Final Review Problems

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The Robertson-Walker metric in spherical coordinates is given by

$$ds^2 = -d\tau^2 + a^2(\tau)\begin{cases} d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\varphi^2), & K = 1 \\ d\psi^2 + \psi^2(d\theta^2 + \sin^2\theta d\varphi^2), & K = 0 \\ d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\varphi^2), & K = -1 \end{cases},$$

where $a(\tau) > 0$ is a positive function of proper time τ and K is the sectional curvature.

Exercise 1. In this problem, assume there is no radiation (or other sources of) pressure in the universe. Given a point p in spacetime, the tangent space to p is 4-dimensional. It splits into a 1-dimensional subspace spanned by u^μ, the unit tangent vector to the worldline of an isotropic observer, and a 3-dimensional subspace spanned by an orthogonal set of unit vectors $\{s_i{}^\mu\}$ tangent to a homogeneous hypersurface passing through the point. Note that $s_i{}^\mu u_\mu = 0$, by definition, and we can write any tensor $S_{\mu\nu}$ in the $\{u, s_i\}$ basis as

$$S_{\mu\nu} = S_{\tau\tau}u_\mu u_\nu + \sum_{i,j} S_{s_i s_j} s_{i,\mu} s_{j,\nu} + \sum_i S_{\tau s_i} u_\mu s_{i,\nu} + \sum_i S_{s_i \tau} s_{i,\mu} u_\nu.$$

a) Let ρ be the average mass density of matter in the universe. Use homogeneity and isotropy to argue that for dust (i.e. matter which exerts no pressure)

$$T_{\mu\nu} = \rho u_\mu u_\nu.$$

b) Argue that the 10 independent equations which arise from Einstein's equation can be reduced to two by homogeneity and isotropy:

$$\begin{aligned} G_{\tau\tau} &= 8\pi\rho \\ G_{**} &= 0, \end{aligned}$$

where (s^μ is any of the $s_i{}^\mu$)

$$\begin{aligned} G_{\tau\tau} &= G_{\mu\nu}u^\mu u^\nu \\ G_{**} &= G_{\mu\nu}s^\mu s^\nu. \end{aligned}$$

Hints: First, argue that the time-space components are 0 and that the space-space components must be the same. Then, project $G_{\mu\nu}$ onto a homogeneous hypersurface and raise an index with the spatial metric. Use homogeneity to argue that the resulting tensor $G^i{}_j$, viewed as a linear map which takes tangent vectors to tangent vectors, must necessarily be a multiple of the identity (by using the spectral theorem for symmetric matrices).

c) Compute the Ricci tensor and Ricci scalar in i) a closed universe (i.e. constant curvature $K = 1$) and ii) in an open universe (constant curvature $K = -1$). The Christoffel symbols and Ricci tensor are

$$\begin{aligned} \Gamma^k_{ij} &= \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) \\ R_{ij} &= \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k{}_{kj} + \Gamma^k_{ij} \Gamma^l{}_{kl} - \Gamma^k_{lj} \Gamma^l_{ki}, \end{aligned}$$

where e.g. $\Gamma^k{}_{jk} = \sum_k \Gamma^k_{jk}$ is the contraction.
Hint: Argue that you only need to calculate R_{00} and R_{11}, and use this to limit the number of Christoffel symbols you need to calculate to 6 (which take only 3 different values!).

d) Using

$$G_{ij} = R_{ij} - \frac{1}{2} \eta_{ij} R,$$

show that the differential equations from (b) become

$$\begin{aligned} 3\frac{\dot{a}^2}{a^2} &= 8\pi\rho - \frac{3K}{a^2} \\ 3\frac{\ddot{a}}{a} &= -4\pi\rho. \end{aligned}$$

It turns out that these also hold for the case of flat spacetime ($K = 0$).
Hint: Note that

$$\begin{aligned} R &= -R_{\tau\tau} + 3R_{**} \\ &= -R_{00} + 3a^{-2} R_{11}. \end{aligned}$$

Exercise 2. Hubble's Law. By analyzing the differential equations for a from Exercise 1, show that $\rho > 0$ implies $\ddot{a} > 0$. Thus, derive Hubble's law,

$$\frac{dR}{d\tau} = HR,$$

where R is the distance between two isotropic observers and $H(\tau) = \dot{a}/a$ is Hubble's constant.

Exercise 3. Critical Energy Density. Define the critical energy density

$$\rho_c = \frac{3H^2}{8\pi}$$

and the ratio of the total energy density to the critical energy density

$$\Omega = \rho/\rho_c.$$

Determine the relation between the value of $K \in \{-1, 0, 1\}$ and the sign of Ω (< 1, > 1, or 1).

Exercise 4. Cosmological measurements today infer $K = 0$. Determine $H(\tau)$ if the energy density ρ is dominated by

a) radiation (assume that $T \propto a^{-1}$),

b) matter,

c) vacuum,

d) kination ($\rho \propto a^{-6}$; energy dominated by kinetic terms of scalar field),

e) or an ultrarelativistic fluid with the equation of state $W = P/\rho$ (assume that the universe expands adiabatically).

Exercise 5. Cosmic Strings. A cosmic string loop of radius R oscillates with period $T = R_0$, so that

$$\rho(\tau, \mathbf{x}) = \mu\delta(r - R_0\cos(\omega t))\delta(z),$$

where $\mu = m/\ell$ is the string tension, we use cylindrical coordinates $\mathbf{x} = (r, \theta, z)$, and $\omega = 2\pi/R_0$.

a) Calculate the quadrupole moment of the string.

b) Calculate the transverse traceless quadrupole moment.

c) Show that the power emitted by the string loop in gravitational waves is

$$P = \Gamma G\mu^2,$$

where Γ is a dimensionless constant. (You should calculate Γ explicitly; in reality, relativistic effects modify the value of Γ to be around 50-100).

d) What is the lifetime of the string loop?

e) A cosmic string loop forms roughly every Hubble doubling and then redshifts like matter before decaying to gravitational waves. This gives a number density of loops in the horizon:

$$\frac{dn}{d\ell} \approx \frac{1}{\tau^{3/2}(\ell + \Gamma\mu\tau)^{5/2}}.$$

Estimate Ω_{GW} from all the loops in the horizon.

Exercise 5. *Hartle 12.9.* Darth Vader follows a few Jedi knights into a black hole. He knows any light pulse he fires will move to smaller and smaller radii. Should he worry that light from his gun will fall back on him before his destruction if he emits it radially?