Some Basics of Derived Algebraic Geometry

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1 Introduction

Derived algebraic geometry grew out of necessity—there is a basic lack of the constructions in classical algebraic geometry that was known already fifty years ago. Given a scheme X, the (derived) category of quasicoherent sheaves $\mathrm{QCoh}(X)$ is a **triangulated category**. In general, a triangulated category comes equipped with a shift endofunctor along with a class of distinguished triangles, where a **triangle** is defined as a collection of morphisms and objects

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1],$$

where $X[1]$ is just the action of the shift functor on X. These triangles are then required to satisfy a slew of compatability conditions, which are actually quite cumbersome to write down. A key example of a triangulated category is the derived category of any abelian category. However, following the introduction of these objects, it was later found that these categories behaved poorly under various geometric constructions; for example, it is not in general possible to glue the derived categories of coherent sheaves on a scheme $X/\mathbb{C}$ (for simplicity) from those obtained from an open cover of X, which is something we would like to have in algebraic geometry. Another example is the Fourier-Mukai-Laumon transform, which identifies the categories

$$\mathscr{D}\text{-mod}(\mathrm{Pic}(X)) \cong \mathrm{QCoh}(\mathrm{LocSys}(X)),$$

where $\mathscr{D}\text{-mod}$ is the category of $\mathscr{D}$-modules, and $\mathrm{LocSys}(X)$ is that of local systems. We recall that a **local system** is just a locally constant sheaf. If we specialize to locally constant sheaves of complex vector spaces of dimension n, we then have, by the Riemann-Hilbert correspondence, that this is equivalent to a rank n vector bundle with flat connection or a monodromy representation $\pi_1(X) \to \mathrm{GL}_n(\mathbb{C})$. Thus, this transform appears in the geometric Langlands program and has deep ties to physics, so its importance should not be understated. However, in order for this equivalence to hold, we need to understand $\mathrm{LocSys}(X)$ as a "derived stack" (even for X a classical scheme). Moreover, derived algebraic geometry provides a natural setting for the study of deformation theory. Lastly, after introducing the infinity-categorical machinery necessary to make the basic definitions, the actual definition of a (derived) stack, which we note subsumes the definition of ordinary stacks, becomes surprisingly simple. The usual definition of stack is extraordinarily complex if we do not use the language of ∞-categories. Most of the paper will consist of making constructions, formulating definitions, and working through examples. We will actually not have to spend much effort on proving various statements. This is a sign that the subject is mathematically well-formulated: Most of the work goes into the definitions, and, having made them, the proofs become almost tautological. Having convinced ourselves that this is a subject worth exploring, we begin with covering some of the more interesting preliminaries.

Nearly all of the material is from [1], except where stated otherwise. The examples are original/from discussions with various other students and seminars over the last year (except for the "Proposition-Example").

After working through some preliminaries in §3, we will define some of the basic geometric objects of derived algebraic geometry in §4. This will lead us to the notion of quasicoherent sheaves on these objects in §5, along with some examples of such. Finally, in §6, we define ind-coherent sheaves and discuss why we would like to make such a definition, mention some key properties of the category of these sheaves, and then discuss the differences between the categories of quasicoherent sheaves and ind-coherent sheaves. These, along with the quaiscoherent sheaves, are in a sense the main objects of study in derived algebraic geometry, so stopping here is natural. I had hoped to terminate the paper with the construction of the !-pullback functor (discussed in §6) for ind-coherent sheaves; however, there was unfortunately not enough space.

2 Notation and Terminology

We work throughout over a field k of characteristic 0. Given an $(\infty, 1)$-category A, we will write $a \in \mathsf{A}$ to denote an object a contained in A. Given another object $a' \in \mathsf{A}$, we denote by $\mathrm{Hom}_{\mathsf{A}}(a, a') \in \mathsf{Spc}$ the mapping space, where Spc is the $(\infty, 1)$-category of ∞-groupoids, in other words, spaces. In the rest of the paper, we will say "category" to mean $(\infty, 1)$-category, since (almost) everything in sight will be one such. We will say "ordinary category" when we want to talk about such objects. For two categories A and B, we denote by $\mathsf{Fun}(\mathsf{A}, \mathsf{B})$ the category of functors $\mathsf{A} \to \mathsf{B}$. We denote by 1-$\mathsf{Cat}$ the category of categories.[1] Denote by Δ the ordinary simplex category, with the usual spanning object definition: $[n]$ is the ordinary category $0 \to 1 \to \cdots \to n$. Denote $[0]$ by $*$; this is the point category. Every other piece of notation is standard or else will be introduced in the main body of the text.

3 Preliminaries

We assume that the reader is familiar with the basics of ∞-categories and especially $(\infty, 1)$-categories; we will only cover some of the crucial constructions that will be used throughout the rest of this paper.

