HW4

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1. (15 points) For some fixed $a > 0$, compute the following.

   (a) The Fourier transform of $f(x) = e^{-a|x|}$ where $x \in \mathbb{R}$.

   (b) The Fourier transform of $f(x) = e^{-a|x|}$ where $x \in \mathbb{R}^n$.

   (Hint: For part (b), you can use the result from part (a), the trivial identity $\int_0^\infty e^{-(a^2+\xi^2)s}ds = \frac{1}{a^2+\xi^2}$, and the Fourier transform of $e^{-b|x|^2}$ we discussed during lectures to derive an integral representation of $e^{-a|x|}$ in terms of $e^{-c|x|^2}$ for some c, and then continue from there.)

a) Note $f(x) = f(-x)$ so

$$\hat{f}(\xi) = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-ax} \cos \xi x \, dx$$

$$\begin{array}{ccc} u & & v \\ \cos \xi x & + & e^{-ax} \\ -\xi \sin \xi x & - & -\frac{1}{a} e^{-ax} \\ -\xi^2 \cos \xi x & & \frac{1}{a^2} e^{-ax} \end{array} \qquad \int u\,dv = uv - \int v\,du$$

$$\int_0^\infty e^{-ax} \cos \xi x \, dx = -\frac{1}{a} \cos \xi x e^{-ax} \Big|_0^\infty + \frac{\xi}{a^2} \sin \xi x e^{-ax} \Big|_0^\infty + \frac{\xi^2}{a^2} \int_0^\infty \cos \xi x e^{-ax} \, dx$$

$$\left(1 + \frac{\xi^2}{a^2}\right) \int_0^\infty e^{-ax} \cos \xi x \, dx = \frac{1}{a} + 0$$

$$\int_0^\infty e^{-ax} \cos \xi x \, dx = \frac{1}{\cancel{a}} \frac{a^{\cancel{2}}}{\xi^2 + a^2} = \frac{a}{\xi^2 + a^2}$$

$$\hat{f}(\xi) = \sqrt{\frac{2}{\pi}} \frac{a}{\xi^2 + a^2}$$

b)

$$\left.\begin{array}{l} \widehat{e^{-c|x|^2}} = (2c)^{-n/2} e^{-\frac{|\xi|^2}{4c}} \qquad \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix^T \xi} dx \end{array}\right\} \text{Hints}$$

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-(a|x| + ix^T\xi)} dx$$

Let $r = |x| \Rightarrow dx = r^{n-1} dr\, dS^{n-1}$ $\qquad x^T \xi = |x||\xi| \cos\theta$

Let $\xi = |\xi| v$

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_0^\infty \int_{S^{n-1}} e^{-(ar + i|x||\xi|\cos\theta)} r^{n-1} dr\, dS^{n-1}$$

$$= (2\pi)^{-n/2} \int_0^\infty (2\pi)^{n/2} \frac{J_{\frac{n-2}{2}}(r|\xi|)}{(r|\xi|)^{\frac{n-2}{2}}} e^{-ar} r^{n-1} dr$$

$$= \int_0^\infty e^{-ar} \frac{J_{\frac{n-2}{2}}(r|\xi|) \, r^{-\frac{n-2}{2}+1}}{|\xi|^{\frac{n-2}{2}}} dr$$

$$= \frac{2^{\frac{n+1}{2}}}{\sqrt{\pi} (a^2 + |\xi|^2)^{\frac{n+1}{2}}} \Gamma\left(\frac{n+1}{2}\right)$$

J is the Bessel function

These are known integrals from integral tables

$\sqrt{\pi}(a^2+|\xi|^2)^{\frac{n+1}{2}}$ ... These are known integrals from integral tables I found

2. (15 points) Let $K(x, y, t)$ be the heat kernel given as

$$K(x,y,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}, \quad x, y \in \mathbb{R}^n, t > 0$$

(a) Show that $K(x, y, t) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times (0, \infty))$, and show it satisfies the heat equation

$$\frac{\partial K(x,y,t)}{\partial t} - \Delta_x K(x,y,t) = 0.$$

(b) Show that for every $x \in \mathbb{R}^n$ and $t > 0$,

$$\int_{\mathbb{R}^n} K(x,y,t)dy = 1.$$

a) $K(x,y,t) = (4\pi t)^{-n/2} e^{\frac{-|x-y|^2}{4t}}$ $\quad \Phi(z,t) = (4\pi t)^{-n/2} e^{\frac{-|z|^2}{4t}}$

$x - y = z$.

