Final Review Problems Solutions

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Exercise 1. A charge is distributed with linear charge density λ over the circumference of a circle of radius R which lies in the (x, y)-plane with center at the origin. Find the potential $V(z)$ on the z-axis in the following cases.

a) λ is uniform.

b) $\lambda = C\sin(n\theta)$, where $n \in \mathbb{N}$, C is a constant, and θ is the polar angle.

c) $\lambda = C\theta$.

a) Since λ is constant and $\mathscr{r} = \sqrt{z^2+R^2}$ for a point z on the z-axis, we have

$$\begin{aligned} V(z) =& \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda}{\mathscr{r}} R\,\mathrm{d}\varphi \\ =& \frac{R\lambda}{2\varepsilon_0\sqrt{z^2+R^2}}. \end{aligned}$$

b) Now, we need to actually do the integral over φ. We get

$$-\frac{RC}{4\pi\varepsilon_0\sqrt{z^2+R^2}} \cdot \frac{1}{n}\cos(n\theta)\Big|_0^{2\pi} = 0.$$

c) We now get

$$V(z) = \frac{\pi RC}{2\varepsilon_0\sqrt{z^2+R^2}}.$$

Exercise 2. A dielectric of arbitrary shape, volume V, and permittivity ε which is close to 1 (i.e. such that $\varepsilon - 1 \ll 1$) is brought into a uniform electric field $\mathbf{E}$. Outside the dielectric, $\varepsilon = 1$. Find the field at a large distance r from the dielectric.

There was a slight typo in this exercise: We should have relative permittivity ε_r which is close to 1 for the dielectric. Then we get that the polarization is

$$\mathbf{P} = (\varepsilon_r - 1)\mathbf{E},$$

so that we have an induced dipole moment

$$\begin{aligned}\mathbf{p} =& \int \mathrm{d}V \mathbf{P} \\ =& V(\varepsilon_r - 1)\mathbf{E}.\end{aligned}$$

The field at large distances will then be the field due to such a dipole moment plus the external field. The former is given by (Griffiths Equation 3.103)

$$\mathbf{E}_{\text{dip}} = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}),$$

where we take $\mathbf{E}$ to point in the z direction (so that $\mathbf{p}$ is in that direction as well).

Exercise 3. The center of a metal sphere of radius a lies on the flat boundary between two dielectric regions of permittivities ε_1 and ε_2. At a distance b from the center of the sphere in the region with permittivity ε_1 is placed a point charge q.

a) Find the potential of the sphere if it is insulated and uncharged.

b) Find the charge induced on the sphere if it is grounded.

a) Before we start solving the problem, let's make some general considerations. Recall from Exercise 2 from the midterm review that if we're in the same situation but in vacuum, then the sphere will be at potential

$$V = \frac{q}{2\pi b\varepsilon_0},$$

since we place a charge $q'' = qa/b$ at the center of the sphere to account for the fact that it isn't grounded. We just need to figure out how this potential will change if we are now in the situation in question. The idea is the same: We should look for an image charge at the center which accounts for the *effective* point charge which is induced by the dielectric (plus the actual point charge). So we need to figure out 1) the charge that accumulates in ε_1 due to the point charge and 2) the charge that accumulates in ε_2. Then, we could just plug that in to the above formula, and we're done.

Since $\rho_b = -\nabla \cdot \mathbf{P}$,

$$\begin{aligned}\rho_b =& -\frac{\varepsilon_r - 1}{\varepsilon_r}\nabla \cdot \mathbf{D} \\ =& -\frac{\varepsilon_r - 1}{\varepsilon_r}\rho_f.\end{aligned}$$

