MASTER PROJECT: Qualitative Properties of the Multilayered Structure - Fluid Interactions Coupled PDE Systems

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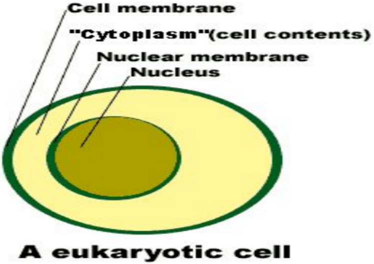
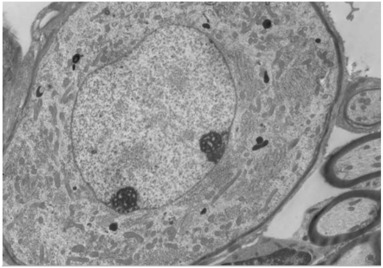
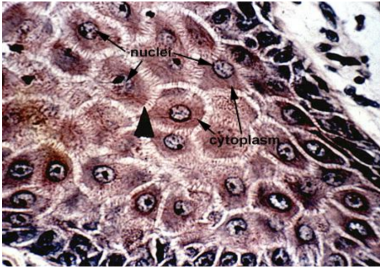
Motivation

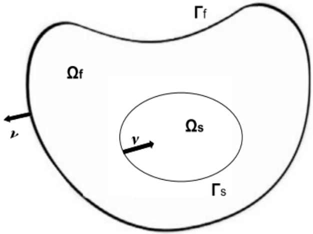

Cell membrane
"Cytoplasm"(cell contents)
Nuclear membrane
Nucleus
A eukaryotic cell




nuclei
cytoplasm

Geometry

- $\Omega_f \subseteq \mathbb{R}^3$, Lipschitz bounded domain
- $\Omega_s \subseteq \mathbb{R}^3$, $\Omega_s \subset\subset \Omega_f$
- $\partial\Omega_s = \Gamma_s$
- $\partial\Omega_f = \Gamma_s \cup \Gamma_f$



Introduction of First Model

$$\begin{cases} u_t - \Delta u = 0 \text{ in } (0, T) \times \Omega_f \\ u|_{\Gamma_f} = 0 \text{ on } (0, T) \times \Gamma_f; \end{cases} \tag{1}$$

$$\begin{cases} \frac{\partial^2}{\partial t^2} h_j - \Delta h_j + h_j = \frac{\partial w}{\partial \nu}|_{\Gamma_j} - \frac{\partial u}{\partial \nu}|_{\Gamma_j} \text{ on } (0, T) \times \Gamma_j \\ h_j|_{\partial\Gamma_j \cap \partial\Gamma_l} = h_l|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } (0, T) \times (\partial\Gamma_j \cap \partial\Gamma_l), \text{ for all } 1 \leq l \leq K \\ \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset \\ \left.\frac{\partial h_j}{\partial n_j}\right|_{\partial\Gamma_j \cap \partial\Gamma_l} = -\left.\frac{\partial h_l}{\partial n_l}\right|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } (0, T) \times (\partial\Gamma_j \cap \partial\Gamma_l), \text{ for all } 1 \leq l \leq K \\ \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset \end{cases} \tag{2}$$

$$\begin{cases} w_{tt} - \Delta w = 0 \text{ on } (0, T) \times \Omega_s \\ w_t|_{\Gamma_j} = \frac{\partial}{\partial t} h_j = u|_{\Gamma_j} \text{ on } (0, T) \times \Gamma_j, \text{ for } j = 1, \ldots, K \end{cases} \tag{3}$$

$$\begin{aligned} &[u(0), h_1(0), \frac{\partial}{\partial t} h_1(0), \ldots, h_K(0), \frac{\partial}{\partial t} h_K(0), w(0), w_t(0)] \\ &\qquad = [u_0, h_{01}, h_{11}, \ldots, h_{0K}, h_{1K}, w_0, w_1] \end{aligned} \tag{4}$$

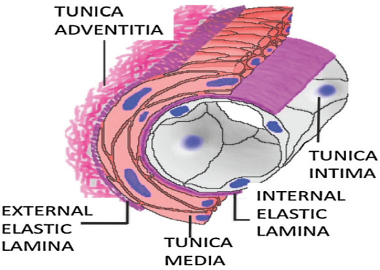
where $\Gamma_s = \cup_{j=1}^K \overline{\Gamma_j}$ and $\Gamma_i \cap \Gamma_j = \emptyset$, for $i \neq j$.

Earlier Works

Generally, single layered or moving boundary FSI Systems

- George Avalos
- Lorena Bociu
- Scott Hansen
- Igor Kukavica
- Irena Lasiecka
- Hyesuk Lee
- Roberto Triggiani
- Amjad Tuffaha
- Justin Webster
- Enrique Zuazua
- etc.

Pioneer Paper Bukač, B. Muha, S. Čanić



- 2014, Fluid-Multi-layered-structure Interaction, Existence of weak solutions
- 2015, Regularity and Regularization effects of the same problem
- 2015, Numerical Results: unconditional stability for the same problem

$$\begin{aligned}\mathbf{H} =&\{[u_0, h_{01}, h_{11}, \dots, h_{0K}, h_{1K}, w_0, w_1] \in L^2(\Omega_f) \times H^1(\Gamma_1) \times L^2(\Gamma_1) \times \dots \\ &\times H^1(\Gamma_K) \times L^2(\Gamma_K) \times H^1(\Omega_s) \times L^2(\Omega_s), \text{ such that for each } 1 \le j \le K: \\ &(i) w_0|_{\Gamma_j} = h_{0j}; \\ &(ii) h_{0j}|_{\partial\Gamma_j \cap \partial\Gamma_l} = h_{0l}|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } \partial\Gamma_j \cap \partial\Gamma_l, \text{ for all } 1 \le l \le K \text{ such that} \\ &\qquad \partial\Gamma_j \cap \partial\Gamma_l \ne \emptyset\}\end{aligned} \tag{5}$$

With the accompanying inner product

$$\begin{aligned}\left\langle \Phi_0, \widetilde{\Phi}_0 \right\rangle_{\mathbf{H}} = \langle u_0, \widetilde{u}_0 \rangle_{\Omega_f} + \sum_{j=1}^{K} \left\langle \nabla h_{0j}, \nabla \widetilde{h}_{0j} \right\rangle_{\Gamma_j} + \sum_{j=1}^{K} \left\langle h_{0j}, \widetilde{h}_{0j} \right\rangle_{\Gamma_j} \\ + \sum_{j=1}^{K} \left\langle h_{1j}, \widetilde{h}_{1j} \right\rangle_{\Gamma_j} + \langle \nabla w_0, \nabla \widetilde{w}_0 \rangle_{\Omega_s} + \langle w_1, \widetilde{w}_1 \rangle_{\Omega_s}\end{aligned} \tag{6}$$

where
$\Phi_0 = [u_0, h_{01}, h_{11}, \dots, h_{0K}, h_{1K}, w_0, w_1] \in \mathbf{H}$ and
$\widetilde{\Phi}_0 = [\widetilde{u}_0, \widetilde{h}_{01}, \widetilde{h}_{11}, \dots, \widetilde{h}_{0K}, \widetilde{h}_{1K}, \widetilde{w}_0, \widetilde{w}_1] \in \mathbf{H}$.

