

# Week 2 Worksheet

## 137A Review; QM in 3D (and some spin)

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January 24, 2024

**Exercise 0.** This is done in Griffiths Chapters 1 and 3.

**Exercise 1.** A Hilbert space is mathematically defined as a *complete* vector space with an inner product. A vector space with an inner product is **complete** if it includes not only all finite sums of vectors in a basis, but also all limits of convergent sequences, i.e. given a sequence  $(v_n)$  of vectors in the Hilbert space,  $v$  is the **limit** of the sequence if  $\lim_{n \rightarrow \infty} \|v_n - v\| = 0$ , where  $\|v\| = \sqrt{v \cdot v}$ .

a) Consider a Hilbert space  $\mathcal{H}$  that consists of all functions  $\psi(x)$  such that

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx < \infty.$$

Show that there are functions in  $\mathcal{H}$  for which  $\hat{x}\psi(x) = x\psi(x)$  is not in  $\mathcal{H}$ .

*Proof.* Let  $\psi(x) = \frac{1}{1 + |x|}$ .

□

b) Consider the function space  $\Omega$  in  $\mathcal{H}$  which consists of all  $\varphi(x)$  that satisfy the set of conditions

$$\int_{-\infty}^{\infty} |\varphi(x)|^2 (1 + |x|)^n dx < \infty,$$

for any  $n \in \{0, 1, 2, \dots\}$ . Show that for any  $\varphi(x)$  in  $\Omega$ ,  $\hat{x}\varphi(x)$  is also in  $\Omega$ .  $\Omega$  is called the **nuclear** space.

*Proof.* By the binomial theorem,

$$\int_{-\infty}^{\infty} |\varphi(x)|^2 (1 + |x|)^n dx = \sum_{i=0}^n \int_{-\infty}^{\infty} |\varphi(x)|^2 \binom{n}{i} |x|^i dx.$$

In particular, for each  $n$ ,  $|\varphi(x)|^2 |x|^n$  has finite integral.

□

c) The **extended** space  $\Omega^\times$  consists of those functions  $\chi(x)$  which satisfy

$$(\chi, \varphi) = \int_{-\infty}^{\infty} \chi^*(x) \varphi(x) dx < \infty,$$

for any  $\varphi$  in  $\Omega$ , where  $(\cdot, \cdot)$  is the inner product on  $\mathcal{H}$ . Which of the following functions belong to  $\Omega$ , to  $\mathcal{H}$ , and/or to  $\Omega^\times$ ?

**Remark.** The collection  $(\Omega, \mathcal{H}, \Omega^\times)$  is called “rigged Hilbert space,” and this is a rigorous way to include all the formalism (e.g. eigenvectors of position are delta functions, and hence can’t belong to an  $L^2$  space) into the Hilbert space formulation of quantum mechanics. Note that  $\Omega \subset \mathcal{H} \subset \Omega^\times$ . Also, note that in order to sit in  $\Omega$ , functions must vanish faster than any power of  $x$  as  $|x| \rightarrow \infty$ . Thus, as long as functions don’t diverge at  $\infty$  more strongly than any power of  $|x|$ , they are in  $\Omega^\times$ . For more details, see Ballentine *Quantum Mechanics*, Chapter 1.

- i)  $\sin(x)$
- ii)  $\sin(x)/x$
- iii)  $x^2 \cos(x)$
- iv)  $e^{-ax}, a > 0$ .
- v)  $\frac{\ln(1 + |x|)}{1 + |x|}$
- vi)  $e^{-x^2}$
- vii)  $x^4 e^{-|x|}$

i) Clearly,  $\sin(x) \notin \mathcal{H}$ , so it’s not in  $\Omega$  either. But it is in  $\Omega^\times$ , since it’s divergence at  $\infty$  is not worse than e.g.  $|x|$ .

ii) Try first to compute

$$\int_{-\infty}^{\infty} \frac{|\sin(x)|^2}{x^2} dx = 2 \int_0^{\infty} \frac{|\sin(x)|^2}{x^2} dx = 2 \int_0^{\epsilon} \frac{|\sin(x)|^2}{x^2} dx + 2 \int_{\epsilon}^{\infty} \frac{|\sin(x)|^2}{x^2} dx.$$

Now, the first term has finite integral, and the second term is less than  $\int_{\epsilon}^{\infty} \frac{2}{x^2} dx$ , which is finite. Thus, it’s in  $\mathcal{H}$ , and it follows that the function is in  $\Omega^\times$ . It’s clearly not in  $\Omega$ , since  $|\sin(x)|^2$  does not have a finite integral.

iii) Write

$$\int_{-\infty}^{\infty} x^4 \cos^2(x) dx = \int_{-\infty}^{\infty} x^4 (1 + \cos(2x)) dx.$$

The second term has either (positive) infinite integral or is finite, but the first term is definitely  $+\infty$ , so this function is not in  $\mathcal{H}$ , so not in  $\Omega$  either. It is in  $\Omega^\times$ , since its divergence is not worse than a power of  $x$ .

iv) Clearly, this isn’t in  $\mathcal{H}$ . It’s also not in  $\Omega^\times$ , since it diverges faster than any power of  $x$  as  $x \rightarrow -\infty$ .

- v) Clearly, this isn't in  $\Omega$ . It is in  $\mathcal{H}$ , though. Explicitly, we could integrate by parts a few times to reduce to

$$\int_{-\infty}^{\infty} \left( \frac{\ln(1+|x|)}{1+|x|} \right)^2 dx = 4 \int_0^{\infty} \frac{1}{(1+x)^2} dx = 4.$$

- vi) This has finite integral, so it's in  $\mathcal{H}$  and  $\Omega^\times$ . It's also in  $\Omega$ , since any power of  $|x|$  times this function is also finite (remember your gaussian integrals!).
- vii) This is in  $\Omega$ , since we can always integrate by parts to get back to an integral over  $e^{-|x|}$ .

