Midterm 1 Review

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Exercise 1. The starship Titanic is approaching a space-iceberg at velocity $V = 3/5$. It immediately fires a 100 kg missile at speed $4/5$ relative to the ship, hoping to break up the ice before they crash.

a) Find all components of the missile's momentum four-vector in the spaceship frame, assuming all motion is along the x-direction.

b) Do (a) in the iceberg frame.

c) The iceberg will break if the momentum impacting it is at least 10^{11} kg m/s. Does the iceberg break?

d) What is the velocity of the missile in the iceberg frame?

Exercise 2. Photon Rockets. Consider a rocket whose propellant is photons: The rocket emits photons to accelerate it—we'll call it a "photon rocket." Suppose such a photon rocket departed Earth with total mass M_0.

a) If the rocket will achieve a final speed with $\gamma = 2$, determine what final fraction M/M_0 of the ship's mass is remaining.

b) Show that the speed of the photon rocket can be written

$$v = \frac{1 - (M/M_0)^2}{1 + (M/M_0)^2}.$$

c) Astronomers on Earth watch the photon rocket recede through a telescope—they can observe the photons emitted in the rocket exhaust. What redshift $f_{\text{observed}}/f_{\text{emitted}}$ do they observe in terms of M_0 and the total mass M of the rocket when the photons are emitted?

Exercise 3. The **canonical stress-energy tensor** for an arbitrary lagrangian density $\mathcal{L}(q_1, \ldots, q_n)$ is defined by

$$T^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_k)} \partial^\nu q_k + \eta^{\mu\nu} \mathcal{L},$$

where η is the Minkowski metric.

a) Starting with the sourceless electromagnetic lagrangian density

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu},$$

show that

$$T^{\mu\nu} = \frac{1}{4\pi} \eta^{\mu\alpha} F_{\alpha\beta} \partial^\nu A^\beta + \eta^{\mu\nu} \mathcal{L}.$$

Hint: The generalized coordinates in relativistic electrodynamics should be thought of as packaged in the single gauge field $q_\mu = A_\mu$.

b) Check that

$$\int T^{00} d^3x = \frac{1}{8\pi} \int (E^2 + B^2) d^3x$$

is the energy contained in the EM field, and that

$$\int T^{0i} d^3x = \frac{1}{4\pi} \int (\mathbf{E} \times \mathbf{B})^i d^3x$$

is the i^{th} component of the momentum contained in the EM field.
Hints: First show that

$$\begin{aligned} T^{00} &= \frac{1}{8\pi}(E^2 + B^2) + \frac{1}{4\pi} \nabla \cdot (V\mathbf{E}) \\ T^{0i} &= \frac{1}{4\pi}(\mathbf{E} \times \mathbf{B})^i + \frac{1}{4\pi} \nabla \cdot (A^i \mathbf{E}), \end{aligned}$$

where $V = A^0$ is the scalar potential. The components of the tensor $F^\alpha{}_\beta$ are derived in the Week 5 Worksheet solutions; to obtain $F_{\alpha\beta}$ or $F^{\alpha\beta}$ from it, just raise or lower one index, e.g. $F_{\alpha\beta} = \eta_{\alpha\mu} F^\mu{}_\beta$.

Exercise 4. Conservation of Energy and Momentum. Although the stress energy tensor of Exercise 3 satisfies $\partial_\mu T^{\mu\nu} = 0$, which is the conservation law corresponding to conservation of energy and momentum in relativistic electrodynamics, it fails to be manifestly Lorentz covariant and traceless. This latter property is required of massless photons. We can fix this by constructing a symmetric, traceless tensor from $T^{\mu\nu}$ as follows.

a) Use the definition of $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ to show that

$$T^{\mu\nu} = -\frac{1}{4\pi}\left(\eta^{\mu\alpha} F_{\alpha\beta} F^{\beta\nu} + \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right) + \frac{1}{4\pi}\eta^{\mu\alpha} F_{\alpha\beta} \partial^\beta A^\nu.$$

b) Using the sourceless Maxwell's equations (in their relativistic form), write the last term as

$$T_D^{\mu\nu} = -\frac{1}{4\pi} \partial_\alpha (F^{\alpha\mu} A^\nu),$$

and define

$$\Theta^{\mu\nu} = T^{\mu\nu} - T_D^{\mu\nu}.$$

c) Check that $\partial_\mu T_D^{\mu\nu} = 0$, so that the conservation law $\partial_\mu T^{\mu\nu} = 0$ implies the conservation law $\partial_\mu \Theta^{\mu\nu} = 0$.

d) Check that the trace $\Theta^\mu{}_\mu = 0$, $\Theta^{\mu\nu} = \Theta^{\nu\mu}$, and Θ defines the "usual" electromagnetic stress-energy tensor with components

$$\begin{aligned}
\Theta^{00} &= \frac{1}{8\pi}(E^2 + B^2) \\
\Theta^{0i} &= \frac{1}{4\pi}(\mathbf{E} \times \mathbf{B})^i \\
\Theta^{ij} &= -\frac{1}{4\pi}\left[E^i E^j + B^i B^j - \frac{1}{2}\delta^{ij}(E^2 + B^2)\right].
\end{aligned}$$

e) It turns out that when we add sources, Θ is the same as in the sourceless case. Show that

$$\partial_\mu \Theta^{\mu\nu} = -F^{\nu\alpha} j_\alpha,$$

and check that the $\nu = 0$ component of this is exactly Poynting's theorem from the Week 5 Worksheet. The space components give conservation of momentum.
Hint: To show the conservation equation, you will need to use both of Maxwell's equations with sources. The first equation is

$$\partial_\mu F^{\mu\nu} = -4\pi j^\nu.$$

The second is the Bianchi identity

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0.$$

Remark. The usual stress energy tensor for electromagnetism is the Θ which we have just obtained above. There is another way to derive it by noticing that it appears from gravity coupled to electromagnetism (we write the Einstein equation for the gravitational lagrangian plus the EM matter lagrangian).

Exercise 4. Do the following parts of Exercise 2 on the Week 7 Worksheet. Consider the **Poincaré upper half-plane** $\mathbb{H}^2 = \{(x, y) | y > 0\}$ with the metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2).$$

Don't bother to do this for review, but in the Week 7 Worksheet you were asked to show that semicircles with center on the x-axis and lines parallel to the y-axis are geodesics. In fact, it's true that these are all of the geodesics; you can assume this for this problem.

a) Check that all geodesics have infinite length in either direction. We say that $\mathbb{H}^2$ is **complete**.

b) Compute the area of the interior of the following object (a 2/3 triangle). Let one side be the arc corresponding to an angle α (measured from the positive x-axis) of the unit semicircle centered at $(0, 0)$. Let the other two sides be vertical lines which extend out from the endpoints of the arc.