

Week 5 Worksheet Solutions

More Electrostatics

Jacob Erlikhman

Exercise 1. a) The potential at a point \mathbf{r} is defined as

$$V(\mathbf{r}) = - \int_{\mathbb{O}}^{\mathbf{r}} \mathbf{E} \cdot d\boldsymbol{\ell},$$

where \mathbb{O} is some reference point. Explain why this is well-defined (i.e. unambiguous, up to the choice of \mathbb{O}).

b) An infinite plate carries a uniform charge density σ . Using your result from Exercise 2, find the potential everywhere.

Hint: Where would you put your reference point \mathbb{O} ?

a) This is due to Stokes' theorem. Suppose we had two different paths that we'd like to take from \mathbb{O} to \mathbf{r} . We need to check that taking the line integral over both paths will yield the same result. But indeed, the difference between taking one path over the other will give

$$\begin{aligned} V_1(\mathbf{r}) - V_2(\mathbf{r}) &= \oint \mathbf{E} \cdot d\boldsymbol{\ell} \\ &= \int \nabla \times \mathbf{E} \cdot d\mathbf{a} = 0 \end{aligned}$$

by Stokes' theorem and the fact that the curl of \mathbf{E} vanishes.

b) The key here is that we can't place our reference point \mathbb{O} at infinity, since our charge distribution also extends to infinity. Thus, let's place it on the charged plate; we may as well take the plate to be contained in the (x, y) -plane and thus put \mathbb{O} at the origin. We now need to calculate

$$- \int_{\mathbf{0}}^{\mathbf{r}} \mathbf{E} \cdot d\boldsymbol{\ell},$$

where $\mathbf{E} = \frac{\sigma}{2\epsilon_0} \hat{z}$. Thus, whatever path we take to get to the point \mathbf{r} , only the vertical distance, i.e. the z -component, of that path will matter. In particular, we may as well take a straight line along \hat{z} and then some line that is parallel to the (x, y) -plane to get to \mathbf{r} . In that case, we need only $r_z = \mathbf{r} \cdot \hat{z}$ to evaluate our integral. We obtain

$$V(\mathbf{r}) = -\frac{\sigma}{2\epsilon_0} r_z.$$

Exercise 2. Consider a uniformly charged spherical shell of radius R and charge Q .

- Find the electric field everywhere using Gauss' law.
 - Find the potential everywhere by direct integration (without using Gauss' law).
Hint: Consider a single point a distance z from the center of the sphere, and use *cylindrical* symmetry.
 - Set up the integral to find the electric field at a point a distance z from the center of the sphere (without using Gauss' law). Consider separately the cases $z < R$ and $z > R$.
- a) Gauss' law says

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{\epsilon_0},$$

while we know that \mathbf{E} should point only radially outwards. Thus, for $r < R$, there is no enclosed charge; hence, $\mathbf{E}(\mathbf{r}) = \mathbf{0}$. On the other hand, for $r > R$, we have

$$E \cdot 4\pi r^2 = \frac{Q}{\epsilon_0},$$

so

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}.$$

- b) Following the hint, we pick a point on the z -axis at a height z from the center of the sphere. Drawing a triangle for yourself and using the cosine law, you should be able to find that

$$r = \sqrt{R^2 + z^2 - 2Rz \cos \theta'}.$$

Plugging this into the integral formula for V , we obtain

$$\frac{Q}{4\pi R^2} \int \frac{\delta(r' - R) d\tau'}{r},$$

where I am working in Gaussian units (equivalently leaving out the $\frac{1}{4\pi\epsilon_0}$ until the end) and have plugged in our result from Exercise 1a. Now, $d\tau' = r'^2 dr' d\cos \theta' d\varphi'$, so we can immediately do the integrals over r' and φ' . This gives us

$$\frac{Q}{2} \int_{-1}^1 \frac{d\cos \theta'}{\sqrt{R^2 + z^2 - 2Rz \cos \theta'}}.$$

The integral over $\cos \theta'$ is easy to take now. We obtain

$$-\frac{Q}{2Rz} \left(\sqrt{R^2 + z^2 - 2Rz} - \sqrt{R^2 + z^2 + 2Rz} \right) = \frac{Q}{2Rz} \left(\sqrt{(R+z)^2} - \sqrt{(R-z)^2} \right).$$

It is at this point where we have to be careful about whether $R > z$ or $R < z$, as it determines some minus signs after we take the square root (because we always want the *positive* square root). Thus,

$$V(z) = \begin{cases} \frac{Q}{R}, & z < R \\ \frac{Q}{z}, & z > R \end{cases}.$$

Writing this in terms of \mathbf{r} and adding back in the $\frac{1}{4\pi\epsilon_0}$, we have

$$V(\mathbf{r}) = \begin{cases} \frac{Q}{4\pi\epsilon_0 R}, & r < R \\ \frac{Q}{4\pi\epsilon_0 r}, & r > R \end{cases}.$$

Checking that $-\nabla V$ gives the right field confirms that this is the correct result.

c) Using (b), we have

$$\mathbf{E} = \int \frac{\rho(\mathbf{r}') d\tau'}{r^2} \hat{\mathbf{r}}.$$

Since \mathbf{E} is a vector, the field at a point z above the sphere due to a single charge element on the sphere will not point in $\hat{\mathbf{r}}$. However, the part of the field that is not pointing in $\hat{\mathbf{r}}$ will be canceled by the same part of a field due to another charge element on the opposite side of the sphere (i.e. the charge element obtained by rotating by π around $\hat{\mathbf{z}}$). So what should we do? The answer is that the field should only point in $\hat{\mathbf{r}}$, so only along $\hat{\mathbf{z}}$. It follows that we should take the *projection* of the field onto $\hat{\mathbf{z}}$, i.e. for each charge element, we should consider the field due to it multiplied by $\cos\alpha$, where α is the angle between $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$. Draw the same triangle you use to compute r , and split it into two right triangles: one with hypotenuse R and the other with hypotenuse r . Then you can compute that $\cos\alpha = \frac{z - R\cos\theta'}{r}$. This gives the result,

$$\mathbf{E} = \frac{Q}{2} \int \frac{(z - R\cos\theta') d\cos\theta}{r^3} \hat{\mathbf{r}},$$

where I've again ignored the $1/4\pi\epsilon_0$.