Functional Settings

For the PDE system given in (1) – (4) if
$\Phi(t) = [u, h_1, \frac{\partial}{\partial t} h_1, \ldots, h_K, \frac{\partial}{\partial t} h_K, w, w_1] \in C([0, T]; \mathbf{H})$ then for
$\mathbf{A} : D(\mathbf{A}) \subseteq \mathbf{H} \to \mathbf{H}$,

$$\frac{d}{dt}\Phi(t) = \mathbf{A}\Phi(t); \quad \Phi(0) = \Phi_0. \tag{7}$$

where $\mathbf{A}$ is given as

$$\mathbf{A} = \begin{bmatrix} \Delta & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & I & \ldots & 0 & 0 & 0 & 0 \\ -\frac{\partial}{\partial \nu}|_{\Gamma_1} & (\Delta - I) & 0 & \ldots & 0 & 0 & \frac{\partial}{\partial \nu}|_{\Gamma_1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & I & 0 & 0 \\ -\frac{\partial}{\partial \nu}|_{\Gamma_K} & 0 & 0 & \ldots & (\Delta - I) & 0 & \frac{\partial}{\partial \nu}|_{\Gamma_K} & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & I \\ 0 & 0 & 0 & \ldots & 0 & 0 & \Delta & 0 \end{bmatrix}; \tag{8}$$

Where the Domain of **A** is given as follows

$$
\begin{aligned}
D(\mathbf{A}) = \{&[u_0, h_{01}, h_{11}, \ldots, h_{0K}, h_{1K}, w_0, w_1] \in \mathbf{H} : \\
&(\mathbf{A.i})\ u_0 \in H^1(\Omega_f),\ h_{1j} \in H^1(\Gamma_j) \text{ for } 1 \le j \le K,\ w_1 \in H^1(\Omega_s); \\
&(\mathbf{A.ii})(a)\ \Delta u_0 \in L^2(\Omega_f),\ \Delta w_0 \in L^2(\Omega_s),\ (b)\ \Delta h_{0j} - \left.\frac{\partial u_0}{\partial \nu}\right|_{\Gamma_j} + \left.\frac{\partial w_0}{\partial \nu}\right|_{\Gamma_j} \in L^2(\Gamma_j) \\
&\quad \text{for } 1 \le j \le K; \text{(c) } \left.\frac{\partial h_{0j}}{\partial n_j}\right|_{\partial\Gamma_j} \in H^{-1/2}(\partial\Gamma_j),\ \text{for } 1 \le j \le K; \\
&(\mathbf{A.iii})\ u_0|_{\Gamma_f} = 0, u_0|_{\Gamma_j} = h_{1j} = w_1|_{\Gamma_j},\ \text{for } 1 \le j \le K; \\
&(\mathbf{A.iv}) \text{ For } 1 \le j \le K : \\
&\ (a)\ h_{1j}|_{\partial\Gamma_j \cap \partial\Gamma_l} = h_{1l}|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } \partial\Gamma_j \cap \partial\Gamma_l,\ \text{for all } 1 \le l \le K \text{ such that} \\
&\qquad \partial\Gamma_j \cap \partial\Gamma_l \ne \emptyset; \\
&\ (b)\ \left.\frac{\partial h_{0j}}{\partial n_j}\right|_{\partial\Gamma_j \cap \partial\Gamma_l} = -\left.\frac{\partial h_{0l}}{\partial n_l}\right|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } \partial\Gamma_j \cap \partial\Gamma_l,\ \text{for all } 1 \le l \le K \\
&\qquad \text{such that } \partial\Gamma_j \cap \partial\Gamma_l \ne \emptyset\}.
\end{aligned}
\tag{9}
$$

Part I

Wellposedness

Theorem ((Wellposedness) G. Avalos, PGG, B. Muha [JDE, 2020])

The operator $\mathbf{A} : D(\mathbf{A}) \subseteq \mathbf{H} \to \mathbf{H}$ *defined above generates a* C_0*-semigroup of contractions. Consequently, the solution* $\Phi(t) = [u, h_1, \frac{\partial}{\partial t} h_1, \ldots, h_K, \frac{\partial}{\partial t} h_K, w, w_1]$ *of the PDE model is given by*

$$\Phi(t) = e^{\mathbf{A}t}\Phi_0 \in C([0, T]; \mathbf{H}),$$

where $\Phi_0 = [u_0, h_{01}, h_{11}, \ldots, h_{0K}, h_{1k}, w_0, w_1] \in D(\mathbf{A})$

G. Avalos, P. G. Geredeli, B. Muha; "Wellposedness, Spectral Analysis and Asymptotic Stability of a Multilayered Heat-Wave-Wave System ", Journal of Differential Equations 269 (2020), pp. 7129-7156.

Sketch of Proof

Wellposedness was shown using the Lumer-Phillips Theorem. To this end, they showed

- $\mathbf{A}$ is dissipative; i.e. $\langle \mathbf{A}\Phi, \Phi\rangle_{\mathbf{H}} \leq 0$ for all $\Phi \in D(\mathbf{A})$
- Use Lax-Milgram to solve to the static resolvent equation $(\lambda I - \mathbf{A})\Phi = \Phi^*$ in $D(\mathbf{A})$; i.e. $\mathbf{A}$ is maximally dissipative
- Recover of the other structure solution variables given initial data; i.e.

$$h_{0j} = \frac{1}{\lambda} h_{1j} + \frac{1}{\lambda} h_{0j}^*, \text{ for } 1 \leq j \leq K,$$
$$w_0 = \frac{1}{\lambda} w_1 + \frac{1}{\lambda} w_0^*.$$

Dissipativity of $\mathbf{A}$

Given data Φ_0 in $D(\mathbf{A})$,

$$\langle \mathbf{A}\Phi_0, \Phi_0 \rangle_{\mathbf{H}} = -\left|\left|\nabla u_0\right|\right|^2_{\Omega_f} + 2i \sum_{j=1}^{K} \mathrm{Im} \left\langle \nabla h_{1j}, \nabla h_{0j} \right\rangle_{\Gamma_j}$$
$$+ 2i \sum_{j=1}^{K} \mathrm{Im} \left\langle h_{1j}, h_{0j} \right\rangle_{\Gamma_j} + 2i\mathrm{Im} \left\langle \nabla w_1, \nabla w_0 \right\rangle_{\Omega_s}, \quad (10)$$

which gives $\mathrm{Re} \langle \mathbf{A}\Phi_0, \Phi_0 \rangle_{\mathbf{H}} \leq 0$. Since Φ_0 was arbitrary, it follows that $\mathbf{A}$ is dissipative.

The author solves the following static problem with the Lax-Milgram Theorem. Given parameter $\lambda > 0$, suppose $\Phi = [u_0, h_{01}, h_{11}, \dots, h_{0K}, h_{1K}, w_0, w_1] \in D(\mathbf{A})$ is a solution of the equation

$$(\lambda I - \mathbf{A})\Phi = \Phi^*, \tag{11}$$

where $\Phi^* = [u_0^*, h_{01}^*, h_{11}^*, \dots, h_{0K}^*, h_{1K}^*, w_0^*, w_1^*] \in \mathbf{H}$.

