Math 8260 Partial Differential Equations HW6

Assigned on March 27, due on April 9 Wednesday by the beginning of the lecture

1. (15 points) (a) Show that there exists at most one $C^2(\overline{U})$ solution to the boundary value problem:

$$\begin{cases} -\Delta u + u = f & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases} \tag{1}$$

   Hint: Energy method.

   (b) Show that the result in (a) does not hold if the PDE is replaced by $-\Delta u - u = f$. Hint: Construct a counter-example on $U = (0, 2\pi)$.

2. (20 points) (Evans 2.5.5 in the new version, or 2.5.4 in the old digital version) We say $v \in C^2(\bar{U})$ is subharmonic if $-\Delta v \leq 0$ in U.

   (a) Prove for subharmonic v that

$$v(x) \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} v(y)dy \quad \text{ for all } B_r(x) \subset U.$$

   (b) Prove that therefore $\max_{\bar{U}} v = \max_{\partial U} v$.

   (c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

   (d) Prove $v := |\nabla u|^2$ is subharmonic whenever u is harmonic.

3. (15 points) (Evans 2.5.6 in the new version, or 2.5.5 in the old digital version) Let U be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on U, such that

$$\max_{\bar{U}} |u| \leq C \left(\max_{\partial U} |g| + \max_{\bar{U}} |f| \right)$$

   whenever u is a smooth solution of

$$\begin{aligned} -\Delta u &= f \text{ in } U \\ u &= g \text{ on } \partial U. \end{aligned}$$

   (Hint: first show that$-\Delta \left(u + \frac{|x|^2}{2n} \lambda \right) \leq 0$ for $\lambda := \max_{\bar{U}} |f|$, then use this fact to continue the proof.)

4. (10 points) Find the Green's function for Laplace's equation on the quarter plane $U = \mathbb{R}_+ \times \mathbb{R}_+ = (0, \infty) \times (0, \infty)$. Use your Green's function to solve Laplace's equation

$$\begin{aligned} \Delta u &= 0 \text{ in } U \\ u &= g \text{ on } \partial U. \end{aligned}$$

   You need NOT prove that $u(x)$ defined this way is a solution.