HW6

Thursday, April 3, 2025 1:19 PM

1. (15 points) (a) Show that there exists at most one $C^2(\overline{U})$ solution to the boundary value problem:

$$\begin{cases} -\Delta u + u = f & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases} \tag{1}$$

Hint: Energy method.

(b) Show that the result in (a) does not hold if the PDE is replaced by $-\Delta u - u = f$. Hint: Construct a counter-example on $U = (0, 2\pi)$.

a) Let $u_1, u_2 \in C^2(\bar{U})$ be solutions to (1). Define $w = u_1 - u_2$. w satisfies

$$\begin{cases} -\Delta w + w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$$

The energy Functional associated with w is as follows

$$E_w = \int_U (-\Delta w + w) w \, dx = \int_U -w\Delta w + w^2 \, dx$$

$$= \int_U \nabla w \cdot \nabla w + w^2 dx - \int_{\partial U} \overset{0}{\cancel{w}} \Delta w \, dx$$

$$= \int_U |\nabla w|^2 + w^2 dx$$

We know $E_w = 0$ since $-\Delta w + w = 0$.

Since $\int_U |\nabla w|^2 dx \geq 0$ and $\int_U w^2 dx \geq 0$, it follows $w = 0$. Thus $u_1 = u_2$

b) Let $u_1 = \cos\theta$ Consider $u_2 = \cos\theta + \sin\theta$

u_1 satisfies

$$\begin{cases} -u_1'' - u_1 = 0 & \text{on } (0, 2\pi) \\ u_1 = 1 & \text{in } \{0, 2\pi\} \end{cases}$$

u_2 satisfies

$$\begin{cases} -u_2'' - u_2 = 0 & \text{on } (0, 2\pi) \\ u_2 = 1 & \text{in } \{0, 2\pi\} \end{cases}$$

2. (20 points) (Evans 2.5.5 in the new version, or 2.5.4 in the old digital version) We say $v \in C^2(\bar{U})$ is subharmonic if $-\Delta v \leq 0$ in U.

(a) Prove for subharmonic v that

$$v(x) \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} v(y) dy \quad \text{for all } B_r(x) \subset U.$$

(b) Prove that therefore $\max_{\bar{U}} v = \max_{\partial U} v$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

(d) Prove $v := |\nabla u|^2$ is subharmonic whenever u is harmonic.

a) Consider $⨍_{\partial B_r(0)} v(y) dS_y = ⨍_{\partial B_1(0)} v(x + zr) dS_z$.

It follows $\frac{d}{dr} ⨍_{\partial B_1(0)} v(x + zr) dS_z = ⨍_{\partial B_1(0)} v'(x+zr) \cdot z \, dS_z$

$$= ⨍_{\partial B_r(0)} v'(y) \cdot \frac{y-x}{r} dS_y$$

$$= ⨍_{\partial B_r(0)} \frac{\partial v}{\partial \nu} dS_y$$

$$= ⨍_{B_r(0)} \Delta v \, dy$$

≥ 0 since $\Delta v \geq 0$

divergence thm

Thus $⨍_{\partial B_r(0)} v(y) dS_y$ is non decreasing wrt r

We see $v(x) = \lim_{t \to 0} ⨍_{\partial B_t(0)} v(y) dS_y \leq ⨍_{\partial B_r(x)} v(y) dS_y$

Therefore $|B_r(x)| v(x) \leq \int_0^r ⨍_{\partial \ldots} v(y) dS_y \, ds$

$$v(x) = \lim_{t\to 0} ⨍_{\partial B_t(x)} v(y)\,dS_y \le ⨍_{\partial B_r(x)} v(y)\,dS_y$$

Therefore
$$|B_r(x)|\,v(x) \le \int_0^r \int_{\partial B_s(x)} v(y)\,dS_y\,ds$$
$$v(x) \le \frac{1}{|B_r(x)|}\int_{B_r(x)} v(y)\,dy$$

b) Let $\max v = M$. Consider $S = \{x \in \bar{U} \mid v(x) = M\}$

Let $x_0 \in S$. It follows
$$M = v(x_0) \le ⨍_{B_r(x_0)} v(y)\,dy \le ⨍_{B_r(x_0)} M\,dy = M.$$

Thus $M = ⨍_{B_r(x_0)} v(y)\,dy \Rightarrow ⨍_{B_r(x_0)} M - v(y)\,dy = 0$

Since $v(y) \le M \;\forall y \in B_r(x_0)$, $0 \le M - v(y)$ and $⨍_{B_r(x_0)} M - v(y)\,dy = 0 \Rightarrow v(y) = M \;\forall y \in B_r(x_0)$.

Thus $y \in S$

Now consider $x' \in \bar{U} \setminus S$. Since $v \in C^2(\bar{U})$, $\exists B_r(x') \subseteq \bar{U}\setminus S \Rightarrow \bar{U}\setminus S$ is open.