Maximality of **A**

In PDE terms, the abstract equation above becomes

$$\begin{cases} \lambda u_0 - \Delta u_0 = u_0^* \text{ in } \Omega_f \\ u_0|_{\Gamma_f} = 0 \text{ on } \Gamma_f; \end{cases} \tag{12}$$

and for $1 \le j \le K$,

$$\begin{cases} \lambda h_{0j} - h_{1j} = h_{0j}^* \text{ in } \Gamma_j \\ \lambda h_{1j} - \Delta h_{0j} + h_{0j} - \frac{\partial w_0}{\partial \nu} + \frac{\partial u_0}{\partial \nu} = h_{1j}^* \text{ in } \Gamma_j \\ u_0|_{\Gamma_j} = h_{1j} = w_1|_{\Gamma_j} \text{ in } \Gamma_j \\ h_{0j}|_{\partial\Gamma_j \cap \partial\Gamma_l} = h_{0l}|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } \partial\Gamma_j \cap \partial\Gamma_l, \text{ for all } 1 \le l \le K \text{ such that} \\ \qquad \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset \\ \left.\frac{\partial h_{0j}}{\partial n_j}\right|_{\partial\Gamma_j \cap \partial\Gamma_l} = -\left.\frac{\partial h_{0l}}{\partial n_l}\right|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } \partial\Gamma_j \cap \partial\Gamma_l, \text{ for all } 1 \le l \le K \text{ such that} \\ \qquad \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset; \end{cases} \tag{13}$$

and also

$$\begin{cases} \lambda w_0 - w_1 = w_0^* \text{ in } \Omega_s \\ \lambda w_1 - \Delta w_0 = w_1^* \text{ in } \Omega_s \end{cases} \tag{14}$$

Maximality of $\mathbf{A}$

Define the sets

$$\begin{aligned}\mathcal{V} =&\{[\psi_1,\ldots,\psi_K] \in H^1(\Gamma_1)\times\cdots\times H^1(\Gamma_k) |\ \text{For all } 1\leq j\leq K,\\ &\psi_j|_{\partial\Gamma_j\cap\partial\Gamma_l} = \psi_l|_{\partial\Gamma_j\cap\partial\Gamma_l},\ \text{for all } 1\leq l\leq K \text{ such that } \partial\Gamma_j\cap\partial\Gamma_l\neq\emptyset\}\end{aligned} \tag{15}$$

and

$$\mathbf{W} \equiv \left\{[\varphi,\psi_1,\ldots,\psi_K,\xi]\in H^1_{\Gamma_f}(\Omega_f)\times\mathcal{V}\times H^1(\Omega_s)\,|\,\varphi|_{\Gamma_j} = \psi_j = \xi|_{\Gamma_j},\ \text{for } 1\leq j\leq K\right\}; \tag{16}$$

If $\Phi = [u_0, h_{01}, h_{11}, \ldots, h_{0K}, h_{1K}, w_0, w_1] \in D(\mathbf{A})$ solves (11), then necessarily its solution components $[u_0, h_{11}, \ldots, h_{1K}, w_1] \in \mathbf{W}$ satisfy for $[\varphi,\psi,\xi]\in\mathbf{H}$,

$$\left\langle \mathbf{B}\begin{bmatrix}\varphi\\ \psi_1\\ \vdots\\ \psi_K\\ \xi\end{bmatrix}, \begin{bmatrix}u_0\\ h_{11}\\ \vdots\\ h_{1K}\\ w_1\end{bmatrix}\right\rangle_{\mathbf{W}^*\times\mathbf{W}} = \mathbf{F}_\lambda\left(\begin{bmatrix}\varphi\\ \psi\\ \xi\end{bmatrix}\right); \tag{17}$$

Maximality of $\mathbf{A}$

where

$$\mathbf{F}_\lambda\left(\begin{bmatrix}\varphi\\ \psi\\ \xi\end{bmatrix}\right) = \langle u_0^*, \varphi\rangle_{\Omega_f} + \sum_{j=1}^{K}\left[\left\langle h_{1j}^*, \psi_j\right\rangle_{\Gamma_j} - \frac{1}{\lambda}\left\langle \nabla h_{0j}^*, \nabla\psi_j\right\rangle_{\Gamma_j} - \frac{1}{\lambda}\left\langle h_{0j}^*, \psi_j\right\rangle_{\Gamma_j}\right] + \langle w_1^*, \xi\rangle_{\Omega_s} - \frac{1}{\lambda}\langle \nabla w_0^*, \nabla\xi\rangle_{\Omega_s} \tag{18}$$

and

$$\left\langle \mathbf{B}\begin{bmatrix}\varphi\\ \psi_1\\ \vdots\\ \psi_K\\ \xi\end{bmatrix}, \begin{bmatrix}\widetilde{\varphi}\\ \widetilde{\psi}_1\\ \vdots\\ \widetilde{\psi}_K\\ \widetilde{\xi}\end{bmatrix}\right\rangle_{\mathbf{W}^*\times\mathbf{W}} = \lambda\langle\varphi,\widetilde{\varphi}\rangle_{\Omega_f} + \langle\nabla\varphi,\nabla\widetilde{\varphi}\rangle_{\Omega_f} + \lambda\left\langle\xi,\widetilde{\xi}\right\rangle_{\Omega_s} + \left\langle\nabla\xi,\nabla\widetilde{\xi}\right\rangle_{\Omega_s} + \sum_{j=1}^{K}\left[\lambda\left\langle\psi_j,\widetilde{\psi}_j\right\rangle_{\Gamma_j} + \frac{1}{\lambda}\left\langle\nabla\psi_j,\nabla\widetilde{\psi}_j\right\rangle_{\Gamma_j} + \frac{1}{\lambda}\left\langle\psi_j,\widetilde{\psi}_j\right\rangle_{\Gamma_j}\right]$$

Lax-Milgram

Theorem (Lax-Milgram)

Let X be a Hilbert space and $a(\cdot,\cdot)$ a continuous X-elliptic bilinear form. Then given $f \in X$, there exists a unique $u \in X$ such that

$$a(u,v) = \langle f, v\rangle_X\,, \quad \text{for every } v \in X.$$

Since it is clear that $\mathbf{B} \in \mathscr{L}(\mathbf{W},\mathbf{W}^*)$ is $\mathbf{W}$-elliptic; by the Lax-Milgram Theorem, the equation (17) has a unique solution

$$[u_0, h_{11}, \ldots, h_{1K}, w_1] \in \mathbf{W}. \tag{19}$$

It was then showed this solution is in $D(\mathbf{A})$.

Intoduction of Second Model

The model was improved from a heat-wave-wave model to a Stokes-wave-Lamé model. The author has shown similar results of wellposedness for this model.

$$\begin{cases} u_t - \text{div } (\nabla u + \nabla^T u) + \nabla p = 0 & \text{in } (0, T) \times \Omega_f \\ \text{div } (u) = 0 & \text{in } (0, T) \times \Omega_f \\ u|_{\Gamma_f} = 0 & \text{on } (0, T) \times \Gamma_f; \end{cases} \tag{20}$$

$$\begin{cases} h_{tt} - \Delta_{\Gamma_s} h = [\nu \cdot \sigma(w)]|_{\Gamma_s} - [\nu \cdot (\nabla u + \nabla^T u)]|_{\Gamma_s} + p\nu & \text{on } (0, T) \times \Gamma_s, \end{cases} \tag{21}$$

$$\begin{cases} w_{tt} - \text{div } \sigma(w) + w = 0 & \text{on } (0, T) \times \Omega_s \\ w_t|_{\Gamma_s} = h_t = u|_{\Gamma_s} & \text{on } (0, T) \times \Gamma_s \end{cases} \tag{22}$$

$$[u(0), h(0), h_t(0), w(0), w_1(0)] = [u_0, h_0, h_1, w_0, w_1] \in \mathbf{H} \tag{23}$$

Here, $\Delta_{\Gamma_s}(\cdot)$ is the Laplace Beltrami operator, and the stress tensor $\sigma(\cdot)$ constitutes the Lamé system of elasticity on the "thick" layer. Namely, for function v in Ω_s,

$$\sigma(v) = 2\mu\epsilon(v) + \lambda[I_3 \cdot \epsilon(v)]I_3,$$

where strain tensor $\epsilon(\cdot)$ is given by

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right), \quad 1 \leq i, j \leq 3$$

