

## HW1

Saturday, January 25, 2025 2:07 PM

1. (10 points) Determine whether the following PDEs are linear, semilinear, quasilinear, or fully nonlinear, and briefly explain the reason.

1.  $\sin(u_x)u_y + \cos(u_y)u_{xx} = 0$ .
2.  $u_x^3 + e^x u_{yy} = \cos(u)$ .
3.  $u_t + x^2 u_x = 0$ .
4.  $u_x u_t + u_x u_y = 0$ .

a) quasilinear

b) semilinear

c) linear

d) fully nonlinear

2. (10 points) (Evans 2.5.1.) Find an explicit formula for a function  $u$  solving the initial-value problem

$$\begin{aligned} u_t + \vec{b} \cdot \nabla u + cu &= 0 && \text{in } \mathbb{R}^n \times (0, \infty) \\ u &= g && \text{on } \mathbb{R}^n \times \{t = 0\}. \end{aligned}$$

Here  $c \in \mathbb{R}$  and  $\vec{b} \in \mathbb{R}^n$  are constants. (Please write down full details).

$$u_t + b \cdot \nabla u + cu = 0$$

$u_t + b \cdot \nabla u = -cu \Rightarrow$  derivatives are related by  $-c$  to  $u$

Fix  $(x, t)$  consider  $z(s) = e^{cs} u(x+sb, t+s)$

$$z'(s) = \frac{d}{ds} (e^{cs} u(x+sb, t+s)) = c e^{cs} u(x+sb, t+s) + (u_t + b \cdot \nabla u) e^{cs} = 0$$

Since  $e^{cs} \neq 0$  and  $u_t + b \cdot \nabla u + cu = 0$ , we see

$z(s)$  is constant in  $s$

$$z(0) = z(-t)$$

$$\begin{aligned} u(x, t) &= e^{-ct} u(x-bt, 0) \\ &= e^{-ct} g(x-bt) \end{aligned}$$

3. (20 points) Solve the following:

(a) Solve the quarter-plane problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}; \quad (x, t) \in (0, \infty) \times (0, \infty) \\ u_x(0, t) &= 0; \quad t \geq 0 \\ u(x, 0) &= g(x); \quad x \geq 0 \\ u_t(x, 0) &= h(x); \quad x \geq 0, \end{aligned}$$

with  $g_x(0) = h_x(0) = 0$ .

(b) Consider the equation from Part (a) with  $c = 2$ ,  $h(x) = 0$ , and

$$g(x) = \begin{cases} x - 2 & 2 \leq x \leq 3 \\ 4 - x & 3 < x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Sketch graphs of  $u(x, 0)$ ,  $u(x, 1)$ , and  $u(x, 2)$ .

(c) Solve the equation from Part (a) with  $c = 2$ ,  $g(x) = 0$  and  $h(x) = \frac{1}{x^2+1}$ . Then sketch a graph of  $u(x, 1)$ .

$$a) \tilde{u}(x, t) = \begin{cases} u(x, t) & x > 0 \\ u(-x, t) & x < 0 \end{cases} \quad \tilde{g}(x) = \begin{cases} g(x) & x > 0 \\ g(-x) & x < 0 \end{cases}$$

$$\tilde{h}(x) = \begin{cases} h(x) & x > 0 \\ h(-x) & x < 0 \end{cases}$$

$$\begin{cases} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0 \\ \tilde{u}(x, 0) = \tilde{g}(x) \\ \tilde{u}_t(x, 0) = \tilde{h}(x) \end{cases} \quad \underline{x - ct > 0} \quad \tilde{u}(x, t) = \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2} \int_{x-ct}^{x+ct} h(y) dy$$

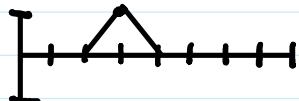
$x - ct < 0$

$$\tilde{u}(x, t) = \frac{1}{2}(g(x+ct) + g(ct-x)) + \frac{1}{2} \int_{x-ct}^0 h(-y) dy + \frac{1}{2} \int_0^{x+ct} h(y) dy$$

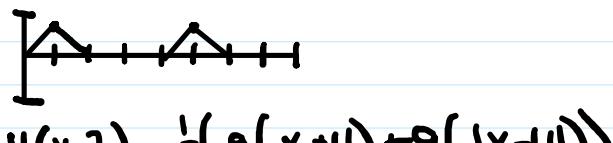
$$\tilde{u}|_{x \geq 0} = u$$

$$\tilde{u}(x, t) = \frac{1}{2}(g(x+ct) + g(|x-ct|)) + \frac{1}{2} \int_{|x-ct|}^{x+ct} h(y) dy$$

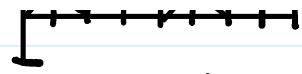
$$b) u(x, 0) = \frac{1}{2}(g(x) + g(|x|)) = g(x)$$



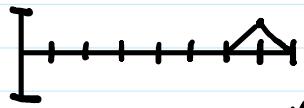
$$u(x, 1) = \frac{1}{2}(g(x+2) + g(|x-2|))$$



