Final Review Problems Solutions

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May 9, 2025

The Robertson-Walker metric in spherical coordinates is given by

$$ds^2 = -d\tau^2 + a^2(\tau)\begin{cases} d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\varphi^2), & K = 1 \\ d\psi^2 + \psi^2(d\theta^2 + \sin^2\theta d\varphi^2), & K = 0 \\ d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\varphi^2), & K = -1 \end{cases},$$

where $a(\tau) > 0$ is a positive function of proper time τ and K is the sectional curvature.

Exercise 1. In this problem, assume there is no radiation (or other sources of) pressure in the universe. Given a point p in spacetime, the tangent space to p is 4-dimensional. It splits into a 1-dimensional subspace spanned by u^μ, the unit tangent vector to the worldline of an isotropic observer, and a 3-dimensional subspace spanned by an orthogonal set of unit vectors $\{s_i{}^\mu\}$ tangent to a homogeneous hypersurface passing through the point. Note that $s_i{}^\mu u_\mu = 0$, by definition, and we can write any tensor $S_{\mu\nu}$ in the $\{u, s_i\}$ basis as

$$S_{\mu\nu} = S_{\tau\tau}u_\mu u_\nu + \sum_{i,j} S_{s_i s_j} s_{i,\mu} s_{j,\nu} + \sum_i S_{\tau s_i} u_\mu s_{i,\nu} + \sum_i S_{s_i \tau} s_{i,\mu} u_\nu.$$

a) Let ρ be the average mass density of matter in the universe. Use homogeneity and isotropy to argue that for dust (i.e. matter which exerts no pressure)

$$T_{\mu\nu} = \rho u_\mu u_\nu.$$

b) Argue that the 10 independent equations which arise from Einstein's equation can be reduced to two by homogeneity and isotropy:

$$\begin{aligned} G_{\tau\tau} &= 8\pi\rho \\ G_{**} &= 0, \end{aligned}$$

where (s^μ is any of the $s_i{}^\mu$)

$$\begin{aligned} G_{\tau\tau} &= G_{\mu\nu}u^\mu u^\nu \\ G_{**} &= G_{\mu\nu}s^\mu s^\nu. \end{aligned}$$

Hints: First, argue that the time-space components are 0 and that the space-space components must be the same. Then, project $G_{\mu\nu}$ onto a homogeneous hypersurface and raise an index with the spatial metric. Use homogeneity to argue that the resulting tensor $G^i{}_j$, viewed as a linear map which takes tangent vectors to tangent vectors, must necessarily be a multiple of the identity (by using the spectral theorem for symmetric matrices).

c) Compute the Ricci tensor and Ricci scalar in i) a closed universe (i.e. constant curvature $K = 1$) and ii) in an open universe (constant curvature $K = -1$). The Christoffel symbols and Ricci tensor are

$$\begin{aligned} \Gamma^k_{ij} =& \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) \\ R_{ij} =& \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k{}_{kj} + \Gamma^k_{ij} \Gamma^l{}_{kl} - \Gamma^k_{lj} \Gamma^l_{ki}, \end{aligned}$$

where e.g. $\Gamma^k{}_{jk} = \sum_k \Gamma^k_{jk}$ is the contraction.
Hint: Argue that you only need to calculate R_{00} and R_{11}, and use this to limit the number of Christoffel symbols you need to calculate to 6 (which take only 3 different values!).

d) Using

$$G_{ij} = R_{ij} - \frac{1}{2} \eta_{ij} R,$$

show that the differential equations from (b) become

$$\begin{aligned} 3\frac{\dot{a}^2}{a^2} =& 8\pi\rho - \frac{3K}{a^2} \\ 3\frac{\ddot{a}}{a} =& -4\pi\rho. \end{aligned}$$

