

# Week 9 Worksheet

## Fine Structure of Hydrogen

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### Exercise 0. Warm up.

- What are the physical effects that make up fine structure?
- Write down  $H_r$ , the relativistic hamiltonian, and explain its physical origin.  
*Hint:* Expand the relativistic energy for  $v \ll c$ .
- Do the same for  $H_{\text{SO}}$ , but don't feel obligated to get every factor of 2 right.  
*Hint:*  $\mathbf{H} = -\boldsymbol{\mu} \cdot \mathbf{B}$ .

**Exercise 1. Darwin Term.** In class, you found that the spin-orbit coupling gave a first-order correction to the energy

$$E_{\text{SO}}^1 = \frac{E_n^2}{mc^2} \frac{n[j(j+1) - \ell(\ell+1) - 3/4]}{\ell(\ell+1/2)(\ell+1)}.$$

- What is  $E_{\text{SO}}^1$  for  $s$ -states, i.e. those with  $\ell = 0$ ?
- There is an additional effect for  $s$ -states called the **Darwin term**:

$$H_D = \frac{\hbar^2}{8m^2c^2} e^2 \nabla^2 V_C,$$

where  $V_C = -e/r$  is the Coulomb potential. Show that this term is of order  $\alpha^2$ .

*Hint:* You can determine  $\alpha$  by using the fundamental constants of the hydrogen atom,  $c$ ,  $\hbar$ ,  $e$ , to form a dimensionless constant.

- This term can be derived from the Dirac equation, but we can get a handle on it using non-relativistic QM. To see where the term comes from, the Dirac equation (relativistic quantum mechanics) predicts that the electron does not have a constant position. Instead, it undergoes a frantic jittering motion due to the creation of virtual position-electron pairs. The lifetime of these is given by the uncertainty principle  $\Delta t \Delta E = \hbar$ , so  $\Delta t = \hbar/mc^2$ . The position of the electron is smeared out due to this motion by the characteristic distance associated to this lifetime, the Compton wavelength:  $\lambda_c \equiv c\Delta t = \hbar/mc \approx 4 \cdot 10^{-11}$  cm. So the potential energy is not at a particular position; rather, it is an average around that point. Suppose that  $r_0$  is the average position, and expand the potential  $V_C(r)$  as a Taylor expansion to second order about  $r_0$ .

- d) Use symmetry to argue that the expectation value of the first term in the expansion is 0.
- e) Use the same symmetry and dimensional analysis to argue that

$$\langle V_C(r) \rangle \approx V_C(r_0) + A \frac{\hbar^2}{m^2 c^2} 4\pi e^2 \delta^{(3)}(\mathbf{r}),$$

where  $A$  is a dimensionless constant. We call the second term the **Darwin term**. Calculate  $A$ , and show that this reproduces  $H_D$  exactly, *assuming that the characteristic length is actually  $3\lambda_c/4$* .

- f) Argue that this term has an expectation value only for  $s$ -states.
- g) Use the fact that  $R_{n0}(0) = 2/(na_0)^{3/2}$ , where  $a_0 = \hbar^2/me^2$ , to show that

$$E_{\text{SO}}^1 + E_D^1$$

is well-defined for  $s$ -states.

**Remark.** You may have noticed that we kind of cheated by assuming that the characteristic length was  $3\lambda_c/4$  rather than just  $\lambda_c$ . Unfortunately, there isn't a good justification for this. The Darwin term and all of fine structure can be derived rigorously from the Dirac equation, and that's all there is to it. The Schrödinger theory just isn't a good theory for dealing with truly relativistic effects.