Final Review Session Solutions

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Exercise 1. The integral form of the Schrödinger equation reads

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int g(\mathbf{r} - \mathbf{r}')V(\mathbf{r}')\psi(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}',$$

where

$$g(\mathbf{r}) = -\frac{m}{2\pi\hbar^2} \cdot \frac{e^{ikr}}{r}$$

is the Green's function for the Schrödinger equation.

a) Use the method of successive approximations to write $\psi(\mathbf{r})$ as a series in the incident wavefunction $\psi_0(\mathbf{r})$.

b) Truncate the Born series you obtain after the second term to get the first Born approximation. Assuming the potential is localized near $\mathbf{r}' = 0$, we can write

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{e^{ikr}}{r} e^{-i\mathbf{k}\cdot\mathbf{r}'}.$$

Using this and the definition of $f(\theta)$,

$$\psi(\mathbf{r}) = Ae^{ikz} + f(\theta)\frac{e^{ikr}}{r},$$

determine $f(\theta)$.

c) In Griffiths, we find that for a potential $V(r) = V_0/r$, $f_{\text{point}}(\theta) = -\frac{2mV_0}{\hbar^2 q^2}$, where $\mathbf{q} = \mathbf{k}' - \mathbf{k}$. If $V(\mathbf{r}) = -e^2 Z/r$ for an electron scattering off a point charge of charge Ze, how would $f(\theta)$ change if instead the electron scatters off a spherical nucleus of radius a, charge Ze, and uniform charge density? Your answer should be of the form

$$f(\theta) = f_{\text{point}}(\theta) \cdot F(q),$$

where $F(q)$ is the **form factor** of the nucleus.

d) If you haven't done so already, calculate $F(q)$ explicitly.

e) From scattering high-energy electrons at nuclei, the actual form factor is measured to be

$$F(q) = \frac{Ze}{(1 + q^2 a_N{}^2)^2},$$

where $a_N \approx 0.26$ fm. If the inverse Fourier transform of $\frac{1}{(1+x^2)^2}$ is $e^{-|x|}$, what does that tell you about the size and charge density of the proton?

a) The idea is to plug in the formula for ψ for the $\psi(\mathbf{r}')$ on the right side. Thus, we obtain

$$\begin{aligned}\psi(\mathbf{r}) =&\psi_0(\mathbf{r}) + \int g(\mathbf{r} - \mathbf{r}')V(\mathbf{r}')\psi_0(\mathbf{r}')\,\mathrm{d}^3r' + \\ &+ \iint g(\mathbf{r} - \mathbf{r}')g(\mathbf{r}' - \mathbf{r}'')V(\mathbf{r}')V(\mathbf{r}'')\psi_0(\mathbf{r}'')\,\mathrm{d}^3r'\,\mathrm{d}^3r'' + \cdots .\end{aligned}$$

b) The Born approximation is then

$$\psi(\mathbf{r}) \approx \psi_0(\mathbf{r}) + \int g(\mathbf{r} - \mathbf{r}')V(\mathbf{r}')\psi_0(\mathbf{r}')\,\mathrm{d}^3r'.$$

Plugging in the suggested approximation for the Green's function, we get

$$\psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ikr}}{r} e^{-i\mathbf{k}\cdot\mathbf{r}'}V(\mathbf{r}')\psi_0(\mathbf{r}')\,\mathrm{d}^3r'.$$

Now, suppose the incident wavefunction is a plane wave e^{ikz} along the $\hat{z}$ direction. It follows then that

$$\psi(\mathbf{r}) \approx e^{ikz} - \frac{m}{2\pi\hbar^2}\int \frac{e^{ikr}}{r} e^{-i\mathbf{k}\cdot\mathbf{r}'}V(\mathbf{r}')e^{i\mathbf{k}'\cdot\mathbf{r}'}\,\mathrm{d}^3r',$$

where we set $\mathbf{k}' = k\hat{z}$. Thus,

$$\psi(\mathbf{r}) = e^{ikz} + \frac{e^{ikr}}{r}\cdot\left(-\frac{m}{2\pi\hbar^2}\int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}'}V(\mathbf{r}')\,\mathrm{d}^3r'\right),$$

so

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}'}V(\mathbf{r}')\,\mathrm{d}^3r'.$$

c) Uniform charge density implies $\rho(r) = \frac{Ze}{\frac{4}{3}\pi a^3}$ for $r \in (0, a)$. Thus,

$$V(\mathbf{r}) = -e\int \frac{1}{|\mathbf{r} - \mathbf{r}''|}\rho(\mathbf{r}'')\,\mathrm{d}^3r''.$$

