HW2

Sunday, February 9, 2025 2:08 PM

1. (20 points) (a) Find the solution of the initial boundary value problem for the one dimensional wave equation with homogeneous Neumann boundary condition

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L, t > 0 \\ u(x,0) = g(x),\ u_t(x,0) = h(x) & 0 < x < L \\ u_x(0,t) = u_x(L,t) = 0 & t > 0. \end{cases}$$

Please include the details as we did for the homogeneous Dirichlet boundary condition during the lecture.

(b) Consider the initial boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < \pi, t > 0 \\ u(x,0) = x^2,\ u_t(x,0) = 0 & 0 < x < \pi \\ u_x(0,t) = u_x(\pi,t) = 0 & t > 0. \end{cases}$$

Find a Fourier series solution.

a) Assume $U(x,t) = XT$

$$XT'' - c^2 X'' T = 0$$

$$\frac{X''}{X} = \frac{T''}{c^2 T} \Rightarrow T \text{ and } X \text{ are constants}$$

$$X'' = \lambda X$$

$\underline{\lambda > 0}$ $X = C_1 e^{\sqrt{\lambda} x} + C_2 e^{-\sqrt{\lambda} x}$

$$X' = C_1 \sqrt{\lambda} e^{\sqrt{\lambda} x} - C_2 \sqrt{\lambda} e^{-\sqrt{\lambda} x}$$

$$X'(0) = 0 = \sqrt{\lambda}(C_1 - C_2)$$

$$\Rightarrow C_1 = C_2 \text{ because } \lambda > 0$$

$$X'(L) = \sqrt{\lambda} C_1 (e^{\sqrt{\lambda} L} - e^{-\sqrt{\lambda} L}) = 0$$

Since $\lambda > 0, L > 0$ $e^{\sqrt{\lambda} L} - e^{-\sqrt{\lambda} L} \neq 0 \Rightarrow C_1 = 0$

$\Rightarrow U(x,t) = 0$

$\underline{\lambda = 0}$ $X = C_1 + C_2 X$

$$X'(0) = 0 = C_2$$

$$X = C_1 \Rightarrow U(x,t) = U(t) = C_1 T(t)$$

$\underline{\lambda < 0}$ $X = C_1 \cos(\sqrt{-\lambda}\, x) + C_2 \sin(\sqrt{-\lambda}\, x)$

$$X' = -\sqrt{-\lambda}\, C_1 \sin(\sqrt{-\lambda}\, x) + \sqrt{-\lambda}\, C_2 \cos(\sqrt{-\lambda}\, x)$$

$X'(0) = 0 = 0 + \sqrt{-\lambda}\, c_2 \Rightarrow c_2 = 0$

$X'(L) = 0 = \sqrt{-\lambda}\, c_1 \sin(\sqrt{-\lambda} L)$

$\Rightarrow \sqrt{-\lambda} L = k\pi$

$\lambda = \frac{-k^2\pi^2}{L^2}$

$X_k = C_k \cos \frac{k\pi x}{L}$

$T'' = \lambda c^2 T$

$\underline{\lambda = 0)}$ $T = C_2 + C_3 t$

$u(x,t) = u(t) = C_1(C_2 + C_3 t) \qquad C_1C_2 = A_0 \quad C_1C_3 = B_0$

$= A_0 + B_0 t$

$\underline{\lambda > 0)}$ $u(x,t) = 0T(t) = 0$

$\underline{\lambda < 0)}$ $\lambda = -\frac{k^2\pi^2}{L^2} \Rightarrow T_k = a_k \cos\frac{k\pi c t}{L} + b_k \sin\frac{k\pi c t}{L}$

Let $a_k C_k = A_k \quad b_k C_k = B_k$

$u(x,t) = B_0 t + A_0 + \sum_{k=1}^{\infty} \cos\frac{k\pi x}{L}\left(A_k \cos\frac{k\pi c t}{L} + B_k \sin\frac{k\pi c t}{L}\right)$

