Week 9 Worksheet
Midterm Review

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Exercise 1. a) ***Griffiths 3.52a.*** Show that the quadrupole term in the multipole expansion,

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0 r^3}\int r'^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')\,\mathrm{d}V',$$

where α is the angle between $\mathbf{r}$ and $\mathbf{r}'$, can be written

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0 r^3}\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j Q_{ij},$$

where $\hat{r}_i\hat{r}_j Q_{ij} = \hat{r}\cdot(Q\hat{r})$ and

$$Q_{ij} = \frac{1}{2}\int[3r'_i r'_j - (r')^2\delta_{ij}]\rho(\mathbf{r}')\,\mathrm{d}V'.$$

We call Q_{ij} the **quadrupole moment** of the charge distribution. Notice that the monopole moment Q is a scalar, the dipole moment $\mathbf{p}$ is a vector, the quadrupole moment Q_{ij} is a second rank tensor (i.e. a matrix), and so on.

b) At the center of a line of charge of length 2ℓ with linear charge density λ is placed a point charge $q = -2\lambda\ell$. Find the quadrupole moment of this system and the potential at large distances.

a) Notice that $\cos\alpha = \hat{r}\cdot\hat{r'}$. Hence, if we plug in the first term of Q_{ij} into V_{quad} above we get

$$\begin{aligned}
V_{\text{quad}} =& \frac{1}{4\pi\varepsilon_0 r^3}\frac{1}{2}\int\left(\sum_{i,j}3r'_i\hat{r}_i\hat{r}_j r'_j - \sum_i \hat{r}_i{}^2 r'^2\right)\rho(\mathbf{r}')\,\mathrm{d}V' \\
=& \frac{1}{8\pi\varepsilon_0 r^3}\int\left(3r'^2\hat{r'}\cdot\hat{r} - r'^2\right)\rho(\mathbf{r}')\,\mathrm{d}V' \\
=& \frac{1}{4\pi\varepsilon_0 r^3}\int r'^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')\,\mathrm{d}V'.
\end{aligned}$$

b) If we place the point charge at the origin, then its a pure monopole and has no quadrupole moment. We need only determine Q_{ij} for the line of charge, then. If we place the line of charge along the x-axis, then $r'_i = 0$ for $i \neq 1$. Hence, all off-diagonal terms, i.e. Q_{ij} with $i \neq j$ are nonzero. Also, notice that $Q_{11} = -2Q_{22} = -2Q_{33}$. Thus, we have

$$Q_{22} = Q_{33} = -\frac{\lambda \ell^3}{3},$$

so that

$$Q_{11} = \frac{2\lambda \ell^3}{3}.$$

We can thus write the matrix

$$(Q_{ij}) = \frac{\lambda \ell^3}{3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Now, the potential at large distances will be given by the dipole contribution from the line, the quadrupole contribution from the line, and the monopole contributions from the point charge and the line. The dipole moment of the line is *zero*, since the integral

$$\begin{aligned} \mathbf{p} &= \int \mathbf{r}' \rho(\mathbf{r}') \, \mathrm{d}V' \\ &= \hat{x} \int_{-\ell}^{\ell} x \, \mathrm{d}x \\ &= 0. \end{aligned}$$

The monopole moment of the line is just

$$\begin{aligned} V_{\text{mon}}(\mathbf{r}) &= \frac{1}{4\pi \varepsilon_0 r} \int_{-\ell}^{\ell} \lambda \, \mathrm{d}x \\ &= \frac{2\lambda \ell}{4\pi \varepsilon_0 r}, \end{aligned}$$

which is exactly opposite to the monopole contribution from the point charge. It follows that the potential at large distances is just the quadrupole potential

$$\frac{1}{4\pi \varepsilon_0 r^3} \sum \hat{r}_i \hat{r}_j Q_{ij} = \frac{\lambda \ell^3}{12\pi_0 r^3} \left(2\hat{r}_1{}^2 - \hat{r}_2{}^2 - \hat{r}_3{}^2 \right).$$

Exercise 2. Two infinite, solid conducting cones have common axis (z), common vertex (O), and equal opening angles 2α (i.e. the equations which define the surfaces of the cones in spherical coordinates are $\theta = \alpha$ and $\theta = \pi - \alpha$, respectively). The potential difference between the cones is V_0, and they are electrically insulated from each other.

a) Start with the azimuthally symmetric laplacian in spherical coordinates:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right),$$

and find the θ equation by separation of variables.
Hint: Write the r equation as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)=\ell(\ell+1)R.$$

b) Recall that the equation has a set of solutions given by the Legendre polynomials, but this is *only one solution to the equation*. Since we have a second order equation, there should be another one. It is given by $\ln(\tan(\theta/2))$. Check that it satisfies the equation from (a). For what values of θ is this solution valid?

c) Find the potential and electric field in the region between the cones.

a) This is done in Griffiths.

b) We plug in $\Theta=\ln\tan(\theta/2)$ to the θ equation at $\ell=0$,

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=0,$$

to obtain

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\sec^2(\theta/2)}{2\tan(\theta/2)}\right)=\frac{\partial}{\partial\theta}\left(\sin\theta\frac{1}{2\sin(\theta/2)\cos(\theta/2)}\right)=\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\sin\theta}\right)=0,$$

so it is indeed a solution. It is valid for $\theta\notin\{0,\pi\}$, i.e. it is valid in situations where the z-axis is excluded from the region where we are trying to calculate V.

c) We try to match the usual solution,

$$V(r,\theta)=\sum\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)P_\ell\cos(\theta)$$

to the boundary conditions, which are given by

$$\begin{aligned}V(r,\alpha)&=V_0/2\\ V(r,\pi-\alpha)&=-V_0/2.\end{aligned}$$

But this clearly has no solution, so we are forced to include the extra solution obtained in (b). This solution does work:

$$\begin{aligned}A\ln\tan(\alpha/2)&=\frac{V_0}{2}\\ A\ln\tan(\pi/2-\alpha/2)&=-\frac{V_0}{2}\end{aligned}$$

has a solution, since $\tan(\pi/2-\alpha/2) = 1/\tan(\alpha/2)$. In particular, this solution is

$$A = \frac{V_0}{2\ln\tan(\alpha/2)}.$$

So the general solution is (we have to set all A_ℓ, B_ℓ to zero)

$$V(r,\theta) = \frac{V_0}{2\ln\tan(\alpha/2)}\ln\tan(\theta/2).$$