Week 15 Worksheet
Waves and Energy

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Exercise 1. a) Starting with Maxwell's equations in vacuum, show that they give you wave equations for $\mathbf{E}$ and $\mathbf{B}$.

b) Show that the waves are transverse: If they travel in the z direction, then $\widetilde{E}_{0z} = 0$ and $\widetilde{B}_{0z} = 0$.

c) Show that $\mathbf{B}$ is perpendicular to $\mathbf{E}$.

Exercise 2. Recalling that magnetic forces do no work, we have that

$$\mathrm{d}W = \mathbf{F} \cdot \mathrm{d}\boldsymbol{\ell} = q\mathbf{E} \cdot \mathbf{v}\, \mathrm{d}t.$$

Thus, the rate at which work is done on the charges in a volume is

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int \mathbf{E} \cdot \mathbf{J}\, \mathrm{d}V.$$

a) Show that

$$\mathbf{E} \cdot \mathbf{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) - \frac{1}{\mu_0} \boldsymbol{\nabla} \cdot \mathbf{E} \times \mathbf{B}$$

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Hints: Use Maxwell's equations along with the identity

$$\boldsymbol{\nabla} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot \boldsymbol{\nabla} \times \mathbf{v} - \mathbf{v} \cdot \boldsymbol{\nabla} \times \mathbf{w}.$$

b) Plug this in to the formula for the rate of work, and derive Poynting's theorem:

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

is the energy per unit time, per unit area, transported by the fields; it is the energy flux density.

c) You showed in class that

$$\begin{aligned} \mathbf{g} &= \mu_0 \varepsilon_0 \mathbf{S} \\ &= \frac{1}{c^2} \mathbf{S} \end{aligned}$$

is the momentum per unit volume stored in the fields. Discuss why $\mathbf{S}$ seems to have two different physical interpretations.

Both of these exercises are done in Griffiths. The only comment I will make is for part (c) of the second exercise. Recall from special relativity

$$E^2 = p^2c^2 + m^2c^4.$$

Since $m = 0$ for an electromagnetic wave, we have the relation

$$E = pc.$$

So there is a proportionality relation between energy and momentum for light. Hence, when we say we have an energy density, we also have a momentum density directly from this formula. It is now straightforward to check that the units work out correctly: Since S has units of energy per unit area per unit time,

$$[S] = \frac{E}{L^2T},$$

this is the same by $E = pc$ as

$$[S] = \frac{M}{T^3},$$

where M denotes units of mass. Now, a momentum per unit volume has units of

$$\frac{M}{L^2T},$$

and we notice that this is exactly a factor of c^2 off from $[S]$.