Week 10 Worksheet
Polarization and Dielectrics

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Exercise 1. a) Consider a line of dipoles of equal moment qd, where the end of one touches the beginning of the next. Suppose the line is of length ℓ. What is the dipole moment of the entire line?

b) Now, suppose you had a cylinder of length ℓ, cross-sectional area A, and uniform polarizaiton $\mathbf{P}$, which is parallel to the axis of the cylinder. What is the surface charge density on the ends of the cylinder?

c) Argue that the bound surface charge for any uniformly polarized object is

$$\sigma_b = \mathbf{P} \cdot \hat{n}.$$

Hint: Consider the cylinder as in (b), with axis along the x-axis, but this time with one of the ends cut at an angle θ relative to the y-axis.

d) What happens if the polarization is *nonuniform*? Consider a sphere with a diverging $\mathbf{P}$. The charge which accumulates at the edges must be σ_b, so what is the net charge inside the sphere?

e) Argue that

$$\int \rho_b \, \mathrm{d}V = -\oint \mathbf{P} \cdot \mathrm{d}\mathbf{a},$$

and use the divergence theorem to obtain that $\rho_b = -\nabla \cdot \mathbf{P}$ for the sphere of (d). Argue that this holds for any object.

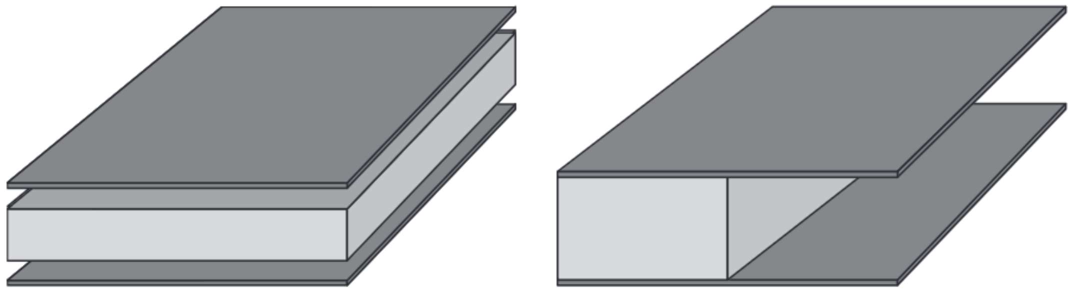
This is done in Griffiths.

Exercise 2. A linear dilectric of susceptibility χ_e is one in which

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}.$$

What is the relation between $\mathbf{D}$ and $\mathbf{E}$ for such dielectrics, and what is the “permittivity,” ε, of the material in terms of χ_e?

Also in Griffiths



(a) (b)

Exercise 3. ***Griffiths 4.19.*** Suppose you have enough linear dielectric matrial of dielectric constant ε_r to half-fill a parallel-plate capacitor. By what fraction is the capacitance increased when you distribute the material as in (a) vs. as in (b) in the figure below? For a fixed potential difference V between the plates, find $\mathbf{E}, \mathbf{D}$, and $\mathbf{P}$ in each region and the free and bound charge on all surfaces in both cases.
Hint: For the second part of the exercise, use $C = Q/V$ to find Q; then, use that to find everything else. Why can't we just use $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$ to find the polarization and hence everything else?

In case (a), we can calculate

$$\begin{aligned} V &= -\int \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} \\ &= -\frac{\sigma}{\varepsilon_0} \cdot \frac{h}{2} - \frac{\sigma}{\varepsilon} \cdot \frac{h}{2} \\ &= -\frac{\sigma h}{2} \left(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon} \right) \\ &= -\frac{\sigma h}{2} \cdot \frac{\varepsilon + \varepsilon_0}{\varepsilon_0 \varepsilon} \\ &= -\frac{h}{2\varepsilon_0} \cdot \frac{\varepsilon_r + 1}{\varepsilon_r}. \end{aligned}$$

