HW1-2

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1. Let H be a Hilbert space and M be a closed subspace of H. Suppose φ is a bounded linear functional defined on M. Use Hilbert space methods to show that φ can be uniquely extended to H, i.e., show that there exists a unique bounded linear functional $\tilde{\varphi}$ on H such that $\tilde{\varphi}(m) = \varphi(m)$, $\forall m \in M$ and $\|\tilde{\varphi}\| = \|\varphi\|$.

Let $\varphi \in M'$ WTS $\exists! \tilde{\varphi} \in H' \ni \tilde{\varphi}|_M = \varphi$ & $\|\tilde{\varphi}\| = \|\varphi\|$

We know this true if $M = \{0\}$ and $\varphi = 0 = \tilde{\varphi}$.

Consider $M \neq \{0\}$

Since $\varphi \in M'$ and M is a closed subspace of a Hilbert space, by the RRT $\exists! z \ni$ $\varphi(x) = \langle x, z \rangle$ $\forall x \in M$ and $\|\varphi\| = \|z\|$.

Consider $\tilde{\varphi}: X \to \mathbb{C}$ $x \mapsto \langle x, z \rangle$. It follows $\tilde{\varphi}|_M = \varphi$ and by RRT $\|\tilde{\varphi}\| = \|z\| = \|\varphi\|$.
The uniqueness of the extension also comes from the RRT.

Consider another extension $\hat{\varphi} \ni$ $\langle \cdot, w \rangle|_M = \langle \cdot, z \rangle$. Then given $m \in M$

$\langle m, w \rangle = \langle m, z \rangle \Rightarrow \langle m, w - z \rangle = 0$

$w - z \in M^\perp$. Thus we can write $w = z + z^\perp$

$\|\hat{\varphi}\|^2 = \|w\|^2 = \|z\|^2 + \|z^\perp\|^2$. However since $\|\hat{\varphi}\| = \|\varphi\| = \|z\|$,

$\|z^\perp\| = 0 \Rightarrow z^\perp = 0$. Thus $w = z$ and $\tilde{\varphi}$ is unique.

<u>Remark</u> This could also be shown by using the norm endowed on H.

2. Let X be a normed linear space and $\mathbf{0} \neq x_0 \in X$. Show that there exists $\varphi \in X'$ such that $\|\varphi\| = \|x_0\|$ and $|\varphi(x_0)| = \|x_0\|^2$.

X nls $x_0 \neq 0 \in X$ WTS $\exists$ $\tilde{\varphi} \in X' \ni \|\tilde{\varphi}\| = \|x_0\|$ & $|\tilde{\varphi}(x_0)| = \|x_0\|^2$

By Thrm 4.3-3 given $x_0 \neq 0$ $\exists$ $\varphi \in X' \ni \|\varphi\| = 1$ & $|\varphi(x_0)| = \|x_0\|$

Consider $\tilde{\varphi}: X \to \mathbb{C}$ $x \mapsto \varphi(x)\|x_0\|$

$\tilde{\varphi}(\alpha x+y) = \varphi(\alpha x+y)\|x_0\|$

$= (\alpha\varphi(x) + \varphi(y))\|x_0\|$ since φ is linear

$= \alpha\tilde{\varphi}(x) + \tilde{\varphi}(y)$

$\tilde{\varphi}(x) = \varphi(x)\|x_0\| \leq |\varphi(x)|\,\|x_0\| \leq \|x\|\,\|x_0\|\,\|\varphi\|'$

Thus $\tilde{\varphi} \in X'$.

It follows $|\tilde{\varphi}(x_0)| = |\varphi(x_0)\|x_0\|| = \|x_0\|^2$ and

$\|\tilde{\varphi}\| = \sup_{\|x\|\leq 1}|\tilde{\varphi}(x)| = \sup_{\|x\|\leq 1}|\varphi(x)\|x_0\|| = \|\varphi\|'\|x_0\|$ ▣

3. Let X be a normed linear space and Y be a closed proper subspace of X. For any $x_0 \in X \backslash Y$, let $d(x_0, Y) := \inf_{y\in Y}\|x_0 - y\|$ be the distance from x_0 to Y. Show that there exists $\varphi \in X'$ such that $\|\varphi\| = 1$, $\varphi(x_0) = d(x_0, Y)$ and $\varphi(y) = 0$ for all $y \in Y$.

