

1. (20 points) We define the convolution of two functions $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$f * g(x) := \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

Establish the following properties of convolution:

(a) $f, g \in L^2(\mathbb{R}^n) \Rightarrow \|f * g\|_{L^\infty} \leq \|f\|_{L^2} \|g\|_{L^2}.$

(b) $f, g \in L^1(\mathbb{R}^n) \Rightarrow \|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}.$

(c) $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \Rightarrow f * g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n).$

(d) (bonus problem, additional 5 points) If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then $\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$

$$\begin{aligned} a) |f * g|^2 &\leq \left(\int_{\mathbb{R}^n} |f(x-y)| |g(y)| dy \right)^2 \\ &\leq \int_{\mathbb{R}^n} |f(x-y)|^2 dy \int_{\mathbb{R}^n} |g(y)|^2 dy \quad \text{C-S} \\ &= \|f\|_{L^2(\mathbb{R}^n)}^2 \|g\|_{L^2(\mathbb{R}^n)}^2 \end{aligned}$$

$$\therefore \|f * g\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}$$

$$\begin{aligned} b) \|f * g\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y)g(y)dy \right| dx \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)| |g(y)| dy dx \\ &= \int_{\mathbb{R}^n} |g(y)| \int_{\mathbb{R}^n} |f(x-y)| dx dy \\ &= \int_{\mathbb{R}^n} |g(y)| \|f\|_{L^1(\mathbb{R}^n)} dy \\ &= \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)} \end{aligned}$$

c) From a & b $f * g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$
WTS $f * g \in C(\mathbb{R}^n)$

Since $f, g \in L^1(\mathbb{R}^n)$, $\exists \{f_k\}, \{g_k\} \subseteq C_c(\mathbb{R}^n)$

$\exists f_k \rightarrow f$ and $g_k \rightarrow g.$

Consider $|f * g(x) - f_k * g_k(x)| \leq \left(\int_{\mathbb{R}^n} |(f - f_k)(x-y)| |(g - g_k)(y)| dy \right)^2$

$$|f * g(x) - f_k * g_k(x)| \leq \left(\int_{\mathbb{R}^n} |(f - f_k)(x-y)| |(g - g_k)(y)| dy \right)^2$$

$$\leq \|f - f_k\|_{L^1(\mathbb{R}^n)}^2 \|g - g_k\|_{L^1(\mathbb{R}^n)}^2 \quad C-S$$

$\rightarrow 0$

Thus $f_k * g_k \rightarrow f * g$ uniformly.

$\therefore f * g \in C(\mathbb{R}^n)$ since $f_k * g_k \in C_c(\mathbb{R}^n)$ as the convolution of continuous functions is continuous

d) Note $\frac{1}{r} + \frac{r-p}{r} + \frac{r-q}{r} = \frac{1}{r} + \frac{1}{p} - \frac{1}{p} + \frac{1}{q} - \frac{1}{q} = 1$

$$|f * g| \leq \int_{\mathbb{R}^n} |f(x-y)| |g(y)| dy$$

$$= \int_{\mathbb{R}^n} |f(x-y)|^{1+\frac{p}{r-p}} |g(y)|^{1+\frac{q}{r-q}} dy$$

$$\leq \left(\int_{\mathbb{R}^n} |f(x-y)|^p |g(y)|^q dy \right)^{1/r} \left(\int_{\mathbb{R}^n} |f(x-y)|^{\frac{r-p}{r}} |g(y)|^{\frac{r-q}{r}} dy \right)^{r-p/r} \quad \text{Hölders}$$

$$\left(\int_{\mathbb{R}^n} |g(y)|^{\frac{r-q}{r}} dy \right)^{r-p/r}$$

$$= \left(\int_{\mathbb{R}^n} |f(x-y)|^p |g(y)|^q dy \right)^{1/r} \left(\int_{\mathbb{R}^n} |f(x-y)|^p dy \right)^{\frac{r-p}{r}} \left(\int_{\mathbb{R}^n} |g(y)|^q dy \right)^{\frac{r-q}{r}}$$

$$= \left(\int_{\mathbb{R}^n} |f(x-y)|^p |g(y)|^q dy \right)^{1/r} \|f\|_p^{\frac{r-p}{r}} \|g\|_q^{\frac{r-q}{r}}$$

$$\|f * g\|_r^r \leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)|^p |g(y)|^q dy \right)^{1/r} \|f\|_p^{\frac{r-p}{r}} \|g\|_q^{\frac{r-q}{r}} dx$$