Functional Settings

With the following Hilbert space

$$\mathbf{H} = \Big\{[u_0, h_0, h_1, w_0, w_1] \in [L^2(\Omega_f)]^3 \times [H^1(\Gamma_s)]^2 \times [L^2(\Gamma_s)]^2 \times [H^1(\Omega_s)]^3 \times [L^2(\Omega_s)]^3 \;\Big|\; \text{div } (u_0) = 0,\ u_0 \cdot \nu|_{\Gamma_f} = 0,\ \text{and } w_0|_{\Gamma_s} = h_0\Big\} \quad (24)$$

with the inner product

$$\left\langle \Phi_0, \widetilde{\Phi}_0 \right\rangle_{\mathbf{H}} = \langle u_0, \widetilde{u}_0\rangle_{\Omega_f} + \left\langle \nabla_{\Gamma_s}(h_0), \nabla_{\Gamma_s}(\widetilde{h}_0) \right\rangle_{\Gamma_s} + \left\langle h_1, \widetilde{h}_1 \right\rangle_{\Gamma_s} \quad (25)$$

$$+ \langle \sigma(w_0), \epsilon(\widetilde{w}_0)\rangle_{\Omega_s} + \langle w_0, \widetilde{w}_0\rangle_{\Omega_s} + \langle w_1, \widetilde{w}_1\rangle_{\Omega_s} \quad (26)$$

where

$$\Phi_0 = [u_0, h_0, h_1, w_0, w_1] \in \mathbf{H}; \widetilde{\Phi}_0 = [\widetilde{u}_0, \widetilde{h}_0, \widetilde{h}_1, \widetilde{w}_0, \widetilde{w}_1] \in \mathbf{H} \quad (27)$$

Elimination of Pressure

Elimination of the pressure will be important to formulate the PDE in (20) – (23) as an ODE. To this end, by using the matching velocity condition and u being divergence free it follows p is harmonic, or

$$\Delta p(t) = 0 \text{ in } \Omega_f. \tag{28}$$

Subsequently, by multiplying by $\nu|_{\Gamma_s}$ and using the matching velocity condition the author obtains the following boundary condition for the pressure variable p:

$$p + \frac{\partial p}{\partial \nu} = \text{div } (\nabla(u) + \nabla^T(u)) \cdot \nu|_{\Gamma_s} + [(\nabla u + \nabla^T u) \cdot \nu - \Delta_{\Gamma_s}(h) - \nu \cdot \sigma(w)|_{\Gamma_s}] \cdot \nu|_{\Gamma_s} \tag{29}$$

Since u is divergence free,

$$\frac{\partial p}{\partial \nu} = [\text{div } (\nabla u + \nabla^T u)] \cdot \nu \text{ on } \Gamma_f.$$

Thus the pressure variable $p(t)$, can formally be written pointwise in time as

$$p(t) = \mathcal{P}_1(u(t)) + \mathcal{P}_2(h(t)) + \mathcal{P}_3(w(t))$$

where each $\mathcal{P}_i(\cdot)$ will be given on the next slide.

Elimination of Pressure

$p(t) = \mathcal{P}_1(u(t)) + \mathcal{P}_2(h(t)) + \mathcal{P}_3(w(t))$

$$\begin{cases} \Delta \mathcal{P}_1(u) = 0 & \text{in } \Omega_f, \\ \mathcal{P}_1(u) = \text{div } (\nabla(u) + \nabla^T(u)) \cdot \nu|_{\Gamma_s} + ([(\nabla u + \nabla^T u)] \cdot \nu) \cdot \nu|_{\Gamma_s} & \text{on } \Gamma_s, \\ \frac{\partial \mathcal{P}_1(u)}{\partial \nu} = \text{div } (\nabla(u) + \nabla^T(u)) \cdot \nu|_{\Gamma_f} & \text{on } \Gamma_f, \end{cases} \tag{30}$$

$$\begin{cases} \Delta \mathcal{P}_2(h) = 0 & \text{in } \Omega_f, \\ \mathcal{P}_2(h) = -\Delta_{\Gamma_s}(h) \cdot \nu|_{\Gamma_s} & \text{on } \Gamma_s, \\ \frac{\partial \mathcal{P}_2(h)}{\partial \nu} = 0 & \text{on } \Gamma_f, \end{cases} \tag{31}$$

and

$$\begin{cases} \Delta \mathcal{P}_3(w) = 0 & \text{in } \Omega_f, \\ \mathcal{P}_3(w) = -[\nu \cdot \sigma(w)|_{\Gamma_s}] \cdot \nu|_{\Gamma_s} & \text{on } \Gamma_s, \\ \frac{\partial \mathcal{P}_3(w)}{\partial \nu} = 0 & \text{on } \Gamma_f. \end{cases} \tag{32}$$

Also note

$$p_0 = \mathcal{P}_1(u_0) + \mathcal{P}_2(h_0) + \mathcal{P}_3(w_0). \tag{33}$$

Functional Settings

It follows, the PDE system given in (20) – (23) may be associated with an abstract ODE in the Hilbert space $\mathbf{H}$; namely,

$$\begin{cases} \frac{d}{dt}\Phi(t) = \mathbf{A}\Phi(t) \\ \Phi(0) = \Phi_0 \end{cases} \tag{34}$$

where $\Phi(t) = [u(t), h(t), h_t(t), w(t), w_1(t)]$, and $\Phi_0 = [u_0, h_0, h_1, w_0, w_1]$. Here, the operator $\mathbf{A} : D(\mathbf{A}) \subseteq \mathbf{H} \to \mathbf{H}$ is defined by

$$\mathbf{A} = \begin{bmatrix} \text{div } (\nabla(\cdot) + \nabla^T(\cdot)) & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ -[\nu \cdot (\nabla(\cdot) + \nabla^T(\cdot))]|_{\Gamma_s} & \Delta_{\Gamma_s}(\cdot) & 0 & \nu \cdot \sigma(\cdot)|_{\Gamma_s} & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & \text{div } \sigma(\cdot) - I & 0 \end{bmatrix} +$$

$$\begin{bmatrix} -\nabla\mathcal{P}_1(\cdot) & -\nabla\mathcal{P}_2(\cdot) & 0 & -\nabla\mathcal{P}_3(\cdot) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mathcal{P}_1(\cdot)\nu & \mathcal{P}_2(\cdot)\nu & 0 & \mathcal{P}_3(\cdot)\nu & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{35}$$

Here, the "pressure" operators $\mathcal{P}_i$ are as defined above.