Let $1 \le i \le j \le n$ $\quad \frac{\partial \Phi}{\partial z_i} = \frac{-z_i}{2t}(4\pi t)^{-n/2} e^{\frac{-|z|^2}{4t}} \in C(\mathbb{R}^n)$

$$\frac{\partial^2 \Phi}{\partial z_i \partial z_j} = \frac{z_i z_j}{4t^2}(4\pi t)^{-n/2} e^{\frac{-|z|^2}{4t}} \in C(\mathbb{R}^n)$$

In general $D^\alpha \Phi = a_\alpha(z) e^{\frac{-|z|^2}{4t}}$ where a_α is a function of polynomials in z. As the product of continuous functions, $D^\alpha \Phi$ is continuous.

$$\frac{\partial}{\partial z_i} D^\alpha \Phi = -\frac{z_i}{2t} a_\alpha(z) e^{-\frac{|x|^2}{4t}} + a_\alpha'(z) e^{\frac{-|x|^2}{4t}} \in C(\mathbb{R}^n).$$

Thus by induction $\Phi \in C^\infty(\mathbb{R}^n)$ in space.

Let $c = (4\pi)^{-n/2}$

$$\frac{\partial \Phi}{\partial t} = -\frac{n}{2} c \frac{1}{t^{n/2+1}} e^{\frac{-|z|^2}{4t}} + \frac{c|z|^2}{4t^{n/2+2}} e^{\frac{-|z|^2}{4t}}$$

$$-\frac{n}{2t}\Phi + \frac{|z|^2}{4t^2}\Phi \quad \in C(\mathbb{R}_+ \setminus \{0\})$$

Similarly to the spacial variables, we will have a polynomial in $1/t$ for higher derivatives. Thus by the same space arguement, $\Phi \in C^\infty(\mathbb{R}_+ \setminus \{0\})$ in time.

$$\frac{\partial \Phi}{\partial z_i} = \frac{-z_i}{2t}(4\pi t)^{-n/2} e^{\frac{-|z|^2}{4t}} = -\frac{z_i}{2t}\Phi$$

$$\frac{\partial^2 \Phi}{\partial z_i^2} = \frac{-z_i}{2t}\left(\frac{-z_i}{2t}\Phi\right) + \frac{-1}{2t}\Phi$$

$$\frac{\partial^2 \Phi}{\partial z_i^2} = \frac{-z_i}{2t}\left(\frac{-z_i}{2t}\Phi\right) + \frac{-1}{2t}\Phi$$
$$= \left(\frac{|z_i|^2}{4t^2} - \frac{1}{2t}\right)\Phi$$

$$\frac{\partial \Phi}{\partial t} - \Delta_x \Phi = \left(\frac{n}{2t} - \frac{|z|^2}{4t^2}\right)\Phi - \sum_1^n \frac{\partial^2 \Phi}{\partial z_i^2}$$
$$= \left(\frac{-n}{2t} + \frac{|z|^2}{4t^2}\right)\Phi - \sum_1^n \left(\frac{|z_i|^2}{4t^2} - \frac{1}{2t}\right)\Phi$$
$$= 0$$

b) WTS $\int_{\mathbb{R}^n} \Phi(z,t)\,dz = 1 \quad t>0 \quad z \in \mathbb{R}^n$

$$(4\pi t)^{-n/2}\int_{\mathbb{R}^n} e^{\frac{-|z|^2}{4t}}\,dz = (4\pi t)^{-n/2}\int_{-\infty}^{\infty} e^{\frac{-z_1^2}{4t}}\,dz_1 \int_{\mathbb{R}^{n-1}} e^{\frac{-|z'|^2}{4t}}\,dz'$$

