HW5

Sunday, November 3, 2024 9:41 AM

1. Let X be a Banach space, $S, T \in \mathcal{B}(X, X)$ and $ST = TS$.

   (a) Show that $r_\sigma(ST) \leqslant r_\sigma(S) r_\sigma(T)$.

   (b) Show (with a counterexample) that the commutativity $ST = TS$ can not be dropped in part (a).

a) $r_\sigma(ST) = \lim (\|(ST)^n\|)^{1/n} = \lim (\|S^n T^n\|)^{1/n} \leq \lim (\|S^n\| \|T^n\|)^{1/n}$

Since $ST = TS$, $(ST)^n = S^n T^n$

$= \lim \|S^n\|^{1/n} \lim \|T^n\|^{1/n}$

$= r_\sigma(S) r_\sigma(T)$

b) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = BA$ Thus $AB \neq BA$

$r_\sigma(A) = 0$ $r_\sigma(B) = 0$ $r_\sigma(AB) = 1$ (max eigval & eigvals of triangular matrix are on diagonal)

$\therefore 1 = r_\sigma(AB) > 0 = r_\sigma(A) r_\sigma(B)$

2. Let H be a Hilbert space. Recall $T \in \mathcal{B}(H, H)$ is called a *normal* operator if $TT^* = T^*T$, where T^* is the (Hilbert) adjoint of T. Show that if T is normal, then

$$r_\sigma(T) = \|T\|.$$

Let $TT^* = T^*T$ $r_\sigma(T) = \lim \|T^n\|^{1/n}$

It sufficies to show $\|T^n\| = \|T\|^n$

Consider base case

$(\leq)$

$\|T^2\| = \sup_{\|x\| \leq 1} \|T^2 x\| \leq \sup_{\|x\| \leq 1} \|T\|^2 \|x\| \leq \|T\|^2$

$(\geq)$

$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle \leq \|T^*Tx\| \|x\|$

Note

$\|T^*Tx\|^2 = \langle T^*Tx, T^*Tx \rangle = \langle TT^*Tx, Tx \rangle = \langle T^*TTx, Tx \rangle = \|T^2 x\|^2$

It folows

$\|T^*Tx\| \|x\| = \|T^2 x\| \|x\| \leq \|T^2\| \|x\|^2$

By taking $\sup_{\|x\| \leq 1}$ we get $\|T\|^2 \leq \|T^2\|$

Let $\|T^k\| = \|T\|^k$ WTS $\|T^{k+1}\| = \|T\|^{k+1}$

$(\leq)$ $\|T^{k+1}\| = \sup_{\|x\|\leq 1} \|T^{k+1}x\| \leq \sup_{\|x\|\leq 1} \|T\|^{k+1}\|x\| \leq \|T\|^{k+1}$

$(\geq)$ WTS $\|T^{k+1}\| \geq \|T\|^{k+1}$ $\forall\ 1 \leq n \leq k$ $n \in \mathbb{N}$

$$\|T^{k+1}\| = \sup_{\|x\|\leq 1} \|T^{k+1}x\| \geq \left\|T^k \frac{Tx}{\|Tx\|}\right\| = \left\|T \frac{Tx}{\|Tx\|}\right\|^k = \frac{1}{\|Tx\|}\|T^2x\|^k$$
$$= \frac{1}{\|Tx\|}\|Tx\|^{2k}$$
$$= \|Tx\|^{2k-1}$$

$\therefore$ $\|T^{k+1}\| \geq \|T\|^{2k-1} \geq \|T\|^{k+1}$

Thus $\|T^n\| = \|T\|^n$ $\forall n \in \mathbb{N}$

$\Rightarrow$ $r_\sigma(T) = \lim \|T^n\|^{1/n} = \lim \|T\|^{n/n} = \|T\|$

3. Let $T : C[0,1] \to C[0,1]$ be defined by

$$(Tf)(x) = \int_0^x f(y)\,dy,\ \forall f \in C[0,1].$$

Find the spectrum of T.

