Week 11 Worksheet
Magnets!

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Exercise 1. The Lorentz force law for a particle of charge q movign with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is given by

$$\mathbf{F}_{\text{mag}} = q\mathbf{v} \times \mathbf{B}.$$

Find an expression for the work that the magnetic force does on the particle.

The work is 0, since $\mathrm{d}\boldsymbol{\ell} = \mathbf{v}\,\mathrm{d}t$, so

$$\mathrm{d}W = \mathbf{F} \cdot \mathrm{d}\boldsymbol{\ell} = q\mathbf{v} \times \mathbf{B} \cdot \mathbf{v}\,\mathrm{d}t = 0.$$

Exercise 2. Find the magnetic field and vector potential due to a current which flows with constant surface density $\mathbf{K}$ which flows along the surface of an infinite cylinder of radius a in the following directions:

a) along the axis of the cylinder;

b) perpendicular to the axis of the cylinder;

c) at an angle α to the axis of the cylinder.

a) The magnetic field can be found using Ampère's law. Clearly, the field will be along $\hat{\varphi}$ if the axis of the cylinder is along the z-axis. Ampères law then gives that the field will be 0 inside the cylinder. Outside, it will be given by

$$\mu_0 K \cdot 2\pi a\ell = \ell B(s) 2\pi s \implies \mathbf{B}(s) = \frac{\mu_0 K a}{s}\hat{\varphi}.$$

To find the vector potential, use

$$\oint \mathbf{A} \cdot \mathrm{d}\boldsymbol{\ell} = \int \mathbf{B} \cdot \mathrm{d}\mathbf{a},$$

which you can obtain from Stokes' theorem and the fact that $\mathbf{B} = \nabla \times \mathbf{A}$. Also, since $\mathbf{B}$ is the curl of $\mathbf{A}$, we see that if $\mathbf{B}$ is along $\hat{\varphi}$, then $\mathbf{A}$ will be along $\hat{z}$. It follows that outside the cylinder we can use a rectangular amperian loop which is placed so that $\mathbf{B}$ is perpendicular to it while $\mathbf{A}$ is parallel to two of its sides (and perpendicular to the other two). This loop will then give that

$$A(s_1) - A(s_2) = \mu_0 K a \ln(\frac{s_2}{s_1}).$$

We can thus set

$$\mathbf{A}(s) = -\mu_0 K a \ln(s)\hat{z}$$

outside the cylinder. Notice that this agrees with everything: $\nabla \times \mathbf{A} = \mathbf{B}$, $\nabla \cdot \mathbf{A} = 0$, and it satisfies the relation above. Inside the cylinder, we can set

$$\mathbf{A}(s) = -\mu_0 K a \ln(a)\hat{z},$$

which clearly satisfies the requirements (and gives us a continuous function for $\mathbf{A}$).

b) This situation is the same as a solenoid. As in that case, the magnetic field is 0 outside (since it has to go to 0 as $s \to \infty$ and an amperian loop gives that $B(s_1) = B(s_2)$ for $s_i > a$). Inside, we instead have that

$$B\ell = \mu_0 K \ell$$

by Ampère's law, so

$$\mathbf{B}(s) = \mu_0 K \hat{z}$$

inside (note that this is constant). Since $\mathbf{B}$ is the curl of $\mathbf{A}$, we have as in (a) that if $\mathbf{B}$ is along $\hat{z}$, $\mathbf{A}$ will be along $\hat{\varphi}$. Using this, we can proceed as in (a) with an amperian loop inside the solenoid that circles around the z-axis. This will give

$$\mu_0 K \cdot \pi s^2 = A \cdot 2\pi s,$$

so that

$$\mathbf{A}(s) = \frac{\mu_0 K s}{2}\hat{\varphi}$$

inside the solenoid. On the other hand, using an amperian loop with radius $s > a$, we have

$$\mu_0 K \cdot \pi a^2 = A \cdot 2\pi s,$$

so

$$\mathbf{A}(s) = \frac{\mu_0 K a^2}{2s}\hat{\varphi}.$$

Note that this form for $\mathbf{A}$ is continuous at the boundary $s = a$, i.e. $\mathbf{A}_{\text{in}}(a) = \mathbf{A}_{\text{out}}(a)$. You can also check that $\nabla \times \mathbf{A} = \mathbf{B}$ everywhere and $\nabla \cdot \mathbf{A} = 0$ everywhere.

c) In this case our $\mathbf{K}$ can be written as

$$\mathbf{K} = K_{\perp}\hat{\varphi} + K_{\parallel}\hat{z}.$$

Now, since *everything is linear*, the solutions in this case will be linear combinations of the solutions in parts (a) and (b), with the appropriate values of K. This statement means that 1. $\mathbf{B} = \nabla \times \mathbf{A}$ is linear, 2. $\nabla \cdot \mathbf{A} = 0$ is linear, and 3. $\mathbf{B}$ satisfies the principle of superposition. Hence, our answers are

$$\begin{aligned}
\mathbf{B}_{\text{out}}(s) &= \mu_0 K_\parallel \frac{a}{s} \hat{\varphi} \\
\mathbf{B}_{\text{in}}(s) &= \mu_0 K_\perp \cdot 2\pi a \hat{z} \\
\mathbf{A}_{\text{in}}(s) &= \frac{\mu_0 K_\perp s}{2} \hat{\varphi} - \mu_0 K_\parallel a \ln(a) \hat{z} \\
\mathbf{A}_{\text{out}}(s) &= \frac{\mu_0 K_\perp a^2}{2s} \hat{\varphi} - \mu_0 K_\parallel a \ln(s) \hat{z}.
\end{aligned}$$