Week 5 Worksheet Solutions
More Electrostatics

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Exercise 1. a) The potential at a point $\mathbf{r}$ is defined as

$$V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell},$$

where $\mathcal{O}$ is some reference point. Explain why this is well-defined (i.e. unambiguous, up to the choice of $\mathcal{O}$).

b) An infinite plate carries a uniform charge density σ. Using your result from Exercise 2, find the potential everywhere.
Hint: Where would you put your reference point $\mathcal{O}$?

a) This is due to Stokes' theorem. Suppose we had two different paths that we'd like to take from $\mathcal{O}$ to $\mathbf{r}$. We need to check that taking the line integral over both paths will yield the same result. But indeed, the difference between taking one path over the other will give

$$\begin{aligned} V_1(\mathbf{r}) - V_2(\mathbf{r}) &= \oint \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} \\ &= \int \boldsymbol{\nabla} \times \mathbf{E} \cdot \mathrm{d}\mathbf{a} = 0 \end{aligned}$$

by Stokes' theorem and the fact that the curl of $\mathbf{E}$ vanishes.

b) The key here is that we can't place our reference point $\mathcal{O}$ at infinity, since our charge distribution also extends to infinity. Thus, let's place it on the charged plate; we may as well take the plate to be contained in the (x, y)-plane and thus put $\mathcal{O}$ at the origin. We now need to calculate

$$-\int_{\mathbf{0}}^{\mathbf{r}} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell},$$

where $\mathbf{E} = \dfrac{\sigma}{2\varepsilon_0}\hat{z}$. Thus, whatever path we take to get to the point $\mathbf{r}$, only the vertical distance, i.e. the z-component, of that path will matter. In particular, we may as well take a straight line along $\hat{z}$ and then some line that is parallel to the (x, y)-plane to get to $\mathbf{r}$. In that case, we need only $r_z = \mathbf{r} \cdot \hat{z}$ to evaluate our integral. We obtain

$$V(\mathbf{r}) = -\frac{\sigma}{2\varepsilon_0} r_z.$$

Exercise 2. Consider a uniformly charged spherical shell of radius R and charge Q.

a) Find the electric field everywhere using Gauss' law.

b) Find the potential everywhere by direct integration (without using Gauss' law).
Hint: Consider a single point a distance z from the center of the sphere, and use *cylindrical* symmetry.

c) Set up the integral to find the electric field at a point a distance z from the center of the sphere (without using Gauss' law). Consider separately the cases $z < R$ and $z > R$.

a) Gauss' law says

$$\int \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{Q}{\varepsilon_0},$$

while we know that $\mathbf{E}$ should point only radially outwards. Thus, for $r < R$, there is no enclosed charge; hence, $\mathbf{E}(\mathbf{r}) = \mathbf{0}$. On the other hand, for $r > R$, we have

$$E \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0},$$

so

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}.$$

b) Following the hint, we pick a point on the z-axis at a height z from the center of the sphere. Drawing a triangle for yourself and using the cosine law, you should be able to find that

$$\mathscr{r} = \sqrt{R^2 + z^2 - 2Rz\cos\theta'}.$$

Plugging this into the integral formula for V, we obtain

$$\frac{Q}{4\pi R^2} \int \frac{\delta(r' - R)\,\mathrm{d}\tau'}{\mathscr{r}},$$

where I am working in Gaussian units (equivalently leaving out the $\frac{1}{4\pi\varepsilon_0}$ until the end) and have plugged in our result from Exercise 1a. Now, $\mathrm{d}\tau' = r'^2\,\mathrm{d}\cos\theta'\,\mathrm{d}\varphi'$, so we can immediately do the integrals over r' and φ'. This gives us

$$\frac{Q}{2} \int_{-1}^{1} \frac{\mathrm{d}\cos\theta'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}.$$

The integral over $\cos\theta'$ is easy to take now. We obtain

$$-\frac{Q}{2Rz}\left(\sqrt{R^2 + z^2 - 2Rz} - \sqrt{R^2 + z^2 + 2Rz}\right) = \frac{Q}{2Rz}\left(\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right).$$

It is at this point where we have to be careful about whether $R > z$ or $R < z$, as it determines some minus signs after we take the square root (because we always want the *positive* square root). Thus,

$$V(z) = \begin{cases} \dfrac{Q}{R}, & z < R \\ \dfrac{Q}{z}, & z > R \end{cases}.$$

Writing this in terms of $\mathbf{r}$ and adding back in the $\frac{1}{4\pi\varepsilon_0}$, we have

$$V(\mathbf{r}) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 R}, & r < R \\ \dfrac{Q}{4\pi\varepsilon_0 r}, & z > R \end{cases}.$$

Checking that $-\boldsymbol{\nabla} V$ gives the right field confirms that this is the correct result.

c) Using (b), we have

$$\mathbf{E} = \int \frac{\rho(\mathbf{r}')\,\mathrm{d}\tau'}{\mathscr{r}^2}\hat{\mathscr{r}}\,.$$

Since $\mathbf{E}$ is a vector, the field at a point z above the sphere due to a single charge element on the sphere will not point in $\hat{r}$. However, the part of the field that is not pointing in $\hat{r}$ will be canceled by the same part of a field due to another charge element on the opposite side of the sphere (i.e. the charge element obtained by rotating by π around $\hat{z}$). So what should we do? The answer is that the field should only point in $\hat{r}$, so only along $\hat{z}$. It follows that we should take the *projection* of the field onto $\hat{z}$, i.e. for each charge element, we should consider the field due to it multiplied by $\cos\alpha$, where α is the angle between $\hat{\mathscr{r}}$ and $\hat{z}$. Draw the same triangle you use to compute $\mathscr{r}$, and split it into two right triangles: one with hypotenuse R and the other with hypotenuse $\mathscr{r}$. Then you can compute that $\cos\alpha = \frac{z - R\cos\theta'}{\mathscr{r}}$. This gives the result,

$$\mathbf{E} = \frac{Q}{2}\int \frac{(z - R\cos\theta')\,\mathrm{d}\cos\theta}{\mathscr{r}^3}\hat{r},$$

where I've again ignored the $1/4\pi\varepsilon_0$.