It turns out that these also hold for the case of flat spacetime ($K = 0$).
Hint: Note that

$$\begin{aligned} R =& -R_{\tau\tau} + 3R_{**} \\ =& -R_{00} + 3a^{-2} R_{11}. \end{aligned}$$

a) By isotropy, the worldlines of the galaxies must coincide with isotropic observers. Since the galaxies are the matter and u_i are the tangent vectors to worldlines of the isotropic observers, we find that the only component of T that is nonzero in the $\{u, s_i\}$ basis is the (u, u)-component. Indeed, the time-space components being nonzero implies that $T^{ij} u_j$ has a spatial component, which is impossible by isotropy. Since the time-space components of T correspond to energy flux through surfaces and isotropy implies that all spatial components of $T^{ij} u_j$ must be the same, then this implies the same (say positive) energy flux is flowing through all surfaces and in all directions, which violates conservation of energy. Similarly, the space-space components describe momentum density/tension, but we assume that this is zero because we are assuming we have dust (and hence no pressure).

b) Certainly, there can be no preferred spatial direction; otherwise, isotropy would be violated. Thus, all the space-space components of the Einstein equations must be the same. Similarly, if $G_{\mu\nu}u^\nu$ had a spatial component, then isotropy is violated. It follows that the time-space components must vanish. Finally, we proceed as in the hint. $G^i{}_j$ is a *symmetric* matrix; hence, there is a basis of simultaneous eigenvectors of the tangent space to the homogeneous hypersurface. Moreover, as a matrix, $G^i{}_j$ can be written in this basis as the concatenation of the eigenvectors. Now, if the matrix had distinct eigenvalues, then there would be a preferred choice of spatial direction, which violates isotropy. It follows that the eigenvalues are identical; thus, $G^i{}_j$ is a multiple of $\delta^i{}_j$. It follows that the off-diagonal space-space components of $G_{\mu\nu}$ vanish. The Einstein equations thus become

$$\begin{aligned} G_{\mu\nu}u^\mu u^\nu =& 8\pi\rho \\ G_{\mu\nu}s^\mu s^\nu =& 0, \end{aligned}$$

where s is (any) unit vector tangent to a homogeneous hypersurface (since s and u are orthogonal).

c) We need only calculate R_{00} and R_{11}, since we can choose $u = (1, 0, 0, 0)$ and $s = (0, 1/a, 0, 0)$, where the factor of $1/a$ comes from the fact that $s^2 = 1$. The relevant Christoffel symbols are

$$\begin{aligned} \Gamma^0_{11} =& \dot{a}a \\ \Gamma^1_{01} =& \frac{\dot{a}}{a} \\ \Gamma^2_{12} =& \cot\psi \\ \Gamma^2_{02} =& \frac{\dot{a}}{a} \\ \Gamma^3_{03} =& \frac{\dot{a}}{a} \\ \Gamma^3_{13} =& \cot\psi \end{aligned}$$

for $K = 1$. The only difference for $K = -1$ is that trig functions of ψ are now hyperbolic functions; everything else is identical (even the signs). We now calculate

$$\begin{aligned} R_{00} = R_{\tau\tau} =& -3\frac{\ddot{a}}{a} \\ R_{11} =& \ddot{a}a + 2\dot{a}^2 + 2, \end{aligned}$$

where in the calculation for R_{11} you should use the trig identity

$$\csc^2\psi = 1 + \cot^2\psi.$$

On the other hand, when $K = -1$, the derivative of $\coth\psi$ is still $-\operatorname{csch}^2\psi$ as in the trigonometric case, so that we pick up an extra minus sign when we use the hyperbolic trig identity

$$1 = \coth^2\psi - \operatorname{csch}^2\psi.$$

Thus,

$$R_{11} = \ddot{a}a + 2\dot{a}^2 + 2K.$$

Now,

$$R_{**} = a^{-2} R_{11}$$

by definition (same reason there's a factor of $1/a$ in the choice of s). Hence,

$$R_{**} = \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2K}{a^2}.$$

We compute

$$\begin{aligned} R =& - R_{\tau\tau} + 3R_{**} \\ =& 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) + \frac{6K}{a^2}. \end{aligned}$$

d) Thus,

$$\begin{aligned} G_{\tau\tau} =& R_{\tau\tau} + \frac{1}{2}R \\ =& \frac{3\dot{a}^2}{a^2} + \frac{3K}{a^2}, \end{aligned}$$

and

$$\begin{aligned} G_{**} =& R_{**} - \frac{1}{2}R \\ =& - 2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{K}{a^2}. \end{aligned}$$

