Week 3 Worksheet Solutions
Identical Particles

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Exercise 1. Symmetries of Many-Particle States.

a) Consider a system of two identical particles. Define a **permutation operator** via

$$P_{12} |\alpha\rangle |\beta\rangle = |\beta\rangle |\alpha\rangle .$$

Show that ${P_{12}}^2 = \mathbb{1}$, the identity operator, and that the eigenvalues of P_{12} are ± 1. Thus, show that its eigenvectors are either totally symmetric or antisymmetric.

b) Generalize part (a) to systems of three identical particles. You should find that you have *six* permutation operators. Assuming the hamiltonian is invariant under each of these operators, is there a complete set of common eigenvectors?

c) ***Griffiths 5.8.*** In the situation of (b), suppose that the particles have access to three distinct one-particle states, $|a\rangle$, $|b\rangle$, and $|c\rangle$. For example, $|abc\rangle$ is an allowed state, as is $|aaa\rangle$. How many states can be constructed if they are (i) bosons or (ii) fermions?

d) Suppose we have a single-particle fermion state $|\alpha\rangle$ and a single-particle bosonic state $|\beta\rangle$. Just like for the harmonic oscillator, we can define **creation operators** ${C_\alpha}^\dagger$ and ${a_\beta}^\dagger$, such that given any state $|\psi\rangle$,

$$\begin{aligned} {C_\alpha}^\dagger |\psi\rangle &= |\alpha\psi\rangle \\ {a_\beta}^\dagger |\psi\rangle &= |\beta\psi\rangle . \end{aligned}$$

The operators ${C_\alpha}^\dagger$ and ${a_\beta}^\dagger$ have the following properties.

$$\begin{aligned} C_\alpha |\alpha\psi\rangle &= |\psi\rangle \\ a_\beta |\beta\psi\rangle &= |\psi\rangle \\ C_\alpha |0\rangle = a_\beta |0\rangle &= 0 \\ {C_\alpha}^\dagger {C_\alpha}^\dagger &= 0 \\ \{C_\alpha, {C_{\alpha'}}^\dagger\} \equiv C_\alpha {C_{\alpha'}}^\dagger + {C_{\alpha'}}^\dagger C_\alpha &= \delta_{\alpha\alpha'} \mathbb{1} \\ \{{C_\alpha}^\dagger, {C_{\alpha'}}^\dagger\} = 0 \\ [a_\beta, {a_{\beta'}}^\dagger] &= \delta_{\alpha\alpha'} \mathbb{1} \\ [{a_\beta}^\dagger, {a_{\beta'}}^\dagger] &= 0, \end{aligned}$$

where $|0\rangle$ denotes a state with no particles at all. To what extent is a bound pair of fermions equivalent to a boson?
Hint: Use the symmetries of many-particle states and the (anti-)commutation relations of the creation/annihilation operators constructed in parts (a)-(d). What algebra must the creation/annihilation operators for the bound pair satisfy?

e) Prove the properties given in (d).
Hints: It may be useful to use the notation $\sim\alpha$ for the α "orbital" being *unoccupied*. To show the first relation for C_α, try to first show that $C_\alpha|\alpha\rangle = |0\rangle$. For the anti-commutator relations, consider separately the cases $\alpha \neq \alpha'$ and whether the α or α' orbitals are occupied.

a) ${P_{12}}^2 = \mathbb{1}$ follows by just applying it twice to the state. Suppose $|\lambda\rangle$ is an eigenvector of P_{12} with eigenvalue $\lambda \in \mathbb{C}$. Then

$$\lambda^2 = 1,$$

so $\lambda \in \{-1, 1\}$. An eigenvector with eigenvalue 1 will be symmetric, while an eigenvector with eigenvalue -1 will be antisymmetric.

b) We want to consider exchanges of three particles, so we will have $3! = 6$ distinct permutations. There is the identity operator, pairwise interchange, P_{12}, P_{23}, P_{13}, and the two cyclic permutations P_{123}, ${P_{123}}^2$. Since the permutation operators are not mutually commuting, we don't have a complete set of common eigenvectors. Instead, the space divides into four **invariant subspaces**, which have the property that any vector in an invariant subspace is transformed by the operators into a vector which is *in the same subspace*. Two of the subspaces are partially symmetric (and hence can be ignored), while the other two are the symmetric subspace and the antisymmetric one.

c) For bosons we have $10 = 6 + 3 + 1$ states. 6 states are symmetrizations of states of type (x, x, y), with $x, y \in \{a, b, c\}$ and $x \neq y$. We get $3 \cdot 2$ states of this type. The other type is (a, b, c), and there is only one state of this kind since we need to include all possible permutations to symmetrize. Finally, there are 3 states of the form (x, x, x).

