Week 5 Worksheet Solutions
Symmetries!

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Exercise 1. In this problem, you will construct the 2×2 matrix corresponding to a finite rotation which places the $\hat{z}$ axis along an arbitrary direction $\hat{r}$.

a) A rotation can be specified by the Euler angles (α, β, γ), or by (θ, φ). The Euler angles represent first a rotation about $\hat{z}$ by an angle α, then a rotation *about the new* y-axis by an angle β, and then a rotation about the *new* z-axis again. Convince yourself that this works.

b) Now, suppose given a rotation specified by the Euler angles (α, β, γ). This is given in quantum mechanics by the matrix

$$e^{-i\gamma S_{z'}/\hbar} e^{-i\beta S_u/\hbar} e^{-i\alpha S_z/\hbar},$$

where the u-axis is the new y-axis after rotating about z, and the z'-axis is the new z-axis after rotating about $\hat{z}$ and $\hat{u}$. Show that this is the same matrix as

$$e^{-i\alpha S_z/\hbar} e^{-i\beta S_y/\hbar} e^{-i\gamma S_z/\hbar}.$$

Hint:[1]

c) Use part (b) with $S_i = \frac{\hbar}{2}\sigma_i$ to calculate the rotation matrix corresponding to placing the $\hat{z}$ axis along $\hat{r}$, where $\hat{r}$ is specified by the two angles (θ, φ). Hint:[2]

d) Calculate the matrix corresponding to a rotation by π about $\hat{x}$.

e) Calculate the matrix corresponding to a 2π rotation about $\hat{z}$. Comment on the answer.

a) Convinced!

[1]Denoting a rotation about the axis r by an angle ζ as $R_r(\zeta)$, we have that $S_u = R_z(\alpha) S_y R_z(-\alpha) = e^{-i\alpha S_z/\hbar} S_y e^{i\alpha S_z/\hbar}$. Now, try to write a similar expression for $R_{z'}(\gamma) = e^{-i\gamma S_{z'}/\hbar}$.

[2]The idea is to Taylor expand each exponential. Think about a simple expression for ${\sigma_i}^n$, where σ_i is the Pauli matrix you need. Finally, one of the results you should get along the way is

$$e^{-i\beta\sigma_y/2} = \cos(\beta/2)\mathbb{1} - i\sigma_y \sin(\beta/2).$$

b) We have $S_u = R_z(\alpha) S_y R_z(-\alpha)$ and $S_{z'} = R_u(\beta) S_z R_u(-\beta)$. First of all, note that

$$\begin{aligned} R_u(\beta) &= \exp(-i\beta R_z(\alpha) S_y R_z(-\alpha)) \\ &= \sum_{n=0}^{\infty} \frac{(-i\beta)^n}{n!} (R_z(\alpha) S_y R_z(-\alpha))^n. \end{aligned}$$

Now, observe that $(R_z(\alpha) S_y R_z(-\alpha))^n = R_z(\alpha) S_y{}^n R_z(-\alpha)$. Thus,

$$\begin{aligned} R_u(\beta) &= R_z(\alpha) \sum_{n=0}^{\infty} \frac{(-i\beta S_y)^n}{n!} R_z(-\alpha) \\ &= R_z(\alpha) e^{-i\beta S_y} R_z(-\alpha) \\ &= R_z(\alpha) R_y(\beta) R_z(-\alpha). \end{aligned}$$

Likewise,

$$R_{z'}(\gamma) = R_u(\beta) R_z(\gamma) R_u(-\beta).$$

Putting everything together, we find

$$R_{z'}(\gamma) R_u(\beta) R_z(\alpha) = R_z(\alpha) R_y(\beta) R_z(\gamma).$$

c) We actually only need two Euler angles to achieve this, α and β, with $\theta = \beta$ and $\varphi = \alpha$. So we get

$$e^{-i\varphi\sigma_z/2} e^{-i\theta\sigma_y/2}.$$

We work one term at a time. The first term is easy since σ_z is diagonal (recall [or immediately prove!] that for a diagonal matrix $D = (d_1, \ldots, d_n)$, $e^D = (e^{d_1}, \ldots, e^{d_n})$), so

$$e^{-i\varphi\sigma_z/2} = \begin{bmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{bmatrix}.$$

For the second term, we get

$$e^{-i\theta\sigma_y/2} = \mathbb{1} \cos(\theta/2) - i\sigma_y \sin(\theta/2).$$

