Final Review Session Solutions

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December 14, 2024

Exercise 1. The integral form of the Schrödinger equation reads

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int g(\mathbf{r} - \mathbf{r}')V(\mathbf{r}')\psi(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}',$$

where

$$g(\mathbf{r}) = -\frac{m}{2\pi\hbar^2} \cdot \frac{e^{ikr}}{r}$$

is the Green's function for the Schrödinger equation.

a) Use the method of successive approximations to write $\psi(\mathbf{r})$ as a series in the incident wavefunction $\psi_0(\mathbf{r})$.

b) Truncate the Born series you obtain after the second term to get the first Born approximation. Assuming the potential is localized near $\mathbf{r}' = 0$, we can write

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{e^{ikr}}{r} e^{-i\mathbf{k}\cdot\mathbf{r}'}.$$

Using this and the definition of $f(\theta)$,

$$\psi(\mathbf{r}) = Ae^{ikz} + f(\theta)\frac{e^{ikr}}{r},$$

determine $f(\theta)$.

c) In Griffiths, we find that for a potential $V(r) = V_0/r$, $f_{\text{point}}(\theta) = -\frac{2mV_0}{\hbar^2 q^2}$, where $\mathbf{q} = \mathbf{k}' - \mathbf{k}$. If $V(\mathbf{r}) = -e^2 Z/r$ for an electron scattering off a point charge of charge Ze, how would $f(\theta)$ change if instead the electron scatters off a spherical nucleus of radius a, charge Ze, and uniform charge density? Your answer should be of the form

$$f(\theta) = f_{\text{point}}(\theta) \cdot F(q),$$

where $F(q)$ is the **form factor** of the nucleus.

d) If you haven't done so already, calculate $F(q)$ explicitly.

e) From scattering high-energy electrons at nuclei, the actual form factor is measured to be

$$F(q) = \frac{Ze}{(1+q^2 {a_N}^2)^2},$$

where $a_N \approx 0.26$ fm. If the inverse Fourier transform of $\frac{1}{(1+x^2)^2}$ is $e^{-|x|}$, what does that tell you about the size and charge density of the proton?

a) The idea is to plug in the formula for ψ for the $\psi(\mathbf{r}')$ on the right side. Thus, we obtain

$$\begin{aligned}\psi(\mathbf{r}) =& \psi_0(\mathbf{r}) + \int g(\mathbf{r}-\mathbf{r}')V(\mathbf{r}')\psi_0(\mathbf{r}')\,\mathrm{d}^3r' + \\ &+ \iint g(\mathbf{r}-\mathbf{r}')g(\mathbf{r}'-\mathbf{r}'')V(\mathbf{r}')V(\mathbf{r}'')\psi_0(\mathbf{r}'')\,\mathrm{d}^3r'\,\mathrm{d}^3r'' + \cdots .\end{aligned}$$

b) The Born approximation is then

$$\psi(\mathbf{r}) \approx \psi_0(\mathbf{r}) + \int g(\mathbf{r}-\mathbf{r}')V(\mathbf{r}')\psi_0(\mathbf{r}')\,\mathrm{d}^3r'.$$

Plugging in the suggested approximation for the Green's function, we get

$$\psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ikr}}{r} e^{-i\mathbf{k}\cdot\mathbf{r}'} V(\mathbf{r}')\psi_0(\mathbf{r}')\,\mathrm{d}^3r'.$$

Now, suppose the incident wavefunction is a plane wave e^{ikz} along the $\hat{z}$ direction. It follows then that

$$\psi(\mathbf{r}) \approx e^{ikz} - \frac{m}{2\pi\hbar^2}\int \frac{e^{ikr}}{r} e^{-i\mathbf{k}\cdot\mathbf{r}'} V(\mathbf{r}') e^{i\mathbf{k}'\cdot\mathbf{r}'}\,\mathrm{d}^3r',$$

where we set $\mathbf{k}' = k\hat{z}$. Thus,

$$\psi(\mathbf{r}) = e^{ikz} + \frac{e^{ikr}}{r}\cdot\left(-\frac{m}{2\pi\hbar^2}\int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}'} V(\mathbf{r}')\,\mathrm{d}^3r'\right),$$

so

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}'} V(\mathbf{r}')\,\mathrm{d}^3r'.$$

c) Uniform charge density implies $\rho(r) = \frac{Ze}{\frac{4}{3}\pi a^3}$ for $r \in (0, a)$. Thus,

$$V(\mathbf{r}) = -e\int \frac{1}{|\mathbf{r}-\mathbf{r}''|}\rho(\mathbf{r}'')\,\mathrm{d}^3r''.$$