The domain $D(\mathbf{A})$ of the generator $\mathbf{A}$ is characterized as follows
$[u_0, h_0, h_1, w_0, w_1] \in D(\mathbf{A}) \Leftrightarrow$
$(\mathbf{A.i})$ $u_0 \in [H^1(\Omega_f)]^3,\ h_1 \in [H^1(\Gamma_s)]^2\ w_1 \in [H^1(\Omega_s)]^3$;
$(\mathbf{A.ii})$ There exists an associated $L^2(\Omega_f)$-function
$p_0 = p_0(u_0, h_0, w_0)$ such that

$$[\text{div } (\nabla u_0 + \nabla^T u_0) - \nabla p_0] \in L^2(\Omega_f)$$

Consequently, p_0 is harmonic and so one has the boundary traces
(a) $\left[p_0|_{\Gamma_f}, \frac{\partial p_0}{\partial \nu}\Big|_{\Gamma_f}\right] \in H^{-1/2}(\Gamma_f) \times H^{-3/2}(\Gamma_f)$;
(b) $(\nabla u_0 + \nabla^T u_0) \cdot \nu \in H^{-3/2}(\Gamma_f)$,
$(\mathbf{A.iii})$ div $\sigma(w_0) \in L^2(\Omega_s)$; consequently, $\nu \cdot \sigma \in H^{-1/2}(\Gamma_s)$,
$(\mathbf{A.iv})$ $\Delta_{\Gamma_s}(h_0) + [\nu \cdot \sigma(w_0)]_{\Gamma_s} - [(\nabla u_0 + \nabla^T u_0) \cdot \nu]|_{\Gamma_s}$
$+[p_0 \nu]|_{\Gamma_s} \in L^2(\Gamma_s)$,
$(\mathbf{A.v})$ $u_0|_{\Gamma_f} = 0,\ u_0|_{\Gamma_s} = h_1 = w_1|_{\Gamma_s}$

Theorem ((Wellposedness) PGG [JEE, 2024])

With reference to the problem $(20)-(23)$, *the operator* $\mathbf{A}: D(\mathbf{A}) \subseteq \mathbf{H} \to \mathbf{H}$, *defined in* (35), *generates a* C_0*-semigroup of contractions on* $\mathbf{H}$. *Consequently, the solution* $\Phi(t) = [u(t), h(t), h_t(t), w(t), w_t(t)]$ *of* $(20)-(23)$, *or equivalently* (34), *is given by*

$$\Phi(t) = e^{\mathbf{A}t}\Phi_0 \in C([0,T];\mathbf{H}),$$

where $\Phi_0 = [u_0, h_0, h_1, w_0, w_1] \in D(\mathbf{A})$.

Geredeli, P.G. An inf-sup approach to C_0-semigroup generation for an interactive composite structure-Stokes PDE dynamics. J. Evol. Equ. 24, 50 (2024). https://doi.org/10.1007/s00028-024-00978-3

Sketch of Proof

Similarly, wellposedness was shown using the Lumer-Phillips Theorem as the problem is still linear. They showed

- $\mathbf{A}$ is dissipative
- Show that $\mathbf{A}$ is maximally dissipative using Babuska-Brezzi to solve to the static resolvent equation $(\lambda I - \mathbf{A})\Phi = \Phi^*$ for $\Phi \in D(\mathbf{A})$

Similar to the previous model, the authors shows **A** is maximally dissipative. It follows, given data Φ in $D(\mathbf{A})$,

$$\begin{aligned}\langle \mathbf{A}\Phi, \Phi\rangle_{\mathbf{H}} = &-\frac{1}{2}\left|\left|\nabla(u_0) + \nabla^T(u_0)\right|\right|^2 + 2i\mathrm{Im}\Big[\langle \nabla_{\Gamma_s}(h_1), \nabla_{\Gamma_s}(h_0)\rangle_{\Gamma_s} \\ &+ \langle \sigma(w_1), \epsilon(w_0)\rangle_{\Omega_s} \\ &+ \langle w_1, w_0\rangle_{\Omega_s} + \langle \sigma(w_1), \epsilon(w_0)\rangle_{\Omega_s} + \langle w_1, w_0\rangle_{\Omega_s} \Big]\end{aligned}$$

which gives Re $\langle \mathbf{A}\Phi, \Phi\rangle_{\mathbf{H}} \leq 0$. Since Φ was arbitrary, it follows that **A** is dissipative.

Maximality of **A**

The author solves the following static problem to show **A** is maximal with the Babuska-Brezzi Theorem. Suppose $\Phi \in D(\mathbf{A})$ is a solution of the equation

$$(\lambda I - \mathbf{A})\Phi = \Phi^*, \tag{36}$$

where $\lambda > 0$, $\Phi = [u_0, h_0, h_1, w_0, w_1]$, and $\Phi^* = [u_0^*, h_0^*, h_1^*, w_0^*, w_1^*] \in \mathbf{H}$.

Babuska-Brezzi

Theorem (Babuska-Brezzi)

Let Σ, V be Hilbert spaces and $a : \Sigma \times \Sigma \to \mathbb{R}$, $b : \Sigma \times V \to \mathbb{R}$, bilinear forms which are continuous. Let

$$Z = \{\sigma \in \Sigma \mid b(\sigma, v) = 0, \quad \text{for every } v \in V\}.$$

Assume that $a(\cdot,\cdot)$ is Z-elliptic, i.e. there exists a constant $\alpha > 0$ such that

$$a(\sigma,\sigma) \geq \alpha ||\sigma||_{\Sigma}^2, \quad \text{for every } \sigma \in Z.$$

Assume further that there exists a constant $\beta > 0$ such that

$$\sup_{\tau \in \Sigma} \frac{b(\tau, v)}{||\tau||_{\Sigma}} \geq \beta ||v||_V.$$

Then if $\kappa \in \Sigma$ and $l \in V$, there exists a unique pair $(\sigma, u) \in \Sigma \times V$ such that

$$\begin{aligned} a(\sigma,\tau) + b(\tau, u) &= \langle \kappa, \tau \rangle_{\Sigma}, \quad \text{for every } \tau \in \Sigma \\ b(\sigma, v) &= \langle l, v \rangle_{V}, \quad \text{for every } v \in V. \end{aligned} \tag{37}$$

Maximailty of **A**

In PDE terms, the resolvent equation will generate the following relations, where p_0 is given via (33):

$$\begin{cases} \lambda u_0 - \text{div }(\nabla u_0 + \nabla^T u_0) + \nabla p_0 = u_0^* \text{ in } \Omega_f \\ \text{div }(u_0) = 0 \text{ in } \Omega_f \\ u_0|_{\Gamma_f} = 0 \text{ on } \Gamma_f; \end{cases} \tag{38}$$

$$\begin{cases} \lambda h_0 - h_1 = h_0^* \text{ in } \Gamma_s \\ \lambda h_1 + [\nu \cdot (\nabla u_0 + \nabla^T u_0)]|_{\Gamma_s} - \Delta_{\Gamma_s}(h_0) - [\nu \cdot \sigma(w_0)]|_{\Gamma_s} - p_0 \nu = h_1^* \text{ in } \Gamma_s \end{cases} \tag{39}$$

$$\begin{cases} \lambda w_0 - w_1 = w_0^* \text{ in } \Omega_s \\ \lambda w_1 - \text{div } \sigma(w_0) + w_0 = w_1^* \text{ in } \Omega_s \\ w_1|_{\Gamma_s} = h_1 = u_0|_{\Gamma_s} \text{ on } \Gamma_s \end{cases} \tag{40}$$

Sketch of Proof

The author used the following outline

- Solve the decomposed stokes flow for $\mathbf{u}$ and p
- Generate a weak formulation for the "thin" (h_1) and "thick" (w_1) variables
- Recover of the other structure solution variables h_0 and w_0 given data $h_0^* \in H^1(\Gamma_s)$ and $w_0^* \in H^1(\Omega_s)$; i.e.

