Midterm 1 Review

Exercise 1. We can represent a counter-clockwise rotation by an angle θ about an axis $\hat{r}$ by the unitary operator

$$U(\theta) = e^{-i\theta\hat{r}\cdot\mathbf{S}/\hbar},$$

where $\mathbf{S}$ is the angular momentum operator. For particles of spin 1/2, $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$.

a) Show that $(\hat{r}\cdot\boldsymbol{\sigma})^2 = \mathbb{1}$, the identity operator.
Hint: Using $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$ and $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}$, show first that

$$\sigma_i\sigma_j = \mathbb{1}\delta_{ij} + i\varepsilon_{ijk}\sigma_k.$$

b) Show that

$$U(\theta) = \mathbb{1}\cos(\theta/2) - i\hat{r}\cdot\boldsymbol{\sigma}\sin(\theta/2).$$

c) Determine the spin operator σ_θ which points in the direction described by (θ, φ) with $\varphi = 0$.
Hint: Do this by rotating σ_z by an angle θ about the y-axis.

d) Redo problem 4.59 from Griffiths: If two electrons are in the spin singlet state, $S_z^{(1)}$ is the component of spin angular momentum of particle 1 along the z-axis, and $S_\theta^{(2)}$ is the spin angular momentum of particle 2 along the $\hat{r} = (\theta, 0)$ axis, show that

$$\left\langle S_z^{(1)} S_\theta^{(2)} \right\rangle = -\frac{\hbar^2}{4}\cos\theta.$$

Exercise 2. ***Griffiths 5.9.*** Consider two non-interacting particles in an infinite square well of width a such that the single particle wavefunction is

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin(n\pi x/a)$$

with energy $E_n = n^2 K$. Construct the ground state and first excited state of the two-particle system if the particles are a) spin-1/2 and b) spin-1. Determine the energy and degeneracies of these states.

Exercise 3. Helium.

a) Consider a singly-ionized helium ion. How much more energy does it take to ionize its bound electron compared to hydrogen?

b) Still with He^+. What is the wavelength of the emitted photon during the electron transition from $n = 2 \to 1$?

c) Now, consider the usual helium-4. Which ground state has higher energy, parahelium (spin singlet) or orthohelium (spin triplet)? Why? ***Griffiths 5.14.*** How would this change if the two electrons are identical bosons?

d) ***Griffiths 5.22.*** Helium-3 is a fermion with spin-1/2 (as compared to helium-4, which is a boson. Why?). At low temperatures, helium-3 can be treated as a Fermi gas. If its mass density is 82 kg/m^3, determine its Fermi temperature.

Exercise 4. Consider a transformation on a physical system represented by a unitary operator U.

a) How do kets transform under U? What about operators?

b) If the hamiltonian H commutes with U, what does that imply about H being invariant under the transformation U? What does this imply about a non-degenerate eigenstate of H?

c) Derive parity selection rules for hydrogen with respect to momentum and angular momentum matrix elements. I.e. determine when

$$\langle n'l'm'|\mathbf{p}|nlm\rangle = 0$$

and

$$\langle n'l'm'|\mathbf{L}|nlm\rangle = 0.$$

Exercise 5. Dilations. Do Exercise 2 on the Week 5 Worksheet: Another symmetry is called **dilation** symmetry. Dilations are given by the transformation $\mathbf{x} \to \mathbf{x}' = e^c\mathbf{x}$, where $c \in \mathbb{R}$. Call its generator D, so that e^{-icD} is the corresponding unitary operator.

Remark. In conformal field theory, the convention is to absorb the factor of i into D, so that e^{-cD} is the dilation operator.

a) Show that the *infinitesimal* transformation

$$e^{i\mathbf{a}\cdot\mathbf{p}}e^{icD}e^{-i\mathbf{a}\cdot\mathbf{p}}e^{-icD}$$

is given by $\mathbb{1} + c\mathbf{a}\cdot[D, \mathbf{p}]$.
Hints: You can reduce to the situation where all the vectors are 1-dimensional (why?). There's a slick way to do this, but the brute force method does work.

b) Calculate $[D, \mathbf{p}]$.
Hint: What coordinate transformation does the above correspond to? In other words, if you write it in the form $\mathbf{x} \to \mathbf{x}'$, what is $\mathbf{x}'$?