Week 5 Worksheet
Relativistic Electrodynamics

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Exercise 1. The Faraday Tensor. Starting from the classical Lorentz force law for a particle of charge q moving with velocity $\mathbf{v}$,

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right), \tag{1}$$

derive a covariant Lorentz force law as follows.

a) Derive an equation for $d\mathbf{p}/d\tau$ in terms of (the components of) the four-velocity u and the fields $\mathbf{E}$ and $\mathbf{B}$.

b) Consider Poynting's theorem,

$$\frac{d\widetilde{U}}{dt} = \mathbf{j} \cdot \mathbf{E} - \nabla \cdot \mathbf{S},$$

where j is the 4-current, $\mathbf{S} = \mathbf{E} \times \mathbf{B}$ is the Poynting vector, and $\widetilde{U}$ denotes the energy density (so $\int_V \widetilde{U}\, d^3r = U$ is the energy contained in a volume V). Give a physical explanation for each term in the theorem (it may help to integrate both sides).

c) Use Poynting's theorem to show that

$$\frac{dp^0}{d\tau} = q\mathbf{E} \cdot \mathbf{u},$$

where u is the 4-velocity.
Hints: The *particle's* energy density is only the first term of Poynting's theorem. What value does the function $\mathbf{j}(\mathbf{r})$ take when $\mathbf{r} \neq \mathbf{r}'$, where $\mathbf{r}'$ is the location of the particle (at a given time)? What does this tell you about the integral $\int \mathbf{j} \cdot \mathbf{E} d^3r$?

d) Combine this and the classical Lorentz force law (1) to obtain a relativistic equation of motion

$$\frac{dp}{d\tau} = qF(u),$$

in terms of a tensor F which acts on u.
Hints: If u is a 4-vector—hence rank 1—and p is also a 4-vector, what rank must F be? To determine the components of F, compare the equation of motion you obtained in terms of u, $\mathbf{E}$, and $\mathbf{B}$ to the tensor equation

$$\frac{dp^\mu}{d\tau} = qF^\mu{}_\nu u^\nu.$$

Use index notation; for example, the cross product can be written as $(\mathbf{a} \times \mathbf{b})^k = \varepsilon_{ijk} a^i b^j$. Note that the "usual" form for F is $(F_{\mu\nu})$, which can be obtained from your result by lowering one index.

Exercise 2. Charge Conservation. Starting from Maxwell's equation

$$\partial_\nu F^{\mu\nu} = 4\pi j^\mu,$$

derive the equation of charge conservation

$$\partial_\mu j^\mu = 0,$$

and show that it corresponds to actual conservation of charge. Make sure to give a physical explanation of your result!
Hint: Is $\partial_\mu \partial_\nu$ a symmetric tensor? What is the contraction of an antisymmetric tensor with a symmetric one, i.e. if A is antisymmetric, and S is symmetric, then what do you know about $S_{\mu\nu} A^{\mu\nu}$?