In the region ε_1, we thus have that the charge which accumulates around q is

$$q_b = -\frac{\varepsilon_r - 1}{\varepsilon_r}q$$

The total charge there is hence

$$q + q_b = \frac{q}{\varepsilon_{1r}},$$

where $\varepsilon_{1r} = \frac{\varepsilon_1}{\varepsilon_0}$. There is also bound surface charge which will accumulate at the interface between the regions (and on both sides of the interface!). We can now follow Example 4.8 in Griffiths. The bound surface charge is

$$\begin{aligned}\sigma_b =& \mathbf{P}\cdot\hat{n}\\ =&(\varepsilon_1-\varepsilon_0)E_z\end{aligned}$$

if we suppose that the boundary region lies in the (x,y)-plane. The total bound surface charge in region ε_1 is then

$$\sigma_{b1} = (\varepsilon_1-\varepsilon_0)\left(\frac{qb}{4\pi\varepsilon_1(r^2+b^2)^{3/2}}-\frac{\sigma_{b1}}{2\varepsilon_0}-\frac{\sigma_{b2}}{\varepsilon_0}\right),$$

where we suppose that the region with ε_1 is above the z-axis. Then

$$\sigma_{b2} = (\varepsilon_2-\varepsilon_0)\left(-\frac{qb}{4\pi\varepsilon_1(r^2+b^2)^{3/2}}-\frac{\sigma_{b1}}{2\varepsilon_0}-\frac{\sigma_{b2}}{\varepsilon_0}\right).$$

We can solve these two equations for σ_{bi} to get

$$\begin{aligned}\sigma_{b1} =& \frac{qb\varepsilon_{2r}(\varepsilon_{1r}-1)}{2\pi(r^2+b^2)^{3/2}\varepsilon_{1r}(\varepsilon_{1r}+\varepsilon_{2r})}\\ \sigma_{b2} =& -\frac{qb(\varepsilon_{2r}-1)}{2\pi(r^2+b^2)^{3/2}(\varepsilon_{1r}+\varepsilon_{2r})}.\end{aligned}$$

Now, add these up, and integrate over the (x,y)-plane (you can just use that the integral of $b(r^2+b^2)^{-3/2}$ is 2π from the example in Griffiths) to get that the total bound charge is

$$q_{\text{tot}} = \frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\frac{q}{\varepsilon_{1r}}.$$

Phew! Now, we can add this up with the contribution due to the charge at the location of q, $\frac{q}{\varepsilon_{1r}}$, to get

$$\frac{2q}{\varepsilon_{1r}+\varepsilon_{2r}}.$$

Thus, we can use the result of Exercise 2 on the midterm review to find that the potential will be

$$V = \frac{q}{2\pi b(\varepsilon_1+\varepsilon_2)}.$$

b) We can again use Exercise 2 from the midterm review. If the sphere is grounded, the charge induced on the sphere will be the same as the image charge. This time, we don't put any charge in the middle of the sphere, so the image charge is just $-qa/b$.

Exercise 4. ***Griffiths 5.13.*** Suppose you have two infinite, parallel line charges λ a distance d apart, which are moving at a constant speed v. How great would v have to be for the magnetic attraction to balance the electrical repulsion? Calculate the number, and comment on the result.

The electric field (per unit length) is given by Gauss' law:

$$2\pi sE = \frac{\lambda}{\varepsilon_0}.$$

This gives us the electric repulsion

$$F = \frac{\lambda^2}{2\pi s\varepsilon_0}.$$

Similarly, the magnetic field is given by Ampère's law:

$$2\pi sB = \mu_0\lambda v.$$

This gives us the magnetic attraction from the Lorentz force law

$$\mathbf{F} = q\mathbf{v}\times\mathbf{B},$$

where we find

$$F = \frac{\mu_0\lambda^2 v^2}{2\pi s}.$$

Setting the two equal, we find

$$\begin{aligned} v =& \frac{1}{\sqrt{\mu_0\varepsilon_0}} \\ =& c. \end{aligned}$$

So it seems that the forces would be balanced only if we were moving at the speed of light. This makes sense, since if in the rest frame (i.e. $v = 0$) the wires repel each other, then they should do so in any other inertial frame which has a speed parallel to the wires. In particular, it shouldn't matter how fast we were moving relative to the wires.

Exercise 5. ***Griffiths 5.16.*** Two long coaxial solenoids each carry a current I, but in opposite directions. The inner solenoid of radius a has n_1 turns per unit length, while the outer one of radius $b > a$ has n_2 turns per unit length. Find $\mathbf{B}$ in each of the three regions:

a) inside the inner solenoid,

b) between the solenoids, and

c) outside both solenoids.

It's easiest to work outside-in. The field outside is $\mathbf{0}$ by the same arguments as for a single solenoid. The field in between the two is then $\mu_0 n_2 I$, while the field in the inner one is $\mu_0 I(n_1 - n_2)$, where the field outside points in $\hat{z}$ and the field inside in $-\hat{z}$, say.

