# Qualitative Properties of the Multilayered Structure - Fluid Interactions Coupled PDE systems

---

A Masters Project
Presented to
the Graduate School of
Clemson University

---

In Partial Fulfillment
of the Requirements for the Degree
Masters of Science
Mathematical Sciences

---

by
Jacob David Manning
December 2025

---

Accepted by:
Dr. Pelin Guven Geredeli, Committee Chair
Dr. Hyesuk Lee
Dr. Quyuan Lin
Dr. Shitao Liu

# Plain Language Abstract

Fluid structure interaction (FSI) partial differential equations (PDEs) show up all over nature and thus are a common mathematical model of study. For example, due to hemodynamic forces generated by blood moving through mammalian arteries, the vascular walls, being composed of viscoelastic materials, undergo large deformations during the blood transport process. As such, there is a coupling of respective blood flow and wall deformation dynamics. This physiological interaction between arterial walls and blood flow plays a crucial role in the physiology and pathophysiology of the human cardiovascular system and can be mathematically realized by FSI PDEs. In such FSI modeling, the blood flow is governed by the fluid flow PDE component (incompressible Stokes or Navier-Stokes); the displacements along the elastic vascular walls are described by the structural PDE component (e.g., Lamé systems of elasticity). "Single layered" FSI models - i.e., FSI models in which only one (three dimensional) elastic PDE appears to describe the structural dynamics - have been studied extensively in the literature. However, many biomedical devices (such as stents) are being developed with the view that vascular wall structures are constituted of composite materials and not of a single layer.

Accordingly, in this project, we revisit two multilayered FSI systems where the coupling of the 3-D fluid (blood flow) and 3-D elastic (structural vascular wall) PDE components is realized via an additional 2-D elastic system on the boundary interface, but looking through the lens of cellular dynamics. The main purpose of this project is to review the methodologies used to show the qualitative properties of those coupled PDE dynamics such as wellposedness and longtime behavior of solutions. Both models that will be covered are a simplification of the Navier-Stokes model. The first one that will be introduced is denoted the canonical model (3D heat, 2D wave, 3D wave). Where as the second is a more realistic and complicated model (3D Stokes, 2D wave, 3D Lamé) which is described again via different PDE dynamics.

# Abstract

In this work, we will conduct a survey of recent developments of two different composite structure multilayered models and investigate the qualitative properties of the wellposedness and stability. The first being a 3-D heat equation is coupled with a 3-D wave equation via a 2-D interface whose dynamics is described by a 2-D wave equation. This model is also notably set up for computational modeling as well. This is a simplification of the Navier-Stokes dynamics that we would expect of such a system and as such does not have any pressure terms that are present in the second model. The authors established wellposedness via a Lumer-Phillips approach and strong stability or asymptotic decay to the zero state for all initial data. This is done by analyzing the spectrum of the generator for the associated $C_0$-semigroup.

The second model uses a different set of PDEs to model the interactions. This updated model is still a simplification, but is more realistic. It is a 3-D Stokes flow equation, coupled with a 3-D elastic dynamics equation with an additional 2-D interface with elastic dynamics. Similarly, as the model is still linear, Lumer-Phillips was used to show wellposedness. Using a nonstandard mixed variational formulation, the original author showed that the PDE system generates a $C_0$-semigroup. The pressure term in the 3D Stokes equation adds a great challenge to the analysis. The author used non-Leray-based elimination of the associated pressure term. The elastic solution was found via a Babuska-Brezzi approach. Long term strong stability was again shown by analyzing the spectrum of the associated operator that generates this $C_0$-semigroup. Of note, the author shows that zero is an eigenvalue of $\mathbf{A}$. From this, the author addresses the issue of asymptotic decay of the solution to the zero state for any initial data taken from the orthogonal complement of the zero eigenspace $\mathrm{Null}(\mathbf{A})^{\perp}$.

# Dedication

I would like to dedicate this project to my wife Rafaela. She has put in just as much work and sacrifice into this degree as I have.

# Acknowledgements

I would like to thank Pelin Guven Geredeli who served as my advisor for my time at Clemson University as well as everyone on my committee. I would also like to recognize the National Science Foundation and acknowledge their partial funding from NSF Grant DMS-2348312 which I received during my time at Clemson.

# Table of Contents

# List of Figures

# Chapter 1

# Introduction

Multilayered fluid-structure interaction (FSI) partial differential equation (PDE) dynamics arise in the context of blood transportation in mammalian arteries and shape deformation of cells in cellular dynamics. The physiological interaction between the arterial walls and blood flowing through plays an important role in cardiovascular systems of cells [12, 15, 31, 38, 39]. For example, in existing literature on FSI models, often only a single layer structural PDE component is presented. We see for example in [36] (Section 9) and [41] that coupled heat-wave systems were examined where the heat equation is a simplification of the fluid flow and the wave equation is a simplification of the elastic structure. Although these simplifications are instructive, there is realism lost by the changes as vascular walls are typically multilayered [18, 23, 39].

In a pioneering paper [39], the authors considered a multilayered FSI PDE model where multilayered FSI is composed of 2D "thick" layer wave equation and 1D "thin" layer wave equation coupled to a 2D fluid PDE across a boundary interface. In this paper it showed that thin structure with mass at the fluid-structure interface regularizes the FSI dynamics. Being inspired, the authors in [8] proposed a new multilayered 3D heat, 2D wave, 3D wave system in order to model eukaryotic cellular dynamics. This "canonical" model will be a topic of focus in this project. As mentioned before, the two 3D heat and wave equations are modeling more complicated flow and structural equations respectively. In the paper, it was shown through a semigroup approach that this system is wellposed by a combination of Lumer-Phillips and Lax-Milgram arguments as well as exhibiting strong stability by analyzing the spectrum of the generator of the semigroup. It should be noted that the authors originally intended to write a follow up paper that focused on a numerical implementation of this model. To this end, the canonical model is framed as a convex polygonal domain representing the nucleus instead of a smooth domain.

Recently, in [24] a more realistic model of 3D Stokes flow, 2D elastic wave, 3D Lamé elastic system was proposed with a smooth nuclear domain. This model will be the other topic of focus for this project. Similar to the canonical model, wellposedness was shown as well as strong stability under mild assumptions later in [25]. In [24] one of the major problems the author overcame was dealing with the pressure term that is introduced with the Stokes flow. It was overcome by solving elliptic boundary value problems given in the fluid pressure variable which appeals to the application of the Lax-Milgram Theorem. As the Stokes fluid system is still linear, the main result still uses a Lumer-Phillips approach. However, as the pressure term adds more complexity, the author used the Babuska-Brezzi Theorem, in order to show the maximal dissipativity of respective generator of the associated $C_0$-semigroup. In [25], the author again analyzed the spectrum of the generator of the semigroup. However, as the origin is on the imaginary axis, and zero is an eigenvalue of the associated generator, the strong stability is restricted to initial data in the orthogonal compliment of the zero eigenspace. Another issue that the author ran into in the spectral analysis is by estimating the 3D "thick" elastic solution variable, an overdetermined eigenvalue problem arises. This forces the geometrical imposition to guarantee that the spectrum does not intersect the imaginary axis. This assumption has been used in similar FSI problems such as [3, 4, 5].

Our main goal is to investigate the requirements to obtain the main results of each PDE model, we hope is to compare and contrast the methodologies used in each paper with respect to the given assumptions. There are major differences between the two models as the Stokes flow adds additional complexity not present in the Canonical model, such as the pressure term in the second model. We have structured this project in a way to highlight the similarities and differences of the methodologies used to show the wellposedness and long term behavior of each model in Part I and Part II respectively.

# Chapter 2

# Notation

For the remainder of the text, norms $||\cdot||_D$ are taken to be $L^2(D)$ for the domain $D$. Inner products in $L^2(D)$ are written $\langle \cdot, \cdot \rangle_D$, and the inner products $L^2(\partial D)$ are written $\langle \cdot, \cdot \rangle_{\partial D}$. The space $H^s(D)$ will denote the Sobolev space of order $s$, defined on a domain $D$, and $H_0^s(D)$ denotes the closure of $C_0^\infty(D)$ in the $H^s(D)$ norm which we denote by $||\cdot||_{H^s(D)}$ or $||\cdot||_{s,D}$. For the sake of simplicity, we denote

$$\mathbf{L}^p(\Omega_f) = [L^p(\Omega_f)]^3,\ \mathbf{L}^p(\Omega_s) = [L^p(\Omega_s)]^3,\ \mathbf{L}^p(\Gamma_s) = [L^p(\Gamma_s)]^2,$$
$$\mathbf{H}^s(\Omega_s) = [H^s(\Omega_s)]^3,\ \mathbf{H}^s(\Gamma_s) = [H^s(\Gamma_s)]^2.$$

Also, we make use of the standard notation for the trace of functions defined on a Lipschitz domain $D$; i.e. for a scalar function $\phi \in H^1(D)$, we denote $\gamma(w)$ to be the trace mapping from $H^1(D)$ to $H^{1/2}(\partial D)$. That is, $\gamma(w) = w|_{\partial D}$, for $w \in C^\infty(D)$. We will also denote pertinent duality pairings as $(\cdot, \cdot)_{X\times X'}$.

# Chapter 3

# Part I: Well Posedness of the Multilayered FSI Systems

## 3.1 Canonical Multilayered FSI PDE Model

Let the fluid geometry $\Omega_f \subseteq \mathbb{R}^3$ be a Lipschitz, bounded domain. The structure domain $\Omega_s \subseteq \mathbb{R}^3$ will be "completely immersed" in $\Omega_f$; with $\Omega_s$ being a convex polyhedral domain. In the figure $\Gamma_f$ is the part of the boundary of $\partial\Omega_f$ which does not come into contact with $\Omega_s$; $\Gamma_s = \partial\Omega_s$ is the boundary interface between $\Omega_f$ and $\Omega_s$ wherein the coupling between the two distinct fluid

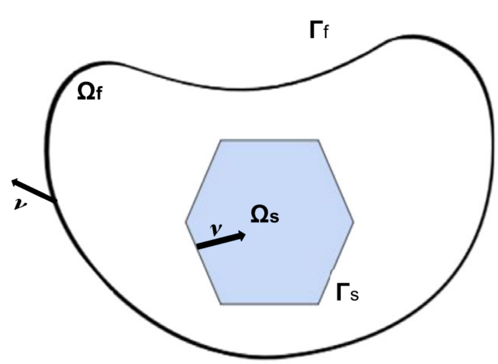


Figure 3.1: Polygonal domain FSI model

and elastic dynamics occurs. (And so $\partial\Omega_f = \Gamma_s \cup \Gamma_f$.) It follows that

$$\Gamma_s = \cup_{j=1}^{K} \overline{\Gamma_j} \tag{3.1}$$

where $\Gamma_i \cap \Gamma_j = \emptyset$, for $i \neq j$. It is further assumed that each $\Gamma_j$ is an open polygonal domain. Moreover, $n_j$ will denote the unit normal vector which is exterior to $\partial\Gamma_j$, $1 \leq j \leq K$. With respect to this geometry, the $\mathbb{R}^3$ wave-$\mathbb{R}^2$ wave-$\mathbb{R}^3$ heat interaction PDE model is given as follow: For $1 \leq j \leq K$,

$$\begin{cases} u_t - \Delta u = 0 \text{ in } (0,T) \times \Omega_f \\ u|_{\Gamma_f} = 0 \text{ on } (0,T) \times \Gamma_f; \end{cases} \tag{3.2}$$

$$\begin{cases} \frac{\partial^2}{\partial t^2} h_j - \Delta h_j + h_j = \frac{\partial w}{\partial \nu}|_{\Gamma_j} - \frac{\partial u}{\partial \nu}|_{\Gamma_j} \text{ on } (0,T) \times \Gamma_j \\ h_j|_{\partial\Gamma_j \cap \partial\Gamma_l} = h_l|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } (0,T) \times (\partial\Gamma_j \cap \partial\Gamma_l), \text{ for all } 1 \leq l \leq K \\ \quad \text{such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset \\ \left.\frac{\partial h_j}{\partial n_j}\right|_{\partial\Gamma_j \cap \partial\Gamma_l} = -\left.\frac{\partial h_l}{\partial n_l}\right|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } (0,T) \times (\partial\Gamma_j \cap \partial\Gamma_l), \text{ for all } 1 \leq l \leq K \\ \quad \text{such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset \end{cases} \tag{3.3}$$

$$\begin{cases} w_{tt} - \Delta w = 0 \text{ on } (0,T) \times \Omega_s \\ w_t|_{\Gamma_j} = \frac{\partial}{\partial t} h_j = u|_{\Gamma_j} \text{ on } (0,T) \times \Gamma_j, \text{ for } j = 1, \ldots, K \end{cases} \tag{3.4}$$

$$[u(0), h_1(0), \frac{\partial}{\partial t} h_1(0), \ldots, h_K(0), \frac{\partial}{\partial t} h_K(0), w(0), w_t(0)] = [u_0, h_{01}, h_{11}, \ldots, h_{0K}, h_{1K}, w_0, w_1] \tag{3.5}$$

Equation $(3.3)_1$ is the dynamic coupling condition and represents a balance of forces on $\Gamma_j$. The left-hand side comes from the inertia and elastic energy of the thin structure, while the right-hand side accounts for the contact forces coming from the 3-D structure and the fluid, respectively. The last term of the left-hand side is added to ensure the uniqueness of the solution and physically means that the structure is anchored and therefore the displacement does not have a translational component. The coupling conditions $(3.3)_2$ and $(3.3)_3$ represent continuity of the displacement and contact force along the interface between sides $\Gamma_i$ and $\Gamma_l$, respectively. Equation $(3.4)_2$ is a kinematic coupling condition and accounts for continuity of the velocity across the interface $\Gamma_j$. It corresponds to the no-slip boundary condition in fluid mechanics. Note that the boundary condition in (3.4) implies that for $t > 0$,

$$w(t)|_{\Gamma_j} - h_j(t) = w(0)|_{\Gamma_j} - h_j(0), \text{ for } j = 1, \ldots, K.$$

Accordingly, the associated space of initial data $\mathbf{H}$ incorporates a compatibility condition. Namely,

$$\begin{aligned}\mathbf{H} =& \{[u_0, h_{01}, h_{11}, \ldots, h_{0K}, h_{1K}, w_0, w_1] \in L^2(\Omega_f) \times H^1(\Gamma_1) \times L^2(\Gamma_1) \times \ldots \\ & \times H^1(\Gamma_K) \times L^2(\Gamma_K) \times H^1(\Omega_s) \times L^2(\Omega_s), \text{ such that for each } 1 \leq j \leq K: \\ & (i) w_0|_{\Gamma_j} = h_{0j}; \\ & (ii) h_{0j}|_{\partial\Gamma_j \cap \partial\Gamma_l} = h_{0l}|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } \partial\Gamma_j \cap \partial\Gamma_l, \text{ for all } 1 \leq l \leq K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset\}\end{aligned} \tag{3.6}$$

Because of the given boundary interface compatibility condition, $\mathbf{H}$ is a Hilbert space with the inner product

$$\begin{aligned}\left\langle \Phi_0, \widetilde{\Phi}_0 \right\rangle_{\mathbf{H}} = \langle u_0, \widetilde{u}_0 \rangle_{\Omega_f} + \sum_{j=1}^{K} \left\langle \nabla h_{0j}, \nabla \widetilde{h}_{0j} \right\rangle_{\Gamma_j} + \sum_{j=1}^{K} \left\langle h_{0j}, \widetilde{h}_{0j} \right\rangle_{\Gamma_j} \\ + \sum_{j=1}^{K} \left\langle h_{1j}, \widetilde{h}_{1j} \right\rangle_{\Gamma_j} + \langle \nabla w_0, \nabla \widetilde{w}_0 \rangle_{\Omega_s} + \langle w_1, \widetilde{w}_1 \rangle_{\Omega_s}\end{aligned} \tag{3.7}$$

where

$$\Phi_0 = [u_0, h_{01}, h_{11}, \ldots, h_{0K}, h_{1K}, w_0, w_1] \in \mathbf{H}; \widetilde{\Phi}_0 = [\widetilde{u}_0, \widetilde{h}_{01}, \widetilde{h}_{11}, \ldots, \widetilde{h}_{0K}, \widetilde{h}_{1K}, \widetilde{w}_0, \widetilde{w}_1] \in \mathbf{H}. \tag{3.8}$$

## 3.2 Preliminaries

With respect to the above setting, the PDE system given above can be recast as an ODE in Hilbert space $\mathbf{H}$. That is, if $\Phi(t) = [u, h_1, \frac{\partial}{\partial t} h_1, \ldots, h_K, \frac{\partial}{\partial t} h_K, w, w_t] \in C([0,T]; \mathbf{H})$ solves the problem for $\Phi_0 \in \mathbf{H}$, then there is a modeling operator $\mathbf{A} : D(\mathbf{A}) \subseteq \mathbf{H} \to \mathbf{H}$ such that $\Phi(\cdot)$ satisfies

$$\frac{d}{dt}\Phi(t) = A\Phi_0; \quad \Phi(0) = \Phi_0. \tag{3.9}$$

In fact, this operator $\mathbf{A} : D(\mathbf{A}) \subseteq \mathbf{H} \to \mathbf{H}$ is defined as follows

$$\mathbf{A} = \begin{bmatrix} \Delta & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 & 0 & 0 \\ -\frac{\partial}{\partial\nu}|_{\Gamma_1} & (\Delta - I) & 0 & \dots & 0 & 0 & \frac{\partial}{\partial\nu}|_{\Gamma_1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & I & 0 & 0 \\ -\frac{\partial}{\partial\nu}|_{\Gamma_K} & 0 & 0 & \dots & (\Delta - I) & 0 & \frac{\partial}{\partial\nu}|_{\Gamma_K} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & I \\ 0 & 0 & 0 & \dots & 0 & 0 & \Delta & 0 \end{bmatrix}; \tag{3.10}$$

$$\begin{aligned} D(\mathbf{A}) = \{&[u_0, h_{01}, h_{11}, \dots, h_{0K}, h_{1K}, w_0, w_1] \in \mathbf{H} : \\ &(\mathbf{A.i})\ u_0 \in H^1(\Omega_f),\ h_{1j} \in H^1(\Gamma_j) \text{ for } 1 \le j \le K,\ w_1 \in H^1(\Omega_s); \\ &(\mathbf{A.ii})(a)\ \Delta u_0 \in L^2(\Omega_f),\ \Delta w_0 \in L^2(\Omega_s),\ (b)\ \Delta h_{0j} - \left.\frac{\partial u_0}{\partial \nu}\right|_{\Gamma_j} + \left.\frac{\partial w_0}{\partial \nu}\right|_{\Gamma_j} \in L^2(\Gamma_j) \\ &\quad \text{for } 1 \le j \le K; \text{(c) } \left.\frac{\partial h_{0j}}{\partial n_j}\right|_{\partial\Gamma_j} \in H^{-1/2}(\partial\Gamma_j),\ \text{for } 1 \le j \le K; \\ &(\mathbf{A.iii})\ u_0|_{\Gamma_f} = 0, u_0|_{\Gamma_j} = h_{1j} = w_1|_{\Gamma_j},\ \text{for } 1 \le j \le K; \\ &(\mathbf{A.iv})\ \text{For } 1 \le j \le K: \\ &\ (a)\ h_{1j}|_{\partial\Gamma_j \cap \partial\Gamma_l} = h_{1l}|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } \partial\Gamma_j \cap \partial\Gamma_l,\ \text{for all } 1 \le l \le K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset; \\ &\ (b)\ \left.\frac{\partial h_{0j}}{\partial n_j}\right|_{\partial\Gamma_j \cap \partial\Gamma_l} = -\left.\frac{\partial h_{0l}}{\partial n_l}\right|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } \partial\Gamma_j \cap \partial\Gamma_l,\ \text{for all } 1 \le l \le K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset\}. \end{aligned} \tag{3.11}$$

## 3.3 Main Result: Existence-Uniqueness of Solution

**Theorem 1.** *The operator $\boldsymbol{A} : D(\boldsymbol{A}) \subseteq \boldsymbol{H} \to \boldsymbol{H}$ defined above generates a $C_0$-semigroup of contractions. Consequently, the solution $\Phi(t) = [u, h_1, \frac{\partial}{\partial t}h_1, \dots, h_K, \frac{\partial}{\partial t}h_K, w, w_t]$ of the PDE model is given by*

$$\Phi(t) = e^{\boldsymbol{A}t}\Phi_0 \in C([0,T]; \boldsymbol{H}),$$

*where* $\Phi_0 = [u_0, h_{01}, h_{11}, \dots, h_{0K}, h_{1k}, w_0, w_1] \in \boldsymbol{H}$

This section is devoted to prove the Hadamard wellposedness of the coupled system given above. The proof hinges on the application of the Lumer-Phillips Theorem ($see Theorem 63$) which assures the existence of a $C_0$-semigroup of contractions $\{e^{\mathbf{A}t}\}_{t\geq 0}$ once it is established that $\mathbf{A}$ is maximally dissipative.

### 3.3.1 Step 1 (Dissipativity of A)

Given data $\Phi_0$ to be in $D(\mathbf{A})$,

$$\begin{aligned}
\langle \mathbf{A}\Phi_0, \Phi_0\rangle_{\mathbf{H}} = {} & \langle \Delta u_0, u_0\rangle_{\Omega_f} + \sum_{j=1}^{K} \langle \nabla h_{1j}, \nabla h_{0j}\rangle_{\Gamma_j} \\
& + \sum_{j=1}^{K} \left\langle \frac{\partial u_0}{\partial \nu}, h_{1j}\right\rangle_{\Gamma_j} + \sum_{j=1}^{K} \left\langle \frac{\partial w_0}{\partial \nu}, h_{1j}\right\rangle_{\Gamma_j} \\
& + \sum_{j=1}^{K} \langle h_{0j}, h_{1j}\rangle_{\Gamma_j} + \sum_{j=1}^{K} \langle (\Delta - I) h_{0j}, h_{1j}\rangle_{\Gamma_j} \\
& + \langle \nabla w_1, \nabla \widetilde{w}_0\rangle_{\Omega_s} + \langle \Delta w_0, w_1\rangle_{\Omega_s} \\
= & - \langle \nabla u_0, \nabla u_0\rangle_{\Omega_f} + \left\langle \frac{\partial}{\partial \nu} u_0, u_0\right\rangle_{\Gamma_s} \\
& + \sum_{j=1}^{K} \langle \nabla h_{1j}, \nabla h_{0j}\rangle_{\Gamma_j} + \sum_{j=1}^{K} \langle h_{1j}, h_{0j}\rangle_{\Gamma_j} \\
& - \sum_{j=1}^{K} \overline{\langle \nabla h_{1j}, \nabla h_{0j}\rangle_{\Gamma_j}} - \sum_{j=1}^{K} \overline{\langle h_{1j}, h_{0j}\rangle_{\Gamma_j}} + \sum_{j=1}^{K} \left\langle \frac{\partial h_{0j}}{\partial n_j}, h_{1j}\right\rangle_{\partial\Gamma_j} \\
& + \sum_{j=1}^{K} \left\langle \frac{\partial u_0}{\partial \nu}, h_{1j}\right\rangle_{\Gamma_j} - \sum_{j=1}^{K} \left\langle \frac{\partial w_0}{\partial \nu}, h_{1j}\right\rangle_{\Gamma_j} \\
& + \langle \nabla w_1, \nabla w_0\rangle_{\Omega_s} - \overline{\langle \nabla w_1, \nabla w_0\rangle_{\Omega_s}} - \left\langle \frac{\partial w_0}{\partial \nu}, w_1\right\rangle_{\Gamma_s}.
\end{aligned} \tag{3.12}$$

(In the last expression, the authors are implicitly using the fact that the unit normal vector $\nu$ is *interior* with respect to $\Gamma_s$.) Note now via domain criterion (**A.iv**), that for fixed index $j$, $1 \leq j \leq K$,

$$\left\langle \frac{\partial h_{0j}}{\partial n_j}, h_{1j}\right\rangle_{\partial\Gamma_j} = \sum_{\substack{1\leq l\leq K \\ \partial\Gamma_j\cap\partial\Gamma_l\neq\emptyset}} - \left\langle \frac{\partial h_{0l}}{\partial n_l}, h_{1l}\right\rangle_{\partial\Gamma_j\cap\partial\Gamma_l}$$

Such relation gives then the inference

$$\sum_{j=1}^{K} \left\langle \frac{\partial h_{0j}}{\partial n_j}, h_{1j} \right\rangle_{\partial \Gamma_j} = 0 \tag{3.13}$$

Applying this relation and domain criterion (**A.iii**) to the inner product,

$$\begin{aligned} \langle \mathbf{A}\Phi_0, \Phi_0 \rangle_{\mathbf{H}} = & -\|\nabla u_0\|^2_{\Omega_f} + 2i \sum_{j=1}^{K} \mathrm{Im} \langle \nabla h_{1j}, \nabla h_{0j} \rangle_{\Gamma_j} \\ & + 2i \sum_{j=1}^{K} \mathrm{Im} \langle h_{1j}, h_{0j} \rangle_{\Gamma_j} + 2i \mathrm{Im} \langle \nabla w_1, \nabla w_0 \rangle_{\Omega_s}, \end{aligned} \tag{3.14}$$

which gives

$$\mathrm{Re} \langle \mathbf{A}\Phi_0, \Phi_0 \rangle_{\mathbf{H}} \leq 0.$$

### 3.3.2 Step 2 (The Maximality of A)

Given parameter $\lambda > 0$, suppose $\Phi = [u_0, h_{01}, h_{11}, \ldots, h_{0K}, h_{1K}, w_0, w_t] \in D(\mathbf{A})$ is a solution of the equation

$$(\lambda I - A)\Phi = \Phi^*, \tag{3.15}$$

where $\Phi^* = [u_0^*, h_{01}^*, h_{11}^*, \ldots, h_{0K}^*, h_{1K}^*, w_0^*, w_t^*] \in \mathbf{H}$. Then in PDE terms, the abstract equation above becomes

$$\begin{cases} \lambda u_0 - \Delta u_0 = u_0^* \text{ in } \Omega_f \\ u_0|_{\Gamma_f} = 0 \text{ on } \Gamma_f; \end{cases} \tag{3.16}$$

and for $1 \leq j \leq K$,

$$\begin{cases} \lambda h_{0j} - h_{1j} = h_{0j}^* \text{ in } \Gamma_j \\ \lambda h_{1j} - \Delta h_{0j} + h_{0j} - \frac{\partial w_0}{\partial \nu} + \frac{\partial u_0}{\partial \nu} = h_{1j}^* \text{ in } \Gamma_j \\ u_0|_{\Gamma_j} = h_{1j} = w_1|_{\Gamma_j} \text{ in } \Gamma_j \\ h_{0j}|_{\partial\Gamma_j \cap \partial\Gamma_l} = h_{0l}|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } \partial\Gamma_j \cap \partial\Gamma_l, \text{ for all } 1 \leq l \leq K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset \\ \left.\frac{\partial h_{0j}}{\partial n_j}\right|_{\partial\Gamma_j \cap \partial\Gamma_l} = -\left.\frac{\partial h_{0l}}{\partial n_l}\right|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } \partial\Gamma_j \cap \partial\Gamma_l, \text{ for all } 1 \leq l \leq K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset; \end{cases} \tag{3.17}$$

and also

$$\begin{cases} \lambda w_0 - w_1 = w_0^* \text{ in } \Omega_s \\ \lambda w_1 - \Delta w_0 = w_1^* \text{ in } \Omega_s. \end{cases} \tag{3.18}$$

With respect to this static PDE system, by multiplying the heat equation above by a test function $\varphi \in H^1_{\Gamma_f}(\Omega_f)$, where

$$H^1_{\Gamma_f}(\Omega_f) = \left\{ \zeta \in H^1(\Omega_f) \,\middle|\, \zeta|_{\Gamma_f} = 0 \right\}.$$

Upon integrating and invoking Green's Theorem, the solution component $u_0$ satisfies the variational relation,

$$\lambda \langle u_0, \varphi \rangle_{\Omega_f} + \langle \nabla u_0, \nabla \varphi \rangle_{\Omega_f} - \left\langle \frac{\partial u_0}{\partial \nu}, \varphi \right\rangle_{\Gamma_s} = \langle u_0^*, \varphi \rangle_{\Omega_f} \text{ for } \varphi \in H^1_{\Gamma_f}(\Omega_f). \tag{3.19}$$

In addition, define the Hilbert space $\mathcal{V}$ by

$$\begin{aligned} \mathcal{V} = &\{[\psi_1, \ldots, \psi_K] \in H^1(\Gamma_1) \times \cdots \times H^1(\Gamma_k) | \text{ For all } 1 \leq j \leq K, \\ &\psi_j|_{\partial\Gamma_j \cap \partial\Gamma_l} = \psi_l|_{\partial\Gamma_j \cap \partial\Gamma_l}, \text{ for all } 1 \leq l \leq K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset\}. \end{aligned} \tag{3.20}$$

Therewith, by multiplying both sides of the $h_{0j}$-wave equation in (3.17) by component $\psi_j$ of $\psi \in \mathcal{V}$, for $1 \leq j \leq K$. Upon integration, for $\psi \in \mathcal{V}$,

$$\begin{bmatrix} \lambda \langle h_{11}, \psi_1 \rangle_{\Gamma_1} - \langle \Delta h_{01}, \psi_1 \rangle_{\Gamma_1} + \langle h_{01}, \psi_1 \rangle_{\Gamma_1} - \left\langle \frac{\partial}{\partial \nu} w_0, \psi_1 \right\rangle_{\Gamma_1} + \left\langle \frac{\partial}{\partial \nu} u_0, \psi_1 \right\rangle_{\Gamma_1} \\ \vdots \\ \lambda \langle h_{1K}, \psi_K \rangle_{\Gamma_K} - \langle \Delta h_{0K}, \psi_K \rangle_{\Gamma_K} + \langle h_{0K}, \psi_K \rangle_{\Gamma_K} - \left\langle \frac{\partial}{\partial \nu} w_0, \psi_K \right\rangle_{\Gamma_K} + \left\langle \frac{\partial}{\partial \nu} u_0, \psi_K \right\rangle_{\Gamma_K} \end{bmatrix}$$
$$= \begin{bmatrix} \langle h_{11}^*, \psi_1 \rangle_{\Gamma_1} \\ \vdots \\ \langle h_{1K}^*, \psi_K \rangle_{\Gamma_K} \end{bmatrix}.$$

