Final Review Session Problems

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Exercise 1. ***Griffiths 8.19.*** Find the lowest bound on the ground state of hydrogen using the variational principle and an exponential trial wavefunction,

$$\psi(\mathbf{r}) = Ae^{-br^2},$$

where A is determined by normalization and b is a variational parameter. Express your answer in eV.

Exercise 2. Consider a 1D harmonic oscillator of angular frequency ω_0 that is perturbed by a time-dependent potential $V(t) = bx\cos(\omega t)$, where x is the displacement of the oscillator from equilibrium. Evaluate $\langle x \rangle$ by time-dependent perturbation theory. Do not assume that the initial state is an eigenstate of the unperturbed system (it could be a linear combination of such eigenstates). Discuss the validity of the result for $\omega \approx \omega_0$ and ω far from ω_0.
Hints: You will need to use time-dependent perturbation theory as developed in the Week 9 Worksheet. This problem is too difficult to solve in full generality, so don't try to do that. Instead, try to consider special cases which elucidate all the physics but don't make the algebra too complicated. For example, you might want to first consider the case that $|\psi(0)\rangle$ is a single eigenstate of the unperturbed hamiltonian. Then, consider upgrading this to more complicated linear combinations, and conjecture what the physics will be in the most general case using the previous results.

Exercise 3. ***Griffiths 11.33*** The spontaneous emission of the 21-cm hyperfine line in hydrogen is a magnetic dipole transition with rate

$$\Gamma = \frac{\omega^3}{3\pi\varepsilon_0\hbar c^3}\left|\left\langle B\left|\frac{\boldsymbol{\mu}_e + \boldsymbol{\mu}_p}{c}\right|A\right\rangle\right|^2,$$

where

$$\boldsymbol{\mu}_e = -\frac{e}{m_e}\mathbf{S}_e$$
$$\boldsymbol{\mu}_p = \frac{5.59e}{2m_p}\mathbf{S}_p.$$

On the Week 11 Worksheet, you showed the triplet has slightly higher energy than the singlet. Calculate (approximately) the lifetime of this transition.

Exercise 4. ***Griffiths 9.18.*** When we turn on an external electric field, it should be possible to ionize the electron in an atom. A crude model for this is to suppose that a particle is in a very deep, one-dimensional finite square well from $x = -a$ to $x = a$.

a) What is the energy of the ground state, measured up from the bottom of the well? Assume that $V_0 \gg \hbar^2/ma^2$.

b) Introduce the perturbation $H' = -\alpha x$, where $\alpha \equiv eE_{\text{ext}}$. Assume that $\alpha a \ll \hbar^2/ma^2$, and sketch the total potential, noting that the electron can tunnel out in the direction of positive x.

c) Calculate

$$\gamma = \frac{1}{\hbar}\int |p(x)|\,\mathrm{d}x,$$

and estimate the time it would take for the particle to escape,

$$\tau = \frac{2x_1}{v}e^{2\gamma},$$

where x_1 is the distance the electron must travel to reach the tipping point of the potential and v is the speed of the electron.

d) Plug in some numbers, e.g. $V_0 = 20$ eV, $E_{\text{ext}} = 7\cdot 10^6$ V/m, $a = 10^{-10}$ m. Calculate τ, and compare it to the age of the universe.

Exercise 5. Semiclassical Approximations. Consider a single particle moving in one dimension. In this problem, you will analyze the semiclassical behavior of such a particle.

a) Make the ansatz

$$\psi = e^{\frac{i}{\hbar}\sigma},$$

and obtain a differential equation for σ from the time-independent Schrödinger equation.

b) Look for solutions to the equation from (a) of the form

$$\sigma = \sigma_0 + \frac{\hbar}{i}\sigma_1 + \left(\frac{\hbar}{i}\right)^2\sigma_2 + \cdots .$$

Proceeding as in perturbation theory, show that the zeroth order in $\hbar$ equation is

$$\frac{1}{2m}\sigma_0'^2 = E - V.$$

c) Solve the equation for σ_0, and give the condition of validity of your solution. Show that this condition can be written

$$\frac{d}{dx}\left(\frac{\lambda}{2\pi}\right) \ll 1,$$

where λ is the de Broglie wavelength of the particle.

d) By writing dp/dx in terms of the classical force F, show that the condition from (c) can be written

$$\frac{m\hbar F}{p^3} \ll 1.$$

Argue that this implies that the semiclassical approximation is not valid near turning points of the potential, where we have to solve Airy's equation and use the connecting solutions.

e) Obtain the first order equation, and show that it has the solution

$$\sigma_1 = -\frac{1}{2}\ln(p).$$

f) Thus, show that, to first order in $\hbar$, you recover the WKB approximation

$$\psi = \frac{C_1}{\sqrt{p}} e^{\frac{i}{\hbar}\int p dx} + \frac{C_2}{\sqrt{p}} e^{-\frac{i}{\hbar}\int p dx}.$$

g) Give a physical interpretation of the $\frac{1}{\sqrt{p}}$ factor which appears in ψ.

Exercise 6. Hard Sphere Scattering Continued. In Example 10.3, Griffiths computes for hard sphere scattering that if $ka \ll 1$, where a is the radius of the sphere, then

$$\sigma \approx 4\pi a^2.$$

Show that in the other limit, $ka \gg 1$,

$$\sigma \approx 2\pi a^2.$$

Exercise 7. Soft Sphere Scattering.

a) Consider scattering from a finite spherical well of depth V_0 and radius a. Show that the s-wave phase shift is

$$\delta_0 = -ka + \arctan\left[\frac{k}{k'}\tan(k'a)\right],$$

where k and k' are the wave numbers outside and inside the well, respectively.

b) Suppose that k is small so that we can ignore ka, and consider varying the depth of the well, i.e. varying k'. Show that whenever

$$k' \approx k'_n \equiv \frac{(2n+1)\pi}{2a},$$

the phase shift becomes

$$\delta_0 = \delta_b + \arctan\left(\frac{\Gamma/2}{E_0 - E}\right), \tag{1}$$

where we can ignore δ_b, the **background phase**, and $\Gamma/2 = \hbar^2 k_n/ma$ (k_n is the value of k when $k' = k'_n$). You should give a reasonable definition of E_0.

c) Assuming the form for δ_0 in Equation 1 and ignoring δ_b, show that in the limit $E_0 \sim E$

$$\sigma = \frac{4\pi}{k^2}\frac{(\Gamma/2)^2}{(E_0 - E)^2 + (\Gamma/2)^2},$$

so that Γ is the width of the bell curve which describes σ. This is called the **Breit-Wigner form** for the cross-section and describes the phenomenon of **resonance**.

d) If we start with V_0 too small to support a bound state, show that k'_1 corresponds to the well developing its first bound state (at zero energy, i.e. at $k = 0$). As the well is deepened further, another zero energy bound state is formed at k'_2.
Hint: It may be helpful at this point to recall your solution for (or solve, if you haven't before) Exercise 4.11 in Griffiths.