Exercise 6. ***Griffiths 6.15.*** If $\mathbf{J}_f = \mathbf{0}$ everywhere, the curl of $\mathbf{H}$ vanishes, so we can express $\mathbf{H}$ as the gradient of a scalar potential W,

$$\mathbf{H} = -\boldsymbol{\nabla} W.$$

Thus,

$$\nabla^2 W = \boldsymbol{\nabla} \cdot \mathbf{M},$$

so W obeys Poisson's equation with $\boldsymbol{\nabla} \cdot \mathbf{M}$ as the "source." As an example, find the field inside a uniformly magnetized sphere by separation of variables.

For a uniformly magnetized sphere of radius R, the magnetization is

$$\mathbf{M} = \begin{cases} M\hat{z}, & r < R \\ \mathbf{0}, & r > R \end{cases}.$$

We can use the boundary condition

$$H^{\perp}_{>R} - H^{\perp}_{<R} = M^{\perp}_{<R} - M^{\perp}_{>R}$$

to get

$$-(\nabla W_{>R})_r + (\nabla W_{<R})_r = M\cos\theta.$$

Also, we have continuity at $r = R$:

$$W_{<R}(R) = W_{>R}(R).$$

Using the general solution for Laplace's equation in spherical coordinates, we have that

$$\sum_{\ell=0}^{\infty} \ell A_\ell r^{\ell-1} P_\ell(\cos\theta) + \sum_{\ell=0}^{\infty} (\ell+1) B_\ell r^{-\ell-2} P_\ell(\cos\theta) = M\cos\theta,$$

and

$$\sum A_\ell R^{2\ell+1} P_\ell(\cos\theta) = \sum B_\ell P_\ell(\cos\theta),$$

which implies that

$$B_\ell = A_\ell R^{2\ell+1},$$

since the Legendre polynomials are orthogonal. Now, we can use the first boundary condition to get that $A_\ell = 0$ for all $\ell \neq 1$, since the right hand side has $\cos\theta = P_1(\cos\theta)$. Thus,

$$A_1 + 2A_1 = M,$$

so

$$A_1 = \frac{M}{3}.$$

We can plug this in to get

$$W = \begin{cases} \frac{M}{3} z, & r < R \\ \frac{MR^3}{3r^3} \cos\theta, & r > R \end{cases}.$$

We can now find **H** inside the sphere, which will be given by

$$\begin{aligned} \mathbf{H} &= -\nabla W \\ &= -\frac{M}{3}\hat{z} \end{aligned}$$

So the field will be

$$\begin{aligned} \mathbf{B} &= \mu_0(\mathbf{H} + \mathbf{M}) \\ &= \mu_0 \frac{2M}{3}\hat{z}, \end{aligned}$$

which is the right answer!

Exercise 7. Find the acceleration a of a freely falling, circular, metal plate in a uniform magnetic field which is parallel to the surface of the ground. The plate is oriented with a diameter parallel to the direction of the magnetic field and the ground; its normal vector is perpendicular to the surface of the ground. The radius of the plate is R and its thickness is $d \ll R$. Its mass is m and the strength of the magnetic field is B.

Let's think about what will physically happen as the plate falls. First of all, it has an acceleration due to gravity given by g. So the electrons inside the conductor will feel a force $q\mathbf{v} \times \mathbf{B}$ which will be toward one side of the plate. Now, notice that this force is the same as the electric force with an electric field $\mathbf{E} = \mathbf{v} \times \mathbf{B}$. Hence, the plate will now polarize as if it is in an external electric field $\mathbf{v} \times \mathbf{B}$. Now, the electric field inside the conductor will be

$$E = \frac{\sigma}{\varepsilon_0}.$$

Since $\sigma = \frac{q}{\pi R^2}$ is uniform, the induced charge on the opposite sides of the plate will then be

$$q = \pi R^2 \varepsilon_0 v B.$$

But $\mathbf{v}$ is changing, so that we actually end up with a current which is flowing from one side of the plate to the other:

$$I = \pi R^2 \varepsilon_0 a B.$$

This then gives another force on the plate which will point *up*. Thus, the acceleration of the plate will be *less than* g. This force will be

$$\begin{aligned} \mathbf{F} &= I \int \mathrm{d}\boldsymbol{\ell} \times \mathbf{B} \\ &= I d B \\ &= \pi R^2 \varepsilon_0 a B^2 d. \end{aligned}$$