$Y < X$ $\quad x_0 \in X\backslash Y$ $\quad$ WTS $\exists\, \varphi \in X' \ni \|\varphi\| = 1$ $\quad \varphi(x_0) = d(x_0, Y)$
$\varphi(y) = 0 \;\; \forall y \in Y$

Let $Y' = \operatorname{span}\{x_0, Y\} \leq X$

$\varphi : Y' \to \mathbb{C}$ $\qquad y' = ax_0 + y \longmapsto a\,d(x_0, Y)$

$\varphi(\alpha y_1' + y_2') = \varphi(\alpha(x_0 + \frac{y_1}{a_1}) + x_0 + \frac{y_2}{a_2}) = \alpha d(x_0, Y) + d(x_0, Y)$
$= \alpha\varphi(y_1') + \varphi(y_2')$

$(\leq)$ $|\varphi(y')| = |\varphi(ax_0 + y)| = a\,d(x_0, Y) \leq a\|x_0 + \frac{y}{a}\| = \|ax_0 + y\|$

Thus $\|\varphi\| \leq 1$

$(\geq)$ Since φ is bdd and X is a nls
by H-B $\exists$ $\tilde{\varphi} \in X' \ni \|\tilde{\varphi}\| = \|\varphi\| \leq 1$

Let $y \in Y' \ni \|y\| \leq 1$

$\|\tilde{\varphi}\| \geq |\tilde{\varphi}(y)| = |\varphi(y)| = d(x_0, Y)$

Thus $\exists\, \tilde{\varphi} \in X'$, $\|\tilde{\varphi}\| = 1$, $\tilde{\varphi}(x_0) = \varphi(x_0) = d(x_0, Y)$,

and $\tilde{\varphi}(y) = \varphi(y) = 0 \;\; \forall y \in Y$

4. Let X be a normed linear space and $x_1, x_2, \cdots, x_n$, $n \in \mathbb{N}$, be linearly independent vectors in X. Show that there exist $\varphi_1, \varphi_2, \cdots, \varphi_n \in X'$ such that

$$\varphi_i(x_j) = \delta_{ij} := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad 1 \leqslant i, j \leqslant n.$$

Let $Y = \text{span}\{x_i\}$. Let $y = \sum a_i x_i \in Y$.

Define $\varphi_i : Y \to \mathbb{C}$ $\quad y \longmapsto a_i$

It follows $y = \sum a_i x_i = \sum \varphi_i(y) x_i$

Each φ_i is well defined by the linear independence of $\{x_i\}$.

I claim $\{\varphi_i\} \subseteq Y'$ since

$\varphi_i(\alpha y_1 + y_2) = \alpha a_{1_i} + a_{2_i} = \alpha \varphi_i(y_1) + \varphi_i(y_2)$

$|\varphi_i(y)| = |a_i|$ thus φ_i is bdd.

Consider $\varphi_i(x_j) = a_i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \delta_{ij}$

5. Let X be a normed linear space and $\{y_n\} \subset X$. Show that $z \in Y = \overline{span\{y_n\}}$ if and only if $\forall \varphi \in X'$, $\varphi(y_n) = 0$, $\forall n \in \mathbb{N}$ implies $\varphi(z) = 0$.

$(\Rightarrow)$ Let $z \in Y$ WTS given $\varphi \in X' \ni \varphi(y_n) = 0 \Rightarrow \varphi(z) = 0$

Let $\varphi \in X' \ni \varphi(y_n) = 0 \ \forall n \in \mathbb{N}$

Since $z \in Y$ we can write $z = \sum a_i y_i$.
By the linearity of φ,

$$\varphi(z) = \varphi(\sum a_i y_i) = \sum a_i \varphi(y_i) = 0$$

$(\Leftarrow)$ Given $\varphi \in X' \ni \varphi(y_n) = 0 \ \forall n \in \mathbb{N} \Rightarrow \varphi(z) = 0$
WTS $z \in \text{span}\{y_n\}$

By our given condition $\varphi \in \{y_n\}^0$ implies $\varphi(z) = 0$

ATC $z \notin Y$ Thus $z \neq \sum a_i y_i$. By H-B thrm $\exists$ $\tilde{\varphi} \in X' \ni \tilde{\varphi}|_Y = \varphi$.