Plugging this in to the expression for $f(\theta)$ from part (b), we get

$$\begin{aligned}f(\theta) =& -\frac{m}{2\pi\hbar^2}\int e^{i\mathbf{q}\cdot\mathbf{r}'}V(\mathbf{r}')\,\mathrm{d}^3r' \\ =&\frac{me}{2\pi\hbar^2}\iint e^{i\mathbf{q}\cdot\mathbf{r}'}\frac{\rho(\mathbf{r}'')}{|\mathbf{r}' - \mathbf{r}''|}\,\mathrm{d}^3r'\,\mathrm{d}^3r''.\end{aligned}$$

Make the substitution $\mathbf{u} = \mathbf{r}' - \mathbf{r}''$, so

$$
\begin{aligned}
f(\theta) =& \frac{me}{2\pi\hbar^2} \iint e^{i\mathbf{q}\cdot\mathbf{u}} e^{i\mathbf{q}\cdot\mathbf{r}''} \frac{\rho(\mathbf{r}'')}{|\mathbf{u}|} \, \mathrm{d}^3u \, \mathrm{d}^3r'' \\
=& f_{\text{point}}(\theta) \frac{1}{2e} \int e^{i\mathbf{q}\cdot\mathbf{r}''} \rho(\mathbf{r}'') \, \mathrm{d}^3r'' \Longrightarrow \\
\Longrightarrow F(q) =& \frac{1}{2e} \int e^{i\mathbf{q}\cdot\mathbf{r}''} \rho(\mathbf{r}'') \, \mathrm{d}^3r''.
\end{aligned}
$$

The integral in the second equality goes as follows. We have

$$
\int \frac{e^{i\mathbf{q}\cdot\mathbf{u}}}{u} \, \mathrm{d}^3u = \int e^{iqu\cos\theta} u \cdot 2\pi \, \mathrm{d}(\cos\theta) \, \mathrm{d}u,
$$

where we can assume that the angle between $\mathbf{q}$ and $\mathbf{u}$ is θ since we're integrating over all θ anyway. Now, do the θ integral to get

$$
\int \frac{2\pi}{iq} (e^{iqu} - e^{-iqu}) \, \mathrm{d}u = \frac{4\pi}{q} \int \sin(qu) \, \mathrm{d}u = \frac{4\pi}{q^2},
$$

and we're done.

d) We plug in the form for ρ above and calculate.

$$
\begin{aligned}
F(q) =& \frac{3}{4\pi a^3} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \int_0^a r^2 \sin\theta e^{iqr\cos\theta} \, \mathrm{d}r \\
=& \frac{3}{2a^3} \int_0^a r^2 \, \mathrm{d}r \frac{1}{iqr} (e^{iqr} - e^{-iqr}) \\
=& \frac{3}{2a^3} \int_0^a \mathrm{d}r \frac{2r}{q} \sin(qr) \\
=& \frac{3}{a^3q^3} \left(\sin(qa) - aq\cos(qa) \right).
\end{aligned}
$$

e) Consider $V(\mathbf{r})$,

$$
\begin{aligned}
V(\mathbf{r}) =& -e \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}^3r' \\
=& -e^2 Z \int \frac{e^{-r/a_N}}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}^3r'.
\end{aligned}
$$

But this is the Fourier transform of $F(q)$ given in the problem statement. Thus, $\rho(\mathbf{r}) = Ze \cdot e^{-r/a_N}$, so we find that the charge density of the nucleus has an *exponential* distribution! The proton is then "smeared out" over all space, but it has a $1/e$ drop off after $r = a_N$. So, we can consider the "size" of the proton to be $\approx a_N$.