If you don't see this right away, try to write out the power series expansion, remembering that $\sigma_y^2 = \mathbb{1}$. Explicitly, we have

$$e^{-i\theta\sigma_y/2} = \sum_{j=0}^{\infty} \left(\frac{-i\theta}{2}\right)^{2j} \frac{1}{(2j)!} \mathbb{1} + \sum_{k=0}^{\infty} \left(\frac{-i\theta}{2}\right)^{2k+1} \frac{1}{(2k+1)!} \sigma_y.$$

Now, $(-i)^{2j} = (-1)^j$, while $(-i)^{2k+1} = -i(-1)^k$. Hence, the first term can be recognized as the power series expansion for $\cos(\theta/2)$, and the second as the power series expansion for $-i\sin(\theta/2)$. Putting it all together, we find

$$e^{-i\varphi\sigma_z/2} e^{-i\theta\sigma_y/2} = \begin{bmatrix} e^{-i\varphi/2} \cos(\theta/2) & -e^{-i\varphi/2} \sin(\theta/2) \\ e^{i\varphi/2} \sin(\theta/2) & e^{i\varphi/2} \cos(\theta/2) \end{bmatrix}.$$

d) This is just $e^{-i\pi\sigma_x/2}$. Notice that ${\sigma_x}^2 = \mathbb{1}$, so that $e^{-i\pi\sigma_x/2} = \cos(\pi/2)\mathbb{1} - i\sin(\pi/2)\sigma_x = -i\sigma_x$.

e) This is $e^{-i\pi\sigma_z} = -\mathbb{1}$. Thus, a 2π rotation about $\hat{z}$ of a spin 1/2 particle returns *negative* the particle state! This is a purely quantum mechanical phenomenon (and can be measured in practice).

Exercise 2. Another symmetry is called **dilation** symmetry. Dilations are given by the transformation $\mathbf{x} \to \mathbf{x}' = e^c\mathbf{x}$, where $c \in \mathbb{R}$. Call its generator D, so that e^{-icD} is the corresponding unitary operator.

a) Show that the *infinitesimal* transformation

$$e^{i\mathbf{a}\cdot\mathbf{p}}e^{icD}e^{-i\mathbf{a}\cdot\mathbf{p}}e^{-icD}$$

is given by $\mathbb{1} + c\mathbf{a}\cdot[D, \mathbf{p}]$.

b) Calculate $[D, \mathbf{p}]$.

a) You could write out all of these exponentials out to second order in a and c (note that it's easier to work in 1 dimension for this whole problem). Another (slicker) way to get the same answer is to use the Baker-Campbell-Hausdorff formula, which says that given any two operators X and Y, we have

$$e^X e^Y = e^Z,$$

where

$$Z = X + Y + \frac{1}{2}[X, Y] + \cdots.$$

The ellipsis above denotes third and higher order terms in X and Y, which we can ignore. Thus, use BCH on

$$e^{-iap}e^{-icD} = \exp\left(-iap - icD + \frac{ac}{2}[D, p] + \cdots\right).$$

Then, use it on

$$e^{iap}e^{icD} = \exp\left(iap + icD + \frac{ac}{2}[D, p] + \cdots\right).$$

Finally, use it on the product to get

$$e^{iap}e^{icD}e^{-iap}e^{-icD} = \exp(ac[D, p] + \cdots).$$

Now, expand out the exponential to get the answer:

$$\mathbb{1} + ac[D, p].$$

The same argument works in 3 dimensions by linearity, so we're done.

b) Note that the operation on the space that corresponds to the transformation given in (a) is:

$$\mathbf{x} \mapsto e^c\mathbf{x} \mapsto e^c\mathbf{x} + \mathbf{a} \mapsto \mathbf{x} + e^{-c}\mathbf{a} \mapsto \mathbf{x} + (e^{-c} - 1)\mathbf{a}.$$

Expanding the final term out to second order (since we went to second order in part (a)), we get

$$\mathbf{x} \mapsto \mathbf{x} + \left(-c + \frac{c^2}{2}\right)\mathbf{a}.$$

By (a), the infinitesimal transformation which corresponds to this is exactly $\mathbb{1} + c\mathbf{a} \cdot [D, \mathbf{p}]$. Since $\mathbf{p}$ is the generator of translations, we see that in order to generate a translation $\mathbf{x} \mapsto \mathbf{x} - c\mathbf{a}$, we need to take $[D, \mathbf{p}] = \mathbf{p}$ itself. If we had defined D to be the generator such that e^{-cD} is the corresponding unitary operator, then we get instead

$$[D, \mathbf{p}] = i\mathbf{p},$$

and this is how it's usually done in conformal field theory.