

HW1

Saturday, January 25, 2025 2:07 PM

1. (10 points) Determine whether the following PDEs are linear, semilinear, quasilinear, or fully nonlinear, and briefly explain the reason.

1. $\sin(u_x)u_y + \cos(u_y)u_{xx} = 0$.

2. $u_x^3 + e^x u_{yy} = \cos(u)$.

3. $u_t + x^2 u_x = 0$.

4. $u_x u_t + u_x u_y = 0$.

a) quasilinear

b) semilinear

c) linear

d) fully nonlinear

2. (10 points) (Evans 2.5.1.) Find an explicit formula for a function u solving the initial-value problem

$$\begin{aligned} u_t + \vec{b} \cdot \nabla u + cu &= 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u &= g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{aligned}$$

Here $c \in \mathbb{R}$ and $\vec{b} \in \mathbb{R}^n$ are constants. (Please write down full details).

$$u_t + b \cdot \nabla u + cu = 0$$

$u_t + b \cdot \nabla u = -cu \Rightarrow$ derivatives are related by $-c$ to u

Fix (x, t) consider $z(s) = e^{cs} u(x + sb, t + s)$

$$z'(s) = \frac{d}{ds} (e^{cs} u(x + sb, t + s)) = c e^{cs} u(x + sb, t + s) + (u_t + b \cdot \nabla u) e^{cs} = 0$$

Since $e^{cs} \neq 0$ and $u_t + b \cdot \nabla u + cu = 0$, we see

$z(s)$ is constant in s

$$z(0) = z(-t)$$

$$\begin{aligned} u(x, t) &= e^{-ct} u(x - bt, 0) \\ &= e^{-ct} g(x - bt) \end{aligned}$$

3. (20 points) Solve the following:

(a) Solve the quarter-plane problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}; & (x, t) &\in (0, \infty) \times (0, \infty) \\ u_x(0, t) &= 0; & t &\geq 0 \\ u(x, 0) &= g(x); & x &\geq 0 \\ u_t(x, 0) &= h(x); & x &\geq 0, \end{aligned}$$

with $g_x(0) = h_x(0) = 0$.

(b) Consider the equation from Part (a) with $c = 2$, $h(x) = 0$, and

$$g(x) = \begin{cases} x-2 & 2 \leq x \leq 3 \\ 4-x & 3 < x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Sketch graphs of $u(x, 0)$, $u(x, 1)$, and $u(x, 2)$.

(c) Solve the equation from Part (a) with $c = 2$, $g(x) = 0$ and $h(x) = \frac{1}{x^2+1}$. Then sketch a graph of $u(x, 1)$.

$$a) \quad \tilde{u}(x, t) = \begin{cases} u(x, t) & x > 0 \\ u(-x, t) & x < 0 \end{cases} \quad \tilde{g}(x) = \begin{cases} g(x) & x > 0 \\ g(-x) & x < 0 \end{cases}$$

$$\tilde{h}(x) = \begin{cases} h(x) & x > 0 \\ h(-x) & x < 0 \end{cases}$$

$$\begin{cases} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0 \\ \tilde{u}(x, 0) = \tilde{g}(x) \\ \tilde{u}_t(x, 0) = \tilde{h}(x) \end{cases}$$

$$\underline{x - ct > 0}$$

$$\tilde{u}(x, t) = \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} h(y) dy$$

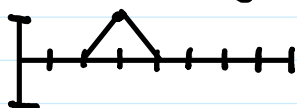
$$\underline{x - ct < 0}$$

$$\tilde{u}(x, t) = \frac{1}{2}(g(x+ct) + g(ct-x)) + \frac{1}{2c} \int_{x-ct}^0 h(-y) dy + \frac{1}{2c} \int_0^{x+ct} h(y) dy$$

$$\tilde{u}|_{x=0} = u$$

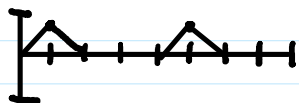
$$u(x, t) = \frac{1}{2}(g(x+ct) + g(|x-ct|)) + \frac{1}{2c} \int_{|x-ct|}^{x+ct} h(y) dy$$

$$b) \quad u(x, 0) = \frac{1}{2}(g(x) + g(|x|)) = g(x)$$

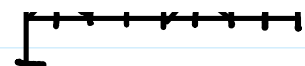


$$g(x) = \begin{cases} x-2 & 2 \leq x \leq 3 \\ 4-x & 3 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

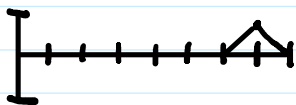
$$u(x, 1) = \frac{1}{2}(g(x+2) + g(|x-2|))$$



