Week 3 Worksheet
Identical Particles

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Exercise 1. Symmetries of Many-Particle States.

a) Consider a system of two identical particles. Define a **permutation operator** via

$$P_{12} |\alpha\rangle |\beta\rangle = |\beta\rangle |\alpha\rangle .$$

Show that ${P_{12}}^2 = \mathbb{1}$, the identity operator, and that the eigenvalues of P_{12} are ± 1. Thus, show that its eigenvectors are either totally symmetric or antisymmetric.

b) Generalize part (a) to systems of three identical particles. You should find that you have *six* permutation operators. Assuming the hamiltonian is invariant under each of these operators, is there a complete set of common eigenvectors?

c) ***Griffiths 5.8.*** In the situation of (b), suppose that the particles have access to three distinct one-particle states, $|a\rangle$, $|b\rangle$, and $|c\rangle$. For example, $|abc\rangle$ is an allowed state, as is $|aaa\rangle$. How many states can be constructed if they are (i) bosons or (ii) fermions?

d) Suppose we have a single-particle fermion state $|\alpha\rangle$ and a single-particle bosonic state $|\beta\rangle$. Just like for the harmonic oscillator, we can define **creation operators** ${C_\alpha}^\dagger$ and ${a_\beta}^\dagger$, such that given any state $|\psi\rangle$,

$$\begin{aligned} {C_\alpha}^\dagger |\psi\rangle &= |\alpha\psi\rangle \\ {a_\beta}^\dagger |\psi\rangle &= |\beta\psi\rangle . \end{aligned}$$

The operators ${C_\alpha}^\dagger$ and ${a_\beta}^\dagger$ have the following properties.

$$\begin{aligned} C_\alpha |\alpha\psi\rangle &= |\psi\rangle \\ a_\beta |\beta\psi\rangle &= |\psi\rangle \\ C_\alpha |0\rangle = a_\beta |0\rangle &= 0 \\ {C_\alpha}^\dagger {C_\alpha}^\dagger &= 0 \\ \{C_\alpha, {C_{\alpha'}}^\dagger\} \equiv C_\alpha {C_{\alpha'}}^\dagger + {C_{\alpha'}}^\dagger C_\alpha &= \delta_{\alpha\alpha'}\mathbb{1} \\ \{{C_\alpha}^\dagger, {C_{\alpha'}}^\dagger\} &= 0 \\ [a_\beta, {a_{\beta'}}^\dagger] &= \delta_{\alpha\alpha'}\mathbb{1} \\ [{a_\beta}^\dagger, {a_{\beta'}}^\dagger] &= 0, \end{aligned}$$

where $|0\rangle$ denotes a state with no particles at all. To what extent is a bound pair of fermions equivalent to a boson?
Hint: Use the symmetries of many-particle states and the (anti-)commutation relations of the creation/annihilation operators constructed in parts (a)-(d). What algebra must the creation/annihilation operators for the bound pair satisfy?

e) Prove the properties given in (d).
Hints: It may be useful to use the notation $\sim \alpha$ for the α "orbital" being *unoccupied*. To show the first relation for C_α, try to first show that $C_\alpha |\alpha\rangle = |0\rangle$. For the anti-commutator relations, consider separately the cases $\alpha \neq \alpha'$ and whether the α or α' orbitals are occupied.

Remark. The algebra satisfied by the bosonic raising and lowering operators is isomorphic to (i.e. the same as) the algebra satisfied by the harmonic oscillator raising and lowering operators.

Exercise 2. ***Griffiths 5.5.***

a) Write down the hamiltonian for two noninteracting identical particles in the infinite square well. Write down the ground states for the three cases: distinguishable, fermions, bosons. Recall that the one-particle wavefunctions are

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right),$$

with energies $E_n = n^2\pi^2\hbar^2/2ma^2$.

b) Find the first three excited states and their energies for each of the three cases (distinguishable, fermions, bosons).

Exercise 3. ***Griffiths 5.9.*** In Exercise 2, we ignored spin (or at least supposed that the particles are in the same spin state).

a) Do it now for particles of spin 1/2. Construct the four lowest-energy configurations, and specify their energies and degeneracies.

b) Do the same for spin 1. (You will need a table of Clebsch-Gordan coefficients.)