For each vector component, it is subsequently integrated by parts while invoking the resolvent relations in (3.17) (and using the domain criterion (**A.iv.b**)). Summing up the components of the resulting vectors, the solution components $[h_{11}, \ldots, h_{1K}] \in \mathcal{V}$ of (3.15) satisfy

$$\sum_{j=1}^{K} \left[ \lambda \langle h_{1j}, \psi_j \rangle_{\Gamma_j} + \frac{1}{\lambda} \langle \nabla h_{1j}, \nabla \psi_j \rangle_{\Gamma_j} + \frac{1}{\lambda} \langle h_{1j}, \psi_j \rangle_{\Gamma_j} + \left\langle \frac{\partial}{\partial \nu} u_0 - \frac{\partial}{\partial \nu} w_0, \psi_j \right\rangle_{\Gamma_j} \right]$$
$$= \sum_{j=1}^{K} \left[ \left\langle h_{1j}^*, \psi_j \right\rangle_{\Gamma_j} - \frac{1}{\lambda} \left\langle \nabla h_{0j}^*, \nabla \psi_j \right\rangle_{\Gamma_j} + \frac{1}{\lambda} \left\langle h_{0j}^*, \psi_j \right\rangle_{\Gamma_j} \right], \text{ for } \psi \in \mathcal{V}. \tag{3.21}$$

Moreover, multiplying the both sides of the wave equation in (3.18) by $\xi \in H^1(\Omega_s)$, and integrating by parts - while using the resolvent relations in (3.18) - the authors note that the solution component $w_1$ of (3.15) satisfies

$$\lambda \langle w_1, \xi \rangle_{\Omega_s} + \frac{1}{\lambda} \langle \nabla w_1, \nabla \xi \rangle_{\Omega_s} + \left\langle \frac{\partial}{\partial \nu} w_0, \xi \right\rangle_{\Gamma_s} = \langle w_1^*, \xi \rangle_{\Omega_s} - \frac{1}{\lambda} \langle \nabla w_0^*, \nabla \xi \rangle_{\Omega_s}, \text{ for } \xi \in H^1(\Omega_s) \tag{3.22}$$

Set now

$$\mathbf{W} \equiv \left\{ [\varphi, \psi_1, \ldots, \psi_K, \xi] \in H^1_{\Gamma_f}(\Omega_f) \times \mathcal{V} \times H^1(\Omega_s) \,\middle|\, \varphi|_{\Gamma_j} = \psi_j = \xi|_{\Gamma_j}, \text{ for } 1 \leq j \leq K \right\};$$
$$||[\varphi, \psi_1, \ldots, \psi_K, \xi]||^2_{\mathbf{W}} = ||\nabla \varphi||^2_{\Omega_f} + \sum_{j=1}^{K} \left[ ||\nabla \psi_j||^2_{\Gamma_j} + ||\psi_j||^2_{\Gamma_j} \right] + ||\nabla \xi||^2_{\Omega_f}. \tag{3.23}$$

With respect to this Hilbert space, upon adding (3.19), (3.21), and (3.22) if $\Phi = [u_0, h_{01}, h_{11}, \ldots, h_{0K}, h_{1K}, w_0, w_1] \in D(\mathbf{A})$ solves (3.15), then necessarily its solution components $[u_0, h_{11}, \ldots, h_{1K}, w_1] \in$

$\mathbf{W}$ satisfy for $[\varphi, \psi, \xi] \in \mathbf{H}$,

$$\lambda \langle u_0, \varphi \rangle_{\Omega_f} + \langle \nabla u_0, \nabla \varphi \rangle_{\Omega_f} + \lambda \langle w_1, \xi \rangle_{\Omega_s} + \frac{1}{\lambda} \langle \nabla w_1, \nabla \xi \rangle_{\Omega_s}$$

$$+ \sum_{j=1}^{K} [\lambda \langle h_1, \psi_j \rangle_{\Gamma_j} + \frac{1}{\lambda} \langle \nabla h_{1j}, \nabla \psi_j \rangle_{\Gamma_j} + \frac{1}{\lambda} \langle h_{1j}, \psi_j \rangle_{\Gamma_j}] = \mathbf{F}_\lambda \left( \begin{bmatrix} \varphi \\ \psi \\ \xi \end{bmatrix} \right); \tag{3.24}$$

where

$$\mathbf{F}_\lambda \left( \begin{bmatrix} \varphi \\ \psi \\ \xi \end{bmatrix} \right) = \langle u_0^*, \varphi \rangle_{\Omega_f} + \sum_{j=1}^{K} [\langle h_{1j}^*, \psi_j \rangle_{\Gamma_j} - \frac{1}{\lambda} \langle \nabla h_{0j}^*, \nabla \psi_j \rangle_{\Gamma_j}$$

$$- \frac{1}{\lambda} \langle h_{0j}^*, \psi_j \rangle_{\Gamma_j}] + \langle w_1^*, \xi \rangle_{\Omega_s} - \frac{1}{\lambda} \langle \nabla w_0^*, \nabla \xi \rangle_{\Omega_s} \tag{3.25}$$

In sum, in order to recover the solution $\Phi = [u_0, h_{01}, h_{11}, \ldots, h_{0K}, h_{1K}, w_0, w_1] \in D(\mathbf{A})$ to (3.15), one can straightaway apply the Lax-Milgram Theorem (see Theorem 58) to the operator $\mathbf{B} \in \mathscr{L}(\mathbf{W}, \mathbf{W}^*)$, given by

$$\left\langle \mathbf{B} \begin{bmatrix} \varphi \\ \psi_1 \\ \vdots \\ \psi_K \\ \xi \end{bmatrix}, \begin{bmatrix} \widetilde{\varphi} \\ \widetilde{\psi}_1 \\ \vdots \\ \widetilde{\psi}_K \\ \widetilde{\xi} \end{bmatrix} \right\rangle_{\mathbf{W}^* \times \mathbf{W}} = \lambda \langle \varphi, \widetilde{\varphi} \rangle_{\Omega_f} + \langle \nabla \varphi, \nabla \widetilde{\varphi} \rangle_{\Omega_f} + \lambda \left\langle \xi, \widetilde{\xi} \right\rangle_{\Omega_s} + \left\langle \nabla \xi, \nabla \widetilde{\xi} \right\rangle_{\Omega_s}$$

$$= \sum_{j=1}^{K} \left[ \lambda \left\langle \psi_j, \widetilde{\psi}_j \right\rangle_{\Gamma_j} + \frac{1}{\lambda} \left\langle \nabla \psi_j, \nabla \widetilde{\psi}_j \right\rangle_{\Gamma_j} + \frac{1}{\lambda} \left\langle \psi_j, \widetilde{\psi}_j \right\rangle_{\Gamma_j} \right]$$

It is clear that $\mathbf{B} \in \mathscr{L}(\mathbf{W}, \mathbf{W}^*)$ is $\mathbf{W}$-elliptic; so by the Lax-Milgram Theorem (see Theorem 58), the equation (3.24) has a unique solution

$$[u_0, h_{11}, \ldots, h_{1K}, w_1] \in \mathbf{W}. \tag{3.26}$$

Subsequently, set

$$\begin{cases} h_{0j} = \frac{h_{1j}+h_{0j}^*}{\lambda}, \text{ for } 1 \leq j \leq K, \\ w_0 = \frac{w_1+w_0^*}{\lambda}. \end{cases} \tag{3.27}$$

In particular, since the data $[u_0^*, h_{01}^*, h_{11}^*, \ldots, h_{0K}^*, h_{1K}^*, w_0^*, w_1^*] \in \mathbf{H}$, then the relations in (3.27) gives that

$$w_0|_{\Gamma_j} = h_{0j},\ 1 \leq j \leq K. \tag{3.28}$$

Further the authors show that the dependent variable $\Phi = [u_0, h_{01}, h_{11}, \ldots, h_{0K}, h_{1K}, w_0, w_1]$, given by the solution of (3.24) and (3.27), is an element of $D(\mathbf{A})$: If $[\varphi, 0, \ldots, 0, 0] \in \mathbf{W}$ in (3.24), where $\varphi \in \mathscr{D}(\Omega_f)$, then

$$\lambda \langle u_0, \varphi\rangle_{\Omega_f} - \langle \Delta u_0, \varphi\rangle_{\Omega_f} = \langle u_0^*, \varphi\rangle_{\Omega_f} \text{ for all } \varphi \in \mathscr{D}(\Omega_f)$$

whence

$$\lambda u_0 - \Delta u_0 = u_0^* \text{ in } L^2(\Omega_f). \tag{3.29}$$

Subsequently, the fact that $\{\Delta u_0, u_0\} \in L^2(\Omega_f) \times H^1(\Omega_F)$ gives

$$\left.\frac{\partial u_0}{\partial \nu}\right|_{\Gamma_s} \in H^{-1/2}(\Gamma_s) \tag{3.30}$$

In turn, using the relations in (3.27), if $[0, 0, \ldots, 0, \xi] \in \mathbf{W}$, where $\xi \in \mathscr{D}(\Omega_s)$, then upon integrating by parts,

$$\lambda \langle w_1, \xi\rangle_{\Omega_s} - \langle \Delta w_0, \xi\rangle_{\Omega_s} = \langle w_1^*, \xi\rangle_{\Omega_s} \text{ for all } \xi \in \mathscr{D}(\Omega_s),$$

and so

$$\lambda w_1 - \Delta w_0 = w_1^* \text{ in } L^2(\Omega_s) \tag{3.31}$$

which gives that $\{\Delta w_0, w_0\} \in L^2(\Omega_s) \times H^1(\Omega_s)$. A subsequent integration by parts yields that

$$\left.\frac{\partial w_0}{\partial \nu}\right|_{\Gamma_s} \in H^{-1/2}(\Gamma_s). \tag{3.32}$$

Moreover, let $\gamma_s^+ \in \mathscr{L}(H^{1/2}(\Gamma_s), H^1(\Omega_s))$ be the right continuous inverse for the Sobolev trace map $\gamma_s \in \mathscr{L}(H^1(\Omega_s), H^{1/2}(\Gamma_s))$; viz.,

$$\gamma_s(f) = f|_{\Gamma_s} \text{ for } f \in C^\infty(\overline{\Omega}_s).$$

Likewise, let $\gamma_f^+ \in \mathscr{L}(H^{1/2}(\Gamma_s), H^1_{\Gamma_f}(\Omega_f))$ denote the right inverse for the Sobolev trace map $\gamma_f \in \mathscr{L}(H^1_{\Gamma_f}(\Omega_f), H^{1/2}(\Gamma_s))$. Also, for given $\psi_j \in H^1_0(\Gamma_j)$, $1 \leq j \leq K$, let

$$(\psi_j)_{\text{ext}}(x) \equiv \begin{cases} \psi_j, \ x \in \Gamma_j \\ 0, \ x \in \Gamma_s \backslash \Gamma_j \end{cases}. \tag{3.33}$$

Then $(\psi_j)_{\text{ext}} \in H^{1/2}(\Gamma_s)$ for all $1 \leq j \leq K$. Specifying the test functions in (3.24), $[\varphi, \psi_1, \ldots, \psi_K, \xi] \in \mathbf{W}$: namely, $\psi_j \in H^1_0(\Gamma_j)$, $1 \leq j \leq K$, and

$$\varphi \equiv \gamma_f^+ \left[ \sum_{j=1}^K (\psi_j)_{\text{ext}} \right], \qquad \xi \equiv \gamma_s^+ \left[ \sum_{j=1}^K (\psi_j)_{\text{ext}} \right]. \tag{3.34}$$

Therewith, from (3.24)

$$\begin{aligned}
&\lambda \langle u_0, \varphi \rangle_{\Omega_f} + \langle \nabla u_0, \nabla \varphi \rangle_{\Omega_f} \\
&\quad + \sum_{j=1}^K \left[ \lambda \langle h_{1j}, \psi_j \rangle_{\Gamma_j} + \frac{1}{\lambda} \langle \nabla h_{1j}, \nabla \psi_j \rangle_{\Gamma_j} + \frac{1}{\lambda} \langle h_{1j}, \psi_j \rangle_{\Gamma_j} \right] \\
&\quad + \lambda \langle w_1, \xi \rangle_{\Omega_s} + \frac{1}{\lambda} \langle \nabla w_1, \nabla \xi \rangle_{\Omega_s} \\
&= \langle u_0^*, \varphi \rangle_{\Omega_f} + \sum_{j=1}^K \left[ \left\langle h_{1j}^*, \psi_j \right\rangle_{\Gamma_j} - \frac{1}{\lambda} \left\langle \nabla h_{0j}^*, \nabla \psi_j \right\rangle_{\Gamma_j} - \frac{1}{\lambda} \left\langle h_{0j}^*, \psi_j \right\rangle_{\Gamma_j} \right] \\
&\quad + \langle w_1^*, \xi \rangle_{\Omega_s} - \frac{1}{\lambda} \langle \nabla w_0^*, \nabla \xi \rangle_{\Omega_s}
\end{aligned}$$

Upon integrating by parts, and invoking the relations in (3.27), as well as (3.29)-(3.32),

$$\left\langle \frac{\partial u_0}{\partial \nu}, \varphi \right\rangle_{\Gamma_s} + \sum_{j=1}^{K} \left[ \lambda \left\langle h_{1j}, \psi_j \right\rangle_{\Gamma_j} - \frac{1}{\lambda} \left\langle \Delta h_{0j}, \psi_j \right\rangle_{\Gamma_j} + \frac{1}{\lambda} \left\langle h_{0j}, \psi_j \right\rangle_{\Gamma_j} \right] - \left\langle \frac{\partial w_0}{\partial \nu}, \xi \right\rangle_{\Gamma_s} = \sum_{j=1}^{K} \left\langle h_{1j}^*, \psi_j \right\rangle_{\Gamma_j}. \tag{3.35}$$

Since each test function component $\psi_j \in H_0^1(\Gamma_j)$ is arbitrary, from this relation and (3.33)-(3.34) that each $h_{0j}$ solves

$$\lambda h_{1j} - \Delta h_{0j} + h_{0j} - \frac{\partial w_0}{\partial \nu} + \frac{\partial u_0}{\partial \nu} = h_{1j}^* \text{ in } \Gamma_j, \ 1 \leq j \leq K. \tag{3.36}$$

In addition, from (3.36), (3.26), (3.30), and (3.32) that $\{\Delta h_{0j}, h_{0j}\} \in [H^1(\Gamma_j)]' \times H^1(\Gamma_j)$, for $1 \leq j \leq K$. Consequently, an integration by parts gives that

$$\frac{\partial h_{0j}}{\partial n_j} \in H^{-1/2}(\partial \Gamma_j), \text{ for } 1 \leq j \leq K. \tag{3.37}$$

Finally: Let given indices $j^*, l^*, 1 \leq j^*, l^* \leq K$, satisfy $\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*} \neq \emptyset$. Let $g$ be a given element in $H_0^{1/2+\varepsilon}(\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*})$. Then one has that $\widetilde{g}_{j^*} \in H^{1/2+\varepsilon}(\partial \Gamma_{j^*})$ and $\widetilde{g}_{l^*} \in H^{1/2+\varepsilon}(\partial \Gamma_{l^*})$, where

$$\widetilde{g}_{j^*}(x) \equiv \begin{cases} g(x), \ x \in \partial \Gamma_{j^*} \cap \partial \Gamma_{l^*} \\ 0, \ x \in \partial \Gamma_{j^*} \backslash (\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*}); \end{cases} \qquad \widetilde{g}_{l^*}(x) \equiv \begin{cases} g(x), \ x \in \partial \Gamma_{j^*} \cap \partial \Gamma_{l^*} \\ 0, \ x \in \partial \Gamma_{l^*} \backslash (\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*}); \end{cases}$$

(see e.g., Theorem 3.33, p. 95 of [37]). Subsequently, by the (limited) surjectivity of the Sobolev trace map on Lipschitz domains - see e.g., Theorem 3.38, p. 102 of [37] there exists $\psi_{j^*} \in H^{1+\varepsilon}(\Gamma_{j^*})$ and $\psi_{l^*} \in H^{1+\varepsilon}(\Gamma_{l^*})$ such that

$$\psi_{j^*}|_{\partial \Gamma_{j^*}} = \widetilde{g}_{j^*} \text{ and } \psi_{l^*}|_{\partial \Gamma_{l^*}} = \widetilde{g}_{l^*}. \tag{3.38}$$

In turn, by the Sobolev Embedding Theorem, define on $\overline{\Gamma}_s$ the function

$$\Upsilon(x) \equiv \begin{cases} \psi_{j^*}(x), \text{ for } x \in \overline{\Gamma}_{j^*} \\ \psi_{l^*}(x), \text{ for } x \in \overline{\Gamma}_{l^*} \\ 0, \text{ for } x \in \overline{\Gamma}_s \backslash (\overline{\Gamma}_{j^*} \cup \overline{\Gamma}_{l^*}), \end{cases} \tag{3.39}$$

then $\Upsilon(x) \in C(\overline{\Gamma}_s)$. Since also $\psi_{j^*} \in H^1(\Gamma_{j^*})$ and $\psi_{l^*} \in H^1(\Gamma_{l^*})$, the authors eventually deduce via an integration by parts that $\Upsilon \in H^1(\Gamma_s)$ (See e.g., the proof of Theorem 2, p. 36 of [22]). With this $H^1$-function in hand, and with aforesaid continuous right inverses $\gamma_s^+ \in \mathscr{L}(H^{1/2}(\Gamma_s), H^1(\Omega_s))$ and $\gamma_f^+ \in \mathscr{L}(H^{1/2}(\Gamma_s), H^1_{\Gamma_f}(\Omega_f))$, specify the vector

$$[\varphi, \psi, \xi] \equiv \left[\gamma_f^+(\Upsilon), 0, \ldots, \psi_{j^*}, 0, \ldots, \psi_{l^*}, \ldots, 0, \gamma_s^+(\Upsilon)\right] \in \mathbf{W}, \tag{3.40}$$

where again, space $\mathbf{W}$ is given in (3.23). With this vector in hand, consider the thin wave equation in (3.36): With respect to the two fixed indices $1 \leq j^*, l^* \leq K$, it follows via (3.36)

$$\begin{aligned} &\lambda \left\langle h_{1j^*}, \psi_{j^*} \right\rangle_{\Gamma_{j^*}} - \left\langle \Delta h_{0j^*}, \psi_{j^*} \right\rangle_{\Gamma_{j^*}} + \left\langle h_{0j^*}, \psi_{j^*} \right\rangle_{\Gamma_{j^*}} \\ &- \left\langle \frac{\partial w_0}{\partial \nu} - \frac{\partial u_0}{\partial v}, \psi_{j^*} \right\rangle_{\Gamma_{j^*}} + \lambda \left\langle h_{1l^*}, \psi_{l^*} \right\rangle_{\Gamma_{l^*}} - \left\langle \Delta h_{0l^*}, \psi_{l^*} \right\rangle_{\Gamma_{l^*}} \\ &+ \left\langle h_{0l^*}, \psi_{l^*} \right\rangle_{\Gamma_{l^*}} - \left\langle \frac{\partial w_0}{\partial \nu} - \frac{\partial u_0}{\partial v}, \psi_{l^*} \right\rangle_{\Gamma_{l^*}} = \left\langle h^*_{1j^*}, \psi_{j^*} \right\rangle_{\Gamma_{j^*}} + \left\langle h^*_{1l^*}, \psi_{l^*} \right\rangle_{\Gamma_{l^*}}. \end{aligned}$$

A subsequent integration by parts, with (3.40) in mind, subsequently yields

$$\begin{aligned} &\lambda \left\langle h_{1j^*}, \psi_{j^*} \right\rangle_{\Gamma_{j^*}} + \left\langle \nabla h_{0j^*}, \nabla \psi_{j^*} \right\rangle_{\Gamma_{j^*}} - \left\langle \frac{\partial h_{0j^*}}{\partial n_{j^*}}, g \right\rangle_{\partial\Gamma_{j^*} \cap \partial\Gamma_{l^*}} + \left\langle h_{0j^*}, \psi_{j^*} \right\rangle_{\Gamma_{j^*}} \\ &+ \lambda \left\langle h_{1l^*}, \psi_{l^*} \right\rangle_{\Gamma_{l^*}} + \left\langle \nabla h_{0l^*}, \nabla \psi_{l^*} \right\rangle_{\Gamma_{l^*}} - \left\langle \frac{\partial h_{0l^*}}{\partial n_{l^*}}, g \right\rangle_{\partial\Gamma_{j^*} \cap \partial\Gamma_{l^*}} + \left\langle h_{0l^*}, \psi_{l^*} \right\rangle_{\Gamma_{l^*}} \\ &+ \left\langle \nabla w_0, \nabla \xi \right\rangle_{\Omega_s} + \left\langle \Delta w_0, \xi \right\rangle_{\Omega_s} + \left\langle \nabla u_0, \nabla \varphi \right\rangle_{\Omega_f} + \left\langle \Delta u_0, \varphi \right\rangle_{\Omega_f} \\ &= \left\langle h^*_{1j^*}, \psi_{j^*} \right\rangle_{\Gamma_{j^*}} + \left\langle h^*_{1l^*}, \psi_{l^*} \right\rangle_{\Gamma_{l^*}} \end{aligned}$$

Invoking (3.29) and (3.31), it follows

$$
\begin{aligned}
& -\left\langle \frac{\partial h_{0j}^*}{\partial n_{j^*}}, g \right\rangle_{\partial\Gamma_{j^*}\cap\partial\Gamma_{l^*}} - \left\langle \frac{\partial h_{0l}^*}{\partial n_{l^*}}, g \right\rangle_{\partial\Gamma_{j^*}\cap\partial\Gamma_{l^*}} + \langle h_{0j^*}, \psi_{j^*}\rangle_{\Gamma_{j^*}} + \langle h_{0l^*}, \psi_{l^*}\rangle_{\Gamma_{l^*}} \\
& + \lambda \langle h_{1j^*}, \psi_{j^*}\rangle_{\Gamma_{j^*}} + \langle \nabla h_{0j^*}, \nabla \psi_{j^*}\rangle_{\Gamma_{j^*}} + \lambda \langle h_{1l^*}, \psi_{l^*}\rangle_{\Gamma_{l^*}} + \langle \nabla h_{0l^*}, \nabla \psi_{l^*}\rangle_{\Gamma_{l^*}} \\
& + \langle \nabla w_0, \nabla \xi\rangle_{\Omega_s} + \lambda \langle w_1, \xi\rangle_{\Omega_s} - \langle w_1^*, \xi\rangle_{\Omega_s} + \langle \nabla u_0, \nabla \varphi\rangle_{\Omega_f} + \lambda \langle u_0, \varphi\rangle_{\Omega_f} - \langle u_0^*, \varphi\rangle_{\Omega_f} \\
& = \langle h_{1j^*}^*, \psi_{j^*}\rangle_{\Gamma_{j^*}} + \langle h_{1l^*}^*, \psi_{l^*}\rangle_{\Gamma_{l^*}}
\end{aligned}
$$

Invoking the relations in (3.27) and the variational equation (3.24), which is satisfied by $[u_0, h_{11}, \ldots, h_{1K}, w_1]$ (where again vector $[\varphi, \psi, \xi]$ is given by (3.40)), it follows

$$
\left\langle \frac{\partial h_{0j^*}}{\partial n_{j^*}}, g \right\rangle_{\partial\Gamma_{j^*}\cap\partial\Gamma_{l^*}} = -\left\langle \frac{\partial h_{0l^*}}{\partial n_{l^*}}, g \right\rangle_{\partial\Gamma_{j^*}\cap\partial\Gamma_{l^*}}, \text{ for all } g \in H_0^{1/2+\varepsilon}(\partial\Gamma_{j^*}\cap\partial\Gamma_{l^*}).
$$

Since $H_0^{1/2+\varepsilon}(\partial\Gamma_{j^*}\cap\partial\Gamma_{l^*})$ is dense in $H^{1/2+\varepsilon}(\partial\Gamma_{j^*}\cap\partial\Gamma_{l^*})$, the authors then deduce that

$$
\frac{\partial h_{0j^*}}{\partial n_{j^*}} = -\frac{\partial h_{0l^*}}{\partial n_{l^*}}, \text{ for } \partial\Gamma_{j^*}\cap\partial\Gamma_{l^*} \neq \emptyset. \tag{3.41}
$$

Collecting (3.26)-(3.32) and (3.36), (3.37), and (3.41), the authors obtained the variable

$$
[u_0, h_{01}, h_{11}, \ldots, h_{0K}, h_{1K}, w_0, w_1] \in D(\mathbf{A}),
$$

and solves the resolvent equation (3.15). This concludes the proof of Theorem 1, upon application of the Lumer-Phillips Theorem (see Theorem 63).

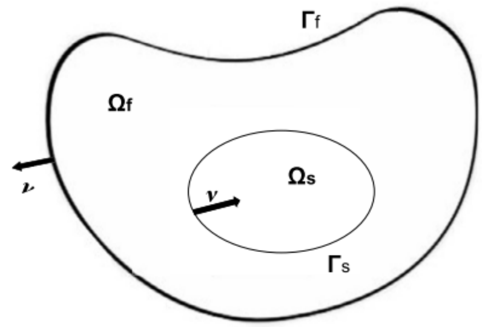


Figure 3.2: Smooth domain FSI model

## 3.4 Stokes Wave Lamé Multilayered FSI PDE Model

Now we start to introduce the second improved model. One of the main differences being the pressure term being added to first PDE in the system. It follows,

$$\begin{cases} u_t - \text{div } (\nabla u + \nabla^T u) + \nabla p = 0 & \text{in } (0,T) \times \Omega_f \\ \text{div } (u) = 0 & \text{in } (0,T) \times \Omega_f \\ u|_{\Gamma_f} = 0 & \text{on } (0,T) \times \Gamma_f; \end{cases} \tag{3.42}$$

$$\begin{cases} h_{tt} - \Delta_{\Gamma_s} h = [\nu \cdot \sigma(w)]|_{\Gamma_s} - [\nu \cdot (\nabla u + \nabla^T u)]|_{\Gamma_s} + p\nu \quad \text{on } (0,T) \times \Gamma_s, \end{cases} \tag{3.43}$$

$$\begin{cases} w_{tt} - \text{div } \sigma(w) + w = 0 & \text{on } (0,T) \times \Omega_s \\ w_t|_{\Gamma_s} = h_t = u|_{\Gamma_s} & \text{on } (0,T) \times \Gamma_s \end{cases} \tag{3.44}$$

$$[u(0), h(0), h_t(0), w(0), w_t(0)] = [u_0, h_0, h_1, w_0, w_1] \in \mathbf{H} \tag{3.45}$$

Here, $\Delta_{\Gamma_s}(\cdot)$ is the Laplace Beltrami operator, and the stress tensor $\sigma(\cdot)$ constitutes the Lamé system of elasticity on the "thick" layer. Namely, for function $v$ in $\Omega_s$,

$$\sigma(v) = 2\mu\epsilon(v) + \lambda[I_3 \cdot \epsilon(v)]I_3,$$

where strain tensor $\epsilon(\cdot)$ is given by

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right), \quad 1 \leq i,\ j \leq 3$$

Also, $\mathbf{H}$ is the finite energy space defined in (3.46) below.

**Remark 1** For the sake of numerical computation, the structure geometry $\Omega_s \subseteq \mathbb{R}^3$, can also be taken to be a convex polyhedral domain with polygonal boundary faces $\Gamma_j$, $1 \leq j \leq K$, where $\Gamma_i \cap \Gamma_j \neq \emptyset$ for $i \neq j$, and

$$\Gamma_s = \cup_{j=1}^{K} \overline{\Gamma_j}.$$

In this case the thin wave equation can be modeled for $j = 1, \ldots, K$ as

$$\begin{cases} \frac{\partial^2}{\partial t^2} h_j - \Delta h_j = [\nu \cdot \sigma(w)]|_{\Gamma_j} - [\nu\dot(\nabla u + \nabla^T u)]|_{\Gamma_j} + pv \text{ on } (0,T) \times \Gamma_j \\ h_j|_{\partial\Gamma_j \cap \partial\Gamma_l} = h_l|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } (0,T) \times (\partial\Gamma_j \cap \partial\Gamma_l), \text{ for all } 1 \leq l \leq K \\ \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset \\ \frac{\partial h_j}{\partial n_j}\Big|_{\partial\Gamma_j \cap \partial\Gamma_l} = -\frac{\partial h_l}{\partial n_l}\Big|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ on } (0,T) \times (\partial\Gamma_j \cap \partial\Gamma_l), \text{ for all } 1 \leq l \leq K \\ \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset \end{cases}$$

where the Laplace Beltrami Operator $\Delta_{\Gamma_s}(\cdot)$ in (3.43) is replaced with the standard Laplace operator with the imposition of additional continuity and boundary conditions in order to satisfy the surface differentiation [8, 10].