3.1 Straightening and Unstraightening

Recall the definition of the over- and under-categories associated to a category C and a functor $F : \mathsf{I} \to \mathsf{C}$. Further, recall that if $\mathsf{I} = *$ and F is defined by an object $c \in \mathsf{C}$, then we denote the over-category of C over F by $\mathsf{C}_{/c}$, and similarly for the under-category. Thus, denote by $\mathsf{Cocart}_{/\mathsf{C}}$ the full subcategory of 1-$\mathsf{Cat}_{/\mathsf{C}}$ whose objects are those (D, F) such that F is a cocartesian fibration. Further, define 0-$\mathsf{Cocart}_{/\mathsf{C}}$ to be the full subcategory of 1-$\mathsf{Cat}_{/\mathsf{C}}$ with the same objects but such that F is a cocartesian fibration in spaces. The key feature of cocartesian fibrations is the following

Proposition 1. There is a canonical equivalence between $(\mathsf{Cocart}_{/\mathsf{C}})_{\text{strict}}$ and $\mathsf{Fun}(\mathsf{C}, 1\text{-}\mathsf{Cat})$, where $(\mathsf{Cocart}_{/\mathsf{C}})_{\text{strict}}$ denotes the 1-full subcategory of $\mathsf{Cocart}_{/\mathsf{C}}$ where we allow as 1-morphisms those functors over C that send cocartesian arrows to cocartesian arrows.

Remark. We skip the proof for brevity; it can be constructed from the remarks in §I.1.1.4.3 in [1]. We call **unstraightening** the operation of passing from a functor $\mathsf{C} \to 1\text{-}\mathsf{Cat}$ to an object of $\mathsf{Cocart}_{/\mathsf{C}}$; likewise, **straightening** is the reverse operation.

[1] We eschew all set theoretic issues here. These are covered in great detail in [3].

There is a similar statement for cartesian fibrations, except cartesian fibrations are mapped to functors $\mathsf{C}^{\mathrm{op}} \to 1\text{-}\mathsf{Cat}$. We likewise define similar notions of straightening/unstraightening for cartesian fibrations. We can use these equivalences to define Kan extension and adjoint functors, which we will now briefly explain. Let $F : \mathsf{C}_0 \to \mathsf{C}_1$ be a functor, and suppose we want to construct its right adjoint. We note that we can view F as a functor $[1] \to 1\text{-}\mathsf{Cat}$ via the equivalence

$$\mathsf{Fun}([1], 1\text{-}\mathsf{Cat}) \underset{1\text{-}\mathsf{Cat} \times 1\text{-}\mathsf{Cat}}{\times} \{\mathsf{C}_0, \mathsf{C}_1\} \cong (\mathsf{Fun}(\mathsf{C}_0, \mathsf{C}_1))^{\mathsf{Spc}},$$

where the superscript Spc denotes the image under the action of the right adjoint of the inclusion $\mathsf{Spc} \hookrightarrow 1\text{-}\mathsf{Cat}$. Obviously this is circular, but note that the left hand side is isomorphic to $\mathrm{Hom}_{1\text{-}\mathsf{Cat}}(\mathsf{C}_0, \mathsf{C}_1) \in \mathsf{Spc}$; after defining adjoints, it will be clear that this is in turn isomorphic to the right hand side as written. Viewing F in this way, we can apply unstraightening and regard it as a cocartesian fibration $\mathsf{M} \to [1]$, where M is some (arbitrary) category. Now, if this fibration happened to be bicartesian, i.e. it was also a cartesian fibration in addition to being a cocartesian one, we could apply straightening again to obtain a new functor $[1]^{\mathrm{op}} \to 1\text{-}\mathsf{Cat}$. The functor thus obtained from $\mathsf{C}_1 \to \mathsf{C}_0$ is the **right adjoint** of F, denoted F^R. Proceeding in an analogous manner, we could define the **left adjoint** of F by inverting all the arrows. Thus, we find that the condition for F to admit a right (resp. left) adjoint is that the associated cocartesian (resp. cartesian) fibration is bicartesian. However, even in the event that it is not bicartesian, we can consider the full subcategory of C_1 such that there exists a cartesian morphism in M over the morphism $0 \to 1$ in $[1]$. Denoting the corresponding subcategory of M by M', we obtain a cartesian fibration $M' \to [1]$, which would give a functor $F^R : \mathsf{C}_1' \to \mathsf{C}_1$. We call this functor the **partially defined right adjoint** of F, and similarly for the case of paritally defined left adjoint. This leads us to the notion of Kan extension.

Example 1. Consider the inclusion $\mathsf{Spc} \hookrightarrow 1\text{-}\mathsf{Cat}$. Viewing it as cocartesian functor $[1] \to 1\text{-}\mathsf{Cat}$, we apply unstraightening to obtain $M \to [1]$. Clearly, this functor is bicartesian, since we started with an inclusion into $1\text{-}\mathsf{Cat}$, so the right adjoint of the inclusion is well-defined.

Suppose we have a functor $F : \mathsf{D} \to \mathsf{C}$ between categories. Consider the functor

$$\mathsf{Fun}(\mathsf{C}, \mathsf{E}) \to \mathsf{Fun}(\mathsf{D}, \mathsf{E}),$$

given by **restriction along** F, i.e. composition with F. We call its partially defined left (resp. right) adjoint the functor of **left** (resp. **right**) **Kan extension along** F and denote it LKE_F (resp. RKE_F). For $\mathsf{C} = *$, the corresponding left (resp. right) Kan extension functor is the functor of **colimit** (resp. **limit**).

