Week 14 Worksheet
Time-Dependent Phenomena

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Exercise 1. General Theory.

a) Consider the Schrödinger equation for time-dependent perturbation theory

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\Psi(t)\right\rangle = [H^0 + \lambda H'(t)]\left|\Psi(t)\right\rangle.$$

Suppose

$$\left|\Psi(t)\right\rangle = \sum_n c_n(t)e^{-iE_n t/\hbar}\left|n\right\rangle,$$

where $\left|n\right\rangle$ are the eigenstates of H^0. Derive the *exact* result

$$i\hbar\frac{\mathrm{d}c_n(t)}{\mathrm{d}t} = \lambda\sum_m \left\langle n\right| H'(t)\left|m\right\rangle e^{i\omega_{nm}t}c_m(t). \tag{1}$$

b) Now, set

$$c_n(t) = \sum_{n=0}^{\infty}\lambda^k c_n^{(k)}(t),$$

and plug it into your result from (b) to obtain the first order, i.e. $\mathcal{O}(\lambda)$, differential equation.

c) Obtain the second order equation.

Remark. Notice that your results for (b) and (c) are *exactly* the same as the two-level results when we begin in a single initial state!

Exercise 2. Sinusoidal Perturbations. In the case that

$$H' = Ke^{-i\omega t} + K^\dagger e^{i\omega t}$$

is sinusoidal and acts up until time t, solve the first order perturbation theory differential equation from Exercise 1(b).

Exercise 3. Spin Resonance. Consider a spin-1/2 particle in a static magnetic field $B_0\hat{z}$, so $H^0 = -\frac{1}{2}\hbar\gamma B_0\sigma_z$. The perturbation is due to a magnetic field B_1 rotating in the (x, y)-plane with angular velocity ω:

$$H'(t) = -\frac{1}{2}\hbar\gamma B_1[\sigma_x \cos(\omega t) + \sigma_y \sin(\omega t)].$$

a) Writing the eigenvectors of σ_z as

$$|+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

rewrite H' in the form given in Exercise 2, using these eigenvectors as a basis.

b) Suppose at $t = 0$ we have the initial state $|i\rangle = |+\rangle$. Find the first order probability for the spin to be down at time t. It is convenient to set $\omega_0 = \gamma B_0$ and $\omega_1 = \gamma B_1$.

c) It turns out that the exact Equation 1 can be solved for such a hamiltonian. The exact answer for (b) is

$$P(t) = \sin^2(\alpha t/2)\left(\frac{\omega_1}{\alpha}\right)^2,$$

where $\alpha^2 = (\omega_0 + \omega)^2 + {\omega_1}^2$; $\alpha/2$ is called the **Rabi flopping frequency**. Using this answer, what is the range of validity of the perturbation theory result, assuming we are not near resonance?

d) Suppose we are near resonance. What is the range of validity of the perturbation theory result? Give a physical explanation of your result.

e) **Challenge.** Solve Equation 1, and derive the formula for $P(t)$.