Newton's law gives

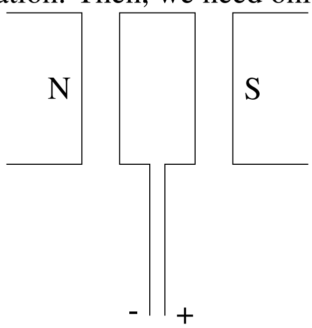
$$ma = mg - \pi R^2 \varepsilon_0 a B^2 d,$$

and we can solve this for a to find

$$a = \frac{g}{1 + \frac{\pi R^2 \varepsilon_0 a B^2 d}{m}}.$$

Exercise 8. Explain how an AC generator works.

The idea is to have a magnet with the north pole facing the south pole and a gap in between. In the gap, we place a coil of wire as shown in the figure below. Now, think about what happens as we rotate this loop about its symmetry axis (the vertical, or y-, axis in the picture). As the magnetic flux through the loop changes, an alternating current will flow through it, which will have a frequency equal to the frequency of rotation. Then, we need only hook up our wires to get an output alternating current.



Exercise 9. ***Griffiths 9.20.*** a) Show that the skin depth in a poor conductor ($\sigma \ll \omega\varepsilon$) is $\frac{2}{\sigma}\sqrt{\varepsilon/\mu}$.

b) Show that the skin depth in a good conductor ($\sigma \gg \omega\varepsilon$) is $\lambda/2\pi$, where λ is the wavelength inside the conductor. Find the skin depth in nanometers for a typical metal ($\sigma \approx 10^7(\Omega\text{m})^{-1}$) in the visible range $\omega \approx 10^{15}$ Hz, assuming $\varepsilon \approx \varepsilon_0$ and $\mu \approx \mu_0$. Why are metals opaque?

c) Show that in a good conductor the magnetic field lags the electric field by $\pi/4$ radians, and find the ratio of their amplitudes.

a) By definition, the skin depth is

$$d = \frac{1}{\kappa},$$

where

$$\kappa = \frac{\omega}{c\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1}.$$

If $\sigma \ll \varepsilon\omega$, then we can expand κ to find

$$\begin{aligned}
\kappa &\approx \frac{\omega}{c\sqrt{2}}\sqrt{1+\frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2 - 1} \\
&\approx \frac{\omega}{2c}\cdot\frac{\sigma}{\varepsilon\omega} \\
&\approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}.
\end{aligned}$$

So

$$d = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}.$$

b) Since $\lambda = 2\pi/k$, if we show that $\kappa = k$ we're done. But indeed,

$$\kappa \approx \frac{\omega}{c\sqrt{2}}\sqrt{\frac{\sigma}{\varepsilon\omega}}\left(1-\frac{\varepsilon\omega}{2\sigma}\right).$$

Similarly,

$$k \approx \frac{\omega}{c\sqrt{2}}\sqrt{\frac{\sigma}{\varepsilon\omega}}\left(1+\frac{\varepsilon\omega}{2\sigma}\right),$$

and we can ignore the small term that goes as $\varepsilon\omega/\sigma$ in both formulas. Hence, they're equal.

Plugging in the numbers, we find

$$\begin{aligned} d =& \frac{1}{\kappa} \\ \approx& 10^{-8} \text{ m}, \end{aligned}$$

or about 10 nm. So that explains why we can't see inside conductors: No light penetrates much farther than a few 10s of nanometers!

c) The phase difference is

$$\begin{aligned} \varphi =& \arctan(\kappa/k) \\ =& \arctan(1) \\ =& \pi/4. \end{aligned}$$

Similarly, the ratio of the amplitudes is given by

$$\begin{aligned} \frac{B_0}{E_0} =& \sqrt{\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}} \\ \approx& \mu\sqrt{\frac{\sigma}{\omega}}. \end{aligned}$$

Plugging in the numbers, we find that this is about 10^{-10}!

