HW2

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1. Let X be a normed linear space and $\varphi_1, \varphi_2, \cdots, \varphi_n$, $n \in \mathbb{N}$, be linearly independent vectors in X'. Show that there exist $x_1, x_2, \cdots, x_n \in X$ such that

$$\varphi_i(x_j) = \delta_{ij} := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad 1 \leqslant i, j \leqslant n.$$

(Compare this problem with Problem 4 in Homework 1)

Let $Y' = \operatorname{span}\{\varphi_i\} \subseteq X'$, $\varphi = \sum_{i=1}^n a_i \varphi_i$

Let l_i be from the canonical mapping of $X \to X''$

$l_{x_i} : Y' \to \mathbb{C} \quad \varphi \mapsto \varphi(x_i)$

We also know from a lemma in class that each l_{x_i} is well defined and the canonical mapping is injective. Since the mapping is injective, $\exists \{x_i\} \subseteq X \ni C(x_i) = l_{x_i}$ for each φ_i as l_{x_i} is determined by actions on L.I. φ_i.

Let $Y = \operatorname{span}\{x_i\}$ and $y = \sum_{j=1}^n b_j x_j$

$\varphi(y) = \sum_{i=1}^n a_i \varphi_i(y) = \varphi(\sum_{j=1}^n b_j x_j) = \sum_{j=1}^n b_j \varphi(x_j)$

Thus $a_i = \varphi(x_i)$ and $b_j = \varphi_j(y)$

$\Rightarrow \varphi(x_j) = \sum_{i=1}^n a_i \varphi_i(x_j) = \sum_{i=1}^n \varphi(x_j) \varphi_i(x_j) \Rightarrow \varphi_i(x_j) = \delta_{ij}$

2. Consider the operator $T : l^1 \to l^1$ defined by

$$T(\{x_1, \cdots, x_n, \cdots\}) = (\{0, x_1, x_2, \cdots, x_n, \cdots\}), \ \forall \{x_n\} \in l^1.$$

Show that $T \in \mathcal{B}(l^1, l^1)$ and find T^*.

Let $x \in l'$ $\quad \|x\| = \sum_{i=1}^\infty |x_i| = 0 + \sum_{i=2}^\infty |x_{i-1}| = \|Tx\|$

Thus $T \in B(l', l')$

$T^* g x = g T x$

Note $(l')' \cong l^\infty$ Thus $g \in l^\infty$

$gTx = \sum_{i=1}^\infty |g_{i+1} x_i| = \sum_{i=2}^\infty |g_i x_{i-1}| = T^* g x$

Thus $T^* : Y' \to X' \quad g \mapsto \{g_2, g_3, \ldots\}$ (left shift)

3. Let X, Y be normed linear spaces and $T \in \mathcal{B}(X, Y)$. Show that if T^{-1} exists and $T^{-1} \in \mathcal{B}(Y, X)$, then $(T^*)^{-1}$ also exists and $(T^*)^{-1} = (T^{-1})^*$.

Since T^{-1} exists T is bijective.

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Since T^{-1} exists, T is bijective.

Let $\varphi_y \in N(T^*)$ and $x \in X$

$0 = T^*\varphi_y x = \varphi_y T x \Rightarrow \varphi_y = 0$ since T is injective and x was arbitrary

Thus T^* is injective

Let $\varphi_x \in X'$ WTS $\exists \varphi_y \in Y' \ni T\varphi_y = \varphi_x$

Let $x \in X$

Define $\varphi_y := \varphi_x T^{-1}$

$T^*\varphi_y x = \varphi_y T x = \varphi_x T^{-1} T x = \varphi_x x$

Thus T^* is surjective.

$\exists$ T^* is a bijection $\Rightarrow \exists (T^*)^{-1}$

WTS $(T^*)^{-1} = (T^{-1})^*$

Let $\varphi \in X'$ $x \in X$ $\varphi_y \in Y$ $y \in Y$

Consider $T^*(T^{-1})^*\varphi\, x = T^*\varphi T^{-1} x = \varphi T T^{-1} x = \varphi\, x$

Similarly $(T^{-1})^* T^* \varphi_y\, y = (T^{-1})^* \varphi_y T y = \varphi_y T^{-1} T y = \varphi_y\, y$

$\therefore (T^{-1})^* T^* = I_Y$ and $T^*(T^{-1})^* = I_X \Rightarrow (T^{-1})^* = (T^*)^{-1}$

4. Recall the space $C^1[0,1]$ defined by

$$C^1[0,1] := \{f \in C[0,1] : \; f' \text{ exists and } f' \text{ is continuous on } [0,1]\}$$

is a Banach space with the norm

$$\|f\|_{C^1} := \max_{x\in[0,1]} |f(x)| + \max_{x\in[0,1]} |f'(x)|.$$

(a) Show that for each $x \in [0,1]$, the functional $\varphi_x : C^1[0,1] \to \mathbb{R}$, defined by

$$\varphi_x(f) := f'(x)$$

is a bounded linear functional on $(C^1[0,1], \|\cdot\|_{C^1})$.