$\dot{u}(x,t) = B_0 + \sum_{k=1}^{\infty} \frac{k\pi c}{L} \cos\frac{k\pi x}{L}\left(B_k \cos\frac{k\pi c t}{L} - A_k \sin\frac{k\pi c t}{L}\right)$

$u(x,0) = A_0 + \sum_{k=1}^{\infty} A_k \cos\frac{k\pi x}{L} = g(x)$

even expansion of g

$A_0 = \frac{1}{L}\int_0^L g(x)\,dx \qquad A_k = \frac{2}{L}\int_0^L g(x)\cos\frac{k\pi x}{L}\,dx$

$\dot{u}(x,0) = h(x) = B_0 + \sum_{k=1}^{\infty} B_k \frac{k\pi c}{L} \cos\frac{k\pi x}{L}$

even expansion of h

$B_0 = \frac{1}{L}\int_0^L h(x)\,dx \qquad B_k = \frac{2}{k\pi c}\int_0^L h(x)\cos\frac{k\pi x}{L}\,dx$

$\therefore u(x,t) = B_0 t + A_0 + \sum_{k=1}^{\infty} \cos\frac{k\pi x}{L}\left(A_k \cos\frac{k\pi c t}{L} + B_k \sin\frac{k\pi c t}{L}\right)$

Where $A_0 = \frac{1}{L}\int_0^L g(x)\,dx \qquad A_k = \frac{2}{L}\int_0^L g(x)\cos\frac{k\pi x}{L}\,dx$

$B_0 = \frac{1}{L}\int_0^L h(x)\,dx \qquad B_k = \frac{2}{k\pi c}\int_0^L h(x)\cos\frac{k\pi x}{L}\,dx$

b) $g(x) = x^2 \quad h(x) = 0 \quad L = \pi \quad c = 1$

From part a

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$$u(x,t) = \frac{1}{\pi}\int_0^\pi x^2\,dx + \sum_{k=1}^{\infty} \cos kx \left(\left[\frac{2}{\pi}\int_0^\pi x^2 \cos kx\,dx\right](\cos kt + 0)\right)$$

$$= \frac{\pi^3}{3} + \sum_{k=1}^{\infty} \cos kx \cos kt \frac{4(-1)^k}{k^2}$$

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(%i4) integrate(x^2,x,0,π)/π;
(%o4) π^2/3
(%i6) declare(k,integer);
(%o6) done
(%i7) 2·integrate(x^2·cos(k·x),x,0,π)/π;
(%o7) 4(-1)^k/k^2
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2. (20 points) Find the solution of the initial initial boundary value problem for the one dimensional homogeneous wave equation with nonhomogeneous Dirichlet boundary condition

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < \pi, t > 0 \\ u(x,0) = 0,\ u_t(x,0) = 0 & 0 < x < \pi \\ u(0,t) = u(\pi,t) = 2t^2 - 3 & t > 0. \end{cases}$$

(Hint: As we did during the lecture, first convert it into a nonhomogeneous wave equation with homogeneous Dirichlet boundary condition. Then you can decompose the problem into a homogeneous equation with homogeneous Dirichelet boundary condition, for which we already have the solution from the lecture, and another part of a nonhomogeneous equation with zero initial condition, for which you can use Duhamel's principle. When you solve the "w" part in Duhamel's principle, you will find that w satisfies a homogeneous equation with homogeneous Dirichelet boundary condition, for which we again have solution.)