Exercise 2. Consider a 1D harmonic oscillator of angular frequency ω_0 that is perturbed by a time-dependent potential $V(t) = bx\cos(\omega t)$, where x is the displacement of the oscillator from equilibrium. Evaluate $\langle x \rangle$ by time-dependent perturbation theory. Discuss the validity of the result for $\omega \approx \omega_0$ and ω far from ω_0.

First, let's figure out what $H'_{nm} = \langle n|V(t)|m\rangle$ is. This will be given by

$$H'_{nm} = b\cos(\omega t)\sqrt{\frac{\hbar}{2m\omega}}\left(\delta_{m,n+1}\sqrt{n+1} + \delta_{m,n-1}\sqrt{n}\right),$$

where we use the form for x in terms of raising and lowering operators. Now, we know that in first order time-dependent perturbation theory,

$$\frac{\mathrm{d}c_n}{\mathrm{d}t} = -\frac{i}{\hbar}\sum H'_{nm}e^{i\omega_{nm}t}c_m,$$

where the c_i are the coefficients of the wavefunction at time $t = 0$, i.e.

$$|\psi(0)\rangle = \sum_m c_m|m\rangle.$$

Plugging in our result for H'_{nm}, we get

$$\begin{aligned}\frac{\mathrm{d}c_n}{\mathrm{d}t} &= -\frac{ib\cos(\omega t)}{\sqrt{2m\hbar\omega_0}}\left(c_{n+1}e^{-i\omega_0 t}\sqrt{n+1} + c_{n-1}e^{i\omega_0 t}\sqrt{n}\right)\\ &= -\frac{ib}{\sqrt{2m\omega_0\hbar}}\left[\left(e^{i(\omega-\omega_0)t} + e^{-i(\omega+\omega_0)t}\right)c_{n+1}\sqrt{n+1} + \left(e^{i(\omega+\omega_0)t} + e^{-i(\omega-\omega_0)t}\right)c_{n-1}\sqrt{n}\right].\end{aligned}$$

Integrating this from 0 to t, we get

$$\begin{aligned}c_n(t) = -\frac{b}{\sqrt{2m\hbar\omega_0}}\Bigg[&\left(\frac{e^{i(\omega-\omega_0)t}}{\omega-\omega_0} - \frac{e^{-i(\omega+\omega_0)t}}{\omega+\omega_0} - \frac{2\omega_0}{\omega^2-{\omega_0}^2}\right)c_{n+1}\sqrt{n+1}+\\ &+\left(\frac{e^{i(\omega+\omega_0)t}}{\omega+\omega_0} - \frac{e^{-i(\omega-\omega_0)t}}{\omega-\omega_0} + \frac{2\omega_0}{\omega^2-{\omega_0}^2}\right)c_{n-1}\sqrt{n}\Bigg].\end{aligned}$$

This is the most general expression for the time-dependence of the coefficients, from which we could then derive the most general expression for the time-dependence of $\langle x\rangle$. Instead, let's consider some more simple cases. The simplest case is $|\psi(0)\rangle = |n\rangle$. Then only the c_{n-1} coefficient acquires a time dependence, which would look like the top line of the equation above. However, let's consider the more general case of the form

$$|\psi(0)\rangle = \alpha|n\rangle + \beta|n+1\rangle.$$

Note that this subsumes the simpler case where $|\psi(0)\rangle$ is an eigenstate of the harmonic oscillator by setting $\beta = 0$. Let's figure out what will happen here. We then have

$$\begin{aligned}c_n(t) &= -\frac{b}{\sqrt{2m\hbar\omega_0}}\left(\frac{e^{i(\omega-\omega_0)t}}{\omega-\omega_0} - \frac{e^{-i(\omega+\omega_0)t}}{\omega+\omega_0} - \frac{2\omega_0}{\omega^2-{\omega_0}^2}\right)\beta\sqrt{n+1}\\ c_{n+1}(t) &= -\frac{b}{\sqrt{2m\hbar\omega_0}}\left(\frac{e^{i(\omega+\omega_0)t}}{\omega+\omega_0} - \frac{e^{-i(\omega-\omega_0)t}}{\omega-\omega_0} + \frac{2\omega_0}{\omega^2-{\omega_0}^2}\right)\alpha\sqrt{n+1},\end{aligned}$$

