Math 8260 Partial Differential Equations HW 1

Assigned on Jan 15th, due on Jan 27th Monday by the beginning of the lecture

1. (10 points) Determine whether the following PDEs are linear, semilinear, quasilinear, or fully nonlinear, and briefly explain the reason.
   1. $\sin(u_x)u_y + \cos(u_y)u_{xx} = 0$.
   2. $u_x^3 + e^x u_{yy} = \cos(u)$.
   3. $u_t + x^2 u_x = 0$.
   4. $u_x u_t + u_x u_y = 0$.

2. (10 points) (Evans 2.5.1.) Find an explicit formula for a function u solving the initial-value problem
$$\begin{aligned} u_t + \vec{b}\cdot\nabla u + cu &= 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t=0\}. \end{aligned}$$
   Here $c \in \mathbb{R}$ and $\vec{b} \in \mathbb{R}^n$ are constants. (Please write down full details).

3. (20 points) Solve the following:
   (a) Solve the quarter-plane problem
$$\begin{aligned} u_{tt} &= c^2 u_{xx}; \quad (x,t) \in (0,\infty)\times(0,\infty) \\ u_x(0,t) &= 0; \quad t \geq 0 \\ u(x,0) &= g(x); \quad x \geq 0 \\ u_t(x,0) &= h(x); \quad x \geq 0, \end{aligned}$$
   with $g_x(0) = h_x(0) = 0$.

   (b) Consider the equation from Part (a) with $c = 2, h(x) = 0$, and
$$g(x) = \begin{cases} x-2 & 2 \leq x \leq 3 \\ 4-x & 3 < x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$
   Sketch graphs of $u(x,0), u(x,1)$, and $u(x,2)$.

   (c) Solve the equation from Part (a) with $c = 2, g(x) = 0$ and $h(x) = \frac{1}{x^2+1}$. Then sketch a graph of $u(x,1)$.

4. (10 points) (Evans 2.5.21)

(a) Assume $E = \left(E^1, E^2, E^3\right)$ and $B = \left(B^1, B^2, B^3\right)$ solve Maxwell's equations

$$\begin{aligned} \vec{E}_t &= \nabla \times \vec{B} \\ \vec{B}_t &= -\nabla \times \vec{E} \\ \nabla \cdot \vec{E} &= 0 \\ \nabla \cdot \vec{B} &= 0. \end{aligned}$$

Show

$$\vec{E}_{tt} - \Delta \vec{E} = 0, \quad \vec{B}_{tt} - \Delta \vec{B} = 0.$$

(b) Assume that $\vec{u} = \left(u^1, u^2, u^3\right)$ solves the evolution equations of linear elasticity

$$\vec{u}_{tt} - \mu \Delta \vec{u} - (\lambda + \mu) D(\operatorname{div} \vec{u}) = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty).$$

Show $w := \operatorname{div} \vec{u}$ and $\vec{w} := \operatorname{curl} \vec{u}$ each solve wave equations, but with different speeds of propagation.

(Hint: you may find that the vector calculus identities page on Wikipedia is very useful)