Consider

$$\tilde{u}(x,t) = 2t^2 - 3 + \frac{x}{\pi}(2t^2 - 3 - 2t^2 + 3)$$

$$= 2t^2 - 3$$

$$\tilde{u}(0,t) = \tilde{u}(\pi,t) = 2t^2 - 3$$

Let

$$v = u - \tilde{u}$$

It follows

$$\begin{cases} v_{tt} - v_{xx} = \overbrace{u_{tt} - u_{xx}}^{0} - 4 = -4 & 0 < x < \pi \quad t > 0 \\ v(x,0) = 0 - -3 = 3 \quad v_t(x,0) = 0 - 0 & 0 < x < \pi \\ v(0,t) = 2t^2 - 3 - (2t^2 - 3) = 0 & \\ v(\pi,t) = 2t^2 - 3 - (2t^2 - 3) = 0 & t > 0 \end{cases}$$

Let $v = v_1 + v_2$ where

$v_1 \{ \partial_{tt} v_1 - \partial_{xx} v_1 = 0 \quad 0 < x < \pi \quad t > 0 \qquad v_2 \{ \partial_{tt} v_2 - \partial_{xx} v_2 = -4 \quad 0 < x < \pi$

Let $V = V_1 + V_2$ where

V_1) $\partial_{tt} V_1 - \partial_{xx} V_1 = 0$ $\quad 0<x<\pi \quad t>0$

$V_1(x,0) = 3 \quad \partial_t V_1(x,0) = 0 \quad 0<x<\pi$

$V_1(0,t) = V_1(\pi,t) = 0 \quad t>0$

V_2) $\partial_{tt} V_2 - \partial_{xx} V_2 = -4$ $\quad 0<x<\pi \quad t>0$

$V_2(x,0) = \partial_t V_2(x,0) = 0 \quad 0<x<\pi$

$V_2(0,t) = V_2(\pi,t) = 0 \quad t>0$

$c=1 \quad L=\pi \quad g(x)=3 \quad h(x)=0$

V_1) $V_1(x,t) = \sum_{k=1}^{\infty} \sin kx [a_k \cos kt]$

$a_k = \frac{2}{\pi}\int_0^{\pi} 3 \sin kx\,dx = \frac{6}{\pi}\left(\frac{1-(-1)^k}{k}\right)$

(%i6) declare(k,integer);

(%o6) done

(%i24) 2·integrate(3·sin(k·x),x,0,π)/π;

(%o24) $\frac{6\left(\frac{1}{k} - \frac{(-1)^k}{k}\right)}{\pi}$

V_2) $\forall s>0 \quad w(x,t;s)$ solves
$$\begin{cases} \partial_{tt} w - \partial_{xx} w = 0 & 0<x<\pi \quad t>0 \\ w(x,s;s) = 0 & 0<x<\pi \\ w_t(x,s;s) = -4 & \\ w(0,t;s) = w(\pi,t;s) = 0 & t>0 \end{cases}$$

By Duhamel's principle

and $V_2(x,t) = \int_0^t w(x,t;s)\,ds$

$c=1 \quad L=\pi \quad h(x)=-4 \quad g(x)=0$

$V_2(x,t) = \int_0^t \sum_{k=1}^{\infty} \sin kx \; b_k \sin ks \, ds$

$b_k = \frac{2}{k\pi}\int_0^{\pi} -4 \sin kx\,dx = \frac{-8}{\pi}\left(\frac{1-(-1)^k}{k^2}\right)$

2·integrate(−4·sin(k·x),x,0,π)/(k·π);

$-\left(\frac{8\left(\frac{1}{k} - \frac{(-1)^k}{k}\right)}{\pi k}\right)$

$V_2(x,t) = \int_0^t \sum_{k=1}^{\infty} \frac{-8}{\pi k^2}(1-(-1)^k) \sin kx \sin ks \, ds$

$= \sum_{k=1}^{\infty} \frac{8}{\pi k^3}(1-(-1)^k) \sin kx (\cos kt - 1)$

integrate(sin(k·x),x,0,t);

Is t positive, negative or zero?positive;

$\frac{1}{k} - \frac{\cos(k\,t)}{k}$

$V = \sum_{k=1}^{\infty} \sin kx (1-(-1)^k)\left(\frac{6}{\pi k}\cos kt + \frac{8}{\pi k^3}(\cos kt - 1)\right)$

$\therefore \; u = 2t^2 - 3 + \sum_{k=1}^{\infty} \sin kx (1-(-1)^k)\left(\frac{6}{\pi k}\cos kt + \frac{8}{\pi k^3}(\cos kt - 1)\right)$

3. (20 points) Find the solution of the following initial value problems. You can use the Kirchhoff's formula directly. (Hint: you can use the spherical coordinator. You should be able to compute the integral explicitly.)