**Exercise 2.** Solve the eigenvalue problem for the 3-D isotropic, harmonic oscillator, whose hamiltonian is  $H = p^2/2m + m\omega^2 x^2/2$ , where  $p^2 = \mathbf{p} \cdot \mathbf{p}$ ,  $x^2 = \mathbf{x} \cdot \mathbf{x}$  is the 3-D dot product.

Remembering (or rederiving) the solution to the 1-D isotropic harmonic oscillator, the energy eigenvalues are  $E_n = (n + 1/2)\hbar\omega$  for  $n \in \{0, 1, 2, \dots\}$ . But the 3-D one is the same except we have 3 directions for  $n$  now,  $n_x, n_y$ , and  $n_z$ . Thus,  $n = \sqrt{n_x^2 + n_y^2 + n_z^2}$ , for arbitrary  $n_x, n_y, n_z \in \{0, 1, 2, \dots\}$ .

**Exercise 3.** A particle of mass  $m$  is placed in a finite spherical well

$$V(r) = \begin{cases} -V_0, & r \leq a \\ 0, & r \geq a \end{cases}.$$

Find the equation that quantizes the energy (you don't need to solve it), by solving the radial Schrödinger equation with  $\ell = 0$ . Explain how you could solve this equation and obtain the energies. Show that there is no bound state if  $V_0 a^2 < \pi^2 \hbar^2 / 8m$ . Hint: <sup>1</sup>

Write the solution as

$$u(r) = \begin{cases} A \cos(kr) + B \sin(kr), & r \leq a \\ C e^{\kappa r} + D e^{-\kappa r}, & r \geq a \end{cases},$$

where

$$k = \sqrt{\frac{2m}{\hbar^2}(E + V_0)}$$

$$\kappa = \sqrt{-\frac{2m}{\hbar^2}E}.$$

Now, the wavefunction is  $u(r)/r$ , so as  $r \rightarrow 0$  the cosine solution blows up, which means  $A = 0$ . On the other hand, as  $r \rightarrow \infty$ , we see that  $C = 0$ , since  $e^{\kappa r}/r$  blows up there. Thus, we are left with

$$u(r) = \begin{cases} A \sin(kr), & r \leq a \\ B e^{-\kappa r}, & r \geq a \end{cases},$$

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<sup>1</sup>Recall that the radial Schrödinger equation is identical to the time-independent, 1-dimensional Schrödinger equation with the wavefunction replaced by  $u(r) = rR(r)$  (where  $\psi(r, \theta, \varphi) = R(r)\Theta(\theta)\Phi(\varphi)$ ) and potential  $V_{\text{eff}}(r) = V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}$ .

where I have renamed  $A$  and  $B$ . It follows that

$$u'(r) = \begin{cases} kA \cos(kr), & r \leq a \\ -\kappa B e^{-\kappa r}, & r \geq a \end{cases}.$$

Now, use continuity of the first derivative and the function at the boundary  $r = a$  to obtain two equations

$$\begin{cases} A \sin(ka) &= B e^{-\kappa a} \\ kA \cos(ka) &= -\kappa B e^{-\kappa a} \end{cases}.$$

Dividing the first equation by the second, we get the transcendental equation

$$\tan(ka) = -\frac{k}{\sqrt{V_0 - k^2}},$$

which can be rewritten in the form

$$\tan(z) = -\frac{1}{\sqrt{\frac{\sqrt{2mV_0}a}{\hbar^2 z^2} - 1}},$$

where  $z = ka$ . To solve it, we should graph the LHS and the RHS on the same graph and look for points of intersection. These will be the allowed  $z$  values, hence the allowed  $k$  values, hence the allowed energies. We can use the same method to see why there's no bound state if  $V_0 a^2 < \pi^2 \hbar^2 / 8m$ . Draw the graph of  $\tan(z)$  superimposed with the graph of  $-1/\sqrt{2mV_0 a^2 / \hbar^2 z^2 - 1}$ . You will see that there can be no solution if  $\sqrt{2mV_0 a^2 / \hbar^2} < \pi/2$ , which implies that there can be no solution for  $2mV_0 a^2 / \hbar^2 < \pi^2/4$ , hence for  $V_0 a^2 < \pi^2 \hbar^2 / 8m$ .

#### Exercise 4. Spin Representations.

- Find the eigenvalues and eigenvectors of  $S_z$ .
  - Do the same for  $S_y$ , and write them in terms of  $|\uparrow\rangle$  and  $|\downarrow\rangle$ , the eigenvectors of  $S_z$ .
  - For a system of two spin 1/2 particles, starting with the “highest weight” state  $|\uparrow\uparrow\rangle$ , find all the states in the triplet.  
*Hint:* Apply the lowering operator.
  - For a system of two spin 1/2 particles, are there any other states than the ones you found in (c)? If so, what are they? What is the action of  $S_-$ ,  $S_+$  on them?
  - Describe how you would approach finding the Clebsch-Gordan coefficients for arbitrary spin systems.
- a) This is done in Griffiths.

b) We have

$$S_y = \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$\frac{\hbar^2}{4} (\lambda^2 + 1),$$

so the eigenvalues are

$$\lambda_{\pm} = \pm \frac{\hbar}{2},$$

as expected. The associated eigenvectors are

$$\begin{aligned} \lambda_+ &\leftrightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \lambda_- &\leftrightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \end{aligned}$$

We can write these as

$$\lambda_{\pm} \leftrightarrow \frac{1}{\sqrt{2}} (|\uparrow\rangle \pm |\downarrow\rangle).$$

c) This is done in Griffiths.

d) Also done in Griffiths.