphi)\|\}$ is bdd and thus by the UBP $\{\|\ell_{x_n}\|\}$ is bdd

Since $\|\ell_{x_n}\| = \|x_n\|$ it follows $\{\|x_n\|\}$ is bdd

6. A normed linear space X is called *weakly complete* if each weak Cauchy sequence in X converges weakly in X. Show that if X is reflexive, then X is weakly complete.

$X \cong X^{**}$ the canonical mapping is surjective

WTS $\forall \{x_n\} \subseteq X \ni \{\varphi(x_n)\}$ is Cauchy in $\mathbb{C}$ $\forall \varphi \in X'$

$X \cong X''$ the canonical mapping is surjective

WTS $\forall \{x_n\} \subseteq X \ni \{\varphi(x_n)\}$ is Cauchy in $\mathbb{C}$ $\forall \varphi \in X'$

Let $x_n \xrightarrow{w} x \in X$

$\exists\ \varphi(x) \in \mathbb{C} \ni \varphi(x_n) \to \varphi(x)$ by completeness of $\mathbb{C}$

Since X is reflexive $\exists x \in X \ni C(x) = l_x$

Consider

$$C(x_n)\varphi = l_{x_n}(\varphi) = \varphi(x_n) \to \varphi(x) = l_x(\varphi) = C(x)\varphi$$

Thus $x_n \xrightarrow{w} x$ and $x \in X$

7. Let X be Banach. Show that a sequence $\{\varphi_n\} \subset X'$ is weak* convergent if and only if

   (a) The sequence $\{\varphi_n\}$ is bounded.

   (b) The sequence $\{\varphi_n(x)\}$ is Cauchy in $\mathbb{C}$ for every x in a dense subspace A of X.

$\varphi_n \xrightarrow{w^*} \varphi$ if $|\varphi_n(x) - \varphi(x)| \to 0$ $\forall x \in X$

$(\Rightarrow)$

Consider for $h > 0$

$$|\varphi_{n+h}(x) - \varphi_n(x)| \leq |\varphi_{n+h}(x) - \varphi(x)| + |\varphi_n(x) - \varphi(x)|$$

$\to 0$ since $\varphi_n \xrightarrow{w^*} \varphi$ $\forall x \in X$ including $x \in A$

Thus $\{\varphi_n(x)\}$ is Cauchy in $\mathbb{C}$ $\forall x \in X$

ATC $\{\varphi_n\}$ is not bdd

Thus $\exists \{\varphi_{n_k}\} \subseteq \{\varphi_n\} \ni \varphi_{n_k}(x) \to \infty$ $\forall x \in X$

However by uniqueness of limits $\varphi_{n_k}(x) \to \varphi(x)$ $\forall x \in X$ $\times$

$(\Leftarrow)$ $\{\varphi_n\}$ bdd & $\{\varphi_n(x)\}$ is Cauchy in $\mathbb{C}$ $\forall x \in A \ni \bar{A} = X$

WTS $\varphi_n \xrightarrow{w^*} \varphi$

Since $\{\varphi_n(x)\}$ is Cauchy in $\mathbb{C}$ $\forall x \in A$ given $\varepsilon > 0$

$\exists\ \varphi(x) \in \mathbb{C}$ and $N \in \mathbb{N} \ni \forall n \geq N$ $|\varphi_n(x) - \varphi(x)| < \varepsilon$.

Similarly since X is Banach and $\bar{A} = X$, $\forall x \in X\ \exists \{x_m\} \subseteq A \ni x_m \to x$

Since $\{\varphi_n\}$ is bdd $\forall n \in \mathbb{N}$ the limiting $\varphi \in X'$ is also bdd.

Similarly since X is Banach and $\bar{A}=X$, $\forall x \in X \; \exists \{x_m\} \subseteq A$ s.t. $x_m \to x$

Since $\{\varphi_n\}$ is bdd $\forall n \in \mathbb{N}$ the limiting $\varphi \in X'$ is also bdd.

Thus

$$|\varphi_n(x) - \varphi(x)| = |\varphi_n(\lim_{m\to\infty} x_m) - \varphi(\lim_{m\to\infty} x_m)|$$

$$= \lim_{m\to\infty} |\varphi_n(x_m) - \varphi(x_m)| < \varepsilon \quad \text{since } \{x_m\} \subseteq A$$

$\therefore |\varphi_n(x) - \varphi(x)| \to 0 \quad \forall x \in X$