# Week 3 Worksheet
# Identical Particles

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January 28, 2024

**Exercise 1. Symmetries of Many-Particle States.**

a) Consider a system of two identical particles. Define a **permutation operator** via

$$P_{12}\left|\alpha\right\rangle\left|\beta\right\rangle = \left|\beta\right\rangle\left|\alpha\right\rangle.$$

Show that ${P_{12}}^2 = \mathbb{1}$, the identity operator, and that the eigenvalues of $P_{12}$ are $\pm 1$. Thus, show that its eigenvectors are either totally symmetric or antisymmetric.

b) Generalize part (a) to systems of three identical particles. You should find that you have *three* permutation operators. Assuming the hamiltonian is invariant under each of these operators, is there a complete set of common eigenvectors?

c) ***Griffiths 5.8.*** In the situation of (b), how many states can be constructed if they are (i) bosons or (ii) fermions?

d) Suppose we have a single-particle fermion state $\left|\alpha\right\rangle$ and a single-particle bosonic state $\left|\beta\right\rangle$. Just like for the harmonic oscillator, we can define **creation operators** ${C_\alpha}^\dagger$ and ${a_\beta}^\dagger$, such that given any state $\left|\psi\right\rangle$,

$$\begin{aligned} {C_\alpha}^\dagger\left|\psi\right\rangle &= \left|\alpha\psi\right\rangle \\ {a_\beta}^\dagger\left|\psi\right\rangle &= \left|\beta\psi\right\rangle. \end{aligned}$$

Show that

$$\begin{aligned} C_\alpha\left|\alpha\psi\right\rangle &= \left|\psi\right\rangle \\ a_\beta\left|\beta\psi\right\rangle &= \left|\psi\right\rangle \\ C_\alpha\left|0\right\rangle = a_\beta\left|0\right\rangle &= 0 \\ {C_\alpha}^\dagger{C_\alpha}^\dagger &= 0 \\ \{C_\alpha, {C_{\alpha'}}^\dagger\} \equiv C_\alpha{C_{\alpha'}}^\dagger + {C_{\alpha'}}^\dagger C_\alpha &= \delta_{\alpha\alpha'}\mathbb{1} \\ \{{C_\alpha}^\dagger, {C_{\alpha'}}^\dagger\} &= 0 \\ [a_\beta, {a_{\beta'}}^\dagger] &= \delta_{\alpha\alpha'}\mathbb{1} \\ [{a_\beta}^\dagger, {a_{\beta'}}^\dagger] &= 0, \end{aligned}$$

where $|0\rangle$ denotes a state with no particles at all.
*Hints*: It may be useful to use the notation $\sim\alpha$ for the $\alpha$ "orbital" being *unoccupied*. To show the first relation for $C_\alpha$, try to first show that $C_\alpha |\alpha\rangle = |0\rangle$. For the anti-commutator relations, consider separately the cases $\alpha \neq \alpha'$ and whether the $\alpha$ or $\alpha'$ orbitals are occupied.

e) To what extent is a bound pair of fermions equivalent to a boson?
*Hint*: Use the symmetries of many-particle states and the (anti-)commutation relations of the creation/annihilation operators constructed in parts (a)-(d).

**Remark.** The algebra satisfied by the bosonic raising and lowering operators is isomorphic to (i.e. the same as) the algebra satisfied by the harmonic oscillator raising and lowering operators.

**Exercise 2.** ***Griffiths 5.5.***

Write down the hamiltonian for two noninteracting identical particles in the infinite square well. Write down the ground states for the three cases: distinguishable, fermions, bosons. Recall that the one-particle wavefunctions are

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right),$$

with energies $E_n = n^2\pi^2\hbar^2/2ma^2$.

Find the first three excited states and their energies for each of the three cases (distinguishable, fermions, bosons).

**Exercise 3.** ***Griffiths 5.9.*** In Exercise 2, we ignored spin (or at least supposed that the particles are in the same spin state).

a) Do it now for particles of spin 1/2. Construct the four lowest-energy configurations, and specify their energies and degeneracies.

b) Do the same for spin 1. (You will need a table of Clebsch-Gordan coefficients.)