Plugging this in to the expression for $f(\theta)$ from part (b), we get

$$\begin{aligned}f(\theta) =& -\frac{m}{2\pi\hbar^2}\int e^{i\mathbf{q}\cdot\mathbf{r}'} V(\mathbf{r}')\,\mathrm{d}^3r' \\ =& \frac{me}{2\pi\hbar^2}\iint e^{i\mathbf{q}\cdot\mathbf{r}'}\frac{\rho(\mathbf{r}'')}{|\mathbf{r}'-\mathbf{r}''|}\,\mathrm{d}^3r'\,\mathrm{d}^3r''.\end{aligned}$$

Make the substitution $\mathbf{u} = \mathbf{r}' - \mathbf{r}''$, so

$$\begin{aligned} f(\theta) =& \frac{me}{2\pi\hbar^2} \iint e^{i\mathbf{q}\cdot\mathbf{u}} e^{i\mathbf{q}\cdot\mathbf{r}''} \frac{\rho(\mathbf{r}'')}{|\mathbf{u}|} \, \mathrm{d}^3u \, \mathrm{d}^3r'' \\ =& f_{\text{point}}(\theta) \frac{1}{2e} \int e^{i\mathbf{q}\cdot\mathbf{r}''} \rho(\mathbf{r}'') \, \mathrm{d}^3r'' \implies \\ \implies F(q) =& \frac{1}{2eZ} \int e^{i\mathbf{q}\cdot\mathbf{r}''} \rho(\mathbf{r}'') \, \mathrm{d}^3r''. \end{aligned}$$

The integral in the second equality goes as follows. We have

$$\int \frac{e^{i\mathbf{q}\cdot\mathbf{u}}}{u} \, \mathrm{d}^3u = \int e^{iqu\cos\theta} u \cdot 2\pi \, \mathrm{d}(\cos\theta) \, \mathrm{d}u,$$

where we can assume that the angle between $\mathbf{q}$ and $\mathbf{u}$ is θ since we're integrating over all θ anyway. Now, do the θ integral to get

$$\int \frac{2\pi}{iq}(e^{iqu} - e^{-iqu}) \, \mathrm{d}u = \frac{4\pi}{q} \int_0^\infty \sin(qu) \, \mathrm{d}u = \frac{4\pi}{q^2},$$

where the final equality follows by regulating the undefined integral with an exponential e^{-au} and then setting $a \to 0$ in the answer.

d) We plug in the form for ρ above and calculate.

$$\begin{aligned} F(q) =& \frac{3}{4\pi a^3} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \int_0^a r^2 \sin\theta e^{iqr\cos\theta} \, \mathrm{d}r \\ =& \frac{3}{2a^3} \int_0^a r^2 \, \mathrm{d}r \frac{1}{iqr}(e^{iqr} - e^{-iqr}) \\ =& \frac{3}{2a^3} \int_0^a \mathrm{d}r \frac{2r}{q} \sin(qr) \\ =& \frac{3}{a^3q^3} \left(\sin(qa) - aq\cos(qa)\right). \end{aligned}$$

e) Note that there were a couple typos in the statement to this problem. It should read that $F(q)$ is measured to be

$$F(q) = \frac{1}{(1 + q^2/a_N{}^2)^2},$$

so that we have

$$F(q) = \frac{1}{2eZ} \int e^{i\mathbf{q}\cdot\mathbf{r}} \rho(\mathbf{r}) \, \mathrm{d}^3r.$$

Up to some numerical constants, this is the Fourier transform of the function $\frac{1}{eZ}\rho(\mathbf{r})$; hence, ρ is given by the inverse Fourier transform of

$$\frac{Ze}{(1 + q^2/a_n{}^2)^2},$$

so

$$\rho(\mathbf{r}) = Zee^{-r/a_N}.$$

Thus, we find that the charge density of the nucleus has an *exponential* distribution! The proton is then "smeared out" over all space, but it has a $1/e$ drop off after $r = a_N$. So, we can consider the "size" of the proton to be $\approx a_N$.

