

Week 13 Worksheet

Partial Wave Analysis

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Exercise 1. Warm up.

- How do the phase shifts δ_ℓ appear in partial wave scattering, and what is their physical significance?
- What is the fundamental assumption on the form of the wavefunctions in the Born approximation?
Hint: If the scattering potential is weak, what approximation can we make?
- Starting from the Lippmann-Schwinger equation,

$$\psi(\mathbf{x}) = \varphi_{\mathbf{k}}(\mathbf{x}) + \int d^3x' G_0(\mathbf{x}, \mathbf{x}', E) V(\mathbf{x}') \psi(\mathbf{x}'),$$

where G_0 is the free particle, time-independent Green's function and

$$\begin{aligned}\varphi_{\mathbf{k}}(\mathbf{x}) &= \langle \mathbf{x} | \mathbf{k} \rangle \\ &= \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}},\end{aligned}$$

explain how you would derive the Born approximation. (Just list the steps, no need to work them out.)

Hint: The Green's function is (note that $E = \hbar^2 k^2 / 2m$)

$$G_0(\mathbf{x}, \mathbf{x}', E) = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|}.$$

Exercise 2. *Griffiths 10.4.* Consider low energy scattering off a spherical delta-function shell:

$$V(r) = \alpha \delta(r - a),$$

where α and a are constants. Calculate the scattering amplitude, the differential cross-section, and the total cross-section. Assume that $ka \ll 1$, and ignore all the terms with $\ell > 0$. It will be helpful to express answers in terms of

$$\beta = \frac{2ma\alpha}{\hbar^2}.$$

Hints and Formulas: Make sure to always use the assumption $ka \ll 1$! This will simplify many of your results. The Hankel functions of the first and second kind for $\ell = 0$ are

$$\begin{aligned} h_0^{(1)}(x) &= -i \frac{e^{ix}}{x} \\ h_0^{(2)}(x) &= i \frac{e^{-ix}}{x}, \end{aligned}$$

where

$$\begin{aligned} h_\ell^{(1)} &= j_\ell + i n_\ell \\ h_\ell^{(2)} &= j_\ell - i n_\ell \end{aligned}$$

in terms of the spherical Bessel functions j_ℓ and n_ℓ . The exterior wavefunction is

$$\psi(r, \theta) = A \sum_{\ell=0}^{\infty} i^\ell (2\ell + 1) \left[j_\ell(kr) + i k a_\ell h_\ell^{(1)}(kr) \right] P_\ell(\cos \theta),$$

and recall that the interior one can be written as a linear combination of spherical Bessel functions (resp. spherical Hankel functions). They are solutions of the radial equation

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + Vu + \frac{\hbar^2}{2m} \frac{\ell(\ell+1)}{r^2} u = Eu,$$

where $u(r) = rR(r)$.