HW4

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1. Let X, Y be normed linear spaces. If $T_1 : X \to Y$ is a closed linear operator and $T_2 \in \mathcal{B}(X, Y)$, show that $T_1 + T_2$ is a closed linear operator.

Let

$\{z_n\} = \{(x_n, (T_1+T_2)x_n)\} \ni x_n \to x$ & $z = (x, (T_1+T_2)x)$

WTS $x \in D(T_1+T_2)$ & $y = (T_1+T_2)x$

$(T_1+T_2)x_n \to y$

$T_1x_n + T_2x_n \to y$

$T_1x_n \to y - T_2x$ since T_2 is bdd

Since T_1 is closed $x \in D(T_1) \subseteq D(T_1+T_2)$

Similarly we can see $T_1x = y - T_2x$ as T_1 is closed

Thus $(T_1+T_2)x = T_1x + T_2x = y - T_2x + T_2x = y$

$\therefore$ T_1+T_2 is closed

2. Let X, X_1, X_2 be Banach spaces and $T_1 : \mathcal{D}(T_1) \to X_1$, $T_2 : \mathcal{D}(T_2) \to X_2$ be closed linear operators with $\mathcal{D}(T_1) \subset \mathcal{D}(T_2) \subset X$. Show that there exists $M > 0$ such that

$$\|T_2x\|_{X_2} \leqslant M\left(\|x\|_X + \|T_1x\|_{X_1}\right), \ \forall x \in \mathcal{D}(T_1).$$

By lemma 4.13-5 since both T_1 & T_2 are closed and the range of both are Banach, $D(T_1)$ and $D(T_2)$ are closed and thus Banach.

By the closed graph thrm, T_2 is bdd.

As $D(T_2) \supseteq D(T_1)$, $\exists M > 0 \ni$

$\|T_2x\|_{X_2} \leq M\|x\|_X \quad \forall x \in D(T_1) \subseteq X$

It follows

$\|T_2x\|_{X_2} \leq M\|x\|_X \leq M(\|x\|_X + \|T_1x\|_{X_1})$

3. Let $A \subset C[0, 1]$ be a closed subspace that only consists of $C^1[0, 1]$ functions. Show that A is finite dimensional.

WTS $\overline{B_1(0)} \subseteq A$ is compact

Let $\{f_n\} \subseteq \overline{B_1(0)}$, $\varepsilon > 0$, $\delta = \frac{\varepsilon}{K}$ where $K = \sup_{n \in \mathbb{N}} \sup_{x \in [0,1]} f_n'(x)$,
and $|x-y| < \delta$

Thus $\{f_n\}$ is uniformly bdd as a subset of $\overline{B_1(0)} \subseteq A$

By the MVT $|f_n(x) - f_n(y)| = f_n'(\zeta)|x-y|$ for some $\zeta \in [0,1]$

It follows $|f_n(x) - f_n(y)| \leq K|x-y| < K\delta = \varepsilon \Rightarrow \{f_n\}$ is equicontinuous

Thus, by Arzelà-Ascoli, $\exists \{f_{n_k}\} \subseteq \{f_n\}$ that converges uniformly.
As $\overline{B_1(0)}$ is closed, the $f_{n_k} \to f \Rightarrow f \in \overline{B_1(0)}$.

$\therefore$ since $\{f_n\}$ was arbitrary and has a convergent subsequence, $\overline{B_1(0)}$ is compact $\Rightarrow \dim A < \infty$

4. (a) Let $T : C[0,1] \to C[0,1]$ be defined by $Tx = vx$, where $v \in C[0,1]$ is fixed. Find the spectrum $\sigma(T)$.

   (b) Find a linear operator $T : C[0,1] \to C[0,1]$ whose spectrum $\sigma(T)$ is a given interval $[a,b]$, where $a, b \in \mathbb{R}$.

a) Note by the Bdd-Inverse Thrm, if $T \in B(X)$ and T is bijective, T^{-1} is bdd. So for $\lambda \in \sigma(T)$, I claim it is sufficient to show that $R_\lambda(T)$ or T_λ is not bijective. Notice if $T \in B(X)$, $T_\lambda \in B(X)$.

