Week 13 Worksheet
Electrodynamics

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Exercise 1. An infinite solenoid with a number of wire loops per unit length n is hooked up to an alternating current $I = I_0 \sin(\omega t)$. Find the electric field inside the solenoid if the radius of the solenoid $a \ll c/\omega$. *Hint*: The z-component of the curl in cylindrical coordinates is

$$(\nabla \times \mathbf{v})_z = \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\varphi) - \frac{\partial v_s}{\partial \varphi}\right].$$

We can find the magnetic field inside the solenoid using Ampère's law as usual. This gives

$$\mathbf{B}_{\text{in}} = \mu_0 n I \hat{z},$$

where z is along the axis of the solenoid. Since

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

we have that

$$\nabla \times \mathbf{E} = -\mu_0 n I_0 \omega \cos(\omega t).$$

It follows that we can set

$$\mathbf{E} = -\frac{\mu_0}{2} n s I_0 \omega \cos(\omega t) \hat{\varphi}.$$

Note that we can't use a component of $\mathbf{E}$ along $\hat{s}$, since if we do that our solution won't satisfy $\nabla \cdot \mathbf{E} = \mathbf{0}$ (which it must, since there is no charge inside the solenoid).

Alternative Solution: Instead of using Maxwell's equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

in its infinitesimal form, we could integrate both sides over a region:

$$\int \nabla \times \mathbf{E} \cdot \mathrm{d}\mathbf{a} = -\frac{\partial \Phi_B}{\partial t},$$

where Φ_B is the magnetic flux through the region of integration. Now, apply Stokes' theorem to get

$$\oint \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\partial \Phi_B}{\partial t}.$$

Using a loop that is concentric with the cylinder (and inside it) gives

$$2\pi s E_\varphi = -\mu_0 n I_0 \omega \cos(\omega t)\pi s^2,$$

since E_φ must be independent of φ by symmetry. Hence,

$$E_\varphi = -\frac{\mu_0}{2} n s I_0 \omega \cos(\omega t),$$

which is the same answer we got above. We still need to show that $E_s = 0$. I think for this we need the form of the divergence for cylindrical coordinates, namely that

$$\boldsymbol{\nabla}\cdot \mathbf{E} = \frac{1}{s}\frac{\partial}{\partial s}(sE_s),$$

where the other two terms vanish. From here, since $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$, we have that this must be 0, which can only hold if $E_s \sim \frac{1}{s}$. But this isn't a valid solution, since it blows up at the origin. Hence, $E_s = 0$.

Exercise 2. A capacitor C is charged up to a voltage V and connected to an inductor L in series at time $t = 0$.

a) ***Griffiths 7.27.*** Find the current in the circuit as a function of time.

b) Show that the total energy of the configuration is constant at any time t, and find this constant.

a) Since the induced emf is

$$\mathcal{E} = -L\frac{\mathrm{d}I}{\mathrm{d}t}$$

and

$$C = \frac{Q}{V},$$

we have that

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = -\frac{I}{LC}.$$

It follows that a general solution for $I(t)$ is given by

$$I(t) = A\sin(\omega t) + B\cos(\omega t),$$

where $\omega = 1/\sqrt{LC}$. Since $I(0) = 0$, we should set $B = 0$. Now, at $t = 0$, we have

$$\begin{aligned} V_0 &= -L\frac{\mathrm{d}I}{\mathrm{d}t}\bigg|_{t=0} \\ &= -LA\omega. \end{aligned}$$

Hence,

$$A = -\frac{V_0}{L\omega}.$$

The solution for I is then

$$I(t) = -V_0\sqrt{\frac{C}{L}}\sin\left(\frac{t}{\sqrt{LC}}\right).$$

b) The energy stored in the inductor is $\frac{1}{2}LI^2$. Similarly, the energy stored in the capacitor is $\frac{1}{2}CV^2$. We already know I, so we need to figure out V. We can do this using $CV = Q$, which implies

$$C\frac{\mathrm{d}V}{\mathrm{d}t} = I.$$

Hence,

$$V(t) = V_0\cos\left(\frac{t}{\sqrt{LC}}\right),$$

where we note that the integration constant is 0 since $V(0) = V_0$. Now, we can compute

$$\frac{1}{2}LI^2 + \frac{1}{2}CV^2 = \frac{1}{2}C{V_0}^2.$$