Midterm 1 Review

Jacob Erlikhman

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Exercise 1. The starship Titanic is approaching a space-iceberg at velocity $V = 3/5$. It immediately fires a 100 kg missile at speed $4/5$ relative to the ship, hoping to break up the ice before they crash.

a) Find all components of the missile's momentum four-vector in the spaceship frame, assuming all motion is along the x-direction.

b) Do (a) in the iceberg frame.

c) The iceberg will break if the momentum impacting it is at least 10^{11} kg m/s. Does the iceberg break?

d) What is the velocity of the missile in the iceberg frame?

a) Set $v' = \frac{4}{5}$. Then $\gamma_{v'} = \frac{5}{3}$, and

$$p'^{\mu} = mu'^{\mu} = \begin{bmatrix} \frac{5}{3}m \\ \frac{4}{3}m \end{bmatrix}.$$

b) In the iceberg frame, the missile's velocity is

$$\begin{aligned} v =& \frac{V + v'}{1 + Vv'} \\ =& \frac{35}{37}. \end{aligned}$$

We compute that

$$\gamma_v = \frac{37}{12},$$

so that

$$p^{\mu} = \begin{bmatrix} \frac{37}{12}m \\ \frac{35}{12}m \end{bmatrix}.$$

c) Its x-momentum is

$$\frac{35}{12}c \cdot 100 \text{ kg} \lesssim 9 \cdot 10^{10} \text{ kg m/s},$$

so the iceberg does not break.

d) We already computed this in (b) as

$$v = \frac{35}{37}.$$

Exercise 2. Photon Rockets. Consider a rocket whose propellant is photons: The rocket emits photons to accelerate it—we'll call it a "photon rocket." Suppose such a photon rocket departed Earth with total mass M_0.

a) If the rocket will achieve a final speed with $\gamma = 2$, determine what final fraction M/M_0 of the ship's mass is remaining.

b) Show that the speed of the photon rocket can be written

$$v = \frac{1-(M/M_0)^2}{1+(M/M_0)^2}.$$

c) Astronomers on Earth watch the photon rocket recede through a telescope—they can observe the photons emitted in the rocket exhaust. What redshift $f_{\text{observed}}/f_{\text{emitted}}$ do they observe in terms of M_0 and the total mass M of the rocket when the photons are emitted?

a) Conservation of energy gives

$$M_0 = E_p + E,$$

where $E_p = p_p$ is the photon energy and E is the ship energy after it has reached the speed corresponding to $\gamma = 2$. On the other hand, conservation of momentum is

$$p_p = p,$$

where p is the ship momentum. Thus,

$$\begin{aligned} m_0 &= p + E \\ {m_0}^2 - 2Em_0 + E^2 &= E^2 - m^2 \\ 1 - 2\gamma\frac{m}{m_0} + \frac{m^2}{{m_0}^2} &= 0. \end{aligned}$$

Solving this for $\frac{m}{m_0}$, we find

$$\frac{m}{m_0} = \gamma - \sqrt{\gamma^2 - 1},$$

where we take the root with the minus sign because $m < m_0$. Plugging in $\gamma = 2$, we find

$$\frac{m}{m_0} = 2 - \sqrt{3}.$$

b) Returning back to the equation

$$1 - 2\gamma\frac{m}{m_0} + \frac{m^2}{{m_0}^2} = 0,$$

we solve it for γ to find

$$\gamma = \frac{1 + \left(\frac{m}{m_0}\right)^2}{\frac{2m}{m_0}}.$$

Solve this for v to get

$$\begin{aligned} v &= \sqrt{\frac{\left[1 + \left(\frac{m}{m_0}\right)^2\right]^2 - \left(\frac{2m}{m_0}\right)^2}{\left[1 + \left(\frac{m}{m_0}\right)^2\right]^2}} \\ &= \frac{1 - \left(\frac{m}{m_0}\right)^2}{1 + \left(\frac{m}{m_0}\right)^2}. \end{aligned}$$

c) Since the gravitational redshift goes as $\frac{1}{c^2}$, we can ignore it—the Doppler shift goes as $\frac{1}{c}$. The Doppler shift is given by

$$\begin{aligned} f_{\text{observed}}/f_{\text{emitted}} &= \sqrt{\frac{1 - v}{1 + v}} \\ &= \sqrt{\frac{2\left(\frac{m}{m_0}\right)^2}{2}} \\ &= \frac{m}{m_0}. \end{aligned}$$

