

# Midterm 1 Review Solutions

**Exercise 1.** We can represent a counter-clockwise rotation by an angle  $\theta$  about an axis  $\hat{r}$  by the unitary operator

$$U(\theta) = e^{-i\theta\hat{r}\cdot\mathbf{S}/\hbar},$$

where  $\mathbf{S}$  is the angular momentum operator. For particles of spin 1/2,  $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$ .

- a) Show that  $(\hat{r} \cdot \boldsymbol{\sigma})^2 = \mathbb{1}$ , the identity operator.

*Hint:* Using  $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$  and  $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}$ , show first that

$$\sigma_i\sigma_j = \mathbb{1}\delta_{ij} + i\epsilon_{ijk}\sigma_k.$$

- b) Show that

$$U(\theta) = \mathbb{1} \cos(\theta/2) - i\hat{r} \cdot \boldsymbol{\sigma} \sin(\theta/2).$$

- c) Determine the spin operator  $\sigma_\theta$  which points in the direction described by  $(\theta, \varphi)$  with  $\varphi = 0$ .

*Hint:* Do this by rotating  $\sigma_z$  by an angle  $\theta$  about the  $y$ -axis.

- d) Redo problem 4.59 from Griffiths: If two electrons are in the spin singlet state,  $S_z^{(1)}$  is the component of spin angular momentum of particle 1 along the  $z$ -axis, and  $S_\theta^{(2)}$  is the spin angular momentum of particle 2 along the  $\hat{r} = (\theta, 0)$  axis, show that

$$\left\langle S_z^{(1)} S_\theta^{(2)} \right\rangle = -\frac{\hbar^2}{4} \cos \theta.$$

- a) Add the commutator to the anti-commutator to get the relation given in the hint. Now,

$$(\hat{r} \cdot \boldsymbol{\sigma})^2 = r_i\sigma_i r_j\sigma_j = r_i r_j (\mathbb{1}\delta_{ij} + i\epsilon_{ijk}\sigma_k) = \mathbb{1},$$

since  $\hat{r}^2 = \sum r_i^2 = 1$  and  $\epsilon_{ijk}$  is antisymmetric under interchange of indices, while  $r_i r_j$  is symmetric.

- b) Write

$$U(\theta) = e^{-i\theta\hat{r}\cdot\mathbf{S}/\hbar},$$

and expand the exponential as a power series:

$$\begin{aligned}
 U(\theta) &= \sum_{n=0}^{\infty} \frac{(-i\theta/2)^n (\hat{r} \cdot \boldsymbol{\sigma})^n}{n!} \\
 &= \sum_{k=0}^{\infty} \frac{(-i\theta/2)^{2k}}{(2k)!} \mathbb{1} + \sum_{k=0}^{\infty} \frac{(-i\theta/2)^{2k+1}}{(2k+1)!} \hat{r} \cdot \boldsymbol{\sigma} \\
 &= \mathbb{1} \cos(\theta/2) - i \hat{r} \cdot \boldsymbol{\sigma} \sin(\theta/2),
 \end{aligned}$$

where we use the power series for sine and cosine in the final line (and note that  $(-i)^{2k} = (-1)^k$  and  $(-i)^{2k+1} = -i(-1)^k$ ).

c) We have determined a nice form for the operator  $U(\theta)$ . Setting  $\hat{r} = \hat{y}$ , we have

$$U_y(\theta) = \mathbb{1} \cos(\theta/2) - i \sigma_y \sin(\theta/2).$$

A coordinate transformation which corresponds to  $U_y(\theta)$  will transform operators written in the original basis by

$$A \mapsto U_y(\theta) A U_y(\theta)^\dagger.$$

Thus, we have

$$\begin{aligned}
 \sigma_\theta &= U_y(\theta) \sigma_z U_y(\theta)^\dagger \\
 &= \sigma_z \cos^2(\theta/2) + i [\sigma_z, \sigma_y] \sin(\theta/2) + \sigma_y \sigma_z \sigma_y \sin^2(\theta/2) \\
 &= \sigma_z \cos \theta + \sigma_x \sin \theta \\
 &= \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}.
 \end{aligned}$$

d) Our spin state is

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle).$$

It's easiest to figure out what to do if we write this as vectors:

$$\frac{1}{\sqrt{2}} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$$

Thus, we get

$$\begin{aligned}
 \langle S_z^{(1)} S_\theta^{(2)} \rangle &= \frac{\hbar^2}{8} \left( \begin{bmatrix} 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \sin \theta \\ -\cos \theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \right) \\
 &= \frac{\hbar^2}{8} (-\cos \theta - \cos \theta) \\
 &= -\frac{\hbar^2}{4} \cos \theta.
 \end{aligned}$$

**Exercise 2. Griffiths 5.9.** Consider two non-interacting particles in an infinite square well of width  $a$  such that the single particle wavefunction is

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin(n\pi x/a)$$

with energy  $E_n = n^2 K$ . Construct the ground state and first excited state of the two-particle system if the particles are a) spin-1/2 and b) spin-1. Determine the energy and degeneracies of these states.

