Midterm 1 Review

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Exercise 1. The starship Titanic is approaching a space-iceberg at velocity $V = 3/5$. It immediately fires a 100 kg missile at speed $4/5$ relative to the ship, hoping to break up the ice before they crash.

a) Find all components of the missile's momentum four-vector in the spaceship frame, assuming all motion is along the x-direction.

b) Do (a) in the iceberg frame.

c) The iceberg will break if the momentum impacting it is at least 10^{11} kg m/s. Does the iceberg break?

d) What is the velocity of the missile in the iceberg frame?

Exercise 2. Photon Rockets. Consider a rocket whose propellant is photons: The rocket emits photons to accelerate it—we'll call it a "photon rocket." Suppose such a photon rocket departed Earth with total mass M_0.

a) If the rocket will achieve a final speed with $\gamma = 2$, determine what final fraction M/M_0 of the ship's mass is remaining.

b) Show that the speed of the photon rocket can be written

$$v = \frac{1 - (M/M_0)^2}{1 + (M/M_0)^2}.$$

c) Astronomers on Earth watch the photon rocket recede through a telescope—they can observe the photons emitted in the rocket exhaust. What redshift $f_{\text{observed}}/f_{\text{emitted}}$ do they observe in terms of M_0 and the total mass M of the rocket when the photons are emitted?

Exercise 3. Conservation of Energy and Momentum. The **canonical stress-energy tensor** for an arbitrary lagrangian density $\mathscr{L}(q_1, \ldots, q_n)$ is defined by

$$T^{\mu\nu} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu q_k)} \partial^\nu q_k - \eta^{\mu\nu} \mathscr{L},$$

where η is the Minkowski metric.

a) Starting with the sourceless electromagnetic lagrangian density (the extra 4π factor is from the fact that this thing is integrated over a sphere to obtain the "usual" form)

$$\mathscr{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu},$$

show that

$$T^{\mu\nu} = -\frac{1}{4\pi} \eta^{\mu\alpha} F_{\alpha\beta} \partial^{\nu} A^{\beta} - \eta^{\mu\nu} \mathscr{L}.$$

Hint: The generalized coordinates in relativistic electrodynamics should be thought of as packaged in the single gauge field $q_\mu = A_\mu$.

b) Check that

$$\int T^{00} d^3x = \frac{1}{8\pi} \int (E^2 + B^2) d^3x$$

is the energy contained in the EM field, and that

$$\int T^{0i} d^3x = \frac{1}{4\pi} \int (\mathbf{E} \times \mathbf{B})^i d^3x$$

is the i^{th} component of the momentum contained in the EM field.
Hint: First show that

$$\begin{aligned} T^{00} &= \frac{1}{8\pi}(E^2 + B^2) + \frac{1}{4\pi} \nabla \cdot (V\mathbf{E}) \\ T^{0i} &= \frac{1}{4\pi}(\mathbf{E} \times \mathbf{B})^i + \frac{1}{4\pi} \nabla \cdot (A^i \mathbf{E}), \end{aligned}$$

where $V = -A^0$ is the scalar potential.

c) Although this tensor satisfies $\partial_\mu T^{\mu\nu} = 0$, which is the relativistic conservation law corresponding to Poynting's theorem, it fails to be manifestly Lorentz covariant. We can fix this by symmetrizing it (we will also want the tensor we obtain to be traceless). First, show that

$$T^{\mu\nu} = \frac{1}{4\pi}\left(\eta^{\mu\alpha} F_{\alpha\beta} F^{\beta\nu} + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) - \frac{1}{4\pi} \eta^{\mu\alpha} F_{\alpha\beta} \partial^{\beta} A^{\nu}.$$

Rewrite the last term as

$$T_D^{\mu\nu} = \frac{1}{4\pi} \partial_\alpha (F^{\alpha\mu} A^\nu),$$

and define

$$\Theta^{\mu\nu} = T^{\mu\nu} - T_D^{\mu\nu}.$$

d) Check that the trace $\sum_\mu \Theta^{\mu\mu} = 0$, $\Theta^{\mu\nu} = \Theta^{\nu\mu}$, and that Θ defines the "usual" EM tensor with components

$$\begin{aligned}\Theta^{00} &= \frac{1}{8\pi}(E^2 + B^2)\\ \Theta^{0i} &= (\mathbf{E} \times \mathbf{B})^i\\ \Theta^{ij} &= -\frac{1}{4\pi}\left[E^i E^j + B^i B^j - \frac{1}{2}\delta^{ij}(E^2 + B^2)\right].\end{aligned}$$

e) It turns out that when we add sources, Θ is the same as in the sourceless case. Show that

$$\partial_\mu \Theta^{\mu\nu} = -F^{\nu\alpha} j_\alpha,$$

and check that the $\nu = 0$ component of this is exactly Poynting's theorem from the Week 5 Worksheet. The space components give conservation of momentum.

Remark. The usual stress energy tensor for electromagnetism is the Θ which we have just obtained above. There is another way to derive it by noticing that it appears from gravity coupled to electromagnetism (we write the Einstein equation for the gravitational lagrangian plus the EM matter lagrangian).

Exercise 4. Consider the **Poincaré upper half-plane** $\mathbb{H}^2 = \{(x, y) | y > 0\}$ with the metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2).$$

a) Compute the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right)$$

for this metric.

b) Consider semicircles in $\mathbb{H}^2$ centered on the x-axis of radius R. Model the semicircles as curves $c(t) = (t, \gamma(t))$, and show that

$$\gamma''(t) = -\frac{\gamma'(t)}{t - c} - \frac{\gamma'(t)^2}{\gamma(t)}.$$

c) Use Exercise 1 to show that in $\mathbb{H}^2$ the (suitably parametrized) semicircles with center on the x-axis and all straight lines parallel to the y-axis are geodesics. In fact, it can be shown that these are *all* the geodesics, but don't bother to do this.

d) Check that all geodesics have infinite length in either direction. We say that $\mathbb{H}^2$ is **complete**.

e) Compute the area of the following **triangle**—a 3-sided figure whose sides are all geodesics. Let one side be the arc corresponding to an angle α (measured from the positive x-axis) of the unit semicircle centered at $(0, 0)$ (assume $\alpha > \pi/2$). Let the other two sides be vertical lines which extend out from the endpoints of the arc.