Exercise 2. Consider a 1D harmonic oscillator of angular frequency ω_0 that is perturbed by a time-dependent potential $V(t) = bx\cos(\omega t)$, where x is the displacement of the oscillator from equilibrium. Evaluate $\langle x\rangle$ by time-dependent perturbation theory. Discuss the validity of the result for $\omega \approx \omega_0$ and ω far from ω_0.

First, let's figure out what $H'_{nm} = \langle n|V(t)|m\rangle$ is. This will be given by

$$H'_{nm} = b\cos(\omega t)\sqrt{\frac{\hbar}{2m\omega}}\left(\delta_{m,n+1}\sqrt{n+1} + \delta_{m,n-1}\sqrt{n}\right),$$

where we use the form for x in terms of raising and lowering operators. Now, we know that in first order time-dependent perturbation theory,

$$\frac{\mathrm{d}c_n}{\mathrm{d}t} = -\frac{i}{\hbar}\sum H'_{nm}e^{i\omega_{nm}t}c_m,$$

where the c_i are the coefficients of the wavefunction at time $t = 0$, i.e.

$$|\psi(0)\rangle = \sum_m c_m|m\rangle.$$

Plugging in our result for H'_{nm}, we get

$$\begin{aligned}\frac{\mathrm{d}c_n}{\mathrm{d}t} &= -\frac{ib\cos(\omega t)}{\sqrt{2m\hbar\omega_0}}\left(c_{n+1}e^{-i\omega_0 t}\sqrt{n+1} + c_{n-1}e^{i\omega_0 t}\sqrt{n}\right)\\ &= -\frac{ib}{\sqrt{2m\omega_0\hbar}}\left[\left(e^{i(\omega-\omega_0)t} + e^{-i(\omega+\omega_0)t}\right)c_{n+1}\sqrt{n+1} + \left(e^{i(\omega+\omega_0)t} + e^{-i(\omega-\omega_0)t}\right)c_{n-1}\sqrt{n}\right].\end{aligned}$$

Integrating this from 0 to t, we get

$$\begin{aligned}c_n(t) = -\frac{b}{\sqrt{2m\hbar\omega_0}}\Bigg[&\left(\frac{e^{i(\omega-\omega_0)t}}{\omega-\omega_0} - \frac{e^{-i(\omega+\omega_0)t}}{\omega+\omega_0} - \frac{2\omega_0}{\omega^2-{\omega_0}^2}\right)c_{n+1}\sqrt{n+1}+\\ &+\left(\frac{e^{i(\omega+\omega_0)t}}{\omega+\omega_0} - \frac{e^{-i(\omega-\omega_0)t}}{\omega-\omega_0} + \frac{2\omega_0}{\omega^2-{\omega_0}^2}\right)c_{n-1}\sqrt{n}\Bigg].\end{aligned}$$

This is the most general expression for the time-dependence of the coefficients, from which we could then derive the most general expression for the time-dependence of $\langle x\rangle$. Instead, let's consider some more simple cases. The simplest case is $|\psi(0)\rangle = |n\rangle$. Then only the c_{n-1} and c_{n+1} coefficients acquire a time dependence. However, let's consider the more general case of the form

$$|\psi(0)\rangle = \alpha|n\rangle + \beta|n+1\rangle.$$

Note that this subsumes the simpler case where $|\psi(0)\rangle$ is an eigenstate of the harmonic oscillator by setting $\beta = 0$. Let's figure out what will happen here. We then have

