Week 2 Worksheet
137A Review; QM in 3D (and some spin)

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January 25, 2024

Exercise 0. This is done in Griffiths Chapters 1 and 3.

Exercise 1. A Hilbert space is mathematically defined as a *complete* vector space with an inner product. A vector space with an inner product is **complete** if it includes not only all finite sums of vectors in a basis, but also all limits of convergent sequences, i.e. given a sequence (v_n) of vectors in the Hilbert space, v is the **limit** of the sequence if $\lim_{n\to\infty} \|v_n - v\| = 0$, where $\|v\| = \sqrt{v \cdot v}$.

a) Consider a Hilbert space $\mathcal{H}$ that consists of all functions $\psi(x)$ such that

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \,\mathrm{d}x < \infty.$$

Show that there are functions in $\mathcal{H}$ for which $\hat{x}\psi(x) = x\psi(x)$ is not in $\mathcal{H}$.

Proof. Let $\psi(x) = \dfrac{1}{1+|x|}$. □

b) Consider the function space Ω in $\mathcal{H}$ which consists of all $\varphi(x)$ that satisfy the set of conditions

$$\int_{-\infty}^{\infty} |\varphi(x)|^2 (1+|x|)^n \,\mathrm{d}x < \infty,$$

for any $n \in \{0, 1, 2, \ldots\}$. Show that for any $\varphi(x)$ in Ω, $\hat{x}\varphi(x)$ is also in Ω. Ω is called the **nuclear** space.

Proof. By the binomial theorem,

$$\int_{-\infty}^{\infty} |\varphi(x)|^2 (1+|x|)^n \,\mathrm{d}x = \sum_{i=0}^{n} \int_{-\infty}^{\infty} |\varphi(x)|^2 \binom{n}{i} |x|^i \,\mathrm{d}x.$$

In particular, for each n, $|\varphi(x)|^2 |x|^n$ has finite integral. □

c) The **extended** space $\Omega^\times$ consists of those functions $\chi(x)$ which satisfy

$$(\chi, \varphi) = \int_{-\infty}^{\infty} \chi^*(x)\varphi(x)\,\mathrm{d}x < \infty,$$

for any φ in Ω, where $(\ ,\ )$ is the inner product on $\mathcal{H}$. Which of the following functions belong to Ω, to $\mathcal{H}$, and/or to $\Omega^\times$?

Remark. The collection $(\Omega, \mathcal{H}, \Omega^\times)$ is called "rigged Hilbert space," and this is a rigorous way to include all the formalism (e.g. eigenvectors of position are delta functions, and hence can't belong to an L^2 space) into the Hilbert space formulation of quantum mechanics. Note that $\Omega \subset \mathcal{H} \subset \Omega^\times$. Also, note that in order to sit in Ω, functions must vanish faster than any power of x as $|x| \to \infty$. Thus, as long as functions don't diverge at ∞ more strongly than any power of $|x|$, they are in $\Omega^\times$. For more details, see Ballentine *Quantum Mechanics*, Chapter 1.

i) $\sin(x)$

ii) $\sin(x)/x$

iii) $x^2\cos(x)$

iv) e^{-ax}, $a > 0$.

v) $\dfrac{\ln(1+|x|)}{1+|x|}$

vi) e^{-x^2}

vii) $x^4 e^{-|x|}$

i) Clearly, $\sin(x) \notin \mathcal{H}$, so it's not in Ω either. But it is in $\Omega^\times$, since it's divergence at ∞ is not worse than e.g. $|x|$.

ii) Try first to compute

$$\int_{-\infty}^{\infty} \frac{|\sin(x)|^2}{x^2}\,\mathrm{d}x = 2\int_{0}^{\infty} \frac{|\sin(x)|^2}{x^2}\,\mathrm{d}x = 2\int_{0}^{\varepsilon} \frac{|\sin(x)|^2}{x^2}\,\mathrm{d}x + 2\int_{\varepsilon}^{\infty} \frac{|\sin(x)|^2}{x^2}\,\mathrm{d}x.$$

Now, the first term has finite integral, and the second term is less than $\int_\varepsilon^\infty \frac{2}{x^2}\,\mathrm{d}x$, which is finite. Thus, it's in $\mathcal{H}$, and it follows that the function is in $\Omega^\times$. It's clearly not in Ω, since $|\sin(x)|^2$ does not have a finite integral.

iii) Write

$$\int_{-\infty}^{\infty} x^4\cos^2(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} x^4(1+\cos(2x))\,\mathrm{d}x.$$

The second term has either (positive) infinite integral or is finite, but the first term is definitely $+\infty$, so this function is not in $\mathcal{H}$, so not in Ω either. It is in $\Omega^\times$, since its divergence is not worse than a power of x.

iv) Clearly, this isn't in $\mathcal{H}$. It's also not in $\Omega^\times$, since it diverges faster than any power of x as $x \to -\infty$.

v) Clearly, this isn't in Ω. It is in $\mathcal{H}$, though. Explicitly, we could integrate by parts a few times to reduce to

$$\int_{-\infty}^{\infty}\left(\frac{\ln(1+|x|)}{1+|x|}\right)^2 \mathrm{d}x = 4\int_0^{\infty}\frac{1}{(1+x)^2}\,\mathrm{d}x = 4.$$

vi) This has finite integral, so it's in $\mathcal{H}$ and $\Omega^{\times}$. It's also in Ω, since any power of $|x|$ times this function is also finite (remember your gaussian integrals!).

vii) This is in Ω, since we can always integrate by parts to get back to an integral over $e^{-|x|}$.