Exercise 10. ***Griffiths 9.39.*** For refraction of light from a medium n_2 into a medium with $n_1 < n_2$, Snell's law has a critical angle

$$\theta_c = \arcsin(n_2/n_1).$$

When the incident angle θ_I is greater than θ_c, there is no refracted ray: We get total internal reflection. However, although no energy penetrates the second medium, there is a nonzero field inside the second medium which is rapidly attenuated. We can use the results from class/the textbook with $k_T = \omega n_2/c$ and

$$\mathbf{k}_T = k_T(\sin\theta_T\hat{x} + \cos\theta_T\hat{z}).$$

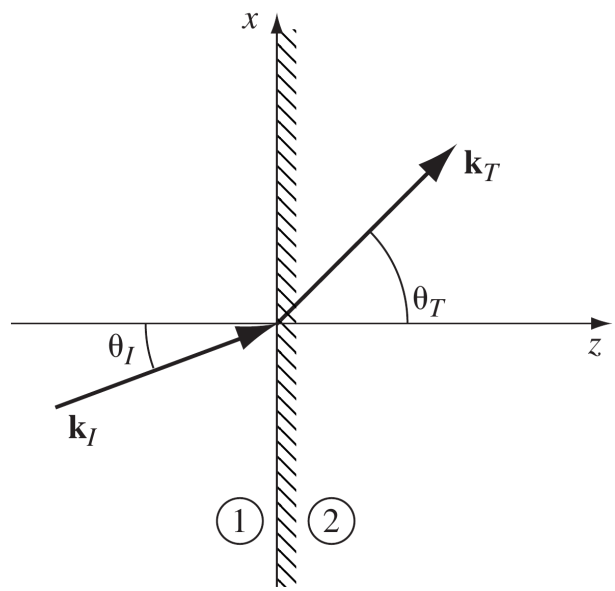
However, we should now take

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I > 1,$$

so that

$$\cos\theta_T = i\sqrt{\sin^2\theta_T - 1}$$

is imaginary.



a) Show that

$$\tilde{\mathbf{E}}_T(\mathbf{r},t) = \tilde{\mathbf{E}}_{0T}e^{-\kappa z}e^{-i(kx-\omega t)},$$

where

$$\kappa = \frac{\omega}{c}\sqrt{(n_1\sin\theta_I)^2 - n_2{}^2} \quad \text{and} \quad k = \frac{\omega n_1}{c}\sin\theta_I.$$

Notice that this is a wave propagating in the x direction and attenuated in the z direction.

b) Noting that

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I}$$

is now imaginary, use Fresnel's equations

$$\begin{aligned}\tilde{E}_{0R} =& \frac{\alpha-\beta}{\alpha+\beta}\tilde{E}_{0I}\\ \tilde{E}_{0T} =& \frac{2}{\alpha+\beta}\tilde{E}_{0I}\end{aligned}$$

to calculate the reflection coefficient for polarization parallel to the plane of incidence.

c) Do the same for polarization perpendicular to the plane of incidence.

d) In the case of polarization perpendicular to the plane of incidence, show that the real evanescent fields are

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) =& E_0 e^{-\kappa z}\cos(kx-\omega t)\hat{y}\\ \mathbf{B}(\mathbf{r},t) =& \frac{E_0}{\omega}e^{-\kappa z}\left[\kappa\sin(kx-\omega t)\hat{x}+k\cos(kx-\omega t)\hat{z}\right].\end{aligned}$$

e) Check that the fields in (d) satisfy Maxwell's equations.

f) For the fields in (d), construct the Poynting vector, and show that, on average, no energy is transmitted in the z direction.

a) We write

$$\tilde{\mathbf{E}}_T(\mathbf{r},t) = \tilde{E}_{0T}e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t},$$

so

$$\tilde{\mathbf{E}}_T(\mathbf{r},t) = \tilde{E}_{0T}\exp[ik_T(x\sin(\theta_T)+z\cos(\theta_T)-\omega t)].$$

We have

$$k_T\sin(\theta_T) = \frac{\omega n_1}{c}\sin(\theta_I)$$

and

$$ik_T\cos(\theta_T) = -\frac{\omega}{c}\sqrt{(n_1\sin\theta_I)^2-n_2{}^2}.$$

Plugging these in to the above formula for $\tilde{\mathbf{E}}$ gives the desired result.