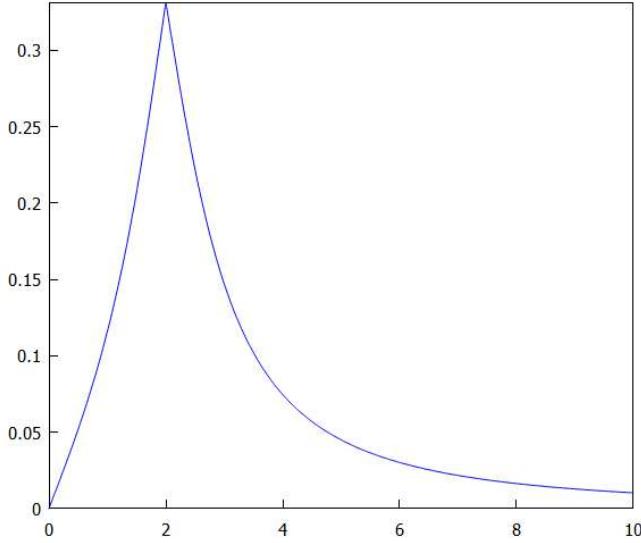
$$g(x) = \begin{cases} x - 2 & 2 \leq x \leq 3 \\ 4 - x & 3 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$



$$u(x, 2) = \frac{1}{2}(g(x+4) + g(|x-4|))$$



$$c) u(x, t) = \frac{1}{4} \int_{|x-2t|}^{x+2t} \frac{1}{y^2+1} dy$$



4. (10 points) (Evans 2.5.21)

(a) Assume  $E = (E^1, E^2, E^3)$  and  $B = (B^1, B^2, B^3)$  solve Maxwell's equations

$$\begin{aligned}\vec{E}_t &= \nabla \times \vec{B} \\ \vec{B}_t &= -\nabla \times \vec{E} \\ \nabla \cdot \vec{E} &= 0 \\ \nabla \cdot \vec{B} &= 0.\end{aligned}$$

Show

$$\vec{E}_{tt} - \Delta \vec{E} = 0, \quad \vec{B}_{tt} - \Delta \vec{B} = 0.$$

(b) Assume that  $\vec{u} = (u^1, u^2, u^3)$  solves the evolution equations of linear elasticity

$$\vec{u}_{tt} - \mu \Delta \vec{u} - (\lambda + \mu) D(\operatorname{div} \vec{u}) = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty).$$

Show  $w := \operatorname{div} \vec{u}$  and  $\vec{w} := \operatorname{curl} \vec{u}$  each solve wave equations, but with different speeds of propagation.

(Hint: you may find that the vector calculus identities page on Wikipedia is very useful)

$$\begin{aligned}a) \quad \vec{E}_{tt} &= \partial_t \nabla \times \vec{B} = \nabla \times \vec{B}_t = \nabla \times (-\nabla \times \vec{E}) \\ &= -\nabla(\nabla \cdot \vec{E}) + \Delta \vec{E} \\ &= \Delta \vec{E}\end{aligned} \quad (\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \Delta \vec{A})$$

$$\therefore \vec{E}_{tt} - \Delta \vec{E} = 0$$

Similarly,

$$\begin{aligned} \mathbf{B}_{tt} &= -\partial_t \nabla \times \mathbf{E} = -\nabla \times \mathbf{E}_t = -\nabla \times \nabla \times \mathbf{B} \\ &= -\nabla(\nabla \cdot \mathbf{B}) + \Delta \mathbf{B} \\ &= \Delta \mathbf{B} \end{aligned}$$

$$\therefore \mathbf{B}_{tt} - \Delta \mathbf{B} = 0$$

b)  $\mathbf{u}$  solves

$$u_{tt} - \mu \Delta u - (\lambda + \mu) \nabla(\nabla \cdot \mathbf{u}) = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty)$$

$$\omega = \nabla \cdot \mathbf{u}$$

$$\Rightarrow w_{tt} - \mu \nabla \cdot \Delta \mathbf{u} - (\lambda + \mu) \nabla \cdot \nabla \cdot \mathbf{w} = 0$$

$$w_{tt} - \mu \Delta w - (\lambda + \mu) \Delta w = 0$$

$$w_{tt} - (\lambda + 2\mu) \Delta w = 0$$

$$(\Delta(\nabla \cdot \mathbf{A}) = \nabla \cdot \Delta \mathbf{A})$$

$$\Rightarrow c^2 = \lambda + 2\mu$$

$$\omega = \nabla \times \mathbf{u}$$

$$w_{tt} - \mu \nabla \times \Delta \mathbf{u} - (\lambda + \mu) \nabla \times \nabla(\nabla \cdot \mathbf{u}) = 0$$

$$w_{tt} - \mu \Delta w - (\lambda + \mu) \nabla \times (\Delta \mathbf{u} + \nabla \times \mathbf{w}) = 0 \quad (\nabla \times \Delta \mathbf{A} = \Delta(\nabla \times \mathbf{A}))$$

$$w_{tt} - \mu \Delta w - (\lambda + \mu)(\Delta w + \nabla \times \nabla \times \mathbf{w}) = 0 \quad (\nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A} = \Delta \mathbf{A})$$

$$w_{tt} - \mu \Delta w - (\lambda + \mu)(\Delta w + \nabla(\nabla \cdot \mathbf{w}) - \Delta \mathbf{w}) = 0$$

$$w_{tt} - \mu \Delta w = 0$$

$$\Rightarrow c^2 = \mu$$

$\lambda + 2\mu \neq \mu$  so different speeds