$$= \|f\|_p^{r-p} \|g\|_q^{r-q} \int_{\mathbb{R}^n} |g(y)|^q \int_{\mathbb{R}^n} |f(x-y)|^p dx dy$$

$$= \|f\|_p^{r-p} \|g\|_q^{r-q} \|g\|_q^q \|f\|_p^p$$

$$= \|f\|_p^r \|g\|_q^r$$

$$\therefore \|f * g\|_r \leq \|f\|_p \|g\|_q$$

2. (20 points) Let $B_r(0)$ denote the open ball of radius r in \mathbb{R}^n , and denote by

$$|B_r(0)| := \int_{B_r(0)} 1 dx, \quad |\partial B_r(0)| := \int_{\partial B_r(0)} 1 dS.$$

Show the following.

(a) $|B_r(0)| = |B_1(0)| r^n$ and $|\partial B_r(0)| = |\partial B_1(0)| r^{n-1}$.

(b) $|\partial B_r(0)| = \frac{n}{r} |B_r(0)|$.

(c) $|\partial B_1(0)| = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$, where Γ denotes the gamma function $\Gamma(k) = \int_0^\infty t^{k-1} e^{-t} dt$. (Hint. Use the integral equality $\int_{\mathbb{R}^n} e^{-|x|^2} dx = \pi^{n/2}$, and evaluate the integral in polar coordinates.)

(d) Show that $\frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} = \frac{n\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$, and then show that $|\partial B_1(0)| = n\alpha(n)$, where $\alpha(n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$.

2. (20 points) Let $B_r(0)$ denote the open ball of radius r in \mathbb{R}^n , and denote by

$$|B_r(0)| := \int_{B_r(0)} 1 dx, \quad |\partial B_r(0)| := \int_{\partial B_r(0)} 1 dS.$$

Show the following.

(a) $|B_r(0)| = |B_1(0)| r^n$ and $|\partial B_r(0)| = |\partial B_1(0)| r^{n-1}$.

(b) $|\partial B_r(0)| = \frac{n}{r} |B_r(0)|$.

(c) $|B_1(0)| = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$, where Γ denotes the gamma function $\Gamma(k) = \int_0^\infty t^{k-1} e^{-t} dt$. (Hint. Use the integral equality $\int_{\mathbb{R}^n} e^{-|x|^2} dx = \pi^{n/2}$, and evaluate the integral in polar coordinates.)

(d) Show that $\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$, and then show that $|\partial B_1(0)| = n\alpha(n)$, where $\alpha(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

a) $|B_r(0)| = \int_0^r \int_0^r \dots \int_0^r 1 ds_1 ds_2 \dots ds_n$

$$= r \int_0^r \int_0^r \dots \int_0^r 1 ds_1 ds_2 \dots ds_n$$

$$\vdots$$

$$= r^n \int_0^1 \dots \int_0^1 1 ds_1 \dots ds_n$$

$$= r^n |B_1(0)|$$

$$|\partial B_r(0)| = n r^{n-1} |B_1(0)| \Rightarrow |\partial B_1(0)| = n |B_1(0)|$$

$$= \frac{n}{r} r^n |B_1(0)|$$

$$= r^{n-1} |\partial B_1(0)|$$

b) $|\partial B_r(0)| = r^{n-1} |\partial B_1(0)| = \frac{n}{r} r^n |B_1(0)| = \frac{n}{r} |B_r(0)|$

c) $\int_{\mathbb{R}^n} e^{-|x|^2} dx = \int_0^\infty \int_{\partial B_r(0)} e^{-r^2} r^{n-1} dS dr \quad r = |x|$

$$= \int_0^\infty e^{-r^2} r^{n-1} dr |\partial B_1(0)|$$

$$= \frac{1}{2} \int_0^\infty e^{-u} u^{n/2-1} du |\partial B_1(0)|$$

$$= \frac{|\partial B_1(0)|}{2} \Gamma\left(\frac{n}{2}\right) = \pi^{n/2}$$

$$\therefore |\partial B_1(0)| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

d) $\Gamma(\alpha+1) = \alpha \Gamma(\alpha)$ so

$$\frac{2\pi^{n/2}}{\Gamma(n/2)} = \frac{\pi^{n/2}}{\frac{1}{n} \Gamma(n/2)} = \frac{n\pi^{n/2}}{\Gamma(n/2+1)}$$

$$r^2 = u$$

$$du = 2r dr$$

$$\frac{r^{n-1}}{2r} = \frac{u^{(n-2)/2}}{2}$$

$$\frac{2\pi^{n/2}}{\Gamma(n/2)} = \frac{\pi^{n/2}}{\frac{1}{2}\Gamma(n/2)} = \frac{\pi^{n/2}}{\frac{1}{n}\Gamma(n/2+1)} = \frac{n\pi^{n/2}}{\Gamma(n/2+1)}$$

