Week 9 Worksheet
Curvature

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Exercise 1. Parallel Transport is Curvature. Let M be a spacetime such that for any two points $p, q \in M$, the parallel transport from p to q does not depend on the curve that joints p and q. You will show that this implies that M is flat, i.e. that the Riemann curvature tensor on M is identically 0. We will do this with the help of the following construction. Consider a parametrized surface $f : U \to M$, where

$$U = \{(s,t) \in \mathbb{R}^2 | s,t \in (-\varepsilon, 1+\varepsilon),\ \varepsilon > 0\}$$

and we force $f(s,0) = f(0,0)$ for all s. Let V_0 be a tangent vector to M at $f(0,0)$, and define a vector field V along f as follows. Set $V(s,0) = V_0$ and $V(s,t)$ to be the parallel transport of V_0 along the curve $c(t) = f(s,t)$.

a) Sketch V.

b) Show that

$$\nabla_t \nabla_s V + R(\partial_t f, \partial_s f)V = 0,$$

where if we set $t = x^1$ and $s = x^2$ this reads

$$\nabla_1 \nabla_2 V^a + R^a{}_{bcd}(\nabla_1 f)^b (\nabla_2 f)^c V^d = 0.$$

Hints: If V is parallel transported along the t-direction, what is $\nabla_t V$? Note that $\nabla_i f = \partial_i f$; why? Recall the definition of R as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z,$$

or, equivalently,

$$\nabla_b \nabla_c v^a - \nabla_c \nabla_b v^a = R^a{}_{bcd} v^d.$$

c) Show that $V(s,1)$ is also the parallel transport of $V(0,1)$ along the curve $c(s) = f(s,1)$, so that $\nabla_s V(s,1) = 0$.

d) Show that

$$R(\partial_t f, \partial_s f)V = 0$$

at the point $(s,t) = (0,1)$.

e) Conclude that $R = 0$ everywhere by arbitrariness of our choices.

Exercise 2. Compute some formulas. Note that ∂_α denotes *ordinary*, rather than covariant, differentiation.

a) Show that

$$\Gamma^{\alpha}_{\beta\alpha} = \partial_\beta(\ln\sqrt{-g}),$$

where the repeated index α is a contraction.

b) Show that the Ricci tensor

$$R_{\alpha\beta} = R^{\gamma}{}_{\gamma\alpha\beta} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,\Gamma^{\mu}_{\alpha\beta}) - \partial_\beta\partial_\alpha\ln(\sqrt{-g}) - \Gamma^{\mu}_{\nu\alpha}\Gamma^{\nu}_{\beta\mu}.$$

c) Check that if V is a vector and $F^{\alpha\beta}$ is an antisymmetric tensor, then

$$\begin{aligned}\nabla_\alpha V^\alpha &= \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}\,V^\alpha)\\ \nabla_\beta F^{\alpha\beta} &= \frac{1}{\sqrt{-g}}\partial_\beta(\sqrt{-g}\,F^{\alpha\beta}).\end{aligned}$$