with all other $c_i(t) = 0$. This means that

$$|\psi(t)\rangle = c_n(t)|n\rangle + c_{n+1}(t)|n+1\rangle.$$

Now, let's try to calculate $\langle x(t)\rangle$. We will get cross-terms corresponding to $c_n^* c_{n+1}$ and $c_{n+1}^* c_n$, with all other terms zero. After a lot of algebra, we should arrive at

$$\frac{\alpha\beta b^2}{2m^2{\omega_0}^2\sqrt{2\hbar}}(n+1)^{3/2}\left(\frac{4\cos(2\omega_0 t)}{\omega^2-{\omega_0}^2} - \frac{8\omega_0\cos[(\omega+\omega_0)t]}{(\omega+\omega_0)(\omega^2-{\omega_0}^2)} + \frac{8\omega_0\cos[(\omega-\omega_0)t]}{(\omega-\omega_0)(\omega^2-{\omega_0}^2)} - \right.$$
$$\left. - \frac{\cos[2(\omega+\omega_0)t]}{2(\omega+\omega_0)^2} - \frac{\cos[2(\omega-\omega_0)t]}{2(\omega-\omega_0)^2} - \frac{8{\omega_0}^2}{(\omega^2-{\omega_0}^2)^2}\right).$$

Similarly, we can think about what will happen if instead we had

$$|\psi(0)\rangle = \gamma|n-1\rangle + \delta|n\rangle.$$

We could get the result really easily from the above by just setting $n \to n-1$, $\alpha \to \gamma$ and $\beta \to \delta$, allowing us to read off the answer.

Returning to the former case, let's consider the limits suggested in the problem. If $\omega \approx \omega_0$, ;let's first consider terms quadratic in $1/(\omega-\omega_0)$. In this case, only the last two terms and the third term in the above result contribute to $\langle x\rangle$. Moreover, both of the cosines will be approximately 1; hence, $\langle x\rangle$ will be constant. This makes sense, since our perturbation that we added will be in phase with the oscillator, adding a constant amount of energy to the system at all times, equivalently, a constant value to the equilibrium position of the oscillator. If we include terms linear in $1/(\omega-\omega_0)$, then we would also include the first two terms as well. Since the cosines in these terms have nonzero arguments, we will end up with small oscillations about the constant $\langle x\rangle$ value from the previous case. Again, this makes sense physically, since a perturbation slightly out of phase with the oscillations will cause oscillations about a *new* equilibrium position. On the other hand, if ω is far from ω_0, the perturbation will be out of phase with the oscillating system. This should introduce chaos into the system so that it "wobbles." This seems consistent with the result above.

Lastly, we could consider the more general case,

$$|\psi(0)\rangle = \alpha|n-1\rangle + \beta|n\rangle + \gamma|n+1\rangle,$$

but I don't think we're going to get any more physics out of it. I expect that for ω near ω_0, we should end up with the same conclusions as in the analysis above, and similarly for ω far from ω_0.

Exercise 3. *Griffiths 11.33* The spontaneous emission of the 21-cm hyperfine line in hydrogen is a magnetic dipole transition with rate

$$\Gamma = \frac{\omega^3}{3\pi\varepsilon_0\hbar c^3}\left|\left\langle B\left|\frac{\boldsymbol{\mu}_e+\boldsymbol{\mu}_p}{c}\right|A\right\rangle\right|^2,$$

where

$$\boldsymbol{\mu}_e = -\frac{e}{m_e}\mathbf{S}_e$$
$$\boldsymbol{\mu}_p = \frac{5.59e}{2m_p}\mathbf{S}_p.$$

On midterm 1, you showed the triplet has slightly higher energy than the singlet. Calculate (approximately) the lifetime of this transition.