With respect to the PDE system given in (3.42) – (3.45), the finite energy Hilbert space $\mathbf{H}$ is given as

$$\mathbf{H} = \{[u_0, h_0, h_1, w_0, w_1] \in \mathbf{L}^2(\Omega_f) \times \mathbf{H}^1(\Gamma_s) \times \mathbf{L}^2(\Gamma_s) \times \mathbf{H}^1(\Omega_s) \times \mathbf{L}^2(\Omega_s) | \text{div } (u_0) = 0,$$
$$u_0 \cdot \nu|_{\Gamma_f} = 0, \text{ and } w_0|_{\Gamma_s} = h_0\} \quad (3.46)$$

with the inner product

$$\left\langle \Phi_0, \widetilde{\Phi}_0 \right\rangle_{\mathbf{H}} = \langle u_0, \widetilde{u}_0 \rangle_{\Omega_f} + \left\langle \nabla_{\Gamma_s}(h_0), \nabla_{\Gamma_s}(\widetilde{h}_0) \right\rangle_{\Gamma_s} + \left\langle h_1, \widetilde{h}_1 \right\rangle_{\Gamma_s} + \langle \sigma(w_0), \epsilon(\widetilde{w}_0) \rangle_{\Omega_s} + \langle w_0, \widetilde{w}_0 \rangle_{\Omega_s} + \langle w_1, \widetilde{w}_1 \rangle_{\Omega_s} \tag{3.47}$$

where

$$\Phi_0 = [u_0, h_0, h_1, w_0, w_1] \in \mathbf{H}; \widetilde{\Phi}_0 = [\widetilde{u}_0, \widetilde{h}_0, \widetilde{h}_1, \widetilde{w}_0, \widetilde{w}_1] \in \mathbf{H}. \tag{3.48}$$

## 3.5 Preliminaries

The PDE system given in (3.42) − (3.45) may be associated with an abstract ODE in Hilbert space $\mathbf{H}$; namely,

$$\begin{cases} \frac{d}{dt}\Phi(t) = \mathbf{A}\Phi(t) \\ \Phi(0) = \Phi_0 \end{cases} \tag{3.49}$$

where $\Phi(t) = [u(t), h(t), h_t(t), w(t), w_t(t)]$, and $\Phi_0 = [u_0, h_0, h_1, w_0, w_1]$. Here, the operator $\mathbf{A} : D(\mathbf{A}) \subseteq \mathbf{H} \to \mathbf{H}$ is defined by

$$\mathbf{A} = \begin{bmatrix} \text{div } (\nabla(\cdot) + \nabla^T(\cdot)) & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ -[\nu \cdot (\nabla(\cdot) + \nabla^T(\cdot))]|_{\Gamma_s} & \Delta_{\Gamma_s}(\cdot) & 0 & \nu \cdot \sigma(\cdot)|_{\Gamma_s} & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & \text{div } \sigma(\cdot) - I & 0 \end{bmatrix};$$

$$\begin{bmatrix} -\nabla\mathcal{P}_1(\cdot) & -\nabla\mathcal{P}_2(\cdot) & 0 & -\nabla\mathcal{P}_3(\cdot) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mathcal{P}_1(\cdot)\nu & \mathcal{P}_2(\cdot)\nu & 0 & \mathcal{P}_3(\cdot)\nu & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.50}$$

Here, the "pressure" operators $\mathcal{P}_i$ are as defined below. The domain $D(\mathbf{A})$ of the generator $\mathbf{A}$ is characterized as follows $[u_0, h_0, h_1, w_0, w_1] \in D(\mathbf{A}) \Leftrightarrow$

($\mathbf{A.i}$) $u_0 \in \mathbf{H}^1(\Omega_f),\ h_1 \in \mathbf{H}^1(\Gamma_s)\, w_1 \in \mathbf{H}^1(\Omega_s)$;

($\mathbf{A.ii}$) There exists an associated $L^2(\Omega_f)$-function $p_0 = p_0(u_0, h_0, w_0)$ such that

$$[\text{div } (\nabla u_0 + \nabla^T u_0) - \nabla p_0] \in L^2(\Omega_f)$$

Consequently, $p_0$ is harmonic and so one has the boundary traces

**(a)** $\left[p_0|_{\Gamma_f}, \frac{\partial p_0}{\partial \nu}\big|_{\Gamma_f}\right] \in H^{-1/2}(\Gamma_f) \times H^{-3/2}(\Gamma_f)$;

**(b)** $(\nabla u_0 + \nabla^T u_0) \cdot \nu \in H^{-3/2}(\Gamma_f)$,

($\mathbf{A.iii}$) div $\sigma(w_0) \in L^2(\Omega_s)$; consequently, $\nu \cdot \sigma \in H^{-1/2}(\Gamma_s)$,

($\mathbf{A.iv}$) $\Delta_{\Gamma_s}(h_0) + [\nu \cdot \sigma(w_0)]_{\Gamma_s} - [(\nabla u_0 + \nabla^T u_0) \cdot \nu]|_{\Gamma_s} + [p_0 \nu]|_{\Gamma_s} \in L^2(\Gamma_s)$,

($\mathbf{A.v}$) $u_0|_{\Gamma_f} = 0,\ u_0|_{\Gamma_s} = h_1 = w_1|_{\Gamma_s}$

Moreover, the PDE system satisfies the following energy relation:

$$\mathcal{E}(T) - \int_0^T \int_{\Omega_f} \left|\left|\nabla u + \nabla^T u\right|\right|^2 = \mathcal{E}(0),$$

where

$$\mathcal{E}(T) = \left|\left|u\right|\right|^2 + \left|\left|\nabla_{\Gamma_s} h\right|\right|^2 + \left|\left|h_t\right|\right|^2 + \left|\left|w\right|\right|^2 + \langle \sigma(w), \epsilon(w) \rangle_{\Omega_s} + \left|\left|w_t\right|\right|^2.$$

It is important to note that the elimination of the pressure variable is very crucial in order to formulate the PDE system as an ODE problem. For this, the author basically applies the divergence operator to the Stokes equation and use the fact that $u$ is solenoidal. This gives that the (pointwise) pressure variable $p(t)$ is harmonic; i.e.,

$$\Delta p(t) = 0 \text{ in } \Omega_f. \tag{3.51}$$

Subsequently, by multiplying by $\nu|_{\Gamma_s}$ and use the matching velocity condition to obtain the following boundary condition for the pressure variable $p$:

$$p + \frac{\partial p}{\partial \nu} = \text{div }(\nabla(u) + \nabla^T(u)) \cdot \nu|_{\Gamma_s} + [(\nabla u + \nabla^T u) \cdot \nu - \Delta_{\Gamma_s}(h) - \nu \cdot \sigma(w)|_{\Gamma_s}] \cdot \nu|_{\Gamma_s} \tag{3.52}$$

Also, since $u$ is divergence free, by taking the inner product of both sides the $u$ equation, with an extension of the normal vector, and subsequently take the trace of this relation on $\Gamma_f$,

$$\frac{\partial p}{\partial \nu} = [\text{div }(\nabla u + \nabla^T u)] \cdot \nu \text{ on } \Gamma_f.$$

Accordingly, the pressure variable $p(t)$, as the solution to the last two equations, can formally be written pointwise in time as

$$p(t) = \mathcal{P}_1(u(t)) + \mathcal{P}_2(h(t)) + \mathcal{P}_3(w(t))$$

where the harmonic functions $\mathcal{P}_1(u(t))$, $\mathcal{P}_2(h(t))$, and $\mathcal{P}_3(w(t))$ solve the following elliptic BVPs:

$$\begin{cases} \Delta\mathcal{P}_1(u) = 0 & \text{in } \Omega_f, \\ \mathcal{P}_1(u) = \text{div }(\nabla(u) + \nabla^T(u)) \cdot \nu|_{\Gamma_s} + ([(\nabla u + \nabla^T u)] \cdot \nu) \cdot \nu|_{\Gamma_s} & \text{on } \Gamma_s, \\ \frac{\partial \mathcal{P}_1(u)}{\partial \nu} = \text{div }(\nabla(u) + \nabla^T(u)) \cdot \nu|_{\Gamma_f} & \text{on } \Gamma_f, \end{cases} \tag{3.53}$$

$$\begin{cases} \Delta\mathcal{P}_2(h) = 0 & \text{in } \Omega_f, \\ \mathcal{P}_2(h) = -\Delta_{\Gamma_s}(h) \cdot \nu|_{\Gamma_s} & \text{on } \Gamma_s, \\ \frac{\partial \mathcal{P}_2(h)}{\partial \nu} = 0 & \text{on } \Gamma_f, \end{cases} \tag{3.54}$$

and

$$\begin{cases} \Delta\mathcal{P}_3(w) = 0 & \text{in } \Omega_f, \\ \mathcal{P}_3(w) = -[\nu \cdot \sigma(w)|_{\Gamma_s}] \cdot \nu|_{\Gamma_s} & \text{on } \Gamma_s, \\ \frac{\partial \mathcal{P}_3(w)}{\partial \nu} = 0 & \text{on } \Gamma_f. \end{cases} \tag{3.55}$$

The construction of these $\mathcal{P}_i$ functions, defined as the solutions to above harmonic equations, allows for the elimination of the pressure term in the original system. As such, the pressure-free system can indeed be associated with the abstract ODE in Hilbert space $\mathbf{H}$, and the associated pressure function $p_0$ in **(A.ii)** can be identified explicitly, via

$$p_0 = \mathcal{P}_1(u_0) + \mathcal{P}_2(h_0) + \mathcal{P}_3(w_0). \tag{3.56}$$

## 3.6 Main Result: Existence-Uniqueness of Solution

The main result of this section is to show that the system (3.42) – (3.45) or equivalently the abstract ODE system (3.49) may be associated with a $C_0$-semigroup $\{e^{\mathbf{A}t}\}_{t\geq 0}$, where $\mathbf{A} : D(\mathbf{A}) \subseteq \mathbf{H} \to \mathbf{H}$ is the matrix operator defined in (3.50). To this end, the author constructs a mixed variational formulation which is necessarily predicated on the "thick" and "thin" structural PDE components. This is quite different than the inf-sup formulations which have been derived for uncoupled Stokes flow; see e.g., [16]. There is no choice in the matter: the proper mixed variational formulation must be driven here by the structural PDE components, although the associated bilinear form, $a(\cdot,\cdot)$, written in Theorem 59 below - necessarily takes into account the presence in (3.42) – (3.45) of Stokes flow. Ultimately, the author will arrive at an inf-sup system of the classical form (6.1), for which they will apply the theorem.

This inf-sup result will be invoked below to recover the "thick" and "thin" structural variables $[h_0, h_1, w_0, w_1]$ of the solution of the abstract resolvent Eq (3.58) below, which is formally a frequency domain version of the time dependent system (3.42) – (3.45). Subsequently, the author will reconstruct (from $[h_1, w_1]$) the fluid and pressure variables $\{u_0, p_0\}$ of the solution to (3.58) and moreover show that this fluid-structure interaction solution is in $D(\mathbf{A})$, where $\mathbf{A}$ is the matrix generator defined in (3.50). In this "post-processing" work, the following Lemma will be required, the proof of which closely follows from that of Proposition 2 of [2].

**Lemma 2** (Elliptic Regularity)**.** *Given a vector-valued function $\mu \in [H^1(\Omega_f)]^d \cap Null(div)$, suppose there exists a scalar-valued function $\rho \in L^2(\Omega_f)$ which satisfies*

$$-\nabla \cdot (\nabla\mu + \nabla\mu^T) + \nabla\rho \in Null(div),$$

*where the subspace Null(div) is defined as*

$$Null(div) = \left\{ f \in [L^2(\Omega_f)]^d \,\middle|\, div\ (f) = 0\ \ in\ \Omega_f \right\}.$$

*Then, $\rho$ is harmonic ($\Delta\rho = 0$ in $\Omega_f$), and one has the following additional boundary regularity for the pair $(\mu, \rho)$:*

$$\rho|_{\partial\Omega_f} \in H^{-1/2}(\partial\Omega_f),\ \left.\frac{\partial\rho}{\partial\nu}\right|_{\partial\Omega_f} \in H^{-3/2}(\partial\Omega_f);$$

$$(\nabla\mu + \nabla\mu^T)\cdot\nu|_{\partial\Omega_f} \in [H^{-1/2}(\partial\Omega_f)]^d$$

$$[\nabla\cdot(\nabla\mu + \nabla\mu^T)]\cdot\nu|_{\partial\Omega_f} \in H^{-3/2}(\partial\Omega_f)$$

We now give cover the main result of the this section:

**Theorem 3.** *With reference to the problem* (3.42) − (3.45)*, the operator $\boldsymbol{A} : D(\boldsymbol{A}) \subseteq \boldsymbol{H} \to \boldsymbol{H}$, defined in* (3.50)*, generates a $C_0$-semigroup of contractions on $\boldsymbol{H}$. Consequently, the solution $\Phi(t) = [u(t), h(t), h_t(t), w(t), w_t(t)]$ of* (3.42) − (3.45)*, or equivalently* (3.49)*, is given by*

$$\Phi(t) = e^{\boldsymbol{A}t}\Phi_0 \in C([0,T]; \boldsymbol{H}),$$

*where $\Phi_0 = [u_0, h_0, h_1, w_0, w_1] \in \boldsymbol{H}$.*

### 3.6.1 Dissipativity

Given the matrix generator $\mathbf{A}$, $\Phi_0 = [u_0, h_0, h_1, w_0, w_1] \in D(\mathbf{A})$, and $p_0$ as in (3.56),

$$\begin{aligned}
\langle \mathbf{A}\Phi, \Phi\rangle_{\mathbf{H}} = & \left\langle \text{div}\ (\nabla(u_0) + \nabla^T(u_0)), u_0 \right\rangle_{\Omega_f} + \langle \nabla_{\Gamma_s}(h_1), \nabla_{\Gamma_s}(h_0)\rangle_{\Gamma_s} \\
& + \left\langle -(\nabla u_0 + \nabla^T u_0)\cdot\nu|_{\Gamma_s}, h_1 \right\rangle_{\Gamma_s} + \langle \Delta_{\Gamma_s}(h_0), h_1\rangle_{\Gamma_s} + \langle \sigma(w_0)\cdot\nu|_{\Gamma_s}, h_1\rangle_{\Gamma_s} \\
& + \langle \sigma(w_1), \epsilon(w_0)\rangle_{\Omega_s} + \langle w_1, w_0\rangle_{\Omega_s} + \langle \text{div}\ \ \sigma(w_0), w_1\rangle_{\Omega_s} - \langle w_0, w_1\rangle_{\Omega_s} \\
& - \langle [\nabla\mathcal{P}_1(u_0) + \nabla\mathcal{P}_2(h_0) + \nabla\mathcal{P}_3(w_0)], u_0\rangle_{\Omega_f} \\
& + \langle [\mathcal{P}_1(u_0)\cdot\nu + \mathcal{P}_2(h_0)\cdot\nu + \mathcal{P}_3(w_0)\cdot\nu], h_1\rangle_{\Gamma_s}.
\end{aligned}$$

Applying Green's Theorem, using the fact that $u_0$ is solenoidal, and $u_0 = 0$ on $\Gamma_f$,

$$\begin{aligned}
\langle \mathbf{A}\Phi, \Phi \rangle_{\mathbf{H}} =& \left\langle (\nabla(u_0) + \nabla^T(u_0)) \cdot \nu, u_0 \right\rangle_{\Gamma_s} - \frac{1}{2} \left| \left| \nabla(u_0) + \nabla^T(u_0) \right| \right|^2 \\
&+ \langle \nabla_{\Gamma_s}(h_1), \nabla_{\Gamma_s}(h_0) \rangle_{\Gamma_s} - \left\langle (\nabla(u_0) + \nabla^T(u_0))\nu, h_1 \right\rangle_{\Gamma_s} \\
&- \langle \nabla_{\Gamma_s}(h_0), \nabla_{\Gamma_s}(h_1) \rangle_{\Gamma_s} + \langle \sigma(w_0) \cdot \nu|_{\Gamma_s}, h_1 \rangle_{\Gamma_s} \\
&+ \langle \sigma(w_1), \epsilon(w_0) \rangle_{\Omega_s} + \langle w_1, w_0 \rangle_{\Omega_s} - \langle \sigma(w_0) \cdot \nu|_{\Gamma_s}, w_1 \rangle_{\Gamma_s} \\
&- \langle \sigma(w_0), \epsilon(w_1) \rangle_{\Omega_s} - \langle w_0, w_1 \rangle_{\Omega_s} .
\end{aligned}$$

Now, using the matching velocity condition given in (**A.v**) it follows that

$$\begin{aligned}
\langle \mathbf{A}\Phi, \Phi \rangle_{\mathbf{H}} =& -\frac{1}{2} \left| \left| \nabla(u_0) + \nabla^T(u_0) \right| \right|^2 + 2i\mathrm{Im}\big\{ \langle \nabla_{\Gamma_s}(h_1), \nabla_{\Gamma_s}(h_0) \rangle_{\Gamma_s} \\
&+ \langle \sigma(w_1), \epsilon(w_0) \rangle_{\Omega_s} \\
&+ \langle w_1, w_0 \rangle_{\Omega_s} + \langle \sigma(w_1), \epsilon(w_0) \rangle_{\Omega_s} + \langle w_1, w_0 \rangle_{\Omega_s} \big\}
\end{aligned}$$

and hence

$$\mathrm{Re} \, \langle \mathbf{A}\Phi, \Phi \rangle_{\mathbf{H}} = -\frac{1}{2} \left| \left| \nabla(u_0) + \nabla^T(u_0) \right| \right|^2 \leq 0,$$

which gives the dissipativity of the operator $\mathbf{A}$.

### 3.6.2 Maximality

This step is the challenging part of the proof. In order to prove the maximality condition, it is shown that the operator $(\lambda I - \mathbf{A})$ is surjective for $\lambda > 0$. That is, it is established that the range condition

$$\mathrm{Range}(\lambda I - \mathbf{A}) = \mathbf{H}, \tag{3.57}$$

where parameter $\lambda > 0$. Let $\Phi^* = [u_0^*, h_0^*, h_1^*, w_0^*, w_1^*] \in \mathbf{H}$, and consider the problem of finding $\Phi = [u_0, h_0, h_1, w_0, w_1] \in D(\mathbf{A})$ which solves

$$(\lambda I - \mathbf{A})\Phi = \Phi^*, \; \lambda > 0 \tag{3.58}$$

In PDE terms, this resolvent equation will generate the following relations, where again $p_0$ is given via (3.56):

$$\begin{cases} \lambda u_0 - \text{div } (\nabla u_0 + \nabla^T u_0) + \nabla p_0 = u_0^* \text{ in } \Omega_f \\ \text{div } (u_0) = 0 \text{ in } \Omega_f \\ u_0|_{\Gamma_f} = 0 \text{ on } \Gamma_f; \end{cases} \tag{3.59}$$

$$\begin{cases} \lambda h_0 - h_1 = h_0^* \text{ in } \Gamma_s \\ \lambda h_1 + [\nu \cdot (\nabla u_0 + \nabla^T u_0)]|_{\Gamma_s} - \Delta_{\Gamma_s}(h_0) - [\nu \cdot \sigma(w_0)]|_{\Gamma_s} - p_0\nu = h_1^* \text{ in } \Gamma_s \end{cases} \tag{3.60}$$

$$\begin{cases} \lambda w_0 - w_1 = w_0^* \text{ in } \Omega_s \\ \lambda w_1 - \text{div } \sigma(w_0) + w_0 = w_1^* \text{ in } \Omega_s \\ w_1|_{\Gamma_s} = h_1 = u_0|_{\Gamma_s} \text{ on } \Gamma_s \end{cases} \tag{3.61}$$

Because of the composite structure of the coupled dynamics, and the matching fluid and structure velocities given in $(3.63)_3$, the static problem $(3.59) - (3.61)$ cannot be solved via mixed variational approaches which are given for uncoupled fluid flows (see pg. 15 and pg. 107 of [42]). Hence, the proof is based on solving an appropriate and nonstandard mixed variational problem formulated for the static PDE system $(3.59) - (3.61)$. It is important to note that the mixed variational formulation is generated and solved with respect to the "thin" and "thick" structure variables $h_1$ and $w_1$. For this, the author mainly appeals to the Babuska-Brezzi approach (see Theorem 59). Once these variables are solved, then the recovery of the other structure solution variables $h_0$ and $w_0$ will be via the relations given in $(3.60)_1$ and $(3.61)_1$ for given data $h_0^* \in \mathbf{H}^1(\Gamma_s)$ and $w_0^* \in \mathbf{H}^1(\Omega_s)$, i.e.,

$$h_0 = \frac{1}{\lambda} h_1 + \frac{1}{\lambda} h_0^* \tag{3.62}$$

$$w_0 = \frac{1}{\lambda} w_1 + \frac{1}{\lambda} w_0^* \tag{3.63}$$

Because of the matching velocities boundary conditions in (3.61),

$$\int_{\Gamma_s} w_1|_{\Gamma_s} d\Gamma_s = \int_{\Gamma_s} h_1 d\Gamma_s = 0.$$

Before starting to generate a weak formulation for the "thin" and "thick" structure variables $h_1$ and $w_1$, the author first proceeds with the fluid component $u_0$: For given $g \in \mathbf{H}^{1/2}(\Gamma_s)$, the unique pair $[u_1(g), p_1(g)] \in \mathbf{H}^1(\Omega_f) \times \hat{L}^2(\Omega_f)$ solves the following static Stokes equation:

$$\begin{cases} \lambda u_1 - \text{div }(\nabla u_1 + \nabla^T u_1) + \nabla p_1 = 0 \text{ in } \Omega_f \\ \text{div }(u_1) = \frac{\int_{\Gamma_s}(g\cdot\nu)d\Gamma_s}{\text{meas}(\Omega_f)} \text{ in } \Omega_f \\ u_1|_{\Gamma_s} = g \text{ on } \Gamma_s \\ u_1|_{\Gamma_f} = 0 \text{ on } \Gamma_f, \end{cases} \tag{3.64}$$

See ([42], pg 22, Theorem 2.4). Here,

$$\hat{L}^2(\Omega_f) = \left\{ f \in L^2(\Omega_f) \,\middle|\, \int_{\Omega_f} f d\Omega_f = 0 \right\}$$

In particular, the compatibility condition holds for the solvability of the above problem, and so $p_1$ is unique up to a constant (see, e.g., [42], pg 31, Theorem 2.4). In a similar way, for a given force term $u_0^* \in \mathbf{L}^2(\Omega_f)$, the unique pair $[u_2(u_0^*), p_2(u_0^*)] \in \mathbf{H}^1(\Omega_f) \times \hat{L}^2(\Omega_f)$ solves the following problem:

$$\begin{cases} \lambda u_2 - \text{div }(\nabla u_2 + \nabla^T u_2) + \nabla p_2 = u_0^* \text{ in } \Omega_f \\ \text{div }(u_2) = 0 \text{ in } \Omega_f \\ u_2|_{\Gamma_f} \text{ on } \Gamma_f \end{cases} \tag{3.65}$$

With the solution maps of (3.64) – (3.65), the unique $\{u_0, p_0\}$ of (3.59) may then be expressed as

$$u_0 = u_1(w_1|_{\Gamma_s}) + u_2(u_0^*); \qquad p_0 = p_1(w_1|_{\Gamma_s}) + p_2(u_0^*) + c_0, \tag{3.66}$$

where $c_0$ is the (presently) unknown constant component of the pressure $p_0$ of (3.59).
Define the space

$$\mathbf{S} = \left\{ (\varphi, \psi) \in \mathbf{H}^1(\Gamma_s) \times \mathbf{H}^1(\Omega_s) \,\middle|\, \varphi = \psi|_{\Gamma_s} \right\}.$$

In order to generate a mixed variational formulation for the static "thin" and "thick" solution variables in (3.60) – (3.61), by respectively multiplying $(3.60)_2$ and $(3.61)_2$ by functions $\phi \in \mathbf{H}^1(\Gamma_s)$, and $\psi \in \mathbf{H}^1(\Omega_s)$ from the space $\mathbf{S}$, use Green's Theorem, and add the subsequent relations, (taking into account (3.62) – (3.63)) to get:

$$\begin{aligned}
&\lambda \langle h_1, \varphi\rangle_{\Gamma_s} + \frac{1}{\lambda}\langle \nabla_{\Gamma_s} h_1, \nabla_{\Gamma_s}\varphi\rangle_{\Gamma_s} + \lambda \langle w_1, \psi\rangle_{\Omega_s} \\
&\quad + \frac{1}{\lambda}\langle \sigma(w_1), \epsilon(\psi)\rangle_{\Omega_s} + \frac{1}{\lambda}\langle w_1, \psi\rangle_{\Omega_s} - c_0 \langle \nu, \varphi\rangle_{\Gamma_s} \\
&\quad + \left\langle \nu\cdot(\nabla u_0 + \nabla^T u_0)|_{\Gamma_s}, \varphi\right\rangle_{\Gamma_s} - \langle (p_1+p_2)\nu, \varphi\rangle_{\Gamma_s} \\
&= -\frac{1}{\lambda}\langle \nabla_{\Gamma_s} h_0^*, \nabla_{\Gamma_s}\varphi\rangle_{\Gamma_s} - \frac{1}{\lambda}\langle \sigma(w_0^*), \epsilon(\psi)\rangle_{\Omega_s} \\
&\quad + \langle h_1^*, \varphi\rangle_{\Gamma_s} + \langle w_1^*, \psi\rangle_{\Omega_s} - \frac{1}{\lambda}\langle w_0^*, \psi\rangle_{\Omega_s}. \qquad (3.67)
\end{aligned}$$

In order to estimate the sum in (3.67), recall that for any $\varphi \in \mathbf{H}^{1/2}(\Gamma_s)$, there is a unique pair $[\widetilde{u}(\varphi), \widetilde{p}(\varphi)] \in \mathbf{H}^1(\Omega_f) \times \hat{L}^2(\Omega_f)$ which solves the BVP

$$\begin{cases}
\lambda\widetilde{u} - \text{div }(\nabla\widetilde{u} + \nabla^T\widetilde{u}) + \nabla\widetilde{p} = 0 \text{ in } \Omega_f \\
\text{div } \widetilde{u} = \frac{\int_{\Gamma_s}(\varphi\cdot\nu)d\Gamma_s}{\text{meas}(\Omega_f)} \text{ in } \Omega_f \\
\widetilde{u}|_{\Gamma_s} = \varphi \text{ on } \Gamma_s \\
\widetilde{u}|_{\Gamma_f} = 0 \text{ on } \Gamma_f.
\end{cases} \qquad (3.68)$$

Appealing to problem (3.68), for the sum in (3.67),

$$\begin{aligned}
&\langle \nu\cdot(\nabla u_0 + \nabla^T u_0)|_{\Gamma_s}, \varphi\rangle_{\Gamma_s} - \langle (p_1+p_2)\nu, \varphi\rangle_{\Gamma_s} \\
&= \left\langle \nabla u_0 + \nabla^T u_0, \nabla\widetilde{u}(\varphi) + \nabla^T\widetilde{u}(\varphi)\right\rangle_{\Omega_f} + \left\langle \text{div }(\nabla u_0 + \nabla^T u_0), \widetilde{u}(\varphi)\right\rangle_{\Omega_f} \\
&\quad - \langle \nabla(p_1+p_2), \widetilde{u}(\varphi)\rangle_{\Omega_f} - \langle p_1+p_2, \text{div } \widetilde{u}(\varphi)\rangle_{\Omega_f} \qquad (3.69) \\
&= \left\langle \nabla u_0 + \nabla^T u_0, \nabla\widetilde{u}(\varphi) + \nabla^T\widetilde{u}(\varphi)\right\rangle_{\Omega_f} + \lambda\langle u_0, \widetilde{u}(\varphi)\rangle_{\Omega_f} \\
&\quad - \langle u_0^*, \widetilde{u}(\varphi)\rangle_{\Omega_f} - \langle p_1+p_2, \text{div } \widetilde{u}(\varphi)\rangle_{\Omega_f}, \qquad (3.70)
\end{aligned}$$

where the author has also used $(3.59)_1$. Then from the last relation and the fluid representation in (3.66),

$$\begin{aligned}
&\langle \nu \cdot (\nabla u_0 + \nabla^T u_0)|_{\Gamma_s}, \varphi \rangle_{\Gamma_s} - \langle (p_1 + p_2)\nu, \varphi \rangle_{\Gamma_s} \\
&\quad = \left\langle \nabla u_1(w_1|_{\Gamma_s}) + \nabla^T u_1(w_1|_{\Gamma_s}), \nabla \widetilde{u}(\varphi) + \nabla^T \widetilde{u}(\varphi) \right\rangle_{\Omega_f} \\
&\qquad + \left\langle \nabla u_2(u_0^*) + \nabla^T u_2(u_0^*), \nabla \widetilde{u}(\varphi) + \nabla^T \widetilde{u}(\varphi) \right\rangle_{\Omega_f} \\
&\qquad + \lambda \left\langle u_1(w_1|_{\Gamma_s}), \widetilde{u}(\varphi) \right\rangle_{\Omega_f} + \lambda \left\langle u_2(u_0^*), \widetilde{u}(\varphi) \right\rangle_{\Omega_f} \\
&\qquad \boxed{- \left\langle p_1(w_1|_{\Gamma_s}), \operatorname{div} \widetilde{u}(\varphi) \right\rangle_{\Omega_f} - \left\langle p_2(u_0^*), \operatorname{div} \widetilde{u}(\varphi) \right\rangle_{\Omega_f}} \\
&\qquad - \left\langle u_0^*, \widetilde{u}(\varphi) \right\rangle_{\Omega_f} . 
\end{aligned} \tag{3.71}$$

