C_0 Semigroups

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This presentation is based on three books, namely "Topics in functional analysis and applications" by Srinivasan Kesavan[1], "Introductory functional analysis with applications" by Erwin Kreyszig[2], and "Semigroups of linear operators and applications to partial differential equations" by Amnon Pazy[3].

Definition

- *A linear operator* $A : D(A) \subseteq X \to Y$ *is said to be* **bounded** *if there exists a* $C > 0$ *such that*

$$||Au||_Y \leq C||u||_X, \quad \textit{for every } u \in D(A)$$

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Preliminary Material

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- *A linear operator $A : D(A) \subseteq X \to Y$ is said to be* **closed** *if the* **graph**

$$G(A) = \{(u, Au) \mid u \in D(A)\} \subseteq X \times Y$$

  is closed as a subspace of $X \times Y$

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Definition

Let X be a Banach space with dual space X'. Denote $x' \in X'$ at $x \in X$ by $\langle x', x \rangle$ or $\langle x, x' \rangle$. Define the following set $F(x) \subseteq X'$ as

$$F(x) = \{x' \mid \langle x', x \rangle = ||x||^2 = ||x'||^2\}$$

(This set is non-empty by the Hahn-Banach theorem.)

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Definition (Maximal Dissipativity)

A linear operator A is called maximally dissipative if it is dissipative and $R(I - A) = X$.

Preliminary Material

Definition

Let $X \neq \{0\}$ be a complex normed space and $A : D(A) \subseteq X \to X$ be a linear operator. With A we associate the operator

$$A_\lambda = \lambda I - A$$

where λ is a complex number and I is the identity operator on $D(A)$. If A_λ has an inverse, we denote it by $R_\lambda(A)$ and call it the resolvent operator of A or, simply, the **resolvent** *of A. If it is clear which operator we are discussing, we will write R_λ.*

Definition

Let $X \neq \{0\}$ be a complex normed space and $A : D(A) \subseteq X \to X$ be a linear operator. A regular value λ of A is a complex number such that

- *R_λ exists,*
- *R_λ is bounded,*
- *R_λ is densely defined.*

The resolvent set $\rho(A)$ of A is the set of all regular values λ of A. Its complement $\sigma(A) = \mathbb{C} \backslash \rho(A)$ in the complex plane $\mathbb{C}$ is called the spectrum of A, and a $\lambda \in \sigma(A)$ is called a spectral value of A.

Motivation

Definition

Recall the limit definition of the derivative

$$u_t = \lim_{h \downarrow 0} \frac{u(t+h) - u(t)}{h}$$

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Example

Consider the following linear first order ODE

$$\begin{cases} u_t = -\alpha u & t > 0,\ \alpha \in \mathbb{R} \\ u(0) = u_0 \end{cases}$$

Motivation

Lemma

$\log(1+x) = x \quad as \quad x \downarrow 0$

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$\log(1+x) = x \quad \textit{as} \quad x \downarrow 0$

Proof

Let $y = \log(1+x)$.
Note $y \to 0$ *as* $x \to 0$
Thus,

$$\begin{aligned} 1 + x &= e^y \\ &= 1 + y + o(y) \\ &\to 1 + y \end{aligned}$$

Now let's get back to the example

Example (Continued)

$$-\alpha = \frac{u_t}{u}$$

Motivation

Example (Continued)

$$
\begin{aligned}
-\alpha &= \frac{u_t}{u} \\
&= \frac{1}{u(t)} \lim_{h \downarrow 0} \frac{u(t+h) - u(t)}{h}
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&= \lim_{h \downarrow 0} \frac{\log |u(t+h)| - \log |u(t)|}{h}
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$$

Therefore $u(t) = u_0 e^{-\alpha t}$

How can we generalize this idea?

