Midterm 1 Review Solutions

Exercise 1. We can represent a counter-clockwise rotation by an angle θ about an axis $\hat{r}$ by the unitary operator

$$U(\theta) = e^{-i\theta\hat{r}\cdot\mathbf{S}/\hbar},$$

where $\mathbf{S}$ is the angular momentum operator. For particles of spin 1/2, $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$.

a) Show that $(\hat{r}\cdot\boldsymbol{\sigma})^2 = \mathbb{1}$, the identity operator.
Hint: Using $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$ and $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}$, show first that

$$\sigma_i\sigma_j = \mathbb{1}\delta_{ij} + i\varepsilon_{ijk}\sigma_k.$$

b) Show that

$$U(\theta) = \mathbb{1}\cos(\theta/2) - i\hat{r}\cdot\boldsymbol{\sigma}\sin(\theta/2).$$

c) Determine the spin operator σ_θ which points in the direction described by (θ, φ) with $\varphi = 0$.
Hint: Do this by rotating σ_z by an angle θ about the y-axis.

d) Redo problem 4.59 from Griffiths: If two electrons are in the spin singlet state, $S_z^{(1)}$ is the component of spin angular momentum of particle 1 along the z-axis, and $S_\theta^{(2)}$ is the spin angular momentum of particle 2 along the $\hat{r} = (\theta, 0)$ axis, show that

$$\left\langle S_z^{(1)} S_\theta^{(2)} \right\rangle = -\frac{\hbar^2}{4}\cos\theta.$$

a) Add the commutator to the anti-commutator to get the relation given in the hint. Now,

$$(\hat{r}\cdot\boldsymbol{\sigma})^2 = r_i\sigma_i r_j\sigma_j = r_i r_j(\mathbb{1}\delta_{ij} + i\varepsilon_{ijk}\sigma_k) = \mathbb{1},$$

since $\hat{r}^2 = \sum r_i{}^2 = 1$ and ε_{ijk} is antisymmetric under interchange of indices, while $r_i r_j$ is symmetric.

b) Write

$$U(\theta) = e^{-i\theta\hat{r}\cdot\mathbf{S}/\hbar},$$

and expand the exponential as a power series:

$$\begin{aligned}
U(\theta) &= \sum_{n=0}^{\infty} \frac{(-i\theta/2)^n (\hat{r}\cdot\boldsymbol{\sigma})^n}{n!} \\
&= \sum_{k=0}^{\infty} \frac{(-i\theta/2)^{2k}}{(2k)!}\mathbb{1} + \sum_{k=0}^{\infty} \frac{(-i\theta/2)^{2k+1}}{(2k+1)!}\hat{r}\cdot\boldsymbol{\sigma} \\
&= \mathbb{1}\cos(\theta/2) - i\hat{r}\cdot\boldsymbol{\sigma}\sin(\theta/2),
\end{aligned}$$

where we use the power series for sine and cosine in the final line (and note that $(-i)^{2k} = (-1)^k$ and $(-i)^{2k+1} = -i(-1)^k$).

c) We have determined a nice form for the operator $U(\theta)$. Setting $\hat{r} = \hat{y}$, we have

$$U_y(\theta) = \mathbb{1}\cos(\theta/2) - i\sigma_y\sin(\theta/2).$$

A coordinate transformation which corresponds to $U_y(\theta)$ will transform operators written in the original basis by

$$A \mapsto U_y(\theta) A U_y(\theta)^\dagger.$$

Thus, we have

$$\begin{aligned}
\sigma_\theta &= U_y(\theta)\sigma_z U_y(\theta)^\dagger \\
&= \sigma_z\cos^2(\theta/2) + i[\sigma_z, \sigma_y]\sin(\theta/2)\cos(\theta/2) + \sigma_y\sigma_z\sigma_y\sin^2(\theta/2) \\
&= \sigma_z[\cos^2(\theta/2) - \sin^2(\theta/2)] + 2\sigma_x\sin(\theta/2)\cos(\theta/2) \\
&= \sigma_z\cos\theta + \sigma_x\sin\theta \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.
\end{aligned}$$

d) Our spin state is

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle).$$

It's easiest to figure out what to do if we write this as vectors:

$$\frac{1}{\sqrt{2}}\left(\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}\right).$$

Thus, we get

$$\begin{aligned}
\left\langle S_z^{(1)} S_\theta^{(2)} \right\rangle &= \frac{\hbar^2}{8}\left(\begin{bmatrix}1 & 0\end{bmatrix}\otimes\begin{bmatrix}0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1\end{bmatrix}\otimes\begin{bmatrix}1 & 0\end{bmatrix}\right)\left(\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}\sin\theta\\-\cos\theta\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}\right) \\
&= \frac{\hbar^2}{8}(-\cos\theta - \cos\theta) \\
&= -\frac{\hbar^2}{4}\cos\theta.
\end{aligned}$$

Exercise 2. ***Griffiths 5.9.*** Consider two non-interacting particles in an infinite square well of width a such that the single particle wavefunction is

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin(n\pi x/a)$$

with energy $E_n = n^2 K$. Construct the ground state and first excited state of the two-particle system if the particles are a) spin-1/2 and b) spin-1. Determine the energy and degeneracies of these states.