Exercise 2. Solve the eigenvalue problem for the 3-D isotropic, harmonic oscillator, whose hamiltonian is $H = p^2/2m + m\omega^2x^2/2$, where $p^2 = \mathbf{p}\cdot\mathbf{p}$, $x^2 = \mathbf{x}\cdot\mathbf{x}$ is the 3-D dot product.

Remembering (or rederiving) the solution to the 1-D isotropic harmonic oscillator, the energy eigenvalues are $E_n = (n+1/2)\hbar\omega$ for $n \in \{0,1,2,\ldots\}$. But the 3-D one is the same except we have 3 directions for n now, n_x, n_y, and n_z. Thus, $n = n_x + n_y + n_z$, for arbitrary $n_x, n_y, n_z \in \{0,1,2,\ldots\}$.

Exercise 3. A particle of mass m is placed in a finite spherical well

$$V(r) = \begin{cases} -V_0, & r \le a \\ 0, & r \ge a \end{cases}.$$

Find the equation that quantizes the energy (you don't need to solve it), by solving the radial Schrödinger equation with $\ell = 0$. Explain how you could solve this equation and obtain the energies. Show that there is no bound state if $V_0a^2 < \pi^2\hbar^2/8m$. Hint: [1]

Write the solution as

$$u(r) = \begin{cases} A\cos(kr) + B\sin(kr)U, & r \le a \\ Ce^{\kappa r} + De^{-\kappa r}, & r \ge a \end{cases},$$

where

$$k = \sqrt{\frac{2m}{\hbar^2}(E+V_0)}$$
$$\kappa = \sqrt{-\frac{2m}{\hbar^2}E}.$$

Now, the wavefunction is $u(r)/r$, so as $r \to 0$ the cosine solution blows up, which means $A = 0$. On the other hand, as $r \to \infty$, we see that $C = 0$, since $e^{\kappa r}/r$ blows up there. Thus, we are left with

$$u(r) = \begin{cases} A\sin(kr), & r \le a \\ Be^{-\kappa r}, & r \ge a \end{cases},$$

[1] Recall that the radial Schrödinger equation is identical to the time-independent, 1-dimensional Schrödinger equation with the wavefunction replaced by $u(r) = rR(r)$ (where $\psi(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi)$) and potential $V_{\text{eff}}(r) = V(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2}$.

where I have renamed A and B. It follows that

$$u'(r) = \begin{cases} kA\cos(kr), & r \leq a \\ -\kappa B e^{-\kappa r}, & r \geq a \end{cases}.$$

Now, use continuity of the first derivative and the function at the boundary $r = a$ to obtain two equations

$$\begin{cases} A\sin(ka) & = Be^{-\kappa a} \\ kA\cos(ka) & = -\kappa Be^{-\kappa a} \end{cases}.$$

Dividing the first equation by the second, we get the transcendental equation

$$\tan(ka) = -\frac{k}{\sqrt{V_0 - k^2}},$$

which can be rewritten in the form

$$\tan(z) = -\frac{1}{\sqrt{\frac{\sqrt{2mV_0 a}}{\hbar z^2} - 1}},$$

where $z = ka$. To solve it, we should graph the LHS and the RHS on the same graph and look for points of intersection. These will be the allowed z values, hence the allowed k values, hence the allowed energies. We can use the same method to see why there's no bound state if $V_0 a^2 < \pi^2\hbar^2/8m$. Draw the graph of $\tan(z)$ superimposed with the graph of $-1/\sqrt{2mV_0a^2/\hbar^2z^2 - 1}$. You will see that there can be no solution if $\sqrt{2mV_0a^2/\hbar^2} < \pi/2$, which implies that there can be no solution for $2mV_0a^2/\hbar^2 < \pi^2/4$, hence for $V_0a^2 < \pi^2\hbar^2/8m$.

Exercise 4. Spin Representations.

a) Find the eigenvalues and eigenvectors of S_z.

b) Do the same for S_y, and write them in terms of $|\uparrow\rangle$ and $|\downarrow\rangle$, the eigenvectors of S_z.

c) For a system of two spin 1/2 particles, starting with the "highest weight" state $|\uparrow\uparrow\rangle$, find all the states in the triplet.
Hint: Apply the lowering operator.

d) For a system of two spin 1/2 particles, are there any other states than the ones you found in (c)? If so, what are they? What is the action of S_-, S_+ on them?

e) Describe how you would approach finding the Clebsch-Gordan coefficients for arbitrary spin systems.

a) This is done in Griffiths.

b) We have

$$S_y = \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$\frac{\hbar^2}{4} \left(\lambda^2 - 1 \right),$$

so the eigenvalues are

$$\lambda_{\pm} = \pm \frac{\hbar}{2},$$

as expected. The associated eigenvectors are

$$\lambda_{+} \leftrightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\lambda_{-} \leftrightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We can write these as

$$\lambda_{\pm} \leftrightarrow \frac{1}{\sqrt{2}} \left(|\uparrow\rangle \pm |\downarrow\rangle \right).$$

c) This is done in Griffiths.

d) Also done in Griffiths.