$$\begin{aligned}
c_{n-1}(t) &= -\frac{b}{\sqrt{2m\hbar\omega_0}}\left(\frac{e^{i(\omega-\omega_0)t}}{\omega-\omega_0} - \frac{e^{-i(\omega+\omega_0)t}}{\omega+\omega_0} - \frac{2\omega_0}{\omega^2-{\omega_0}^2}\right)\alpha\sqrt{n}\\
c_{n}(t) &= -\frac{b}{\sqrt{2m\hbar\omega_0}}\left(\frac{e^{i(\omega-\omega_0)t}}{\omega-\omega_0} - \frac{e^{-i(\omega+\omega_0)t}}{\omega+\omega_0} - \frac{2\omega_0}{\omega^2-{\omega_0}^2}\right)\beta\sqrt{n+1}\\
c_{n+1}(t) &= -\frac{b}{\sqrt{2m\hbar\omega_0}}\left(\frac{e^{i(\omega+\omega_0)t}}{\omega+\omega_0} - \frac{e^{-i(\omega-\omega_0)t}}{\omega-\omega_0} + \frac{2\omega_0}{\omega^2-{\omega_0}^2}\right)\alpha\sqrt{n+1}\\
c_{n+2}(t) &= -\frac{b}{\sqrt{2m\hbar\omega_0}}\left(\frac{e^{i(\omega+\omega_0)t}}{\omega+\omega_0} - \frac{e^{-i(\omega-\omega_0)t}}{\omega-\omega_0} + \frac{2\omega_0}{\omega^2-{\omega_0}^2}\right)\beta\sqrt{n+2},
\end{aligned}$$

with all other $c_i(t) = 0$. This means that

$$|\psi(t)\rangle = c_{n-1}(t)\,|n-1\rangle + c_n(t)|n\rangle + c_{n+1}(t)|n+1\rangle + c_{n+2}(t)\,|n+2\rangle\,.$$

Now, let's try to calculate $\langle x(t)\rangle$. For a general $|\psi\rangle = \sum c_n\,|n\rangle$, we have

$$\langle x\rangle = \sum_n \sqrt{\frac{\hbar}{2m\omega}}c_n^*\left(\sqrt{n+1}c_{n+1} + \sqrt{n}c_{n-1}\right).$$

Since our $|\psi\rangle$ only has 4 nonzero c_i,

$$\begin{aligned}
\langle x\rangle =&\sqrt{\frac{\hbar}{2m\omega}}\Big[c_n^*\left(\sqrt{n+1}c_{n+1} + \sqrt{n}c_{n-1}\right) + c_{n-1}^*\sqrt{n}c_n + c_{n+1}^*\left(\sqrt{n+2}c_{n+2} + \sqrt{n+1}c_n\right) + c_{n+2}^*\sqrt{n+2}c_{n+1}\\
=&2\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\mathrm{Re}\,[c_n^*c_{n+1}] + \sqrt{n}\mathrm{Re}\,[c_n^*c_{n-1}] + \sqrt{n+2}\mathrm{Re}\,[c_{n+2}^*c_{n+1}]\right).
\end{aligned}$$

So we only need to calculate $c_n^*c_{n+1}$, $c_n^*c_{n-1}$, and $c_{n+2}^*c_{n+1}$. The first is given by (after a lot of algebra)

$$\begin{aligned}
c_n^*c_{n+1} = \frac{\alpha\beta^*b^2}{2m\hbar\omega_0}(n+1)\Bigg(&\frac{2e^{2i\omega_0 t}}{\omega^2-{\omega_0}^2} - \frac{e^{-2i(\omega-\omega_0)t}}{(\omega-\omega_0)^2} - \frac{e^{2i(\omega+\omega_0)t}}{(\omega+\omega_0)^2} - \frac{4\omega_0 e^{i(\omega+\omega_0)t}}{(\omega+\omega_0)(\omega^2-{\omega_0}^2)}+\\
&+\frac{4\omega_0 e^{-i(\omega-\omega_0)t}}{(\omega-\omega_0)(\omega^2-{\omega_0}^2)} - \frac{4{\omega_0}^2}{(\omega^2-{\omega_0}^2)^2}\Bigg)
\end{aligned}$$

The real part of this is then

$$\begin{aligned}
\mathrm{Re}\,[c_n^*c_{n+1}] = \frac{\mathrm{Re}\,[\alpha\beta^*]b^2}{2m\hbar\omega_0}(n+1)\Bigg(&\frac{2\cos(2\omega t)}{\omega^2-{\omega_0}^2} - \frac{\cos[2(\omega-\omega_0)t]}{(\omega-\omega_0)^2} - \frac{\cos[2(\omega+\omega_0)t]}{(\omega+\omega_0)^2}-\\
&-\frac{4\omega_0\cos[(\omega+\omega_0)t]}{(\omega+\omega_0)(\omega^2-{\omega_0}^2)} + \frac{4\omega_0\cos[(\omega-\omega_0)t]}{(\omega-\omega_0)(\omega^2-{\omega_0}^2)} - \frac{4{\omega_0}^2}{(\omega^2-{\omega_0}^2)^2}\Bigg)
\end{aligned}$$