Exercise 3. The **canonical stress-energy tensor** for an arbitrary lagrangian density $\mathscr{L}(q_1, \ldots, q_n)$ is defined by

$$T^{\mu\nu} = -\frac{\partial\mathscr{L}}{\partial(\partial_\mu q_k)}\partial^\nu q_k + \eta^{\mu\nu}\mathscr{L},$$

where η is the Minkowski metric.

a) Starting with the sourceless electromagnetic lagrangian density

$$\mathscr{L} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu},$$

show that

$$T^{\mu\nu} = \frac{1}{4\pi}\eta^{\mu\alpha}F_{\alpha\beta}\partial^\nu A^\beta + \eta^{\mu\nu}\mathscr{L}.$$

Hint: The generalized coordinates in relativistic electrodynamics should be thought of as packaged in the single gauge field $q_\mu = A_\mu$.

b) Check that

$$\int T^{00} d^3x = \frac{1}{8\pi} \int (E^2 + B^2) d^3x$$

is the energy contained in the EM field, and that

$$\int T^{0i} d^3x = \frac{1}{4\pi} \int (\mathbf{E} \times \mathbf{B})^i d^3x$$

is the i^{th} component of the momentum contained in the EM field.
Hints: First show that

$$\begin{aligned} T^{00} &= \frac{1}{8\pi}(E^2 + B^2) + \frac{1}{4\pi}\nabla \cdot (V\mathbf{E}) \\ T^{0i} &= \frac{1}{4\pi}(\mathbf{E} \times \mathbf{B})^i + \frac{1}{4\pi}\nabla \cdot (A^i\mathbf{E}), \end{aligned}$$

where $V = A^0$ is the scalar potential. The components of the tensor $F^\alpha{}_\beta$ are derived in the Week 5 Worksheet solutions; to obtain $F_{\alpha\beta}$ or $F^{\alpha\beta}$ from it, just raise or lower one index, e.g. $F_{\alpha\beta} = \eta_{\alpha\mu} F^\mu{}_\beta$.

a) First, write

$$\mathscr{L} = -\frac{1}{16\pi}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu),$$

and compute

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\beta)} &= -\frac{1}{4\pi}(\partial^\mu A^\beta - \partial^\beta A^\mu) \\ &= -\frac{1}{4\pi} F^{\mu\beta}. \end{aligned}$$

where one factor of two comes from product rule and the other comes from the two terms in each factor of $\mathscr{L}$. Thus,

$$\begin{aligned} -\frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\beta)} \partial^\nu A_\beta &= \frac{1}{4\pi} F^{\mu\beta} \partial^\nu A_\beta \\ &= \frac{1}{4\pi} F^\mu{}_\beta \partial^\nu A^\beta \\ &= \frac{1}{4\pi} \eta^{\mu\alpha} F_{\alpha\beta} \partial^\nu A^\beta. \end{aligned}$$

b) We plug in to find

$$T^{00} = \frac{1}{4\pi} F_{0\beta} \partial_t A^\beta - \mathscr{L},$$

where $\partial_t = \partial_0 = -\partial^0$. Now, you can compute that

$$\begin{aligned}\mathscr{L} &= -\frac{1}{16\pi}\operatorname{tr}(F^2)\\ &= \frac{1}{8\pi}(E^2 - B^2).\end{aligned}$$

Thus,

$$T^{00} = -\frac{1}{4\pi}\mathbf{E}\cdot\partial_t\mathbf{A} - \frac{1}{8\pi}(E^2 - B^2).$$

Now, recall that

$$\mathbf{E} = -\nabla V - \partial_t\mathbf{A},$$

so

$$\partial_t\mathbf{A} = -\mathbf{E} - \nabla V,$$

which along with the sourceless Maxwell's equation

$$\nabla\cdot\mathbf{E} = 0$$

gives the result for T^{00}.

For T^{0i}, we compute

$$\begin{aligned}T^{0i} &= -\frac{1}{4\pi}F_{0\beta}\partial^i A^\beta\\ &= \frac{1}{4\pi}\mathbf{E}\cdot\partial^i\mathbf{A}.\end{aligned}$$

Now,

$$(\mathbf{E}\times\mathbf{B})^i = E^j B^k \varepsilon^{ijk} = E^j\partial^m A^n\varepsilon^{ijk}\varepsilon^{mnk} = E^j\partial^m A^n(\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm}) = \mathbf{E}\cdot\partial^i\mathbf{A} - \mathbf{E}\cdot\nabla A^i.$$

On the other hand,

$$\nabla\cdot(A^i\mathbf{E}) = \mathbf{E}\cdot\nabla A^i,$$

since $\nabla\cdot\mathbf{E} = 0$. Combining these three calculations gives the result.

