

# Week 3 Worksheet Solutions

## Identical Particles

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### Exercise 1. Symmetries of Many-Particle States.

- a) Consider a system of two identical particles. Define a **permutation operator** via

$$P_{12} |\alpha\rangle |\beta\rangle = |\beta\rangle |\alpha\rangle .$$

Show that  $P_{12}^2 = \mathbb{1}$ , the identity operator, and that the eigenvalues of  $P_{12}$  are  $\pm 1$ . Thus, show that its eigenvectors are either totally symmetric or antisymmetric.

- b) Generalize part (a) to systems of three identical particles. You should find that you have *six* permutation operators. Assuming the hamiltonian is invariant under each of these operators, is there a complete set of common eigenvectors?
- c) **Griffiths 5.8.** In the situation of (b), suppose that the particles have access to three distinct one-particle states,  $|a\rangle$ ,  $|b\rangle$ , and  $|c\rangle$ . For example,  $|abc\rangle$  is an allowed state, as is  $|aaa\rangle$ . How many states can be constructed if they are (i) bosons or (ii) fermions?
- d) Suppose we have a single-particle fermion state  $|\alpha\rangle$  and a single-particle bosonic state  $|\beta\rangle$ . Just like for the harmonic oscillator, we can define **creation operators**  $C_\alpha^\dagger$  and  $a_\beta^\dagger$ , such that given any state  $|\psi\rangle$ ,

$$C_\alpha^\dagger |\psi\rangle = |\alpha\psi\rangle$$

$$a_\beta^\dagger |\psi\rangle = |\beta\psi\rangle .$$

The operators  $C_\alpha^\dagger$  and  $a_\beta^\dagger$  have the following properties.

$$C_\alpha |\alpha\psi\rangle = |\psi\rangle$$

$$a_\beta |\beta\psi\rangle = |\psi\rangle$$

$$C_\alpha |0\rangle = a_\beta |0\rangle = 0$$

$$C_\alpha^\dagger C_\alpha^\dagger = 0$$

$$\{C_\alpha, C_{\alpha'}^\dagger\} \equiv C_\alpha C_{\alpha'}^\dagger + C_{\alpha'}^\dagger C_\alpha = \delta_{\alpha\alpha'} \mathbb{1}$$

$$\{C_\alpha^\dagger, C_{\alpha'}^\dagger\} = 0$$

$$[a_\beta, a_{\beta'}^\dagger] = \delta_{\alpha\alpha'} \mathbb{1}$$

$$[a_\beta^\dagger, a_{\beta'}^\dagger] = 0,$$

where  $|0\rangle$  denotes a state with no particles at all. To what extent is a bound pair of fermions equivalent to a boson?

*Hint:* Use the symmetries of many-particle states and the (anti-)commutation relations of the creation/annihilation operators constructed in parts (a)-(d). What algebra must the creation/annihilation operators for the bound pair satisfy?

e) Prove the properties given in (d).

*Hints:* It may be useful to use the notation  $\sim \alpha$  for the  $\alpha$  “orbital” being *unoccupied*. To show the first relation for  $C_\alpha$ , try to first show that  $C_\alpha |\alpha\rangle = |0\rangle$ . For the anti-commutator relations, consider separately the cases  $\alpha \neq \alpha'$  and whether the  $\alpha$  or  $\alpha'$  orbitals are occupied.

### Exercise 2.

Write down the hamiltonian for two noninteracting identical particles in the infinite square well. Write down the ground states for the three cases: distinguishable, fermions, bosons. Recall that the one-particle wavefunctions are

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right),$$

with energies  $E_n = n^2\pi^2\hbar^2/2ma^2$ .

Find the first three excited states and their energies for each of the three cases (distinguishable, fermions, bosons).

**Exercise 3.** In Exercise 2, we ignored spin (or at least supposed that the particles are in the same spin state).

- Do it now for particles of spin 1/2. Construct the four lowest-energy configurations, and specify their energies and degeneracies.
- Do the same for spin 1 (you will need the Clebsch-Gordan table from bCourses).