Week 5 Worksheet Solutions
Relativistic Electrodynamics

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Exercise 1. The Faraday Tensor. Starting from the classical Lorentz force law for a particle of charge q moving with velocity $\mathbf{v}$,

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right), \tag{1}$$

derive a covariant Lorentz force law as follows.

a) Derive an equation for $d\mathbf{p}/d\tau$ in terms of (the components of) the four-velocity u and the fields $\mathbf{E}$ and $\mathbf{B}$.

b) Consider Poynting's theorem,

$$\frac{d\widetilde{U}}{dt} = \mathbf{j} \cdot \mathbf{E} - \nabla \cdot \mathbf{S},$$

where j is the 4-current, $\mathbf{S} = \mathbf{E} \times \mathbf{B}$ is the Poynting vector, and $\widetilde{U}$ denotes the energy density (so $\int_V \widetilde{U}\, d^3r = U$ is the energy contained in a volume V). Give a physical explanation for each term in the theorem (it may help to integrate both sides).

c) Use Poynting's theorem to show that

$$\frac{dp^0}{d\tau} = q\mathbf{E} \cdot \mathbf{u},$$

where u is the 4-velocity.
Hints: The *particle's* energy density is only the first term of Poynting's theorem. What value does the function $\mathbf{j}(\mathbf{r})$ take when $\mathbf{r} \neq \mathbf{r}'$, where $\mathbf{r}'$ is the location of the particle (at a given time)?

d) Combine this and the classical Lorentz force law (1) to obtain a relativistic equation of motion

$$\frac{dp}{d\tau} = qF(u),$$

in terms of a tensor F which acts on u.
Hints: If u is a 4-vector—hence rank 1—and p is also a 4-vector, what rank must F be? To determine

the components of F, compare the equation of motion you obtained in terms of u, $\mathbf{E}$, and $\mathbf{B}$ to the tensor equation

$$\frac{dp^\mu}{d\tau} = qF^\mu{}_\nu u^\nu.$$

Use index notation; for example, the cross product can be written as $(\mathbf{a} \times \mathbf{b})^k = \varepsilon_{ijk} a^i b^j$. Note that the "usual" form for F is $(F_{\mu\nu})$, which can be obtained from your result by lowering one index.

a) Since $\mathbf{u} = \gamma\mathbf{v}$ and $d\mathbf{p}/dt = \frac{1}{\gamma} d\mathbf{p}/d\tau$,

$$\frac{d\mathbf{p}}{d\tau} = q(\gamma\mathbf{E} + \mathbf{u} \times \mathbf{B}).$$

b) If we integrate both sides over a volume V, then Poynting's theorem says that the rate of change of the energy in the volume V is given by $\int \mathbf{j} \cdot \mathbf{E}$ minus

$$\int_V \nabla \cdot \mathbf{S} d^3x = \int_{\partial V} \mathbf{S} \cdot d\mathbf{a},$$

which is the energy flux that leaves the volume, carried away by the electromagnetic fields. So we should interpet the second term in the theorem $\nabla \cdot \mathbf{S}$ as denoting the energy flux density per unit time stored in the fields, while the first term is the energy density carried by the moving charges (currents).

c) We need to be a bit careful here, since the current is localized at the particle. When we integrate Poynting's theorem, we need to keep in mind that there is secretly a delta function (actually, three delta functions) living in the current. Explicitly, $\mathbf{j}(\mathbf{r}) = \delta^3(\mathbf{r} - \mathbf{r}')q\mathbf{v}$, where $\mathbf{r}'$ is the location of the particle. Since the energy carried by the particles is the first term $\int \mathbf{j} \cdot \mathbf{E} d^3r = q\mathbf{v} \cdot \mathbf{E}$, we have

$$\begin{aligned} \frac{dU}{d\tau} &= \gamma q \mathbf{v} \cdot \mathbf{E} \\ &= q\mathbf{u} \cdot \mathbf{E}. \end{aligned}$$

d) So we now have an equation for $dp^\mu/d\tau$ for any $\mu \in \{0, 1, 2, 3\}$. To calculate $F^\mu{}_\nu$, just fix

$$(F^\mu{}_\nu) = \begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}.$$

To get the first row, it is easy to see that by setting the (0,0)-component 0 and the other three to the components of $\mathbf{E}$, we get $qFu = q\mathbf{E} \cdot \mathbf{u}$. Similarly, by using the classical Lorentz force law, we see that the first column must have its last three components as the components of $\mathbf{E}$ as well. To get the other 9 components, we need the relation $F'\mathbf{v} = \mathbf{v} \times \mathbf{B}$, where $F' = (F^i{}_j)$ is the 3×3 submatrix of $F = (F^\mu{}_\nu)$ that we're interested in. Writing this in index notation, we have

$$F^i{}_j v^j = \varepsilon_{ijk} v^j B^k,$$

where we have cleverly used the same index j on both sides of the equation. By staring at this for a while, you should be able to see that we can write

$$F^i{}_j = \varepsilon_{ijk} B^k,$$

this immediately shows that F is antisymmetric, so we only need to figure out three components, $F^1{}_2$, $F^2{}_3$, and $F^1{}_3$, which you can do by plugging in those indices to the above equation.

Exercise 2. Charge Conservation. Starting from Maxwell's equation

$$\partial_\nu F^{\mu\nu} = 4\pi j^\mu,$$

derive the equation of charge conservation

$$\partial_\mu j^\mu = 0,$$

and show that it corresponds to actual conservation of charge. Make sure to give a physical explanation of your result!
Hint: Is $\partial_\mu \partial_\nu$ a symmetric tensor? What is the contraction of an antisymmetric tensor with a symmetric one, i.e. if A is antisymmetric, and S is symmetric, then what do you know about $S_{\mu\nu} A^{\mu\nu}$?

Take ∂_μ on both sides of Maxwell's equation. Since $\partial_\mu \partial_\nu$ is symmetric under $\mu \leftrightarrow \nu$ and $F^{\mu\nu}$ is antisymmetric, we must have

$$\partial_\mu \partial_\nu F^{\mu\nu} = -\partial_\mu \partial_\nu F^{\mu\nu} = 0 = \partial_\mu j^\mu.$$

Writing this out in components, it says that

$$\partial_t j^0 = \nabla \cdot \mathbf{j}.$$

Since $j^0 = 4\pi\rho$, it says (upon integration) that the current flux leaving a volume is exactly the rate of change of the charge inside that volume, i.e.

$$\frac{dQ}{dt} = \int_{\partial V} \mathbf{j} \cdot d\mathbf{a},$$

where ∂V is the boundary of the volume V and Q is the charge contained in V. But this is exactly what charge conservation is: If the charge is changing in some region of space, then it must be going somewhere outside that region.