Final Review Problems

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Exercise 1. A charge is distributed with linear charge density λ over the circumference of a circle of radius R which lies in the (x, y)-plane with center at the origin. Find the potential $V(z)$ on the z-axis in the following cases.

a) λ is uniform.

b) $\lambda = C \sin(n\theta)$, where $n \in \mathbb{N}$, C is a constant, and θ is the polar angle.

c) $\lambda = C\theta$.

Exercise 2. A dielectric of arbitrary shape, volume V, and relative permittivity ε_r which is close to 1 (i.e. such that $\varepsilon_r - 1 \ll 1$) is brought into a uniform electric field $\mathbf{E}$. Outside the dielectric, $\varepsilon_r = 1$. Find the field at a large distance r from the dielectric.

Exercise 3. The center of a metal sphere of radius a lies on the flat boundary between two dielectric regions of permittivities ε_1 and ε_2. At a distance b from the center of the sphere in the region with permittivity ε_1 is placed a point charge q.

a) Find the potential of the sphere if it is insulated and uncharged.
Hint: If you find the total charge which is induced in the dielectrics, then you can use Exercise 2 from the midterm review problems to determine the potential of the sphere. You should get

$$V = \frac{q}{2\pi b(\varepsilon_1 + \varepsilon_2)}.$$

b) Find the charge induced on the sphere if it is grounded.

Exercise 4. ***Griffiths 5.13.*** Suppose you have two infinite, parallel line charges λ a distance d apart, which are moving at a constant speed v. How great would v have to be for the magnetic attraction to balance the electrical repulsion? Calculate the number, and comment on the result.

Exercise 5. ***Griffiths 5.16.*** Two long coaxial solenoids each carry a current I, but in opposite directions. The inner solenoid of radius a has n_1 turns per unit length, while the outer one of radius $b > a$ has n_2 turns per unit length. Find $\mathbf{B}$ in each of the three regions:

a) inside the inner solenoid,

b) between the solenoids, and

c) outside both solenoids.

Exercise 6. ***Griffiths 6.15.*** If $\mathbf{J}_f = \mathbf{0}$ everywhere, the curl of $\mathbf{H}$ vanishes, so we can express $\mathbf{H}$ as the gradient of a scalar potential W,

$$\mathbf{H} = -\boldsymbol{\nabla} W.$$

Thus,

$$\nabla^2 W = \boldsymbol{\nabla} \cdot \mathbf{M},$$

so W obeys Poisson's equation with $\boldsymbol{\nabla} \cdot \mathbf{M}$ as the "source." As an example, find the field inside a uniformly magnetized sphere by separation of variables.

Exercise 7. Find the acceleration a of a freely falling, circular, metal plate in a uniform magnetic field which is parallel to the surface of the ground. The plate is oriented with a diameter parallel to the direction of the magnetic field and the ground; its normal vector is perpendicular to the surface of the ground. The radius of the plate is R and its thickness is $d \ll R$. Its mass is m and the strength of the magnetic field is B.

Exercise 8. Explain how an AC generator works.

Exercise 9. ***Griffiths 9.20.***

a) Show that the skin depth in a poor conductor ($\sigma \ll \omega\varepsilon$) is $\frac{2}{\sigma}\sqrt{\varepsilon/\mu}$.

b) Show that the skin depth in a good conductor ($\sigma \gg \omega\varepsilon$) is $\lambda/2\pi$, where λ is the wavelength inside the conductor. Find the skin depth in nanometers for a typical metal ($\sigma \approx 10^7(\Omega\mathrm{m})^{-1}$) in the visible range $\omega \approx 10^{15}$ Hz, assuming $\varepsilon \approx \varepsilon_0$ and $\mu \approx \mu_0$. Why are metals opaque?

c) Show that in a good conductor the magnetic field lags the electric field by $\pi/4$ radians, and find the ratio of their amplitudes.

