Week 9 Worksheet
Time-Dependent Phenomena

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October 22, 2025

Exercise 1. General Theory.

a) Consider the Schrödinger equation for time-dependent perturbation theory

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\Psi(t)\right\rangle = [H^0 + \lambda H^1(t)]\left|\Psi(t)\right\rangle.$$

Suppose

$$\left|\Psi(t)\right\rangle = \sum_n c_n(t)e^{-iE_nt/\hbar}\left|n\right\rangle,$$

where $|n\rangle$ are the eigenstates of H^0. Derive the *exact* result

$$i\hbar\dot{c}_n(t) = \lambda\sum_m \langle n|\, H^1(t)\,|m\rangle\, e^{i\omega_{nm}t}c_m(t). \tag{1}$$

b) Now, set

$$c_n(t) = \sum_{k=0}^{\infty}\lambda^k c_n^{(k)}(t),$$

and plug it into Equation 1 to obtain the first order, i.e. $\mathcal{O}(\lambda)$, differential equation.

c) Obtain the second order equation.

Remark. Notice that your results for (b) and (c) are *exactly* the same as the two-level results when we begin in a single initial state!

Exercise 2. Sinusoidal Perturbations. In the case that

$$H^1 = Ke^{-i\omega t} + K^\dagger e^{i\omega t}$$

is sinusoidal and acts up until time t, solve the first order perturbation theory differential equation from Exercise 1(b).

Exercise 3. Spin Resonance. Consider a spin-1/2 particle in a static magnetic field $B_0\hat{z}$, so $H^0 = -\frac{1}{2}\hbar\gamma B_0\sigma_z$. The perturbation is due to a magnetic field $\mathbf{B}_1$ rotating in the (x, y)-plane with angular velocity ω:

$$H^1(t) = -\frac{1}{2}\hbar\gamma B_1[\sigma_x\cos(\omega t) + \sigma_y\sin(\omega t)].$$

a) Writing the eigenvectors of σ_z as

$$|+\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \qquad |-\rangle = \begin{bmatrix}0\\1\end{bmatrix},$$

rewrite H^1 in the form given in Exercise 2, using these eigenvectors as a basis.

b) Suppose at $t = 0$ we have the initial state $|i\rangle = |+\rangle$. Find the first order probability for the spin to be down at time t. It is convenient to set $\omega_0 = \gamma B_0$ and $\omega_1 = \gamma B_1$.
Hint: Note that

$$e^{i\theta} - 1 = 2i\sin(\theta/2)e^{i\theta/2}.$$

This makes it easy to calculate $|e^{i\theta} - 1|^2$.

c) It turns out that the exact Equation 1 can be solved for such a hamiltonian (see part (e)). The exact answer for (b) is

$$P(t) = \sin^2(\alpha t/2)\left(\frac{\omega_1}{\alpha}\right)^2,$$

where $\alpha^2 = (\omega_0 + \omega)^2 + {\omega_1}^2$; $\alpha/2$ is called the **Rabi flopping frequency**. Using this answer, what is the range of validity of the perturbation theory result, assuming we are not near resonance?
Hint: Note that γ can be negative! For example, what is the sign of γ for an electron?

d) Suppose we are near resonance. What is the range of validity of the perturbation theory result? Give a physical explanation of your result.

e) **Challenge.** Solve Equation 1, and derive the formula for $P(t)$.