1. Let X be a normed linear space and $\varphi_1, \varphi_2, \cdots, \varphi_n$, $n \in \mathbb{N}$, be linearly independent vectors in X'. Show that there exist $x_1, x_2, \cdots, x_n \in X$ such that

$$\varphi_i(x_j) = \delta_{ij} := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad 1 \leqslant i, j \leqslant n.$$

(Compare this problem with Problem 4 in Homework 1)

2. Consider the operator $T : l^1 \to l^1$ defined by

$$T\left(\{x_1, \cdots, x_n, \cdots\}\right) = \left(\{0, x_1, x_2, \cdots, x_n, \cdots\}\right), \ \forall \{x_n\} \in l^1.$$

Show that $T \in \mathcal{B}(l^1, l^1)$ and find T^*.

3. Let X, Y be normed linear spaces and $T \in \mathcal{B}(X, Y)$. Show that if T^{-1} exists and $T^{-1} \in \mathcal{B}(Y, X)$, then $(T^*)^{-1}$ also exists and $(T^*)^{-1} = (T^{-1})^*$.

4. Recall the space $C^1[0,1]$ defined by

$$C^1[0,1] := \{f \in C[0,1] : \ f' \text{ exists and } f' \text{ is continuous on } [0,1]\}$$

is a Banach space with the norm

$$\|f\|_{C^1} := \max_{x \in [0,1]} |f(x)| + \max_{x \in [0,1]} |f'(x)|.$$

   (a) Show that for each $x \in [0,1]$, the functional $\varphi_x : C^1[0,1] \to \mathbb{R}$, defined by

$$\varphi_x(f) := f'(x)$$

   is a bounded linear functional on $(C^1[0,1], \|\cdot\|_{C^1})$.

   (b) Fix $g : [0,1] \to [0,1]$ in $C^1[0,1]$ and define T_g on $C^1[0,1]$ by $T_g(f) = f \circ g$. Show that T_g is a bounded linear operator on $C^1[0,1]$ and find $T_g^*(\varphi_x)$.

5. Show that every Hilbert space is reflexive.

6. Let X be a normed linear space. Show that if X is reflexive, then for any $\varphi \in X'$, there exists $x \in X$ with $\|x\| = 1$ such that $\|\varphi\| = \varphi(x)$.

7. Show that a Banach space X is reflexive if and only if its dual space X' is reflexive.