Let $\{f_n\} \subseteq C[0,1]$ be bounded $(\|f_n\| < M\ \forall n \in \mathbb{N})$

WTS $Tf_{n_k} \to Tf$ Note $C[0,1]$ is Banach

Let $\varepsilon > 0$ $\delta = \frac{\varepsilon}{M}$ $|u-u'| < \delta$ $u, u' \in [0,1]$

$$|(Tf_n)u - (Tf_n)u'| = \left|\int_0^u f_n(y)dy - \int_0^{u'} f(y)dy\right|$$
$$\leq M|u-u'|$$
$$< M\delta$$
$$= \varepsilon$$

Thus by A.A. $\exists$ $\{Tf_{n_k}\} \subseteq \{Tf_n\}$ that converges uniformly in $C[0,1]$

$\therefore$ T is compact.

Since T is compact and $C[0,1]$ is Banach, for $\lambda \neq 0$

Let $f \in \ker T_\lambda$

$$(T - \lambda I)f = \int_0^x f(y)dy - \lambda f(x) = 0$$
$$\int_0^x f(y)dy = \lambda f(x)$$

$(T-\lambda I)f = \int_0^x f(y)dy - \lambda f(x)$

$$\int_0^x f(y)dy = \lambda f(x)$$

$$0 = \int_0^0 f(y)dy = \lambda f(0)$$

$\Rightarrow f(0) = 0$ since $\lambda \neq 0$

Since $\int_0^x f(y)dy \in C'[0,1]$, $f \in C'[0,1]$

$$\frac{d}{dx}\int_0^x f(y)dy = \lambda \frac{d}{dx} f(x)$$

$$f(x) = \lambda f'(x)$$

$$\Rightarrow f(x) = f(0)e^{x/\lambda} = 0$$

$$\Rightarrow \ker T_\lambda = \{0\} \Rightarrow \lambda \notin P(T)$$

$\therefore \sigma(T) = \{0\}$ since $\sigma(T) \neq \emptyset$

4. Let $\{a_n\} \subset \mathbb{C}$ be a sequence of scalars such that $a_n \to 0$, $n \to \infty$. Define $T : l^2 \to l^2$ by $T(\{x_1, x_2, x_3, \cdots\}) = \{a_1x_1, a_2x_2, a_3x_3, \cdots\}$. Show that T is compact.

Define $T_n : l^2 \to l^2$ $x \mapsto \{a_1x_1, \dots a_nx_n, 0, \dots\}$

Since $\dim R(T_n) < \infty$ $\forall n \in \mathbb{N}$, $T_n \in \mathcal{K}(l^2)$ $\forall n \in \mathbb{N}$

Consider

$$\|T - T_n\|^2 = \sup_{\|x\|\le 1} \|Tx - T_nx\|^2 = \sup_{\|x\|\le 1} \|\{a_{n+1}x_{n+1}, \dots\}\|^2$$

$$= \sup_{\|x\|\le 1} \sum_{k=n+1}^\infty |a_kx_k|^2$$

$$\le \sup_{\|x\|\le 1} \sup_{j>n} |a_j|^2 \sum_{k=n+1}^\infty |x_k|^2$$

$$\le \sup_{j>n} |a_j|^2 \sup_{\|x\|\le 1} \|x\|$$

$$= \sup_{j>n} |a_j|^2$$

$$\to 0 \text{ as } n \to \infty$$

Thus $T_n \to T$ in norm

$\therefore T \in \mathcal{K}(l^2)$ by thm in class

5. Let H be a Hilbert space and $T \in \mathcal{K}(H, H)$. Suppose $\{e_n\}$ is an orthonormal basis of H, show that $\langle Te_n, e_n\rangle \to 0$, $n \to \infty$.

Since $\{e_n\}$ is orthonormal, $\|e_n\| = 1$ $\forall n \in \mathbb{N} \Rightarrow \{e_n\}$ is bdd

By B.A., as H is reflexive, $\overline{B_1(0)}$ is W.S.C.

Thus $\exists$ $\{e_{n_k}\} \subset \{e_n\}$ $\ni$ $e_{n_k} \xrightarrow{w} z \in \overline{B_1(0)}$

By B.A., as H is reflexive, $\overline{B_1(0)}$ is W.S.C.