(b) Fix $g : [0,1] \to [0,1]$ in $C^1[0,1]$ and define T_g on $C^1[0,1]$ by $T_g(f) = f \circ g$. Show that T_g is a bounded linear operator on $C^1[0,1]$ and find $T_g^*(\varphi_x)$.

a) Let $f_1, f_2 \in C'$ $\alpha \in \mathbb{R}$

a) Let $f_1, f_2 \in C^1$ $\alpha \in \mathbb{R}$

$\varphi_x(\alpha f_1 + f_2) = \frac{d}{dx}[\alpha f_1(x) + f_2(x)] = \alpha f_1'(x) + f_2'(x) = \alpha \varphi_x(f_1) + \varphi_x(f_2)$

$|\varphi_x f| = |f'(x)| \leq \max_{x \in \Omega} |f'(x)| \leq \|f\|_{C^1}$

b) Let $f_1, f_2 \in C^1$ $\alpha \in \mathbb{R}$

$T_g(\alpha f_1 + f_2) = (\alpha f_1 + f_2) \circ g = \alpha f_1 \circ g + f_2 \circ g = \alpha T_g(f_1) + T_g(f_2)$

$|T_g f| = |f(g(x))| \leq \max_{x \in \Omega} |f(x)| \leq \|f\|_{C^1}$

WTF $T_g^*(\varphi_x)$

$T^*_g \varphi_x f = \varphi_x T_g f = \varphi_x f(g(x)) = f'(g(x)) g'(x)$

Thus $T^*_g \varphi_x f = f'(g(x)) g'(x)$

5. Show that every Hilbert space is reflexive.

WTS C is surjective $\Rightarrow CH = H''$

Define $A : H' \to H$ $\quad f \mapsto z$

Where x is the Riez representative of f.

Notice for $\alpha, \beta \in \mathbb{C}$ $f_1, f_2 \in H'$

$(\alpha f_1 + \beta f_2) x = \alpha f_1(x) + \beta f_2(x) = \alpha \langle x, z_1 \rangle + \beta \langle x, z_2 \rangle$
$= \langle x, \bar{\alpha} z_1 + \bar{\beta} z_2 \rangle$

Thus $A(\alpha f_1 + \beta f_2) = \bar{\alpha} A f_1 + \bar{\beta} A f_2$.

Define $\langle f, g \rangle_1 = \langle Ag, Af \rangle$

- $\langle x + y, h \rangle_1 = \langle Ah, A(x+y) \rangle$
  $= \langle Ah, Ax \rangle + \langle Ah, Ay \rangle$
  $= \langle x, h \rangle_1 + \langle y, h \rangle_1$
- $\langle x, y \rangle_1 = \langle Ay, Ax \rangle = \overline{\langle Ax, Ay \rangle} = \overline{\langle y, x \rangle_1}$
- $\langle x, x \rangle_1 = 0 = \langle Ax, Ax \rangle \Rightarrow Ax = 0 \Rightarrow x = 0$
- $\langle \alpha x, y \rangle_1 = \langle Ay, A\alpha x \rangle = \langle Ay, \bar{\alpha} Ax \rangle = \alpha \langle Ay, Ax \rangle = \alpha \langle x, y \rangle_1$

H' is complete as $\mathbb{C}$ is complete.

Thus $(H', \langle \cdot, \cdot \rangle_1)$ is a Hilbert space.

Let $g \in H''$ and $g(f) \underset{RRT}{=} \langle f, f_0 \rangle_1 = \langle Af_0, Af \rangle = \langle Af_0, z \rangle$
$= f(Af_0)$

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$= f(Af_0)$

Thus $g = CAf_0$ where C is the canonical mapping.

$\therefore$ C is surjective $\Rightarrow H \cong H''$

6. Let X be a normed linear space. Show that if X is reflexive, then for any $\varphi \in X'$, there exists $x \in X$ with $\|x\| = 1$ such that $\|\varphi\| = \varphi(x)$.

Since X is reflexive, $X'' \cong X$ or the dual of X' is isomorphic to X.

If $\varphi = 0$ $\forall x \in X \ \exists \ \|x\| = 1$ & $\|\varphi\| = 0 = \varphi(x)$

By a corollary of H-B $\forall \varphi \in X' \setminus \{0\} \ \exists \ l_x \in X'' \ \ni$ $\|l_x\| = 1$ and $l_x \varphi = \|\varphi\|$.

Since X is reflexive $\forall \ l_x \in X'' \ \exists \ x \in X \ \ni$ $\|l_x\| = \|x\|$ and $l_x \varphi = \varphi(x)$

Thus $\exists \ x \in X \ \ni \ \|x\| = 1$ and $\varphi(x) = \|\varphi\|$

7. Show that a Banach space X is reflexive if and only if its dual space X' is reflexive.

$(\Rightarrow)$ WTS $(X^*)^{**} \cong X^*$

Given $X^{**} \cong X$

We can write $(X^*)^{**} = (X^{**})^* \cong (X)^*$

thus X^* is reflexive.

$(\Leftarrow)$ WTS $X \cong X^{**}$ Given $X^* \cong (X^*)^{**}$

Let $l_x \in X^{**}$ $\psi \in X^{***}$ $\varphi \in X^* \ \ni \ C_1 \varphi = \psi$

$\psi l_x = l_x \varphi = \varphi(x) \Rightarrow \exists \ x \in X \ \ni \ Cx = l_x$

$\therefore$ C is surjective $\Rightarrow X \cong X^{**}$

$\therefore$ C is surjective $\Rightarrow$ X = X...