Week 4 Worksheet Solutions
Electrostatics

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Exercise 1. Using Dirac delta functions in the appropriate coordinates if necessary, express the following charge distributions as three-dimensional charge densities $\rho(\mathbf{r})$.

a) In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R.

b) In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b.

c) In cylindrical coordinates, a charge Q spread uniformly over a flat circular disc of negligible thickness and radius R.

d) Same as (c), but in spherical coordinates.

a)

$$\rho(\mathbf{r}) = \frac{Q}{4\pi R^2}\delta(r-R).$$

b) Place the cylinder axis along the z-axis. Then,

$$\rho(\mathbf{r}) = \frac{\lambda}{2\pi}\delta(s-b).$$

c) Place the disc in the plane $z=0$. Then,

$$\rho(\mathbf{r}) = \frac{Q}{\pi R^2}\delta(z).$$

d) This problem was ridiculously tricky, so I'm sorry for including it. This solution is beyond the scope of this course, so feel free to ignore it.

We want to set $\theta = \frac{\pi}{2}$, but we need to account for the r that appears in the θ part of the measure. In particular, the measure in spherical coordinates looks like

$$\mathrm{d}V = \mathrm{d}r\,(r\,\mathrm{d}\theta)(r\sin\theta\,\mathrm{d}\varphi),$$

so if we want to get rid of the θ part of the measure with a delta function, we better divide by r as well. Thus, we should have

$$\rho(\mathbf{r}) = \frac{Q}{\pi R^2 r}\delta(\theta - \pi/2).$$

Another way to see this is as follows. The point is that we start with the result from (c), and then consider the plate placed along $z = 0$. Now, recall that $z = r\cos\theta$, so we want

$$\delta(r\cos\theta).$$

We now need to use the fact that $\delta(f(x))|f'(x')| = \delta(x - x')$ if f has a root at x'. This follows from the following observation. Consider

$$\int \delta(f(x))g(x)|f'(x)|\,\mathrm{d}x = \int \delta(u)g(u)\,\mathrm{d}u,$$

where we set $u = f(x)$. Then we should obtain that $\delta(f(x))$ can be uniquely defined to be given by $\delta(x - x')/|f'(x')|$ if f has a root at x'. The idea is that we need to break up the domain in the initial integral to take out the $f'(x') = 0$ point. Since $r\cos\theta$ has a root at $\frac{\pi}{2}$ (and no other roots; remember, $\theta \in [0, \pi]$), we have

$$\delta(r\cos\theta) = \delta(\theta - \pi/2)/r,$$

since $r \geq 0$. Note that we want θ in the delta function; r should be allowed to vary as usual. This gives us the result,

$$\rho(\mathbf{r}) = \frac{Q}{\pi R^2 r}\delta(\theta - \pi/2).$$

Exercise 2. Two infinite parallel plates carry equal and opposite uniform charge densities $\pm\sigma$. Put the positively charged plate in the (x, y)-plane and the negatively charged one at $z = 1$ above it. Find the electric field in each of three regions: $z < 0$, $0 < z < 1$, and $z > 1$.

Use a "Gaussian pillbox" and the infinitude of the plates. The latter implies that given one such plate, the electric field will point directly perpendicular to it. Thus, we can use a Gaussian pillbox and Gauss' law to determine the electric field. Since it is everywhere perpendicular to one such plate, the electric field due to it will be

$$2E \cdot A = \pm\frac{\sigma A}{\varepsilon_0},$$

where A is the area of plate enclosed by the pillbox. The reason the factor of two appears is because the pillbox has two sides through which electric field exits, and both of them have area A; moreover, the electric field will either point "out" through both sides or "in" through both sides, depending on whether the plate is positively or negatively charged, respectively. Hence, we have

$$\mathbf{E} = \pm\frac{\sigma}{2\varepsilon_0}\hat{z}.$$

Now, suppose given two such plates with opposite charge densities. Then in between them the electric field will be $\frac{\sigma}{\varepsilon_0}\hat{z}$, for $z < 0$ it will be $\mathbf{0}$, and for $z > 0$ it will be $\mathbf{0}$ as well.