Week 12 Worksheet Solutions
Scattering

Jacob Erlikhman

April 17, 2024

Exercise 1. Warm up.

a) How do the phase shifts δ_ℓ appear in partial wave scattering, and what is their physical significance?

b) What is the fundamental assumption on the form of the wavefunctions in the Born approximation? *Hint*: If the scattering potential is weak, what approximation can we make?

c) Starting from the Lippmann-Schwinger equation,

$$\psi(\mathbf{x}) = \varphi_{\mathbf{k}}(\mathbf{x}) + \int \mathrm{d}^3x'\; G_0(\mathbf{x}, \mathbf{x}', E) V(\mathbf{x}')\psi(\mathbf{x}'),$$

where G_0 is the free particle, time-independent Green's function and

$$\begin{aligned}\varphi_{\mathbf{k}}(\mathbf{x}) &= \langle \mathbf{x} | \mathbf{k} \rangle \\ &= \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}},\end{aligned}$$

explain how you would derive the Born approximation.
Hints: The Green's function is (note that $E = \hbar^2 k^2/2m$)

$$G_0(\mathbf{x}, \mathbf{x}', E) = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}.$$

This is done in Griffiths. See also the final review.

Exercise 2. Spin-spin Interaction. Consider two spin-1/2 particles that interact in a potential of the form

$$V(r) = V_0(r) + V_s(r)\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}.$$

Suppose that both the orbital and spin interactions are short range in the interparticle separation r (i.e. vanish faster than $1/r$ as $r \to \infty$).

a) The first Born approximation for the scattering amplitude is given by

$$f(\mathbf{k},\mathbf{k}') = -\frac{4\pi^2 m}{\hbar^2}\left\langle \mathbf{k}'|V|\mathbf{k}\right\rangle.$$

Use a Fourier transform to express the scattering amplitude in terms of

$$\int e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_0} V_0(r_0)\,\mathrm{d}^3 r_0,$$

and a similar expression for $V_s(r_0)$.
Hint: Don't forget to account for the incoming and outgoing spins!

b) You computed on midterm 1 that the eigenvectors of $\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}$ are the singlet and triplet states, with eigenvalues -3 and 1, respectively. If the incoming particles have parallel spins, is a spin flip possible? Why or why not? Explain why the scattering is elastic or inelastic in this case, and then calculate it.

c) Calculate the scattering amplitude for incident particles with opposite spins. You should be able to split it into two channels: an elastic one and an inelastic one.

a) The Fourier transform of

$$\begin{aligned}
\left\langle \mathbf{k}'|V|\mathbf{k}\right\rangle &= \int \mathrm{d}^3 r_0 \left\langle \mathbf{k}'|V|\mathbf{r}_0\right\rangle\left\langle \mathbf{r}_0|\mathbf{k}\right\rangle \\
&= \int \mathrm{d}^3 r_0\, V(r_0)\left\langle \mathbf{k}'|\mathbf{r}_0\right\rangle\left\langle \mathbf{r}_0|\mathbf{k}\right\rangle \\
&= \left(\frac{1}{(2\pi)^{3/2}}\right)^2 \int e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_0} V(r_0)\,\mathrm{d}^3 r_0.
\end{aligned}$$

Thus, the scattering amplitude is

$$f(\mathbf{k},\mathbf{k}') = -\frac{m}{2\pi\hbar^2}\left(\int e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_0} V_0(r_0)\,\mathrm{d}^3 r_0 \left\langle f|i\right\rangle + \int e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_0} V_s(r_0)\,\mathrm{d}^3 r_0 \left\langle f|\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}|i\right\rangle\right),$$

where $|i\rangle$ ($|f\rangle$) denote the spin states of the incoming (resp. outgoing) particles. Note that the incoming spin state space is *four*-dimensional, as we should account for the spins of *both* particles (each of which has a two-dimensional state space). Indeed, the spins of both particles can change from their initial configurations to some different final configurations.

b) Note that parallel spins are eigenstates of $\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}$; thus, it is impossible for a scattered particle to change spin, so the scattering will be purely elastic. Since these have eigenvalue 1, we get

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\left(\int e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_0} V_0(r_0)\,\mathrm{d}^3 r_0 + \int e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_0} V_s(r_0)\,\mathrm{d}^3 r_0\right).$$

c) If the spins are not parallel, then the scattered wave can have either the same spins or opposite spins (spin-flip). This is because the singlet and mixed triplet states are superpositions of the antiparallel configurations. It follows that we have an elastic channel (no spin-flip), as well as an inelastic one (spin-flip). We compute

$$
\begin{aligned}
\langle \uparrow\downarrow | \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)} | \uparrow\downarrow \rangle &= -1 \\
\langle \uparrow\downarrow | \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)} | \downarrow\uparrow \rangle &= 2,
\end{aligned}
$$

This is most easily seen by writing e.g. $|\uparrow\downarrow\rangle$ as a linear combination of triplet and singlet states. We then end up with two amplitudes.

$$
\begin{aligned}
f_{\uparrow\downarrow,\uparrow\downarrow}(\theta) &= -\frac{m}{2\pi\hbar^2}\left(\int e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_0} V_0(r_0)\, \mathrm{d}^3 r_0 - \int e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_0} V_s(r_0)\, \mathrm{d}^3 r_0\right) \\
f_{\uparrow\downarrow,\downarrow\uparrow}(\theta) &= -2\frac{m}{2\pi\hbar^2}\int e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_0} V_s(r_0)\, \mathrm{d}^3 r_0.
\end{aligned}
$$

Note that the *kinetic* energy of the flipped spin states is still the same as the original states; only potential energy can change via the potentials V_0, V_s.