3.2 (Symmetric) Monoidal Structures

Recall that we define a **monoidal category** to be a functor $\mathsf{A} : \Delta^{\mathrm{op}} \to 1\text{-}\mathsf{Cat}$ such that $\mathsf{A}([0]) = *$ and for any $n \in \mathbb{N}$ the functor given by the n-tuple of maps in Δ

$$[1] \to [n], \qquad 0 \mapsto i, 1 \mapsto i+1, \qquad i \in \{0, \ldots, n-1\},$$

defines an equivalence

$$\mathsf{A}([n]) \cong \mathsf{A}([1]) \times \cdots \times \mathsf{A}([1]).$$

We call the map in the definition $[1] \to 2$, $0 \mapsto 0$, $1 \mapsto 2$ the **monoidal operation** $\mathsf{A}[1] \times \mathsf{A}[1] \to \mathsf{A}[1]$. On objects, we denote this functor by $a, b \mapsto a \otimes b$.

Proposition and Example. Let $\mathsf{C} \in 1\text{-}\mathsf{Cat}$. Then $\mathsf{A} = \mathsf{Fun}(\mathsf{C}, \mathsf{C})$ **acquires a natural monoidal structure**, which means there is a monoidal operation $\mathsf{A} \times \mathsf{A} \to \mathsf{A}$ which is obtained from a monoidal category B such that $\mathsf{A} = \mathsf{B}[1]$.

Proof. Define the monoidal category

$$\mathsf{Fun}(\mathsf{C}, \mathsf{C})^{\otimes} : \Delta^{\mathrm{op}} \to 1\text{-}\mathsf{Cat}$$

as follows. It sends

$$[n] \mapsto \mathsf{Cart}_{/[n]^{\mathrm{op}}} \underset{1\text{-}\mathsf{Cat} \times \cdots \times 1\text{-}\mathsf{Cat}}{\times} \{\mathsf{C} \times \cdots \times \mathsf{C}\},$$

where the functor

$$\mathsf{Cart}_{/[n]^{\mathrm{op}}} \to \mathsf{Cart}_{/* \sqcup \cdots \sqcup *} \cong 1\text{-}\mathsf{Cat} \times \cdots \times 1\text{-}\mathsf{Cat}$$

is just restriction along

$$* \sqcup \cdots \sqcup * = ([n]^{\mathrm{op}})^{\mathsf{Spc}} \to [n]^{\mathrm{op}}.$$

By Corollary 2.4.4 of [1] Chapter I.10, The category

$$\mathsf{Cart}_{/[1]^{\mathrm{op}}} \underset{1\text{-}\mathsf{Cat} \times 1\text{-}\mathsf{Cat}}{\times} \{\mathsf{C} \times \mathsf{C}\} \cong \mathsf{Fun}(\mathsf{C}, \mathsf{C}).$$

Thus, it remains to show that the functor $\mathsf{Fun}(\mathsf{C}, \mathsf{C})^{\otimes}$ satisfies the conditions given above, so that it is in fact a monoidal category. But this immediately follows from induction on n; indeed, consider the image of [2]:

$$[2] \mapsto \mathsf{Cart}_{/[2]^{\mathrm{op}}} \underset{1\text{-}\mathsf{Cat} \times 1\text{-}\mathsf{Cat} \times 1\text{-}\mathsf{Cat}}{\times} \{\mathsf{C} \times \mathsf{C} \times \mathsf{C}\}.$$

Unwinding the definitions, we find that this is indeed isomorphic to two copies of the image of [1]. Explicitly, we note that the functor associated to $[2]^{\mathrm{op}} = 0 \leftarrow 1 \leftarrow 2$ can be viewed as two copies of $[1]^{\mathrm{op}} \times [1]^{\mathrm{op}}$, and we can similarly break up the product of the 1-Cat's and the product of the C's. Proceeding in this way, we arrive at the desired conclusion. $\square$

We can make a similar definition for a *symmetric* monoidal category by replacing Δ^{op} by the ordinary category Fin_* of finite pointed sets. Now, let $1\text{-}\mathsf{Cat}^{\mathrm{Mon}}$ be the full subcategory of $\mathsf{Fun}(\Delta^{\mathrm{op}}, 1\text{-}\mathsf{Cat})$ of monoidal categoires. Unstraightening then defines a fully faithful embedding

$$\begin{aligned} 1\text{-}\mathsf{Cat}^{\mathrm{Mon}} &\hookrightarrow (\mathsf{Cocart}_{/\Delta^{\mathrm{op}}})_{\mathrm{strict}}. \\ \mathsf{A} &\mapsto \mathsf{A}^{\Delta^{\mathrm{op}}}. \end{aligned}$$

Equip the point category $*$ with a monoidal structure via $*([n]) = *$. We thus define an **associate algebra** in a monoidal category A as a **right-lax monoidal functor**, i.e. a functor between objects in $\mathsf{Cocart}_{/\Delta^{\mathrm{op}}}$ that maps morphisms that are cocartesian over morphisms in Δ^{op} of the kind which define monoidal categories to morphisms with the same property,

$$\mathcal{A} : \Delta^{\mathrm{op}} = *^{\Delta^{\mathrm{op}}} \to \mathsf{A}^{\Delta^{\mathrm{op}}}.$$

Similarly, one defines commutative algebra objects in symmetric monoidal categories.