The solution to this problem is in the Week 3 Worksheet Solutions.

Exercise 3. Helium.

a) Consider a singly-ionized helium ion. How much more energy does it take to ionize its bound electron compared to hydrogen?

b) Still with He^+. What is the wavelength of the emitted photon during the electron transition from $n = 2 \to 1$?

c) Now, consider the usual helium-4. Which ground state has higher energy, parahelium (spin singlet) or orthohelium (spin triplet)? Why? ***Griffiths 5.14.*** How would this change if the two electrons are identical bosons?

d) ***Griffiths 5.22.*** Helium-3 is a fermion with spin-1/2 (as compared to helium-4, which is a boson. Why?). At low temperatures, helium-3 can be treated as a Fermi gas. If its mass density is 82 kg/m^3, determine its Fermi temperature.

a) The only difference between singly-ionized helium and hydrogen is the number of protons. The energy levels for hydrogen-like atoms come from a potential in the Schrödinger equation which is proportional to Ze^2. Now, we have to remember that in hydrogen the energy levels go as Z^2. Explicitly,

$$\begin{aligned} E_n &= -\frac{1}{2}\frac{\alpha^2 Z^2 mc^2}{n^2} \\ &= -E_0 \frac{Z^2}{n^2}, \end{aligned}$$

where $E_0 = 13.6$ eV. Thus, helium-3 is 4 times harder to ionize than hydrogen.

b) Recall that $E = pc$ for light; hence,

$$E = \frac{hc}{\lambda}.$$

Since $E_2 - E_1 = 3E_0$, we have

$$\lambda = \frac{hc}{3E_0}.$$

Since $hc = 1240$ eV $\cdot$ nm, we have

$$\lambda \approx 30 \text{ nm}.$$

c) Since the triplet is symmetric, the spatial wavefunction which is associated to orthohelium must be antisymmetric. Since the lowest energy state is symmetric (both electrons in $n = 1$), it follows that the triplet has higher energy than the singlet. If they were both bosons, then we'd have the opposite.

d) Helium-3 is a fermion because it has an odd number of fermions. The Fermi energy is given by (make sure you can derive this!)

$$E_F = \frac{\pi^2\hbar^2}{2m}\left(\frac{3n}{\pi}\right)^{2/3},$$

where $n = N/V$ is the number density. Now, $mn = \rho$ and $E_F = kT_F$, where k is Boltzmann's constant, so

$$T_F = \frac{\hbar^2}{2}\left(\frac{3\rho\pi^2}{m^{5/2}}\right)^{2/3}.$$

We can now plug in numbers:

$$T_F \approx 4\text{ K}.$$

Exercise 4. Consider a transformation on a physical system represented by a unitary operator U.

a) How do kets transform under U? What about operators?

b) If the hamiltonian H commutes with U, what does that imply about H being invariant under the transformation U? What does this imply about a non-degenerate eigenstate of H?

c) Derive parity selection rules for hydrogen with respect to momentum and angular momentum matrix elements. I.e. determine when

$$\langle n'l'm'|\mathbf{p}|nlm\rangle = 0$$

and

$$\langle n'l'm'|\mathbf{L}|nlm\rangle = 0.$$

a) Kets just transform as

$$|\psi\rangle \mapsto U\,|\psi\rangle\,.$$

Operators transform as

$$A \mapsto UAU^\dagger,$$

since we want to force $\langle A\rangle$ to be unchanged after action by U. Indeed, since

$$\langle A\rangle = \langle\psi|\,A\,|\psi\rangle\,,$$

we have

$$\langle A\rangle \mapsto \langle\psi|\,U^\dagger A' U\,|\psi\rangle\,,$$

so we need

$$A' = UAU^\dagger.$$

b) If the hamiltonian commutes with the symmetry U, then that means

$$UH = HU \implies UHU^{\dagger} = H.$$

Thus, if we have a nondegenerate eigenket of H with

$$H\left|\psi\right\rangle = \lambda\left|\psi\right\rangle,$$

then

$$HU\left|\psi\right\rangle = UH\left|\psi\right\rangle = U\lambda\left|\psi\right\rangle = \lambda U\left|\psi\right\rangle,$$

so that $U\left|\psi\right\rangle$ is also an eigenket of H.

c) We know that under parity $\hat{x} \mapsto -\hat{x}$, where $\hat{x}$ denotes the position operator. Indeed, since an operator is determined by its action on a basis, we can take the basis $\{|x\rangle\}$ and consider the action of $\pi\hat{x}\pi$ on it.

