HW3

Tuesday, February 18, 2025 11:16 AM

1. (10 points) Fix some $x_0 \in \mathbb{R}^n$ and $t_0 > 0$. For each $s \in [0, t_0]$ denote by

$$\Omega_s = \{(x,t) : 0 \le t \le s, |x - x_0| \le c(t_0 - t)\}$$

as well as its side boundary

$$\mathcal{S}_{side} = \{(x,t) : 0 \le t \le s, |x - x_0| = c(t_0 - t)\}.$$

Let $\nu = (\nu_1, \dots, \nu_n, \nu_{n+1})$ be the unit normal vector on $\mathcal{S}_{side}$. Show that $\nu_{n+1}^2 = c^2(\nu_1^2 + \cdots + \nu_n^2)$. (This is an exercise left during the lecture.)

Consider $F(x,t) = |x - x_0| - c(t_0 - t)$

Let $\hat{V}_x = \{V_i\}_{i=1}^{n} = \left\{\frac{\partial F}{\partial x^{(i)}}\right\}_{i=1}^{n}$ note $\left\|\frac{\partial F}{\partial x^{(i)}}\right\| = \left\|\frac{x^i - x_0^i}{|x^i - x_0^i|}\right\| = 1$

so $\|\hat{V}_x\|^2 = n$

Let $V_x = \frac{1}{\sqrt{n}} \hat{V}_x$ so that $\|V_x\| = 1$. It follows

$$\nu = \frac{(V_x, \hat{V}_{n+1})}{\|(V_x, \hat{V}_{n+1})\|} = \frac{1}{\sqrt{1 + c^2}} \left(\left\{ \frac{x^i - x_0^i}{|x^i - x_0^i|} \right\}_{i=1}^{n}, c \right)$$

$$\nu_{n+1}^2 = \frac{c^2}{1 + c^2} \qquad \sum_{i=1}^{n} \nu_i^2 = \frac{1}{1+c^2} \|V_x\|^2 = \frac{1}{1+c^2}$$

Thus $c^2 \sum_{i=1}^{n} \nu_i^2 = \frac{c^2}{1+c^2} = \nu_{n+1}^2$

2. (15 points) Let u solve the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = g, \ u_t = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{1}$$

Suppose g, h have compact support. The kinetic energy is $k(t) := \frac{1}{2} \int_{\mathbb{R}} u_t^2(x,t)\, dx$ and the potential energy is $p(t) := \frac{1}{2} \int_{\mathbb{R}} u_x^2(x,t)\, dx$. Prove $k(t) = p(t)$ for all large enough time t.

$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy$$

$$u_x(x,t) = \frac{1}{2}\left(g'(x+t) + g'(x-t)\right) + \frac{1}{2}\frac{d}{dx}\int_{x-t}^{x+t} h(y)\,dy$$

$$= \frac{1}{2}\left(g'(x+t) + g'(x-t)\right) + \frac{1}{2}\left(h(x+t) - h(x-t)\right)$$

$$u_t(x,t) = \frac{1}{2}\left(g'(x+t) - g'(x-t)\right) + \frac{1}{2}\frac{d}{dt}\int_{x-t}^{x+t} h(y)\,dy$$

$$= \frac{1}{2}\left(g'(x+t) - g'(x-t)\right) + \frac{1}{2}\left(h(x+t) + h(x-t)\right)$$

Since g, h have compact support $\exists\, G, H \subset \mathbb{R} \ni$ $g, h = 0$ in G^c, H^c and G, H are compact.

Also $\operatorname{supp}(g') \subseteq G$ which is compact.

$$\frac{d}{dt}E(t) = \frac{1}{2}\frac{d}{dt}\int u_x^2 + u_t^2\,dx = \int u_x u_{xt} + u_t u_{tt}\,dx$$

$$= \int \text{[illegible]}\,dx$$

$$\frac{d}{dt}E(t) = \frac{1}{2}\frac{d}{dt}\int u_x^2 + u_t^2\,dx = \int u_x u_{xt} + u_t u_{tt}\,dx$$

$$= \int u_x u_{xt} + u_t u_{xx}\,dx$$

$$= \int u_x u_{xt}\,dx + [u_t u_x]_{-\infty}^{\infty} - \int u_{tx} u_x\,dx$$

$$= [u_t u_x]_{-\infty}^{\infty}$$

$= 0$ since h and g have compact support

$\therefore E(0) = E(t)$ (constant in time)

Since energy is conserved we can choose a large t so that $u(x,t)$ is seperated into two waves centered around $x-t$ and $x+t$.