Finally, one can define modules over (symmetric) monoidal categories. Moreover, for a given monoidal category A, one can define the category of left A-modules via

$$\mathsf{A}\text{-}\mathsf{mod} = 1\text{-}\mathsf{Cat}^{\mathrm{Mon}^+} \underset{1\text{-}\mathsf{Cat}^{\mathrm{Mon}}}{\times} \{\mathsf{A}\},$$

where $1\text{-}\mathsf{Cat}^{\mathrm{Mon}^+}$ denotes the category of pairs of a monoidal category equipped with a module; see §3.4 in Chapter I.3 of [1] for the rest of the definitions. The rest of the preliminary machinery we require will be eschewed,[2] except to mention the definition of a differential-graded or dg category. We simply define $\mathsf{dgCat}_{\mathrm{cont}}$ to be the category

$$\mathsf{Vect}\text{-}\mathsf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}},$$

i.e. the category of Vect-modules that are stable, cocomplete (have filtered colimits), and continuous (preserve filtered colimits). Since it is a module (sub)category, it comes equipped with a symmetric monoidal structure that desceneds from the one on the modules themselves.

3.3 Ind-completion

In this last preliminary section, we will quickly describe the Ind functor. By definition, given any $\mathsf{C} \in \mathsf{dgCat}^{\mathrm{non\text{-}cocmpl}}$,

$$\mathrm{Ind}(\mathsf{C}) = \mathsf{Fun}_{\mathrm{ex}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Sptr}),$$

where the subscript ex denotes exact functors, i.e. those which preserve pullbacks (equivalently, pushouts), including the empty one, and Sptr is the **category of spectra**; it is by definition the unit object in the symmetric monoidal category $1\text{-}\mathsf{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$. Note that this object is cocomplete by definition; thus, it is actually an object of dgCat, the essential image of the functor $\mathsf{dgCat}_{\mathrm{cont}} \to \mathsf{dgCat}^{\mathrm{non\text{-}cocmpl}}$.

4 Basic Objects of Derived Algebraic Geometry

4.1 Affine Schemes and Prestacks

Armed with the necessary prerequisites, we can finally begin a study of DAG proper. As mentioned in the introduction, one of the advantages that we will discover is that definitions of the principal objects of study in DAG are actually quite simple. In contrast, schemes in classical algebraic geometry are defined as locally ringed spaces; the definition is quite heavy, taking up about half a page in Hartshorne. Taking Grothendieck's functor of points approach, this definition reduces to the category of functors

$$(\mathsf{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathsf{Set},$$

where $(\mathsf{Sch}^{\mathrm{aff}})^{\mathrm{op}}$ is the category of commutative rings. The functors which correspond to schemes are those which have a "Zariski atlas."

[2] It can be found in Chapter I.1 of [1] or in [3].

We now subsume all of this by passing to the derived language. For the rest of the paper, everything in sight will be derived (derived stack, derived scheme, etc.), so we will drop this adjective. If we wish to refer to usual schemes or stacks, we will add the adjective "classical." In particular, $\mathsf{Sch}^{\mathrm{aff}}$ will denote the category of derived affine schemes, with ${}^{\mathrm{cl}}\mathsf{Sch}^{\mathrm{aff}}$ denoting the category of classical affine schemes, and similarly for other categories to be defined presently.

Finally, we define the category of affine schemes to be

$$\mathsf{Sch}^{\mathrm{aff}} = (\mathsf{ComAlg}(\mathsf{Vect}^{\geq 0}))^{\mathrm{op}},$$

where $\mathsf{Vect}^{\geq 0}$ denotes the obvious subcategory of Vect under its natural t-structure. Explicitly, one can think of objects in this subcategory as complexes of vector spaces with only negative cohomologies. We thus define the category of prestacks to be

$$\mathsf{PreStk} = \mathsf{Fun}((\mathsf{Sch}^{\mathrm{aff}})^{\mathrm{op}}, \mathsf{Spc}).$$

On the one hand, this definition is quite simple; however, it is too general to be able to prove anything of use for these most general sorts of prestacks. Thus, in the next section we will study various subclasses of these objects.

4.2 Finiteness Conditions

Consider the full subcategory $\mathsf{Vect}^{\geq -n, \leq 0} \subset \mathsf{Vect}^{\leq 0}$, which has the obvious definition. This fully faithful embedding has a left adjoint, $\tau^{\geq -n}$, called **truncation**. Moreover, we claim

Lemma 1. There is a symmetric monoidal structure on $\mathsf{Vect}^{\geq -n, \leq 0}$ that makes $\tau^{\geq -n}$ a symmetric monoidal functor. Moreover, this structure is unique.

