Contents

1 Notation

Definition 1 (4.1.1 Kreyszig).

- *A linear operator* $A : D(A) \subseteq X \to Y$ *is said to be* ***bounded*** *if there exists a* $C > 0$ *such that*

$$||Au||_Y \leq C||u||_X, \quad \text{for every } u \in D(A)$$

Otherwise it is said to be ***unbounded****.*

- *A linear operator* $A : D(A) \subseteq X \to Y$ *is said to be* ***densely defined*** *if* $\overline{D(A)} = X$

- *A linear operator* $A : D(A) \subseteq X \to Y$ *is said to be* ***closed*** *if the* ***graph***

$$G(A) = \{(u, Au) \mid u \in D(A)\} \subseteq X \times Y$$

is closed as a subspace of $X \times Y$

Definition 2 (7.2 Kreyszig).
Let $X \neq \{0\}$ *be a complex normed space and* $T : \mathscr{D}(T) \subseteq X \to X$ *be a linear operator. With* T *we associate the operator*

$$T_\lambda = T - \lambda I$$

where λ *is a complex number and* I *is the identity operator on* $\mathscr{D}(T)$*. If* T_λ *has an inverse, we denote it by* $R_\lambda(T)$ *and call it the resolvent operator of* T *or, simply, the* ***resolvent*** *of* T*. If it is clear which operator we are discussing, we will write* R_λ*.*

Definition 3 (7.2-1 (Regular value, resolvent set, spectrum)).
Let $X \neq \{0\}$ *be a complex normed space and* $T : \mathscr{D}(T) \subseteq X \to X$ *be a linear operator. A regular value* λ *of* T *is a complex number such that*

- $R_\lambda(T)$ *exists,*

- $R_\lambda(T)$ *is bounded,*

- $R_\lambda(T)$ *is densely defined.*

The resolvent set $\rho(T)$ *of* T *is the set of all regular values* λ *of* T*. Its complement* $\sigma(T) = \mathbb{C}\backslash\rho(T)$ *in the complex plane* $\mathbb{C}$ *is called the spectrum of* T*, and a* $\lambda \in \sigma(T)$ *is called a spectral value of* T*. Furthermore, the spectrum* $\sigma(T)$ *is partitioned into three disjoint sets as follows.*
The ***point spectrum*** *or discrete spectrum* $\sigma_p(T)$ *is the set such that* $R_\lambda(T)$ *does not exist. A* $\lambda \in \sigma_p(T)$ *is called an eigenvalue of* T*.*
The ***continuous spectrum*** $\sigma_c(T)$ *is the set such that* $R_\lambda(T)$ *exists and is densely defined, but it unbounded.*
The ***residual spectrum*** $\sigma_r(T)$ *is the set such that* $R_\lambda(T)$ *exists, but is not densely defined (may or may not be bounded).*

2 3.1 Weak Solutions of Elliptic Boundary Value Problems

Theorem 1 (3.1.4 (Lax-Milgram))**.**
Let V be a Hilbert space and $a(\cdot,\cdot)$ a continuous V-elliptic bilinear form. Then given $f \in V$, there exists a unique $u \in V$ such that

$$a(u,v) = (f,v), \quad \text{for every } v \in V.$$

If $a(\cdot,\cdot)$ is also symmetric then the functional $J : V \to \mathbb{R}$ defined by

$$J(v) = \frac{1}{2}a(v,v) - (f,v)$$

attains its minimum at u.

Theorem 2 (3.1.5 (Babuska-Brezzi))**.**
Let Σ, V be Hilbert spaces and $a : \Sigma \times \Sigma \to \mathbb{R}$, $b : \Sigma \times V \to \mathbb{R}$, bilinear forms which are continuous. Let

$$Z = \{\sigma \in \Sigma \mid b(\sigma,v) = 0, \quad \text{for every } v \in V\}.$$

Assume that $a(\cdot,\cdot)$ is Z-elliptic, i.e. there exists a constant $\alpha > 0$ such that

$$a(\sigma,\sigma) \geq \alpha||\sigma||_{\Sigma}^2, \quad \text{for every } \sigma \in Z.$$

Assume further that there exists a constant $\beta > 0$ such that

$$\sup_{\tau \in \Sigma} \frac{b(\tau,v)}{||\tau||_{\Sigma}} \geq \beta||v||_V.$$