For the $x-t$ we see $u_x = \frac{1}{2}(g'(x-t) - h(x-t))$

$u_t = \frac{1}{2}(-g'(x-t) + h(x-t))$

$\Rightarrow u_x = -u_t$

For the $x+t$ we see $u_x = \frac{1}{2}(g'(x+t) + h(x+t))$

$u_t = \frac{1}{2}(g'(x+t) + h(x+t))$

$\Rightarrow u_x = u_t$

In both cases $u_x^2 = u_t^2 \Rightarrow p(t) = k(t)$ for large enough t

3. (15 points) Use energy method to show the following (nonlinear) wave equation only has a zero solution

$$\begin{cases} u_{tt} - \Delta u + u + u^3 = 0 & x \in \Omega, 0 < t < T; \\ u(x,0) = u_t(x,0) = 0 & x \in \Omega; \\ u(x,t) = 0 & x \in \partial\Omega, 0 < t < T \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is an open smooth bounded domain. (Hint: Recall that for our original wave equation $u_{tt} - \Delta u = 0$, the energy is defined as $\frac{1}{2}\int_\Omega u_t^2 + |\nabla u|^2 dx$. Think about why the energy is defined in that way, and here you need to construct a new suitable energy $E(t)$ for our nonlinear wave equation and prove it is zero at $t = 0$ and $\frac{d}{dt}E(t) = 0$.)

$$E(t) = \frac{1}{2}\int_\Omega u_t^2 + |\nabla u|^2 + u^2 + \frac{1}{2}u^4\,d\Omega$$

$$\frac{d}{dt}E(t) = \int_\Omega u_t u_{tt} + \nabla u \cdot \nabla u_t + u u_t + u^3 u_t\,d\Omega$$

$$= \int_\Omega u_t(\Delta u - u - u^3) + \nabla u \cdot \nabla u_t + u u_t + u^3 u_t\,d\Omega$$

$$= \int_\Omega u_t \Delta u + \nabla u \cdot \nabla u_t\,d\Omega$$

$= \int_\Omega u_t \Delta u + \nabla u \cdot \nabla u_t \, d\Omega$

$= \cancel{\int_{\partial\Omega} u_t \nabla u \cdot \hat{n}} \, dS - \int_\Omega \nabla u_t \cdot \nabla u \, d\Omega + \int_\Omega \nabla u_t \cdot \nabla u \, d\Omega$

$= 0$

Thus $E(t) = E(0) = \frac{1}{2}\int_\Omega 0 + 0 + 0 + 0 \, d\Omega = 0$

$\therefore \; u(x,t) = 0$

4. (20 points) Let $Q = \Omega \times (0, T]$ and $u \in C^{2,1}(Q) \cap C(\bar{Q})$ satisfy

$$u_t - \Delta u + cu \leqslant 0 \quad \text{in } Q$$

where $c \geqslant 0$ is a constant.

(a) If $u \geqslant 0$, show that the weak maximum principle holds for u, i.e., u achieves its maximum at the parabolic boundary of Q. (Hint: the proof is similar to the one we did during the lecture.)

(b) Give a counterexample to show the weak maximum principle may not hold without the condition $u \geqslant 0$. (Hint: consider $\Omega = (-1, 1)$ and consider $u = -x^2 + at - b$ with some suitable a and b.)

a) Assume at some point in the interior of Q, (x_0, t_0) a maximum is achieved, say $u(x_0,t_0) = M$. Also assume $u_t - \Delta u + cu < 0$ in Q

At (x_0, t_0) since it is a max, $u_t(x_0,t_0) \geq 0$ and $\Delta u(x_0,t_0) \leq 0$. It follows

$0 \leq u_t(x_0,t_0) - \Delta u(x_0,t_0) < -cM$

$\Rightarrow \quad cM < 0$

$\divideontimes$ since $u \geq 0$ and $c \geq 0$ $\therefore (x_0,t_0) \in \Gamma$

Next consider $u_t - \Delta u + cu \leq 0$ in Q. It follows $\forall \varepsilon > 0$

$V_\varepsilon(x,t) = u(x,t) - \varepsilon t$

$V_\varepsilon \in C^{2,1}(Q) \cap C(\bar{Q}) \qquad \partial_t V_\varepsilon - \Delta V_\varepsilon + cV_\varepsilon = u_t - \Delta u^2 - \varepsilon + cu - c\varepsilon t < 0$

$\max_Q u = \max_Q (V_\varepsilon + \varepsilon t)$

$\leq \max_Q V_\varepsilon + \varepsilon T$

$\leq \max_\Gamma V_\varepsilon + \varepsilon T$ By applying the previous case

$\leq \max_\Gamma u + \varepsilon T$

as $\varepsilon \to 0 \quad \max_Q u \leq \max_\Gamma u$

Also, as $\Gamma \subseteq \bar{Q}$ $\quad \max_{\Gamma} u \le \max_{\bar{Q}} u$

$\therefore \max_{\Gamma} u = \max_{\bar{Q}} u$

b) $-x^2+at-b<0$ $\quad \Omega = (-1,1)$

$u_t = a$ $\quad \Delta u = -2$ $\quad c=1$

$a+2-x^2+at-b \le 0$

$-x^2+(1+t)a-b \le 0$ $\quad \cancel{b}-a\cancel{t}+a+a\cancel{t}-\cancel{b}<0$

$a<0$

$-x^2+at-b<0$

$b>0$

$t > \frac{b}{a}$

$-x^2-t-1<0 \ \forall x \in \mathbb{R} \ t>0$

max is at $(0,t)$ $\quad 0 \in (-1,1)$ $\quad 0 \notin \{-1,1\}$

I tried to draw a picture