Since $p_1(w_1|_{\Gamma_s})$ and $p_2(u_0^*)$ are each in $\hat{L}^2(\Omega_f)$, then the boxed term of (3.71) disappears. Combining (3.67) – (3.71),

$$\begin{aligned}
&\lambda \langle h_1, \varphi \rangle_{\Gamma_s} + \frac{1}{\lambda} \left\langle \nabla_{\Gamma_s} h_1, \nabla_{\Gamma_s} \varphi \right\rangle_{\Gamma_s} + \lambda \left\langle w_1, \psi \right\rangle_{\Omega_s} + \frac{1}{\lambda} \left\langle \sigma(w_1), \epsilon(\psi) \right\rangle_{\Omega_s} \\
&\quad + \frac{1}{\lambda} \left\langle w_1, \psi \right\rangle_{\Omega_s} - c_0 \left\langle \nu, \varphi \right\rangle_{\Gamma_s} + \lambda \left\langle u_1(w_1|_{\Gamma_s}), \widetilde{u}(\varphi) \right\rangle_{\Omega_f} \\
&\quad + \left\langle \nabla u_1(w_1|_{\Gamma_s}) + \nabla^T u_1(w_1|_{\Gamma_s}), \nabla \widetilde{u}(\varphi) + \nabla^T \widetilde{u}(\varphi) \right\rangle_{\Omega_f} \\
&\quad = - \left\langle \nabla u_2(u_0^*) + \nabla^T u_2(u_0^*), \nabla \widetilde{u}(\varphi) + \nabla^T \widetilde{u}(\varphi) \right\rangle_{\Omega_f} \\
&\quad - \lambda \left\langle u_2(u_0^*), \widetilde{u}(\varphi) \right\rangle_{\Omega_f} + \left\langle u_0^*, \widetilde{u}(\varphi) \right\rangle_{\Omega_f} \\
&\quad - \frac{1}{\lambda} \left\langle \nabla_{\Gamma_s} h_0^*, \nabla_{\Gamma_s} \varphi \right\rangle_{\Gamma_s} - \frac{1}{\lambda} \left\langle \sigma(w_0^*), \epsilon(\psi) \right\rangle_{\Omega_s} \\
&\quad + \left\langle h_1^*, \varphi \right\rangle_{\Gamma_s} + \left\langle w_1^*, \psi \right\rangle_{\Omega_s} - \frac{1}{\lambda} \left\langle w_0^*, \psi \right\rangle_{\Omega_s} .
\end{aligned} \tag{3.72}$$

The last relation now gives the following mixed variational formulation in terms of the "thin" and "thick" structure variables $h_1$ and $w_1$: Namely,

$$\begin{aligned}
\mathbf{a}([h_1, w_1], [\varphi, \psi]) + \mathbf{b}([\varphi, \psi], c_0) &= \mathbf{F}([\varphi, \psi]), \text{ for all } [\varphi, \psi] \in \mathbf{S} \\
\mathbf{b}([h_1, w_1], r) &= 0, \text{ for all } r \in \mathbb{R}.
\end{aligned} \tag{3.73}$$

Here, the bilinear forms $\mathbf{a}(\cdot,\cdot) : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ and $\mathbf{b}(\cdot,\cdot) : \mathbf{S} \times \mathbb{R} \to \mathbb{R}$ are respectively given as

$$\begin{aligned}\mathbf{a}([\phi,\xi],[\widetilde{\phi},\widetilde{\xi}]) =& \lambda \left\langle \phi, \widetilde{\phi} \right\rangle_{\Gamma_s} + \frac{1}{\lambda} \left\langle \nabla_{\Gamma_s}\phi, \nabla_{\Gamma_s}\widetilde{\phi} \right\rangle_{\Gamma_s} \\ &+ \lambda \left\langle \xi, \widetilde{\xi} \right\rangle_{\Omega_s} + \frac{1}{\lambda} \left\langle \sigma(\xi), \epsilon(\widetilde{\xi}) \right\rangle_{\Omega_s} + \frac{1}{\lambda} \left\langle \xi, \widetilde{\xi} \right\rangle_{\Omega_s} \\ &+ \left\langle \nabla u_1(\xi|_{\Gamma_s}) + \nabla^T u_1(\xi|_{\Gamma_s}), \nabla \widetilde{u}(\widetilde{\phi}) + \nabla^T \widetilde{u}(\widetilde{\phi}) \right\rangle_{\Omega_f} \\ &+ \lambda \left\langle u_1(\xi|_{\Gamma_s}), \widetilde{u}(\widetilde{\phi}) \right\rangle_{\Omega_f},\end{aligned}$$

and

$$\mathbf{b}([\widetilde{\phi},\widetilde{\xi}], r) = -r \left\langle v, \widetilde{\phi} \right\rangle_{\Gamma_s},$$

and the functional $\mathbf{F}(\cdot)$ is defined as

$$\begin{aligned}\mathbf{F}([\widetilde{\phi},\widetilde{\xi}]) =& - \left\langle \nabla u_2(u_0^*) + \nabla^T u_2(u_0^*), \nabla \widetilde{u}(\widetilde{\phi}) + \nabla^T \widetilde{u}(\widetilde{\phi}) \right\rangle_{\Omega_f} \\ &- \frac{1}{\lambda} \left\langle \nabla_{\Gamma_s} h_0^*, \nabla_{\Gamma_s} \widetilde{\phi} \right\rangle_{\Gamma_s} - \frac{1}{\lambda} \left\langle \sigma(w_0^*), \epsilon(\widetilde{\xi}) \right\rangle_{\Omega_s} \\ &- \lambda \left\langle u_2(u_0^*), \widetilde{u}(\widetilde{\phi}) \right\rangle_{\Omega_f} + \left\langle u_0^*, \widetilde{u}(\widetilde{\phi}) \right\rangle_{\Omega_f} \\ &+ \left\langle h_1^*, \widetilde{\phi} \right\rangle_{\Gamma_s} + \left\langle w_1^*, \widetilde{\xi} \right\rangle_{\Omega_s} - \frac{1}{\lambda} \left\langle w_0^*, \widetilde{\xi} \right\rangle_{\Omega_s}.\end{aligned}$$

The remainder of the proof hinges on properly applying Theorem 59. It is clear that the bilinear forms $\mathbf{a}(\cdot,\cdot)$ and $\mathbf{b}(\cdot,\cdot)$ are continuous and moreover $\mathbf{a}(\cdot,\cdot)$ is S-elliptic. In order to conclude that the variational problem (3.73) has a unique solution, the author needed to show that the bilinear form $\mathbf{b}(\cdot,\cdot)$ satisfies the "inf-sup" condition given in Theorem 59. For this, consider the following problem:

Given $r \in \mathbb{R}$, let $z \in \mathbf{H}^1(\Gamma_s)$ satisfy

$$\Delta_{\Gamma_s} z = \operatorname{sgn}(r)\nu \text{ on } \Gamma_s$$

It is easily seen that $||\nabla_{\Gamma_s} z||_{\Gamma_s} \leq C||\nu||_{\Gamma_s}$. Now, taking into account that $\gamma : H^1(\Omega_s) \to H^{1/2}(\Gamma_s)$ is a surjective map, and so it has a continuous right inverse $\gamma^+(z)$,

$$\begin{aligned}
\sup_{[\eta,\varsigma]} \frac{\mathbf{b}([\eta,\varsigma],r)}{|||[\eta,\varsigma]|||_{\mathbf{S}}} &\geq \frac{\mathbf{b}([z,\gamma^+(z)],r)}{||z||_{\mathbf{H}^1(\Gamma_s)}} \\
&= \frac{-r\int_{\Gamma_s} \nu \cdot z d\Gamma_s}{||z||_{\mathbf{H}^1(\Gamma_s)}} \\
&= -r \ \mathrm{sgn}(r) \frac{\int_{\Gamma_s} \Delta_{\Gamma_s} z \cdot z d\Gamma_s}{||z||_{\mathbf{H}^1(\Gamma_s)}} \\
&= |r| \frac{\int_{\Gamma_s} |\nabla_{\Gamma_s} z|^2 d\Gamma_s}{||z||_{\mathbf{H}^1(\Gamma_s)}} \\
&= |r| \ ||z||_{\mathbf{H}^1(\Gamma_s)}
\end{aligned}$$

which yields that the inf-sup condition holds wth the constant $\beta = ||z||_{\mathbf{H}^1(\Gamma_s)}$. Consequently, the existence and uniqueness of the solution $[h_1, w_1] \in \mathbf{S}$ and $c_0 \in \mathbb{R}$ to the mixed variational problem (3.73) follows from Theorem 59.

Now, the unique attainment of the solution components $[h_1, w_1] \in \mathbf{S}$ allows the subsequent recovery of the solution variables $h_0$ and $w_0$ via the resolvent relations (3.62) – (3.63), with

$$h_0 = w_0|_{\Gamma_s},$$

since the data $\Phi^* = [u_0^*, h_0^*, h_1^*, w_0^*, w_1^*] \in \mathbf{H}$. Moreover, having the "thick" structure component

$$w_1 \in \mathbf{H}^1(\Omega_s) \text{ with } w_1|_{\Gamma_s} = h_1 \in \mathbf{H}^1(\Gamma_s),$$

and $u_0^* \in \mathbf{L}^2(\Omega_f)$, the fluid solution component $u_0$ and the pressure term $p_0$ can be given via the expressions in (3.66). That is, the unique pair $\{u_0, p_0\} \in \mathbf{H}^1(\Omega_f) \times L^2(\Omega_f)$ solves the Stokes system

$$\begin{cases}
\lambda u_0 - \mathrm{div}\ (\nabla u_0 + \nabla^T u_0) + \nabla p_0 = u_0^* \text{ in } \Omega_f \\
\mathrm{div}\ (u_0) = 0 \text{ in } \Omega_f \\
u_0|_{\Gamma_f} = 0 \text{ on } \Gamma_f \\
u_0|_{\Gamma_s} = w_1 \text{ on } \Gamma_s.
\end{cases} \tag{3.74}$$

To conclude the proof of maximality, it is shown that the derived solution $[u_0, h_0, h_1, w_0, w_1] \in D(\mathbf{A})$ and satisfies the resolvent Eqs. (3.59) – (3.61). First, by (3.66), it is clear that $\{u_0, p_0\} \in \mathbf{H}^1(\Omega_f) \times L^2(\Omega_f)$ where $u_0 = u_1 + u_2$; $p_0 = p_1 + p_2 + c_0$ satisfies (3.59). Consequently, by Lemma 2,

$$p_0|_{\Gamma_f} \in H^{-1/2}(\Gamma_f), \quad \left.\frac{\partial p_0}{\partial \nu}\right|_{\Gamma_f} \in H^{-3/2}(\Gamma_f),$$

and

$$(\nabla u_0 + \nabla u_0^T) \cdot \nu|_{\Gamma_f} \in [H^{-1/2}(\Gamma_f)]^d;\ [\nabla \cdot (\nabla u_0 + \nabla^T u_0)]\nu|_{\Gamma_f} \in H^{-3/2}(\Gamma_f).$$

Moreover, if in (3.73), $\phi = 0$, and $\psi \in [\mathcal{D}(\Omega_s)]^3$, then

$$\begin{aligned} &\lambda \langle w_1, \psi \rangle_{\Omega_s} + \frac{1}{\lambda} \langle \sigma(w_1), \epsilon(\psi) \rangle_{\Omega_s} + \frac{1}{\lambda} \langle w_1, \psi \rangle_{\Omega_s} \\ &= -\frac{1}{\lambda} \langle \sigma(w_0^*), \epsilon(\psi) \rangle_{\Omega_s} + \langle w_1^*, \psi \rangle_{\Omega_s} - \frac{1}{\lambda} \langle w_0^*, \psi \rangle_{\Omega_s}. \end{aligned}$$

Thus, via the resolvent relation $w_0 = \frac{1}{\lambda}[w_1 + w_0^*]$,

$$\lambda w_1 - \text{div}\ \sigma(w_0) + w_0 = w_1^* \text{ in } \mathbf{L}^2(\Omega_s), \tag{3.75}$$

which gives (3.61). Since also $w_0 \in \mathbf{H}^1(\Omega_s)$, an energy method yields in turn,

$$\nu \cdot \sigma(w_0) \in H^{-1/2}(\Gamma_s) \tag{3.76}$$

via (3.75). This gives, for all $[\varphi, \psi] \in \mathbf{S}$, (after using the resolvent relation $h_0 = \frac{1}{\lambda}(h_1 + h_0^*)$):

$$-\lambda \langle w_1, \psi \rangle_{\Omega_s} + \langle \text{div}\ \sigma(w_0), \psi \rangle_{\Omega_s} - \langle w_0, \psi \rangle_{\Omega_s} = - \langle w_1^*, \psi \rangle_{\Omega_s},$$

and

$$\begin{aligned}
&\lambda\langle h_1, *\,\varphi\rangle_{\Gamma_s} + \langle\nabla_{\Gamma_s} h_0, \nabla_{\Gamma_s}\varphi\rangle_{\Gamma_s} + \lambda\,\langle w_1, \psi\rangle_{\Omega_s} + \langle\sigma(w_0), \epsilon(\psi)\rangle_{\Omega_s} \\
&\quad + \langle w_0, \psi\rangle_{\Omega_s} - c_0\,\langle\nu, \varphi\rangle_{\Gamma_s} - \langle(p_1 + p_2)\nu, \varphi\rangle_{\Gamma_s} + \langle(p_1 + p_2)\nu, \varphi\rangle_{\Gamma_s} \\
&\quad + \left\langle\nabla u_0 + \nabla^T u_0, \nabla\widetilde{u}(\varphi) + \nabla^T\widetilde{u}(\varphi)\right\rangle_{\Omega_f} + \lambda\,\langle u_0, \widetilde{u}(\varphi)\rangle_{\Omega_f} \\
&= \langle h_1^*, \varphi\rangle_{\Gamma_s} + \langle w_1^*, \psi\rangle_{\Omega_s} + \langle u_0^*, \widetilde{u}(\varphi)\rangle_{\Omega_f}\,.
\end{aligned}$$

Integrating by parts (having in hand the boundary trace in (3.76)) and adding the two relations now gives

$$\begin{aligned}
&\lambda\langle h_1,\varphi\rangle_{\Gamma_s} + \langle\nabla_{\Gamma_s} h_0, \nabla_{\Gamma_s}\varphi\rangle_{\Gamma_s} - \langle\nu\cdot\sigma(w_0), \varphi\rangle_{\Gamma_s} \\
&\quad - \langle p\nu, \varphi\rangle_{\Gamma_s} + \langle\nabla p\nu, \varphi\rangle_{\Omega_f} + \lambda\,\langle u_0, \widetilde{u}(\varphi)\rangle_{\Omega_f} \\
&\quad - \left\langle \mathrm{div}\,[\nabla u_0 + \nabla^T u_0], \widetilde{u}(\varphi)\right\rangle_{\Omega_f} + \left\langle\nu\cdot[\nabla u_0 + \nabla^T u_0], \varphi\right\rangle_{\Omega_s} \\
&= \langle h_1^*, \varphi\rangle_{\Gamma_s} + \langle u_0^*, \widetilde{u}(\varphi)\rangle_{\Omega_f} \;\text{ for all } [\varphi, \psi] \in \mathbf{S}.
\end{aligned}$$

This finally gives the inference that

$$\begin{aligned}
&\lambda\langle h_1,\varphi\rangle_{\Gamma_s} + \langle\nabla_{\Gamma_s} h_0, \nabla_{\Gamma_s}\varphi\rangle_{\Gamma_s} - \langle\nu\cdot\sigma(w_0), \varphi\rangle_{\Gamma_s} \\
&\quad - \langle p\nu, \varphi\rangle + \left\langle\nu\cdot[\nabla u_0 + \nabla^T u_0], \varphi\right\rangle_{\Omega_s} \\
&= \langle h_1^*, \varphi\rangle_{\Gamma_s}\,, \text{ for all } \varphi \in [\mathcal{D}(\Omega_s)]^3
\end{aligned}$$

which gives the “thin” structural equation in (3.60), and the proof of Theorem 3 now finishes.

# Chapter 4

# Part II: Asymptotic Decay of the FSI Systems

## 4.1 Revisiting the Canonical FSI PDE Model

We turn our attention now back to the PDE described in (3.2) – (3.5) with the associated Hilbert space, inner product, and generator with domain ((3.6), (3.7), (3.10) and (3.11) respectively). In this framework the authors also showed strong stability for any initial data taken in the domain of the generator. This is summarized in the following theorem.

**Theorem 4.** *For the modeling generator $\boldsymbol{A} : D(\boldsymbol{A}) \subseteq \boldsymbol{H} \to \boldsymbol{H}$ of* (3.2)−(3.5)*, one has $\sigma(\boldsymbol{A}) \cap i\mathbb{R} = \emptyset$. Consequently, the $C_0$-semigroup $\left\{e^{\boldsymbol{A}t}\right\}_{t\geq 0}$ given in Theorem* 1 *is strongly stable. That is, the solution $\Phi(t)$ of the PDE* (3.2) − (3.5) *tends asymptotically to the zero state for all initial data $\Phi_0 \in \boldsymbol{H}$*

The proof of this theorem will be independent of the compactness or non-compactness of the resolvent of **A**. It will hinge on an ultimate appeal to the following well known result from the paper by W. Arendt and C. J. K. Batty (1988)[1]:

**Theorem 5.** *Let $T(t)_{t\geq 0}$ be a bounded $C_0$-semigroup on a reflexive Banach space $X$, with generator $\boldsymbol{A}$. Assume that $\sigma_p(\boldsymbol{A}) \cap i\mathbb{R} = \emptyset$, where $\sigma_p(\boldsymbol{A})$ is the point spectrum of $\boldsymbol{A}$. If $\sigma(\boldsymbol{A}) \cap i\mathbb{R}$ is countable then $T(t)_{t\geq 0}$ is strongly stable.*

The proof of Theorem 4 entails the elimination of all three parts of the spectrum of the generator

$\mathbf{A}$ from the imaginary axis. For this, we will go through the necessary analysis done by the authors on the spectrum in the following section.

## 4.2 Main Result: Asymptotic Decay of the Solution

### 4.2.1 A is Boundedly Invertible

In order to satisfy the conditions of Theorem 5, the authors will prove that $\sigma(\mathbf{A}) \cap i\mathbb{R} = \emptyset$ which is equivalent to show that

$$i\mathbb{R} \subseteq \rho(\mathbf{A})$$

To do this, start with the following proposition:

**Proposition 6.** *With generator $\boldsymbol{A} : D(\boldsymbol{A}) \subseteq \boldsymbol{H} \to \boldsymbol{H}$ given in* (3.10) − (3.11)*, the point $0 \in \rho(\boldsymbol{A})$. That is, $\boldsymbol{A}$ is boundedly invertible.*

*Proof.* Given $\Phi^* = [u_0^*, h_{01}^*, h_{11}^*, \ldots, h_{0K}^*, h_{1K}^*, w_0^*, w_1^*] \in \mathbf{H}$, the authors take up the task of finding $\Phi = [u_0, h_{01}, h_{11}, \ldots, h_{0K}, h_{1K}, w_0, w_1] \in D(\mathbf{A})$ which solves

$$\mathbf{A}\Phi = \Phi^* \tag{4.1}$$

or,

$$\begin{bmatrix} \Delta u_0 \\ h_{11} \\ -\frac{\partial u_0}{\partial \nu}\big|_{\Gamma_1} + (\Delta - I)h_{01} + \frac{\partial w_0}{\partial \nu}\big|_{\Gamma_1} \\ \vdots \\ h_{1K} \\ -\frac{\partial u_0}{\partial \nu}\big|_{\Gamma_K} + (\Delta - I)h_{0K} + \frac{\partial w_0}{\partial \nu}\big|_{\Gamma_K} \\ w_1 \\ \Delta w_0 \end{bmatrix} = \begin{bmatrix} u_0^* \\ h_{01}^* \\ h_{11}^* \\ \vdots \\ h_{0K}^* \\ h_{1K}^* \\ w_0^* \\ w_1^* \end{bmatrix} \tag{4.2}$$

From the thin and thick wave component of this equation note that

$$w_1 = w_0^* \in H^1(\Omega_s) \tag{4.3}$$

$$h_{1j} = h_{0j}^* \in H^1(\Gamma_j), \text{ for } 1 \leq j \leq K \tag{4.4}$$

Moreover, from the heat and thick wave components of (4.2), and the domain criterion ($\mathbf{A}.iii$), the solution component $u_0$ should satisfy the following BVP:

$$\begin{cases} \Delta u_0 = u_0^* \\ u_0|_{\Gamma_f} = 0 \\ u_0|_{\Gamma_s} = w_0^*|_{\Gamma_s} \end{cases} \tag{4.5}$$

Solving this BVP, and estimating its solution, in part by the Sobolev Trace Theorem, it follows

$$||u_0||_{H^1_{\Gamma_f}(\Omega_f)} + ||\Delta u_0||_{\Omega_f} \leq C\left[||u_0^*||_{\Omega_f} + ||w_0^*||_{H^1(\Omega_s)}\right]. \tag{4.6}$$

In turn, the use of this estimate in an integration by parts gives

$$\left|\left|\frac{\partial u_0}{\partial \nu}\right|\right|_{H^{-1/2}(\Gamma_f)} \leq C\left[||u_0^*||_{\Omega_f} + ||w_0^*||_{H^1(\Omega_s)}\right]. \tag{4.7}$$

In addition, with the space $\mathcal{V}$ as in (3.20), set

$$\chi \equiv \left\{[\psi, \xi] \in \mathcal{V} \times H^1(\Omega_s) \;\middle|\; \psi_j = \xi|_{\Gamma_j} \text{ for } 1 \leq j \leq K\right\}. \tag{4.8}$$

with this space in hand, and with the thin-wave and thick-wave components of equation (4.2) in mind, consider the variational relation

$$\begin{aligned} &\langle \nabla w_0, \nabla \xi\rangle_{\Omega_s} + \sum_{j=1}^{K}[\langle \nabla h_{0j}, \nabla \psi_j\rangle_{\Gamma_j} + \langle h_{0j}, \psi_j\rangle_{\Gamma_j}] \\ &= - \langle w_1^*, \xi\rangle_{\Omega_s} - \sum_{j=1}^{K}\left[\left\langle h_{1j}^*, \psi_j\right\rangle_{\Gamma_j} + \left\langle \frac{\partial u_0}{\partial \nu}, \psi_j\right\rangle_{\Gamma_j}\right], \end{aligned} \tag{4.9}$$

for every $[\psi, \xi] \in \chi$ where the term $\frac{\partial u_0}{\partial \nu}\big|_{\Gamma_s}$ is from (4.7). Since the bilinear form $b(\cdot, \cdot) : \chi \to \mathbb{R}$, given by

$$b\left([\psi,\xi],\left[\widetilde{\psi},\widetilde{\xi}\right]\right)=\left\langle\nabla\xi,\nabla\widetilde{\xi}\right\rangle_{\Omega_s}+\sum_{j=1}^{K}\left[\left\langle\nabla\psi_j,\nabla\widetilde{\psi}_j\right\rangle_{\Gamma_j}+\left\langle\psi_j,\widetilde{\psi}_j\right\rangle_{\Gamma_j}\right] \tag{4.10}$$

for every $[\psi,\xi]$, $\left[\widetilde{\psi},\widetilde{\xi}\right]\in\chi$, is continuous and $\chi$-elliptic, then by Lax-Milgram (see Theorem 58), there exists a unique solution

$$\varphi=[(h_{01},h_{02},\ldots,h_{0K}),w_0]\in\chi \tag{4.11}$$

to the variational relation (4.9). To show that the obtained $[u_0,[h_{01},h_{11},\ldots,h_{0K},h_{1K}],w_0,w_1]\in\mathbf{H}$ is in $D(\mathbf{A})$ and satisfies the equation (4.2):

Proceeding very much as in the proof of Theorem 1, take in (4.9)

$$[\psi,\xi]=[[0,0,\ldots,0],\varphi],$$

where $\varphi\in\mathscr{D}(\Omega_s)$. This gives

$$\langle\nabla w_0,\nabla\xi\rangle_{\Omega_s}=-\langle w_1^*,\xi\rangle_{\Omega_s},$$

whence the authors obtain

$$-\Delta w_0=-w_1^* \text{ in } \Omega_s \tag{4.12}$$

with

$$\begin{aligned}||\Delta w_0||_{\Omega_s}+\left|\left|\frac{\partial w_0}{\partial\nu}\right|\right|_{H^{-1/2}(\Gamma_s)}&\le C\left[||w_1^*||_{\Omega_s}+||w_0||_{H^1(\Omega_s)}\right]\\&\le C\,||[u_0^*,[h_{01}^*,h_{11}^*,\ldots,h_{0K}^*,h_{1K}^*],w_0^*,w_1^*]||_{\mathbf{H}}\end{aligned} \tag{4.13}$$

after using (4.11). In turn, using aforesaid right continuous inverse $\gamma_s^+\in\mathscr{L}(H^{1/2}(\Gamma_s),H^1(\Omega_s))$, let in (4.9), be the test functions

$$[\psi, \xi] = \left[ [(\psi_1)_{\text{ext}}, \dots, (\psi_K)_{\text{ext}}], \gamma_s^+ \left( \sum_{j=1}^{K} (\psi_j)_{\text{ext}} \right) \right] \in \chi,$$

where each $\psi_j \in H_0^1(\Gamma_j)$ for $1 \leq j \leq K$ and each $(\psi_j)_{\text{ext}}$ is as in (3.33). Applying this function to (4.9), integrating by parts and invoking (4.12),

$$\begin{aligned} &- \langle \Delta w_0, \xi \rangle_{\Omega_s} - \left\langle \frac{\partial w_0}{\partial \nu}, \xi|_{\Gamma_s} \right\rangle_{\Gamma_s} + \sum_{j=1}^{K} \left[ \langle \nabla h_{0j}, \nabla \psi_j \rangle_{\Gamma_j} + \langle h_{0j}, \psi_j \rangle_{\Gamma_j} \right] \\ &= - \sum_{j=1}^{K} \left[ \left\langle \frac{\partial u_0}{\partial \nu}, \psi_j \right\rangle_{\Gamma_j} + \langle h_{1j}^*, \psi_j \rangle_{\Gamma_j} \right] - \langle w_1^*, \xi \rangle_{\Omega_s}. \end{aligned}$$

Again, as each $\psi_j \in H_0^j(\Gamma_j)$ is arbitrary, it is deduced that each $h_{0j}$ solves the thin-wave equation

$$-\Delta h_{0j} + h_{0j} - \frac{\partial w_0}{\partial \nu} + \frac{\partial u_0}{\partial \nu} = -h_{1j}^*, \text{ in } \Gamma_j,\ 1 \leq j \leq K. \tag{4.14}$$

A subsequent integration by parts, and invocation of (4.7), (4.11), and (4.13), give for $1 \leq j \leq K$,

$$||\Delta h_{0j}||_{\Gamma_j} + \left|\left| \frac{\partial h_{0j}}{\partial n_j} \right|\right|_{H^{-1/2}(\partial \Gamma_j)} \leq C \, ||[u_0^*, [h_{01}^*, h_{11}^*, \dots, h_{0K}^*, h_{1K}^*], w_0^*, w_1^*]||_{\mathbf{H}}. \tag{4.15}$$

Now, proceeding as in the final stage of the proof of Theorem 1: Let fixed indices $j^*, l^*, 1 \leq j^*, l^* \leq K$ satisfy $\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*} \neq \emptyset$. Given the function $g \in H_0^{1/2+\epsilon}(\partial \Gamma_{j^*} \cap \partial \Gamma_{l^*})$, invoke the associated functions $\psi_{j^*} \in H^{1+\epsilon}(\Gamma_{j^*})$ and $\psi_{l^*} \in H^{1+\epsilon}(\Gamma_{l^*})$ as in (3.38), also $\Upsilon \in H^1(\Gamma_s)$ as in (3.39). With these functions, and said continuous right inverse $\gamma_s^+ \in \mathscr{L}(H^{1/2}(\Gamma_s), H^1(\Omega_s))$, consider the test function

$$[\psi, \xi] = [[0, \dots, \psi_{j^*}, 0, \dots, 0, \psi_{l^*}, \dots, 0], \gamma_s^+(\Upsilon)] \in \chi.$$

Applying this test function to the variational relation (4.9), and subsequently invoking (4.12), it follows

$$\begin{aligned} &- \left\langle \frac{\partial w_0}{\partial \nu}, \xi|_{\Gamma_s} \right\rangle_{\Gamma_s} + \langle \nabla h_{0j^*}, \nabla \psi_{j^*} \rangle_{\Gamma_{j^*}} + \langle h_{0j^*}, \psi_{j^*} \rangle_{\Gamma_{j^*}} + \langle \nabla h_{0l^*}, \nabla \psi_{l^*} \rangle_{\Gamma_{l^*}} + \langle h_{0l^*}, \psi_{l^*} \rangle_{\Gamma_{l^*}} \\ &= - \langle h_{1j^*}, \psi_{j^*} \rangle_{\Gamma_{j^*}} - \left\langle \frac{\partial u_0}{\partial \nu}, \psi_{j^*} \right\rangle_{\Gamma_{j^*}} - \langle h_{1l^*}, \psi_{l^*} \rangle_{\Gamma_{l^*}} - \left\langle \frac{\partial u_0}{\partial \nu}, \psi_{l^*} \right\rangle_{\Gamma_{l^*}}. \end{aligned}$$

Integrating by parts with respect to the thin wave components, and invoking (4.14) and (3.38), the authors get

$$\left\langle \frac{\partial h_{0j^*}}{\partial n_{j^*}}, g \right\rangle_{\partial\Gamma_{j^*}\cap\partial\Gamma_{l^*}} + \left\langle \frac{\partial h_{0l^*}}{\partial n_{l^*}}, g \right\rangle_{\partial\Gamma_{j^*}\cap\partial\Gamma_{l^*}} = 0.$$

