Week 14 Worksheet Solutions Time-Dependent Phenomena

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Exercise 1. General Theory.

a) Consider the Schrödinger equation for time-dependent perturbation theory

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\Psi(t)\rangle = [H^0 + \lambda H'(t)] |\Psi(t)\rangle.$$

Suppose

$$|\Psi(t)\rangle = \sum_{n} c_n(t) e^{-iE_n t/\hbar} |n\rangle,$$

where $|n\rangle$ are the eigenstates of H^0 . Derive the *exact* result

$$i\hbar \frac{\mathrm{d}c_n(t)}{\mathrm{d}t} = \lambda \sum_m \langle n | H'(t) | m \rangle e^{i\omega_{nm}t} c_m(t). \tag{1}$$

b) Now, set

$$c_n(t) = \sum_{k=0}^{\infty} \lambda^k c_n^{(k)}(t),$$

and plug it into your result from (b) to obtain the first order, i.e. $\mathfrak{O}(\lambda)$, differential equation.

c) Obtain the second order equation.

Remark. Notice that your results for (b) and (c) are *exactly* the same as the two-level results when we begin in a single initial state!

a) Plug in the form for $|\Psi\rangle$ into the Schrödinger equation:

$$i\hbar \sum_{n} (\dot{c}_n - i\omega_n) e^{-iE_n t/\hbar} |n\rangle = \sum_{n} c_n e^{-iE_n t/\hbar} (E_n + \lambda H') |n\rangle.$$

Now, take the inner product with $|n\rangle$ (and change the dummy summation variable from n to m):

$$i\hbar(\dot{c}_n - i\omega_n)e^{-iE_nt/\hbar} = c_n E_n e^{-iE_nt/\hbar} + \sum_m \langle n|\lambda H'|m\rangle c_m e^{-iE_mt/\hbar}.$$

Notice that the second term on the LHS is equal to the first term on the RHS, so they cancel. Thus,

$$i\hbar\dot{c}_n(t) = \sum_m \lambda \langle n|H'|m\rangle c_m(t)e^{-i\omega_{nm}t},$$

as desired, where

$$\omega_{nm} = \frac{E_n - E_m}{\hbar}.$$

b) The order λ result is just

$$\dot{c}_n^{(1)}(t) = -\frac{i}{\hbar} \sum_m \langle n | H' | m \rangle c_m^{(0)} e^{-i\omega_{nm}t}.$$

c) This is exactly the same:

$$\dot{c}_n^{(k+1)}(t) = -\frac{i}{\hbar} \sum_m \langle n| H' | n \rangle c_m^{(k)}(t) e^{-i\omega_{nm}t}.$$

Exercise 2. Sinusoidal Perturbations. In the case that

$$H' = Ke^{-i\omega t} + K^{\dagger}e^{i\omega t}$$

is sinusoidal and acts up until time t, solve the first order perturbation theory differential equation from Exercise 1(b).

We plug in

$$\dot{c}^{(1)} = -\frac{i}{\hbar} \left[\sum_{m} K_{nm} c_{m}^{(0)} e^{it(\omega_{nm} - \omega)} + \sum_{m} K_{nm}^{\dagger} c_{m}^{(0)} e^{it(\omega_{nm} + \omega)} \right].$$

Integrating from 0 to t, we find

$$c^{(1)}(t) = -\frac{1}{\hbar} \left[\sum_{m} \frac{K_{nm} c_{m}^{(0)}}{\omega_{nm} - \omega} \left(e^{it(\omega_{nm} - \omega)} - 1 \right) + \sum_{m} \frac{K_{nm}^{\dagger} c_{m}^{(0)}}{\omega_{nm} + \omega} \left(e^{it(\omega_{nm} + \omega)} - 1 \right) \right].$$

Exercise 3. Spin Resonance. Consider a spin-1/2 particle in a static magnetic field $B_0\hat{z}$, so $H^0 = -\frac{1}{2}\hbar\gamma B_0\sigma_z$. The perturbation is due to a magnetic field B_1 rotating in the (x, y)-plane with angular velocity ω :

$$H'(t) = -\frac{1}{2}\hbar\gamma B_1[\sigma_x\cos(\omega t) + \sigma_y(\sin(\omega t))].$$

a) Writing the eigenvectors of σ_z as

$$|+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

rewrite H' in the form given in Exercise 2, using these eigenvectors as a basis.

- b) Suppose at t=0 we have the initial state $|i\rangle=|+\rangle$. Find the first order probability for the spin to be down at time t. It is convenient to set $\omega_0=\gamma B_0$ and $\omega_1=\gamma B_1$.
- c) It turns out that the exact Equation 1 can be solved for such a hamiltonian. The exact answer for (b) is

$$P(t) = \sin^2(\alpha t/2) \left(\frac{\omega_1}{\alpha}\right)^2,$$

where $\alpha^2 = (\omega_0 + \omega)^2 + {\omega_1}^2$; $\alpha/2$ is called the **Rabi flopping frequency**. Using this answer, what is the range of validity of the perturbation theory result, assuming we are not near resonance?