$(z' = (z_2, \ldots, z_n))$

$$= (4\pi t)^{-(n-1)/2}\int_{\mathbb{R}^{n-1}} e^{\frac{-|z'|^2}{4t}}\,dz'$$

note $\int_{-\infty}^{\infty} e^{\frac{-x^2}{4t}}\,dx = \sqrt{4\pi t}$

Integral table 5

$$\ldots$$
$$= 1$$

3. (20 points) Suppose $g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and set

$$u(x,t) = \int_{\mathbb{R}^n} K(x,y,t)g(y)dy$$

where $K(x,y,t)$ denotes the heat kernel as defined in problem 2. Show that for all $t > 0$:

(a) $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-n/2}\|g\|_{L^1(\mathbb{R}^n)}$

(b) $\|u(\cdot,t)\|_{L^1(\mathbb{R}^n)} \leq \|g\|_{L^1(\mathbb{R}^n)}$

(c) $\|u_{x_i}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-1/2}\|g\|_{L^\infty(\mathbb{R}^n)}$.

(d) $\|u_{x_i}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{n+1}{2}}\|g\|_{L^1(\mathbb{R}^n)}$

a) $|u(x,t)| = |\int K\, g\, dy| \leq (4\pi t)^{-n/2}\int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4t}}|g|\,dy$

$\sup e^{\frac{-|x-y|^2}{4t}} = 1$ when $x = y$

$$\leq C t^{-n/2}\|g\|_{L^1}$$

Thus $\|u(x,t)\|_{L^\infty} \leq C t^{-n/2}\|g\|_{L^1}$

b) $\|u(x,t)\|_{L^1} = \int_{\mathbb{R}^n}|u(x,t)|\,dx = \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} K(x,y,t)g(y)\,dy\right|dx$

$$\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y,t)\,|g(y)|\,dy\,dx$$

$$\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y,t)\,|g(y)|\,dy\,dx$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(z,t)\,dz\,|g(y)|\,dy$$
$$= \|g\|_{L^1}$$

c) $$\frac{\partial u}{\partial x_i} = (4\pi t)^{-n/2}\int_{\mathbb{R}^n} \frac{\partial}{\partial x_i} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy$$
$$= \frac{1}{2t}(4\pi t)^{-n/2}\int_{\mathbb{R}} (x_i-y_i) e^{-\frac{(x_i-y_i)^2}{4t}} g(y)\int_{\mathbb{R}^{n-1}} e^{-\frac{|x'-y'|^2}{4t}}\,dy\,dy'$$
$$\|u_{x_i}(x,t)\|_{L^\infty} \le \frac{1}{2t}(4\pi t)^{-n/2}\int_{\mathbb{R}} |x_i-y_i| e^{-\frac{|x_i-y_i|^2}{4t}} \|g\|_\infty \int_{\mathbb{R}^{n-1}} e^{-\frac{|x'-y'|^2}{4t}}\,dy\,dy'$$
$$= (2t)^{-1}(4\pi t)^{-n/2}\int_{\mathbb{R}^{n-1}} e^{-\frac{|z'|^2}{4t}}\,dz'\int_{\mathbb{R}}|z_i| e^{-\frac{|z_i|^2}{4t}}\,dz_i\,\|g\|_\infty$$

Note $\int_{\mathbb{R}} |x| e^{-\frac{|x|^2}{t}}\,dx = 2\int_0^\infty x e^{-\frac{x^2}{4t}}\,dx = 4t$