Similarly, the second is

$$c_n^* c_{n-1} = \frac{b^2\beta^*\alpha}{2m\hbar\omega_0}\sqrt{(n+1)n}\Bigg(\frac{1}{(\omega-\omega_0)^2}+\frac{4{\omega_0}^2}{(\omega^2-{\omega_0}^2)^2}+\frac{1}{(\omega+\omega_0)^2}-\frac{2\cos(2\omega t)}{\omega^2-\omega_0^2}- \\ -\frac{4\omega_0\cos[(\omega-\omega_0)t]}{(\omega-\omega_0)(\omega^2-{\omega_0}^2)}+\frac{4\omega_0\cos[(\omega+\omega_0)t]}{(\omega+\omega_0)(\omega^2-{\omega_0}^2)}\Bigg).$$

Lastly, we have

$$c_{n+2}^* c_{n+1} = \frac{b^2\beta^*\alpha}{2m\hbar\omega_0}\sqrt{(n+1)(n+2)}\Bigg(\frac{1}{(\omega+\omega_0)^2}+\frac{1}{(\omega-\omega_0)^2}-\frac{2\cos(2\omega t)}{\omega^2-{\omega_0}^2}+\frac{4\omega_0\cos[(\omega+\omega_0)t]}{(\omega+\omega_0)(\omega^2-{\omega_0}^2)}- \\ -\frac{4\omega_0\cos[(\omega-\omega_0)t]}{(\omega-\omega_0)(\omega^2-{\omega_0}^2)}+\frac{4{\omega_0}^2}{(\omega^2-{\omega_0}^2)^2}\Bigg)$$

Note that the answer we have obtained is for a system in a mixing of two eigenmodes of the harmonic oscillator. This means that the initial system had two associated frequencies, $n\omega$ and $(n+1)\omega$, so it's not surprising that we have obtained an answer which causes the mixing to extend to nearby states with (at first glance) no patterns. However, we can notice that when we are near resonance, $\omega \sim \omega_0$, then the dominant terms in $\langle x\rangle$ are given by several frequencies: 2ω, $\omega+\omega_0$, and $2(\omega+\omega_0)$.

On the otherhand, if we instead suppose the initial state was *in a single energy eiegenstate*, $|\psi\rangle = |n\rangle$, then c_{n-1} and c_{n+1} are as in the previous case (with α set equal to 1), while $c_n = 1$ is constant for all time. This follows from the general case above by setting $\beta = 0$. Thus, we see that the dominant frequency near resonance is $\omega+\omega_0$, and the position of equilibrium is shifted from $x = 0$. On the other hand, when ω is far from ω_0, we have two frequencies $\omega-\omega_0$ and $\omega+\omega_0$, which correspond to the eigenmodes of the classical driven harmonic oscillator! So this seems commensurate with the classical equation of motion. Note that this makes sense, because expectation values are expected to obey classical equations of motion.

Lastly, we could consider the more general case,

$$|\psi(0)\rangle = \alpha|n-1\rangle + \beta|n\rangle + \gamma|n+1\rangle,$$

but I don't think we're going to get any more physics out of it. I expect that for ω near ω_0, we should end up with the same conclusions as in the analysis above for $\gamma = 0$, and similarly for ω far from ω_0.

Exercise 3. *Griffiths 11.33* The spontaneous emission of the 21-cm hyperfine line in hydrogen is a magnetic dipole transition with rate

$$\Gamma = \frac{\omega^3}{3\pi\varepsilon_0\hbar c^3}\left|\left\langle B\left|\frac{\boldsymbol{\mu}_e+\boldsymbol{\mu}_p}{c}\right|A\right\rangle\right|^2,$$

where

$$\begin{aligned}\boldsymbol{\mu}_e &= -\frac{e}{m_e}\mathbf{S}_e\\ \boldsymbol{\mu}_p &= \frac{5.59e}{2m_p}\mathbf{S}_p.\end{aligned}$$

On midterm 1, you showed the triplet has slightly higher energy than the singlet. Calculate (approximately) the lifetime of this transition.