Proof. Let $V_1 \to V_1$ be a morphism in $\mathsf{Vect}^{\leq 0}$ such that $\tau^{\geq -n}(V_1) \to \tau^{\geq -n}(V_1)$ is an isomorphism. Then we immediately have

$$\tau^{\geq -n}(V_1 \otimes V_2) \xrightarrow{\cong} \tau^{\geq -n}(V_1 \otimes V_2)$$

as well, just by the definitions. Thus, $\mathsf{Vect}^{\geq -n, \leq 0}$ acquires a uniquely defined monoidal structure in which $\tau^{\geq -n}$ is symmetric monoidal. □

Thus, it follows from the definitions that the embedding

$$\mathsf{Vect}^{\geq -n, \leq 0} \hookrightarrow \mathsf{Vect}^{\leq 0}$$

has a natural right-lax symmetric monoidal structure. Therefore, this embedding induces a fully faithful functor

$$\mathsf{ComAlg}(\mathsf{Vect}^{\geq -n, \leq 0}) \to \mathsf{ComAlg}(\mathsf{Vect}^{\leq 0}) \tag{1}$$

whose essential image is those objects in the codomain that belong to $\mathsf{Vect}^{\geq -n, \leq 0}$ when regarded as plan objects of $\mathsf{Vect}^{\leq 0}$ (i.e. under the image of the natural forgetful functor). Similarly, the functor (1) admits a left adjoint $\tau^{\geq -n}$ which is now truncation of commutative algebra objects, and it is compatible with the originally defined truncation functor in the obvious way (i.e. it fits into a commutative diagram with it and the forgetful functors that forget the commutative algebra structures). Finally, we say that $S \in \mathsf{Sch}^{\mathrm{aff}}$ is n-**coconnective** if $S = \operatorname{Spec} A$ with A lying in the essential image of the functor (1). Equivalently, $H^{-i}(A) = 0$ for $i > n$. We denote the full subcategory of $\mathsf{Sch}^{\mathrm{aff}}$ spanned by n-coconnective objects by ${}^{\leq n}\mathsf{Sch}^{\mathrm{aff}}$. By definition, we thus obtain

Proposition 2. For $n = 0$, we recover the category of classical affine schemes

$$^{\mathrm{cl}}\mathsf{Sch}^{\mathrm{aff}} = {}^{\leq 0}\mathsf{Sch}^{\mathrm{aff}}.$$

We can similarly define n-**coconnective** prestacks. Namely, a prestack $\mathcal{Y} \in \mathsf{PreStk}$ is n-coconnective if it is in the essential image of the functor $\mathrm{LKE}_{{}^{\leq n}\mathsf{Sch}^{\mathrm{aff}} \hookrightarrow \mathsf{Sch}^{\mathrm{aff}}}$. We remark that again by the way we have defined everything, if $\mathscr{Y}$ is representable by an affine scheme S, then the prestack $\tau^{\leq n}(\mathcal{Y})$ is representable by the affine scheme $\tau^{\leq n}(S)$, where $\tau^{\leq n}$ for prestacks is defined exactly as in the case for schemes and

$${}^{\leq n}\mathsf{PreStk} := \mathsf{Fun}(({}^{\leq n}\mathsf{Sch}^{\mathrm{aff}})^{\mathrm{op}}, \mathsf{Spc}).$$

We say that an object $S = \operatorname{Spec} A \in {}^{<\infty}\mathsf{Sch}^{\mathrm{aff}}$ is **of finite type** if $H^0(A)$ is of finite type over k and each $H^{-i}(A)$ is finitely generated as a module over $H^0(A)$. Note that in the case of classical affine schemes, this reduces exactly to the notion of the affine scheme being of finite type over k. In the derived setting, schemes of finite type similarly belong to "finite-dimensional" geometry.

Let $\mathcal{Y} \in {}^{\leq n}\mathsf{PreStk}$ for some n. We say $\mathcal{Y}$ is **locally of finite type** if it is the left Kan extension of its own restriction along the embedding

$${}^{\leq n}\mathsf{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathsf{Sch}^{\mathrm{aff}},$$

where the category on the left denotes the subcategory of finite type affine schemes. We state the next result without proof, as its proof depends on an induction on n argument using deformation theory, which is beyond the scope of this paper.

Proposition 3. Let S be an n-coconnective affine scheme. It is of finite type if and only if the prestack it represents is n-coconnective locally of finite type.

Taking this on faith, we see that the condition of begin locally of finite type for prestacks is well-formed due to this proposition.

Finally, we say that an affine scheme is **almost of finite type** if ${}^{\leq n}S$ is of finite type for each n. Similarly, a prestack $\mathcal{Y}$ is **locally almost of finite type** if the following conditions hold.

1. $\mathcal{Y}$ is **convergent**, i.e. for each affine scheme S, the map

   $$\mathcal{Y}(S) \to \lim_n \mathcal{Y}(\tau^{\leq n}(S))$$

   is an isomorphism.

2. For each $n \in \mathbb{N}$, we have ${}^{\leq n}\mathcal{Y}$ is n-coconnective locally of finite type.

We will develop the theory of ind-coherent sheaves for exactly this class of prestacks, so they are of crucial importance.

4.3 Stacks

We will define stacks by specifying certain descent conditions which prestacks must satisfy. The notion of a stack is just the interaction of ag given Grothendieck topology (e.g. flat, ppf, étale, or Zariski) on the category of affine schemes with the notion of a prestack.

A map of affine schemes $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is called **flat** if $H^0(B)$ is flat as a module over $H^0(A)$ and the following equivalent conditions hold.

1. The natural map
$$H^0(B) \underset{H^0(A)}{\otimes} H^i(A) \to H^i(B)$$
is an isomorphism for each i.

2. For any A-module M, the natural map
$$H^0(B) \underset{H^0(A)}{\otimes} H^i(M) \to H^i(B \otimes_A M)$$
is an isomorphism for each i.

