

# Week 11 Worksheet Solutions

## Bouncing Ball

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**Exercise 1.** A ball of mass  $m$  bounces elastically on the floor.

- a) What is the potential as a function of the height  $x$  above the floor?
- b) Solve the Schrödinger equation. You don't need to normalize your solution.

*Hint:* You should get Airy's differential equation,  $\psi''(z) - z\psi(z) = 0$ . One way to manipulate the Schrödinger equation into such a form is to notice that for  $\psi''(x) - \alpha^3 x \psi(x) = 0$ ,  $z = \alpha x$  does the trick. The solutions of this equation are the Airy functions,  $\text{Ai}(z)$  and  $\text{Bi}(z)$ . The graphs of these functions are below.

- c) Calculate (approximately) the first 4 energies, using  $g = 10 \text{ m/s}^2$  and  $m = 0.100 \text{ kg}$ .
- d) Now, analyze this problem using the WKB approximation. Find the allowed energies  $E_n$  in terms of  $m$ ,  $g$ , and  $\hbar$ .

*Hint:* The connecting WKB wavefunctions are

$$\psi(x) = \begin{cases} \frac{2D}{\sqrt{p(x)}} \sin\left(\frac{1}{\hbar} \int_x^{x_2} p(x') dx' + \frac{\pi}{4}\right), & x < x_2 \\ \frac{D}{\sqrt{p(x)}} \exp\left(-\frac{1}{\hbar} \int_{x_2}^x |p(x')| dx'\right), & x > x_2 \end{cases}.$$

- e) Plug in the values from (c), and compare the WKB calculation to the “exact” one for the first four energies.
- f) How large would  $n$  have to be to give the ball an average height of 1 meter above the ground?
- a) We need to be careful here. The ball cannot go below the floor ( $x = 0$ ), so the potential must be

$$V(x) = \begin{cases} mgx, & x \geq 0 \\ \infty, & x < 0 \end{cases}.$$

b) The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\psi''(x) + mgx\psi(x) = E\psi(x).$$

Rewrite this as

$$\psi''(x) - \frac{2m^2g}{\hbar^2}(x - E/mg)\psi(x) = 0.$$

Now, make the substitution

$$z = \alpha(x - E/mg),$$

where

$$\alpha \equiv \sqrt[3]{\frac{2m^2g}{\hbar^2}},$$

so that the Schrödinger equation becomes

$$\psi''(z) - z\psi(z) = 0.$$

Note that we can freely replace the  $z/\alpha + E/mg$  in the argument of  $\psi$  by  $z$ , since we are just looking for the solutions  $\psi$ . If we solve the ODE with the new argument  $z$ , the solution will already account for our replacement. Now, the solutions of this equation will be Airy functions:

$$\psi(x) = C_1 \text{Ai}(z) + C_2 \text{Bi}(z),$$

where  $C_i \in \mathbb{C}$  are constants. Now, use the boundary conditions. One boundary condition is given by the floor. Since  $V(x) = \infty$  for  $x < 0$ , and since the wavefunction must be continuous at 0, it follows that  $\psi(x = 0) = 0$ . For a classical bouncing ball dropped from a height  $h$ , the maximum  $x$  value it can take is  $h$ . However, since the potential is finite for  $x > h$ , a quantum bouncing ball can tunnel all the way to  $\infty$ ; hence,  $x$  can take any positive value. Looking at the graphs of the Airy functions, Bi diverges as  $z \rightarrow \infty$ , i.e. as  $x \rightarrow \infty$ , so that  $C_2 = 0$ . Thus, the solution is

$$\psi(x) = C_1 \text{Ai}(\alpha x - \alpha E/mg).$$

c) On the other hand, the first boundary condition gives that  $\psi(x = 0) = 0$ ; hence,

$$\text{Ai}(-\alpha E/mg) = 0.$$

Again, we look at the graph of  $\text{Ai}(z)$ , where we can see that  $-\alpha E/mg$  must be a zero of Ai, of which there are countably many. Denoting these zeroes by  $-z_n$  (so that  $z_n > 0$ ), we have

$$\begin{aligned} E_n &= \frac{mgz_n}{\alpha} \\ &= z_n \sqrt[3]{\frac{mg^2\hbar^2}{2}} \end{aligned}$$

are the quantized energies of the bouncing ball.

We now calculate (all energies are in Joules)

$$E_1 = 8.6 \cdot 10^{-23}$$

$$E_2 = 1.5 \cdot 10^{-22}$$

$$E_3 = 2.0 \cdot 10^{-22}$$

$$E_4 = 2.5 \cdot 10^{-22}.$$

- d) After all that work, we now get to the easy part. The boundary condition  $\psi(0) = 0$  gives us the quantization condition

$$\int_0^{x_2} p(x') dx' = \pi \hbar (n - 1/4),$$

where  $x_2$  is set by the energy, i.e.  $x_2 = E/mg$ . Likewise,  $p(x) = \sqrt{2mE - 2m^2gx}$ , so that

$$\begin{aligned} (2mE - 2m^2gx)^{3/2} \Big|_0^{E/mg} \left( -\frac{1}{3m^2g} \right) &= \pi \hbar (n - \pi/4) \implies \\ \implies (2mE)^{3/2} &= 3m^2g\pi \hbar (n - \pi/4) \implies \\ \implies E_n &= \frac{(3m^2g\pi \hbar (n - \pi/4))^{2/3}}{2m}. \end{aligned}$$

- e) Again, we calculate (in Joules)

$$E_1 = 8.5 \cdot 10^{-23}$$

$$E_2 = 1.2 \cdot 10^{-21}$$

$$E_3 = 1.8 \cdot 10^{-21}$$

$$E_4 = 2.5 \cdot 10^{-22}.$$

These are within an order of magnitude to the exact solutions, which is still really close for the amount of work necessary!

- f)  $E/mg = 1$  m, so  $E = 1$  J. In order to get the ball that high up, we'd have to take

$$\begin{aligned} n &= \frac{(2m)^{3/2}}{3m^2g\pi \hbar} + \pi/4 \\ &= 9.5 \cdot 10^{32}! \end{aligned}$$

Notice that we plugged in the maximum height as 1 m, when we were supposed to assume an *average* height of 1 m. But a bouncing ball will have an average height that is near the maximum (since it spends most of its time there), and the true average will have an  $n$  value different from  $10^{33}$  by a relatively small number that would be impossible to measure. So we can take this  $n$  for the average height as well.