$$h_0 = \frac{1}{\lambda} h_1 + \frac{1}{\lambda} h_0^* \tag{41}$$

$$w_0 = \frac{1}{\lambda} w_1 + \frac{1}{\lambda} w_0^* \tag{42}$$

Decomposition of Stokes Flow

Decomposition of the Stokes flow into two parts
Zero force and Dirichlet boundary data $g \in H^{1/2}(\Gamma_s)$: solution is $[u_1(g), p_1(g)]$

$$\begin{cases} \lambda u_1 - \text{div } (\nabla u_1 + \nabla^T u_1) + \nabla p_1 = 0 \text{ in } \Omega_f \\ \text{div } (u_1) = \frac{\int_{\Gamma_s} (g \cdot \nu) d\Gamma_s}{\text{meas}(\Omega_f)} \text{ in } \Omega_f \\ u_1|_{\Gamma_s} = g \text{ on } \Gamma_s \\ u_1|_{\Gamma_f} = 0 \text{ on } \Gamma_f, \end{cases} \tag{43}$$

With force term u_0^*, zero Dirichlet data and zero divergence: solution is $[u_2(u_0^*), p_2(u_0^*)]$

$$\begin{cases} \lambda u_2 - \text{div } (\nabla u_2 + \nabla^T u_2) + \nabla p_2 = u_0^* \text{ in } \Omega_f \\ \text{div } (u_2) = 0 \text{ in } \Omega_f \\ u_2|_{\Gamma_f} \text{ on } \Gamma_f \end{cases} \tag{44}$$

The unique $\{u_0, p_0\}$ of (38) may then be expressed as

$$u_0 = u_1(g|_{\Gamma_s}) + u_2(u_0^*); \qquad p_0 = p_1(g|_{\Gamma_s}) + p_2(u_0^*) + c_0, \tag{45}$$

where c_0 is the (presently) unknown constant component of the pressure p_0 of (38).

Define the space

$$\mathbf{S} = \left\{ (\varphi, \psi) \in [H^1(\Gamma_s)]^2 \times [H^1(\Omega_s)]^3 \mid \varphi = \psi|_{\Gamma_s} \right\}.$$

The last relation now gives us the following mixed variational formulation in terms of the "thin" and "thick" structure variables h_1 and w_1: Namely,

$$\begin{aligned} \mathbf{a}([h_1, w_1], [\varphi, \psi]) + \mathbf{b}([\varphi, \psi], c_0) = \mathbf{F}([\varphi, \psi]), \text{ for all } [\varphi, \psi] \in \mathbf{S} \\ \mathbf{b}([h_1, w_1], r) = 0, \text{ for all } r \in \mathbb{R}. \end{aligned} \tag{46}$$

Where
$\mathbf{a}(\cdot, \cdot) : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$, $\mathbf{b}(\cdot, \cdot) : \mathbf{S} \times \mathbb{R} \to \mathbb{R}$, and the functional $\mathbf{F}(\cdot)$ are defined as follows

$$\begin{aligned}
\mathbf{a}([\phi,\xi],[\widetilde{\phi},\widetilde{\xi}]) =& \lambda\left\langle \phi,\widetilde{\phi}\right\rangle_{\Gamma_s} + \frac{1}{\lambda}\left\langle \nabla_{\Gamma_s}\phi, \nabla_{\Gamma_s}\widetilde{\phi}\right\rangle_{\Gamma_s} \\
&+ \lambda\left\langle \xi,\widetilde{\xi}\right\rangle_{\Omega_s} + \frac{1}{\lambda}\left\langle \sigma(\xi),\epsilon(\widetilde{\xi})\right\rangle_{\Omega_s} + \frac{1}{\lambda}\left\langle \xi,\widetilde{\xi}\right\rangle_{\Omega_s} \\
&+ \left\langle \nabla u_1(\xi|_{\Gamma_s}) + \nabla^T u_1(\xi|_{\Gamma_s}), \nabla\widetilde{u}(\widetilde{\phi}) + \nabla^T\widetilde{u}(\widetilde{\phi})\right\rangle_{\Omega_f} \\
&+ \lambda\left\langle u_1(\xi|_{\Gamma_s}),\widetilde{u}(\widetilde{\phi})\right\rangle_{\Omega_f}, \\
\mathbf{b}([\widetilde{\phi},\widetilde{\xi}],r) =& -r\left\langle v,\widetilde{\phi}\right\rangle_{\Gamma_s},
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{F}([\widetilde{\phi},\widetilde{\xi}]) =& -\left\langle \nabla u_2(u_0^*) + \nabla^T u_2(u_0^*), \nabla\widetilde{u}(\widetilde{\phi}) + \nabla^T\widetilde{u}(\widetilde{\phi})\right\rangle_{\Omega_f} \\
&- \frac{1}{\lambda}\left\langle \nabla_{\Gamma_s}h_0^*, \nabla_{\Gamma_s}\widetilde{\phi}\right\rangle_{\Gamma_s} - \frac{1}{\lambda}\left\langle \sigma(w_0^*),\epsilon(\widetilde{\xi})\right\rangle_{\Omega_s} \\
&- \lambda\left\langle u_2(u_0^*),\widetilde{u}(\widetilde{\phi})\right\rangle_{\Omega_f} + \left\langle u_0^*,\widetilde{u}(\widetilde{\phi})\right\rangle_{\Omega_f} \\
&+ \left\langle h_1^*,\widetilde{\phi}\right\rangle_{\Gamma_s} + \left\langle w_1^*,\widetilde{\xi}\right\rangle_{\Omega_s} - \frac{1}{\lambda}\left\langle w_0^*,\widetilde{\xi}\right\rangle_{\Omega_s}.
\end{aligned}$$

Given $r \in \mathbb{R}$, let $z \in [H^1(\Gamma_s)]^2$ satisfy

$$\Delta_{\Gamma_s} z = \text{sgn}(r)\nu \text{ on } \Gamma_s$$

It is easily seen that $||\nabla_{\Gamma_s} z||_{\Gamma_s} \leq C||\nu||_{\Gamma_s}$. Now, taking into account that $\gamma : H^1(\Omega_s) \to H^{1/2}(\Gamma_s)$ is a surjective map, and so it has a continuous right inverse $\gamma^+(z)$, we have

$$\begin{aligned}
\sup_{[\eta,\varsigma]} \frac{\mathbf{b}([\eta,\varsigma], r)}{||[\eta,\varsigma]||_{\mathbf{S}}} &\geq \frac{\mathbf{b}([z,\gamma^+(z)], r)}{||z||_{[H^1(\Gamma_s)]^2}} \\
&= \frac{-r\int_{\Gamma_s} \nu \cdot z d\Gamma_s}{||z||_{[H^1(\Gamma_s)]^2}} \\
&= -r \text{ sgn}(r) \frac{\int_{\Gamma_s} \Delta_{\Gamma_s} z \cdot z d\Gamma_s}{||z||_{[H^1(\Gamma_s)]^2}} \\
&= |r| \frac{\int_{\Gamma_s} |\nabla_{\Gamma_s} z|^2 d\Gamma_s}{||z||_{[H^1(\Gamma_s)]^2}} \\
&= |r| \; ||z||_{[H^1(\Gamma_s)]^2}
\end{aligned}$$

which yields that the inf-sup condition holds wth the constant $\beta = ||z||_{[H^1(\Gamma_s)]^2}$.

By the Babuska-Brezzi Theorem, it is then shown that this unique solution is in $D(\mathbf{A})$.