The solution to this problem is in the Week 3 Worksheet Solutions.

**Exercise 3. Helium.**

- Consider a singly-ionized helium ion. How much more energy does it take to ionize its bound electron compared to hydrogen?
- Still with  $\text{He}^+$ . What is the wavelength of the emitted photon during the electron transition from  $n = 2 \rightarrow 1$ ?
- Now, consider the usual helium-4. Which ground state has higher energy, parahelium (spin singlet) or orthohelium (spin triplet)? Why? **Griffiths 5.14**. How would this change if the two electrons are identical bosons?
- Griffiths 5.22**. Helium-3 is a fermion with spin-1/2 (as compared to helium-4, which is a boson. Why?). At low temperatures, helium-3 can be treated as a Fermi gas. If its mass density is  $82 \text{ kg/m}^3$ , determine its Fermi temperature.
- The only difference between singly-ionized helium and hydrogen is the number of protons. The energy levels for hydrogen-like atoms come from a potential in the Schrödinger equation which is proportional to  $Ze^2$ . Now, we have to remember that in hydrogen the energy levels go as  $Z^2$ . Explicitly,

$$\begin{aligned} E_n &= -\frac{1}{2} \frac{\alpha^2 Z^2 mc^2}{n^2} \\ &= -E_0 \frac{Z^2}{n^2}, \end{aligned}$$

where  $E_0 = 13.6 \text{ eV}$ . Thus, helium-3 is 4 times harder to ionize than hydrogen.

- Recall that  $E = pc$  for light; hence,

$$E = \frac{hc}{\lambda}.$$

Since  $E_2 - E_1 = 3E_0$ , we have

$$\lambda = \frac{hc}{3E_0}.$$

Since  $hc = 1240 \text{ eV} \cdot \text{nm}$ , we have

$$\lambda \approx 30 \text{ nm}.$$

- c) Since the triplet is symmetric, the spatial wavefunction which is associated to orthohelium must be antisymmetric. Since the lowest energy state is symmetric (both electrons in  $n = 1$ ), it follows that the triplet has higher energy than the singlet. If they were both bosons, then we'd have the opposite.
- d) Helium-3 is a fermion because it has an odd number of fermions. The Fermi energy is given by (make sure you can derive this!)

$$E_F = \frac{\pi^2 \hbar^2}{2m} \left( \frac{3n}{\pi} \right)^{2/3},$$

where  $n = N/V$  is the number density. Now,  $mn = \rho$  and  $E_F = kT_F$ , where  $k$  is Boltzmann's constant, so

$$T_F = \frac{\hbar^2}{2} \left( \frac{3\rho\pi^2}{m^{5/2}} \right)^{2/3}.$$

We can now plug in numbers:

$$T_F \approx 4 \text{ K}.$$

**Exercise 4.** Consider a transformation on a physical system represented by a unitary operator  $U$ .

- a) How do kets transform under  $U$ ? What about operators?
- b) If the hamiltonian  $H$  commutes with  $U$ , what does that imply about  $H$  being invariant under the transformation  $U$ ? What does this imply about a non-degenerate eigenstate of  $H$ ?
- c) Derive parity selection rules for hydrogen with respect to momentum and angular momentum matrix elements. I.e. determine when

$$\langle n'l'm' | \mathbf{p} | nlm \rangle = 0$$

and

$$\langle n'l'm' | \mathbf{L} | nlm \rangle = 0.$$

- a) Kets just transform as

$$|\psi\rangle \mapsto U |\psi\rangle.$$

Operators transform as

$$A \mapsto UAU^\dagger,$$

since we want to force  $\langle A \rangle$  to be unchanged after action by  $U$ . Indeed, since

$$\langle A \rangle = \langle \psi | A | \psi \rangle,$$

we have

$$\langle A \rangle \mapsto \langle \psi | U^\dagger A' U | \psi \rangle,$$

so we need

$$A' = UAU^\dagger.$$

- b) If the hamiltonian commutes with the symmetry  $U$ , then that means

$$UH = HU \implies UHU^\dagger = H.$$

Thus, if we have a nondegenerate eigenket of  $H$  with

$$H|\psi\rangle = \lambda|\psi\rangle,$$

then

$$HU|\psi\rangle = UH|\psi\rangle = U\lambda|\psi\rangle = \lambda U|\psi\rangle,$$

so that  $U|\psi\rangle$  is also an eigenket of  $H$ .