Since $g \in H_0^{1/2+\epsilon}(\partial\Gamma_{j^*} \cap \partial\Gamma_{l^*})$ is arbitrary, a density argument yields

$$\left\langle \frac{\partial h_{0j^*}}{\partial n_{j^*}}, g \right\rangle_{\partial\Gamma_{j^*}\cap\partial\Gamma_{l^*}} = -\left\langle \frac{\partial h_{0l^*}}{\partial n_{l^*}}, g \right\rangle_{\partial\Gamma_{j^*}\cap\partial\Gamma_{l^*}}, \text{ for all } j^*,\ l^*,\ 1 \le j^*,\ l^* \le K \tag{4.16}$$

such that $\partial\Gamma_{j^*} \cap \partial\Gamma_{l^*} \neq \emptyset$. Collecting (4.3), (4.4), (4.6), (4.7), (4.11), (4.12), and (4.14) – (4.16), it follows that the obtained $[u_0, [h_{01}, h_{11}, \ldots, h_{0K}, h_{1K}], w_0, w_1] \in D(\mathbf{A})$ satisfies the equation (4.1) for arbitrary $\Phi^* \in \mathbf{H}$. Since also $\mathbf{A} : D(\mathbf{A}) \subseteq \mathbf{H} \to \mathbf{H}$ is dissipative (and so injective), the authors conclude that $\mathbf{A}$ is boundedly invertible. □

### 4.2.2 Analysis of the Continuous and Point Spectrum $\sigma_c(\mathbf{A})$ and $\sigma_p(\mathbf{A})$

**Lemma 7.** *The point $\sigma_p(\boldsymbol{A})$ and continuous specta $\sigma_c(\boldsymbol{A})$ of $\boldsymbol{A}$ have empty intersection with $i\mathbb{R}$.*

*Proof.* To prove this, it will be enough to show that $i\mathbb{R}\backslash\{0\}$ has empty intersection with the approximate spectrum of $\mathbf{A}$; see e.g., Theorem 2.27, pg. 128 of [28]. To this end, given $\beta \neq 0$, suppose that $i\beta$ is in the approximate spectrum of $\mathbf{A}$. Then there exist sequences

$$\{\Phi_n\} = \left\{ \begin{bmatrix} u_n \\ h_{1n} \\ \xi_{1n} \\ \vdots \\ h_{Kn} \\ \xi_{Kn} \\ w_{0n} \\ w_{1n} \end{bmatrix} \right\} \subseteq D(\mathbf{A}); \quad \{(i\beta I - \mathbf{A})\Phi_n\} = \left\{ \begin{bmatrix} u_n^* \\ \varphi_{1n}^* \\ \psi_{1n}^* \\ \vdots \\ \varphi_{Kn}^* \\ \psi_{Kn}^* \\ w_{0n}^* \\ w_{1n}^* \end{bmatrix} \right\} \subseteq \mathbf{H}, \tag{4.17}$$

which satisfy for $n \in \mathbb{N}$

$$||\Phi_n||_{\mathbf{H}} = 1, \quad ||(i\beta I - \mathbf{A})\Phi_n||_{\mathbf{H}} < \frac{1}{n} \tag{4.18}$$

As such, each $\Phi_n$ solves the following static system:

$$\begin{cases} i\beta u_n - \Delta u_n = u_n^* \text{ in } \Omega_f \\ u_n|_{\Gamma_f} = 0 \text{ on } \Gamma_f \end{cases} \tag{4.19}$$

For $1 \leq j \leq K$,

$$\begin{cases} i\beta h_{jn} - \xi_{jn} = \varphi_{jn}^* \text{ in } \Gamma_j \\ -\beta^2 h_{jn} - \Delta h_{jn} + h_{jn} + \frac{\partial u_n}{\partial \nu} - \frac{\partial w_{0n}}{\partial \nu} = \psi_{jn}^* + i\beta\varphi_{jn}^* \text{ in } \Gamma_j \end{cases} \tag{4.20}$$

Also

$$\begin{cases} i\beta w_{0n} - w_{1n} = w_{0n}^* \text{ in } \Omega_s \\ -\beta^2 w_{0n} - \Delta w_{0n} = w_{1n}^* + i\beta w_{0n}^* \text{ in } \Omega_s \end{cases} \tag{4.21}$$

and again for $1 \leq j \leq K$,

$$\begin{cases} u_n|_{\Gamma_j} = \xi_{jn} = w_{1n}|_{\Gamma_j} \\ \frac{\partial h_{nj}}{\partial n_j}\Big|_{\partial\Gamma_j \cap \partial\Gamma_l} = -\frac{\partial h_{nl}}{\partial n_l}\Big|_{\partial\Gamma_j \cap \partial\Gamma_l} \text{ for all } 1 \leq l \leq K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset. \end{cases} \tag{4.22}$$

Now the left part of the proof of Lemma 7 will be given in five steps:

**Step 1: (Estimating the heat component of $\Phi_n$)**

Proceeding as earlier in establishing the dissipativity of $\mathbf{A} : D(\mathbf{A}) \subseteq \mathbf{H} \to \mathbf{H}$, (see relations (3.12) and (3.14)), denote

$$\Phi_n^* = (i\beta I - \mathbf{A})\Phi_n$$

then from the relation

$$\langle (i\beta I - \mathbf{A})\Phi_n, \Phi_n \rangle_{\mathbf{H}} = \langle \Phi_n^*, \Phi_n \rangle_{\mathbf{H}},$$

then

$$||\nabla u_n||_{\Omega_f}^2 = \mathrm{Re}\,\langle \Phi_n^*, \Phi_n \rangle_{\mathbf{H}}. \tag{4.23}$$

From (4.18),

$$\lim_{n\to\infty} u_n = 0 \text{ in } H^1(\Omega_f). \tag{4.24}$$

In turn, via the thin wave resolvent condition in (4.20) and boundary conditions in (4.22), it follows for $1 \leq j \leq K$

$$h_{jn} = \frac{i}{\beta} u_n|_{\Gamma_j} - \frac{i}{\beta}\varphi_{jn}^*|_{\Gamma_j} \text{ in } \Gamma_j.$$

From this relation, invoke (4.24), the Sobolev trace map, and (4.18), to have

$$\lim_{n\to\infty} h_{jn} = 0 \text{ in } H^{1/2}(\Gamma_j) \tag{4.25}$$

for $1 \leq j \leq K$. Moreover, an integration by parts, with respect to the heat equation (4.19), gives the estimate

$$\begin{aligned}\left|\left|\frac{\partial u_n}{\partial \nu}\right|\right|_{H^{-1/2}(\Gamma_f)} &\leq C[||\nabla u_n||_{\Omega_f} + ||\Delta u_n||_{\Omega_f}] \\ &\leq C[||\nabla u_n||_{\Omega_f} + ||i\beta u_n - u_n^*||_{\Omega_f}].\end{aligned}$$

Now, invoking (4.23) and (4.18) gives

$$\lim_{n\to\infty} \frac{\partial u_n}{\partial \nu} = 0 \text{ in } H^{-1/2}(\Gamma_j) \tag{4.26}$$

**Step 2:**

Start here by defining the "Direchlet" map $D_s : L^2(\Gamma_s) \to L^2(\Omega_s)$ via

$$D_s g = f \leftrightarrow \begin{cases} \Delta f = 0 \text{ in } \Omega_s \\ f|_{\Gamma_s} = g \text{ on } \Gamma_s \end{cases}$$

By the Lax-Milgram Theorem (see Theorem 58)

$$D_s \in \mathscr{L}(H^{1/2}(\Gamma_s), H^1(\Omega_s)). \tag{4.27}$$

Therewith, considering the resolvent relations in (4.21), we set

$$z_n \equiv w_{0n} + \frac{i}{\beta} D_s[u_n|_{\Gamma_s} + w_{0n}^*|_{\Gamma_s}], \tag{4.28}$$

and so from (4.21) $z_n$ satisfies the following BVP:

$$\begin{cases} -\beta^2 z_n - \Delta z_n = w_{1n}^* + i\beta w_{0n}^* - i\beta D_s[u_n|_{\Gamma_s} + w_{0n}^*|_{\Gamma_s}] \text{ in } \Omega_s \\ z_n|_{\Gamma_s} = 0 \text{ on } \Gamma_s. \end{cases} \tag{4.29}$$

Since $\Omega_s$ is convex, then $z_n \in H^2(\Omega_S)$ (See Theorem 3.2.1.2, pg. 147 of [29]). In consequence, the authors can apply the static version of the well-known wave identity which is often used in PDE control theory - see (Proposition 7 (ii) of [27]), [20], [43]. To wit, let $m(x)$ by any $[C^2(\overline{\Omega_s})]^3$-vector field with associated Jacobian matrix

$$[M(x)]_{ij} = \frac{\partial m_i(x)}{\partial x_j},\ 1 \leq i,\ j \leq 3$$

Therewith, it follows that

$$\begin{aligned} \int_{\Omega_s} M\nabla z_n \cdot \nabla z_n d\Omega_s = &-\text{Re} \int_{\Gamma_s} \frac{\partial z_n}{\partial \nu} m \cdot \nabla \overline{z_n} d\Gamma_s - \frac{\beta^2}{2} \int_{\Gamma_s} |z_n|^2 m \cdot \nu d\Gamma_s \\ &+ \frac{1}{2} \int_{\Gamma_s} |\nabla z_n|^2 m \cdot \nu d\Gamma_s + \frac{1}{2} \int_{\Omega_s} [|\nabla z_n|^2 - \beta^2 |z_n|^2] \text{div } (m) d\Omega_s \\ &+ \text{Re} \int_{\Omega_s} \left[F_\beta^* - i\beta D_s[u_n|_{\Gamma_s} + w_{0n}^*|_{\Gamma_s}]\right] m \cdot \nabla \overline{z_n} d\Omega_s, \end{aligned} \tag{4.30}$$

where

$$F_\beta^* = (\mathrm{Re}\, w_{1n}^* - \beta I_m w_{0n}^*) + i(I_m w_{1n}^* + \beta \mathrm{Re}\, w_{0n}^*) \tag{4.31}$$

Again, relation (3.21) holds for any $C^2$-vector field $m(x)$. The authors now specify it to be the smooth vector field of Lemma 1.5.1.9, pg. 40 of [29]. Namely, for some $\delta > 0$, the $C^\infty$ vector field $m(x)$ satisfies

$$-m(x) \cdot \nu \geq \delta \text{ a.e. on } \Gamma_s \tag{4.32}$$

Specifying this vector field in (4.30), and considering that $z_n|_{\Gamma_s} = 0$, it follows that

$$\begin{aligned} -\frac{1}{2}\int_{\Gamma_s} \left|\frac{\partial z_n}{\partial \nu}\right|^2 m \cdot \nu d\Gamma_s = \int_{\Omega_s} M\nabla z_n \cdot \nabla z_n d\Omega_s + \frac{1}{2}\int_{\Omega_s} [\beta^2 |z_n|^2 - |\nabla z_n|^2] d\Omega_s \\ - \mathrm{Re}\int_{\Omega_s} \left[F_\beta^* - i\beta D_s[u_n|_{\Gamma_s} + w_{0n}^*|_{\Gamma_s}]\right] m \cdot \nabla \overline{z_n} d\Omega_s. \end{aligned} \tag{4.33}$$

Estimating this relation via (4.18), (4.24), (4.28), and (4.27) and the Sobolev trace map,

$$\int_{\Gamma_s} \left|\frac{\partial z_n}{\partial \nu}\right|^2 d\Gamma_s \leq C_{\delta,\beta,m}, \tag{4.34}$$

where positive constant $C_{\delta,\beta,m}$ is independent of $n \in \mathbb{N}$.

**Step 3: (An energy estimate for $h_{jn}$)**

By multiplying both sides of the thin wave $h_{jn}$-equation (4.20) by $h_{jn}$, integrating and subsequently integrating by parts it follows for $1 \leq j \leq K$,

$$\begin{aligned} \int_{\Gamma_j} |\nabla h_{jn}|^2 d\Gamma_j = \int_{\Gamma_j} \frac{\partial w_{0n}}{\partial \nu} h_{jn} d\Gamma_j + (\beta^2 - 1)\int_{\Gamma_j} |h_{jn}|^2 d\Gamma_j - \int_{\Gamma_j} \frac{\partial u_n}{\partial \nu} h_{jn} d\Gamma_j \\ + \int_{\Gamma_j} (\psi_{jn}^* + i\beta\varphi_{jn}^*) h_{jn} d\Gamma_j \end{aligned} \tag{4.35}$$

Here, the authors are also implicitly using $D(\mathbf{A})$-criterion **(A.iv)**. For the first term on the right hand side: upon combining the regularity for $D_s$ in (4.27) with an integration by parts, it follows that

$$\frac{\partial}{\partial \nu} D_s \in \mathscr{L}(H^{1/2}(\Gamma_s), H^{-1/2}(\Omega_s)) \tag{4.36}$$

This gives the estimate, via the decomposition (4.28),

$$\left\|\frac{\partial w_{0n}}{\partial \nu}\right\|_{H^{-1/2}(\Gamma_s)} \leq C\left[\left\|\frac{\partial z_n}{\partial \nu}\right\|_{H^{-1/2}(\Gamma_s)} + \left\|i\beta\frac{\partial}{\partial \nu}D_s[u_n|_{\Gamma_s} + w_{0n}^*|_{\Gamma_s}]\right\|_{H^{-1/2}(\Gamma_s)}\right] \leq C_\beta \tag{4.37}$$

after also using (4.18), (4.24), the Sobolev trace map, and (4.34). Applying this estimate to the right hands side of (4.35), along with (4.25), (4.26), and (4.18)

$$\lim_{n\to\infty} h_{jn} = 0 \text{ in } H^1(\Gamma_j),\ 1 \leq j \leq K. \tag{4.38}$$

**Step 4:**

From the previous step the limit in (4.38) when applied to the equation

$$\left.\frac{\partial w_{0n}}{\partial \nu}\right|_{\Gamma_j} = -\Delta h_{jn} + (1-\beta^2)h_{jn} + \frac{\partial u_n}{\partial \nu} - (\psi_{jn}^* + i\beta\varphi_{jn}^*) \text{ in } \Gamma_j,\ 1 \leq j \leq K,$$

gives

$$\lim_{n\to\infty} \left.\frac{\partial w_{0n}}{\partial \nu}\right|_{\Gamma_j} = 0 \text{ in } H^{-1}(\Gamma_j). \tag{4.39}$$

In obtaining this limit, along with (4.38), the authors are also using (4.26) and (4.18). In turn, via an interpolation it follows for $1 \leq j \leq K$,

$$\begin{aligned}\left\|\frac{\partial z_n}{\partial \nu}\right\|_{H^{-1/2}(\Gamma_j)} &\leq C\left\|\frac{\partial z_n}{\partial \nu}\right\|_{H^{-1}(\Gamma_j)}^{1/2}\left\|\frac{\partial z_n}{\partial \nu}\right\|_{L^2(\Gamma_j)}^{1/2} \\ &= C\left\|i\beta\frac{\partial}{\partial \nu}D_s[u_n|_{\Gamma_s} + w_{0n}^*|_{\Gamma_s}]\right\|_{H^{-1}(\Gamma_s)}^{1/2}\left\|\frac{\partial w_{0n}}{\partial \nu} + \frac{\partial z_n}{\partial \nu}\right\|_{L^2(\Gamma_j)}^{1/2}\end{aligned} \tag{4.40}$$

Applying (4.36), (4.18), (4.39), and (4.34) to the right hand side of (4.40), and upon summing up over $j$,

$$\lim_{n\to\infty} \frac{\partial z_n}{\partial \nu} = 0 \text{ in } H^{-1/2}(\Gamma_s) \tag{4.41}$$

**Step 5:**

By (4.18) it is shown that $\{z_n\}$ of (4.28) converges weakly to, say, $z$ in $H_0^1(\Omega_s)$. With this limit in mind, by multiplying both sides of the wave equation in (4.29) by given $\eta \in H^1(\Omega_s)$ and integrating by parts, it follows

$$\begin{aligned} -\beta^2 \langle z_n, \eta\rangle_{\Omega_s} + \langle \nabla z_n, \nabla \eta\rangle_{\Omega_s} + \left\langle \frac{\partial z_n}{\partial \nu}, \eta \right\rangle_{\Gamma_s} & \\ = \langle w_{1n}^* + i\beta w_{0n}^* - i\beta D_s[u_n|_{\Gamma_s} + w_{0n}^*|_{\Gamma_s}], \eta\rangle_{\Omega_s} & \text{ for all } \eta \in H^1(\Omega_s). \end{aligned}$$

Taking the limit of both sides of this equation, while taking into account (4.18), (4.24), (4.27), the Sobolev trace map, and (4.41), it follows that $z \in H_0^1(\Omega_s)$ satisfies the variational problem

$$-\beta^2 \langle z, \eta\rangle_{\Omega_s} + \langle \nabla z, \nabla \eta\rangle_{\Omega_s} = 0, \text{ for all } \eta \in H^1(\Omega_s)$$

That is, $z$ satisfies the overdetermined eigenvalue problem

$$\begin{cases} -\Delta z = \beta^2 z \text{ in } \Omega_s \\ z|_{\Gamma_s} = \left.\frac{\partial z}{\partial \nu}\right|_{\Gamma_s} = 0 \end{cases}$$

which gives that

$$z = 0 \text{ in } \Omega_s$$

Combining this convergence with (4.28), (4.24), (4.18), and (4.27),

$$\lim_{n\to\infty} w_{0n} = 0 \text{ in } H^1(\Omega_s) \tag{4.42}$$

**Completion of the Proof of Lemma 7**

The resolvent relations in (4.20), (4.21) and the convergences (4.25), (4.42) give also

$$\begin{cases} \lim_{n\to\infty} \xi_{jn} = 0 \text{ in } L^2(\Gamma_j),\ 1 \leq j \leq K \\ \lim_{n\to\infty} w_{1n} = 0 \text{ in } H^1(\Omega_s) \end{cases} \tag{4.43}$$

Collecting now (4.24), (4.38), (4.42), and (4.43) it follows that

$$\lim_{n\to\infty} \Phi_n = 0 \text{ in } \mathbf{H},$$

which contradicts (4.18) and finishes the proof of Lemma 7. □

### 4.2.3 Analysis of the Residual Spectrum $\sigma_r(\mathbf{A})$

In what follows, the Hilbert space adjoint of $\mathbf{A} : D(\mathbf{A}) \subseteq \mathbf{H} \to \mathbf{H}$ will be important which can be readily computed:

**Proposition 8.** *The Hilbert space adjoint $\boldsymbol{A}^* : D(\boldsymbol{A}^*) \subseteq \boldsymbol{H} \to \boldsymbol{H}$ of the thick wave-thin wave-heat generator is given as,*

$$\boldsymbol{A}^* = \begin{bmatrix} \Delta & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & \dots & 0 & 0 & 0 & 0 \\ -\frac{\partial}{\partial\nu}|_{\Gamma_1} & (I-\Delta) & 0 & \dots & 0 & 0 & -\frac{\partial}{\partial\nu}|_{\Gamma_1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -I & 0 & 0 \\ -\frac{\partial}{\partial\nu}|_{\Gamma_K} & 0 & 0 & \dots & (I-\Delta) & 0 & -\frac{\partial}{\partial\nu}|_{\Gamma_K} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & -I \\ 0 & 0 & 0 & \dots & 0 & 0 & -\Delta & 0 \end{bmatrix};$$

*where*

$$
\begin{aligned}
&D(\boldsymbol{A}^*) = \{[u_0, h_{01}, h_{11}, \ldots, h_{0K}, h_{1K}, w_0, w_1] \in \boldsymbol{H} : \\
&\quad (\boldsymbol{A}^*\boldsymbol{.i})\ u_0 \in H^1(\Omega_f),\ h_{1j} \in H^1(\Gamma_j) \text{ for } 1 \le j \le K,\ w_1 \in H^1(\Omega_s); \\
&\quad (\boldsymbol{A}^*\boldsymbol{.ii})(a)\ \Delta u_0 \in L^2(\Omega_f),\ \Delta w_0 \in L^2(\Omega_s),\ (b)\ \Delta h_{0j} - \left.\frac{\partial u_0}{\partial \nu}\right|_{\Gamma_j} - \left.\frac{\partial w_0}{\partial \nu}\right|_{\Gamma_j} \in L^2(\Gamma_j) \\
&\qquad \text{for } 1 \le j \le K;\ (c)\ \left.\frac{\partial h_{0j}}{\partial n_j}\right|_{\partial\Gamma_j} \in H^{-1/2}(\partial\Gamma_j),\ \text{for } 1 \le j \le K; \\
&\quad (\boldsymbol{A}^*\boldsymbol{.iii})\ u_0|_{\Gamma_f} = 0, u_0|_{\Gamma_j} = h_{1j} = w_1|_{\Gamma_j},\ \text{for } 1 \le j \le K; \\
&\quad (\boldsymbol{A}^*\boldsymbol{.iv})\ \text{For } 1 \le j \le K: \\
&\quad (a)\ h_{1j}|_{\partial\Gamma_j\cap\partial\Gamma_l} = h_{1l}|_{\partial\Gamma_j\cap\partial\Gamma_l} \text{ on } \partial\Gamma_j \cap \partial\Gamma_l,\ \text{for all } 1 \le l \le K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset; \\
&\quad (b)\ \left.\frac{\partial h_{0j}}{\partial n_j}\right|_{\partial\Gamma_j\cap\partial\Gamma_l} = -\left.\frac{\partial h_{0l}}{\partial n_l}\right|_{\partial\Gamma_j\cap\partial\Gamma_l} \text{ on } \partial\Gamma_j \cap \partial\Gamma_l,\ \text{for all } 1 \le l \le K \text{ such that } \partial\Gamma_j \cap \partial\Gamma_l \neq \emptyset\}.
\end{aligned}
$$

Lastly, consider the following corollary regarding the residual spectrum $\sigma_r(\mathbf{A})$:

**Corollary 9.** *The residual spectrum $\sigma_r(\boldsymbol{A})$ of $\boldsymbol{A}$ does not intersect the imaginary axis.*

*Proof.* Given the form of the adjoint operator $\mathbf{A}^* : D(\mathbf{A}^*) \subseteq \mathbf{H} \to \mathbf{H}$ in Proposition 8, then proceeding identically as in the proof of Lemma 7 it follows that

$$\sigma_p(\mathbf{A}^*) \cap i\mathbb{R} = \sigma_c(\mathbf{A}^*) \cap i\mathbb{R} = \emptyset$$

which finishes the proof of Corollary 9. □

Now having established the above results for the spectrum of $\mathbf{A}$, the authors finish the proof of Theorem 4:

**Proof of Theorem** 4

Then by combining the above results, Proposition 6, Lemma 7, and Corollary 9 and remembering that $\{e^{\mathbf{A}t}\}_{t\ge 0}$ is a contraction semigroup, the strong stability result follows immediately from the application of Theorem 5.

## 4.3 Revisiting the Stokes-Wave-Lamé FSI PDE Model

We turn our attention now back to the PDE described in (3.42) − (3.45) with the associated Hilbert space, inner product, and generator with domain ((3.50), (3.46), and (3.47) respectively). In order to establish the strong stability result, one of the key tools that the author uses to show that zero is the resolvent set of the operator $\{\mathbf{A}|_{\mathbf{N}^\perp}\}$ is the Babuska-Brezzi Theorem (see Theorem 59). Here $\mathbf{N}$ represents the zero eigenspace of the semigroup generator which is associated to the above PDE system (see the definition of $\mathbf{A} : \mathbf{H} \to \mathbf{H}$ in (3.50) above.) Also, $\lambda = 0$ is indeed an eigenvalue of $\mathbf{A}$, see Theorem 13 below. Moreover, the orthogonal complement $\mathbf{N}^\perp$ is characterized as

$$\mathbf{N}^\perp = \left\{ [u_0, h_0, h_1, w_0, w_1] \in \mathbf{H} \,\middle|\, \int_{\Gamma_s} (\nu \cdot h_0) d\Gamma_s = 0 \right\}, \tag{4.44}$$

We note that the author splits $\mathbf{H}$ in 3.46 into the two subspaces $\mathbf{N}$ and $\mathbf{N}^\perp$.

**Remark 10.** *Given the form of the adjoint* $\boldsymbol{A}^* : D(\boldsymbol{A}^*) \subseteq \boldsymbol{H} \to \boldsymbol{H}$*, in Proposition* 16 *below, it is readily seen that* $\lambda > 0$ *is also an eigenvalue of* $\boldsymbol{A}^*$ *with*

$$Null(\boldsymbol{A}^*) = Null(\boldsymbol{A}) = \boldsymbol{N}.$$

*Consequently, one has "invariance of the flow" on* $\boldsymbol{N}^\perp$*. That is, if* $\Phi_0 \in \boldsymbol{N}^\perp$*, then*

$$e^{\boldsymbol{A}t}\Phi_0 \in C([0,T]; \boldsymbol{N}^\perp).$$

## 4.4 Main Result: Asymptotic Decay of Solution

This section is devoted to addressing the issue of asymptotic behavior of the solution whose existence-uniqueness is guaranteed by Theorem 3. In this regard, it is shown that the system given in (3.42) − (3.45) is strongly stable in the space $\mathbf{N}^\perp$ (see (4.44) for definition). Before giving the main result, the author states the following assumption which is crucial in the proof:

**Assumption 11.** *Given a fixed* $\beta \neq 0$*, suppose that the function* $w_0 \in \boldsymbol{H}^1(\Omega_s)$ *satisfies the following* *static PDE problem:*

$$\begin{cases} -\beta^2 w_0 - div\ \sigma(w_0) + w_0 = 0 & in\ \Omega_s \\ w_0|_{\Gamma_s} = 0 & on\ \Gamma_s \\ \nu \cdot \sigma(w_0) = -c_0 \nu & on\ \Gamma_s. \end{cases}$$

*Then, the solution of this overdetermined problem is* $w_0 = 0$*, and so necessarily* $c_0 = 0$

Such overdetermined eigenvalue boundary value problems also play a part in considering stability properties of other fluid-structure PDE systems. See, e.g. [3, 5, 7]. Now, we give the stability result:

**Theorem 12.** *Let Assumption* 11 *hold. Then, with reference to the dynamical system* (3.42) – (3.45)*, for any* $[\widetilde{u}_0, \widetilde{h}_0, \widetilde{h}_1, \widetilde{w}_0, \widetilde{w}_1] \in \boldsymbol{N}^{\perp}$*, the corresponding solution* $[u(t), h(t), h_t(t), w(t), w_t(t)] \in C([0,T]; \boldsymbol{N}^{\perp})$ *of* (3.42) – (3.45) *satisfies*

$$\lim_{t\to\infty} ||[u(t), h(t), h_t(t), w(t), w_t(t)]||_{\boldsymbol{N}^{\perp}} = 0.$$

The domain $D(\mathbf{A})$ of the semigroup generator is not compactly embedded in $\mathbf{H}$, and so classical weak stability approaches are inapplicable here. See e.g., ([14] page 378, Corollary 3.1). The proof of Theorem 12 is based on the well known spectral criterion given by W. Arendt and C.J.K. Batty [1] (See Theorem 5). The application of Theorem 5 on the dynamical system (3.42) – (3.45) relies on analyzing the spectral properties of the operator $\mathbf{A}$ defined in (3.50). The analysis will be given in a few steps:

### 4.4.1 Zero Eigenvalue for the Generator A and Explicit Characterization of $\mathbf{N} = Null(\mathbf{A})$

In order to analyze the long term dynamics of the system (3.42) – (3.45) and apply Theorem 5 it is important to avoid steady states so as to reasonably consider the possibility of finite energy solutions, tending to the zero state at infinity. For this reason, in the process of applying Theorem 5, it is shown that zero is an eigenvalue for the operator $\mathbf{A}$. Moreover, the author gives an explicit characterization for the corresponding zero eigenspace $\mathbf{N} = Null(\mathbf{A})$ and its orthogonal complement

$\mathbf{N}^{\perp} = Null(\mathbf{A})^{\perp}$.