Part II

Stability Types

- Strong Stability
- Polynomial Stability
- Exponential Stability

Strong Stability

Theorem ((Strong Stability) G. Avalos, PGG, B. Muha [JDE, 2020])

For the modeling generator $\mathbf{A} : D(\mathbf{A}) \subseteq \mathbf{H} \to \mathbf{H}$ *of* $(1)-(4)$, *one has* $\sigma(\mathbf{A}) \cap i\mathbb{R} = \emptyset$. *Consequently, the* C_0*-semigroup* $\left\{ e^{\mathbf{A}t} \right\}_{t \geq 0}$ *given in Theorem* 1 *is strongly stable. That is, the solution* $\Phi(t)$ *of the PDE* $(1)-(4)$ *tends asymptotically to the zero state for all initial data* $\Phi_0 \in \mathbf{H}$

G. Avalos, P. G. Geredeli, B. Muha; "Wellposedness, Spectral Analysis and Asymptotic Stability of a Multilayered Heat-Wave-Wave System ", Journal of Differential Equations 269 (2020), pp. 7129-7156.

Sketch of Proof

Theorem ((Strong Stability) [Arendt-Batty, 1988])

Let $T(t)_{t\geq 0}$ be a bounded C_0-semigroup on a reflexive Banach space X, with generator $\mathbf{A}$. Assume that $\sigma_p(\mathbf{A}) \cap i\mathbb{R} = \emptyset$, where $\sigma_p(\mathbf{A})$ is the point spectrum of $\mathbf{A}$. If $\sigma(\mathbf{A}) \cap i\mathbb{R}$ is countable then $T(t)_{t\geq 0}$ is strongly stable.

Recall: $\sigma(\mathbf{A}) = \sigma_p(\mathbf{A}) \cup \sigma_c(\mathbf{A}) \cup \sigma_r(\mathbf{A})$
Note that $\sigma(\mathbf{A}) \cap i\mathbb{R} = \emptyset$ is equivalent to showing $i\mathbb{R} \subseteq \rho(\mathbf{A})$
To this end the authors checked

- $0 \in \rho(\mathbf{A})$
- The continuous spectrum
- The eigenvalues of $\mathbf{A}^*$

$0 \in \rho(\mathbf{A})$

Given $\Phi^* = [u_0^*, h_{01}^*, h_{11}^*, \ldots, h_{0K}^*, h_{1K}^*, w_0^*, w_1^*] \in \mathbf{H}$, the problem is to find $\Phi = [u_0, h_{01}, h_{11}, \ldots, h_{0K}, h_{1K}, w_0, w_1] \in D(\mathbf{A})$ which solves

$$\mathbf{A}\Phi = \Phi^* \tag{47}$$

It follows

$$w_1 = w_0^* \in H^1(\Omega_s) \tag{48}$$

$$h_{1j} = h_{0j}^* \in H^1(\Gamma_j), \text{ for } 1 \le j \le K \tag{49}$$

and

$$\begin{cases} \Delta u_0 = u_0^* \\ u_0|_{\Gamma_f} = 0 \\ u_0|_{\Gamma_s} = w_0^*|_{\Gamma_s} \end{cases} \tag{50}$$

Recall the set (to be used on the next slide)

$$\begin{aligned} \mathcal{V} =&\{[\psi_1, \ldots, \psi_K] \in H^1(\Gamma_1) \times \cdots \times H^1(\Gamma_k) | \text{ For all } 1 \le j \le K, \\ &\psi_j|_{\partial\Gamma_j \cap \partial\Gamma_l} = \psi_l|_{\partial\Gamma_j \cap \partial\Gamma_l}, \text{ for all } 1 \le l \le K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \ne \emptyset\} \end{aligned} \tag{51}$$

$0 \in \rho(\mathbf{A})$

And define the set

$$\chi \equiv \left\{ [\psi, \xi] \in \mathcal{V} \times H^1(\Omega_s) \mid \psi_j = \xi|_{\Gamma_j} \text{ for } 1 \le j \le K \right\}. \tag{52}$$

Testing with these equations, it follows

$$\begin{aligned} &\langle \nabla w_0, \nabla \xi \rangle_{\Omega_s} + \sum_{j=1}^{K} \left[\langle \nabla h_{0j}, \nabla \psi_j \rangle_{\Gamma_j} + \langle h_{0j}, \psi_j \rangle_{\Gamma_j} \right] \\ &= - \langle w_1^*, \xi \rangle_{\Omega_s} - \sum_{j=1}^{K} \left[\left\langle h_{1j}^*, \psi_j \right\rangle_{\Gamma_j} + \left\langle \frac{\partial u_0}{\partial \nu}, \psi_j \right\rangle_{\Gamma_j} \right], \end{aligned} \tag{53}$$

Since the bilinear form $b(\cdot, \cdot) : \chi \to \mathbb{R}$, given by

$$b([\psi, \xi], [\widetilde{\psi}, \widetilde{\xi}]) = \left\langle \nabla \xi, \nabla \widetilde{\xi} \right\rangle_{\Omega_s} + \sum_{j=1}^{K} \left[\left\langle \nabla \psi_j, \nabla \widetilde{\psi}_j \right\rangle_{\Gamma_j} + \left\langle \psi_j, \widetilde{\psi}_j \right\rangle_{\Gamma_j} \right] \tag{54}$$

for every $[\psi, \xi]$, $\left[\widetilde{\psi}, \widetilde{\xi}\right] \in \chi$, is continuous and χ-elliptic, then by Lax-Milgram, there exists a unique solution

$$\varphi = [(h_{01}, h_{02}, \ldots, h_{0K}), w_0] \in \chi \tag{55}$$

(Then shown to be in $D(\mathbf{A})$)

Assume to the contrary. Then since $\sigma_c(\mathbf{A}) \subseteq \sigma_{\text{app}}(\mathbf{A})$ there exists a sequence $\{\Phi_n\} = \{[u_n, h_{1n}, \xi_{1n}, ..., h_{Kn}, \xi_{Kn}, w_{0n}, w_{1n}]\} \subseteq D(\mathbf{A})$ which satisfy for $n \in \mathbb{N}$

$$||\Phi_n||_{\mathbf{H}} = 1 \text{ and } ||(i\beta I - \mathbf{A})\Phi_n||_{\mathbf{H}} < \frac{1}{n}$$

$\beta,\ i\beta \notin \sigma_r(\mathbf{A}),\ \beta \neq 0$

It is known that for a closed and densely defined operator $\mathbf{A}$, if $\lambda \in \sigma_r(\mathbf{A})$ then $\overline{\lambda} \in \sigma_p(\mathbf{A}^*)$. It follows $\mathbf{A}^* : D(\mathbf{A}^*) \subseteq \mathbf{H} \to \mathbf{H}$ is given by

$$\mathbf{A}^* = \begin{bmatrix} \Delta & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & \dots & 0 & 0 & 0 & 0 \\ -\frac{\partial}{\partial\nu}|_{\Gamma_1} & (I-\Delta) & 0 & \dots & 0 & 0 & -\frac{\partial}{\partial\nu}|_{\Gamma_1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -I & 0 & 0 \\ -\frac{\partial}{\partial\nu}|_{\Gamma_K} & 0 & 0 & \dots & (I-\Delta) & 0 & -\frac{\partial}{\partial\nu}|_{\Gamma_K} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & -I \\ 0 & 0 & 0 & \dots & 0 & 0 & -\Delta & 0 \end{bmatrix};$$

$D(\mathbf{A}) = D(\mathbf{A}^*)$, $i\beta$ is not an eigenvalue of $\mathbf{A}^*$ so $i\beta \notin \sigma_r(\mathbf{A})$

Conclusion

By the spectral analysis, the authors concluded there is strong stability for any initial data in $D(\mathbf{A})$.