Thus,

$$\begin{aligned} C &= \frac{Q}{V} \\ &= C_{\text{vac}} \cdot \frac{2\varepsilon_r}{\varepsilon_r + 1}. \end{aligned}$$

In case (b), we by the same method calculate that

$$\begin{aligned} V_{\text{vac}} &= -\frac{\sigma h}{\varepsilon_0} \\ V_{\text{di}} &= -\frac{\sigma h}{\varepsilon}, \end{aligned}$$

where V_{vac} is the potential in the region where there is no dielectric and V_{di} is the potential in the region where there is dielectric. To compute C, then, we can add the two contributions to each of the capacitances:

$$\begin{aligned} C =& C_1 + C_2 \\ =& \frac{A\varepsilon_0}{h} + \frac{A\varepsilon}{h} \\ =& C_{\text{vac}}(1+\varepsilon_r). \end{aligned}$$

In case (a), then, with fixed V there is a free charge of magnitude

$$\begin{aligned} Q =& CV \\ =& C_{\text{vac}}\frac{2\varepsilon_r V}{\varepsilon_r + 1}. \end{aligned}$$

We can now use Gauss' law for the displacement,

$$\oint \mathbf{D} \cdot \mathrm{d}\mathbf{a} = Q_{f,\text{enc}},$$

to solve for the displacement, which we know is given by

$$\mathbf{D} = \varepsilon \mathbf{E}.$$

So in case (a) in the region where there is no dielectric, the displacement is given by

$$D \cdot A = \frac{A\varepsilon_0}{h} \cdot \frac{2\varepsilon_r V}{\varepsilon_r + 1} \implies D = \frac{V\varepsilon_0}{h} \cdot \frac{\varepsilon_r}{2(\varepsilon_r + 1)}.$$

Hence, the electric field is

$$E = \frac{D}{\varepsilon_0},$$

where both the electric field and displacement point from the positively charged plate to the negatively charged one. Where there is dielectric, the displacement is the same, while the electric field is

$$\begin{aligned} E =& \frac{D}{\varepsilon} \\ =& \frac{D}{\varepsilon_0 \varepsilon_r}. \end{aligned}$$

The polarization is then 0 where there is no dielectric and given by

$$P = \varepsilon_0 \chi_e E$$

where there is dielectric. Now, $\chi_e = \varepsilon_r - 1$, so

$$P = \frac{\varepsilon_r - 1}{2(\varepsilon_r + 1)} \cdot \frac{\varepsilon_0 V}{h}.$$

Since P is constant, there is no ρ_b, while $\sigma_b = \mathbf{P} \cdot \hat{n}$, so

$$\sigma_b = -P$$

on the bottom of the dielectric and

$$\sigma_b = P$$

on the top of it.

In case (b), the total charge is

$$Q = C_{\text{vac}} V(1 + \varepsilon_r),$$

so in the region where there is no dielectric, the displacement can again be obtained by Gauss' law. It will be given by

$$DA = \frac{A\varepsilon_0}{h} V(1 + \varepsilon_r),$$

so

$$D = \frac{\varepsilon_0 V}{h}(1 + \varepsilon_r).$$

The electric field in this region will then be given by

$$\begin{aligned} E =& \frac{D}{\varepsilon_0} \\ =& \frac{V}{h}(1 + \varepsilon_r). \end{aligned}$$

The displacement is the same in the region where there is dielectric, so there the electric field is

$$\begin{aligned} E =& \frac{D}{\varepsilon} \\ =& \frac{V}{h\varepsilon_r}(1 + \varepsilon_r). \end{aligned}$$

From this, we can obtain P, which will be

$$\begin{aligned} P =& \varepsilon_0 \chi_e E \\ =& \frac{\varepsilon_0 V}{h\varepsilon_r}(1 - {\varepsilon_r}^2). \end{aligned}$$

As in the previous case, there is no ρ_b, and $\sigma_b = \pm P$.