Week 6 Worksheet
Electrostatic Energy

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Exercise 1. In this exercise, you will find the energy contained in the electric field $\mathbf{E}$.

a) How much work does it take to assemble a configuration of two point charges?

b) What about n point charges?

c) Show that your result from (b) can be written

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i),$$

where q_i is the charge of the i^{th} charge and $\mathbf{r}_i$ is its position vector.

d) Generalize your result from (c) to a *continuous* charge distribution with (not necessarily uniform) charge density ρ.

e) Using the differential form of Gauss' law, $\rho = \varepsilon_0 \boldsymbol{\nabla}\cdot\mathbf{E}$, rewrite this in terms of $\mathbf{E}$ and V. Integrate by parts to transfer the derivative $\boldsymbol{\nabla}$ to V instead of $\mathbf{E}$, and argue that your answer is exactly

$$W = \frac{\varepsilon_0}{2}\int E^2 \,\mathrm{d}V,$$

where the integral is taken over *all space*.
Hint: Integration by parts in this context can be achieved by using the product rule on $\boldsymbol{\nabla}\cdot(\mathbf{v}f)$ (also the divergence theorem will help). Notice the similarity to 1-dimensional integration by parts, which can be derived from the identity $\frac{\mathrm{d}}{\mathrm{d}x}(fg) = f'g + fg'$.

Exercise 2. ***Griffiths 2.39.*** Two spherical cavities of radii a and b, respectively, are hollowed out from the interior of a neutral conducting sphere of radius R. At the center of each cavity is a point charge, q_a and q_b, respectively.

a) Find the surface charge densities σ_a, σ_b, and σ_R.

b) What is the field outside the conductor?

c) What is the field within each cavity?

d) What is the force on q_a and q_b?

e) Use the previous exercise to find the energy of this configuration.