To see the integrals of $T^{0\mu}$ kill the second term, we need to use the divergence theorem and the principle of relativity. Namely, we can always assume that the fields are localized in some region of space: Even if they extend out to infinity, the far-away fields cannot affect the localized region we're interested in because of the finite speed of light. Thus, the integrals of the divergences become the flux through the boundary, but this is necessarily 0 because the fields are localized (and we take a large enough region of integration).

Exercise 4. Conservation of Energy and Momentum. Although the stress energy tensor of Exercise 3 satisfies $\partial_\mu T^{\mu\nu} = 0$, which is the conservation law corresponding to conservation of energy and momentum in relativistic electrodynamics, it fails to be manifestly Lorentz covariant and traceless. This latter property is required of massless photons. We can fix this by constructing a symmetric, traceless tensor from $T^{\mu\nu}$ as follows.

a) Use the definition of $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ to show that

$$T^{\mu\nu} = -\frac{1}{4\pi}\left(\eta^{\mu\alpha} F_{\alpha\beta} F^{\beta\nu} + \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right) + \frac{1}{4\pi}\eta^{\mu\alpha} F_{\alpha\beta}\partial^\beta A^\nu.$$

b) Using the sourceless Maxwell's equations (in their relativistic form), write the last term as

$$T_D^{\mu\nu} = -\frac{1}{4\pi}\partial_\alpha (F^{\alpha\mu} A^\nu),$$

and define

$$\Theta^{\mu\nu} = T^{\mu\nu} - T_D^{\mu\nu}.$$

c) Check that $\partial_\mu T_D^{\mu\nu} = 0$, so that the conservation law $\partial_\mu T^{\mu\nu} = 0$ implies the conservation law $\partial_\mu \Theta^{\mu\nu} = 0$.

d) Check that the trace $\Theta^\mu{}_\mu = 0$, $\Theta^{\mu\nu} = \Theta^{\nu\mu}$, and Θ defines the "usual" electromagnetic stress-energy tensor with components

$$\begin{aligned}
\Theta^{00} &= \frac{1}{8\pi}(E^2 + B^2) \\
\Theta^{0i} &= \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})^i \\
\Theta^{ij} &= -\frac{1}{4\pi}\left[E^i E^j + B^i B^j - \frac{1}{2}\delta^{ij}(E^2 + B^2)\right].
\end{aligned}$$

e) It turns out that when we add sources, Θ is the same as in the sourceless case. Show that

$$\partial_\mu \Theta^{\mu\nu} = -F^{\nu\alpha} j_\alpha,$$

and check that the $\nu = 0$ component of this is exactly Poynting's theorem from the Week 5 Worksheet. The space components give conservation of momentum.
Hint: To show the conservation equation, you will need to use both of Maxwell's equations with sources. The first equation is

$$\partial_\mu F^{\mu\nu} = -4\pi j^\nu.$$

The second is the Bianchi identity

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0.$$

Remark. The usual stress energy tensor for electromagnetism is the Θ which we have just obtained above. There is another way to derive it by noticing that it appears from gravity coupled to electromagnetism (we write the Einstein equation for the gravitational lagrangian plus the EM matter lagrangian).

a) Note that $\partial^\nu A^\beta = -F^{\beta\nu} + \partial^\beta A^\nu$. Plugging this in to $T^{\mu\nu}$ and writing out $\mathscr{L}$ gives

$$T^{\mu\nu} = -\frac{1}{4\pi}\left(\eta^{\mu\alpha}F_{\alpha\beta}F^{\beta\nu} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) + \frac{1}{4\pi}\eta^{\mu\alpha}F_{\alpha\beta}\partial^\beta A^\nu.$$

b) We have

$$\partial_\alpha(F^{\alpha\mu}A^\nu) = \partial_\alpha F^{\alpha\mu}A^\nu + F^{\alpha\mu}\partial_\alpha A^\nu.$$

Now, $\partial_\alpha F^{\alpha\mu} = 0$ is one of Maxwell's equations, and

$$\eta^{\mu\alpha}F_{\alpha\beta}\partial^\beta = F^\mu{}_\beta\partial^\beta = F^{\mu\beta}\partial_\beta = -F^{\beta\mu}\partial_\beta = -F^{\alpha\mu}\partial_\alpha.$$

This gives the desired rewriting of the final term of $T^{\mu\nu}$.