$$\pi\hat{x}\pi\left|x\right\rangle = \pi\hat{x}\left|-x\right\rangle = -x\left|x\right\rangle,$$

so we must have $\pi\hat{x}\pi = -\hat{x}$. Similarly, momentum is odd under parity (dropping all the hats now):

$$[x, \pi p\pi] = x\pi p\pi - \pi p\pi x = -\pi x p\pi + \pi p x\pi = -\pi[x, p]\pi = -i\hbar,$$

so we must have $\pi p\pi = -p$. Now, we can write

$$\left\langle n'l'm'\middle|\mathbf{p}\middle|nlm\right\rangle = \left\langle n'l'm'\middle|\pi\pi\mathbf{p}\pi\pi\middle|nlm\right\rangle = -(-1)^{l+l'}\left\langle n'l'm'\middle|\mathbf{p}\middle|nlm\right\rangle.$$

Thus, this is exactly 0 for even values of $l + l'$. Similarly, we have

$$\left\langle n'l'm'\middle|\mathbf{L}\middle|nlm\right\rangle = (-1)^{l+l'}\left\langle n'l'm'\middle|\mathbf{L}\middle|nlm\right\rangle,$$

which is 0 for odd values of $l + l'$, since $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ is even under parity (since both $\mathbf{x}$ and $\mathbf{p}$ are odd).

Exercise 5. Dilations. Do Exercise 2 on the Week 5 Worksheet: Another symmetry is called **dilation** symmetry. Dilations are given by the transformation $\mathbf{x} \to \mathbf{x}' = e^{c}\mathbf{x}$, where $c \in \mathbb{R}$. Call its generator D, so that e^{-icD} is the corresponding unitary operator.

Remark. In conformal field theory, the convention is to absorb the factor of i into D, so that e^{-cD} is the dilation operator.

a) Show that the *infinitesimal* transformation

$$e^{i\mathbf{a}\cdot\mathbf{p}}e^{icD}e^{-i\mathbf{a}\cdot\mathbf{p}}e^{-icD}$$

is given by $\mathbb{1} + c\mathbf{a} \cdot [D, \mathbf{p}]$.
Hints: You can reduce to the situation where all the vectors are 1-dimensional (why?). There's a slick way to do this, but the brute force method does work.

b) Calculate $[D, \mathbf{p}]$.
Hint: What coordinate transformation does the above correspond to? In other words, if you write it in the form $\mathbf{x} \to \mathbf{x}'$, what is $\mathbf{x}'$?

a) You could write out all of these exponentials out to second order in a and c (note that it's easier to work in 1 dimension for this whole problem). Another (slicker) way to get the same answer is to use the Baker-Campbell-Hausdorff formula, which says that given any two operators X and Y, we have

$$e^X e^Y = e^Z,$$

where

$$Z = X + Y + \frac{1}{2}[X, Y] + \cdots.$$

The ellipsis above denotes third and higher order terms in X and Y, which we can ignore. Thus, use BCH on

$$e^{-iap}e^{-icD} = \exp\left(-iap - icD + \frac{ac}{2}[D, p] + \cdots\right).$$

Then, use it on

$$e^{iap}e^{icD} = \exp\left(iap + icD + \frac{ac}{2}[D, p] + \cdots\right).$$

Finally, use it on the product to get

$$e^{iap}e^{icD}e^{-iap}e^{-icD} = \exp(ac[D, p] + \cdots).$$

Now, expand out the exponential to get the answer:

$$\mathbb{1} + ac[D, p].$$

The same argument works in 3 dimensions by linearity, so we're done.

b) Note that the operation on the space that corresponds to the transformation given in (a) is:

$$\mathbf{x} \mapsto e^c\mathbf{x} \mapsto e^c\mathbf{x} + \mathbf{a} \mapsto \mathbf{x} + e^{-c}\mathbf{a} \mapsto \mathbf{x} + (e^{-c} - 1)\mathbf{a}.$$

Expanding the final term out to second order (since we went to second order in part (a)), we get

$$\mathbf{x} \mapsto \mathbf{x} - c\mathbf{a}.$$

By (a), the infinitesimal transformation which corresponds to this is exactly $\mathbb{1} + c\mathbf{a} \cdot [D, \mathbf{p}]$. Since $\mathbf{p}$ is the generator of translations, we see that in order to generate a translation $\mathbf{x} \mapsto \mathbf{x} - c\mathbf{a}$, we need to take $[D, \mathbf{p}] = -\mathbf{p}$. If we had defined D to be the generator such that e^{-cD} is the corresponding unitary operator, then we get instead

$$[D, \mathbf{p}] = i\mathbf{p},$$

and this is how it's usually done in conformal field theory.