For fermions, only 1 state is possible, since any state with repeating letters cannot be anti-symmetrized. Hence, the only type of allowed state is the anti-symmetrization of (a, b, c).

d) First some comments. In the Standard Model, quarks (which are fermions) can bind together to form particles. Some of these obey fermionic statistics, like the proton, while some obey bosonic statistics, like the pion. So two *distinguishable* fermions can bind together to form a boson in principle, and this is really what happens in our world! I say distinguishable here because in the case of quark pairs they are always different kinds of quarks. Let's see how this works out from the formalism.

If we consider only the single bound state by itself, i.e. the one-particle state, then there is no difference. Indeed, the spin is an integer, and there is no other requirement for bosons until we consider multi-particle states. Thus, consider a two-particle system made up of two such bound pairs. Under interchange of the pairs, we are effectively interchanging two fermions twice; thus, the composite system will be multiplied by $(-1)^2 = 1$. It follows that under interchange of pairs, the system obeys bosonic statistics. This clearly generalizes to systems of many pairs. Consider now the commutation relations between the creation and annihilation operators which make such a state. We need to consider the algebra generated by $D_{\alpha\alpha'} = C_\alpha C'_\alpha$ and $D_{\alpha\alpha'}{}^{+} = C_\alpha{}^\dagger C_{\alpha'}{}^\dagger$. Notice that $D_{\alpha\alpha'}{}^\dagger \neq D_{\alpha\alpha'}{}^{+}$! First of all, note that $(D^\dagger)^2 = D^2 = 0$, which is distinct from the bosonic creation operators. What about (anti-)commutators? Consider

$$[D_{12}{}^\dagger, D_{12}].$$

This is given by (after using the identity $[A, B] = \{A, B\} - 2BA$)

$$C_1{}^\dagger C_1 - 2C_1{}^\dagger C_1 C_2 C_2{}^\dagger - 2C_1{}^\dagger C_1 C_2{}^\dagger C_2 - 2C_1 C_2 C_1^\dagger C_2{}^\dagger + C_2 C_2{}^\dagger - 2C_1 C_1{}^\dagger C_2 C_2{}^\dagger.$$

Now, use the fact that C_i and C_j anti-commute, as do $C_i{}^\dagger$ and $C_j{}^\dagger$, to find that the first two terms with 4 operators cancel. On the other hand, notice that the last two terms with 4 operators are

$$-2C_1\{C_1{}^\dagger, C_2\}C_2{}^\dagger = 0.$$

Hence, we're left with

$$C_1{}^\dagger C_1 + C_2 C_2{}^\dagger.$$

Clearly, this is different than the algebra for the bosonic creation/annihilation operators! Already, we can say that we have something that's not quite a boson, just based on how we create and annihilate these new particles. This begs the question: Why in the Standard Model do quarks form actual bosons? That's because the Standard Model is a quantum field theory; everything is not so simple as it seems from this calculation!

e) Consider the state $C_\alpha |\alpha\rangle = (\langle\alpha| C_\alpha{}^\dagger)^\dagger$. Note that the state $\langle\alpha| C_\alpha{}^\dagger$ satisfies

$$\langle\alpha| C_\alpha{}^\dagger |\psi\rangle = \delta_{0,\psi}.$$

Hence, $(\langle\alpha| C_\alpha{}^\dagger)^\dagger = C_\alpha |\alpha\rangle = |0\rangle$. If instead of $|\alpha\rangle$ we use $|\alpha\psi\rangle$, we end up with the desired relation.

The proof for a_β is identical.