Definition (C_0 Semigroups)

Let X be a Banach space and $\{S(t)\}_{t\geq 0}$ be a family of bounded linear operators on X. $\{S(t)\}_{t\geq 0}$ is said to be a C_0 semigroup if the following are true:

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- $S(t+s) = S(t)S(s)$, *for all* $t, s \geq 0$
- *For every* $u \in X$

$$S(t)u \to u \quad \text{as } t \downarrow 0$$

C_0 Semigroup Definition

Definition

Let $\{S(t)\}_{t\geq 0}$ *be a* C_0 *semigroup on* X. *The* **infinitesimal generator** *of the semigroup is a linear operator* A *given by*

$$D(A) = \left\{ u \in X \mid \lim_{t \downarrow 0} \frac{S(t)u - u}{t} \text{ exists} \right\}$$

$$Au = \lim_{t \downarrow 0} \frac{S(t)u - u}{t}, \ u \in D(A)$$

Some C_0 Semigroup Properties

What are some properties of these semigroups?

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Theorem

Let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup on X. Then there exists $M \geq 1$ and ω such that

$$||S(t)|| \leq Me^{\omega t}, \quad \textit{for all } t \geq 0$$

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Definition

If $M = 1$ and $\omega = 0$, so that $||S(t)|| \leq 1$ for all $t \geq 0$, we say that $\{S(t)\}$ is a **contraction semigroup**.

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Theorem

Let $\{S(t)\}_{t\geq 0}$ be a C_0 semigroup and let A be its infinitesimal generator. Let $u \in D(A)$. Then

$$S(t)u \in C^1([0,\infty);X) \cap C([0,\infty);X)$$

and

$$\frac{d}{dt}(S(t)u) = AS(t)u = S(t)Au$$

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Next consider

$$\frac{S(t)u - S(t-h)u}{h} = S(t-h)\frac{S(h)u-u}{h}$$

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$$\frac{S(t)u - S(t-h)u}{h} - S(t)Au = S(t-h)\left(\frac{S(h)u - u}{h} - Au \right) + (S(t-h) - S(t))Au$$

Proof (Proof Continued)

$$\left\|S(t-h)\left(\frac{S(h)u-u}{h}-Au\right)\right\| \leq Me^{\omega t}\left\|\frac{S(h)u-u}{h}-Au\right\|$$

Proof (Proof Continued)

$$\left\| S(t-h)\left(\frac{S(h)u-u}{h} - Au\right)\right\| \le Me^{\omega t}\left\|\frac{S(h)u-u}{h} - Au\right\|$$
$$\to 0 \quad as\, h \downarrow 0$$

Proof (Proof Continued)

$$\begin{aligned}\left\|S(t-h)\left(\frac{S(h)u-u}{h}-Au\right)\right\| &\leq Me^{\omega t}\left\|\frac{S(h)u-u}{h}-Au\right\| \\ &\to 0 \quad as\, h\downarrow 0 \\ \|(S(t-h)-S(t))Au\| &\to 0 \quad as\, h\downarrow 0\end{aligned}$$

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Thus,

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Similarly, by the boundedness of $S(t)$, the map $t \mapsto S(t)Au$ is continuous so $S(t)u \in C^1([0,\infty);X)$.

What does that tell us?

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Remark

If A is the infinitesimal generator of a C_0 semigroup $\{S(t)\}$ then we know by the above theorem that

$$u(t) = S(t)u_0$$

defines the unique solution of the initial value problem

$$\begin{cases} \frac{du(t)}{dt} & = Au(t),\ t \geq 0 \\ u(0) & = u_0 \end{cases}$$

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- *For every $\lambda > 0$, $R_\lambda(A)$ is a bounded linear operator and*

$$||R_\lambda(A)|| \leq \frac{1}{\lambda}$$

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This is a more general result and will not be shown.

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Theorem

- *If A is dissipative and there is a $\lambda_0 > 0$ such that $R(A_{\lambda_0}) = X$, then A is the infinitesimal generator of a C_0 semigroup of contractions on X.*
- *If A is the infinitesimal generator of a C_0 semigroup of contractions on X then $R(A_\lambda) = X$ for all $\lambda > 0$ and A is dissipative.*

Proof

Let $\lambda > 0$, the dissipativeness of A implies that

$$||A_\lambda x|| = ||\lambda x - Ax|| \geq \lambda||x|| \quad \text{for every } \lambda > 0 \text{ and } x \in D(A).$$

Since $R(A_{\lambda_0}) = X$, it follows when $\lambda = \lambda_0$ that R_{λ_0} is a bounded linear operator and thus closed. This implies A is closed. If $R(A_\lambda) = X$ for every $\lambda > 0$ then $\rho(A) \supseteq (0, \infty)$ and $||R_\lambda|| \leq \lambda^{-1}$. It follows by the Hille-Yosida theorem that A is the infinitesimal generator of a C_0 semigroup of contractions on X.