- c) We know that under parity  $\hat{x} \mapsto -\hat{x}$ . Indeed, since an operator is determined by its action on a basis, we can take the basis  $|x\rangle$  and consider the action of  $\pi\hat{x}\pi$  on it. We would then find

$$\pi x \pi |x\rangle = \pi x |-x\rangle = -x |x\rangle,$$

so we must have  $\pi x \pi = -x$ . Similarly, momentum is odd under parity:

$$[x, \pi p \pi] = x \pi p \pi - \pi p \pi x = -\pi x p \pi + \pi p x \pi = -\pi[x, p]\pi = -i\hbar,$$

so we must have  $\pi p \pi = -p$ . Now, we can write

$$\langle n'l'm'|\mathbf{p}|nlm\rangle = \langle n'l'm'|\pi\pi\mathbf{p}\pi\pi|nlm\rangle = -(-1)^{l+l'}\langle n'l'm'|\mathbf{p}|nlm\rangle.$$

Thus, this is exactly 0 for even values of  $l + l'$ . Similarly, we have

$$\langle n'l'm'|\mathbf{L}|nlm\rangle = (-1)^{l+l'}\langle n'l'm'|\mathbf{L}|nlm\rangle,$$

which is 0 for odd values of  $l + l'$ .

**Exercise 5. Dilations.** Do Exercise 2 on the Week 5 Worksheet: Another symmetry is called **dilation** symmetry. Dilations are given by the transformation  $\mathbf{x} \rightarrow \mathbf{x}' = e^c \mathbf{x}$ , where  $c \in \mathbb{R}$ . Call its generator  $D$ , so that  $e^{-icD}$  is the corresponding unitary operator.

**Remark.** In conformal field theory, the convention is to absorb the factor of  $i$  into  $D$ , so that  $e^{-cD}$  is the dilation operator.

- a) Show that the *infinitesimal* transformation

$$e^{i\mathbf{a}\cdot\mathbf{p}} e^{icD} e^{-i\mathbf{a}\cdot\mathbf{p}} e^{-icD}$$

is given by  $\mathbb{1} + c\mathbf{a} \cdot [D, \mathbf{p}]$ .

*Hints:* You can reduce to the situation where all the vectors are 1-dimensional (why?). There's a slick way to do this, but the brute force method does work.

- b) Calculate  $[D, \mathbf{p}]$ .

*Hint:* What coordinate transformation does the above correspond to? In other words, if you write it in the form  $\mathbf{x} \rightarrow \mathbf{x}'$ , what is  $\mathbf{x}'$ ?

- a) You could write out all of these exponentials out to second order in  $a$  and  $c$  (note that it's easier to work in 1 dimension for this whole problem). Another (slicker) way to get the same answer is to use the Baker-Campbell-Hausdorff formula, which says that given any two operators  $X$  and  $Y$ , we have

$$e^X e^Y = e^Z,$$

where

$$Z = X + Y + \frac{1}{2}[X, Y] + \dots.$$

The ellipsis above denotes third and higher order terms in  $X$  and  $Y$ , which we can ignore. Thus, use BCH on

$$e^{-iap} e^{-icD} = \exp\left(-iap - icD + \frac{ac}{2}[D, p] + \dots\right).$$

Then, use it on

$$e^{iap} e^{icD} = \exp\left(iap + icD + \frac{ac}{2}[D, p] + \dots\right).$$

Finally, use it on the product to get

$$e^{iap} e^{icD} e^{-iap} e^{-icD} = \exp(ac[D, p] + \dots).$$

Now, expand out the exponential to get the answer:

$$\mathbb{1} + ac[D, p].$$

The same argument works in 3 dimensions by linearity, so we're done.

- b) Note that the operation on the space that corresponds to the transformation given in (a) is:

$$\mathbf{x} \mapsto e^c \mathbf{x} \mapsto e^c \mathbf{x} + \mathbf{a} \mapsto \mathbf{x} + e^{-c} \mathbf{a} \mapsto \mathbf{x} + (e^{-c} - 1)\mathbf{a}.$$

Expanding the final term out to second order (since we went to second order in part (a)), we get

$$\mathbf{x} \mapsto \mathbf{x} + \left(-c + \frac{c^2}{2}\right) \mathbf{a}.$$

By (a), the infinitesimal transformation which corresponds to this is exactly  $\mathbb{1} + c\mathbf{a} \cdot [D, \mathbf{p}]$ . Since  $\mathbf{p}$  is the generator of translations, we see that in order to generate a translation  $\mathbf{x} \mapsto \mathbf{x} - c\mathbf{a}$ , we need to take  $[D, \mathbf{p}] = \mathbf{p}$  itself. If we had defined  $D$  to be the generator such that  $e^{-cD}$  is the corresponding unitary operator, then we get instead

$$[D, \mathbf{p}] = i\mathbf{p},$$

and this is how it's usually done in conformal field theory.