Week 2 Worksheet
Math Review

Jacob Erlikhman

Exercise 0. Warm up. a) Write down the divergence theorem.

b) Write down Stokes' theorem.

c) Suppose in the divergence theorem I let the volume I was integrating over be given by the *open* ball:

$$B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | {x_1}^2 + {x_2}^2 + {x_3}^2 < 1\}.$$

What does the divergence theorem say in this case? Does it make sense? Why or why not?

Exercise 1. a) What does the gradient tell you about a function? Why?
Hint: If $\boldsymbol{\nabla} f(\mathbf{x}) = \mathbf{w}$, argue or show that

$$D_{\mathbf{v}} f(\mathbf{x}) = \mathbf{v} \cdot \mathbf{w},$$

where $D_{\mathbf{v}} f(\mathbf{x})$ is the directional derivative of f at $\mathbf{x}$ in the direction $\mathbf{v}$.

Remark. Notice that this result holds in *any dimension* $n \in \mathbb{N}$.

b) What does the curl tell you about a vector field? Why?
Hint: Draw and calculate the curls of some example vector fields, like $-y\hat{x} + x\hat{y}$ or $x\hat{y}$. Now, try the vector fields $x\hat{x} + y\hat{y} + z\hat{z}$, $\hat{z}$, and $z\hat{z}$.

c) Use (a) and (b) to give an intuitive explanation of why the curl of a gradient is always 0.

d) Show that $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = 0$ directly.

Exercise 2. *Griffiths 1.13*. Let $\mathbf{d}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z), and let d be its length. Show that

a) $\boldsymbol{\nabla}(d^2) = 2\mathbf{d}$,

b) $\boldsymbol{\nabla}(1/d) = -\hat{d}/d^2$.

c) What is the general formula for $\boldsymbol{\nabla}(d^n)$?

d) You computed these formulas in cartesian coordinates. Do they hold in other coordinate systems? Why or why not?

Remark. To prove this would require quite a bit of work or more tools than we have at our disposal. However, you should be able to come up with an intuitive argument.

Exercise 3. The Stokes' and divergence theorems have generalizations to any dimension $n \in \mathbb{N}$. In this problem, you'll get an idea of what those are.

a) The divergence theorem directly generalizes to any dimension. Write down the generalization.

b) Focus for the divergence theorem on the 1- and 2-dimensional cases. Do either of these look familiar?

c) How could you generalize the Stokes' theorem to other dimensions?
Hint: Think about how the cross product is obtained in 3d, and try to generalize this first.

d) Using your results from (c), write down the generalization of Stokes' theorem in dimensions 1 and 2. Do either of them look familiar? You should obtain that in dimensions 1 and 2 the divergence and Stokes' theorems give *the same theorem*. Give an intuitive explanation for why that is.