Exercise 10. ***Griffiths 9.39.*** For refraction of light from a medium n_2 into a medium with $n_1 < n_2$, Snell's law has a critical angle

$$\theta_c = \arcsin(n_2/n_1).$$

When the incident angle θ_I is greater than θ_c, there is no refracted ray: We get total internal reflection. However, although no energy penetrates the second medium, there is a nonzero field inside the second medium which is rapidly attenuated. We can use the results from class/the textbook with $k_T = \omega n_2/c$ and

$$\mathbf{k}_T = k_T(\sin\theta_T\hat{x} + \cos\theta_T\hat{z}).$$

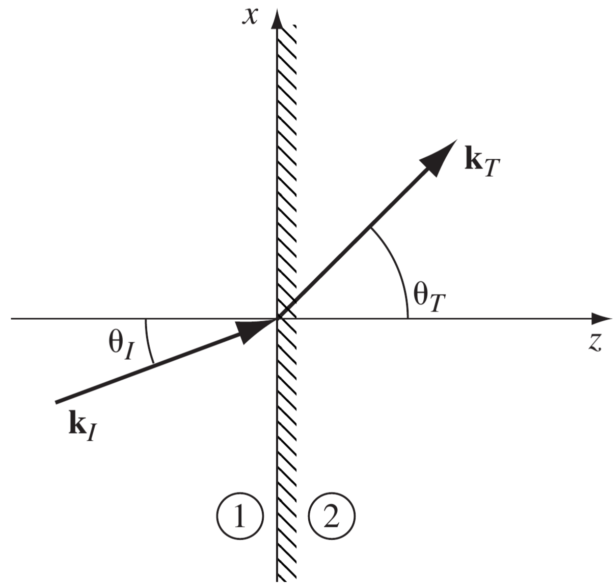
However, we should now take

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I > 1,$$

so that

$$\cos\theta_T = i\sqrt{\sin^2\theta_T - 1}$$

is imaginary.



a) Show that

$$\tilde{\mathbf{E}}_T(\mathbf{r},t) = \tilde{\mathbf{E}}_{0T} e^{-\kappa z} e^{-i(kx-\omega t)},$$

where

$$\kappa = \frac{\omega}{c}\sqrt{(n_1 \sin\theta_I)^2 - n_2{}^2} \quad \text{and} \quad k = \frac{\omega n_1}{c}\sin\theta_I.$$

Notice that this is a wave propagating in the x direction and attenuated in the z direction.

b) Noting that

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I}$$

is now imaginary, use Fresnel's equations

$$\begin{aligned}\tilde{E}_{0R} &= \frac{\alpha-\beta}{\alpha+\beta}\tilde{E}_{0I}\\ \tilde{E}_{0T} &= \frac{2}{\alpha+\beta}\tilde{E}_{0I}\end{aligned}$$

to calculate the reflection coefficient for polarization parallel to the plane of incidence.

c) Do the same for polarization perpendicular to the plane of incidence.

d) In the case of polarization perpendicular to the plane of incidence, show that the real evanescent fields are

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= E_0 e^{-\kappa z}\cos(kx-\omega t)\hat{y}\\ \mathbf{B}(\mathbf{r},t) &= \frac{E_0}{\omega}e^{-\kappa z}\left[\kappa\sin(kx-\omega t)\hat{x} + k\cos(kx-\omega t)\hat{z}\right].\end{aligned}$$

e) Check that the fields in (d) satisfy Maxwell's equations.

f) For the fields in (d), construct the Poynting vector, and show that, on average, no energy is transmitted in the z direction.

Exercise 11. The Dirac Monopole. Consider a half-infinite string of magnetic dipoles, equivalently, a half-infinite solenoid, denoted L.

a) Show that the vector potential outside the string is

$$\mathbf{A}(\mathbf{r}) = -\frac{g}{4\pi}\int_L \mathrm{d}\boldsymbol{\ell} \times \boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}}\right),$$

where g is a constant.

b) Show that the curl of $\mathbf{A}$ is directed radially outward from the end of the string, varies inversely with distance squared from the end of the string, and has total outward flux g.

Remark. The result of (b) shows that the magnetic field outside of the solenoid is that given by a magnetic monopole of exactly charge g. On the other hand, it can be shown (try for yourself!) that changing the position of the string changes $\mathbf{A}$ by a gauge transformation. Explicitly, if we have two different strings L, L', then the integral taken along the closed path $C = L - L'$ will give

$$\mathbf{A}_{L'}(\mathbf{r}) = \mathbf{A}_L(\mathbf{r}) + \frac{g}{4\pi}\boldsymbol{\nabla}\Omega_C(\mathbf{r}),$$

where Ω_C is the solid angle subtended by the contour C at the observation point $\mathbf{r}$. This means that the string itself is not observable, which is consistent with the fact that physical effects due to the monopole should not depend on the theoretical artifice used to create it (the string). In 1930, Dirac famously showed that the existence of magnetic monopoles *implies* the quantization of electric (and magnetic) charge! This is why people have been interested in magnetic monopoles to this day.