

Week 11 Worksheet

Magnets!

Jacob Erlikhman

Exercise 1. The Lorentz force law for a particle of charge q moving with velocity \mathbf{v} in a magnetic field \mathbf{B} is given by

$$\mathbf{F}_{\text{mag}} = q\mathbf{v} \times \mathbf{B}.$$

Find an expression for the work that the magnetic force does on the particle.

The work is 0, since $d\ell = \mathbf{v} dt$, so

$$dW = \mathbf{F} \cdot d\ell = q\mathbf{v} \times \mathbf{B} \cdot \mathbf{v} dt = 0.$$

Exercise 2. Find the magnetic field and vector potential due to a current which flows with constant surface density \mathbf{K} which flows along the surface of an infinite cylinder of radius a in the following directions:

- a) along the axis of the cylinder;
 - b) perpendicular to the axis of the cylinder;
 - c) at an angle α to the axis of the cylinder.
- a) The magnetic field can be found using Ampère's law. Clearly, the field will be along $\hat{\phi}$ if the axis of the cylinder is along the z -axis. Ampère's law then gives that the field will be 0 inside the cylinder. Outside, it will be given by

$$\mu_0 K \cdot 2\pi a \ell = \ell B(s) 2\pi s \implies \mathbf{B}(s) = \frac{\mu_0 K a}{s} \hat{\phi}.$$

To find the vector potential, use

$$\oint \mathbf{A} \cdot d\ell = \int \mathbf{B} \cdot d\mathbf{a},$$

which you can obtain from Stokes' theorem and the fact that $\mathbf{B} = \nabla \times \mathbf{A}$. Also, since \mathbf{B} is the curl of \mathbf{A} , we see that if \mathbf{B} is along $\hat{\phi}$, then \mathbf{A} will be along \hat{z} . It follows that outside the cylinder we can use a rectangular amperian loop which is placed so that \mathbf{B} is perpendicular to it while \mathbf{A} is parallel to two of its sides (and perpendicular to the other two). This loop will then give that

$$A(s_1) - A(s_2) = \mu_0 K a \ln\left(\frac{s_2}{s_1}\right).$$

We can thus set

$$\mathbf{A}(s) = -\mu_0 K a \ln(s) \hat{z}$$

outside the cylinder. Notice that this agrees with everything: $\nabla \times \mathbf{A} = \mathbf{B}$, $\nabla \cdot \mathbf{A} = 0$, and it satisfies the relation above. Inside the cylinder, we can set

$$\mathbf{A}(s) = -\mu_0 K a \ln(a) \hat{z},$$

which clearly satisfies the requirements (and gives us a continuous function for \mathbf{A}).

- b) This situation is the same as a solenoid. As in that case, the magnetic field is 0 outside (since it has to go to 0 as $s \rightarrow \infty$ and an amperian loop gives that $B(s_1) = B(s_2)$ for $s_i > a$). Inside, we instead have that

$$B\ell = \mu_0 K \ell$$

by Ampère's law, so

$$\mathbf{B}(s) = \mu_0 K \hat{z}$$

inside (note that this is constant). Since \mathbf{B} is the curl of \mathbf{A} , we have as in (a) that if \mathbf{B} is along \hat{z} , \mathbf{A} will be along $\hat{\phi}$. Using this, we can proceed as in (a) with an amperian loop inside the solenoid that circles around the z -axis. This will give

$$\mu_0 K \cdot \pi s^2 = A \cdot 2\pi s,$$

so that

$$\mathbf{A}(s) = \frac{\mu_0 K s}{2} \hat{\phi}$$

inside the solenoid. On the other hand, using an amperian loop with radius $s > a$, we have

$$\mu_0 K \cdot \pi a^2 = A \cdot 2\pi s,$$

so

$$\mathbf{A}(s) = \frac{\mu_0 K a^2}{2s} \hat{\phi}.$$

Note that this form for \mathbf{A} is continuous at the boundary $s = a$, i.e. $\mathbf{A}_{\text{in}}(a) = \mathbf{A}_{\text{out}}(a)$. You can also check that $\nabla \times \mathbf{A} = \mathbf{B}$ everywhere and $\nabla \cdot \mathbf{A} = 0$ everywhere.

- c) In this case our \mathbf{K} can be written as

$$\mathbf{K} = K_{\perp} \hat{\phi} + K_{\parallel} \hat{z}.$$

Now, since *everything is linear*, the solutions in this case will be linear combinations of the solutions in parts (a) and (b), with the appropriate values of K . This statement means that 1. $\mathbf{B} = \nabla \times \mathbf{A}$ is linear, 2. $\nabla \cdot \mathbf{A} = 0$ is linear, and 3. \mathbf{B} satisfies the principle of superposition. Hence, our answers are

$$\begin{aligned}\mathbf{B}_{\text{out}}(s) &= \mu_0 K_{\parallel} \frac{a}{s} \hat{\phi} \\ \mathbf{B}_{\text{in}}(s) &= \mu_0 K_{\perp} \hat{z} \\ \mathbf{A}_{\text{in}}(s) &= \frac{\mu_0 K_{\perp} s}{2} \hat{\phi} - \mu_0 K_{\parallel} a \ln(a) \hat{z} \\ \mathbf{A}_{\text{out}}(s) &= \frac{\mu_0 K_{\perp} a^2}{2s} \hat{\phi} - \mu_0 K_{\parallel} a \ln(s) \hat{z}.\end{aligned}$$

Exercise 3. Griffiths 5.13. Suppose you have two infinite, parallel line charges λ a distance d apart, which are moving at a constant speed v . How great would v have to be for the magnetic attraction to balance the electrical repulsion? Calculate the number, and comment on the result.

Hint: The speed of light is

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}.$$

Nobody got to this exercise, and it's quite good. I think it will benefit you more if I leave it without a solution, and you can work on it during your studying for the final.