# Midterm 1 Review

**Exercise 1.** We can represent a counter-clockwise rotation by an angle $\theta$ about an axis $\hat{r}$ by the unitary operator

$$U(\theta) = e^{-i\theta\hat{r}\cdot\mathbf{S}/\hbar},$$

where $\mathbf{S}$ is the angular momentum operator. For particles of spin 1/2, $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$.

a) Show that $(\hat{r}\cdot\boldsymbol{\sigma})^2 = \mathbb{1}$, the identity operator.
*Hint*: Using $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$ and $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}$, show first that

$$\sigma_i\sigma_j = \mathbb{1}\delta_{ij} + i\varepsilon_{ijk}\sigma_k.$$

b) Show that

$$U(\theta) = \mathbb{1}\cos(\theta/2) - i\hat{r}\cdot\boldsymbol{\sigma}\sin(\theta/2).$$

c) Determine the spin operator $\sigma_\theta$ which points in the direction described by $(\theta, \varphi)$ with $\varphi = 0$.
*Hint*: Do this by rotating $\sigma_z$ by an angle $\theta$ about the $y$-axis.

d) Redo problem 4.59 from Griffiths: If two electrons are in the spin singlet state, $S_z^{(1)}$ is the component of spin angular momentum of particle 1 along the $z$-axis, and $S_\theta^{(2)}$ is the spin angular momentum of particle 2 along the $\hat{r} = (\theta, 0)$ axis, show that

$$\left\langle S_z^{(1)} S_\theta^{(2)} \right\rangle = -\frac{\hbar^2}{4}\cos\theta.$$

**Exercise 2.** ***Griffiths 5.9.*** Consider two non-interacting particles in an infinite square well of width $a$ such that the single particle wavefunction is

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin(n\pi x/a)$$

with energy $E_n = n^2 K$. Construct the ground state and first excited state of the two-particle system if the particles are a) spin-1/2 and b) spin-1. Determine the energy and degeneracies of these states.

**Exercise 3. Helium.**

a) Consider a singly-ionized helium ion. How much more energy does it take to ionize its bound electron compared to hydrogen?

b) Still with $He^+$. What is the wavelength of the emitted photon during the electron transition from $n = 2 \to 1$?

c) Now, consider the usual helium-4. Which ground state has higher energy, parahelium (spin singlet) or orthohelium (spin triplet)? Why? ***Griffiths 5.14.*** How would this change if the two electrons are identical bosons?

d) ***Griffiths 5.22.*** Helium-3 is a fermion with spin-1/2 (as compared to helium-4, which is a boson. Why?). At low temperatures, helium-3 can be treated as a Fermi gas. If its mass density is 82 kg/m$^3$, determine its Fermi temperature.

**Exercise 4.** Consider a transformation on a physical system represented by a unitary operator $U$.

a) How do kets transform under $U$? What about operators?

b) If the hamiltonian $H$ commutes with $U$, what does that imply about $H$ being invariant under the transformation $U$? What does this imply about a non-degenerate eigenstate of $H$?

c) Derive parity selection rules for hydrogen with respect to momentum and angular momentum matrix elements. I.e. determine when

$$\langle n'l'm'|\mathbf{p}|nlm\rangle = 0$$

and

$$\langle n'l'm'|\mathbf{L}|nlm\rangle = 0.$$

**Exercise 5. Dilations.** Do Exercise 2 on the Week 5 Worksheet: Another symmetry is called **dilation** symmetry. Dilations are given by the transformation $\mathbf{x} \to \mathbf{x}' = e^c\mathbf{x}$, where $c \in \mathbb{R}$. Call its generator $D$, so that $e^{-icD}$ is the corresponding unitary operator.

**Remark.** In conformal field theory, the convention is to absorb the factor of $i$ into $D$, so that $e^{-cD}$ is the dilation operator.

a) Show that the *infinitesimal* transformation

$$e^{i\mathbf{a}\cdot\mathbf{p}}e^{icD}e^{-i\mathbf{a}\cdot\mathbf{p}}e^{-icD}$$

is given by $\mathbb{1} + c\mathbf{a} \cdot [D, \mathbf{p}]$.
*Hints*: You can reduce to the situation where all the vectors are 1-dimensional (why?). There's a slick way to solve this problem (use the Baker-Campbell-Hausdorff formula), but the brute force method does work.

b) Calculate $[D, \mathbf{p}]$.
*Hint*: What coordinate transformation does the above correspond to? In other words, if you write it in the form $\mathbf{x} \to \mathbf{x}'$, what is $\mathbf{x}'$?