Call the triplet state(s) $|1\rangle$ and the singlet state $|0\rangle$. We then have (remembering that $m_p \gg m_e$)

$$\begin{aligned}\Gamma \approx& \frac{\omega^3 e^2}{3\pi\varepsilon_0\hbar c^5}\left|\left\langle 0\left|\frac{e^2}{{m_e}^2}\mathbf{S}_e\right|1\right\rangle\right|^2\\ =&\frac{\omega^3 e^4}{3\pi\varepsilon_0\hbar c^5 {m_e}^2}\left|\langle 0|\mathbf{S}_e|1\rangle\right|^2.\end{aligned}$$

Evaluating the matrix element, we get

$$\langle 0|\mathbf{S}_e|1\rangle = \frac{\hbar}{2\sqrt{2}}\Big[\big(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\big)\,\boldsymbol{\sigma}_e|\uparrow\uparrow\rangle\Big] = -\frac{\hbar}{2\sqrt{2}}(\hat{x} + i\,\hat{y}),$$

which you can obtain by using any triplet. Note that the first arrow is the electron and the second the proton. Thus,

$$\left|\langle 0|\mathbf{S}_e|1\rangle\right|^2 = \frac{\hbar^2}{4}.$$

Plugging this in to Γ, we get

$$\begin{aligned}\Gamma =& \frac{\omega^3 e^2}{3\pi\varepsilon_0\hbar c^5 {m_e}^2}\cdot\frac{\hbar^2}{4}\\ =&\alpha\frac{4\hbar^2\omega^3}{12c^4{m_e}^2}.\end{aligned}$$

Since $\omega = 2\pi c/\lambda$, we can plug in and evaluate ($\lambda = 21$ cm). We find

$$\Gamma \approx 10^{-14}\ \mathrm{s}^{-1},$$

so $T \approx 10^{14}$ s or 10^7 years.

Exercise 4. Consider a dynamical variable ξ that can take only two values, 1 or -1 (for example, σ_z is such an operator for a spin 1/2 particle). Denote the corresponding eigenvectors as $|+\rangle$ and $|-\rangle$. Now, consider the following states.

a) The one-parameter family of pure states

$$|\theta\rangle = \sqrt{\frac{1}{2}}(|+\rangle + e^{i\theta}|-\rangle)$$

for any real θ.

b) The nonpure state

$$\rho = \frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|).$$

Show that $\langle\xi\rangle = 0$ in all of these states. What, if any, are the physical differences between these various states, and how could they be measured?

We compute

$$\begin{aligned}\langle\xi\rangle =& \frac{1}{2}(\langle+|\xi|+\rangle + \langle-|\xi|-\rangle)\\ =&0.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle\xi\rangle =& \operatorname{tr}(\rho\xi)\\ =&\frac{1}{2}(\langle+|\xi|+\rangle + \langle-|\xi|-\rangle)\\ =&0.\end{aligned}$$

How could we measure the differences between these states? One option, in the case of $|\theta\rangle$, is to consider a particle in a superposition of $|0\rangle$ and $|\pi\rangle$. Then the particle would be an eigenstate of ξ, namely $|+\rangle$. Similarly, we could do the same thing with $|\theta\rangle$ and $|\theta+\pi\rangle$ for any θ. Obviously, if we didn't choose a π-shifted (moduli 2π) state to superpose with, we wouldn't get anything interesting. Thus, one difference between the states of type (a) is that they have a unique (up to a multiple of 2π) state with which you can superpose to get an eigenstate of ξ. Thus, one way to detect the difference between the states is to initiate various particles in superpositions of them, and measure $\langle\xi\rangle$.

On the other hand, the density matrix ρ is in a superposition of the eigenstates of ξ, and *there's no way to change that* using the states given above. However, we could detect if we're in ρ, versus one of the θ states by the procedure above. Indeed, the density matrix associated to the state $|\theta\rangle$ is given by

$$\rho_\theta = \frac{1}{2}\left(2\rho + e^{i\theta}|-\rangle\langle+| + e^{-i\theta}|+\rangle\langle-|\right).$$

Using the superposition trick from the last paragraph, we could detect θ by superposing with a π-shifted θ state. However, if we started with just ρ, we would find that there is no such state, so that $\langle\xi\rangle = 0$ for any chosen superposition.