**Theorem 13.** *With reference to the PDE system* (3.42) − (3.45) *and the corresponding generator* $\boldsymbol{A}$ *defined in* (3.50)*, the point zero is an eigenvalue for* $\boldsymbol{A}$*. Moreover, the following explicit characterizations follow:*

$$\boldsymbol{N} = Null(\boldsymbol{A}) = span\left\{\phi \in \boldsymbol{H} \,\middle|\, \phi = [0, h_0(1), 0, w_0(1), 0]^T\right\} \tag{4.45}$$

*Here, for any scalar* $\alpha$*, the pair*

$$[h_0(\alpha), w_0(\alpha)] \in \boldsymbol{S} = \left\{[f, g] \in \boldsymbol{H}^1(\Gamma_s) \times \boldsymbol{H}^1(\Omega_s) \,\middle|\, f = g|_{\Gamma_s}\right\}$$

*satisfies the variational relation*

$$\langle \nabla_{\Gamma_s}(h_0), \nabla_{\Gamma_s}(f)\rangle_{\Gamma_s} + \langle \sigma(w_0), \epsilon(g)\rangle_{\Omega_s} + \langle w_0, g\rangle_{\Omega_s} = \alpha \langle \nu, f\rangle_{\Gamma_s}, \ \textit{for all } [f, g] \in \boldsymbol{S}$$

*Also,*

$$\boldsymbol{N}^{\perp} = Null(\boldsymbol{A})^{\perp} = \left\{\widetilde{\phi} = [\widetilde{u}_0, \widetilde{h}_0, \widetilde{h}_1, \widetilde{w}_0, \widetilde{w}_1]^T \in \boldsymbol{H} \,\middle|\, \int_{\Gamma_s} (\nu \cdot \widetilde{h}_0) d\Gamma_s = 0\right\} \tag{4.46}$$

*Proof.* Suppose $\Phi = [u_0, h_0, h_1, w_0, w_1] \in D(\mathbf{A})$ is a solution of

$$\mathbf{A}\phi = 0, \tag{4.47}$$

where $\mathbf{A}$, is as given in (3.50). With pressure term $p_0$ as defined in (3.56), this equation generates the following PDEs:

$$\begin{cases} \text{div } (\nabla u_0 + \nabla^T u_0) - \nabla p_0 = 0 & \text{in } \Omega_f \\ \text{div } (u_0) = 0 & \text{in } \Omega_f \\ h_1 = 0 & \text{in } \Gamma_s \\ -[\nu \cdot (\nabla u_0 + \nabla^T u_0)]|_{\Gamma_s} + \Delta_{\Gamma_s}(h_0) + [\nu \cdot \sigma(w_0)]|_{\Gamma_s} + p_0 \nu = 0 & \text{in } \Gamma_s \\ w_1 = 0 & \text{in } \Omega_s \\ \text{div } \sigma(w_0) - w_0 = 0 & \text{in } \Omega_s \\ u_0|_{\Gamma_f} = 0, \ u_0|_{\Gamma_s} = h_1 = 0. \end{cases} \tag{4.48}$$

Multiplying (4.47) by $\Phi$,

$$\begin{aligned} \langle \mathbf{A}\Phi, \Phi \rangle_{\mathbf{H}} &= \left\langle \text{div } (\nabla(u_0) + \nabla^T(u_0)), u_0 \right\rangle_{\Omega_f} + \langle \nabla_{\Gamma_s}(h_1), \nabla_{\Gamma_s}(h_0) \rangle_{\Gamma_s} \\ &+ \left\langle -(\nabla u_0 + \nabla^T u_0) \cdot \nu|_{\Gamma_s}, h_1 \right\rangle_{\Gamma_s} + \langle \Delta_{\Gamma_s}(h_0), h_1 \rangle_{\Gamma_s} + \langle \sigma(w_0) \cdot \nu|_{\Gamma_s}, h_1 \rangle_{\Gamma_s} \\ &+ \langle \sigma(w_1), \epsilon(w_0) \rangle_{\Omega_s} + \langle w_1, w_0 \rangle_{\Omega_s} + \langle \text{div } \sigma(w_0), w_1 \rangle_{\Omega_s} - \langle w_0, w_1 \rangle_{\Omega_s} \\ &- \langle [\nabla \mathcal{P}_1(u_0) + \nabla \mathcal{P}_2(h_0) + \nabla \mathcal{P}_3(w_0)], u_0 \rangle_{\Omega_f} \\ &+ \langle [\mathcal{P}_1(u_0) \cdot \nu + \mathcal{P}_2(h_0) \cdot \nu + \mathcal{P}_3(w_0) \cdot \nu], h_1 \rangle_{\Gamma_s} = 0. \end{aligned}$$

Applying Green's Theorem, using the fact that $u_0$ is solenoidal, $u_0 = 0$ on $\Gamma_f$, and $h_1 = w_1|_{\Gamma_s} = 0$ on $\Gamma_s$ the author concludes

$$\begin{aligned} \langle \mathbf{A}\Phi, \Phi \rangle_{\mathbf{H}} = -\frac{1}{2} \left|\left|\nabla(u_0) + \nabla^T(u_0)\right|\right|^2 + \langle \sigma(w_1), \epsilon(w_0) \rangle_{\Omega_s} + \langle w_1, w_0 \rangle_{\Omega_s} - \langle \sigma(w_0), \epsilon(w_1) \rangle_{\Omega_s} \\ - \langle w_0, w_1 \rangle_{\Omega_s} = 0, \end{aligned}$$

or,

$$\langle \mathbf{A}\Phi, \Phi \rangle_{\mathbf{H}} = -\frac{1}{2} \left|\left|\nabla(u_0) + \nabla^T(u_0)\right|\right|^2 + 2i\text{Im}\{\langle \sigma(w_1), \epsilon(w_0) \rangle_{\Omega_s} + \langle w_1, w_0 \rangle_{\Omega_s}\} = 0.$$

Hence

$$\operatorname{Re}\langle \mathbf{A}\Phi, \Phi\rangle_{\mathbf{H}} = -\frac{1}{2}\left|\left|\nabla(u_0) + \nabla^T(u_0)\right|\right|^2 \tag{4.49}$$

which, considering also the boundary condition $u_0|_{\Gamma_f} = 0$, gives

$$u_0 = 0 \text{ in } \Omega_f \tag{4.50}$$

In turn, from $(4.48)_1$ it follows that

$$p_0 = c_0 \text{ (constant)}.$$

Now, define the space

$$\mathbf{S} = \left\{[f, g] \in \mathbf{H}^1(\Gamma_s) \times \mathbf{H}^1(\Omega_s) \;\middle|\; f = g|_{\Gamma_s}\right\}.$$

Multiplying $(4.48)_4$ by $f$ and $(4.48)_6$ by $g$,

$$\begin{aligned} &-\langle \nabla_{\Gamma_s}(h_0), \nabla_{\Gamma_s}(f)\rangle_{\Gamma_s} + \langle \nu \cdot \sigma(w_0)|_{\Gamma_s}, f\rangle_{\Gamma_s} + \langle c_0\nu, f\rangle_{\Gamma_s} \\ &\quad - \langle \sigma(w_0) \cdot \nu|_{\Gamma_s}, g\rangle_{\Gamma_s} - \langle \sigma(w_0), \epsilon(g)\rangle_{\Omega_s} - \langle w_0, g\rangle_{\Omega_s} = 0. \end{aligned} \tag{4.51}$$

Using the fact that $f = g|_{\Gamma_s}$, and taking the variational form in terms of the solution variables $\{h_0, w_0\}$ :

$$\mathbf{a}([h_0, w_0], [f, g]) = \mathbf{F}([f, g]) \text{ for all } [f, g] \in \mathbf{S}, \tag{4.52}$$

where the bilinear form $\mathbf{a}(\cdot, \cdot) : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ is defined as

$$\mathbf{a}([h_0, w_0], [f, g]) = \langle \nabla_{\Gamma_s}(h_0), \nabla_{\Gamma_s}(f)\rangle_{\Gamma_s} + \langle \sigma(w_0), \epsilon(g)\rangle_{\Omega_s} + \langle w_0, g\rangle_{\Omega_s}$$

and

$$\mathbf{F}([f, g]) = c_0 \langle \nu, f\rangle_{\Gamma_s}.$$

Since it can easily be seen that the bilinear form $\mathbf{a}(\cdot,\cdot)$ is continuous and $\mathbf{S}$-elliptic, the application of Lax-Milgram Theorem (see Theorem 58) gives the existence and uniqueness of a solution $[h_0, w_0] \in \mathbf{S}$ to the variational equation (4.52). To conclude the proof of Theorem 13, it is important to show that the derived solution $[h_0, w_0] \in D(\mathbf{A})$ and satisfies the equations in (4.48). To this end, by taking $g \in \mathscr{D}(\Omega_s)$, and $f = 0$ in (4.51) then

$$\langle -\text{div } (w_0) + w_0, g \rangle_{\Omega_s} = 0, \text{ for all } g \in \mathscr{D}(\Omega_s),$$

and hence

$$-\text{div } \sigma(w_0) + w_0 = 0, \text{ in } L^2(\Omega_s).$$

In consequence, it follows

$$||\sigma(w_0)\cdot\nu||_{H^{-1/2}(\Gamma_s)} \leq C\,||w_0||_{H^1(\Omega_s)} \leq C|c_0|. \tag{4.53}$$

In turn: let $\gamma_0^+ \in \mathscr{L}(H^{1/2}(\Gamma_s), H^1(\Omega_s))$ be the right inverse of the Dirichlet trace map $\gamma_0 : H^1(\Omega_s) \to H^{1/2}(\Gamma_s)$. Therewith, setting

$$[f, g] \equiv [f, \gamma_0^+(f)]$$

in (4.51) where $f \in \mathbf{H}^1(\Gamma_s)$,

$$\langle \nabla_{\Gamma_s}(h_0), \nabla_{\Gamma_s}(f) \rangle_{\Gamma_s} + \left\langle \sigma(w_0), \epsilon(\gamma_0^+(f)) \right\rangle_{\Omega_s} + \left\langle w_0, \gamma_0^+(f) \right\rangle_{\Omega_s} = c_0 \langle \nu, f \rangle_{\Gamma_s}.$$

or

$$\langle \nabla_{\Gamma_s}(h_0), \nabla_{\Gamma_s}(f) \rangle_{\Gamma_s} + \left\langle \sigma(w_0), \epsilon(\gamma_0^+(f)) \right\rangle_{\Omega_s} + \left\langle w_0, \gamma_0^+(f) \right\rangle_{\Omega_s} + \left\langle \text{div } \sigma(w_0) - w_0, \gamma_0^+(f) \right\rangle_{\Gamma_s} = c_0 \langle \nu, f \rangle_{\Gamma_s}.$$

Application of the Green's Theorem gives then

$$\langle -\Delta_{\Gamma_s}(h_0) + \nu \cdot \sigma(w_0) \rangle_{\Gamma_s} = c_0 \langle \nu, f \rangle_{\Gamma_s} = 0$$

or

$$-\Delta_{\Gamma_s}(h_0) + \nu \cdot \sigma(w_0) = c_0 \nu \text{ in } L^2(\Gamma_s) \tag{4.54}$$

In sum, it follows that the vector $\phi = [0, h_0(1), 0, w_0(1), 0]^T \in D(\mathbf{A})$ solves

$$\mathbf{A}\Phi = 0,$$

and is indeed the zero eigenvector with the corresponding zero eigenspace $\mathbf{N} = Null(\mathbf{A})$ could be characterized as in (4.45). Subsequently, recalling the inner product introduced in (3.47), and using the relations (4.53) – (4.54), the orthogonal complement $\mathbf{N}^{\perp}$ follows as in (4.46). This completes the proof of Theorem 13. □

The proof of Theorem 12 will rely on the ultimate application of the spectral criterion of W. Arendt and C. Batty for strong decay (see Theorem 5). This will entail the elimination of all three parts of the spectrum of the generator $\mathbf{A}$ from the imaginary axis: In this connection, the author now proceeds with the analysis of the point spectrum $\sigma_p(\mathbf{A})$.

### 4.4.2 Analysis of the Point Spectrum $\sigma_p(\mathbf{A})$

**Lemma 14.** *Let Assumption* 11 *hold. With reference to the PDE system* (3.42) – (3.45) *and the corresponding generator* $\boldsymbol{A}$ *defined in* (3.50), *given for* $\beta \neq 0$, $i\beta \notin \sigma_p(\boldsymbol{A})$

*Proof.* Suppose $\Phi = [u_0, h_0, h_1, w_0, w_1] \in D(\mathbf{A})$ satisfies the relation

$$(i\beta I - \mathbf{A})\Phi = 0. \tag{4.55}$$

In PDE terms, it follows then

$$\begin{cases} i\beta u_0 - \text{div } (\nabla u_0 + \nabla^T u_0) + \nabla p_0 = 0 & \text{in } \Omega_f \\ \text{div } (u_0) = 0 & \text{in } \Omega_f \\ u_0|_{\Gamma_f} = 0 & \text{on } \Gamma_f \end{cases} \tag{4.56}$$

$$\begin{cases} i\beta h_0 - h_1 = 0 & \text{in } \Gamma_s \\ i\beta h_1 + [\nu \cdot (\nabla u_0 + \nabla^T u_0)]|_{\Gamma_s} - \Delta_{\Gamma_s}(h_0) - [\nu \cdot \sigma(w_0)]|_{\Gamma_s} - p_0\nu = 0 & \text{in } \Gamma_s \end{cases} \tag{4.57}$$

$$\begin{cases} i\beta w_0 - w_1 = 0 & \text{in } \Omega_s \\ i\beta w_1 - \text{div } \sigma(w_0) + w_0 = 0 & \text{in } \Omega_s \\ w_1|_{\Gamma_s} = h_1 = u_0|_{\Gamma_s} & \text{on } \Gamma_s \end{cases} \tag{4.58}$$

with $p_0$ being the associated pressure of the PDE system. Therewith, the usual energy method and the relation (4.49) gives

$$\text{Re}\,\langle (i\beta I - \mathbf{A})\Phi, \Phi\rangle_{\mathbf{H}} = -\text{Re}\,\langle \mathbf{A}\Phi, \Phi\rangle_{\mathbf{H}} = \frac{1}{2}\left|\left|\nabla(u_0) + \nabla^T(u_0)\right|\right|^2 = 0 \tag{4.59}$$

which, considering also the boundary condition $u_0|_{\Gamma_f} = 0$, gives

$$u_0 = 0 \text{ in } \Omega_f. \tag{4.60}$$

Also, taking into account (4.60) in $(4.58)_3$,

$$w_1|_{\Gamma_s} = h_1 = 0 \text{ on } \Gamma_s. \tag{4.61}$$

Let $p_0 = q_0 + c_0$, where $q_0 \in \hat{L}^2(\Omega_f)$, where

$$\hat{L}^2(\Omega_f) = \left\{ f \in L^2(\Omega_f) \,\middle|\, \int_{\Omega_f} f d\Omega_f = 0 \right\}, \tag{4.62}$$

and $c_0$ is a constant. Then by (4.60) and $(4.56)_1$,

$$q_0 = 0 \text{ in } \Omega_f. \tag{4.63}$$

Also, from (4.61) and $(4.57)_1$, it follows

$$h_0 = 0 \text{ in } \Gamma_s. \tag{4.64}$$

Now, consider (4.61) and (4.64) in the thin layer equation $(4.57)_2$, as well as $(4.58)_1$ and $(4.58)_2$, it follows that if $\Phi = [u_0, h_0, h_1, w_0, w_1] \in D(\mathbf{A})$ is an eigenfunction corresponding to the eigenvalue $i\beta$ $(\beta \neq 0)$, then $w_0$ solves the following overdetermined eigenvalue problem:

$$\begin{cases} -\beta^2 w_0 - \text{div } \sigma(w_0) + w_0 = 0 & \text{in } \Omega_s \\ w_0|_{\Gamma_s} = 0 & \text{on } \Gamma_s \\ \nu \cdot \sigma(w_0) = -c_0 \nu \text{ on } \Gamma_s. \end{cases} \tag{4.65}$$

Exploiting Assumption 11 for the problem (4.65) gives that $w_0 = 0$ and $c_0 = 0$, which then yield that $\sigma_p(\mathbf{A}) \cap i\mathbb{R} = \emptyset$. This completes the proof of Lemma 14. □

### 4.4.3 Analysis of the Residual Spectrum $\sigma_r(\mathbf{A})$

**Lemma 15.** *Let Assumption* 11 *hold. With reference to the PDE system* (3.42) – (3.45) *and the corresponding generator* $\boldsymbol{A}$ *defined in* (3.50)*, given* $\beta \neq 0$, $i\beta \notin \sigma_r(\boldsymbol{A})$.

*Proof.* The proof of Lemma 15 relies on the wellknown fact that for any closed, densely defined operator $\mathbf{A}$, if $\lambda \in \sigma_r(\mathbf{A})$ then $\overline{\lambda} \in \sigma_p(\mathbf{A}^*)$. (See e.g., [28], page 127). Accordingly, the author first gives a representation of the adjoint operator $\mathbf{A}^* : D(\mathbf{A}^*) \to \mathbf{H}$. In fact, a standard computation yields:

**Proposition 16.** *For the generator operator* $\boldsymbol{A}$ *defined in* (3.50)*, the Hilbert space adjoint* $\boldsymbol{A}^*$ :

*$D(\boldsymbol{A}^*) \to \boldsymbol{H}$ is given by*

$$\boldsymbol{A}^* = \begin{bmatrix} div\ (\nabla(\cdot)+\nabla^T(\cdot)) & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 \\ -[\nu\cdot(\nabla(\cdot)+\nabla^T(\cdot))]|_{\Gamma_s} & -\Delta_{\Gamma_s}(\cdot) & 0 & -\nu\cdot\sigma(\cdot)|_{\Gamma_s} & 0 \\ 0 & 0 & 0 & 0 & -I \\ 0 & 0 & 0 & -div\ \sigma(\cdot)+I & 0 \end{bmatrix} +$$

$$\begin{bmatrix} -\nabla\mathcal{P}_1(\cdot) & \nabla\mathcal{P}_2(\cdot) & 0 & \nabla\mathcal{P}_3(\cdot) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mathcal{P}_1(\cdot)\nu & -\mathcal{P}_2(\cdot)\nu & 0 & -\mathcal{P}_3(\cdot)\nu & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.66}$$

*The domain $D(\boldsymbol{A}^*)$ of $\boldsymbol{A}^*$ is characterized as follows $[\widetilde{u}_0,\widetilde{h}_0,\widetilde{h}_1,\widetilde{w}_0,\widetilde{w}_1] \in D(\boldsymbol{A}^*) \Leftrightarrow$*

***(A.i)*** $\widetilde{u}_0 \in \boldsymbol{H}^1(\Omega_f)$, $\widetilde{h}_1 \in \boldsymbol{H}^1(\Gamma_s)$ $\widetilde{w}_1 \in \boldsymbol{H}^1(\Omega_s)$;

***(A.ii)*** *There exists an associated $L^2(\Omega_f)$-function $\widetilde{p}_0 = \widetilde{p}_0(\widetilde{u}_0,\widetilde{h}_0,\widetilde{w}_0)$ such that*

$$[div\ (\nabla\widetilde{u}_0+\nabla^T\widetilde{u}_0)-\nabla\widetilde{p}_0] \in L^2(\Omega_f)$$

*Consequently, $\widetilde{p}_0$ is harmonic and so*

***(a)*** $\left[\widetilde{p}_0|_{\Gamma_f}, \frac{\partial\widetilde{p}_0}{\partial\nu}\Big|_{\Gamma_f}\right] \in H^{-1/2}(\Gamma_f)\times H^{-3/2}(\Gamma_f)$;

***(b)*** $(\nabla\widetilde{u}_0+\nabla^T\widetilde{u}_0)\cdot\nu \in H^{-3/2}(\Gamma_f)$,

***(A.iii)*** *$div\ \sigma(\widetilde{w}_0) \in L^2(\Omega_s)$; consequently, $\nu\cdot\sigma(\widetilde{w}_0) \in H^{-1/2}(\Gamma_s)$,*

***(A.iv)*** $-\Delta_{\Gamma_s}(\widetilde{h}_0)+[\nu\cdot\sigma(\widetilde{w}_0)]_{\Gamma_s} - [(\nabla\widetilde{u}_0+\nabla^T\widetilde{u}_0)\cdot\nu]|_{\Gamma_s} + [\widetilde{p}_0\nu]|_{\Gamma_s} \in L^2(\Gamma_s)$,

***(A.v)*** $\widetilde{u}_0|_{\Gamma_f}=0$, $\widetilde{u}_0|_{\Gamma_s}=\widetilde{h}_1=\widetilde{w}_1|_{\Gamma_s}$

It is readily discerned that $Null(\mathbf{A}) = Null(\mathbf{A}^*)$. In turn, $\lambda = 0$ is an eigenvalue of $\mathbf{A}^*$. Moreover, under Assumption 11, $i\beta$ ($\beta \neq 0$) is not an eigenvalue for the adjoint operator $\mathbf{A}^*$ and hence it is not in the residual spectrum of $\mathbf{A}$. This completes the proof of Lemma 15. □

### 4.4.4 Analysis of the Continuous Spectrum $\sigma_c(\mathbf{A})$

**Lemma 17.** *Let Assumption* 11 *hold. With reference to the PDE system* (3.42) – (3.45) *and the corresponding generator* $\boldsymbol{A}$ *defined in* (3.50), *for a given* $\beta \neq 0$, $i\beta \notin \sigma_c(\boldsymbol{A})$.

*Proof.* The proof is based on a contradiction argument. To start with, assume that $i\beta$ ($\beta \neq 0$) is in the continuous spectrum $\sigma_c(\mathbf{A})$ of $\mathbf{A}$. Since $\sigma_c(\mathbf{A}) \subseteq \sigma_{app}(\mathbf{A})$ (approximate spectrum) (see e.g., [28], page 128) then there exist sequences

$$\begin{aligned} &\{\Phi_n\} = \{[u_{0n}, h_{0n}, h_{1n}, w_{0n}, w_{1n}]^T\} \subseteq D(\mathbf{A}); \\ &\{(i\beta I - A)\Phi_n\} \equiv \{\Phi_n^*\} = \{[u_{0n}^*, h_{0n}^*, h_{1n}^*, w_{0n}^*, w_{1n}^*]^T\} \subseteq \mathbf{H}, \end{aligned} \tag{4.67}$$

which satisfy for $n = 1, 2, \ldots,$

$$||\Phi_n||_{\mathbf{H}} = 1 \text{ and } ||(i\beta I - A)\Phi_n||_{\mathbf{H}} < \frac{1}{n}. \tag{4.68}$$

In PDE terms, each $\Phi_n$ solves the following static system, where again $p_n$ is given via (3.56):

$$\begin{cases} i\beta u_{0n} - \text{div } (\nabla u_{0n} + \nabla^T u_{0n}) + \nabla p_n = u_{0n}^* & \text{in } \Omega_f \\ \text{div } (u_{0n}) = 0 & \text{in } \Omega_f \\ u_{0n}|_{\Gamma_f} = 0 & \text{on } \Gamma_f \end{cases} \tag{4.69}$$

$$\begin{cases} i\beta h_{0n} - h_{1n} & \text{in } \Gamma_s \\ i\beta h_{1n} + [\nu \cdot (\nabla u_{0n} + \nabla^T u_{0n})]|_{\Gamma_s} - \Delta_{\Gamma_s}(h_{0n}) - [\nu \cdot \sigma(w_{0n})]|_{\Gamma_s} - p_n \nu = h_{1n}^* & \text{in } \Gamma_s \end{cases} \tag{4.70}$$

$$\begin{cases} i\beta w_{0n} - w_{1n} = w_{0n}^* & \text{in } \Omega_s \\ i\beta w_{1n} - \text{div } \sigma(w_{0n}) + w_{0n} = w_{1n}^* & \text{in } \Omega_s \\ w_{1n}|_{\Gamma_s} = h_{1n} = u_{0n}|_{\Gamma_s} & \text{on } \Gamma_s \end{cases} \tag{4.71}$$

**Step I: Estimates for the fluid variable $\{u_{0n}\}$ and thin-layer solution variables $\{h_{0n}, h_{1n}\}$**

To start, via the integration by parts, dissipativity relation (4.49), and (4.68),

$$\text{Re}\,\langle \Phi_n^*, \Phi_n \rangle_{\mathbf{H}} = \text{Re}\,\langle (i\beta I - \mathbf{A})\Phi_n, \Phi_n \rangle_{\mathbf{H}} = \frac{1}{2} \left|\left|\nabla(u_{0n}) + \nabla^T(u_{0n})\right|\right|^2 = \mathcal{O}\left(\frac{1}{n}\right), \tag{4.72}$$

and hence

$$||u_{0n}||_{H^1(\Omega_f)} \to 0. \tag{4.73}$$

In turn, by (4.71) – (4.72) and the Sobolev Trace Theorem

$$u_{0n}|_{\Gamma_s} = w_{1n}|_{\Gamma_s} = h_{1n} = \mathcal{O}\left(\frac{1}{n}\right). \tag{4.74}$$

Let $p_n = q_n + c_n$, where $q_n \in \hat{L}^2(\Omega_f)$ (as defined in (4.62)), and $c_n$ is a constant. Then, by the Stokes theory [42], $\{u_{0n}, q_n\} \in \mathbf{H}^1(\Omega_f) \times \hat{L}^2(\Omega_f)$ uniquely solve

$$\begin{cases} -\text{div } (\nabla u_{0n} + \nabla^T u_{0n}) + \nabla q_n = -i\beta u_{0n} + u_{0n}^* & \text{in } \Omega_f \\ \text{div } (u_{0n}) = 0 & \text{in } \Omega_f \\ u_{0n}|_{\Gamma_s} = u_{0n}^*|_{\Gamma_s} & \text{on } \Gamma_s \\ u_{0n}|_{\Gamma_f} = 0 & \text{on } \Gamma_f, \end{cases} \tag{4.75}$$

and so the following estimate holds:

$$||q_n||_{L^2(\Omega_f)} \leq C[||\Phi_n^*||_{\mathbf{H}} + ||u_{0n}||_{H^1(\Omega_f)}] = \mathcal{O}\left(\frac{1}{n}\right). \tag{4.76}$$

Subsequently, an energy method yields that

$$\{\nu \cdot (\nabla u_{0n} + \nabla^T u_{0n}) - q_n \cdot \nu\} \in H^{-1/2}(\Gamma_f) \tag{4.77}$$

with

$$\left|\left|\nu \cdot (\nabla u_{0n} + \nabla^T u_{0n}) - q_n \cdot \nu\right|\right|_{H^{-1/2}(\Gamma_f)} \leq C[||u_{0n}^* - i\beta u_{0n}||_{H^1(\Omega_f)} + ||u_{0n}||_{H^1(\Omega_f)}] = \mathcal{O}\left(\frac{1}{n}\right), \tag{4.78}$$

after using (4.68) and (4.72). Moreover, since by the thin-layer resolvent relation in $(4.70)_1$ and the boundary condition $(4.71)_3$

$$h_{0n} = -\frac{i}{\beta}h_{1n} - \frac{i}{\beta}h^*_{0n} = -\frac{i}{\beta}u_{0n} - \frac{i}{\beta}h^*_{0n}. \tag{4.79}$$

Using again (4.68) and (4.72),

$$||h_{0n}||_{H^{1/2}(\Gamma_s)} = \mathcal{O}\left(\frac{1}{n}\right). \tag{4.80}$$

**<u>Step II:</u> Estimates for the term** $\{\nu \cdot \sigma(w_{0n})\}$

Start by invoking the "Dirichlet" map $D_s : L^2(\Gamma_s) \to L^2(\Omega_s)$ via

$$D_s g = f \Leftrightarrow \begin{cases} \text{div } \sigma(f) = 0 & \text{in } \Omega_s \\ f|_{\Gamma_s} = g & \text{on } \Gamma_s. \end{cases}$$

By the Lax-Milgram Theorem (see Theorem 58)

$$D_s \in \mathscr{L}(H^{1/2}(\Gamma_s), H^1(\Omega_s)). \tag{4.81}$$

Now, with the Dirichlet map $D_s$ in hand, make the change of variable

$$z_n \equiv w_{0n} + \frac{i}{\beta}D_s[u_{0n} + w^*_{0n}|_{\Gamma_s}]. \tag{4.82}$$

Considering the resolvent relations in $(4.71)_{1-2}$, $z_n$ then solves the following boundary value problem (BVP):

$$\begin{aligned} -\beta^2 z_n - \text{div } \sigma(z_n) &= F_\beta \text{ in } \Omega_s \\ z_n|_{\Gamma_s} &= 0 \text{ on } \Gamma_s \end{aligned} \tag{4.83}$$

where

$$F_\beta = w^*_{1n} + i\beta w^*_{0n} - w_{0n} - i\beta D_s[u_{0n} + w^*_{0n}|_{\Gamma_s}] \tag{4.84}$$

Since this BVP has homogeneous boundary data the author then has the estimate (see e.g., [21], page 296, Theorem 6.3-6):

$$||z_n||_{H^2(\Omega_s)} \leq ||w_{1n}^* + i\beta w_{0n}^* - w_{0n} - i\beta D_s[u_{0n} + w_{0n}^*|_{\Gamma_s}]||_{\Omega_s} \leq C_{1,\beta} \tag{4.85}$$

after using (4.68). Consequently, there is the trace mapping - see, e.g. [40],

$$\left|\left|\frac{\partial z_n}{\partial \nu}\right|\right|_{\Gamma_s} \leq C\,||z_n||_{H^2(\Omega_s)} \leq C_{2,\beta} \tag{4.86}$$

(again, after using (4.85).) With this trace estimate in hand, invoke the following known expression for $\{\sigma(z_n)\cdot\nu|_{\Gamma_s}\}$ in terms of the normal and tangential derivatives (see [4], page 18, Proposition A.1):

$$\begin{aligned}\sigma(z_n)\cdot\nu = \lambda\left[\frac{\partial z_n}{\partial \nu}\cdot\nu + \frac{\partial z_n}{\partial \tau}\cdot\tau + \frac{\partial z_n}{\partial e}\cdot e\right]\nu + 2\mu\frac{\partial z_n}{\partial \nu} \\ + \mu\left[\frac{\partial z_n}{\partial \tau}\cdot\nu - \frac{\partial z_n}{\partial \nu}\cdot\tau\right]\tau + \mu\left[\frac{\partial z_n}{\partial e}\cdot\nu - \frac{\partial z_n}{\partial \nu}\cdot e\right]e\end{aligned} \tag{4.87}$$

Here, unit (tangent) vectors $\{e,\tau\}$ and $\nu$ constitute an orthonormal system on $\mathbb{R}^3$. Since $z_n = 0$ on $\Gamma_s$ then (4.87) is simplified to

$$\sigma(z_n)\cdot\nu = \lambda\left(\frac{\partial z_n}{\partial \nu}\cdot\nu\right) + 2\mu\frac{\partial z_n}{\partial \nu} - \mu\left(\frac{\partial z_n}{\partial \nu}\cdot\tau\right)\tau - \mu\left(\frac{\partial z_n}{\partial \nu}\cdot e\right)e. \tag{4.88}$$

Applying the estimate (4.86) to the RHS of (4.88) now yields

$$||\sigma(z_n)\cdot\nu||_{\Gamma_s} \leq C_{3,\beta}. \tag{4.89}$$

Moreover, an integration by parts yields the inference that

$$\nu\cdot\sigma(D_s(\cdot)) \in \mathscr{L}(H^{1/2}(\Gamma_s), H^{-1/2}(\Gamma_s)).$$

Combining this boundedness with (4.68), and (4.72) it follows that

$$||\nu\cdot(D_s[u_{0n}|_{\Gamma_s} + w_{0n}^*|_{\Gamma_s}])||_{H^{-1/2}(\Gamma_s)} = \mathcal{O}\left(\frac{1}{n}\right) \tag{4.90}$$

Applying now (4.89) – (4.90) to the relation (4.82),

$$||\nu \cdot \sigma(w_{0n})||_{H^{-1/2}(\Gamma_s)} \leq C. \tag{4.91}$$

**<u>Step III:</u> Estimates for the term** $\{c_n\}$

At this step, recall the definition of the pressure term $p_n = q_n + c_n$, and read off the equation $(4.70)_2$ to have

$$\begin{aligned}||-c_n\nu||_{H^{-1}(\Gamma_s)} &= || - i\beta h_{1n} + h^*_{1n} - [\nu \cdot (\nabla u_{0n} + \nabla^T u_{0n})]|_{\Gamma_s} + \Delta_{\Gamma_s}(h_{0n}) + [\nu \cdot \sigma(w_{0n})]|_{\Gamma_s} \\ &\quad + q_n\nu||_{H^{-1}(\Gamma_s)},\end{aligned}$$

whence it follows via (4.68), (4.76), (4.78), and (4.91)

$$|c_n| \leq C, \tag{4.92}$$

where again $C > 0$ depends on $\lambda, \beta, \mu$.