c) This follows easily from the form of $T_D^{\mu\nu}$ given in part (a):

$$\partial_\mu T_D^{\mu\nu} = -\frac{1}{4\pi}\partial_\mu(\eta^{\mu\alpha}F_{\alpha\beta}\partial^\beta A^\nu) = -\frac{1}{4\pi}(\partial^\alpha F_{\alpha\beta}\partial^\beta A^\nu + F_{\alpha\beta}\partial^\alpha\partial^\beta A^\nu) = 0,$$

where the first term is 0 by Maxwell's equation again and the second term is 0 by the fact that F is antisymmetric and $\partial^\alpha\partial^\beta$ is symmetric.

d) The trace is

$$\Theta^\mu{}_\mu = \Theta^{\mu\nu}\eta_{\nu\mu} = -\frac{1}{4\pi}(F_{\nu\beta}F^{\beta\nu} + F_{\alpha\beta}F^{\alpha\beta}) = 0$$

by antisymmetry of F.

The second term of $\Theta^{\mu\nu}$ is symmetric because $\eta^{\mu\nu}$ is. For the first term, we compute

$$\eta^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu} = F^\nu{}_\beta F^{\beta\mu} = F_\beta{}^\mu F^{\nu\beta} = \eta^{\mu\alpha}F_{\alpha\beta}F^{\beta\nu},$$

where in the last equality we use antisymmetry of F. Thus, this term is symmetric as well, so Θ is symmetric.

To find the components of Θ, it is sufficient to find the components of T_D and of T; we have found the latter in Exercise 3. For the former, we compute

$$\begin{aligned}T_D^{00} &= -\frac{1}{4\pi}\partial_\alpha(F^{\alpha 0}A^0)\\ &= -\frac{1}{4\pi}[-\partial_t(F^{00}A^0) - \nabla\cdot(A^0\mathbf{E})]\\ &= \frac{1}{4\pi}\nabla\cdot(V\mathbf{E}).\end{aligned}$$

Combining this with the result for T^{00}, we find $T^{00} - T^{00}_D = \frac{1}{8\pi}(E^2 + B^2)$. Similarly,

$$\begin{aligned} T^{0i}_D &= -\frac{1}{4\pi}\partial_\alpha(F^{\alpha 0}A^i) \\ &= \frac{1}{4\pi}\nabla \cdot (A^i \mathbf{E}). \end{aligned}$$

Again we find that $T^{0i} - T^{0i}_D = \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})^i$. To find the i, j components, recall from the Week 5 Worksheet that

$$F^{ij} = F^i{}_j = F_{ij} = \varepsilon^{ijk}B_k.$$

Thus,

$$\begin{aligned} \Theta^{ij} &= -\frac{1}{4\pi}\left(\eta^{i\alpha}F_{\alpha\beta}F^{\beta j} + \frac{1}{4}\delta^{ij}F_{\alpha\beta}F^{\alpha\beta}\right) \\ &= -\frac{1}{4\pi}\left[F_{i\beta}F^{\beta j} - \frac{1}{2}\delta^{ij}(E^2 - B^2)\right] \\ &= -\frac{1}{4\pi}\left[E^iE^j + F_{ik}F^{kj} - \frac{1}{2}\delta^{ij}(E^2 - B^2)\right]. \end{aligned}$$

Using the form for F^{ij} above, we find

$$F_{ik}F^{kj} = \varepsilon_{ikm}\varepsilon^{kjn}B^mB_n = -B^nB^m\left(\delta^{ij}\delta_{mn} - \delta^i{}_n\delta^j{}_m\right) = -\delta^{ij}B^2 + B^iB^j.$$

Plugging this into our result for Θ^{ij}, we find

$$\Theta^{ij} = -\frac{1}{4\pi}\left[E^iE^j + B^iB^j - \frac{1}{2}\delta^{ij}\left(E^2 + B^2\right)\right],$$

as desired.

e) We compute

$$\begin{aligned} \partial_\mu\Theta^{\mu\nu} &= -\frac{1}{4\pi}\left[-4\pi j_\alpha F^{\alpha\nu} + F_{\alpha\beta}\partial^\alpha F^{\beta\nu} + \frac{1}{4}(\partial^\nu F_{\alpha\beta}F^{\alpha\beta} + F_{\alpha\beta}\partial^\nu F^{\alpha\beta})\right] \\ &= -\frac{1}{4\pi}\left[-4\pi j_\alpha F^{\alpha\nu} + F_{\alpha\beta}\partial^\alpha F^{\beta\nu} + \frac{1}{2}F_{\alpha\beta}\partial^\nu F^{\alpha\beta}\right]. \end{aligned}$$

Now, we can use the Bianchi identity, also known as the second (relativistic) Maxwell's equation,