- d) Suppose we are near resonance. What is the range of validity of the perturbation theory result? Give a physical explanation of your result.
- e) Challenge. Solve Equation 1, and derive the formula for P(t).
- a) We can write

$$H'(t) = -\frac{1}{2}\hbar\gamma B_1 \begin{bmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{bmatrix},$$

so that

$$K = -\frac{1}{2}\hbar\gamma B_1 \begin{bmatrix} 0 & 1\\ 0 & 0 \end{bmatrix}.$$

b) Since we have a single initial state, the complicated expression from Exercise 2 reduces to two terms. Further, since $K \mid + \rangle = 0$, we are left with only one term. Thus, we find

$$c^{(1)} = \frac{\omega_1}{2(\omega_0 + \omega)} \left(e^{it(\omega_0 + \omega)} - 1 \right),$$

where we notice that

$$E_{-} = \frac{\hbar\omega_0}{2}$$

$$E_{+} = -\frac{\hbar\omega_0}{2},$$

so $\omega_{-+} = \omega_0$. Now, notice that we can pull out a factor of $e^{it(\omega_0 + \omega)/2}$ to get a single phase factor:

$$c^{(1)} = i \frac{\omega_1}{(\omega_0 + \omega)} \sin\left(\frac{\omega_0 + \omega}{2}t\right) e^{it(\omega_0 + \omega)/2}.$$

This trick makes it easy to calculate the first order probability:

$$|c^{(1)}|^2 = \frac{{\omega_1}^2}{(\omega_0 + \omega)^2} \sin^2\left(\frac{\omega_0 + \omega}{2}t\right).$$

c) Consider the second term in the exact answer

$$\frac{{\omega_1}^2}{\alpha^2} = \frac{{\omega_1}^2}{(\omega_0 + \omega)^2 \left[1 + \frac{{\omega_1}^2}{(\omega_0 + \omega)^2}\right]}.$$

We can expand the term in brackets as a binomial expansion if and only if $|\omega_1| \ll |\omega_0 + \omega|$, which would give us the perturbation theory result. Thus, this is the range of validity of that result. Note that resonance would correspond to $\omega_0 + \omega \sim 0$, so this wouldn't apply in that case.

d) We have to play a different game if we're near resonance. In that case, the exact result becomes

$$P_n(t) \sim \sin^2\left(\frac{\omega_1}{2}t\right).$$

In order to determine what happens to the perturbation theory result, recall that for small θ , $\sin \theta \sim \theta$. Thus,

$$|c^{(1)}(t)|^2 \sim \frac{{\omega_1}^2}{4}t^2.$$

Clearly, these will match well if and only if $|\omega_1 t| \ll 1$. So perturbation theory works well for short times, no matter how big B_1 is. This is because the effect of the perturbation is a product of both its strength *and* its duration.

e) Note that

$$\langle -|H'|+\rangle = -\frac{\hbar\omega_1}{2}e^{i\omega t},$$

and $\langle +|H'|-\rangle$ is just the complex conjugate. Thus, we want to solve the system of differential equations

$$\begin{cases} \dot{c}_{-} = i \frac{\omega_{1}}{2} e^{it(\omega_{0} + \omega)} c_{+} \\ \dot{c}_{+} = i \frac{\omega_{1}}{2} e^{-it(\omega_{0} + \omega)} c_{-} \end{cases}.$$

This is most easily done by differentiating one of the equations once and substituting. We obtain

$$\ddot{c}_{-} = i(\omega_0 + \omega)\dot{c}_{-} + i\frac{\omega_1}{2}e^{it(\omega_0 + \omega)}\dot{c}_{+},$$

so that

$$\ddot{c}_{-} - i(\omega_0 + \omega)\dot{c}_{-} + \frac{{\omega_1}^2}{4}c_{-} = 0.$$

To solve this equation, assume solutions of the form e^{kt} , with $k \in \mathbb{C}$. Plugging this in, we obtain an equation for k.

$$k^2 - ik(\omega_0 + \omega) - \frac{{\omega_1}^2}{4} = 0.$$

This has solution

$$k_{\pm} = \frac{i(\omega_0 + \omega) \pm i\alpha}{2},$$

where $\alpha/2$ is the Rabi flopping frequency. Thus,

$$c_{-} = Ae^{k-t} + Be^{k+t}.$$

Now, the initial condition $c_+(0) = 1$ gives that

$$\dot{c}_{-}(0) = i \frac{\omega_1}{2}.$$

Since

$$\dot{c}_{-}(t) = k_{-}Ae^{k_{-}t} + k_{+}Be^{k_{+}t}$$

and $c_{-}(0) = 0$, we have two equations for A and B.

$$\begin{cases} A = -B \\ \frac{\omega_1}{2} = A \frac{\omega_0 + \omega - \alpha}{2} + B \frac{\omega_0 + \omega + \alpha}{2} \end{cases}.$$

These have the solution

$$\begin{cases} A = -\frac{\omega_1}{2\alpha} \\ B = \frac{\omega_1}{2\alpha} \end{cases}.$$

Thus,

$$c_{-}(t) = \frac{i\omega_1}{\alpha} e^{it(\omega_0 + \omega)/2} \sin(\alpha t/2),$$

so that

$$|c_{-}|^{2} = P(t) = \frac{{\omega_{1}}^{2}}{\alpha^{2}} \sin^{2}(\alpha t/2),$$

which exactly matches the result quoted in part (c).