Given a morphism $f : S' \to S$ of affine schemes, we will say that it is ppf, or flat of finite presentation, (resp. smooth, étale, open embedding Zariski) if the following conditions hold.

1. f is flat.

2. The map of classical affine schemes ${}^{\mathrm{cl}}S' \to {}^{\mathrm{cl}}S$ is of finite presentation (resp. smooth, étale, open embedding, disjoint union of open embeddings).

Now, let $\mathcal{Y}$ be a prestack. We say that it **satisfies flat** (resp. **ppf, smooth, étale, Zariski**) **descent** if whenever $f : S' \to S$ is a morphism of affine schemes that is a flat covering, the map

$$\mathcal{Y}(S) \to \mathrm{Tot}(\mathcal{Y}(S'^{\bullet}/S))$$

is an isomorphism, where $S'^{\bullet}/S$ is the Čech nerve of the map f. The Čech nerve of a map in some arbitrary category C with cartesian products is defined as follows. Since it has cartesian products, there is a functor for each $c \in \mathsf{C}$

$$\begin{aligned} \mathsf{Fin}^{\mathrm{op}} &\to \mathsf{C} \\ I &\mapsto c^I, \end{aligned}$$

where Fin is the ordinary category of finite sets. Composing this with the functor $\Delta \to \mathsf{Fin}$, we get a functor $\Delta^{\mathrm{op}} \to \mathsf{C}$. Now, suppose D is some other category with fiber products, and $d \in \mathsf{D}$. Define $\mathsf{C} = \mathsf{D}_{/d}$, so that cartesian products in C are the fiber products in D over d. Given an object $c \in \mathsf{C}$ we thus get a functor

$$\Delta^{\mathrm{op}} \to \mathsf{C} \to \mathsf{D}.$$

It is this functor that is called the **Čech nerve** of the morphism $c \to d$; it's denoted $c^{\bullet}/d$. Finally, the notation Tot just denotes taking the limit in spaces over Δ. We call prestacks $\mathcal{Y}$ that satisfy this descent condition **stacks**, and denote the corresponding full subcategory of PreStk just by Stk.

4.4 Schemes

We call a prestack X a **scheme** if the following conditions hold.

1. X satisfies étale descent.
2. The diagonal map $X \to X \times X$ is **affine schematic**, i.e. for every $S \in (\mathsf{Sch}^{\mathrm{aff}})_{/X\times X}$, the prestack $S \underset{X\times X}{\times} X$ is representable by an affine scheme. Moreover, for each $S \in (\mathsf{Sch}^{\mathrm{aff}})_{/X\times X}$, the induced map of classical schemes ${}^{\mathrm{cl}}(S \underset{X\times X}{\times} X) \to {}^{\mathrm{cl}}S$ is a closed embedding.
3. There exists a collection of affine schemes S_i and maps $f_i : S_i \to X$, called a **Zariski atlas**, such that each f_i is an open embedding, and, for every $S \in (\mathsf{Sch}^{\mathrm{aff}})_{/X}$, the images of the maps ${}^{\mathrm{cl}}(S \times_X S_i) \to {}^{\mathrm{cl}}S$ cover ${}^{\mathrm{cl}}S$.

There is quite a lot we could say about schemes and their relation to the classical objects of algebraic geometry; however, we will restrict ourselves to discussing the finiteness conditions in the case of schemes, in the interest of space.

Since schemes are prestacks, we can import the notion of coconnectivity to them. For example, by changing the category PreStk in the definition of a scheme to the category ${}^{\leq n}\mathsf{PreStk}$, we obtain the notion of an n-coconnective scheme. Similarly, again since schemes are by definition prestacks, the finiteness conditions locally of finite type and locally almost of finite type carry over to the setting of schemes.

We could further define the notion of an Artin stack, which geometrically corresponds to a non-separated scheme (the schemes defined above correspond to the classical separated schemes), but we will not do this here.

5 Quasicoherent Sheaves

One could argue that the objects of study in algebraic geometry are not schemes and their generalizations but rather the categories of (quasi)coherent sheaves on these objects. We will now define these categories for the objects of the previous section.

Recall that we have a canonically defined functor

$$\begin{aligned}(\mathsf{AssocAlg}(\mathsf{Vect}))^{\mathrm{op}} &\to \mathsf{dgCat}_{\mathrm{cont}}\\ A &\mapsto A\text{-}\mathrm{mod}\,.\end{aligned}$$

Note that this is not the same as the category of A-modules A-mod. The latter category was defined in §2. The former category is defined as follows. Consider the category of right-lax functors

$$*^{+} \to \mathsf{A}^{+},$$

where $*^{+}$ and A^{+} are objects of $1\text{-}\mathsf{Cat}^{\mathsf{Mon}^{+}}$. A^{+} is any object which we view as a module over itself, i.e. over its underlying category, while $*^{+}$ is the point category viewed as a module. Now, from this category of right-lax functors defined above, consider the forgetful functor to $\mathsf{AssocAlg}(\mathsf{A})$, where A is the underlying category of A^{+}, i.e. the value of $\mathsf{A}^{+}([0]^{+})$ (recall that the plus just denotes that instead of considering functors from Δ or Δ^{op}, we consider functors from Δ^{+} or $(\Delta^{+})^{\mathrm{op}}$, whose objects are of the form $0 \to 1 \to \cdots \to n \to +$, and whose morphisms satisfy a certain set of conditions based on how they act on $+$). The fiber over this forgetful functor is then $\mathsf{A}\text{-}\mathrm{mod}$, the **category of** A**-modules in** A).

