

Week 14 Worksheet Solutions

Time-Dependent Phenomena

Jacob Erlikhman

December 14, 2024

Exercise 1. General Theory.

- a) Consider the Schrödinger equation for time-dependent perturbation theory

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = [H^0 + \lambda H^1(t)] |\Psi(t)\rangle .$$

Suppose

$$|\Psi(t)\rangle = \sum_n c_n(t) e^{-iE_n t/\hbar} |n\rangle ,$$

where $|n\rangle$ are the eigenstates of H^0 . Derive the *exact* result

$$i\hbar \frac{dc_n(t)}{dt} = \lambda \sum_m \langle n | H^1(t) | m \rangle e^{i\omega_{nm}t} c_m(t). \quad (0.1)$$

- b) Now, set

$$c_n(t) = \sum_{k=0}^{\infty} \lambda^k c_n^{(k)}(t),$$

and plug it into your result from (b) to obtain the first order, i.e. $\mathcal{O}(\lambda)$, differential equation.

- c) Obtain the second order equation.

Remark. Notice that your results for (b) and (c) are *exactly* the same as the two-level results when we begin in a single initial state!

- a) Plug in the form for $|\Psi\rangle$ into the Schrödinger equation:

$$i\hbar \sum_n (\dot{c}_n - i\omega_n) e^{-iE_n t/\hbar} |n\rangle = \sum_n c_n e^{-iE_n t/\hbar} (E_n + \lambda H^1) |n\rangle .$$

Now, take the inner product with $|n\rangle$ (and change the dummy summation variable from n to m):

$$i\hbar(\dot{c}_n - i\omega_n)c_n e^{-iE_n t/\hbar} = c_n E_n e^{-iE_n t/\hbar} + \sum_m \langle n | \lambda H^1 | m \rangle c_m e^{-iE_m t/\hbar}.$$

Notice that the second term on the LHS is equal to the first term on the RHS, so they cancel. Thus,

$$i\hbar\dot{c}_n(t) = \sum_m \lambda \langle n | H^1 | m \rangle c_m(t) e^{-i\omega_{nm}t},$$

as desired, where

$$\omega_{nm} = \frac{E_n - E_m}{\hbar}.$$

b) The order λ result is just

$$\dot{c}_n^{(1)}(t) = -\frac{i}{\hbar} \sum_m \langle n | H^1 | m \rangle c_m^{(0)} e^{-i\omega_{nm}t}.$$

c) This is exactly the same:

$$\dot{c}_n^{(k+1)}(t) = -\frac{i}{\hbar} \sum_m \langle n | H^1 | m \rangle c_m^{(k)}(t) e^{-i\omega_{nm}t}.$$

Exercise 2. Sinusoidal Perturbations. In the case that

$$H^1 = K e^{-i\omega t} + K^\dagger e^{i\omega t}$$

is sinusoidal and acts up until time t , solve the first order perturbation theory differential equation from Exercise 1(b).

We plug in

$$\dot{c}^{(1)} = -\frac{i}{\hbar} \left[\sum_m K_{nm} c_m^{(0)} e^{it(\omega_{nm}-\omega)} + \sum_m K_{nm}^\dagger c_m^{(0)} e^{it(\omega_{nm}+\omega)} \right].$$

Integrating from 0 to t , we find

$$c^{(1)}(t) = -\frac{1}{\hbar} \left[\sum_m \frac{K_{nm} c_m^{(0)}}{\omega_{nm} - \omega} \left(e^{it(\omega_{nm}-\omega)} - 1 \right) + \sum_m \frac{K_{nm}^\dagger c_m^{(0)}}{\omega_{nm} + \omega} \left(e^{it(\omega_{nm}+\omega)} - 1 \right) \right].$$

Exercise 3. Spin Resonance. Consider a spin-1/2 particle in a static magnetic field $B_0 \hat{z}$, so $H^0 = -\frac{1}{2}\hbar\gamma B_0 \sigma_z$. The perturbation is due to a magnetic field B_1 rotating in the (x, y) -plane with angular velocity ω :

$$H^1(t) = -\frac{1}{2}\hbar\gamma B_1 [\sigma_x \cos(\omega t) + \sigma_y \sin(\omega t)].$$

- a) Writing the eigenvectors of σ_z as

$$|+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

rewrite H^1 in the form given in Exercise 2, using these eigenvectors as a basis.

- b) Suppose at $t = 0$ we have the initial state $|i\rangle = |+\rangle$. Find the first order probability for the spin to be down at time t . It is convenient to set $\omega_0 = \gamma B_0$ and $\omega_1 = \gamma B_1$.
- c) It turns out that the exact Equation 1 can be solved for such a hamiltonian. The exact answer for (b) is

$$P(t) = \sin^2(\alpha t/2) \left(\frac{\omega_1}{\alpha} \right)^2,$$

where $\alpha^2 = (\omega_0 + \omega)^2 + \omega_1^2$; $\alpha/2$ is called the **Rabi flopping frequency**. Using this answer, what is the range of validity of the perturbation theory result, assuming we are not near resonance?

