Week 14 Worksheet
Black Holes

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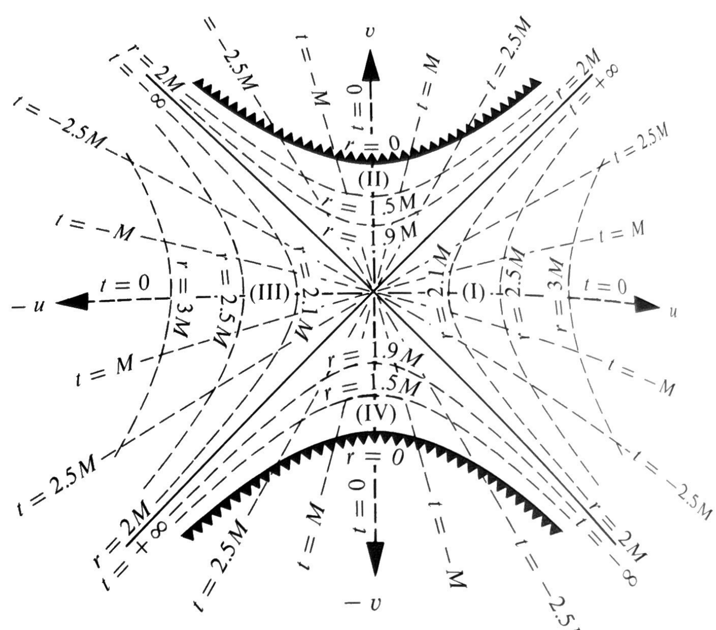
The Kruskal coordinates V, U are defined by

$$\begin{aligned}\left(\frac{r}{2M}-1\right)e^{r/2M} &= U^2 - V^2\\ \frac{t}{2M} = \ln\left(\frac{V+U}{U-V}\right) &= 2\tanh^{-1}(V/U).\end{aligned}$$

The Schwarzschild metric in Kruskal coordinates is

$$ds^2 = \frac{32M^3 e^{-r/2M}}{r}(-dV^2 + dU^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

The Kruskal diagram of a black hole is



Kruskal-Szekeres

Exercise.

a) Identify the worldlines of photons traveling radially in a Kruskal diagram.

b) Show that the worldline of a photon traveling nonradially makes an angle of less than 45 degrees with the vertical axis of the Kruskal diagram.

c) Use part (b) to show that particles with finite mass always move at an angle less than 45 degrees with the vertical axis.

d) If someone falls past the radius $r = 2M$, he or she will always hit the singularity at $r = 0$.

e) Once someone has fallen past $r = 2M$, he or she can't send messages to friends located at $r > 2M$ but can still receive messages.

f) Show that once someone falling in reaches the gravitational radius $r = 2M$, then *no matter what he or she does subsequently—no matter in what direction, how long, and how hard he or she blasts his or her rocket engines*—he or she will be killed by the singularity at $r = 0$ in a proper time of

$$\tau < 1.5 \cdot 10^{-5} \frac{M}{M_\odot} \text{ seconds,}$$

where $M_\odot = 2 \cdot 10^{30}$ kg is the mass of the Sun (and $G = 6.7 \cdot 10^{-11}$ m^3/kg s^2).
Hint: Note that

$$\left(\frac{dr}{d\tau}\right)^2 = e^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right).$$

For what kind of motion does this equation hold?