Week 13 Worksheet Solutions
Partial Wave Analysis

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December 3, 2025

Exercise 1. Warm up.

a) How do the phase shifts δ_ℓ appear in partial wave scattering, and what is their physical significance?

b) What is the fundamental assumption on the form of the wavefunctions in the Born approximation? *Hint*: If the scattering potential is weak, what approximation can we make?

c) Starting from the Lippmann-Schwinger equation,

$$\psi(\mathbf{x}) = \varphi_{\mathbf{k}}(\mathbf{x}) + \int \mathrm{d}^3x' \, G_0(\mathbf{x}, \mathbf{x}', E) V(\mathbf{x}') \psi(\mathbf{x}'),$$

where G_0 is the free particle, time-independent Green's function and

$$\begin{aligned} \varphi_{\mathbf{k}}(\mathbf{x}) &= \langle \mathbf{x} | \mathbf{k} \rangle \\ &= \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}, \end{aligned}$$

explain how you would derive the Born approximation. (Just list the steps, no need to work them out.)
Hint: The Green's function is (note that $E = \hbar^2 k^2/2m$)

$$G_0(\mathbf{x}, \mathbf{x}', E) = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}.$$

This is done/explained in Griffiths.

Exercise 2. *Griffiths 10.4.* Consider low energy scattering off a spherical delta-function shell:

$$V(r) = \alpha\delta(r - a),$$

where α and a are constants. Calculate the scattering amplitude, the differential cross-section, and the total cross-section. Assume that $ka \ll 1$, and ignore all the terms with $\ell > 0$. It will be helpful to express answers in terms of

$$\beta = \frac{2ma\alpha}{\hbar^2}.$$

Hints and Formulas: Make sure to always use the assumption $ka \ll 1$! This will simplify many of your results. The Hankel functions of the first and second kind for $\ell = 0$ are

$$\begin{aligned} h_0^{(1)}(x) &= -i\frac{e^{ix}}{x} \\ h_0^{(2)}(x) &= i\frac{e^{-ix}}{x}, \end{aligned}$$

where

$$\begin{aligned} h_\ell^{(1)} &= j_\ell + i n_\ell \\ h_\ell^{(2)} &= j_\ell - i n_\ell \end{aligned}$$

in terms of the spherical Bessel functions j_ℓ and n_ℓ. The exterior wavefunction is

$$\psi(r,\theta) = A\sum_{\ell=0}^{\infty} i^\ell(2\ell+1)\left[j_\ell(kr) + ika_\ell h_\ell^{(1)}(kr)\right]P_\ell(\cos\theta),$$

and recall that the interior one can be written as a linear combination of spherical Bessel functions (resp. spherical Hankel functions). They are solutions of the radial equation

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + Vu + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}u = Eu,$$

where $u(r) = rR(r)$.

Since we keep only the $\ell = 0$ terms, we have

$$\psi(r,\theta,\varphi) = \begin{cases} A\left[\frac{\sin(kr)}{kr} + a_0\frac{e^{ikr}}{r}\right], & r > a \\ B\frac{\sin(kr)}{kr}, & r < a, \end{cases}$$

since for $r < a$ we can drop the n_ℓ part of the solution. We know that ψ must be continuous at $r = a$ and the discontinuity of ψ' (derivative with respect to the r coordinate) is controlled by the delta function. Thus, continuity gives

$$A\left[\frac{\sin(ka)}{ka} + a_0\frac{e^{ika}}{a}\right] = B\frac{\sin(ka)}{ka}.$$

The discontinuity can be evaluated by means of the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\int u''dr + \int \alpha\delta(r-a)u(r)dr = E\int u\,dr,$$

where the integrals are taken over a small interval of size 2ε about $r = a$. Taking $\varepsilon \to 0$, we find that the integral on the RHS vanishes, while on the LHS we are left with

$$-\frac{\hbar^2}{2m}\lim_{\varepsilon\to 0}[u'(a+\varepsilon) - u'(a-\varepsilon)] + \alpha u(a) = 0.$$

At this point, we do a lot of algebra and use the approximations $\sin(ka) \approx ka$ and $\cos(ka) \approx 1$ when $ka \ll 1$ to find

$$a_0 = -\frac{\beta a}{1+\beta}.$$

Since $f(\theta) \approx a_0$, we're done (noting that $\sigma = 4\pi|f|^2$ since f is constant).