Finally, we can write the differential equations for G as

$$\begin{aligned} \frac{3\dot{a}^2}{a^2} &= 8\pi\rho - \frac{3K}{a^2} \\ -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{K}{a^2} &= 0. \end{aligned}$$

Using the first equation to replace the factor of $\dot{a}^2/a^2 + K/a^2$ in the second, we can rewrite the second one as

$$2\frac{\ddot{a}}{a} = -\frac{8\pi\rho}{3}$$

or, alternatively, as

$$\frac{3\ddot{a}}{a} = -4\pi\rho.$$

Exercise 2. Hubble's Law. By analyzing the differential equations for a from Exercise 1, show that $\rho > 0$ implies $\ddot{a} < 0$. Thus, derive Hubble's law,

$$\frac{dR}{d\tau} = HR,$$

where R is the distance between two isotropic observers and $H(\tau) = \dot{a}/a$ is Hubble's constant.

Since $\rho > 0$ and $a > 0$, we see that the first equation of Exercise 1(d) shows $\ddot{a} < 0$. Let the coordinate distance between the two observers be R_0, so that

$$R(\tau) = a(\tau)R_0.$$

Then

$$\frac{dR}{d\tau} = \frac{da}{d\tau}R_0,$$

or

$$\frac{dR}{d\tau} = \frac{da}{d\tau}\frac{R(\tau)}{a}.$$

Exercise 3. Critical Energy Density. Define the critical energy density

$$\rho_c = \frac{3H^2}{8\pi}$$

and the ratio of the total energy density to the critical energy density

$$\Omega = \rho/\rho_c.$$

Determine the relation between the value of $K \in \{-1, 0, 1\}$ and the sign of Ω (< 1, > 1, or 1).

The Friedmann equation reads

$$H^2 = \frac{8\pi\rho}{3} - \frac{K}{a^2},$$

so

$$\begin{aligned} 1 =& 8\pi\frac{\rho}{3H^2} - \frac{K}{a^2H^2} \\ 1 =& \Omega - \frac{K}{a^2H^2}. \end{aligned}$$

Thus, we have

$$\Omega - 1 = \frac{K}{a^2H^2}.$$

Since $a^2H^2 > 0$, we see that the sign of K determines the sign of $\Omega - 1$.

Exercise 4. Cosmological measurements today infer $K = 0$. Determine $H(\tau)$ if the energy density ρ is dominated by

a) radiation (assume that $T \propto a^{-1}$),

b) matter,

c) vacuum,

d) kination ($\rho \propto a^{-6}$; energy dominated by kinetic terms of scalar field),

e) or an ultrarelativistic fluid with the equation of state $W = P/\rho$ (assume that the universe expands adiabatically).

a) Since $K = 0$, we have

$$H^2 = \frac{8\pi\rho}{3}.$$

Recall from thermodynamics the energy density of a photon gas $\rho \propto T^4$. Since $T \propto a^{-1}$, $\rho \propto a^{-4}$; hence,

$$H = Ca^{-2},$$

where C is some constant. Since

$$H = \frac{\dot{a}}{a}$$

by definition, we have

$$\int a da = \int C dt,$$

so

$$a = \sqrt{2Ct}.$$

We plug in for H to find

$$\begin{aligned} H =& Ca^{-2} \\ =& \frac{1}{2t}, \end{aligned}$$

which is independent of C!

b) For matter, $\rho = \frac{M}{V} \propto \frac{1}{a^3}$. Hence,

$$H = Ca^{-3/2}.$$

We again solve

$$Ca^{-3/2} = \frac{\dot{a}}{a}$$

for a:

$$\begin{aligned} \int \sqrt{a} da =& \int C dt \\ \frac{2}{3}a^{3/2} =& Ct \\ a =& \left(\frac{3}{2}Ct\right)^{2/3}. \end{aligned}$$