(a)
$$\begin{cases} u_{tt} - \Delta u = 0 & x = (x_1, x_2, x_3) \in \mathbb{R}^3, t>0 \\ u(x,0) = 0, \; u_t(x,0) = x_1^2 + x_2^2 + x_3^2 & x = (x_1, x_2, x_3) \in \mathbb{R}^3 \end{cases}$$

(b)
$$\begin{cases} u_{tt} - \Delta u = 0 & x = (x_1, x_2, x_3) \in \mathbb{R}^3, t>0 \\ u(x,0) = x_1^2 + x_2^2, \; u_t(x,0) = 0 & x = (x_1, x_2, x_3) \in \mathbb{R}^3 \end{cases}$$

$u(x,t) = ⨍_{\partial B_t(x)} \left(g(y) + \nabla g(y)(y-x) + t h(y)\right) dS_y$

a) $g(x) = 0 \qquad h(x) = x_1^2 + x_2^2 + x_3^2$

a) $g(x)=0 \qquad h(x)=x_1^2+x_2^2+x_3^2$

$$u(x,t) = ⨍_{\partial B_t(x)} t(y_1^2+y_2^2+y_3^2)\,dS_y \qquad y_i = x_i + t z_i$$

$$= \frac{t}{4\pi}\int_{\partial B_1(0)} (x_1+tz_1)^2+(x_2+tz_2)^2+(x_3+tz_3)^2\,dS_z$$

$$= \frac{t}{4\pi}\int_{\partial B_1(0)} x_1^2+x_2^2+x_3^2+2t(x_1z_1+x_2z_2+x_3z_3)+t^2(z_1^2+z_2^2+z_3^2)\,dS_z$$

$$= \frac{t}{4\pi}\left(4\pi(x_1^2+x_2^2+x_3^2)+0+4\pi t^2\right)$$

Note $\int_{\partial B_1(0)} z_i\,dS=0$ by symmetry and $\int_{\partial B_1(0)} z_i^2\,dS=\frac{4\pi}{3}$

stirands ↑

$$= t(x_1^2+x_2^2+x_3^2+t^2)$$

b) $g(x)=x_1^2+x_2^2 \qquad h(x)=0$

$$u(x,t)= ⨍_{\partial B_t(x)} g(y)+\nabla g(y)(y-x)\,dS_y \qquad y_i=x_i+tz_i$$

$$= \frac{1}{4\pi}\int_{\partial B_1(0)} (x_1+tz_1)^2+(x_2+tz_2)^2+\left[2(x_1+tz_1),\,2(x_2+tz_2),\,0\right]\begin{bmatrix} tz_1\\ tz_2\\ tz_3\end{bmatrix} dS_z$$

$$= \frac{1}{4\pi}\int_{\partial B_1(0)} x_1^2+x_2^2+2t(x_1z_1+x_2z_2)+t^2(z_1^2+z_2^2)$$

$$+2t(x_1z_1+x_2z_2)+2t^2(z_1^2+z_2^2)\,dS_z$$

Similarly to part a,

$\int_{\partial B_1(0)} z_i\,dS=0$ by symmetry and $\int_{\partial B_1(0)} z_i^2\,dS=\frac{4\pi}{3}$

$$= \frac{1}{4\pi}\left(4\pi(x_1^2+x_2^2)+0+\cancel{3}t^2\,\frac{8\pi}{\cancel{3}}\right)$$

$$= x_1^2+x_2^2+2t^2$$