We have that

$$\langle 0| C_\alpha{}^\dagger |\psi\rangle = 0$$

for any ψ. Hence,

$$\langle 0| C_\alpha{}^\dagger = 0 \implies C_\alpha |0\rangle = (\langle 0| C_\alpha{}^\dagger)^\dagger = 0.$$

The property $(C_\alpha{}^\dagger)^2 = 0$ follows from the fact that fermions are antisymmetric under interchange.

Consider first the case $\alpha \neq \alpha'$. Then it's clear we get 0 if either the α orbital is empty or the α' orbital is occupied. So it's sufficient to consider its effect on a vector $|\alpha \cdots \sim \alpha'\rangle$. This gives

$$\begin{aligned}\{C_\alpha{}^\dagger, C_\alpha\} |\alpha \cdots \sim \alpha'\rangle &= C_\alpha |\alpha'\alpha \cdots\rangle + C_{\alpha'}{}^\dagger |\cdots \sim \alpha \sim \alpha'\rangle \\ &= - C_\alpha |\alpha\alpha' \cdots\rangle + C_{\alpha'}{}^\dagger |\cdots \sim \alpha \sim \alpha'\rangle \\ &= - |\alpha' \cdots \sim \alpha\rangle + |\alpha' \cdots \sim \alpha\rangle = 0.\end{aligned}$$

For the case $\alpha = \alpha'$, consider separately the cases of the α orbital being occupied or empty. We have

$$\{C_\alpha{}^\dagger, C_\alpha\} |\alpha \cdots\rangle = 0 + C_\alpha{}^\dagger |\cdots \sim \alpha\rangle = |\alpha \cdots\rangle$$
$$\{C_\alpha{}^\dagger, C_\alpha\} |\cdots \sim \alpha\rangle = C_\alpha |\alpha \cdots\rangle + 0 = |\cdots \sim \alpha\rangle .$$

Thus, $\{C_\alpha{}^\dagger, C_\alpha\}$ is the identity operator.

The key to figuring out this relation is to show that the creation operators and annihilation operators are the same as those for the harmonic oscillator. For example, consider $a_\beta{}^\dagger$. If it acts on $|0\rangle$, we get $|\beta\rangle$, but if it acts on $|\beta\rangle$, we get *something proportional to* $|2\beta\rangle$, since we can have multiple bosons in the same state. The key to figuring out this proportionality factor is to demand that $a_\beta{}^\dagger a_\beta$ acts as the **number operator** for the state β. So

$$a_\beta{}^\dagger a_\beta |n_1, n_2, \ldots, n_\beta, \ldots\rangle = n_\beta |n_1, n_2, \ldots, n_\beta, \ldots\rangle .$$

From the relation

$$\langle n_1, n_2, \ldots, n_\beta, \ldots| a_\beta{}^\dagger a_\beta |n_1, n_2, \ldots, n_\beta, \ldots\rangle = n_\alpha,$$

we find

$$a_\beta |n_1, n_2, \ldots, n_\beta, \ldots\rangle = \sqrt{n_\beta} |n_1, n_2, \ldots, n_\beta - 1, \ldots\rangle .$$

Now, we can determine the proportionality factor for $a_\beta{}^\dagger$. Taking

$$a_\beta a_\beta{}^\dagger |n_1, n_2, \ldots, n_\beta, \ldots\rangle = \sqrt{n_\beta + 1} c |n_1, n_2, \ldots, n_\beta, \ldots\rangle ,$$

where c is the proportionality factor to be determined. Hit this again with $a_\beta{}^\dagger$ to find

$$a_\beta{}^\dagger a_\beta a_\beta{}^\dagger |n_1, n_2, \ldots, n_\beta, \ldots\rangle = \sqrt{n_\beta + 1} c^2 |n_1, n_2, \ldots, n_\beta + 1, \ldots\rangle .$$

We can rewrite the LHS of this equation by noting that the first two operators are the number operator, so that

$$a_\beta{}^\dagger a_\beta a_\beta{}^\dagger |n_1, n_2, \ldots, n_\beta, \ldots\rangle = (n_\beta + 1) c |n_1, n_2, \ldots, n_\beta + 1, \ldots\rangle .$$

Thus,

$$c = \sqrt{n_\beta + 1}.$$

This immediately gives the desired relation, since it is that satisfied by the harmonic oscillator algebra. Whew!

The last identity follows since interchange of the two bosonic states which we create will not force a sign change, so the commutator will vanish.