Then if $\kappa \in \Sigma$ and $l \in V$, there exists a unique pair $(\sigma,u) \in \Sigma \times V$ such that

$$\begin{aligned} a(\sigma,\tau) + b(\tau,u) &= (\kappa,\tau), \quad \text{for every } \tau \in \Sigma \\ b(\sigma,v) &= (l,v), \quad \text{for every } v \in V. \end{aligned}$$

3 4.3 C_0 Semigroups

Definition 4 (4.3.1).
Let X be a Banach space and $\{S(t)\}_{t\geq 0}$ be a family of bounded linear operators on X. It is said to be a C_0 semigroup if the following are true:

- *$S(0) = I$, the identity of X*
- *$S(t+s) = S(t)S(s)$, for all $t, s \geq 0$*
- *For every $u \in X$*

$$S(t)u \to u \quad \text{as } t \downarrow 0$$

Theorem 3 (4.3.1).
Let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup on X. Then there exists $M \geq 1$ and ω such that

$$||S(t)|| \leq Me^{\omega t}, \quad \text{for all } t \geq 0$$

Definition 5 (4.3.2).
*If $M = 1$ and $\omega = 0$, so that $||S(t)|| \leq 1$ for all $t \geq 0$, we say that $\{S(t)\}$ is a **contraction semigroup**.*

Definition 6 (4.3.3).
*Let $\{S(t)\}_{t\geq 0}$ be a C_0 semigroup on X. The **infinitesimal generator** of the semigroup is a linear operator A given by*

$$D(A) = \left\{ u \in X \mid \lim_{t\downarrow 0} \frac{S(t)u - u}{t} \text{ exists} \right\}$$
$$Au = \lim_{t\downarrow 0} \frac{S(t)u - u}{t}, \; u \in D(A)$$

Theorem 4 (4.3.2).
Let $\{S(t)\}_{t\geq 0}$ be a C_0 semigroup and let A be its infinitesimal generator. Let $u \in D(A)$. Then

$$S(t)u \in C^1([0, \infty); X) \cap C([0, \infty); X)$$

and

$$\frac{d}{dt}(S(t)u) = AS(t)u = S(t)Au$$

Proof.
Let $u \in D(A)$. Then

$$\left(\frac{S(h) - I}{h} \right) S(t)u = S(t) \left(\frac{S(h) - I}{h} \right) \to S(t)Au$$

as $h \downarrow 0$, by the definition of A. Thus $S(t)u \in D(A)$ and

$$AS(t)u = S(t)Au = D^{+}S(t)u.$$

Next consider

$$\frac{S(t)u - S(t-h)u}{h} = S(t-h)\frac{S(h)u - u}{h}$$

Hence

$$\frac{S(t)u - S(t-h)u}{h} - S(t)Au = S(t-h)\left(\frac{S(h)u - u}{h} - Au\right) + (S(t-h) - S(t))Au.$$

But

$$\begin{aligned} \left|\left|S(t-h)\left(\frac{S(h)u - u}{h} - Au\right)\right|\right| &\leq Me^{\omega t}\left|\left|\frac{S(h)u - u}{h} - Au\right|\right| \\ &\to 0 \quad \text{as } h \downarrow 0 \end{aligned}$$

and

$$||(S(t-h) - S(t))Au|| \to 0 \quad \text{as } h \downarrow 0$$

by the boundedness of $S(t)$.
Thus,

$$D^{-}S(t)u = S(t)Au = D^{+}S(t)u$$

and thus

$$\frac{d}{dt}(S(t)u) = AS(t)u = S(t)Au$$

Similarly, by the boundedness of $S(t)$, the map $t \mapsto S(t)Au$ is continuous so $S(t)u \in C^1([0,\infty); X)$ $\square$

Remark 1 (4.3.3)**.** *If A is the infinitesimal generator of a C_0 semigroup $\{S(t)\}$ then we know by the above theorem that*

$$u(t) = S(t)u_0$$

defines the unique solution of the initial value problem

$$\left.\begin{aligned} \tfrac{du(t)}{dt} &= Au(t),\ t \geq 0 \\ u(0) &= u_0 \end{aligned}\right\}$$

4 4.4 The Hille-Yosida Theorem

Theorem 5 (4.4.3 (Hille Yosida))**.**
A linear unbounded operator A on a Banach space X is the infinitesimal generator of a contraction semigroup if and only if

- *A is closed*

- *A is densely defined*

- *For every $\lambda > 0$, $(\lambda I - A)^{-1}$ is a bounded linear operator and*

$$||(\lambda I - A)^{-1}|| \leq \frac{1}{\lambda}$$

5 Pazy Stuff

Definition 7.
Let X be a Banach space with dual space X'. Denote $x' \in X'$ at $x \in X$ by $\langle x', x\rangle$ or $\langle x, x'\rangle$. Define the following set $F(x) \subseteq X'$ as

$$F(x) = \{x' \mid \langle x', x\rangle = ||x||^2 = ||x'||^2\}$$

(This set is non-empty by the Hahn-Banach theorem.)