Example 2. Let A be a commutative algebra object in Vect, and consider the category $A\text{-mod} = A\text{-mod}(\mathsf{Vect})$, the category of A-modules in Vect. Objects in this category are complexes of vector spaces with multiplication maps $A \times \cdots \times A \times M \to M$ satisfying a homotopy-coherent set of conditions. In particular, we have a map $A \times M \to M$, so that everything in the complex is an A-module in the usual sense.

Now, compose the functor $A \mapsto A\text{-mod}$ with the forgetful functors

$$\mathsf{ComAlg}(\mathsf{Vect}^{\leq 0}) \to \mathsf{ComAlg}(\mathsf{Vect}) \to \mathsf{AssocAlg}(\mathsf{Vect})$$

to obtain a functor

$$\mathrm{QCoh}_{\mathsf{Sch}^{\mathrm{aff}}} : (\mathsf{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathsf{dgCat}_{\mathrm{cont}}.$$

We would like to extend this functor from affine schemes to prestacks (hence also to schemes). By definition, we can set for $\mathcal{Y} \in \mathsf{PreStk}$

$$\mathrm{QCoh}(\mathcal{Y}) = \lim_{y: S \to \mathcal{Y}} \mathrm{QCoh}(S),$$

where we take the limit over the category opposite to $(\mathsf{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$. Thus, an object $\mathscr{F} \in \mathrm{QCoh}(\mathcal{Y})$ is an assignment

$$(y : S \to \mathcal{Y}) \rightsquigarrow \mathscr{F}_{S,y} \in \mathrm{QCoh}(S)$$
$$(f : S' \to S) \in (\mathsf{Sch}^{\mathrm{aff}})_{/\mathcal{Y}} \rightsquigarrow (\mathscr{F}_{S', y\circ f} \cong f^*(\mathscr{F}_{S,y})) \in \mathrm{QCoh}(S'),$$

where f^* is the 2-categorical left adjoint of $f_* = \mathrm{QCoh}_{\mathsf{Sch}^{\mathrm{aff}}}(f)$. Moreover, this assignment satisfies a homotopy-coherent system of compatibilities for the composition of morphisms in $(\mathsf{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$, which again elucidates the utility of the infinity-categorical framework.

For $X \in \mathsf{Sch}_{\mathrm{aft}}$, i.e. for X an almost finite type scheme, we can consider the full subcategory $\mathrm{Coh}(X)$ of $\mathrm{QCoh}(X)$ consisting of bounded complexes with coherent cohomologies. This requires explanation, and this is most easily done by studying some examples.

Example 3. Let $S = \operatorname{Spec} A$ be an affine scheme. A coherent module (in classical commutative algebra) is just a finitely generated module such that every surjection from a finite rank free module has finitely generated kernel. By Example 2 above note that we can think of an A-module as an actual complex M_* of modules, and when we ask it to have coherent cohomology we literally mean that each $H^i(M)$ is coherent as a module.

Example 4. Suppose now X is a general scheme, not necessarily affine, so $\mathrm{QCoh}(X) = \lim \mathrm{QCoh}(\operatorname{Spec} A)$ is defined as a limit as above. Thus, for each A, we give an A-module M_x for every morphism $x : \operatorname{Spec} A \to X$ (which can be thought of as the pullback x^*M) and an isomorphism $M_{xf} = f^*(M_x)$ for every morphism $f : \operatorname{Spec} A \to \operatorname{Spec} B$ over X, and higher morphisms between these, and so on. Notice that these isomorphisms generalize the "compatibilities on overlaps" from classical algebraic geometry. In this setting, we can again just ask that the modules M_x have coherent cohomology.

One natural question to ask at this point is how to define the structure sheaf of a scheme X. Recall that in classical algebraic geometry, the structure sheaf can be specified by a Zariski open cover of a classical scheme Y by the requirement $y^*\mathscr{O}_Y = B$ where $y : \operatorname{Spec} B \hookrightarrow Y6$ are the inclusion maps of the open cover. In the notation of Example 4, then, for X a derived scheme, the structure sheaf $\mathscr{O}_X$ should be specified exactly by the condition $x^*\mathscr{O}_X = A$ for every morphism $x : \operatorname{Spec} A \to X$.

6 Introduction to Ind-coherent Sheaves

In this section, we will define the category of ind-coherent sheaves on a scheme X and examine some of its properties, particularly why it is in some sense the more appropriate category to study when we want to think of $\mathscr{O}_X$-modules.