**<u>Step IV:</u> Estimate for** $\{\nabla_{\Gamma_s} h_{0n}\}$

By multiplying both sides of equation $(4.70)_2$ by $h_{0n}$, integrating in space, and then using the integration by parts to get

$$\begin{aligned}||\nabla_{\Gamma_s} h_{0n}||^2_{\Gamma_s} &= -i\beta \langle h_{1n}, h_{0n}\rangle_{\Gamma_s} + \langle h^*_{1n}, h_{0n}\rangle_{\Gamma_s} - \left\langle [\nu \cdot (\nabla u_{0n} + \nabla^T u_{0n})]|_{\Gamma_s}, h_{0n}\right\rangle_{\Gamma_s} \\ &\quad + \langle [\nu \cdot \sigma(w_{0n})]|_{\Gamma_s}, h_{0n}\rangle_{\Gamma_s} + \langle p_n\nu, h_{0n}\rangle_{\Gamma_s}.\end{aligned}$$

Invoking again (4.68), (4.72), (4.74), (4.76), (4.91) – (4.92), and moreover using the resolvent relation $(4.70)_1$ for the second term on the RHS of the last equality, it follows

$$||\nabla_{\Gamma_s} h_{0n}||^2_{\Gamma_s} = \mathcal{O}\left(\frac{1}{n}\right). \tag{4.93}$$

Now, collecting all the estimates obtained in (4.73), (4.74), (4.76), and (4.93) it follows

$$\begin{cases} u_{0n} \to 0 & \text{in } \mathbf{H}^1(\Omega_f) \\ q_n \to 0 & \text{in } L^2(\Omega_f) \\ q_n|_{\Gamma_s} \to 0 & \text{in } H^{-1/2}(\Gamma_s) \\ w_{0n}|_{\Gamma_s} \to 0 & \text{in } \mathbf{H}^{1/2}(\Gamma_s) \\ h_{0n} \to 0 & \text{in } \mathbf{H}^1(\Gamma_s) \\ h_{1n} \to 0 & \text{in } \mathbf{H}^{1/2}(\Gamma_s). \end{cases} \tag{4.94}$$

Moreover, from (4.89), the sequence $\{\nu \cdot \sigma(z_n)\}$ has a weakly convergent subsequence (still denoted as itself) such that $\{\nu \cdot \sigma(z_n)\}$ converges (weakly) in $L^2(\Gamma_s)$ and since $L^2(\Gamma_s) \hookrightarrow H^{-1}(\Gamma_s)$ is compact, it follows that $\{\nu \cdot \sigma(z_n)\}$ converges strongly in $H^{-1}(\Gamma_s)$.

Lastly, from (4.92), it is shown that $\{c_n\}$ converges strongly to, say $c^*$. Recall the definition of $z_n$ given in (4.82) and by invoking the resolvent relation $(4.70)_2$ together with the limits in (4.94), the author concludes

$$\lim_{n\to\infty} \nu \cdot \sigma(z_n) = -c^* \nu. \tag{4.95}$$

To finish the proof, consider again the BVP, given in (4.83): Initially, the relations (4.68) and (4.82) provide for the weak convergence to say $z$, i.e.

$$z_n \to z \text{ in } \mathbf{H}^1(\Omega_s).$$

Hence, if by passing to the limit in (4.83) when $n \to \infty$, then it follows that $z \in \mathbf{H}^1(\Omega_s)$ satisfies the following problem:

$$-\beta^2 \langle z, \Psi \rangle_{\Omega_s} + \langle \sigma(z), \epsilon(\Psi) \rangle_{\Omega_s} + \langle c^* \nu, \Psi \rangle_{\Omega_s} + \langle z, \Psi \rangle_{\Omega_s} = 0, \text{ for all } \Psi \in \mathbf{H}^1(\Omega_s).$$

That is, $\{-\beta^2, z\}$ solves the following overdetermined eigenvalue problem:

$$\begin{cases} -\beta^2 z - \text{div } \sigma(z) + z = 0 & \text{in } \Omega_s \\ z = 0 & \text{on } \Gamma_s \ \cdot \\ \frac{\partial z}{\partial \nu} = -c^* \nu & \text{on } \Gamma_s. \end{cases} \tag{4.96}$$

Now, under Assumption 11, the only solution to problem (4.96) is $z = 0$ for $c^* = 0$. Then from (4.82), (4.73), and (4.68), we see that $w_{0n} \to 0$. This convergence, and those in (4.94), contradicts the assumption that

$$||\Phi_n||_{\mathbf{H}} = 1. \tag{4.97}$$

As a result, for any $\beta \neq 0$, $i\beta \notin \sigma_c(\mathbf{A})$, and this completes the proof of Lemma 17. $\square$

To proceed with the proof of Theorem 12, recall by Theorem 13 that zero is an eigenvalue for the generator $\mathbf{A}$. It is shown in fact that $\lambda = 0$ is in the resolvent set of $\mathbf{A}|_{\mathbf{N}^\perp}$:

**Lemma 18.** *$\lambda = 0$ is in the resolvent set $\rho(\boldsymbol{A}|_{\boldsymbol{N}^\perp})$ of $\boldsymbol{A}|_{\boldsymbol{N}^\perp} : D(\boldsymbol{A}|_{\boldsymbol{N}^\perp}) \to \boldsymbol{N}^\perp$. That is*

$$(\boldsymbol{A}|_{\boldsymbol{N}^\perp})^{-1} \in \mathscr{L}(\boldsymbol{N}^\perp).$$

*Proof.* As before, the author uses the denotations

$$\Phi = [u_0, h_0, h_1, w_0, w_1] \in D(\mathbf{A}) \cap \mathbf{N}^\perp, \quad \Phi^* = [u_0^*, h_0^*, h_1^*, w_0^*, w_1^*] \in \mathbf{N}^\perp.$$

and consider solving the following relation

$$\mathbf{A}\Phi = \Phi^*. \tag{4.98}$$

Then, in PDE terms, this equation generates the following static system

$$\begin{cases} \text{div } (\nabla u_0 + \nabla^T u_0) - \nabla q = u_0^* & \text{in } \Omega_f \\ \text{div } (u_0) = 0 & \text{in } \Omega_f \\ h_1 = h_0^* & \text{in } \Gamma_s \\ -[\nu \cdot (\nabla u_0 + \nabla^T u_0)]|_{\Gamma_s} + \Delta_{\Gamma_s}(h_0) + [\nu \cdot \sigma(w_0)]|_{\Gamma_s} + (q + c_0^*)\nu = h_1^* & \text{in } \Gamma_s \\ w_1 = w_0^* & \text{in } \Omega_s \\ \text{div } \sigma(w_0) - w_0 = w_1^* & \text{in } \Omega_s \\ u_0|_{\Gamma_f} = 0, \; u_0|_{\Gamma_s} = h_1 = w_1|_{\Gamma_s} \end{cases} \tag{4.99}$$

where $p = q + c_0^*$ is the associated pressure as described in $D(\mathbf{A})$. Firstly, the fluid component $u_0$ of (4.99) and the pressure term $q$ can be recovered via the Stokes theory y (See [42], pg 22, Theorem 2.4) and the pair $\{u_0, q\} \in [\mathbf{H}^1(\Omega_f) \cap Null(\text{div})] \times \hat{L}^2(\Omega_f)$ solves the following static problem

$$\begin{cases} -\text{div } (\nabla u_0 + \nabla^T u_0) + \nabla q = -u_0^* & \text{in } \Omega_f \\ \text{div } (u_0) = 0 & \text{in } \Omega_f \\ u_0|_{\Gamma_s} = h_0^* \end{cases} \tag{4.100}$$

with the estimate

$$\left|\left|\nu \cdot (\nabla u_0 + \nabla^T u_0) - q\nu\right|\right|_{H^{-1/2}(\Gamma_s)} + \left|\left|\nabla u_0 + \nabla^T u_0\right|\right|_{\Omega_f} + ||q||_{\Omega_f} \leq C \, ||\Phi^*||_{\mathbf{H}} \, . \tag{4.101}$$

Now, with the associated pressure $p = q + c_0^*$, the constant component $c_0^*$ is to be determined. Now the author turns their attention to the thick and thin elastic PDE component in (4.99). Define the space

$$\mathbf{S} = \left\{(\varphi, \psi) \in \mathbf{H}^1(\Gamma_s) \times \mathbf{H}^1(\Omega_s) \,\middle|\, \varphi = \psi|_{\Gamma_s}\right\}.$$

In order to generate a mixed variational formulation for the static “thin” and “thick” solution variables in (4.99), by respectively multiplying $(4.99)_4$ and $(4.99)_6$ by functions $\varphi \in \mathbf{H}^1(\Gamma_s)$, and

$\psi \in \mathbf{H}^1(\Omega_s)$ in the space $\mathbf{S}$, use Green's Theorem, and add the subsequent relations to get:

$$\begin{aligned}
&- \langle \nabla_{\Gamma_s} h_0, \nabla_{\Gamma_s} \varphi \rangle_{\Gamma_s} + \langle \nu \cdot \sigma(w_0)|_{\Gamma_s}, \varphi \rangle_{\Gamma_s} + \langle c_0^* \nu, \varphi \rangle_{\Gamma_s} \\
&- \langle \nu \cdot \sigma(w_0)|_{\Gamma_s}, \psi \rangle_{\Gamma_s} - \langle \sigma(w_0), \epsilon(\psi) \rangle_{\Omega_s} - \langle w_0, \psi \rangle_{\Omega_s} \\
= &\langle h_1^*, \varphi \rangle_{\Gamma_s} + \langle w_1^*, \psi \rangle_{\Omega_s} + \left\langle \nu \cdot (\nabla u_0 + \nabla^T u_0)|_{\Gamma_s}, \varphi \right\rangle_{\Gamma_s} - \langle q\nu, \varphi \rangle_{\Gamma_s} .
\end{aligned} \tag{4.102}$$

The last relation now gives the following mixed variational formulation in terms of the variables $h_0$ and $w_0$: Namely,

$$\begin{cases}
\mathbf{a}([h_0, w_0], [\varphi, \psi]) + \mathbf{b}([\varphi, \psi], c_0^*) = \mathbf{F}([\varphi, \psi]) & \text{for all } [\varphi, \psi] \in \mathbf{S} \\
\mathbf{b}([h_0, w_0], r) = 0 & \text{for all } r \in \mathbb{R}.
\end{cases} \tag{4.103}$$

Here, the bilinear forms $\mathbf{a}(\cdot, \cdot) : \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ and $\mathbf{b}(\cdot, \cdot) : \mathbf{S} \times \mathbb{R} \to \mathbb{R}$ are respectively given as

$$\begin{aligned}
\mathbf{a}([\phi, \xi], [\widetilde{\phi}, \widetilde{\xi}]) &= \langle \nabla_{\Gamma_s} h_0, \nabla_{\Gamma_s} \varphi \rangle_{\Gamma_s} - \langle \nu \cdot \sigma(w_0)|_{\Gamma_s}, \varphi \rangle_{\Gamma_s} \\
&\quad + \langle \nu \cdot \sigma(w_0)|_{\Gamma_s}, \psi \rangle_{\Gamma_s} + \langle \sigma(w_0), \epsilon(\psi) \rangle_{\Omega_s} + \langle w_0, \psi \rangle_{\Omega_s} \\
\mathbf{b}([\widetilde{\phi}, \widetilde{\xi}], r) &= -r \int_{\Gamma_s} \nu \cdot \widetilde{\phi} d\Gamma_s,
\end{aligned}$$

and the functional $\mathbf{F}(\cdot)$ is defined as

$$\mathbf{F}([\widetilde{\phi}, \widetilde{\xi}]) = - \langle h_1^*, \varphi \rangle_{\Gamma_s} - \langle w_1^*, \psi \rangle_{\Omega_s} - \left\langle \nu \cdot (\nabla u_0 + \nabla^T u_0)|_{\Gamma_s}, \varphi \right\rangle_{\Gamma_s} + \langle q\nu, \varphi \rangle_{\Gamma_s} .$$

In order to solve this variational formulation, the author appeals to the Babuska-Brezzi Theorem (see Theorem 59). It is clear that the bilinear forms $\mathbf{a}(\cdot, \cdot)$ and $\mathbf{b}(\cdot, \cdot)$ are continuous, and moreover $\mathbf{a}(\cdot, \cdot)$ is $\mathbf{S}$-elliptic. In order to conclude that the variational problem (4.103) has a unique solution, it is important to show that the bilinear form $\mathbf{b}(\cdot, \cdot)$ satisfies the "inf-sup" condition given in Theorem 59. For this, consider the following problem:
Given $r \in \mathbb{R}$, let $\eta \in \mathbf{H}^1(\Gamma_s)$ satisfy

$$\Delta_{\Gamma_s} \eta = sgn(r) \nu \text{ in } \Gamma_s$$

It is easily seen that $||\nabla_{\Gamma_s} \eta||_{\Gamma_s} \leq C \, ||\nu||_{\Gamma_s}$. Now, taking into account that $\gamma : \mathbf{H}^1(\Omega_s) \to \mathbf{H}^{1/2}(\Gamma_s)$

is a surjective map, and so it has a continuous right inverse $\gamma^+(\eta)$,

$$\begin{aligned}
\sup_{[\theta,\varsigma]\in\mathbf{S}} \frac{\mathbf{b}([\theta,\varsigma],r)}{||[\theta,\varsigma]||_{\mathbf{S}}} &\geq \frac{\mathbf{b}([\theta,\gamma^+(\eta)],r)}{||[\theta,\gamma^+(\eta)]||_{\mathbf{S}}} \\
&= \frac{-r\int_{\Gamma_s}\nu\cdot\eta d\Gamma_s}{||[\theta,\gamma^+(\eta)]||_{\mathbf{S}}} \\
&= -r\ sgn(r)\frac{\int_{\Gamma_s}\Delta_{\Gamma_s}\eta\cdot\eta d\Gamma_s}{||[\theta,\gamma^+(\eta)]||_{\mathbf{S}}} \\
&= |r|\frac{\int_{\Gamma_s}|\nabla_{\Gamma_s}\eta|^2 d\Gamma_s}{||[\theta,\gamma^+(\eta)]||_{\mathbf{S}}} \\
&\geq C|r|\frac{\int_{\Gamma_s}|\nabla_{\Gamma_s}\eta|^2 d\Gamma_s}{||\eta||_{\mathbf{H}^1(\Gamma_s)}} \\
&= C|r|\ ||\eta||_{\mathbf{H}^1(\Gamma_s)}
\end{aligned}$$

which yields that the inf-sup condition holds with the constant $\beta = C\,||\eta||_{\mathbf{H}^1(\Gamma_s)}$. Consequently, the existence and uniqueness of the solution $[h_0,w_0]\in\mathbf{S}$ and $c_0^*\in\mathbb{R}$ to the mixed variational problem (4.103) follows from Theorem 59, and satisfy

$$||[h_0,w_0]||_{\mathbf{S}} + |c_0^*| \leq C\,||\Phi^*||_{\mathbf{H}}\,. \tag{4.104}$$

Subsequently, by taking $[\varphi,\psi]=[0,\psi]$ in (4.103) where $\psi\in[\mathscr{D}(\Omega_s)]^3$, and can infer that the obtained $w_0$ solves:

$$-\text{div}\ \sigma(w_0) + w_0 = -w_1^* \text{ in } \Omega_s. \tag{4.105}$$

In turn, via an energy method,

$$||\nu\cdot\sigma(w_0)||_{H^{-1/2}(\Gamma_s)} \leq C\,||\Phi_0^*||_{\mathbf{H}}\,. \tag{4.106}$$

With this estimate in hand, $\{[h_0,w_0],c_0^*\}$ solves

$$\begin{aligned}
&\langle\nabla_{\Gamma_s}h_0,\nabla_{\Gamma_s}\varphi\rangle_{\Gamma_s} + \langle\sigma(w_0),\epsilon(\psi)\rangle_{\Omega_s} + \langle w_0,\psi\rangle_{\Omega_s} - \langle c_0^*\nu,\varphi\rangle_{\Gamma_s} \\
&= -\langle h_1^*,\varphi\rangle_{\Gamma_s} - \langle w_1^*,\psi\rangle_{\Omega_s} - \left\langle \nu\cdot(\nabla u_0+\nabla^T u_0)|_{\Gamma_s},\varphi\right\rangle_{\Gamma_s} + \langle q\nu,\varphi\rangle_{\Gamma_s}\,.
\end{aligned}$$

An integration by parts and consideration of (4.105) then yields

$$-\Delta_{\Gamma_s}(h_0) + [\nu \cdot (\nabla u + \nabla^T u_0)]|_{\Gamma_s} - [\nu \cdot \sigma(w_0)]|_{\Gamma_s} - p\nu = h_1^* \text{ in } \Gamma_s.$$

By reading off the last equation it follows

$$\left|\left|-\Delta_{\Gamma_s}(h_0) + [\nu \cdot (\nabla u + \nabla^T u_0)]|_{\Gamma_s} - [\nu \cdot \sigma(w_0)]|_{\Gamma_s} - p\nu\right|\right|_{\Gamma_s} \leq C\,||\Phi^*||_{\mathbf{H}}\,.$$

Finally, the pressure term $p$ can be reconstructed via the maps $(\mathcal{P}_i)$'s, as defined in (3.56), which then yields that $\Phi = [u_0, h_0, h_1, w_0, w_1] \in D(\mathbf{A}) \cap \mathbf{N}^{\perp}$ indeed solves (4.98). Hence $0 \in \rho(\mathbf{A}|_{\mathbf{B}^{\perp}})$. $\square$

As a result, combining Theorem 13, Lemma 14, Lemma 15, Lemma 17, and 18, it follows that $\sigma(\mathbf{A}|_{\mathbf{N}^{\perp}}) \cap i\mathbb{R} = \emptyset$. Thus, by the spectral criteria for stability in Theorem 5, the proof of Theorem 12 is complete.

# Chapter 5

# Conclusions and Discussion of Future Work

As we have discussed, multilayered FSI PDEs arise in the context of blood transportation and cellular dynamics. To this end, the models that have been discussed in this project gives a better understanding to what is actually happening in the real world. The canonical model provides a great foundation both numerically and analytically for future work, which was improved upon in the following Stokes flow model. Along with the additionally complexity added in the second model, there were new challenges such as the pressure term. This called for elimination using elliptic boundary value problems and restrictions for the strong decay to avoid intersection with the imaginary axis. Something that was not covered in this project was that in [10], the authors showed that the canonical model displayed rational decay.

It has also been theorized that in all of the aforementioned models, the models do not exhibit exponential decay. For example, in [46], exponential decay has only been shown in simpler models in lower dimensions or for the canonical model in 2D using eigenfunctions to generate solutions explicitly. Thus most effort goes towards establishing rational decay, see [6, 10, 41, 44, 46] which is not as strong as exponential decay. Rational decay has not been shown for Stokes flow model and thus is an open problem being actively worked on.

Finally, as both of these models are simplifications of the Navier-Stokes model, future work would be forced to tackle the full physical model. One of the major issues that is introduced with the full Navier-Stokes model is that the problem suddenly becomes nonlinear. This would cause major issues as the methodology used in the proofs that we covered is based on linear $C_0$-semigroups. Thus changing to the Navier-Stokes model would require a different approach with potentially unforeseen difficulties.

# Chapter 6

# Appendix

Below are a few foundational definitions and theorems that are related to the work covered in this project. The material was sourced from S. Kesavan's "Topics in Functional Analysis and Applications" [32], Amon Pazy's "Semigroups of linear operators and applications to partial differential equations" [33], and Erwin Kreyszig's "Introductory functional analysis with applications" [34].

## Operations with Distributions

**Definition 19.** *The space of* ***test-functions*** *denoted as* $\mathscr{D}(\Omega)$*, where* $\Omega$ *is any open set in* $\mathbb{R}^n$*, is a subset of* $C^\infty(\Omega)$ *that have compact support contained within* $\Omega$*.*

**Definition 20.** *A sequence of functions* $\{\phi_m\}$ *in* $\mathscr{D}(\Omega)$ *is said to converge to* $0$ *if there exists a fixed compact set* $K \subset \Omega$ *such that supp*$(\phi_m) \subset K$ *for all* $m$ *and* $\phi_m$ *and all its derivatives converge uniformly to zero on* $K$*.*

**Definition 21.** *A linear functional* $T$ *on* $\mathscr{D}(\Omega)$ *is said to be a* ***distribution*** *on* $\Omega$ *if whenever* $\phi_m \to 0$ *in* $\mathscr{D}(\Omega)$*, we have* $T(\phi_m) \to 0$*.*
*The space of distributions, which is the dual of the space of test-functions, is denoted by* $\mathscr{D}'(\Omega)$*. In the case* $\Omega = \mathbb{R}^n$*, the symbol* $\mathscr{D}'$ *will also be used.*

**Definition 22.** *Let* $x \in \mathbb{R}^n$ *with coordinates* $(x_1, ..., x_n)$*. A multi-index is a* $n$*-tuple*

$$\alpha = (\alpha_1, ..., \alpha_n), \quad \alpha_i \geq 0, \quad \alpha_i \text{ integers.}$$

*Associated to a multi-index $\alpha$, we have the following symbols*

$$|\alpha| = \alpha_1 + ... + \alpha_n$$

$$\alpha! = \alpha_1!...\alpha_n!$$

$$x^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n}, \quad x \in \mathbb{R}^n$$

*We say that two multi-indices $\alpha$ and $\beta$ are related by $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for all $1 \leq i \leq n$. Finally we set*

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}$$

**Definition 23.** *Let $T \in \mathscr{D}'(\mathbb{R})$. If $T = T_f$, $f$ a $C^1$ function, then $f'$ is locally integrable. Thus for $\phi \in \mathscr{D}(\mathbb{R})$*

$$T_{f'}(\phi) = \int_{\mathbb{R}} f'\phi = -\int_{\mathbb{R}} f\phi' = -T_f(\phi')$$

*Generalizing this, define for any $T \in \mathscr{D}'(\mathbb{R})$,*

$$T'(\phi) = -T(\phi'), \quad \phi \in \mathscr{D}(\mathbb{R})$$

*Thus $T' \in \mathscr{D}'(\mathbb{R})$. For if $\phi_n \to 0$ in $\mathscr{D}(\mathbb{R})$ then $\{\phi_n\}$ is also a sequence in $\mathscr{D}(\mathbb{R})$ converging to zero. Hence $T'(\phi_n) = -T(\phi_n')$ converges to zero. Upon iterating this:*

$$T''(\phi) = -T'(\phi') = T(\phi'')$$

*and more generally*

$$T^{(k)}(\phi) = (-1)^k T(\phi^{(k)})$$

*In general if $T \in \mathscr{D}'(\Omega)$, $\Omega \subset \mathbb{R}^n$ an open set, then we define, for any multi-index $\alpha$, the distribution*

*$D^\alpha T$ by*

$$(D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi), \quad \phi \in \mathscr{D}(\Omega)$$

## Definitions and Basic Properties

**Definition 24.** *Let $m > 0$ be an integer and let $1 \leq p \leq \infty$. The **Sobolev space** $W^{m,p}(\Omega)$ is defined by*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega)\, \forall\, |\alpha| \leq m\}$$

*We provide the norm*

$$||u||_{m,p,\Omega} = \sum_{|\alpha| \leq m} ||D^\alpha u||_{L^p(\Omega)}$$

*Equivalently $1 < p < \infty$ (Note not equal equivalent)*

$$||u||_{m,p,\Omega} = \left( \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u|^p \right)^{1/p} = \left( \sum_{|\alpha| \leq m} ||D^\alpha u||^p_{L^p(\Omega)} \right)^{1/p}$$

**Remark 25.** *As new notation is added the logical extension of previous notation will be assumed.*

- *In the case $p = 2$, the spaces are Hilbert spaces with notation as follows,*

$$H^m(\Omega) = W^{m,2}(\Omega)$$
$$||u||_{m,\Omega} = ||u||_{m,2,\Omega}$$

- *A semi-norm consists of the $L^p$-norms of the highest order derivatives with notation as follows,*

$$|u|_{m,p,\Omega} = \sum_{|\alpha| = m} ||D^\alpha u||_{L^p(\Omega)}$$

- *The space $L^p(\Omega)$ can be seen as a special case of the Sobolev class. We denote the $L^p(\Omega)$ norm as the semi-norm by $|\cdot|_{0,p,\Omega}$ (as they are the same).*

- *In $H^m(\Omega)$ there is a natural inner-product defined as*

$$(u,v)_{m,\Omega} = \sum_{|\alpha|\le m} \int_\Omega D^\alpha u D^\alpha v, \quad \text{for } u,v \in H^m(\Omega)$$

- *When $\Omega = \mathbb{R}^n$ the space $H^m(\mathbb{R}^n)$ can be defined via the Fourier transform. Let $u \in H^m(\mathbb{R}^n)$. By definition $D^\alpha u \in L^2(\mathbb{R}^n) \quad \forall |\alpha| \le m$. Hence the Fourier transform of $D^\alpha u$ is well defined as follows*

$$\widehat{D^\alpha u} = (2\pi i)^{|\alpha|} \xi^\alpha \hat{u}.$$

*Thus $\xi^\alpha \hat{u} \in L^2(\mathbb{R}^n)$ for all $|\alpha| \le m$. Conversely if $u \in L^2(\mathbb{R}^n) \ni \xi^\alpha \hat{u} \in L^2(\mathbb{R}^n)$ for all $|\alpha| \le m$, we have $D^\alpha u \in L^2(\mathbb{R}^n)$ for all $|\alpha| \le m$ and so $u \in H^m(\mathbb{R}^n)$*

**Lemma 26.** *There exists positive constants $M_1$ and $M_2$ depending only on $m$ and $n$ such that*

$$M_1(1+|\xi|^2)^m \le \sum_{|\alpha|\le m} |\xi^\alpha|^2 \le M_2(1+|\xi|^2)^m$$

*for all $\xi \in \mathbb{R}^n$*

**Theorem 27.** *For every $1 \le p \le \infty$, the space $W^{1,p}(\Omega)$ is a Banach space. If $1 < p < \infty$, it is reflexive and if $1 \le p < \infty$, it is separable. In particular $H^1(\Omega)$ is a separable Hilbert space.*

**Remark 28.**

- *The results of this theorem can be proved by the same way for any integer $m \ge 2$. In the future, unless absolutely necessary, theorems for the spaces $W^{1,p}(\Omega)$ will be covered. The extensions to higher order spaces will often be obvious.*

- *In the course of the proof of the preceding theorem it was shown: "if $u_m \to u$ in $L^p(\Omega)$ and $\frac{\partial u_m}{\partial x_i} \to v_i$ in $L^p(\Omega)$ for each $1 \le i \le n$, then $u \in W^{1,p}(\Omega)$ and $\frac{\partial u}{\partial x_i} = v_i$". Indeed, we can weaken the hypotheses even further. What was needed to pass to the limit in the second set of integrals and obtain the first set of integrals was only the weak convergence of $\left\{\frac{\partial u_m}{\partial x_i}\right\}$ (weak $*$ if $p = \infty$). Since bounded sequences have weakly convergent (weak $*$ convergent when $p = \infty$) subsequences, it is enough to know that $u_m \to u$ in $L^p(\Omega)$ and $\left\{\frac{\partial u_m}{\partial x_i}\right\}$ are bounded $(1 < p \le \infty)$ to deduce $u \in W^{1,p}(\Omega)$.*

- *$L^p(\Omega)$ is made up of equivalence classes of functions. Thus by saying "u is a continuous function" in $L^p(\Omega)$ it is meant that the corresponding equivalence class has a continuous representative function.*

**Remark 29.** *If $1 \leq p < \infty$, we know that $\mathscr{D}(\Omega)$ is dense in $L^p(\Omega)$. Also, if $\phi \in \mathscr{D}(\Omega)$ then every derivative of $\phi$ is also in $\mathscr{D}(\Omega) \subset W^{m,p}(\Omega)$ for any $m$ and $p$. We define $W_0^{m,p}(\Omega)$ as the closure of $\mathscr{D}$ in $W^{m,p}(\Omega)$. Thus $W_0^{m,p}(\Omega)$ is a closed subspace and its elements can be approximated in the $W^{m,p}(\Omega)$ norm by $C^\infty$ functions with compact support. Generally this is a strict subspace unless $\Omega = \mathbb{R}^n$.*