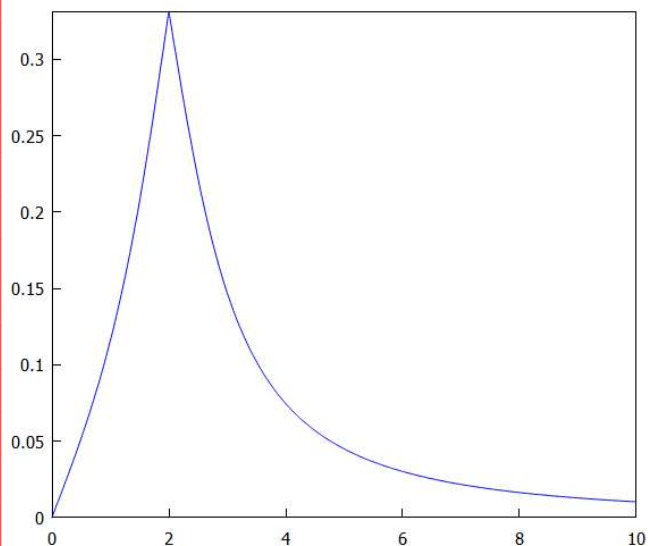
$$u(x, 2) = \frac{1}{2}(g(x+4) + g(|x-4|))$$



$$u(x,2) = \frac{1}{2}(g(x+4) + g(x-4))$$



$$c) u(x,t) = \frac{1}{4} \int_{|x-2|}^{x+2} \frac{1}{y^2+1} dy$$



4. (10 points) (Evans 2.5.21)

(a) Assume $E = (E^1, E^2, E^3)$ and $B = (B^1, B^2, B^3)$ solve Maxwell's equations

$$\begin{aligned}\vec{E}_t &= \nabla \times \vec{B} \\ \vec{B}_t &= -\nabla \times \vec{E} \\ \nabla \cdot \vec{E} &= 0 \\ \nabla \cdot \vec{B} &= 0.\end{aligned}$$

Show

$$\vec{E}_{tt} - \Delta \vec{E} = 0, \quad \vec{B}_{tt} - \Delta \vec{B} = 0.$$

(b) Assume that $\vec{u} = (u^1, u^2, u^3)$ solves the evolution equations of linear elasticity

$$\vec{u}_{tt} - \mu \Delta \vec{u} - (\lambda + \mu) D(\operatorname{div} \vec{u}) = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty).$$

Show $w := \operatorname{div} \vec{u}$ and $\vec{w} := \operatorname{curl} \vec{u}$ each solve wave equations, but with different speeds of propagation.

(Hint: you may find that the vector calculus identities page on Wikipedia is very useful)

$$\begin{aligned}a) \quad E_{tt} &= \partial_t \nabla \times B = \nabla \times B_t = \nabla \times (-\nabla \times E) \\ &= -\nabla(\nabla \cdot E) + \Delta E \\ &= \Delta E\end{aligned} \quad (\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \Delta A)$$

$$\therefore E_{tt} - \Delta E = 0$$

Similarly,

$$\begin{aligned} B_{tt} &= -\partial_t \nabla \times E = -\nabla \times E_t = -\nabla \times \nabla \times B \\ &= -\nabla(\nabla \cdot B) + \Delta B \\ &= \Delta B \end{aligned}$$

$$\therefore B_{tt} - \Delta B = 0$$

b) u solves

$$u_{tt} - \mu \Delta u - (\lambda + \mu) \nabla(\nabla \cdot u) = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty)$$

$$w = \nabla \cdot u$$

$$\Rightarrow w_{tt} - \mu \nabla \cdot \Delta u - (\lambda + \mu) \nabla \cdot \nabla w = 0$$

$$w_{tt} - \mu \Delta w - (\lambda + \mu) \Delta w = 0$$

$$w_{tt} - (\lambda + 2\mu) \Delta w = 0$$

$$(\Delta(\nabla \cdot A) = \nabla \cdot \Delta A)$$

$$\Rightarrow c^2 = \lambda + 2\mu$$

$$w = \nabla \times u$$

$$w_{tt} - \mu \nabla \times \Delta u - (\lambda + \mu) \nabla \times \nabla(\nabla \cdot u) = 0$$

$$w_{tt} - \mu \Delta w - (\lambda + \mu) \nabla \times (\Delta u + \nabla \times w) = 0 \quad (\nabla \times \Delta A = \Delta(\nabla \times A))$$

$$w_{tt} - \mu \Delta w - (\lambda + \mu) (\Delta w + \nabla \times \nabla \times w) = 0 \quad (\nabla(\nabla \cdot A) - \nabla \times \nabla \times A = \Delta A)$$

$$w_{tt} - \mu \Delta w - (\lambda + \mu) (\cancel{\Delta w} + \nabla(\cancel{\nabla \cdot w}) - \Delta w) = 0$$

$$w_{tt} - \mu \Delta w = 0$$

$$\Rightarrow c^2 = \mu$$

$\lambda + 2\mu \neq \mu$ so different speeds