By definition, we let $\mathrm{IndCoh}(X) = \mathrm{Ind}(\mathrm{Coh}(X))$, where Ind is the functor defined in §2. By construction, we have a naturally defined functor

$$\Psi_X : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$$

obtained by the ind-extension fo the inclusion $\mathrm{Coh}(X) \hookrightarrow \mathrm{QCoh}(X)$. If X is a smooth classical scheme, then Ψ_X is an equivalence; indeed, this follows from the fact that if X is classical, $\mathrm{QCoh}(X) \cong \mathrm{Ind}(\mathrm{QCoh}(X)^{\mathrm{perf}})$, where the subscript perf denotes the subcategory of perfect complexes. Further, $\mathrm{QCoh}(X)^{\mathrm{perf}} = \mathrm{Coh}(X)$ tautologically as subcategories of $\mathrm{QCoh}(X)$, so the equivalence follows.

Call X **eventually coconnective** if its structure sheaf is coherent.

Unfortunately, we do not have enough space[3] to get to the construction of $f^!$, a pullback functor $\mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$ associated to a morphism $f : X \to Y$ of schemes. Explicitly, this functor is adjoint[4] to f_*, which is inherent in the data of $\mathrm{IndCoh}(X)$ or $\mathrm{QCoh}(X)$. This pullback functor is the reason for constructing IndCoh. This is because, for example, the right adjoint to the $*$-direct image for QCoh is not necessarily continuous. Continuity is absolutely crucial, as the operation of tensor product of dg categories is only functorial with respect to *continuous* functors. Thus, if we want to "do algebra," we are forced to stay within the realm of continuous functors.

Faced with this lack of space, we can at least heuristically describe the interaction between QCoh and IndCoh as well as their differences. A key property of IndCoh is that the functor

$$\mathrm{IndCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{X} \times \mathcal{Y})$$

is an equivalence if either $\mathcal{X}$ or $\mathcal{Y}$ is a scheme. In fact, it suffices to require that $\mathcal{X}$ or $\mathcal{Y}$ be an "inf-scheme," which is a much more general object than a scheme. The general story is that the pushforwards and pullbacks associated to IndCoh are well-behaved on a much larger class of algebro-geometric objects than the same functors associated to QCoh.

Even though we will not be able to discuss the details of IndCoh and QCoh, we can learn a lot from the following example(s), with which we will conclude the paper.

Example 5. It was shown in [4] that $\mathrm{IndCoh}(X)$ is equivalent to the homotopy category of injective complexes on X. To see that QCoh and IndCoh are different, it is enough to consider the ring of dual numbers $A = k[x]/(x^2)$ and $X = \mathrm{Spec}\, A$ (as an ordinary affine scheme). Note that in this case $\mathrm{QCoh}(X)$ is just the (usually defined) derived category of quasicoherent sheaves on X: Its objects are (not necessarily bounded) complexes of quasicoherent sheaves on X, equivalently A-modules, which are *only defined up to quasi-isomorphism*. This last point is key, as we do not have this requirement for $\mathrm{IndCoh}(X)$. We can resolve A

[3] I'd need at least five, but probably close to seven more pages to get to the construction of the pullback functor. This is because I'd have to introduce the t-structures on both $\mathrm{IndCoh}(X)$ and $\mathrm{QCoh}(X)$, describe the pushforwards for both categories, describe the pullback functors associated to these pushforwards, and then describe how all of these structures interact with each other. I wasn't able to think of a clever way to condense these approximately 20 pages of material into two or so pages, so I'm deciding to leave it out.

[4] whether it is left or right adjoint depends on properties of f

via an injective resolution

$$\cdots \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} A,$$

so that this complex is an element of $\mathrm{IndCoh}(X)$. Obviously, this complex is not the dg algebra k, $\cdots \to 0 \to k$. However, note that its cohomology is 0 everywhere (the image and kernel of each of the maps are both (x), except at the last term, where the cohomology is k); thus, it is quasi-isomorphic to the complex $\cdots \to 0 \to k$, which is the dg algebra associated to k. By the notion of $\mathrm{QCoh}(X)$ above, this implies that it is k in $\mathrm{QCoh}(X)$. This shows that these two categories are really not the same in general.

Consider another

Example 6. Let $X = \mathrm{pt} \times_{\mathbb{A}^1} \mathrm{pt}$. We can compute this as $X = \mathrm{Spec}(k \otimes^L_{k[t]} k)$, which gives $X = \mathrm{Spec}\, k[\varepsilon]$. Indeed, resolve k by a free resolution

$$k[t] \xrightarrow{\cdot t} k[t] \xrightarrow{1 \mapsto 1} k.$$

Cut the last term, and, since this is a free resolution and k is free, we can just tensor with k to get

$$\begin{aligned} k \otimes^L_{k[t]} k =& k[t] \otimes k \xrightarrow{\cdot t \otimes} k[t] \otimes k \\ \cong& k \xrightarrow{1 \mapsto 0} k. \end{aligned}$$

Now, notice that $k \to k \overset{1 \mapsto \varepsilon}{\cong} k\varepsilon \xrightarrow{\cdot \varepsilon} k \cong k[\varepsilon]$ where $\deg \varepsilon = -1$.

Now, let $A = k[\varepsilon]$, so that $A\text{-}\mathsf{mod} = \mathrm{IndCoh}(X)$, while $\mathrm{QCoh}(X)$ is just the subcategory consisting of objects on which the generator ε acts locally nilpotently. This just encodes the fact $\mathrm{QCoh}(X)$ consists of bounded modules.

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