Justification of claim

$\lambda \in \sigma_p(T) \Rightarrow T_\lambda$ is not injective

$\lambda \in \sigma_c(T) \Rightarrow R_\lambda(T)$ is not bdd $\Rightarrow T_\lambda$ is not bijective by thrm

$\lambda \in \sigma_r(T) \Rightarrow R_\lambda(T)$ is not densely defined so T_λ is not surjective

Since if $\overline{R(T_\lambda)} \neq Y$ $\exists$ $y \in Y \ni \nexists x \in D(T_\lambda) \ni T_\lambda x = y$
so T_λ is not surjective

Consider T given in the problem,

$\|Tx\| = \|vx\| = \sup_{t \in [0,1]} |v(t)x(t)| = |v(\zeta)||x(\zeta)|$ where $\zeta \in [0,1]$ reaches sup

Thus $T \in B(C[0,1])$ and since $C[0,1]$ is Banach we will look at $\lambda \in \mathbb{C} \ni T_\lambda$ is not bijective.

Let $f \in C[0,1]$ and $\lambda = v(t_0)$ for $t_0 \in [0,1]$

$T_\lambda f(t) = (v(t)-\lambda)f(t) = 0$ at t_0 $\forall f \Rightarrow T_\lambda$ is not bijective

If $\lambda \notin R(v)$ $T_\lambda f(t) = v(t)f(t) - \lambda f(t) \neq 0$ unless $f=0$

Similarly $R_\lambda f(t) = \frac{f(t)}{v(t)-\lambda}$

$\therefore \lambda \in \sigma(T)$ $\forall \lambda \in R(v) = v([0,1]) = [\min v, \max v]$

As v is cont on compact set max and min exist

b) As seen in 4a,

$\sigma(T) = v([0,1]) = [\min v, \max v]$ $\min/\max v \in \mathbb{R}$

5. Let T be the left shift operator on l^p sequences. Namely, $T : l^p \to l^p$ is defined as

$$T(\{x_1, x_2, x_3, \cdots\}) = \{x_2, x_3, x_4, \cdots\}, \ \forall x = \{x_1, x_2, x_3, \cdots\} \in l^p, \ 1 \leqslant p \leqslant \infty.$$

(a) For $p = \infty$. If $|\lambda| > 1$ show that $\lambda \in \rho(T)$; and if $|\lambda| \leqslant 1$, show that λ is an eigenvalue and find the corresponding eigenspace.

(b) For $1 \leqslant p < \infty$. If $|\lambda| = 1$, is λ an eigenvalue of T?

a) Let $x \in \ker(T_\lambda)$

Let $|\lambda| > 1$ $T_\lambda x = 0$

$Tx - \lambda x = 0$

$\Rightarrow x_{k+1} = \lambda x_k$ $\forall k \in \mathbb{N}$

$\Rightarrow x_k = \lambda^{k-1} x_1$

Since $|\lambda| > 1$ and $x \in l^\infty$

$x = 0$

Similarly for $y \in \ker R_\lambda(x)$

$R_\lambda(x)\, y = 0$

$y = T_\lambda 0 = T0 - \lambda 0 = 0$

Thus $y = 0$

$\therefore$ for $|\lambda| > 1$ T_λ is bijective so $\lambda \in \rho(T)$

Let $|\lambda| \leq 1$ given $x \in \ker T_\lambda$

It follows $x_k = \lambda^{k-1} x_1$

since $|\lambda| \leq 1$ $\ldots \leq x_k \leq \ldots \leq x_1 < \infty$ $\forall k \in \mathbb{N}$

Thus $\exists\, x \in \ker T_\lambda \ni x \neq 0$ so T_λ is not injective

$\Rightarrow R_\lambda(T)$ DNE

Eigspace $x \neq 0$ $\lambda \neq 0$

$Tx = \lambda x$

x would follow the aforementioned recursion

If $\lambda = 0$

$Tx = \lambda x \Rightarrow x = \{x_1, 0\}$

b) Consider $x \in \ker T_\lambda$ $|\lambda| = 1$

$Tx = \lambda x$

$x_{k+1} = \lambda x_k \Rightarrow x_k = \lambda^{k-1} x_1$

$\sum_{k=1}^{N} |x_k| = \sum_{k=1}^{N} |x_1||\lambda^{k-1}| \le \sum_{k=1}^{N} |x_1||\lambda|^{k-1} \to \infty$ as $N \to \infty$

$\therefore$ For $|\lambda| = 1$, $|\lambda|$ is not an eigenvalue for $1 \le p < \infty$.