Thus $\exists \{e_{n_k}\} \subseteq \{e_n\} \ni e_{n_k} \xrightarrow{w} z \in \overline{B_1(0)}$.

Note $\langle e_i, e_j \rangle = \delta_{ij}$

As

$\langle e_{n_k}, x \rangle \to z$ as $k \to \infty$ $\forall x \in H$

Choosing $x = e_{n_{k+1}} \in H$ $\langle e_{n_k}, e_{n_{k+1}} \rangle = 0$ $\forall k \in \mathbb{N}$

$\therefore e_{n_k} \xrightarrow{w} 0$

Note we don't need the subsequence since $\langle e_n, e_{n+1} \rangle = 0$ $\forall n \in \mathbb{N}$

Consider

$|\langle Te_n, e_n \rangle| \leq \|Te_n\| \|e_n\| = \|Te_n\|$

as $T \in \mathcal{K}(H)$ and $e_n \xrightarrow{w} 0$, $Te_n \to 0$ as $k \to \infty$

$\therefore \langle Te_n, e_n \rangle \to 0$ as $n \to \infty$

6. Let X be a normed linear space and $T \in \mathcal{K}(X, X)$. Suppose $S \in \mathcal{B}(X, X)$, show that both TS and ST are compact linear operators on X.

Let $\{x_n\} \subseteq X \ni \|x_n\| \leq M$ $\forall n \in \mathbb{N}$

(TS) WTS $\{Sx_n\}$ is bdd

Consider $\|Sx_n\| \leq \|S\| \|x_n\| \leq M \|S\|$

Thus $\{Sx_n\}$ is bdd. Since T is compact

$\exists \{TSx_{n_k}\} \subseteq \{TSx_n\}$ that is convergent.

$\therefore$ TS is a compact operator.

(ST) Since T is compact $\exists \{Tx_{n_k}\} \subseteq \{Tx_n\}$ that is convergent.

By the bddness of S $\lim STx_{n_k} = S \lim Tx_{n_k}$ which is convergent.

$\therefore$ ST is compact.

7. Let H be a Hilbert space and $T \in \mathcal{B}(H, H)$. Let T^* denote the Hilbert adjoint operator of T. Show that T is compact if and only if T^*T is compact.

Let $\{x_n\} \subseteq H \ni \|x_n\| \leq M$ $\forall n \in \mathbb{N}$

($\Rightarrow$) Since $T \in \mathcal{K}(H) \subseteq B(H)$ $T^* \in B(H)$

As shown in #6 T^*T (ST) is compact

As shown in #6 T^*T (ST) is compact

($\Leftarrow$) Since T^*T is compact,

$\exists \{T^*Tx_{n_k}\} \subseteq \{T^*Tx_n\} \ni T^*Tx_{n_k} \to T^*Tx \in H$

$\Rightarrow x \in H$

WTS $\{Tx_{n_k}\} \subseteq \{Tx_n\} \ni Tx_{n_k} \to Tx$

Consider $\|Tx_{n_k} - Tx\|^2 = \langle T^*T(x_{n_k}-x), x_{n_k}-x \rangle$

$\leq \|T^*T(x_{n_k}-x)\| \, \|x_{n_k}-x\|$

$\leq \|T^*T(x_{n_k}-x)\|(\|x_{n_k}\|+\|x\|)$

$\to 0$ as $k \to \infty$ as $\|x_{n_k}\|+\|x\| < \infty$

since $T^*Tx_{n_k} \to T^*Tx$

Thus T is compact

8. Let $T : l^2 \to l^2$ be defined by

$$Tx = \left\{ \frac{x_2}{1}, \frac{x_3}{2}, \frac{x_4}{3}, \cdots \right\}, \ \forall x = \{x_1, x_2, x_3, \cdots\} \in l^2.$$

Show that T is compact and the point spectrum $\sigma_p(T) = \{0\}$.

Let $x \in l^2$

Let L be the left shift operator and

$\tilde{T} : l^2 \to l^2 \quad x \mapsto \{\frac{x_1}{1}, \frac{x_2}{2}, \ldots\}$

It follows by inspection $T = \tilde{T}L$

Thus by #4 $\tilde{T}$ is compact since $\frac{1}{n} \to 0$ and

by #6 $\tilde{T}L$ is compact if L is bdd.

