Week 5 Worksheet
Symmetries!

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Exercise 1. In this problem, you will construct the 2×2 matrix corresponding to a finite rotation which places the $\hat{z}$ axis along an arbitrary direction $\hat{r}$.

a) A rotation can be specified by the Euler angles (α, β, γ), or by (θ, φ). The Euler angles represent first a rotation about $\hat{z}$ by an angle α, then a rotation *about the new* y-axis by an angle β, and then a rotation about the *new* z-axis again. Convince yourself that this works.

b) Now, suppose given a rotation specified by the Euler angles (α, β, γ). This is given in quantum mechanics by the matrix

$$e^{-i\gamma S_{z'}/\hbar} e^{-i\beta S_u/\hbar} e^{-i\alpha S_z/\hbar},$$

where the u-axis is the new y-axis after rotating about z, and the z'-axis is the new z-axis after rotating about $\hat{z}$ and $\hat{u}$. Show that this is the same matrix as

$$e^{-i\alpha S_z/\hbar} e^{-i\beta S_y/\hbar} e^{-i\gamma S_z/\hbar}.$$

Hint: Denoting a rotation about the axis r by an angle ζ as $R_r(\zeta)$, we have that $S_u = R_z(\alpha) S_y R_z(-\alpha) = e^{-i\alpha S_z/\hbar} S_y e^{i\alpha S_z/\hbar}$. Now, try to write a similar expression for $R_{z'}(\gamma) = e^{-i\gamma S_{z'}/\hbar}$.

c) Use part (b) with $S_i = \frac{\hbar}{2}\sigma_i$ to calculate the rotation matrix corresponding to placing the $\hat{z}$ axis along $\hat{r}$, where $\hat{r}$ is specified by the two angles (θ, φ).
Hints: The idea is to Taylor expand each exponential. Think about a simple expression for ${\sigma_i}^n$, where σ_i is the Pauli matrix you need. Finally, one of the results you should get along the way is

$$e^{-i\beta\sigma_y/2} = \cos(\beta/2)\mathbb{1} - i\sigma_y \sin(\beta/2).$$

d) ***Griffiths 6.32(f).*** Calculate the matrix corresponding to a rotation by π about $\hat{x}$.

e) ***Griffiths 6.32(g).*** Calculate the matrix corresponding to a 2π rotation about $\hat{z}$. Comment on the answer.

Exercise 2. Another symmetry is called **dilation** symmetry. Dilations are given by the transformation $\mathbf{x} \to \mathbf{x}' = e^c \mathbf{x}$, where $c \in \mathbb{R}$. Call its generator D, so that e^{-icD} is the corresponding unitary operator.

Remark. In conformal field theory, the convention is to absorb the factor of $-i$ into D, so that e^{cD} is the dilation operator.

a) Show that the *infinitesimal* transformation

$$e^{i\mathbf{a}\cdot\mathbf{p}}e^{icD}e^{-i\mathbf{a}\cdot\mathbf{p}}e^{-icD}$$

is given by $\mathbb{1} + c\mathbf{a} \cdot [D, \mathbf{p}]$.
Hints: You can reduce to the situation where all the vectors are 1-dimensional (why?). There's a slick way to do this, but the brute force method does work.

b) Calculate $[D, \mathbf{p}]$.
Hints: What coordinate transformation does the above correspond to? In other words, if you write it in the form $\mathbf{x} \to \mathbf{x}'$, what is $\mathbf{x}'$? Think of *another* transformation which combined with part (a) will let you use the form for $\mathbf{x}'$ to determine $[D, \mathbf{p}]$.