Exercise 5. In the homework, you showed that the most general density matrix for a spin 1/2 particle is $\rho = \frac{1}{2}(1+\mathbf{a}\cdot\boldsymbol{\sigma})$, where $\mathbf{a}$ is some 3-vector. If the system has a magnetic moment $\boldsymbol{\mu} = \frac{1}{2}\gamma\hbar\boldsymbol{\sigma}$ and is in a constant magnetic field $\mathbf{B}$, calculate $\rho(t)$. Describe the result geometrically in terms of the variation of the vector $\mathbf{a}$.

The problem tells us that the hamiltonian is given by $H = -\frac{1}{2}\gamma\hbar\boldsymbol{\sigma}\cdot\mathbf{B}$. We derived the equation of motion for ρ on the homework,

$$i\hbar\frac{\mathrm{d}\rho}{\mathrm{d}t} = [H,\rho].$$

Thus,

$$\begin{aligned}i\hbar\frac{\mathrm{d}\rho}{\mathrm{d}t} =& -\frac{1}{4}\gamma\hbar\mathbf{B}\cdot[\boldsymbol{\sigma}, 1+\mathbf{a}\cdot\boldsymbol{\sigma}]\\ =& -\frac{1}{4}\gamma\hbar\sum_{i,j}B_i[\sigma_i, a_j\sigma_j]\\ =& -\frac{i}{2}\gamma\hbar\sum_{i,j,k}B_i a_j\varepsilon_{ijk}\sigma_k.\end{aligned}$$

Thus,

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\gamma \sum_{i,j,k} B_i a_j \varepsilon_{ijk} \sigma_k \Longrightarrow$$

$$\Longrightarrow \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} \cdot \boldsymbol{\sigma} = \gamma \mathbf{a} \times \mathbf{B} \cdot \boldsymbol{\sigma}.$$

Notice that if we instead consider the equation

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = \gamma \mathbf{a} \times \mathbf{B},$$

its solutions will be solutions of the above equation. Indeed, these are the only solutions, since the differential equation above is the same as

$$\left(\dot{\mathbf{a}} - \frac{1}{2} \gamma \mathbf{a} \times \mathbf{B} \right) \cdot \boldsymbol{\sigma} = 0,$$

and the Pauli matrices σ_i form an orthonormal basis of the space of traceless 2×2 matrices. It follows that the only solution is

$$\dot{\mathbf{a}} = \gamma \mathbf{a} \times \mathbf{B},$$

Now, notice further that this equation has exactly the same form as the equation of motion for a charged particle in a magnetic field:

$$q\mathbf{v} \times \mathbf{B} = m\dot{\mathbf{v}},$$

where $\mathbf{a}$ is now playing the role of velocity. Thus, solutions for $\mathbf{a}$ will be (the velocities of) solutions to the problem of a charged particle in a magnetic field. This is standard. If you haven't seen this before, here is a quick overview. Set $\mathbf{B} = B\hat{z}$. Thus,

$$\mathbf{a} \times \mathbf{B} = (a_y B, -a_x B, 0).$$

We then have three equations

$$\begin{aligned} \dot{a}_x =& \gamma B a_y \\ \dot{a}_y =& -\gamma B a_x \\ \dot{a}_z =& 0. \end{aligned}$$

We are thus down to a two-dimensional problem. Define $\omega = \gamma B$; this is the cyclotron frequency. Define the complex variable $\xi = a_x + i a_y$. We then get

$$\dot{\xi} = -i\omega\xi.$$

Its solution is easy:

$$\xi(t) = Ae^{-i\omega t}.$$

Thus,

$$\mathbf{a}(t) = A(\cos(\omega t), -\sin(\omega t), a_z),$$

where a_z is a fixed constant for all time and A is some normalization. So $\mathbf{a}(t)$ rotates in the xy-plane at frequency $\omega = \gamma B$. Returning back to the original problem (so $\mathbf{B}$ is not necessarily along $\hat{z}$), then $\mathbf{a}(t)$ will rotate in the plane perpendicular to $\mathbf{B}$ at angular frequency γB and be constant in the direction of $\mathbf{B}$. We're done.