**Theorem 30.** *Let $1 \leq p < \infty$. Then for any integer $m \geq 0$,*

$$W^{m,p}(\mathbb{R}^n) = W_0^{m,p}(\mathbb{R}^n).$$

**Theorem 31.** *Let $1 \leq p < \infty$. Then $\mathscr{D}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.*

## Extension Theorems

**Definition 32.** *Let $x \in \mathbb{R}^n$, $\quad x = (x_1, ..., x_n)$. We set $x' = (x_1, ..., x_{n-1})$ and write $x = (x', x_n)$. Define the sets*

$$Q_+ = \{x \in \mathbb{R}^n \mid |x'| < 1, 0 < x_n < 1\}$$
$$Q = \{x \in \mathbb{R}^n \mid |x'| < 1, |x_n| < 1\}$$

*Where $|x'|$ is the Euclidean norm of $|x'|$ in $\mathbb{R}^{n-1}$*

**Definition 33.** *We say that an open set $\Omega$ is of class $C^k$ ($k$ an integer $\geq$ 1) if for every $x \in \partial\Omega$, there exists a neighborhood $U$ of $x$ in $\mathbb{R}^n$ and a map $T : Q \to U$ such that*

- *$T$ is a bijection*
- *$T \in C^k(\bar{Q}), \quad T^{-1} \in C^k(\bar{U})$*
- *$T(Q_+) = U \cap \Omega, \quad T(Q_0) = U \cap \partial\Omega$*

*Where $Q_+, Q$ are defined above and*

$$Q_0 = \{x \in Q \mid x_n = 0\}$$

*We say that $\Omega$ is of class $C^\infty$ if it is of class $C^k$ for every integer $k \geq 1$.*

**Corollary 34.** *If $\Omega$ is of class $C^1$ and has $\partial\Omega$ bounded, then $C^\infty(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$, $1 \leq p < \infty$.*

**Theorem 35** (Poincaré's Inequality)**.** *Let $\Omega$ be a bounded open set in $\mathbb{R}^n$. Then there exists a positive constant $C = C(\Omega, p)$ such that*

$$|u|_{0,p,\Omega} \leq C|u|_{1,p,\Omega} \quad \textit{for every } u \in W_0^{1,p}(\Omega).$$

*In particular, $u \to |u|_{1,p,\Omega}$ defines a norm on $W_0^{1,p}(\Omega)$, which is equivalent to the norm $||\cdot||_{1,p,\Omega}$. On $H_0^1(\Omega)$, the bilinear form*

$$(u, v) \mapsto \int_\Omega \sum_{i=1}^n \frac{\partial u}{\partial x_l} \frac{\partial v}{\partial x_i},$$

*defines an inner-product giving rise to the norm $|\cdot|_{1,\Omega}$, equivalent to the norm $||\cdot||_{l,\Omega}$.*

## Embedding Theorems

**Theorem 36.** *Let $m \geq 1$ be an integer and $1 \leq p < \infty$. Then*

- *if $\frac{1}{p} - \frac{m}{n} > 0$, $W^{m,p}(\Omega) \subset L^q(\Omega)$, $\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$,*
- *if $\frac{1}{p} - \frac{m}{n} = 0$, $W^{m,p}(\Omega) \subset L^q(\Omega)$, for $q \in [p, \infty)$,*
- *if $\frac{1}{p} - \frac{m}{n} < 0$, $W^{m,p}(\Omega) \subset L^\infty(\Omega)$,*

*and in the latter case, i.e. when $m > (n/p)$ if we set*

$$k = \left[m - \frac{n}{p}\right], \theta = \left(m - \frac{n}{p}\right) - k,$$

*$[\cdot]$, denoting the integral part of a real number, we have*

$$|D^\alpha u|_{0,\infty,\mathbb{R}^n} \leq C||u||_{m,p,\mathbb{R}^n} \quad \textit{for } |\alpha| \leq k$$

*and*

$$|D^\alpha u(x) - D^\alpha(y)| \leq C|x-y|^\theta ||u||_{m,p,\mathbb{R}^n} \ \textit{a.e.} \ (x,y),$$

*for* $|\alpha| = k$. *(If* $|\alpha| < k$, *the previous inequality holds with* $\theta = 1$ *by virtue of the previous inequality. In particular we have the continuous inclusion*

$$W^{m,p}(\Omega) \to C^k(\Omega), \ m > (n/p).$$

## Compactness Theorems

**Definition 37.** *Let* $1 \leq p < n$. *Then we define the exponent* $p^*$ *by*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n} \ \textit{or} \ p^* = \frac{np}{n-p}$$

**Theorem 38** (Rellich-Kondrasov)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set of class* $C^1$. *Then the following inclusions are compact.*

- *if* $p < n$, $W^{1,p}(\Omega) \to L^q(\Omega), \quad 1 \leq q < p^*$,
- *if* $p = n$, $W^{1,n}(\Omega) \to L^q(\Omega), \quad 1 \leq q < \infty$,
- *if* $p > n$, $W^{1,p}(\Omega) \to C(\bar{\Omega})$.

## Dual Spaces, Fractional Order Spaces and Trace Spaces

**Definition 39.** *Let* $1 \leq p < \infty$. *Let* $p'$ *be the conjugate exponent of* $p$. *The dual of the space* $W_0^{m,p}(\Omega)$ *where* $m \geq 1$ *is an integer, is denoted by* $W^{-m,p'}(\Omega)$. *If* $p = 2, H^{-m}(\Omega)$ *is the dual of the space* $H_0^m(\Omega)$.

**Theorem 40.** *Let* $F \in W^{-1,p'}(\Omega)$. *Then there exist functions* $f_0, f_1, ..., f_n \in L^{p'}(\Omega)$ *such that*

$$F(v) = \int_\Omega f_0 v + \sum_{i=1}^n \int_\Omega f_i \frac{\partial v}{\partial x_i}, \ v \in W_0^{1,p}(\Omega)$$

*and*

$$||F|| = \max_{0 \le i \le n} |f_i|_{0,p',\Omega}$$

*Further, if $\Omega$ is bounded, we assume $f_0 = 0$.*

## Trace Theory

**Theorem 41.** *(This was switched from $\mathbb{R}^n$ to $\Omega$) Let $s > 0$ be a real number. Then*

$$H^{-s}(\Omega) = \left\{ u \in \mathcal{S}'(\Omega) \,\middle|\, (1 + |\xi|^2)^{-s/2}\hat{u}(\xi) \in L^2(\Omega) \right\}$$

**Theorem 42.** *(This was switched from $\mathbb{R}^n$ to $\Omega$) Let $s_1 < s_2$ and $s = \theta s_1 + (1-\theta)s_2, \quad \theta \in (0,1)$. If $u \in H^{s_2}(\Omega)$, then*

$$||u||_{H^s(\Omega)} \le ||u||^{\theta}_{H^{s_1}(\Omega)} ||u||^{1-\theta}_{H^{s_2}(\Omega)}$$

**Remark 43.** *(This was switched from $\mathbb{R}^n$ to $\Omega$) Let $s_1 < s_2$ and $t_1 < t_2$. Let $\theta \in (0,1)$ and set $s = \theta s_1 + (1-\theta)s_2$, $t = \theta t_1 + (1-\theta)t_2$. Assume that $T$ is a linear operator such that*

$$T \in \mathscr{L}(H^{s_1}(\Omega), H^{t_1}(\Omega)) \cap \mathscr{L}(H^{s_2}(\Omega), H^{t_2}(\Omega))$$

*Then it is true that $T \in \mathscr{L}(H^s(\Omega), H^t(\Omega))$ and we also have*

$$||T||_{\mathscr{L}(H^s(\Omega),H^t(\Omega))} \le ||T||^{\theta}_{\mathscr{L}(H^{s_1}(\Omega),H^{t_1}(\Omega))} ||T||^{1-\theta}_{\mathscr{L}(H^{s_2}(\Omega),H^{t_2}(\Omega))}$$

**Definition 44.** *Let $V = (H^1(\Omega))^3$, where $\Omega \subset \mathbb{R}^3$ is a bounded open set of class $C^1$. If $v \in V$, let $v = (v_1, v_2, v_3)$ be its components. For $1 \le i, j \le 3$ we define*

$$\epsilon_{ij}(v) = \frac{1}{2}\left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

*We denote by $|| \cdot ||_V$ the usual product norm on $V$.*

**Theorem 45** (Korn's Inequality)**.** *Let $\Omega$ be a bounded open subset of $\mathbb{R}^3$, of class $C^1$. Then there* *exists a constant $C > 0$, depending only on $\Omega$, such that*

$$\int_\Omega \sum_{i,j=1}^{3} |\epsilon_{ij}(v)|^2 + \int_\Omega \sum_{i=1}^{3} |v_i|^2 \geq C||v||_V^2$$

*for every $v \in V$.*

**Theorem 46.** *(This was changed from $\mathbb{R}$ case to $\Omega$) Let $\Omega \subset \mathbb{R}^n$ be bounded. Then there exists a continuous linear map $\gamma_0 : H^1(\Omega) \to L^2(\mathbb{R}^{n-1})$ which is such that if $v$ is continuous on $\Omega$ then*

$$\gamma_0(v) = v|_{\mathbb{R}^{n-1}}$$

**Theorem 47.** *The range of the map $\gamma_0$ is the space $H^{1/2}(\mathbb{R}^{n-1})$.*

**Remark 48.** *(This was changed from $\mathbb{R}$ case to $\Omega$) Similarly we can prove that $\gamma_0$ maps $H^m(\Omega)$ onto $H^{m-1/2}(\mathbb{R}^{n-1})$. Also if $u \in H^2(\Omega)$ we can show that $\frac{\partial u}{\partial x_n}(x', 0)$ is in $L^2(\mathbb{R}^{n-1})$ and again $\frac{\partial u}{\partial x_n}(x', 0) \in H^{1/2}(\mathbb{R}^{n-1})$. We can then extend $-\frac{\partial u}{\partial x_n}(x', 0)$ to a continuous linear map $\gamma_1 : H^2(\Omega) \to L^2(\mathbb{R}^{n-1})$ whose range is $H^{1/2}(\mathbb{R}^{n-1})$. More generally we have a series of continuous linear maps $\{\gamma_i\}$ into $L^2(\mathbb{R}^{n-1})$. such that the map $\gamma = (\gamma_0, \gamma_1, ..., \gamma_{m-1})$ maps $H^m(\Omega)$ into $(L^2(\mathbb{R}^{n-1}))^m$ and the range in the space*

$$\prod_{j=0}^{m-1} H^{m-j-1/2}(\mathbb{R}^{n-1})$$

*Looking at the kernel of the map $\gamma_0$. If $u$ is continuous on $\bar{\Omega}$ and $u$ vanishes on $\Gamma$, then $u \in H_0^1(\Omega)$. A few steps not included here lead to $H_0^1(\Omega) = \ker(\gamma_0)$*

**Theorem 49** (Trace Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class $C^{m+1}$ with boundary $\Gamma$. Then there exists a trace map $\gamma = (\gamma_0, \gamma_1, ..., \gamma_{m-1})$ from $H^m(\Omega)$ into $(L^2(\Omega))^m$ such that*

- *If $v \in C^\infty(\bar{\Omega})$, then $\gamma_0(v) = v|_\Gamma$, $\gamma_1(v) = \frac{\partial v}{\partial \nu}|_\Gamma$,..., and $\gamma_{m-1}(v) = \frac{\partial^{m-1}}{\partial \nu^{m-1}}(v)|_\Gamma$, where $\nu$ is the unit exterior normal to the boundary $\Gamma$.*

- *The range of $\gamma$ is the space*

$$\prod_{j=0}^{m-1} H^{m-j-1/2}(\Gamma)$$

- *The kernel of $\gamma$ is $H_0^m(\Omega)$.*

**Theorem 50** (Green's Theorem, or, Green's Formula)**.** *Let $\Omega$ be a bounded open set of $\mathbb{R}^n$ set of class $C^1$ lying on the same side of its boundary $\Gamma$. Let $u, v \in H^1(\Omega)$. Then for $1 \leq i \leq n$,*

$$\int_\Omega u \frac{\partial v}{\partial x_i} = -\int_\Omega \frac{\partial u}{\partial x_i} v + \int_\Gamma (\gamma_0 u)(\gamma_0 v)\nu_i.$$

## $C_0$ Semigroups

**Definition 51.** *Let $X$ be a Banach space and $\{S(t)\}_{t\geq 0}$ be a family of bounded linear operators on $X$. It is said to be a $C_0$ semigroup if the following are true:*

- *$S(0) = I$, the identity of $X$*

- *$S(t+s) = S(t)S(s)$, for all $t, s \geq 0$*

- *For every $u \in X$*

$$S(t)u \to u \quad \textit{as } t \downarrow 0$$

**Theorem 52.** *Let $\{S(t)\}_{t\geq 0}$ be a $C_0$-semigroup on $X$. Then there exists $M \geq 1$ and $\omega$ such that*

$$||S(t)|| \leq Me^{\omega t}, \quad \textit{for all } t \geq 0$$

**Definition 53.** *If $M = 1$ and $\omega = 0$, so that $||S(t)|| \leq 1$ for all $t \geq 0$, we say that $\{S(t)\}$ is a* ***contraction semigroup****.*

**Definition 54.** *Let $\{S(t)\}_{t\geq 0}$ be a $C_0$ semigroup on $X$. The* ***infinitesimal generator*** *of the semigroup is a linear operator $A$ given by*

$$D(A) = \left\{ u \in X \mid \lim_{t\downarrow 0} \frac{S(t)u - u}{t} \textit{ exists} \right\}$$
$$Au = \lim_{t\downarrow 0} \frac{S(t)u - u}{t}, \; u \in D(A)$$

**Theorem 55.** *Let $\{S(t)\}_{t\geq 0}$ be a $C_0$ semigroup and let $A$ be its infinitesimal generator. Let $u \in D(A)$. Then*

$$S(t)u \in C^1([0,\infty);X) \cap C([0,\infty);X)$$

*and*

$$\frac{d}{dt}(S(t)u) = AS(t)u = S(t)Au$$

**Remark 56.** *If $A$ is the infinitesimal generator of a $C_0$ semigroup $\{S(t)\}$ then we know by the above theorem that*

$$u(t) = S(t)u_0$$

*defines the unique solution of the initial value problem*

$$\left.\begin{array}{ll} \frac{du(t)}{dt} & = Au(t),\ t \geq 0 \\ u(0) & = u_0 \end{array}\right\}$$

## Crucial Existence and Uniqueness Theorems

**Theorem 57** (Hille Yosida)**.** *A linear unbounded operator $A$ on a Banach space $X$ is the infinitesimal generator of a contraction semigroup if and only if*

- *$A$ is closed*
- *$A$ is densely defined*
- *For every $\lambda > 0$, $(\lambda I - A)^{-1}$ is a bounded linear operator and*

$$||(\lambda I - A)^{-1}|| \leq \frac{1}{\lambda}$$

**Theorem 58** (Lax-Milgram)**.** *Let $V$ be a Hilbert space and $a(\cdot,\cdot)$ a continuous $V$-elliptic bilinear* *form. Then given* $f \in V$, *there exists a unique* $u \in V$ *such that*

$$a(u, v) = (f, v), \quad \textit{for every } v \in V.$$

*If* $a(\cdot, \cdot)$ *is also symmetric then the functional* $J : V \to \mathbb{R}$ *defined by*

$$J(v) = \frac{1}{2} a(v, v) - (f, v)$$

*attains its minimum at* $u$.

**Theorem 59** ((Babuska-Brezzi))**.** *Let* $\Sigma$, $V$ *be Hilbert spaces and* $a : \Sigma \times \Sigma \to \mathbb{R}$, $b : \Sigma \times V \to \mathbb{R}$, *bilinear forms which are continuous. Let*

$$Z = \{\sigma \in \Sigma \mid b(\sigma, v) = 0, \quad \textit{for every } v \in V\}.$$

*Assume that* $a(\cdot, \cdot)$ *is* $Z$*-elliptic, i.e. there exists a constant* $\alpha > 0$ *such that*

$$a(\sigma, \sigma) \geq \alpha ||\sigma||_{\Sigma}^2, \quad \textit{for every } \sigma \in Z.$$

*Assume further that there exists a constant* $\beta > 0$ *such that*

$$\sup_{\tau \in \Sigma} \frac{b(\tau, v)}{||\tau||_{\Sigma}} \geq \beta ||v||_V.$$

*Then if* $\kappa \in \Sigma$ *and* $l \in V$, *there exists a unique pair* $(\sigma, u) \in \Sigma \times V$ *such that*

$$\begin{aligned} a(\sigma, \tau) + b(\tau, u) &= (\kappa, \tau), \quad \textit{for every } \tau \in \Sigma \\ b(\sigma, v) &= (l, v), \quad \textit{for every } v \in V. \end{aligned} \tag{6.1}$$

**Definition 60.** *Let* $X$ *be a Banach space with dual space* $X'$. *Denote* $x' \in X'$ *at* $x \in X$ *by* $\langle x', x \rangle$ *or* $\langle x, x' \rangle$. *Define the following set* $F(x) \subseteq X'$ *as*

$$F(x) = \left\{ x' \,\middle|\, \langle x', x \rangle = ||x||^2 = ||x'||^2 \right\}$$

*(This set is non-empty by the Hahn-Banach theorem.)*

**Definition 61** (Dissipativity). *A linear operator $A$ is dissipative if for every $x \in D(A)$ there is a $x' \in F(x)$ such that $Re\langle Ax, x' \rangle \leq 0$*

**Definition 62** (Maximal Dissipativity). *A linear operator $A$ is called maximally dissipative if it is dissipative and $R(I - A) = X$.*

**Theorem 63** (Lumer-Phillips). *If $A$ is dissipative and there is a $\lambda_0 > 0$ such that $R(\lambda_0 I - A) = X$, then $A$ is the infinitesimal generator of a $C_0$ semigroup of contractions on $X$. If $A$ is the infinitesimal generator of a $C_0$ semigroup of contractions on $X$ then $R(\lambda I - A) = X$ for all $\lambda > 0$ and $A$ is dissipative.*

## Operator and Spectrum Definitions

**Definition 64.**

- *A linear operator $A : D(A) \subseteq X \to Y$ is said to be **bounded** if there exists a $C > 0$ such that*

  $$||Au||_Y \leq C||u||_X, \quad \textit{for every } u \in D(A)$$

  *Otherwise it is said to be **unbounded**.*

- *A linear operator $A : D(A) \subseteq X \to Y$ is said to be **densely defined** if $\overline{D(A)} = X$*

- *A linear operator $A : D(A) \subseteq X \to Y$ is said to be **closed** if the **graph***

  $$G(A) = \{(u, Au) \mid u \in D(A)\} \subseteq X \times Y$$

  *is closed as a subspace of $X \times Y$*

**Definition 65.** *Let $X \neq \{0\}$ be a complex normed space and $T : \mathscr{D}(T) \subseteq X \to X$ be a linear operator. With $T$ we associate the operator*

$$T_\lambda = T - \lambda I$$

*where $\lambda$ is a complex number and $I$ is the identity operator on $\mathscr{D}(T)$. If $T_\lambda$ has an inverse, we* *denote it by $R_\lambda(T)$ and call it the resolvent operator of $T$ or, simply, the* ***resolvent*** *of $T$. If it is clear which operator we are discussing, we will write $R_\lambda$.*

**Definition 66** (Regular value, resolvent set, spectrum)**.** *Let $X \neq \{0\}$ be a complex normed space and $T : \mathscr{D}(T) \subseteq X \to X$ be a linear operator. A regular value $\lambda$ of $T$ is a complex number such that*

- *$R_\lambda(T)$ exists,*
- *$R_\lambda(T)$ is bounded,*
- *$R_\lambda(T)$ is densely defined.*

*The resolvent set $\rho(T)$ of $T$ is the set of all regular values $\lambda$ of $T$. Its complement $\sigma(T) = \mathbb{C}\backslash\rho(T)$ in the complex plane $\mathbb{C}$ is called the spectrum of $T$, and a $\lambda \in \sigma(T)$ is called a spectral value of $T$. Furthermore, the spectrum $\sigma(T)$ is partitioned into three disjoint sets as follows.*

*The* ***point spectrum*** *or discrete spectrum $\sigma_p(T)$ is the set such that $R_\lambda(T)$ does not exist. A $\lambda \in \sigma_p(T)$ is called an eigenvalue of $T$.*

*The* ***continuous spectrum*** *$\sigma_c(T)$ is the set such that $R_\lambda(T)$ exists and is densely defined, but it unbounded.*

*The* ***residual spectrum*** *$\sigma_r(T)$ is the set such that $R_\lambda(T)$ exists, but is not densely defined (may or may not be bounded).*

# Bibliography


[1] W. Arendt and C. J. K. Batty, Tauberian theorems and stability of one-parameter semigroups, *Trans. Amer. Math. Soc.* 306, (1988) pp. 837–852.

[2] G. Avalos and M. Dvorak A new maximality argument for a coupled fluid-structure interaction with implications for a divergence-free finite element method, *Applicationes Mathematicae,* Vol. 35 (3), (2008), 259-280

[3] G. Avalos; The strong stability and instability of a fluid-structure semigroup, *Appl, Math. Optim.* (55) (2007), pp. 163-184.

[4] G. Avalos and R. Triggiani; Boundary feedback stabilization of a coupled parabolic-hyperbolic Stokes-Lam-e PDE system, *Journal of Evolution Equations* (9) (2009), pp. 341-370.

[5] G. Avalos and R. Triggiani, The Coupled PDE System Arising in Fluid-Structure Interaction, Part I: Explicit Semigroup Generator and its Spectral Properties, *Contemporary Mathematics*, Volume 440, (2007), pp. 15-54.

[6] George Avalos, Irena Lasiecka, and Roberto Triggiani. Heat-wave interaction in 2–3 dimensions: optimal rational decay rate. *J. Math. Anal. Appl.*, 437(2):782–815, 2016.

[7] G. Avalos and R. Triggiani. Semigroup well-posedness in the energy space of a parabolic-hyperbolic coupled Stokes-Lamé PDE system of fluid-structure interaction. *Discrete Contin. Dyn. Syst. Ser. S*, 2(3): (2009), pp. 417–447.

[8] G. Avalos, P. G. Geredeli, B. Muha; "Wellposedness, Spectral Analysis and Asymptotic Stability of a Multilayered Heat-Wave-Wave System ", *Journal of Differential Equations* 269 (2020), pp. 7129-7156.

[9] G. Avalos, P. G. Geredeli; "A Resolvent Criterion Approach to Strong Decay of a Multilayered Lamé-Heat System", *Applicable Analysis,* https://doi.org/10.1080/00036811.2021.1954165 (2021).

[10] G. Avalos, P. G. Geredeli and B. Muha; Rational Decay of A Multilayered Structure-Fluid PDE System, *Journal of Mathematical Analysis and Applications 514 (2), https://doi.org/10.1016/j.jmaa.2022.126284* (2022)

[11] V. Barbu, Z. Grujić, I. Lasiecka and A. Tuffaha, "Weak and strong solutions in nonlinear fluid-structure interactions", *Contemporary Mathematics* 440: Fluids and Waves, American Mathematical Society, Providence Rhode Island (2007), pp. 55-82.

[12] Y. Bazilevs, M-C. Hsu, Y. Zhang, W. Wang, T. Kvamsdal, S. Hentschel, J. Isaksen; "Computational fluid-structure interaction: methods and application to cerebral aneurysms.", *Biomech Model Machanobiol* 9 (2010), pp. 481-498.

[13] A. Benadallah, I. Lasiecka; Exponential Decay Rates for a Full von Karman System of Dynamic Thermoelasticity, *Journal of Differential Equations* 160 (2000), pp. 51-93.

[14] C.D. Benchimol; A note on weak stability of contraction semigroups, *SIAM J. Control and Optimization,* Vol. 16 (3), May 1978, pp. 373-379

[15] T. Bodnar, G. P. Galdi, and S. Necasova, editors. *Fluid-Structure Interaction and Biomedical Applications.* Birkhäuser/Springer, Basel, 2014.

[16] F. Brezzi and M. Fortin; (1991) *Mixed and Hybrid Finite Element Methods,* Springer-Verlag Newyork Inc

[17] M. Bukac, S. Canic, and B. Muha. A partitioned scheme for fluid-composite structure interaction problems. *Journal of Computational Physics*, 281(0): (2015) pp. 493-517.

[18] M. Bukac, S. Canic, B. Muha; "A nonlinear fluid-structure interaction problem in compliant arteries treated with vascular stents,"*Applied Mathematics and Optimization,* 73 (3), (2016), pp. 433-473.

[19] A. Chambolle, B. Desjardins, M.J. Esteban and C. Grandmont, Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate, *J. Math. Fluid Mech* 7, no. 3, (2005), pp. 368-404.

[20] G. Chen, A note on the boundary stabilization of the wave equation, *SIAM J. Control Optim.* 19: 106–113, 1981.

[21] Philippe G. Ciarlet. *Mathematical elasticity. Vol. I*: Three-Dimensional Elasticity North-Holland Publishing Co., Amsterdam, 1988.

[22] P. G. Ciarlet, *Numerical Analysis of the Finite Element Method*, *Séminaire de Mathématique Supérieures*, Les Presses de L'Université de Montréal (1976).

[23] D. Coutand and S. Shkoller, "Motion of an elastic solid inside an incompressible viscous fluid", *Arch. Ration. Mech. Anal.* 176 (2005), pp. 25-102.

[24] Geredeli, P.G. An inf-sup approach to $C_0$-semigroup generation for an interactive composite structure-Stokes PDE dynamics. J. Evol. Equ. 24, 50 (2024). https://doi.org/10.1007/s00028-024-00978-3

[25] Pelin G. Geredeli, Spectral analysis and asymptotic decay of the solutions to multilayered structure-Stokes fluid interaction PDE system, Journal of Differential Equations, Volume 427, 2025, Pages 1-25, ISSN 0022-0396, https://doi.org/10.1016/j.jde.2025.01.080.

[26] Q. Du, M. D. Gunzburger, L. S. Hou and J. Lee, "Analysis of a linear fluid-structure interaction problem", *Discrete and Continuous Dynamical Systems*, Volume 9, No. 3 (May 2003), pp. 633-650.

[27] George Avalos, Pelin G. Geredeli, Uniform stability for solutions of a structural acoustics PDE model with no added dissipative feedback, Math. Methods Appl. Sci. 39 (18) (2016) 5497-5512

[28] B. Friedman, Principles and Techniques of Applied Mathematics, Dover Publications, Inc., New York, 1990.

[29] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman Publishing Inc., Boston 1985.

[30] M. A. Horn; Implications of Sharp Trace Regularity Results on Boundary Stabilization of the System of Linear Elasticity, *Journal of Mathematical Analysis and Applications* 223, (1998), pp. 126-150.

[31] M-C. Hsu, Y. Bazilevs; "Blood vessel tissue prestress modeling for vascular fluid–structure interaction simulation" *Finite Elements in Analysis and Design,* 47, (2011), pp. 593-599.

[32] S. Kesavan, Topics in Functional Analysis and Applications, New Age International Limited Publishers, New Delhi (1989).

[33] Amon Pazy, Semigroups of linear operators and applications to partial differential equations, Vol. 44, Springer Science & Business Media (2012).

[34] Erwin Kreyszig, Introductory functional analysis with applications, John Wiley & Sons (1991).

[35] H. Koch and E. Zuazua. A hybrid system of PDE's arising in multi-structure interaction: coupling of wave equations in $n$ and $n-1$ space dimensions. In *Recent trends in partial differential equations*, volume 409 of *Contemp. Math.*, pages 55–77. Amer. Math. Soc., Providence, RI, 2006.

[36] Jacques-Louis and Lions. *Quelques méthodes de résolution des problemes aux limites non linéaires*, volume 31. Dunod Paris, 1969.

[37] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, NewYork (2000).

[38] B. Muha. A note on optimal regularity and regularizing effects of point mass coupling for a heat-wave system. *Journal of Mathematical Analysis and Applications*, 425 (2), 2015, pp. 1134-1147.

[39] B. Muha and S. Čanić. Existence of a solution to a fluid–multi-layered-structure interaction problem. *J. Differential Equations*, 256(2), (2014), pp. 658–706.

[40] J. Nečas, Direct Methods in the Theory of Elliptic Equations, Springer, New York, 2012 (translated by Gerard Tronel and Alois Kufner).

[41] J. Rauch, X. Zhang, and E. Zuazua. Polynomial decay for a hyperbolic-parabolic coupled system. *J. Math. Pures Appl. (9)*, 84(4), (2005), pp. 407-470.

[42] R. Temam; *Navier Stokes Equations, Theory and Numerical Analysis*, AMS Chelsea Publishing, Providence, R.I (2001).

[43] R. Triggiani, Wave equation on a bounded domain with boundary dissipation: an operator approach, *J. Math. Anal. Appl.* 137: 438-461, 1989.

[44] Xu Zhang and Enrique Zuazua. Long-time behavior of a coupled heat-wave system arising in fluid-structure interaction. *Arch. Ration. Mech. Anal.*, 184(1):49–120, 2007.

[45] Y. Tomilov, A resolvent approach to stability of operator semigroups, *J. Operator Theory,* 46,(2001), pp. 63–98.

[46] S. Micu and E. Zuazua, "Boundary controllability of a linear hybrid system arising in the control of noise ", SIAM J. Control Optim. , 35(1997), no.5, 1614-1637.