b) We have that

$$R = \left|\frac{\alpha-\beta}{\alpha+\beta}\right|^2.$$

Since $\alpha = ia$ is imaginary and β is real, we have

$$\frac{ia-\beta}{ia+\beta} = \frac{(ia-\beta)^2}{a^2+\beta^2},$$

so

$$
\begin{aligned}
R =& \frac{|\beta^2 - a^2 - 2ia\beta|^2}{(a^2+\beta^2)^2} \\
=& \frac{(\beta^2-a^2)^2 + 4a^2\beta^2}{(a^2+\beta^2)^2} \\
=& 1.
\end{aligned}
$$

c) Use the result of Exercise 3 on Homework 12. Here, the reflection Fresnel equation reads

$$E_R = \frac{1-\alpha\beta}{1+\alpha\beta} E_I.$$

So we have

$$
\begin{aligned}
R =& \left|\frac{1-ia\beta}{1+ia\beta}\right|^2 \\
=& \left|\frac{(1-ia\beta)^2}{1+a^2\beta^2}\right|^2 \\
=& \frac{|1-a^2\beta^2-2ia\beta|^2}{1+a^2\beta^2} \\
=& 1.
\end{aligned}
$$

d) Take the real part of the result of (a), where we set $\tilde{\mathbf{E}}_0 = E_0\hat{y}$, to get the result for $\mathbf{E}$. To compute $\mathbf{B}$, we should take the real part of

$$\tilde{\mathbf{B}} = \frac{1}{c}\hat{k} \times \tilde{\mathbf{E}}.$$

Note that we can't use the real electric field, since $\mathbf{k}_T$ is itself complex. Taking the cross product, we find

$$\tilde{\mathbf{B}} = \frac{E_0 k_T}{c} e^{-\kappa z} e^{i(kx-\omega t)}[\sin(\theta_T)\hat{z} - \cos(\theta_T)\hat{x}].$$

Taking the real part, we get

$$\frac{E_0 e^{-\kappa z}}{c}[k_T \sin\theta_T \cos(kx-\omega t)\hat{z} + ik_T\cos\theta_T \sin(kx-\omega t)\hat{x}].$$

Use again that

$$k_T \sin(\theta_T) = \frac{\omega n_1}{c}\sin(\theta_I)$$

and

$$ik_T\cos(\theta_T) = -\frac{\omega}{c}\sqrt{(n_1\sin\theta_I)^2 - n_2{}^2}.$$

Plugging these in to the above formula gives the desired result for $\mathbf{B}$.

e) Notice that the divergence of $\mathbf{E}$ vanishes easily. For $\mathbf{B}$, we have

$$\begin{aligned}\nabla\cdot\mathbf{B} =&\frac{E_0}{\omega}e^{-\kappa z}[\kappa k\cos(kx-\omega t)-\kappa k\cos(kx-\omega t)\\ =&0.\end{aligned}$$

The curl of $\mathbf{E}$ gives

$$\nabla\times\mathbf{E} = E_0 e^{-\kappa z}[\kappa\cos(kx-\omega t)\hat{x}-k\sin(kx-\omega t)\hat{z}],$$

which is exactly equal to negative the time derivative of $\mathbf{B}$. The final Maxwell's equation reads

$$\nabla\times\mathbf{B} = \frac{{n_2}^2}{c^2}\frac{\partial\mathbf{E}}{\partial t}.$$

The curl of $\mathbf{B}$ is

$$-\frac{E_0}{\omega}e^{-\kappa z}\sin(kx-\omega t)(\kappa^2-k^2)\hat{y}.$$

We also have

$$\frac{{n_2}^2}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \frac{{n_2}^2}{c^2}E_0\omega e^{-\kappa z}\sin(kx-\omega t)\hat{y},$$

so we need to show that

$$k^2-\kappa^2 = \frac{{n_2}^2\omega^2}{c^2}.$$

Indeed,

$$k^2-\kappa^2 = \frac{\omega^2}{c^2}\left({n_1}^2\sin^2\theta_I-{n_1}^2\sin^2\theta_I+{n_2}^2\right) = \frac{{n_2}^2\omega^2}{c^2},$$

as desired.