- d) Suppose we are near resonance. What is the range of validity of the perturbation theory result? Give a physical explanation of your result.
- e) **Challenge.** Solve Equation 1, and derive the formula for $P(t)$.

- a) We can write

$$H^1(t) = -\frac{1}{2}\hbar\gamma B_1 \begin{bmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{bmatrix},$$

so that

$$K = -\frac{1}{2}\hbar\gamma B_1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

- b) Since we have a single initial state, the complicated expression from Exercise 2 reduces to two terms. Further, since $K|+\rangle = 0$, we are left with only one term. Thus, we find

$$c^{(1)} = \frac{\omega_1}{2(\omega_0 + \omega)} \left(e^{it(\omega_0 + \omega)} - 1 \right),$$

where we notice that

$$E_- = \frac{\hbar\omega_0}{2}$$

$$E_+ = -\frac{\hbar\omega_0}{2},$$

so $\omega_{-+} = \omega_0$. Now, notice that we can pull out a factor of $e^{it(\omega_0+\omega)/2}$ to get a single phase factor:

$$c^{(1)} = i \frac{\omega_1}{(\omega_0 + \omega)} \sin\left(\frac{\omega_0 + \omega}{2}t\right) e^{it(\omega_0+\omega)/2}.$$

This trick makes it easy to calculate the first order probability:

$$|c^{(1)}|^2 = \frac{\omega_1^2}{(\omega_0 + \omega)^2} \sin^2\left(\frac{\omega_0 + \omega}{2}t\right).$$

c) Consider the second term in the exact answer

$$\frac{\omega_1^2}{\alpha^2} = \frac{\omega_1^2}{(\omega_0 + \omega)^2 \left[1 + \frac{\omega_1^2}{(\omega_0 + \omega)^2}\right]}.$$

We can expand the term in brackets as a binomial expansion if and only if $|\omega_1| \ll |\omega_0 + \omega|$, which would give us the perturbation theory result. Thus, this is the range of validity of that result. Note that resonance would correspond to $\omega_0 + \omega \sim 0$, so this wouldn't apply in that case.

d) We have to play a different game if we're near resonance. In that case, the exact result becomes

$$P_n(t) \sim \sin^2\left(\frac{\omega_1}{2}t\right).$$

In order to determine what happens to the perturbation theory result, recall that for small θ , $\sin \theta \sim \theta$. Thus,

$$|c^{(1)}(t)|^2 \sim \frac{\omega_1^2}{4}t^2.$$

Clearly, these will match well if and only if $|\omega_1 t| \ll 1$. So perturbation theory works well for short times, no matter how big B_1 is. This is because the effect of the perturbation is a product of both its strength *and* its duration.

e) Note that

$$\langle - | H^1 | + \rangle = -\frac{\hbar\omega_1}{2}e^{i\omega t},$$

and $\langle + | H^1 | - \rangle$ is just the complex conjugate. Thus, we want to solve the system of differential equations

$$\begin{cases} \dot{c}_- = i \frac{\omega_1}{2} e^{it(\omega_0+\omega)} c_+ \\ \dot{c}_+ = i \frac{\omega_1}{2} e^{-it(\omega_0+\omega)} c_- \end{cases}.$$

This is most easily done by differentiating one of the equations once and substituting. We obtain

$$\ddot{c}_- = i(\omega_0 + \omega)\dot{c}_- + i \frac{\omega_1}{2} e^{it(\omega_0+\omega)} \dot{c}_+,$$

so that

$$\ddot{c}_- - i(\omega_0 + \omega)\dot{c}_- + \frac{\omega_1^2}{4}c_- = 0.$$

To solve this equation, assume solutions of the form e^{kt} , with $k \in \mathbb{C}$. Plugging this in, we obtain an equation for k .

$$k^2 - ik(\omega_0 + \omega) - \frac{\omega_1^2}{4} = 0.$$

This has solution

$$k_{\pm} = \frac{i(\omega_0 + \omega) \pm i\alpha}{2},$$

where $\alpha/2$ is the Rabi flopping frequency. Thus,

$$c_- = Ae^{k_-t} + Be^{k_+t}.$$

Now, the initial condition $c_+(0) = 1$ gives that

$$\dot{c}_-(0) = i\frac{\omega_1}{2}.$$

Since

$$\dot{c}_-(t) = k_-Ae^{k_-t} + k_+Be^{k_+t}$$

and $c_-(0) = 0$, we have two equations for A and B .

$$\begin{cases} A = -B \\ \frac{\omega_1}{2} = A\frac{\omega_0 + \omega - \alpha}{2} + B\frac{\omega_0 + \omega + \alpha}{2} \end{cases}.$$

These have the solution

$$\begin{cases} A = -\frac{\omega_1}{2\alpha} \\ B = \frac{\omega_1}{2\alpha} \end{cases}.$$

Thus,

$$c_-(t) = \frac{i\omega_1}{\alpha} e^{it(\omega_0 + \omega)/2} \sin(\alpha t/2),$$

so that

$$|c_-|^2 = P(t) = \frac{\omega_1^2}{\alpha^2} \sin^2(\alpha t/2),$$

which exactly matches the result quoted in part (c).