Week 14 Worksheet Solutions
Time-Dependent Phenomena

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Exercise 1. General Theory.

a) Consider the Schrödinger equation for time-dependent perturbation theory

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\Psi(t)\rangle = [H^0 + \lambda H'(t)] |\Psi(t)\rangle .$$

Suppose

$$|\Psi(t)\rangle = \sum_n c_n(t) e^{-iE_n t/\hbar} |n\rangle ,$$

where $|n\rangle$ are the eigenstates of H^0. Derive the *exact* result

$$i\hbar \frac{\mathrm{d}c_n(t)}{\mathrm{d}t} = \lambda \sum_m \langle n| H'(t) |m\rangle e^{i\omega_{nm}t} c_m(t). \tag{1}$$

b) Now, set

$$c_n(t) = \sum_{n=0}^{\infty} \lambda^k c_n^{(k)}(t),$$

and plug it into your result from (b) to obtain the first order, i.e. $\mathcal{O}(\lambda)$, differential equation.

c) Obtain the second order equation.

Remark. Notice that your results for (b) and (c) are *exactly* the same as the two-level results when we begin in a single initial state!

a) Plug in the form for $|\Psi\rangle$ into the Schrödinger equation:

$$i\hbar \sum_n (\dot{c}_n - i\omega_n) e^{-iE_n t/\hbar} |n\rangle = \sum_n c_n e^{-iE_n t/\hbar} (E_n + \lambda H') |n\rangle .$$

Now, take the inner product with $|n\rangle$ (and change the dummy summation variable from n to m):

$$i\hbar(\dot{c}_n - i\omega_n)e^{-iE_nt/\hbar} = c_nE_ne^{-iE_nt/\hbar} + \sum_m \langle n|\,\lambda H'\,|m\rangle\, c_me^{-iE_mt/\hbar}.$$

Notice that the second term on the LHS is equal to the first term on the RHS, so they cancel. Thus,

$$i\hbar\dot{c}_n(t) = \sum_m \lambda\, \langle n|\, H'\,|m\rangle\, c_m(t)e^{-i\omega_{nm}t},$$

as desired, where

$$\omega_{nm} = \frac{E_n - E_m}{\hbar}.$$

b) The order λ result is just

$$\dot{c}_n^{(1)}(t) = -\frac{i}{\hbar}\sum_m \langle n|\, H'\,|m\rangle\, c_m^{(0)}e^{-i\omega_{nm}t}.$$

c) This is exactly the same:

$$\dot{c}_n^{(k+1)}(t) = -\frac{i}{\hbar}\sum_m \langle n|\, H'\,|n\rangle\, c_m^{(k)}(t)e^{-i\omega_{nm}t}.$$

Exercise 2. Sinusoidal Perturbations. In the case that

$$H' = Ke^{-i\omega t} + K^\dagger e^{i\omega t}$$

is sinusoidal and acts up until time t, solve the first order perturbation theory differential equation from Exercise 1(b).

We plug in

$$\dot{c}^{(1)} = -\frac{i}{\hbar}\left[\sum_m K_{nm}c_m^{(0)}e^{it(\omega_{nm}-\omega)} + \sum_m K_{nm}^\dagger c_m^{(0)}e^{it(\omega_{nm}+\omega)}\right].$$

Integrating from 0 to t, we find

$$c^{(1)}(t) = -\frac{1}{\hbar}\left[\sum_m \frac{K_{nm}c_m^{(0)}}{\omega_{nm}-\omega}\left(e^{it(\omega_{nm}-\omega)} - 1\right) + \sum_m \frac{K_{nm}^\dagger c_m^{(0)}}{\omega_{nm}+\omega}\left(e^{it(\omega_{nm}+\omega)} - 1\right)\right].$$

Exercise 3. Spin Resonance. Consider a spin-1/2 particle in a static magnetic field $B_0\hat{z}$, so $H^0 = -\frac{1}{2}\hbar\gamma B_0\sigma_z$. The perturbation is due to a magnetic field B_1 rotating in the (x, y)-plane with angular velocity ω:

$$H'(t) = -\frac{1}{2}\hbar\gamma B_1[\sigma_x\cos(\omega t) + \sigma_y(\sin(\omega t)].$$

a) Writing the eigenvectors of σ_z as

$$|+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

rewrite H' in the form given in Exercise 2, using these eigenvectors as a basis.

b) Suppose at $t = 0$ we have the initial state $|i\rangle = |+\rangle$. Find the first order probability for the spin to be down at time t. It is convenient to set $\omega_0 = \gamma B_0$ and $\omega_1 = \gamma B_1$.

c) It turns out that the exact Equation 1 can be solved for such a hamiltonian. The exact answer for (b) is

$$P(t) = \sin^2(\alpha t/2)\left(\frac{\omega_1}{\alpha}\right)^2,$$

where $\alpha^2 = (\omega_0 + \omega)^2 + {\omega_1}^2$; $\alpha/2$ is called the **Rabi flopping frequency**. Using this answer, what is the range of validity of the perturbation theory result, assuming we are not near resonance?

d) Suppose we are near resonance. What is the range of validity of the perturbation theory result? Give a physical explanation of your result.

