Week 15 Worksheet
Waves and Energy

Jacob Erlikhman

Exercise 1. a) Starting with Maxwell's equations in vacuum, show that they give you wave equations for $\mathbf{E}$ and $\mathbf{B}$.

b) Show that the waves are transverse: If they travel in the z direction, then $\widetilde{E}_{0z} = 0$ and $\widetilde{B}_{0z} = 0$.

c) Show that $\mathbf{B}$ is perpendicular to $\mathbf{E}$.

Exercise 2. Recalling that magnetic forces do no work, we have that

$$\mathrm{d}W = \mathbf{F} \cdot \mathrm{d}\boldsymbol{\ell} = q\mathbf{E} \cdot \mathbf{v}\, \mathrm{d}t.$$

Thus, the rate at which work is done on the charges in a volume is

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int \mathbf{E} \cdot \mathbf{J}\, \mathrm{d}V.$$

a) Show that

$$\mathbf{E} \cdot \mathbf{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\boldsymbol{\nabla} \cdot \mathbf{E} \times \mathbf{B}$$

.

Hints: Use Maxwell's equations along with the identity

$$\boldsymbol{\nabla} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot \boldsymbol{\nabla} \times \mathbf{v} - \mathbf{v} \cdot \boldsymbol{\nabla} \times \mathbf{w}.$$

b) Plug this in to the formula for the rate of work, and derive Poynting's theorem:

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$$

is the energy per unit time, per unit area, transported by the fields; it is the energy flux density.

c) You showed in class that

$$\begin{aligned} \mathbf{g} &= \mu_0 \varepsilon_0 \mathbf{S} \\ &= \frac{1}{c^2}\mathbf{S} \end{aligned}$$

is the momentum per unit volume stored in the fields. Discuss why $\mathbf{S}$ seems to have two different physical interpretations.

Both of these exercises are done in Griffiths. The only comment I will make is for part (c) of the second exercise. Recall from special relativity

$$E^2 = p^2c^2 + m^2c^4.$$

Since $m = 0$ for an electromagnetic wave, we have the relation

$$E = pc.$$

So there is a proportionality relation between energy and momentum for light. Hence, when we say we have an energy density, we also have a momentum density directly from this formula. It is now straightforward to check that the units work out correctly: Since S has units of energy per unit area per unit time,

$$[S] = \frac{E}{L^2T},$$

this is the same by $E = pc$ as

$$[S] = \frac{M}{T^3},$$

where M denotes units of mass. Now, a momentum per unit volume has units of

$$\frac{M}{L^2T},$$

and we notice that this is exactly a factor of c^2 off from $[S]$. Another comment is that this is physically sensible. Since the energy is carried out by the fields, and the fields travel at speed c, it follows that we get an extra factor of c when we convert from $\mathbf{S}$ to $\mathbf{g}$. So one factor comes from $E = pc$, and the other comes from the fact that the wave itself travels at speed c.