It follows $\tilde{\varphi}(z) \neq \tilde{\varphi}(\sum a_i y_i) = \sum a_i \tilde{\varphi}(y_i) = \sum a_i \varphi(y_i) = 0 \Rightarrow \tilde{\varphi}(z) = 0$

It follows $\tilde{\varphi}(z) \neq \tilde{\varphi}(\sum a_i y_i) = \sum a_i \tilde{\varphi}(y_i) = \sum a_i \varphi(y_i) = 0 \Rightarrow \tilde{\varphi}(z) = 0$ $\times$

$\therefore z \in Y$

6. Let X be a normed linear space and $A, B \subset X$ be two nonempty convex subsets such that $A \cap B = \varnothing$. Assume that A is closed and B is compact, show that A and B can be strictly separated by a hyperplane, i.e., there exists $c \in \mathbb{R}$ and $\varphi \in X'$ such that $\varphi(a) < c < \varphi(b)$, $\forall a \in A$, $b \in B$.

Consider $d(A,B) = \inf\{\|a-b\| \mid a \in A, b \in B\}$

I claim that $d(A,B) > 0$ and the minimum is achieved.

Consider for fixed $a \in A$ $f_a(b) = \|a-b\|$

By the compactness of B and continuity of the norm, we know $\exists b_a \in B$ that reaches a minimum value since f_a is bounded as the image of a compact set.

Similarly, for $a \in A$ define $g_b(a) = \|a - b_a\|$.

As A is closed and $g_b(a)$ is continuous we know $g_b(a)$ also reaches a minimum.

ATC $d(A,B) = 0$. Let $\varepsilon > 0$. Then $\exists \{a_n\}, \{b_n\} \subseteq A, B$ $\ni \|a_n - b_n\| < \varepsilon$.

Since B is compact $\exists \{b_{n_k}\}, b_{n_k} \to b_a \Rightarrow b_a \in B$.

However since A is closed, this also implies $b_a \in A$.

$\times$ since $A \cap B = \varnothing$.

Define $C := \{a - b \mid a \in A, b \in B\}$.

As the sum of two convex sets, C is also convex. Cruically, $0 \notin C$ since $d(A,B) > 0$.

By the corollary $\exists \varphi \in X' \ni \varphi(c) \leq \varphi(c') \; \forall c \in C \; c' \in C^c$

$\varphi(a-b) \leq \varphi(0) = 0$ as noted $a - b \neq 0 \Rightarrow \varphi(c) < \varphi(0)$

Thus $\varphi(a) < \varphi(b)$.

Let $C = \frac{\varphi(a)+\varphi(b)}{2}$.

7. Let X be a normed linear space and $Y \leqslant X$ be a subspace such that $\overline{Y} \neq X$. Show that there exists $\varphi \in X'$, $\varphi \neq \mathbf{0}$, such that $\varphi(y) = 0$, $\forall y \in Y$.

$Y < X$ WTS $\exists \varphi \in X'$ $\varphi(y) = 0$ $\forall y \in Y$

Let $\varphi \in Y'$ $y \mapsto d(y, Y)$

Clearly φ is bdd and linear since

$\varphi(\alpha y_1 + y_2) = d(\alpha y_1 + y_2, Y) = 0 = \alpha d(y_1, Y) + d(y_2, Y)$

and $|\varphi(y)| = |d(y, Y)| = 0 \leq \|y\|$

Thus by H-B $\exists$ $\tilde{\varphi} \in X'$ $\ni$ $\tilde{\varphi}|_Y = \varphi$

We see $\tilde{\varphi}(y) = \varphi(y) = d(y, Y) = 0$ $\forall y \in Y$.

Define $\tilde{\varphi} : X \to \mathbb{C}$ $x \mapsto d(x, Y)$

We see $\tilde{\varphi}(x) \neq 0$ $\forall x \in X \setminus Y$ thus $\tilde{\varphi} \neq 0$