f) We compute

$$\mathbf{S} = \frac{{E_0}^2}{\mu_0\omega}e^{-2\kappa z}[k\cos^2(kx-\omega t)\hat{x}-\kappa\cos(kx-\omega t)\sin(kx-\omega t)\hat{z}].$$

So we want to compute the time average of the component S_z. This will be proportional to

$$\frac{1}{T}\int_0^T \mathrm{d}t\cos(kx-\omega t)\sin(kx-\omega t) = \frac{1}{2T}\int_0^T \mathrm{d}t\sin[2(kx-\omega t)] = \frac{1}{4\omega T}\cos[2(kx-\omega t)]\Big|_0^T = 0,$$

where by assumption T is a period, so that $\cos(kx-n\omega T)=\cos(kx)$ and $\sin(kx)=\sin(kx-\omega T)$.

Exercise 11. The Dirac Monopole. Consider a half-infinite string of magnetic dipoles, equivalently, a half-infinite solenoid, denoted L.

a) Show that the vector potential outside the string is

$$\mathbf{A}(\mathbf{r}) = -\frac{g}{4\pi}\int_L \mathrm{d}\boldsymbol{\ell} \times \nabla\left(\frac{1}{\imath}\right),$$

where g is a constant.

b) Show that the curl of $\mathbf{A}$ is directed radially outward from the end of the string, varies inversely with distance squared from the end of the string, and has total outward flux g.

Remark. The result of (b) shows that the magnetic field outside of the solenoid is that given by a magnetic monopole of exactly charge g. On the other hand, it can be shown (try for yourself!) that changing the position of the string changes $\mathbf{A}$ by a gauge transformation. Explicitly, if we have two different strings L, L', then the integral taken along the closed path $C = L - L'$ will give

$$\mathbf{A}_{L'}(\mathbf{r}) = \mathbf{A}_L(\mathbf{r}) + \frac{g}{4\pi}\nabla\Omega_C(\mathbf{r}),$$

where Ω_C is the solid angle subtended by the contour C at the observation point $\mathbf{r}$. This means that the string itself is not observable, which is consistent with the fact that physical effects due to the monopole should not depend on the theoretical artifice used to create it (the string). In 1930, Dirac famously showed that the existence of magnetic monopoles *implies* the quantization of electric (and magnetic) charge! This is why people have been interested in magnetic monopoles to this day.

a) We can use the dipole term in the multipole expansion for $\mathbf{A}$ to find

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{r}}{r^2},$$

for a dipole $\mathbf{m}$ at the origin. In general, for a single dipole along the string,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\hat{z}\times\hat{\imath}}{\imath^2}.$$

So $\mathbf{A}$ will be the sum or, more precisely, the integral of all such dipoles. Noting that

$$\nabla\left(\frac{1}{\imath}\right) = -\frac{\hat{\imath}}{\imath^2},$$

we can write this integral as

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 m}{4\pi}\int_{-\infty}^{0} \mathrm{d}z \cdot \hat{z}\times\nabla\left(\frac{1}{\imath}\right),$$

so that the constant $g = \mu_0 m$, where m is the dipole moment of a single dipole on the string.

b) Take the curl of the integrand, and use "BAC-CAB:"

$$\nabla\times\left(\hat{z}\times\nabla\left(\frac{1}{\imath}\right)\right) = \hat{z}\nabla^2\left(\frac{1}{\imath}\right) - \frac{\partial}{\partial z}\nabla\left(\frac{1}{\imath}\right).$$

Now, the first term is proportional to the delta function

$$\delta^3(\boldsymbol{\mathscr{r}}),$$

which will vanish inside the integral, since $\boldsymbol{\mathscr{r}}$ will never be $\mathbf{0}$ over the integration region (for nonzero $\mathbf{r}$, which we can assume). On the other hand, the second term makes the integral easy:

$$\int_{-\infty}^{0} \mathrm{d}z \frac{\partial}{\partial z}\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \left.\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right|_{z=-\infty}^{z=0} = \frac{\hat{r}}{r^2},$$

so the magnetic field is indeed radially outward and varies as $\frac{1}{r^2}$. Let's compute the flux.

$$\int \frac{\hat{r}}{R^2} \cdot \mathrm{d}\mathbf{a} = 4\pi,$$

where we've integrated over a sphere of radius R centered at the origin. Hence, the flux is exactly g.