Strong Stability

Theorem ((Strong Stability) PGG [JDE, 2025])

With reference to problem $(20) - (23)$, *zero is an eigenvalue for the generator* $\mathbf{A}: D(\mathbf{A}) \subseteq \mathbf{H} \to \mathbf{H}$. *Consequently the solution* $\{e^{\mathbf{A}t}\}|_{[Null(\mathbf{A})]^{\perp}}$ *decays to the zero state for any initial data* $\Phi_0 = [u_0, h_0, h_1, w_0, w_1] \in [Null(\mathbf{A})]^{\perp}$.

Pelin G. Geredeli, Spectral analysis and asymptotic decay of the solutions to multilayered structure-Stokes fluid interaction PDE system, Journal of Differential Equations, Volume 427, 2025, Pages 1-25, ISSN 0022-0396, https://doi.org/10.1016/j.jde.2025.01.080.

Sketch of Proof

As a reminder

Theorem ((Strong Stability) [Arendt-Batty, 1988])

Let $T(t)_{t\geq 0}$ be a bounded C_0-semigroup on a reflexive Banach space X, with generator $\mathbf{A}$. Assume that $\sigma_p(\mathbf{A}) \cap i\mathbb{R} = \emptyset$, where $\sigma_p(\mathbf{A})$ is the point spectrum of $\mathbf{A}$. If $\sigma(\mathbf{A}) \cap i\mathbb{R}$ is countable then $T(t)_{t\geq 0}$ is strongly stable.

It is then checked

- Zero is an eigenvalue
- The continuous spectrum
- The eigenvalues of $\mathbf{A}^*$

Zero is an Eigenvalue

For $\Phi = [u_0, h_0, h_1, w_0, w_1] \in D(\mathbf{A})$, it follows $\mathbf{A}\Phi = 0$ implies

$$\text{div } (\nabla u_0 + \nabla^T u_0) - \nabla p = 0 \text{ in } \Omega_f$$
$$\boxed{h_1 = 0} \text{ in } \Gamma_s$$
$$-\nu \cdot (\nabla u_0 + \nabla^T u_0)|_{\Gamma_s} + \Delta_{\Gamma_s}(h_0) + \nu \cdot \sigma(w_0)|_{\Gamma_s} + p\nu = 0 \text{ in } \Gamma_s$$
$$\boxed{w_1 = 0} \text{ in } \Omega_s$$
$$\text{div } \sigma(w_0) - w_0 = 0 \text{ in } \Omega_s$$

Dissipativity shows

$$0 = \text{Re}\,\langle \mathbf{A}\Phi, \Phi \rangle = \frac{1}{2}||\nabla u_0 + \nabla^T u_0||^2 \implies \boxed{u_0 = 0}$$
$$\boxed{p = c_0 = \text{constant}}$$

Zero is an Eigenvalue

Define $S = \left\{[f,g] \in H^1(\Gamma_s) \times H^1(\Omega_s) \mid f = g|_{\Gamma_s}\right\}$.
Testing the previous equations with f, g,

$$\begin{aligned}\mathbf{B}([h_0, w_0],[f,g]) &= \langle \nabla_{\Gamma_s}(h_0), \nabla_{\Gamma_s}(f)\rangle_{\Gamma_s} + \langle \sigma(w_0), \epsilon(g)\rangle_{\Omega_s} + \langle w_0, g\rangle_{\Omega_s} \\ &= \langle c_0 \nu, f\rangle_{\Gamma_s}\end{aligned}$$

As $B(\cdot,\cdot)$ is continuous and S-elliptic, using Lax-Milgram finds $\{h_0, w_0\} \in S$. (Then in $D(\mathbf{A})$)

$$\text{Null}(\mathbf{A}) = \text{Span}\left\{\begin{bmatrix} 0 \\ h_0 \\ 0 \\ w_0 \\ 0 \end{bmatrix}\right\}$$

$$\text{Null}(\mathbf{A})^{\perp} = \left\{[\tilde{u}_0, \tilde{h}_0, \tilde{h}_1, \tilde{w}_0, \tilde{w}_1] \in \mathbf{H} \mid \int_{\Gamma_s} \nu \cdot \tilde{h}_0 \, d\Gamma_s = 0\right\}$$

$\beta,\ i\beta \notin \sigma_p(\mathbf{A}),\ \beta \neq 0$

For $\Phi = [u_0, h_0, h_1, w_0, w_1]$,

$$[i\beta I - \mathbf{A}]\Phi = 0$$

Thus we see there is an overdetermined eigenvalue problem

$$\begin{aligned} -\beta^2 w_0 - \text{div } \sigma(w_0) + w_0 = 0 \text{ in } \Omega_s \\ w_0|_{\Gamma_s} = 0 \text{ on } \Gamma_s \\ \nu \cdot \sigma(w_0) = -c_0\nu \text{ on } \Gamma_s \end{aligned}$$

Assumption for (fixed) given β, assume that the only solution to the above overdetermined problem is $w_0 = 0$ (and necessarily $c_0 = 0$).

$\beta,\ i\beta \notin \sigma_c(\mathbf{A}),\ \beta \neq 0$

Assume to the contrary. Then since $\sigma_c(\mathbf{A}) \subseteq \sigma_{\text{app}}(\mathbf{A})$ there exists a sequence $\{\Phi_n\} = \{[u_{0n}, h_{0n}, h_{1n}, w_{0n}, w_{1n}]\} \subseteq D(\mathbf{A})$ such that

$$||\Phi_n|| = 1 \text{ and } ||(i\beta I - \mathbf{A})\Phi_n||_{\mathbf{H}} < \frac{1}{n}$$

$\beta,\ i\beta \notin \sigma_r(\mathbf{A}),\ \beta \neq 0$

It is known that for a closed and densely defined operator $\mathbf{A}$, if $\lambda \in \sigma_r(\mathbf{A})$ then $\overline{\lambda} \in \sigma_p(\mathbf{A}^*)$. It follows $\mathbf{A}^* : D(\mathbf{A}^*) \subseteq \mathbf{H} \to \mathbf{H}$ is given by

$$\mathbf{A}^* = \begin{bmatrix} \mathrm{div}\,(\nabla(\cdot) + \nabla^T(\cdot)) & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 \\ -[\nu \cdot (\nabla(\cdot) + \nabla^T(\cdot))]|_{\Gamma_s} & -\Delta_{\Gamma_s}(\cdot) & 0 & -\nu \cdot \sigma(\cdot)|_{\Gamma_s} & 0 \\ 0 & 0 & 0 & 0 & -I \\ 0 & 0 & 0 & -\mathrm{div}\,\sigma(\cdot) + I & 0 \end{bmatrix} + \begin{bmatrix} -\nabla\mathcal{P}_1(\cdot) & \nabla\mathcal{P}_2(\cdot) & 0 & \nabla\mathcal{P}_3(\cdot) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mathcal{P}_1(\cdot)\nu & -\mathcal{P}_2(\cdot)\nu & 0 & -\mathcal{P}_3(\cdot)\nu & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{56}$$

$D(\mathbf{A}) = D(\mathbf{A}^*)$, $i\beta$ is not an eigenvalue of $\mathbf{A}^*$ so $i\beta \notin \sigma_r(\mathbf{A})$

Conclusion

It is important to note that $0 \in \rho(\mathbf{A}|_{\mathrm{Null}(\mathbf{A})^\perp})$. In other words, the resolvent of $\mathbf{A}|_{\mathrm{Null}(\mathbf{A})^\perp}$.

By the spectral analysis, the author showed that there is strong stability for any initial data taken in $\mathrm{Null}(\mathbf{A})^\perp$.

Future Work and Open Questions

Topics

- Stokes Model polynomial decay
- Lack of exponential decay
- Navier-Stokes model

Thank You!