Exercise 2.

Write down the hamiltonian for two noninteracting identical particles in the infinite square well. Write down the ground states for the three cases: distinguishable, fermions, bosons. Recall that the one-particle wavefunctions are

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right),$$

with energies $E_n = n^2\pi^2\hbar^2/2ma^2$.

Find the first three excited states and their energies for each of the three cases (distinguishable, fermions, bosons).

a) $H = \frac{p_1{}^2}{2m} + \frac{p_2{}^2}{2m} + V(x_1, x_2)$, where

$$V(x_1, x_2) = \begin{cases} 0, & x_1, x_2 \in [0, a] \\ \infty, & x_1 > a \text{ or } x_2 > a \end{cases}.$$

Distinguishable is just $\psi_1(x_1)\psi_1(x_2)$ (ground), $\psi_1(x_1)\psi_2(x_2)$ (1st and 2nd excited, along with the same with $x_1 \leftrightarrow x_2$), and $\psi_2(x_1)\psi_2(x_2)$ (3rd excited). Their energies are $2E_1$, $5E_1$, $5E_1$, and $8E_1$, respectively.

b) For bosons, we get almost the same thing. The 1st and 2nd excited states now merge into $(\psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2))/\sqrt{2}$. Thus, the ground state is the same, the 1st excited state is the one in the previous sentence, the second excited state is $\psi_2(x_1)\psi_2(x_2)$, and the third is $(\psi_3(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_3(x_2))/\sqrt{2}$. Their energies are $2E_1$, $5E_1$, $8E_1$, and $10E_1$, respectively.

c) For fermions, any manifestly symmetric state is now not allowed. Thus, the ground state is $(\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2))/\sqrt{2}$, the first excited is $(\psi_1(x_1)\psi_3(x_2) - \psi_3(x_1)\psi_1(x_2))/\sqrt{2}$, the second excited is $(\psi_2(x_1)\psi_3(x_2) - \psi_3(x_1)\psi_2(x_2))/\sqrt{2}$, and the third excited is $(\psi_1(x_1)\psi_4(x_2) - \psi_4(x_1)\psi_1(x_2))/\sqrt{2}$. Their energies are $5E_1$, $10E_1$, $13E_1$, and $17E_1$, respectively.

Exercise 3. In Exercise 2, we ignored spin (or at least supposed that the particles are in the same spin state).

a) Do it now for particles of spin 1/2. Construct the four lowest-energy configurations, and specify their energies and degeneracies.

b) Do the same for spin 1 (you will need the Clebsch-Gordan table from bCourses).

a) If you remember (or look up) the triplet and singlet states, you'll find that there are 3 symmetric ones and 1 antisymmetric one. Hence, we can pair a symmetric wavefunctiton with the antisymmetric spin state to get an antisymmetric state; conversely, we can pair an antisymmetric wavefunction with a symmetric spin state (in 3 different ways) to also get an antisymmetric state. Thus, the ground state for spin 1/2 particles is $\psi_1(x_1)\psi_1(x_2)$ paired with the singlet. It has multiplicity 1 and energy $2E_1$. The next highest state is $(\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2))/\sqrt{2}$ paired with the triplet *or* $(\psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2))/\sqrt{2}$ paired with the singlet. Hence, these states all have energy $5E_1$ and multiplicity 4. Continuing with the game, the next highest energy state is $\psi_2(x_1)\psi_2(x_2)$ paired with the singlet. This has multiplicity 1 and energy $8E_1$. Finally, the 3rd highest energy states are $(\psi_1(x_1)\psi_3(x_2) - \psi_3(x_1)\psi_1(x_2))/\sqrt{2}$ paired with the triplet or $(\psi_1(x_1)\psi_3(x_2) + \psi_3(x_1)\psi_1(x_2))/\sqrt{2}$ paired with the singlet, which again have multiplicity 4 and energy $10E_1$.

b) We take a look at the Clebsch-Gordan table. To play the same game, we want to find the states that are antisymmetric and those that are symmetric. Looking at the 1×1 subtable, I find 6 symmetric spin states and 3 antisymmetric. Other than this, the game is the same as in part (a). We get the same wavefunctions, but the multiplicities are instead 6, 9, 6, 9.