Call the triplet state(s) $|1\rangle$ and the singlet state $|0\rangle$. We then have (remembering that $m_p \gg m_e$)

$$\begin{aligned}\Gamma \approx& \frac{\omega^3 e^2}{3\pi\varepsilon_0 \hbar c^5}\left|\left\langle 0\left|\frac{e^2}{{m_e}^2}\mathbf{S}_e\right|1\right\rangle\right|^2 \\ =& \frac{\omega^3 e^4}{3\pi\varepsilon_0\hbar c^5 {m_e}^2}\left|\langle 0|\mathbf{S}_e|1\rangle\right|^2.\end{aligned}$$

Evaluating the matrix element, we get

$$\langle 0|\mathbf{S}_e|1\rangle = \frac{\hbar}{2\sqrt{2}}\Big[\big(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\big)\boldsymbol{\sigma}_e|\uparrow\uparrow\rangle\Big] = -\frac{\hbar}{2\sqrt{2}}(\hat{x} + i\,\hat{y}),$$

which you can obtain by using any triplet. Note that the first arrow is the electron and the second the proton. Thus,

$$\left|\langle 0|\mathbf{S}_e|1\rangle\right|^2 = \frac{\hbar^2}{4}.$$

Plugging this in to Γ, we get

$$\begin{aligned}\Gamma =& \frac{\omega^3 e^2}{3\pi\varepsilon_0\hbar c^5 {m_e}^2}\cdot\frac{\hbar^2}{4} \\ =& \alpha\frac{4\hbar^2\omega^3}{12 c^4 {m_e}^2}.\end{aligned}$$

Since $\omega = 2\pi c/\lambda$, we can plug in and evaluate ($\lambda = 21$ cm). We find

$$\Gamma \approx 10^{-14}\ \mathrm{s}^{-1},$$

so $T \approx 10^{14}$ s or 10^7 years.

Exercise 4. Consider a dynamical variable ξ that can take only two values, 1 or -1 (for example, σ_z is such an operator for a spin 1/2 particle). Denote the corresponding eigenvectors as $|+\rangle$ and $|-\rangle$. Now, consider the following states.

a) The one-parameter family of pure states

$$|\theta\rangle = \sqrt{\frac{1}{2}}(|+\rangle + e^{i\theta}|-\rangle)$$

for any real θ.

b) The nonpure state

$$\rho = \frac{1}{2}(|+\rangle\langle +| + |-\rangle\langle -|).$$

Show that $\langle\xi\rangle = 0$ in all of these states. What, if any, are the physical differences between these various states, and how could they be measured?

We compute

$$\begin{aligned}\langle\xi\rangle =&\frac{1}{2}(\langle+|\xi|+\rangle + \langle-|\xi|-\rangle)\\ =&0.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle\xi\rangle =& \operatorname{tr}(\rho\xi)\\ =&\frac{1}{2}(\langle+|\xi|+\rangle + \langle-|\xi|-\rangle)\\ =&0.\end{aligned}$$

How could we measure the differences between these states? One option, in the case of $|\theta\rangle$, is to consider a particle in a superposition of $|0\rangle$ and $|\pi\rangle$. Then the particle would be an eigenstate of ξ, namely $|+\rangle$. Similarly, we could do the same thing with $|\theta\rangle$ and $|\theta+\pi\rangle$ for any θ. Obviously, if we didn't choose a π-shifted (moduli 2π) state to superpose with, we wouldn't get anything interesting. Thus, one difference between the states of type (a) is that they have a unique (up to a multiple of 2π) state with which you can superpose to get an eigenstate of ξ. Thus, one way to detect the difference between the states is to initiate various particles in superpositions of them, and measure $\langle\xi\rangle$.

On the other hand, the density matrix ρ is in a superposition of the eigenstates of ξ, and *there's no way to change that* using the states given above. However, we could detect if we're in ρ, versus one of the θ states by the procedure above. Indeed, the density matrix associated to the state $|\theta\rangle$ is given by

$$\rho_\theta = \frac{1}{2}\left(2\rho + e^{i\theta}|-\rangle\langle+| + e^{-i\theta}|+\rangle\langle-|\right).$$

Using the superposition trick from the last paragraph, we could detect θ by superposing with a π-shifted θ state. However, if we started with just ρ, we would find that there is no such state, so that $\langle\xi\rangle = 0$ for any chosen superposition.