Thus S is both open and closed $\Rightarrow S = \emptyset$ or $S = \bar{U}$.

If $S = \emptyset \Rightarrow v(x) < M \;\forall x \in \bar{U}$ which is impossible by the existance of supremum of a set.

If $S = \bar{U}$ then v is a constant function $\therefore \max_{\bar{U}} v = \max_{\partial U} v$

If v is not a constant function, then we have reached a contradiction and thus
$$\max_{\bar{U}} v = \max_{\partial U} v$$

c)
$$\begin{aligned}
-\Delta \phi(u) = -\nabla\cdot\nabla\phi(u) &= -\nabla\cdot\{\phi'(u)u_{x_i}\}_{i=1}^n \\
&= -\sum_{i=1}^n \phi'(u)u_{x_ix_i} + \phi''(u)u_{x_i}^2 \\
\text{Since } \Delta u = 0 \quad &= -\phi'(u)\underbrace{\Delta u}_{0} - \sum_{i=1}^n \phi''(u)u_{x_i}^2 \\
&\le 0 \quad \text{since } u_{x_i}^2 \ge 0 \;\forall i \text{ and } \phi''(u) \ge 0 \text{ by convexity}
\end{aligned}$$

Thus v is subharmonic

d) WTS $|\nabla\cdot|^2$ is convex $\Rightarrow v$ is subharmonic when u is harmonic by part c). Note $(\Delta u)_{x_i} = \Delta(u_{x_i})$ so ∇u is also harmonic

Let $\lambda \in [0,1]$ $\quad f, g \in C^2(\bar{U})$
$$\begin{aligned}
|\nabla(\lambda f + (1-\lambda)g)|^2 = |\lambda\nabla f + (1-\lambda)\nabla g|^2 &\le |\lambda\nabla f|^2 + |(1-\lambda)\nabla g|^2 \\
&= \lambda^2|\nabla f|^2 + (1-\lambda)^2|\nabla g|^2 \\
&\le \lambda|\nabla f|^2 + (1-\lambda)|\nabla g|^2 \quad \text{since } \lambda^2 \le \lambda \text{ and } (1-\lambda)^2 \le (1-\lambda)
\end{aligned}$$

Thus $|\nabla\cdot|^2$ is convex.

3. (15 points) (Evans 2.5.6 in the new version, or 2.5.5 in the old digital version) Let U be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on U, such that

$$\max_{\bar{U}} |u| \le C \left(\max_{\partial U} |g| + \max_{\bar{U}} |f| \right)$$

whenever u is a smooth solution of

$$-\Delta u = f \text{ in } U$$
$$u = g \text{ on } \partial U.$$

(Hint: first show that $-\Delta \left(u + \frac{|x|^2}{2n} \lambda \right) \le 0$ for $\lambda := \max_{\bar{U}} |f|$, then use this fact to continue the proof.)

Let $v = u + \frac{|x|^2}{2n}\lambda$ Note $\Delta \sum_{i=1}^{n} x_i^2 = 2n$

$$-\Delta v = -\Delta\left(u + \frac{|x|^2}{2n}\lambda\right) = f - \lambda \le 0$$

Thus it follows (2b)

$$u \le v \le \max_{\bar{U}} v \overset{\downarrow}{=} \max_{\partial U} v \le \max_{\partial U} u + \max_{\partial U} \frac{|x|^2}{2n} \max_{\bar{U}} |f|$$

(since $\frac{|x|^2}{2n}\lambda \ge 0$)

$$\le \max_{\partial U} |g| + \max_{\bar{U}} \frac{R^2}{2n} |f|$$

Where R is chosen such that $|x|^2 \le R^2$. This can be chosen because U is bdd.

Let $C = \max\{1, R^2/2n\}$.

It follows $u \le C\left(\max_{\partial U}|g| + \max_{\bar{U}}|f|\right)$

Similarly, if $\bar{v} = -u + \frac{|x|^2}{2n}\lambda$, $-\Delta \bar{v} = -\Delta\left(-u + \frac{|x|^2}{2n}\lambda\right) = f - \lambda \le 0$

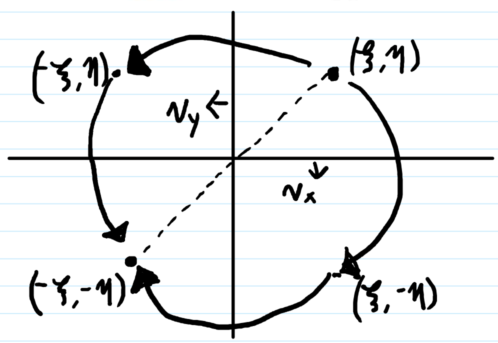
Then $-u \le \bar{v} \le \max_{\bar{U}} \bar{v} = \max_{\partial U} \bar{v} \le \max_{\partial U} |g| + \frac{R^2}{2n}\lambda$

$\therefore \max_{\bar{U}} |u| \le C\left(\max_{\partial U}|g| + \max_{\bar{U}}|f|\right)$

4. (10 points) Find the Green's function for Laplace's equation on the quarter plane $U = \mathbb{R}_+ \times \mathbb{R}_+ = (0,\infty) \times (0,\infty)$. Use your Green's function to solve Laplace's equation

$$\Delta u = 0 \text{ in } U$$
$$u = g \text{ on } \partial U.$$

You need NOT prove that $u(x)$ defined this way is a solution.