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Consider the set

$$\Lambda = \{\lambda \,|\, 0 < \lambda < \infty,\ R(A_\lambda) = X\}.$$

Let $\lambda \in \Lambda$. By the previous inequality, $\lambda \in \rho(A)$. Since $\rho(A)$ is open, the intersection of $B_r(\lambda) \cap \mathbb{R} \subseteq \Lambda$ and thus Λ is open.

Proof (Proof Continued)

On the other hand, let $\{\lambda_n\} \subseteq \Lambda$ *and* $\lambda_n \to \lambda > 0$. *For every* $y \in X$ *there exists an* $x_n \in D(A)$ *such that*

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From the inequality it follows that $||x_n|| \leq \lambda_n^{-1}||y|| \leq C$ *for some* $C > 0$*. Now,*

$$\begin{aligned} \lambda_m||x_n - x_m|| &\leq ||\lambda_m(x_n - x_m) - A(x_n - x_m)|| \\ &= |\lambda_n - \lambda_m| \, ||x_n|| \\ &\leq C|\lambda_n - \lambda_m| \to 0 \end{aligned}$$

We see $\{x_n\}$ *is Cauchy. Let* $x_n \to x$*. It follows* $Ax_n \to \lambda x - y$*. Since* A *is closed,* $x \in D(A)$ *and* $A_\lambda x = y$*. Thus* $R(A_\lambda) = X$ *and* $\lambda \in \Lambda$ *which implies that* Λ *is closed.*

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We see $\{x_n\}$ *is Cauchy. Let* $x_n \to x$. *It follows* $Ax_n \to \lambda x - y$. *Since* A *is closed,* $x \in D(A)$ *and* $A_\lambda x = y$. *Thus* $R(A_\lambda) = X$ *and* $\lambda \in \Lambda$ *which implies that* Λ *is closed. By assumption,* $\Lambda \neq \emptyset$, *therefore* $\Lambda = (0, \infty)$

Proof (Proof Continued)

If A is the infinitesimal generator of a C_0 semigroup of contractions, $S(t)$, on X, then by the Hille-Yosida theorem $\rho(A) \supseteq (0, \infty)$ and therefore $R(A_\lambda) = X$ for all $\lambda > 0$.

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If A is the infinitesimal generator of a C_0 semigroup of contractions, $S(t)$, on X, then by the Hille-Yosida theorem $\rho(A) \supseteq (0, \infty)$ and therefore $R(A_\lambda) = X$ for all $\lambda > 0$. Furthermore if $x \in D(A)$, $x' \in F(x)$ then

$$|\langle S(t)x, x' \rangle| \leq ||S(t)x|| \, ||x'|| \leq ||x||^2$$

Proof (Proof Continued)

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$$|\langle S(t)x, x' \rangle| \leq ||S(t)x|| \, ||x'|| \leq ||x||^2$$

and therefore,

$$Re\langle S(t)x - x, x' \rangle = Re\langle S(t)x, x' \rangle - ||x||^2 \leq 0.$$

By dividing the previous line by $t > 0$ and letting $t \downarrow 0$ yields

$$Re\langle Ax, x' \rangle \leq 0.$$

By using our definition of maximally dissipative operators we can rewrite the Lumer-Phillips theorem as follows

Lumer-Phillips

By using our definition of maximally dissipative operators we can rewrite the Lumer-Phillips theorem as follows

Theorem

A densely defined operator A is the infinitesimal generator of a C_0 semigroup of contractions if and only if it is maximal dissipative.

Bibliography

[1] Srinivasan Kesavan. *Topics in functional analysis and applications*. Wiley Eastern Ltd., 1989.

[2] Erwin Kreyszig. *Introductory functional analysis with applications*. John Wiley & Sons, 1991.

[3] Amnon Pazy. *Semigroups of linear operators and applications to partial differential equations*. Vol. 44. Springer Science & Business Media, 2012.