Consider $\|Lx\| = \sum_{i=2}^{\infty} |x_i|^2 \leq \sum_{i=1}^{\infty} |x_i|^2 = \|x\|$

$\therefore$ L is bdd $\Rightarrow$ T is compact.

Since l^2 is a Hilbert space, hence Banach,

$\forall \lambda \neq 0$ and $x \in \ker T_\lambda$

$T_\lambda x = 0$ $\qquad x_2 = \lambda x_1 \qquad \frac{x_4}{3} = 2\lambda^3 x_1$

$Tx = \lambda x$ $\qquad \frac{x_3}{2} = \lambda^2 x_1 \qquad \frac{x_5}{4} = 3!\lambda^4 x_1$

$\frac{x_{n+1}}{n} = \lambda x_n$

$x_n = (n-2)!\lambda^{n-1}x_1 \quad \forall n \geq 2$

Note

$(n-2)!\lambda^{n-1} = (n-2)!\, e^{(n-1)\ln\lambda} = \sum \frac{(n-2)!((n-1)\ln\lambda)^k}{k!} \to \infty$ as $n \to \infty$

Note

$(n-2)!\,\lambda^{n-1} = (n-2)!\,e^{(n-1)\ln\lambda} = \sum_{k=0}^{\infty} \frac{(n-2)!\,((n-1)\ln\lambda)^k}{k!} \to \infty$ as $n \to \infty$

Thus for $x \in \ell^2$, $x_1 = 0 \Rightarrow x = 0$

$\therefore \lambda \in \rho(T) \Rightarrow \sigma(T) = \{0\}$ as $\sigma(T) \neq \emptyset$

9. (*Extra credits*) Let $T : L^2[0,1] \to L^2[0,1]$ be defined by $(Tf)(x) = \int_0^1 K(x,y)f(y)dy$ where $K(x,y)$ satisfies $\int_0^1 \int_0^1 |K(x,y)|^2 dxdy < \infty$. Show that T is compact.

Let $\{f_n\} \subseteq L^2[0,1]$ be bdd $\|f_n\|_y \leq N$ $\forall n \in \mathbb{N}$

Let $\varepsilon > 0$ $\delta > 0$. Let $|x - x'| < \delta$ (Note $K \in L^2 \times L^2$ and $\|\cdot\|_y$ is for a L^2)

It follows

$\|Tf_n(x) - Tf_n(x')\|_x = \|\int_0^1 K(x,y)f_n(y)dy - \int_0^1 K(x',y)f_n(y)dy\|_x$

$= \|\int_0^1 (K(x,y) - K(x',y))f_n(y)dy\|_x$

$\leq \|\,\|\int_0^1 K(x,y) - K(x',y)dy\|\,\|f_n\|_y\|_x$ ← C-S

$\leq N \int_0^1 |\int_0^1 K(x,y) - K(x',y)dy|^2 dx$

$\leq N \int_0^1 \int_0^1 |K(x,y) - K(x',y)|^2 dydx$

This is finite by triangle inequality and assumption

Since $K \in L^2 \times L^2$ $\exists$ $\{k_n\} \subseteq C[0,1] \times C[0,1]$ $\ni$ $k_n \nearrow K$

$= N \int_0^1 \int_0^1 |\lim (k_n(x,y) - k_n(x',y))|^2 dydx$

also because lim from below

Since $\{k_n\} \subseteq C[0,1] \times C[0,1]$

$\lim (k_n(x,y) - k_n(x',y)) < \sqrt{\frac{\varepsilon}{N}}$ for $|x - x'| < \delta$

$\therefore \|Tf_n(x) - Tf_n(x')\|_x \leq \varepsilon \Rightarrow \{Tf_n\}$ is equicontinuous

As shown in class this sequence is also uniformly bdd

$\therefore$ by A.A. $\exists$ $\{Tf_{n_k}\} \subseteq \{Tf_n\} \ni Tf_{n_k} \to Tf \in L^2 \Rightarrow T$ is compact.