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0$$

to rewrite this as

$$\partial_\mu\Theta^{\mu\nu} = j_\alpha F^{\alpha\nu} - \frac{1}{8\pi}(F_{\alpha\beta}\partial^\alpha F^{\beta\nu} - F_{\alpha\beta}\partial^\beta F^{\nu\alpha}).$$

We attack each term in the parentheses separately.

$$F_{\alpha\beta}\partial^\alpha F^{\beta\nu} = F_{\alpha\beta}(\partial^\alpha\partial^\beta A^\nu - \partial^\alpha\partial^\nu A^\beta) = -F_{\alpha\beta}\partial^\alpha\partial^\nu A^\beta,$$

where the final equality follows by antisymmetry of F and symmetry of $\partial^\alpha\partial^\beta$. On the other hand

$$F_{\alpha\beta}\partial^\beta F^{\nu\alpha} = F_{\alpha\beta}\partial^\beta\partial^\nu A^\alpha = -F_{\alpha\beta}\partial^\alpha\partial^\nu A^\beta,$$

again by antisymmetry of F and by replacing dummy indices in the last equality. But this is exactly equal to the first term. Thus, they cancel, and

$$\begin{aligned}\partial_\mu\Theta^{\mu\nu} &= F^{\alpha\nu} j_\alpha \\ &= -F^{\nu\alpha} j_\alpha.\end{aligned}$$

We plug in $\nu = 0$ to find

$$\partial_\mu\Theta^{\mu 0} = -\mathbf{E}\cdot\mathbf{j}.$$

The LHS is

$$\partial_\mu\Theta^{\mu 0} = \frac{1}{8\pi}\partial_t(E^2 + B^2) + \frac{1}{4\pi}\nabla\cdot(\mathbf{E}\times\mathbf{B}).$$

Now, we have $\frac{1}{8\pi}(E^2 + B^2) = u$, and $\frac{1}{4\pi}\mathbf{E}\times\mathbf{B} = \mathbf{S}$, so the conservation equation reads

$$\partial_t u = -\mathbf{E}\cdot\mathbf{j} - \nabla\cdot\mathbf{S},$$

which is exactly Poynting's theorem.

Exercise 4. Do the following parts of Exercise 2 on the Week 7 Worksheet. Consider the **Poincaré upper half-plane** $\mathbb{H}^2 = \{(x, y) | y > 0\}$ with the metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2).$$

Don't bother to do this for review, but in the Week 7 Worksheet you were asked to show that semicircles with center on the x-axis and lines parallel to the y-axis are geodesics. In fact, it's true that these are all of the geodesics; you can assume this for this problem.

a) Check that all geodesics have infinite length in either direction. We say that $\mathbb{H}^2$ is **complete**.

b) Compute the area of the following **triangle**—a 3-sided figure whose sides are all geodesics. Let one side be the arc corresponding to an angle α (measured from the positive x-axis) of the unit semicircle centered at $(0, 0)$. Let the other two sides be vertical lines which extend out from the endpoints of the arc. We still call this a triangle because the vertical lines meet at infinity.

a) The length of a geodesic $c(t)$ is

$$\int_a^b \sqrt{c^i g_{ij} c^j}\, dt.$$

If our geodesic is a semicircle $c(t) = (t, \gamma(t))$ with center $(C, 0)$ on the x-axis, we find its length to be

$$\int_a^b \left(\frac{1}{\gamma} + \frac{\dot{\gamma}}{\gamma}\right) dt.$$

For $\gamma(t) = \sqrt{R^2 - (t-C)^2}$,

$$\dot{\gamma}(t) = -\frac{t-C}{\sqrt{R^2 - (t-C)^2}},$$

so its length is

$$\int_{C-R}^{C+r} \left(\frac{1}{\sqrt{R^2 - (t-C)^2}} - \frac{t-C}{R^2 - (t-C)^2}\right) dt = \left.\frac{\ln(R^2 - (t-C)^2)}{2}\right|_{C-R}^{C+R} + \alpha \to \infty,$$

where α is a finite constant. If instead we have a vertical line (C, t), its length is

$$\int_0^\infty \frac{1}{y^2} dy = \infty.$$

b) The area element is given by

$$\frac{1}{y^2} dx dy,$$

so we need to compute

$$\int_{\cos\alpha}^1 dx \int_{\sqrt{1-x^2}}^\infty \frac{dy}{y^2} = \int_{\cos\alpha}^1 \frac{dx}{\sqrt{1-x^2}} = -\arccos x\Big|_{\cos\alpha}^1 = \alpha.$$