Exercise 5. In the homework, you showed that the most general density matrix for a spin 1/2 particle is $\rho = \frac{1}{2}(1+\mathbf{a}\cdot\boldsymbol{\sigma})$, where $\mathbf{a}$ is some 3-vector. If the system has a magnetic moment $\boldsymbol{\mu} = \frac{1}{2}\gamma\hbar\boldsymbol{\sigma}$ and is in a constant magnetic field $\mathbf{B}$, calculate $\rho(t)$. Describe the result geometrically in terms of the variation of the vector $\mathbf{a}$.

The problem tells us that the hamiltonian is given by $H = -\frac{1}{2}\gamma\hbar\boldsymbol{\sigma}\cdot\mathbf{B}$. We derived the equation of motion for ρ on the homework,

$$i\hbar\frac{\mathrm{d}\rho}{\mathrm{d}t} = [H,\rho].$$

Thus,

$$\begin{aligned}i\hbar\frac{\mathrm{d}\rho}{\mathrm{d}t} =& -\frac{1}{4}\gamma\hbar\mathbf{B}\cdot[\boldsymbol{\sigma}, 1+\mathbf{a}\cdot\boldsymbol{\sigma}]\\ =& -\frac{1}{4}\gamma\hbar\sum_{i,j}B_i[\sigma_i, a_j\sigma_j]\\ =& -\frac{i}{4}\gamma\hbar\sum_{i,j,k}B_i a_j\varepsilon_{ijk}\sigma_k.\end{aligned}$$

Thus,

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{1}{4}\gamma \sum_{i,j,k} B_i a_j \varepsilon_{ijk} \sigma_k \Longrightarrow$$
$$\Longrightarrow \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} \cdot \boldsymbol{\sigma} = \frac{1}{2}\gamma \mathbf{a} \times \mathbf{B} \cdot \boldsymbol{\sigma}.$$

Notice that if we instead consider the equation

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = \frac{1}{2}\gamma \mathbf{a} \times \mathbf{B},$$

its solutions will be solutions of the above equation. Indeed, these are the only solutions, since the differential equation above is the same as

$$\left(\dot{\mathbf{a}} - \frac{1}{2}\gamma \mathbf{a} \times \mathbf{B}\right) \cdot \boldsymbol{\sigma} = 0.$$

It follows that the only solution is

$$\dot{\mathbf{a}} = \frac{1}{2}\gamma \mathbf{a} \times \mathbf{B},$$

Now, notice further that this equation has exactly the same form as the equation of motion for a charged particle in a magnetic field:

$$q\mathbf{v} \times \mathbf{B} = m\dot{\mathbf{v}},$$

where $\mathbf{a}$ is now playing the role of velocity. Thus, solutions for $\mathbf{a}$ will be (the velocities of) solutions to the problem of a charged particle in a magnetic field. This is standard. If you haven't seen this before, here is a quick overview. Set $\mathbf{B} = B\hat{z}$. Thus,

$$\mathbf{a} \times \mathbf{B} = (a_y B, -a_x B, 0).$$

We then have three equations

$$\begin{aligned}
\dot{a}_x &= \frac{\gamma}{2} B a_y \\
\dot{a}_y &= -\frac{\gamma}{2} B a_x \\
\dot{a}_z &= 0.
\end{aligned}$$

We are thus down to a two-dimensional problem. Define $\omega = \gamma B/2$; this is the cyclotron frequency. Define the complex variable $\xi = a_x + i a_y$. We then get

$$\dot{\xi} = -i\omega\xi.$$

Its solution is easy:

$$\xi(t) = Ae^{-i\omega t}.$$

Thus,

$$\mathbf{a}(t) = A(\cos(\omega t), -\sin(\omega t), a_z),$$

where a_z is a fixed constant for all time and A is some normalization. So $\mathbf{a}(t)$ rotates in the xy-plane at frequency $\omega = \gamma B/2$. Returning back to the original problem (so $\mathbf{B}$ is not necessarily along $\hat{z}$), then $\mathbf{a}(t)$ will rotate in the plane perpendicular to $\mathbf{B}$ at angular frequency $\gamma B/2$ and be constant in the direction of $\mathbf{B}$. We're done.