6. Let $T : l^2 \to l^2$ be the right shift operator, namely,

$$T(x) = \{0, x_1, x_2, \cdots\}, \ \forall x = \{x_1, x_2, \cdots\} \in l^2.$$

Find $\sigma_p(T)$, $\sigma_c(T)$, and $\sigma_r(T)$.

$\lambda \in \sigma_p(T) \Rightarrow T_\lambda$ is not injective

$\lambda \in \sigma_c(T) \Rightarrow R_\lambda(T)$ is not bdd

$\lambda \in \sigma_r(T) \Rightarrow R_\lambda(T)$ is not densely defined

$|\lambda| < 1$ $x \in l^\infty$ $\overline{R(T_\lambda)} \neq l^2$

$T_\lambda x = Tx - \lambda x$ $\exists\, y \in l^2 \ni \nexists\, x \in l^2$ $T_\lambda x = y$

$|-\lambda x_1| < |x_1|$ $\forall x_1 \in \mathbb{C}$ Let $\varepsilon = \frac{|1-\lambda|}{2|x_1|} > 0$ since $|\lambda| < 1$

$|x_1 - \lambda x_1| = |(1-\lambda)x_1| = |1-\lambda||x_1| > |\frac{1-\lambda}{2}||x_1| = \varepsilon$

Thus for $|\lambda| < 1$ $\exists\, \varepsilon > 0 \ni |x_1 - \lambda x_1| > \varepsilon$ so $R(T_\lambda)$ is not densely defined.

Thus $|\lambda| < 1 \Rightarrow \lambda \in \sigma_r(T)$

Let $x \in \ker T_\lambda$

If $\lambda = 0$

$Tx = x \Rightarrow x_{k+1} = x_k$ $\forall k \in \mathbb{N} \Rightarrow x = 0$ since $(Tx)_1 = 0$

If $\lambda \neq 0$

$0 = T_\lambda x = Tx - \lambda x$ since $\lambda \neq 0$, $x = 0$

$Tx=\lambda x \Rightarrow x_{k+1}=\lambda x_k \ \forall k\in\mathbb{N} \Rightarrow x=0$ since $(Tx)_1=0$

If $\lambda\neq 0$

$0=Tx-\lambda x = Tx=\lambda x$ since $\lambda\neq 0$ $x=0$

Thus $\sigma_p(T)=\emptyset$

By looking at the Neumann series

$(T-\lambda I)^{-1}=\sum\left(\frac{T}{\lambda}\right)^n$ which only converges if $\|\frac{T}{\lambda}\|<1$

or $\|T\|<|\lambda|$. as $\|T\|$ has been shown to be 1,

$\forall\lambda\ni|\lambda|>1$, $\lambda\in\rho(T)$.

For $|\lambda|=1$ Let $y\in\overline{R(T_\lambda)}^{\perp}$ I claim $y=0 \Rightarrow \overline{R(T_\lambda)}=\ell^2$

$0=\langle y, Tx-\lambda x\rangle=\langle y,Tx\rangle-\langle y,\lambda x\rangle=\langle y,Tx\rangle-\bar{\lambda}\langle y,x\rangle\leq\langle y,Tx-x\rangle=\langle y,T_1x\rangle=0$

since $x\neq 0$ $x_{n+1}\neq x_n \Rightarrow$ $y=0$

As seen before $\|T\|=1$ and $(T-\lambda I)^{-1}$ is unbounded

$\forall$ $\lambda\ni|\lambda|\leq 1$ in particular for $|\lambda|=1$. Thus $|\lambda|=1\Rightarrow\lambda\in\sigma_c(T)$