$G(x,y;\xi,\eta) = \Phi(y-x;\xi,\eta) - \phi^x(y;\xi,\eta)$

$\Phi(y-x;\xi,\eta) = \frac{-1}{2\pi}\log|z|$ where $|z| = \sqrt{(x-\xi)^2+(y-\eta)^2}$

$$\phi^x(y;\xi,\eta) = \Phi(y-x;-\xi,\eta) + \Phi(y-x;\xi,-\eta) - \Phi(y-x;-\xi,-\eta)$$
$$= \frac{-1}{2\pi}\left(\log\sqrt{(x+\xi)^2+(y-\eta)^2} + \log\sqrt{(x-\xi)^2+(y+\eta)^2} - \log\sqrt{(x+\xi)^2+(y+\eta)^2}\right)$$

$$= -\frac{1}{2\pi}\left(\log\sqrt{(x+\xi)^2+(y-\eta)^2} + \log\sqrt{(x-\xi)^2+(y+\eta)^2} - \log\sqrt{(x+\xi)^2+(y+\eta)^2}\right)$$

Consider $\Delta \log((x\pm\xi)^2+(y\pm\eta)^2) = \nabla\cdot\left(\frac{2(x\pm\xi)}{(x\pm\xi)^2+(y\pm\eta)^2}, \frac{2(y\pm\eta)}{(x\pm\xi)^2+(y\pm\eta)^2}\right)$

$$= \frac{2}{(x\pm\xi)^2+(y\pm\eta)^2} + \frac{2}{(x\pm\xi)^2+(y\pm\eta)^2} - \frac{4(x\pm\xi)^2}{((x\pm\xi)^2+(y\pm\eta)^2)^2} - \frac{4(y\pm\eta)^2}{((x\pm\xi)^2+(y\pm\eta)^2)^2}$$

$$= \frac{4((x\pm\xi)^2+(y\pm\eta))}{((x\pm\xi)^2+(y\pm\eta)^2)^2} - \frac{4(x\pm\xi)^2}{((x\pm\xi)^2+(y\pm\eta)^2)^2} - \frac{4(y\pm\eta)^2}{((x\pm\xi)^2+(y\pm\eta)^2)^2}$$

$$= 0$$

Thus $\phi^x(y)$ is harmonic no matter the sign of ξ and η.

$$\phi^x(y;\xi,\eta)|_{\xi=0} = \Phi(y-x;0,\eta) + \Phi(y-x;0,-\eta) - \Phi(y-x;0,-\eta)$$
$$= \Phi(y-x;0,\eta)$$

Similarly $\phi^x(y;\xi,\eta)|_{\eta=0} = \Phi(y-x;\xi,0)$

So $\Delta\phi^x(y;\xi,\eta) = 0$ in $\mathcal{U}$ and $\phi^x(y;\xi,\eta) = \Phi(y-x;\xi,\eta)$ on $\partial\mathcal{U}$

It follows $G(x,y;\xi,\eta) = \Phi(y-x;\xi,\eta) - (\Phi(y-x;-\xi,\eta) + \Phi(y-x;\xi,-\eta) - \Phi(y-x;-\xi,-\eta))$

Thus $u(\xi,\eta) = \int_{\partial\mathcal{U}} \frac{\partial G}{\partial \nu} u(y)\,dy$

$\nabla\Phi$ done earlier

Consider $\frac{\partial G}{\partial \nu}\Big|_{x=0} = -\frac{\partial G}{\partial x}\Big|_{x=0} = \frac{1}{2\pi}\left(\frac{-2\xi}{\xi^2+(y-\eta)^2} + \frac{2\xi}{\xi^2+(y-\eta)^2}\right)$

Similarly $\frac{\partial G}{\partial \nu}\Big|_{y=0} = -\frac{\partial G}{\partial y}\Big|_{y=0} = \frac{1}{2\pi}\left(\frac{-2\eta}{(x-\xi)^2+\eta^2} + \frac{2\eta}{(x+\xi)^2+\eta^2}\right)$

$$\therefore u(\xi,\eta) = \int_0^\infty g(0,y)\frac{1}{2\pi}\left(\frac{-2\xi}{\xi^2+(y-\eta)^2} + \frac{2\xi}{\xi^2+(y-\eta)^2}\right)dy + \int_0^\infty g(x,0)\frac{1}{2\pi}\left(\frac{-2\eta}{(x-\xi)^2+\eta^2} + \frac{2\eta}{(x+\xi)^2+\eta^2}\right)dx$$