e) **Challenge.** Solve Equation 1, and derive the formula for $P(t)$.

a) We can write

$$H'(t) = -\frac{1}{2}\hbar\gamma B_1 \begin{bmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{bmatrix},$$

so that

$$K = -\frac{1}{2}\hbar\gamma B_1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

b) Since we have a single initial state, the complicated expression from Exercise 2 reduces to two terms. Further, since $K\,|+\rangle = 0$, we are left with only one term. Thus, we find

$$c^{(1)} = \frac{\omega_1}{2(\omega_0 + \omega)}\left(e^{it(\omega_0+\omega)} - 1\right),$$

where we notice that

$$E_- = \frac{\hbar\omega_0}{2}$$
$$E_+ = -\frac{\hbar\omega_0}{2},$$

so $\omega_{-+} = \omega_0$. Now, notice that we can pull out a factor of $e^{it(\omega_0+\omega)/2}$ to get a single phase factor:

$$c^{(1)} = i\frac{\omega_1}{(\omega_0+\omega)}\sin\left(\frac{\omega_0+\omega}{2}t\right)e^{it(\omega_0+\omega)/2}.$$

This trick makes it easy to calculate the first order probability:

$$|c^{(1)}|^2 = \frac{{\omega_1}^2}{(\omega_0+\omega)^2}\sin^2\left(\frac{\omega_0+\omega}{2}t\right).$$

c) Consider the second term in the exact answer

$$\frac{{\omega_1}^2}{\alpha^2} = \frac{{\omega_1}^2}{(\omega_0+\omega)^2\left[1+\frac{{\omega_1}^2}{(\omega_0+\omega)^2}\right]}.$$

We can expand the term in brackets as a binomial expansion if and only if $|\omega_1| \ll |\omega_0+\omega|$, which would give us the perturbation theory result. Thus, this is the range of validity of that result. Note that resonance would correspond to $\omega_0+\omega \sim 0$, so this wouldn't apply in that case.

d) We have to play a different game if we're near resonance. In that case, the exact result becomes

$$P_n(t) \sim \sin^2\left(\frac{\omega_1}{2}t\right).$$

In order to determine what happens to the perturbation theory result, recall that for small θ, $\sin\theta \sim \theta$. Thus,

$$|c^{(1)}(t)|^2 \sim \frac{{\omega_1}^2}{4}t^2.$$

Clearly, these will match well if and only if $|\omega_1 t| \ll 1$. So perturbation theory works well for short times, no matter how big B_1 is. This is because the effect of the perturbation is a product of both its strength *and* its duration.

e) Note that

$$\langle -|\, H'\, |+\rangle = -\frac{\hbar\omega_1}{2}e^{i\omega t},$$

and $\langle +|\, H'\, |-\rangle$ is just the complex conjugate. Thus, we want to solve the system of differential equations

$$\begin{cases} \dot{c}_- = i\frac{\omega_1}{2}e^{it(\omega_0+\omega)}c_+ \\ \dot{c}_+ = i\frac{\omega_1}{2}e^{-it(\omega_0+\omega)}c_- \end{cases}.$$

This is most easily done by differentiating one of the equations once and substituting. We obtain

$$\ddot{c}_- = i(\omega_0+\omega)\dot{c}_- + i\frac{\omega_1}{2}e^{it(\omega_0+\omega)}\dot{c}_+,$$

so that

$$\ddot{c}_{-} - i(\omega_0 + \omega)\dot{c}_{-} + \frac{{\omega_1}^2}{4}c_{-} = 0.$$

To solve this equation, assume solutions of the form e^{kt}, with $k \in \mathbb{C}$. Plugging this in, we obtain an equation for k.

$$k^2 - ik(\omega_0 + \omega) - \frac{{\omega_1}^2}{4} = 0.$$

This has solution

$$k_{\pm} = \frac{i(\omega_0 + \omega) \pm i\alpha}{2},$$

where $\alpha/2$ is the Rabi flopping frequency. Thus,

$$c_{-} = Ae^{k_{-}t} + Be^{k_{+}t}.$$

Now, the initial condition $c_{+}(0) = 1$ gives that

$$\dot{c}_{-}(0) = i\frac{\omega_1}{2}.$$

Since

$$\dot{c}_{-}(t) = k_{-}Ae^{k_{-}t} + k_{+}Be^{k_{+}t}$$

and $c_{-}(0) = 0$, we have two equations for A and B.

$$\begin{cases} A = -B \\ \frac{\omega_1}{2} = A\frac{\omega_0+\omega-\alpha}{2} + B\frac{\omega_0+\omega+\alpha}{2} \end{cases}.$$

These have the solution

$$\begin{cases} A = -\frac{\omega_1}{2\alpha} \\ B = \frac{\omega_1}{2\alpha} \end{cases}.$$

Thus,

$$c_{-}(t) = \frac{i\omega_1}{\alpha}e^{it(\omega_0+\omega)/2}\sin(\alpha t/2),$$

so that

$$|c_{-}|^2 = P(t) = \frac{{\omega_1}^2}{\alpha^2}\sin^2(\alpha t/2),$$

which exactly matches the result quoted in part (c).