Math 8260 Partial Differential Equations HW5

Assigned on March 12st, due on March 26th Wednesday by the beginning of the lecture

1. (20 points) We define the convolution of two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ by

$$f * g(x) := \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

Establish the following properties of convolution:

   (a) $f, g \in L^2(\mathbb{R}^n) \Rightarrow \|f * g\|_{L^\infty} \le \|f\|_{L^2}\|g\|_{L^2}$.

   (b) $f, g \in L^1(\mathbb{R}^n) \Rightarrow \|f * g\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}$.

   (c) $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \Rightarrow f * g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

   (d) (bonus problem, additional 5 points) If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then $\|f * g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}$.

2. (20 points) Let $B_r(0)$ denote the open ball of radius r in $\mathbb{R}^n$, and denote by

$$|B_r(0)| := \int_{B_r(0)} 1dx, \quad |\partial B_r(0)| := \int_{\partial B_r(0)} 1dS.$$

Show the following.

   (a) $|B_r(0)| = |B_1(0)|\, r^n$ and $|\partial B_r(0)| = |\partial B_1(0)|\, r^{n-1}$.

   (b) $|\partial B_r(0)| = \frac{n}{r}\,|B_r(0)|$.

   (c) $|\partial B_1(0)| = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$, where Γ denotes the gamma function $\Gamma(k) = \int_0^\infty t^{k-1}e^{-t}dt$. (Hint. Use the integral equality $\int_{\mathbb{R}^n} e^{-|x|^2}dx = \pi^{n/2}$, and evaluate the integral in polar coordinates.)

   (d) Show that $\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$, and then show that $|\partial B_1(0)| = n\alpha(n)$, where $\alpha(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

3. (20 points) (Evans 2.5.3.) Modify the proof of the mean value formulas to show for $n \ge 3$ that

$$u(0) = \frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)} gdS + \frac{1}{n(n-2)\alpha(n)}\int_{B_r(0)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) fdx,$$

provided

$$\begin{aligned} -\Delta u &= f \text{ in } B_r(0) \\ u &= g \text{ in } \partial B_r(0). \end{aligned}$$

Here we assume $f \in C(\overline{B_r(0)})$ (the closed ball), and $g \in C(\partial B_r(0))$.