Definition 8 (Dissipativity)**.**
A linear operator A is dissipative if for every $x \in D(A)$ there is a $x' \in F(x)$ such that $Re\langle Ax, x'\rangle \leq 0$

Definition 9 (Maximal Dissipativity)**.**
A linear operator A is called maximally dissipative if it is dissipative and $R(I - A) = X$.

Theorem 6 (1.4.3 Lumer-Phillips)**.**

- *If A is dissipative and there is a $\lambda_0 > 0$ such that $R(\lambda_0 I - A) = X$, then A is the infinitesimal generator of a C_0 semigroup of contractions on X.*
- *If A is the infinitesimal generator of a C_0 semigroup of contractions on X then $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative.*

Proof.
Let $\lambda > 0$, the dissipativeness of A implies that $||\lambda x - Ax|| \geq \lambda||x||$ (proof skipped) for every $\lambda > 0$ and $x \in D(A)$. Since $R(\lambda_0 I - A) = X$, it follows when $\lambda = \lambda_0$ that $(\lambda_0 I - A)^{-1}$ is a bounded linear operator and thus closed. This implies $\lambda_0 I - A$ is closed and therefore A is also closed. If $R(\lambda I - A) = X$ for every $\lambda > 0$ then $\rho(A) \subseteq \mathbb{R}$ and $||R_\lambda(A)|| \leq \lambda^{-1}$. It follows by the Hille-Yosida theorem that A is the infinitesimal generator of a C_0 semigroup of contractions on X.
Consider the set $\Lambda = \{\lambda \,|\, 0 < \lambda < \infty,\, R(\lambda I - A) = X\}$. Let $\lambda \in \Lambda$. By previous inequality, $\lambda \in \rho(A)$. Since $\rho(A)$ is open, the intersection of $B_r(\lambda) \cap \mathbb{R} \subseteq \Lambda$ and thus Λ is open.
On the other hand, let $\{\lambda_n\} \subseteq \Lambda$ and $\lambda_n \to \lambda > 0$. For every $y \in X$ there exists an $x_n \in D(A)$ such that

$$\lambda_n x_n - Ax_n = y$$

From the inequality it follows that $||x_n|| \leq \lambda_n^{-1}||y|| \leq C$ for some $C > 0$. Now,

$$\begin{aligned}\lambda_m||x_n - x_m|| &\leq ||\lambda_m(x_n - x_m) - A(x_n - x_m)|| \\ &= |\lambda_n - \lambda_m|\,||x_n|| \\ &\leq C|\lambda_n - \lambda_m| \to 0\end{aligned}$$

Therefore $\{x_n\}$ is Cauchy. Let $x_n \to x$. It follows $Ax_n \to \lambda x - y$. Since A is closed, $x \in D(A)$ and $\lambda x - Ax = y$. Therefore $R(\lambda I - A) = X$ and $\lambda \in \Lambda$. Thus Λ is closed. By assumption, $\Lambda \neq \emptyset$, therefore $\Lambda = (0, \infty)$
If A is the infinitesimal generator of a C_0 semigroup of contractions, $S(t)$, on X, then by the Hille-Yosida theorem $\rho(A) \supseteq (0, \infty)$ and therefore $R(\lambda I - A) = X$ for all $\lambda > 0$. Furthermore if $x \in D(A), x^* \in F(x)$ then

$$|\langle S(t)x, x^* \rangle| \leq ||S(t)x||\, ||x^*|| \leq ||x||^2$$

and therefore,

$$Re\langle S(t)x - x, x^* \rangle = Re\langle S(t)x, x^* \rangle - ||x||^2 \leq 0.$$

By dividing the previous line by $t > 0$ and letting $t \downarrow 0$ yields

$$Re\langle Ax, x^* \rangle \leq 0.$$

□

Theorem 7 (Lumer-Phillips)**.**
A densely defined operator A is the infinitesimal generator of a C_0 semigroup of contractions if and only if it is maximal dissipative.

6 Bibliography