Thus

$$|\partial B_r(0)| = \frac{n\pi^{n/2}}{\Gamma(n/2+1)} = n\alpha(n)$$

3. (20 points) (Evans 2.5.3.) Modify the proof of the mean value formulas to show for $n \geq 3$ that

$$u(0) = \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} g dS + \frac{1}{n(n-2)\alpha(n)} \int_{B_r(0)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f dx,$$

provided

$$-\Delta u = f \text{ in } B_r(0)$$

$$u = g \text{ in } \partial B_r(0).$$

Here we assume $f \in C(\overline{B_r(0)})$ (the closed ball), and $g \in C(\partial B_r(0))$.

$$\text{Let } \phi(s) = \int_{\partial B_s(0)} u(y) dS_y = \int_{\partial B_s(0)} u(x+sz) dS_z \quad y = sz$$

$$\begin{aligned} \phi'(s) &= \int_{\partial B_s(0)} \nabla u(s z) \cdot z dS_z = \int_{\partial B_s(0)} \nabla u(y) \cdot \frac{y}{s} dS_y \\ &= \int_{\partial B_s(0)} \nabla u(y) \cdot \nu dS_y \\ &\stackrel{\text{2b) \& div thrm}}{=} \frac{s}{n} \int_{B_s(0)} \Delta u(y) dy \end{aligned}$$

Let $\varepsilon > 0 \ni \frac{s^n}{\varepsilon^n} < M < \infty$ for fixed s

$$\begin{aligned} \phi(r) - \phi(\varepsilon) &= \int_{\varepsilon}^r \phi'(s) ds = \int_{\varepsilon}^r \frac{s}{n} \int_{B_s(0)} \Delta u(y) dy ds \\ &= \frac{1}{n\alpha(n)} \int_{\varepsilon}^r \frac{1}{s^{n-1}} \int_{B_s(0)} f(y) dy ds \\ &= \frac{1}{n\alpha(n)} \left(\frac{-1}{n-2} \frac{1}{s^{n-2}} \int_{B_s(0)} f(y) dy \right) \Big|_{\varepsilon}^r \\ &\quad + \int_{\varepsilon}^r \frac{1}{s^{n-1}} \int_{\partial B_s(0)} f(y) dy ds \\ &= \frac{1}{n(n-2)\alpha(n)} \left(\frac{-1}{r^{n-2}} + \frac{1}{\varepsilon^{n-2}} \right) \int_{B_s(0)} f(y) dy \\ &\quad + \frac{1}{n\alpha(n)} \int_{\varepsilon}^r \int_{\partial B_s(0)} \frac{f(y)}{s^{n-1}} dy ds \end{aligned}$$

$$\frac{1}{\varepsilon^{n-2}} \int_{B_s(0)} f(y) dy \leq \frac{\|f\|_{\infty} |\partial B_s(0)|}{\varepsilon^{n-2}} = \frac{\|f\|_{\infty} \pi^{n/2} s^n}{\Gamma(n/2+1) \varepsilon^{n-2}} < \frac{\|f\|_{\infty} \pi^{n/2} M}{\Gamma(n/2+1)} \varepsilon^2 \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

Thus

Thus

$$\phi(r) \rightarrow \phi(0) + \frac{1}{n(n-2)\omega(n)} \left(\frac{1}{r^{n-2}} \int_{\partial B_r(0)} f(y) dy - \frac{1}{|x|^{n-2}} \int_{\partial B_r(0)} f(y) dy \right)$$

as $r \rightarrow 0$

$$\phi(r) = \int_{\partial B_r(0)} u(y) dS_y = \int_{\partial B_r(0)} u(y) dS_y + \frac{1}{n(n-2)\omega(n)} \left(\frac{1}{r^{n-2}} \int_{\partial B_r(0)} f(y) dy - \frac{1}{|x|^{n-2}} \int_{\partial B_r(0)} f(y) dy \right)$$

$$\therefore \int_{\partial B_r(0)} u(y) dS_y = u(0) = \int_{\partial B_r(0)} g dS_y + \frac{1}{n(n-2)\omega(n)} \int_{\partial B_r(0)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f(x) dx$$