By symmetry & Integral tables

$$\|u_{x_i}(x,t)\|_\infty \le (2t)^{-1}(4\pi t)^{-n/2}\, 4t\, (4\pi t)^{\frac{n-1}{2}}\|g\|_\infty$$
$$= C t^{-1/2}\|g\|_\infty$$

d) $$|u_{x_i}(x,t)| \le \frac{1}{2t}(4\pi t)^{-n/2}\int_{\mathbb{R}^n}|x-y|\, e^{-\frac{|x-y|^2}{4t}}|g(y)|\,dy \qquad |x_i-y_i| \le |x-y|$$
$$\le \frac{1}{2t}(4\pi t)^{-n/2}\int_{\mathbb{R}^n}|z| e^{-\frac{|z|^2}{4t}}|g(x-z)|\,dz$$

Consider $h(r) = r e^{-\frac{r^2}{4t}} \qquad r = |z|$

$$h'(r) = e^{-\frac{r^2}{4t}} - r\cdot\frac{2r}{4t}e^{-\frac{r^2}{4t}} = e^{-\frac{r^2}{4t}}\left(1-\frac{r^2}{2t}\right)$$

$h'(r) = 0$ @ $r = \sqrt{2t}$

$h(\sqrt{2t}) = \sqrt{2t}\, e^{-1/2}$

$$|u_{x_i}(x,t)| \le \frac{\sqrt{2t}}{2t}e^{-1/2}(4\pi t)^{-n/2}\int_{\mathbb{R}^n}|g(x-z)|\,dz$$
$$= C t^{-\left(\frac{n+1}{2}\right)}\|g\|_{L^1}$$

4. (10 points) For a fixed constant $c \in \mathbb{R}$, find an explicit solution for

$$u_t - \Delta u + cu = f \quad (x,t) \in \mathbb{R}^n \times (0,\infty),$$
$$u(x,0) = g(x) \quad x \in \mathbb{R}^n.$$

Let $u = u_1 + u_2 \ \ni$

u_1: $\begin{cases} u_t - \Delta u + cu = 0 & x\in\mathbb{R}^n\ t>0 \\ u(x,0) = g(x) \end{cases}$ u_2: $\begin{cases} u_t - \Delta u + cu = f & x\in\mathbb{R}^n\ t>0 \\ u(x,0) = 0 \end{cases}$

$u_1)\ \begin{cases} u_t - \Delta u + cu = 0 & x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = g(x) \end{cases}$ $\quad u_2)\ \begin{cases} u_t - \Delta u + cu = f & x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = 0 \end{cases}$

Let $\hat{u} = F(u)$. $\quad \hat{g} = F(g)$

$u_1)$ $\frac{d}{dt}\hat{u} - i^2|\xi|^2\hat{u} + c\hat{u} = 0$

$$\frac{d}{dt}\hat{u} + (|\xi|^2 + c)\hat{u} = 0$$

$$\Rightarrow \hat{u}(\xi,t) = C e^{-(|\xi|^2 + c)t}$$

$$\hat{u}(\xi,0) = \hat{g}(\xi) = C$$

$$\hat{u}(\xi,t) = \hat{g}(\xi) e^{-(|\xi|^2 + c)t}$$

$$F^{-1}(\hat{u})(x,t) = e^{-ct}\,\Phi(x,t) * g(x)$$

$$= e^{-ct}\int_{\mathbb{R}^n} (4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy$$

$u_2)$ By the Duhamel principle

$u_2 = \int_0^t w(x, t-s)\,ds$ where

w solves $\begin{cases} w_t - \Delta w + cw = 0 & x \in \mathbb{R}^n \quad t > s \\ w(x,0) = f(x,0) & x \in \mathbb{R}^n \end{cases}$

Thus $u_2 = \int_0^t e^{-c(t-s)} \int_{\mathbb{R}^n} (4\pi(t-s))^{-n/2} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,0)\,dy\,ds$

$$\therefore\ u(x,t) = e^{-ct}\int_{\mathbb{R}^n} (4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy + \int_0^t e^{-c(t-s)} \int_{\mathbb{R}^n} (4\pi(t-s))^{-n/2} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,0)\,dy\,ds$$