Thus,

$$H = \frac{2}{3t}.$$

c) For vacuum, we have $\rho = C$ is a constant. Thus,

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi C}{3}},$$

so

$$\begin{aligned}\int \frac{da}{a} &= \int \sqrt{\frac{8\pi C}{3}} dt \\ \ln(a/a_0) &= \sqrt{\frac{8\pi C}{3}} t \\ a &= a_0 e^{\sqrt{\frac{8\pi}{3}C}t}.\end{aligned}$$

d) Since $\rho \propto a^{-6}$, $H = Ca^{-3}$. So

$$\begin{aligned}Ca^{-3} &= \frac{\dot{a}}{a} \\ Ca^{-3} &= \frac{1}{3t}.\end{aligned}$$

Thus,

$$H = \frac{1}{3t}.$$

e) The second law of thermodynamics gives

$$d(\rho V) = TdS - PdV.$$

We can rewrite this as

$$TdS = d[V(\rho + p)] - VdP.$$

For constant entropy $dS = 0$, we find

$$VdP = d[\rho V(1 + W)].$$

Now, $V \propto a^3$, so

$$a^3 dP = d[\rho a^3(1 + W)].$$

W is constant, so

$$a^3 dP = a^3(1 + W)d\rho + 3a^2\rho(1 + W)da.$$

We use the equation of state again to find that the $a^3 W d\rho$ on the RHS cancels the LHS. We're thus left with

$$-a d\rho = 3\rho(1 + W)da.$$

Solving this for ρ, we find

$$\frac{\rho}{\rho_0} = \left(\frac{a}{a_0}\right)^{-3(1+W)}.$$

We can check that this is the right equation: For matter, $W = 0$, so $\rho \propto a^{-3}$. For radiation, $W = 1/3$, so $\rho \propto a^{-4}$. For vacuum, $W = -1$, so $\rho \propto a^0$ which is constant. Since $H \propto \sqrt{\rho}$,

$$\frac{\dot{a}}{a} = Ca^{-\frac{3}{2}(1+W)}.$$

Solving this gives

$$H = \frac{2}{3t(1+W)}.$$

Exercise 5. Cosmic Strings. A cosmic string loop of radius R oscillates with period $T = R_0$, so that

$$\rho(\tau, \mathbf{x}) = \mu\delta(r - R_0\cos(\omega t))\delta(z),$$

where $\mu = m/\ell$ is the string tension, we use cylindrical coordinates $\mathbf{x} = (r, \theta, z)$, and $\omega = 2\pi/R_0$.

a) Calculate the quadrupole moment of the string.

b) Calculate the transverse traceless quadrupole moment.

c) Show that the power emitted by the string loop in gravitational waves is

$$P = \Gamma G\mu^2,$$

where Γ is a dimensionless constant. (You should calculate Γ explicitly; in reality, relativistic effects modify the value of Γ to be around 50-100).

d) What is the lifetime of the string loop?

e) A cosmic string loop forms roughly every Hubble doubling and then redshifts like matter before decaying to gravitational waves. This gives a number density of loops in the horizon:

$$\frac{dn}{d\ell} \approx \frac{1}{\tau^{3/2}(\ell + \Gamma\mu\tau)^{5/2}}.$$

Estimate Ω_{GW} from all the loops in the horizon.

a) By definition, the quadrupole moment is

$$I^{ij}(t, \mathbf{x}) = \int \rho(t, \mathbf{x}')x'^i x'^j d^3x'.$$

Since in cylindrical coordinates

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, \end{aligned}$$

we see that the I^{xy} and I^{yx} components are 0. Similarly, the symmetry of ρ around the z-axis implies that $I^{xx} = I^{yy}$. On the other hand, $I^{iz} = 0$ because of the delta function. So we only need to calculate one matrix coefficient, say I^{xx}.

