Math 8260 Partial Differential Equations HW4

assigned on Feb 27, due on March 10th Monday by the beginning of the lecture

1. (15 points) For some fixed $a > 0$, compute the following.
   (a) The Fourier transform of $f(x) = e^{-a|x|}$ where $x \in \mathbb{R}$.
   (b) The Fourier transform of $f(x) = e^{-a|x|}$ where $x \in \mathbb{R}^n$.
   (Hint: For part (b), you can use the result from part (a), the trivial identity $\int_0^\infty e^{-(a^2+\xi^2)s} ds = \frac{1}{a^2+\xi^2}$, and the Fourier transform of $e^{-b|x|^2}$ we discussed during lectures to derive an integral representation of $e^{-a|x|}$ in terms of $e^{-c|x|^2}$ for some c, and then continue from there.)

2. (15 points) Let $K(x, y, t)$ be the heat kernel given as

$$K(x, y, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}, \quad x, y \in \mathbb{R}^n, t > 0$$

   (a) Show that $K(x, y, t) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times (0, \infty))$, and show it satisfies the heat equation

$$\frac{\partial K(x, y, t)}{\partial t} - \Delta_x K(x, y, t) = 0.$$

   (b) Show that for every $x \in \mathbb{R}^n$ and $t > 0$,

$$\int_{\mathbb{R}^n} K(x, y, t) dy = 1.$$

3. (20 points) Suppose $g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and set

$$u(x, t) = \int_{\mathbb{R}^n} K(x, y, t) g(y) dy$$

   where $K(x, y, t)$ denotes the heat kernel as defined in problem 2. Show that for all $t > 0$:
   (a) $\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-n/2} \|g\|_{L^1(\mathbb{R}^n)}$
   (b) $\|u(\cdot, t)\|_{L^1(\mathbb{R}^n)} \leq \|g\|_{L^1(\mathbb{R}^n)}$
   (c) $\|u_{x_i}(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-1/2} \|g\|_{L^\infty(\mathbb{R}^n)}$.
   (d) $\|u_{x_i}(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{n+1}{2}} \|g\|_{L^1(\mathbb{R}^n)}$

4. (10 points) For a fixed constant $c \in \mathbb{R}$, find an explicit solution for

$$\begin{aligned} u_t - \Delta u + cu &= f \quad (x, t) \in \mathbb{R}^n \times (0, \infty), \\ u(x, 0) &= g(x) \quad x \in \mathbb{R}^n. \end{aligned}$$