Math 8260 Partial Differential Equations HW3

Assigned on Feb 10th, due on Feb 19th Wednesday by the beginning of the lecture

1. (10 points) Fix some $x_0 \in \mathbb{R}^n$ and $t_0 > 0$. For each $s \in [0, t_0]$ denote by

$$\Omega_s = \{(x,t) : 0 \le t \le s, |x - x_0| \le c(t_0 - t)\}$$

as well as its side boundary

$$\mathcal{S}_{side} = \{(x,t) : 0 \le t \le s, |x - x_0| = c(t_0 - t)\}.$$

Let $\nu = (\nu_1, \dots, \nu_n, \nu_{n+1})$ be the unit normal vector on $\mathcal{S}_{side}$. Show that $\nu_{n+1}^2 = c^2(\nu_1^2 + \cdots + \nu_n^2)$. (This is an exercise left during the lecture.)

2. (15 points) Let u solve the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = g, \ u_t = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{1}$$

Suppose g, h have compact support. The kinetic energy is $k(t) := \frac{1}{2} \int_{\mathbb{R}} u_t^2(x,t)\, dx$ and the potential energy is $p(t) := \frac{1}{2} \int_{\mathbb{R}} u_x^2(x,t)\, dx$. Prove $k(t) = p(t)$ for all large enough time t.

3. (15 points) Use energy method to show the following (nonlinear) wave equation only has a zero solution

$$\begin{cases} u_{tt} - \Delta u + u + u^3 = 0 & x \in \Omega, 0 < t < T; \\ u(x,0) = u_t(x,0) = 0 & x \in \Omega; \\ u(x,t) = 0 & x \in \partial\Omega, 0 < t < T \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is an open smooth bounded domain. (Hint: Recall that for our original wave equation $u_{tt} - \Delta u = 0$, the energy is defined as $\frac{1}{2} \int_\Omega u_t^2 + |\nabla u|^2 dx$. Think about why the energy is defined in that way, and here you need to construct a new suitable energy $E(t)$ for our nonlinear wave equation and prove it is zero at $t = 0$ and $\frac{d}{dt} E(t) = 0$.)

4. (20 points) Let $Q = \Omega \times (0, T]$ and $u \in C^{2,1}(Q) \cap C(\bar{Q})$ satisfy

$$u_t - \Delta u + cu \leqslant 0 \quad \text{in } Q$$

where $c \geqslant 0$ is a constant.

(a) If $u \geqslant 0$, show that the weak maximum principle holds for u, i.e., u achieves its maximum at the parabolic boundary of Q. (Hint: the proof is similar to the one we did during the lecture.)

(b) Give a counterexample to show the weak maximum principle may not hold without the condition $u \geqslant 0$. (Hint: consider $\Omega = (-1, 1)$ and consider $u = -x^2 + at - b$ with some suitable a and b.)