$$\begin{aligned} I^{xx} =& \int \mu\delta(r - R_0\cos(\omega t))\delta(z)r^2\cos^2(\theta)rdrd\theta dz \\ =& \pi\mu{R_0}^3\cos^3(\omega t), \end{aligned}$$

and

$$I = I^{xx}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b) The transverse traceless quadrupole moment is $Q = I - \frac{1}{3}\operatorname{tr}(I)\mathbb{1}$. The trace of I is

$$\operatorname{tr}(I) = 2I^{xx},$$

so

$$Q = \frac{1}{3}I^{xx}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

c) The power emitted due to the quadrupole is

$$P = \frac{G}{5}\left\langle \sum_{i,j}(\dddot{Q}^{ij})^2 \right\rangle.$$

We calculate

$$\sum_{i,j}(\dddot{Q}^{ij})^2 = \frac{2}{3}(\dddot{I}^{xx})^2.$$

Now, note that

$$\cos^3(\omega t) = \frac{3\omega^3\cos(\omega t) + \cos(3\omega t)}{4},$$

which allows us to easily calculate three time derivatives:

$$\frac{d^3}{dt^3}\cos^3(\omega t) = \frac{3\omega^3\sin(\omega t) + (3\omega)^3\sin(3\omega t)}{4}.$$

Thus,

$$(\dddot{I}^{xx})^2 = \pi^2\mu^2{R_0}^6\omega^6\frac{9}{16}[\sin(\omega t) + 9\sin(3\omega t)]^2.$$

Taking the time average over a period, we find that this is

$$369\pi^2\mu^2 \cdot \pi^6,$$

where we use the fact that $\omega = 2\pi/R_0$. Plugging this in to the formula for P gives

$$P = G\mu^2\left(\frac{369}{5}\pi^8\right).$$

In reality, the quadrupole moment is not valid when the string is relativistic, which it is for parts of its period. Simulations show that $\Gamma \approx$ 50-100, whereas our value of Γ is too large.

d) Since

$$P = -\frac{dE}{dt},$$

we can calculate T as

$$T = -\int_{E_0}^{0} \frac{dt}{dE} dE,$$

where E_0 is the maximum energy of the string. The energy of the string is

$$\int \rho r dr d\theta dz = 2\pi\mu R_0 \cos(\omega t),$$

and this has maximum value

$$E_0 = 2\pi\mu R_0.$$

Thus, we can evaluate

$$\begin{aligned} T &= \frac{E_0}{\Gamma G\mu^2} \\ &= \frac{2\pi R_0}{\Gamma G\mu}. \end{aligned}$$

e) We need to determine ρ_c and ρ_{GW}. Since in the radiation dominated era $H = \frac{1}{2t}$, we can calculate

$$\begin{aligned} \rho_c &= \frac{3H^2}{8\pi G} \\ &= \frac{3}{32\pi G t^2}. \end{aligned}$$

On the other hand,

$$\rho_{\mathrm{GW}} = Ptn = Pt\int_0^{\ell_{\max}} \frac{dn}{d\ell} d\ell.$$

Now, $Pt = \Gamma G\mu^2 t$, and we can calculate

$$\begin{aligned}\ell_{\max} =&2\pi R_{\max}\\ =&2\pi R_0.\end{aligned}$$

Thus, we can evaluate the integral to obtain

$$\begin{aligned}\rho_{\mathrm{GW}} =&\frac{2}{3}\Gamma G\mu^2 t^{-1/2}\left[(\Gamma G\mu t)^{-3/2} - (R_0 + \Gamma G\mu t)^{-3/2}\right]\\ =&\frac{2}{3t^2}\sqrt{\frac{\mu}{\Gamma G}}\left(1 - \left(\frac{R_0}{\Gamma G\mu t} + 1\right)^{-3/2}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\Omega_{\mathrm{GW}} =&\frac{\rho_{\mathrm{GW}}}{\rho_c}\\ =&\frac{64\pi}{9}\sqrt{\frac{\mu G}{\Gamma}}\left(1 - \left(\frac{R_0}{\Gamma G\mu t} + 1\right)^{-3/2}\right).\end{aligned}$$

Exercise 6. ***Hartle 12.9.*** Darth Vader follows a few Jedi knights into a black hole. He knows any light pulse he fires will move to smaller and smaller radii. Should he worry that light from his gun will fall back on him before his destruction if he emits it radially?

Since light travels along lines which are 45 degrees with respect to the vertical in a Kruskal diagram (see the Week 14 Worksheet), while Darth Vader must necessarily travel along lines within the light cones (hence with smaller angles with respect to the vertical axis), it is not possible for him to be hit by the laser.