Week 6 Worksheet Solutions
Electrostatic Energy

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Exercise 1. In this exercise, you will find the energy contained in the electric field $\mathbf{E}$.

a) How much work does it take to assemble a configuration of two point charges?

b) What about n point charges?

c) Show that your result from (b) can be written

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i),$$

where q_i is the charge of the i^{th} charge and $\mathbf{r}_i$ is its position vector.

d) Generalize your result from (c) to a *continuous* charge distribution with (not necessarily uniform) charge density ρ.

e) Using the differential form of Gauss' law, $\rho = \varepsilon_0 \nabla \cdot \mathbf{E}$, rewrite this in terms of $\mathbf{E}$ and V. Integrate by parts to transfer the derivative ∇ to V instead of $\mathbf{E}$, and argue that your answer is exactly

$$W = \frac{\varepsilon_0}{2}\int E^2 \, \mathrm{d}V,$$

where the integral is taken over *all space*.
Hint: Integration by parts in this context can be achieved by using the product rule on $\nabla \cdot (\mathbf{v} f)$ (also the divergence theorem will help). Notice the similarity to 1-dimensional integration by parts, which can be derived from the identity $\frac{\mathrm{d}}{\mathrm{d}x}(fg) = f'g + fg'$.

a) To bring in the first charge is free, since there are no forces acting on it. The force on the second charge will be

$$F = \frac{q_1 q_2}{r^2},$$

where r is the distance between q_1 and q_2 (I work in Gaussian units again, until we get the answer). Hence, the work will be

$$\begin{aligned} W &= -\int \mathbf{F} \cdot \mathrm{d}\boldsymbol{\ell} \\ &= -q_1 q_2 \int_{\infty}^{r} \mathrm{d}r' \frac{1}{r'^2} \\ &= \frac{q_1 q_2}{r}. \end{aligned}$$

Putting in the factor of $\frac{1}{4\pi\varepsilon_0}$, we get

$$W = \frac{q_1 q_2}{4\pi\varepsilon r}.$$

b) Let's think about what happens to the n^{th} charge. There will already be $n-1$ charges in place, and the n^{th} one will feel a force due to each other charge. Notice that we can rewrite our answer for $n = 2$ as

$$W = q_2 V_1(\mathbf{r}_2),$$

where $V_1(\mathbf{r}_2)$ is the potential due to charge 1 at the location of charge 2 (which is given by $\mathbf{r}_2$). So we can see that the n^{th} charge will be affected by such potentials due to the $n-1$ other charges that we already brought in. It follows that we can write the work to bring in the n^{th} charge as

$$W = q_n \sum_{i=1}^{n-1} V_i(\mathbf{r}_n).$$

The same thing will happen to the k^{th} charge for $k < n$, and we merely need to sum over all the individual pieces of work from $k = 1$ to $k = n$. It follows that the total work is

$$W = \sum_{k=1}^{n} q_k \sum_{i=1}^{k-1} V_i(\mathbf{r}_k).$$

c) We want to rewrite this as

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i).$$

What is V in this result? I claim that it is the potential due to *all the charges other than q_i*. Unlike in our previous result where we only count up to the charges we have already in place, here, we also count up the potential due to the charges that haven't yet been brought in. But notice that this means we overcount by exactly a factor of 2. Indeed, given two charges, note that the formula above gives a term

$$\frac{q_1 q_2}{r_{12}}$$

and also a term

$$\frac{q_2 q_1}{r_{21}}.$$

But of course $r_{12} = r_{21}$, so we get two identical terms when we want to have only one. For $n = 3$, we get a similar double count. Thus, this must be equal to our previous result.

d) If the charge distribution is now over some volume, the little piece of charge q_i becomes $\mathrm{d}q$. But given a charge density ρ, we can write

$$\mathrm{d}q = \rho\,\mathrm{d}V,$$

so that our result becomes (we also turn the sum into an integral)

$$W = \frac{1}{2}\int \rho(\mathbf{r})V(\mathbf{r})\,\mathrm{d}V.$$

e) Plugging in, we have

$$W = \frac{\varepsilon_0}{2}\int \nabla\cdot\mathbf{E}V\,\mathrm{d}V.$$

Now, notice that

$$\nabla\cdot(\mathbf{E}V) = V\nabla\cdot\mathbf{E} + \mathbf{E}\cdot\nabla V.$$

We can use this to rewrite the above integral as

$$W = \frac{\varepsilon_0}{2}\int \big(\nabla\cdot(\mathbf{E}V) - \mathbf{E}\cdot\nabla V\big)\,\mathrm{d}V.$$

Now, we can use the divergence theorem on the first term to obtain

$$W = \frac{\varepsilon_0}{2}\left(\int V\mathbf{E}\cdot\mathrm{d}\mathbf{a} - \int \mathbf{E}\cdot\nabla V\,\mathrm{d}V\right).$$

Notice that since $-\nabla V = \mathbf{E}$, the second term is exactly the integral we want, but integrated over what region? We needed to only include the region which contained our charge distribution, but we may as well have picked any large volume, since ρ will be 0 outside of the charged volume. In particular, we can take our integration volume to be all space. What will happen to the first integral? Notice that the surface area of the sphere over which we are computing our flux will go as r^2, while VE goes as r^{-3}. Hence, as $r \to \infty$, we have that the flux will go to 0. It follows that as long as we integrate over all space, the first integral vanishes, and we are left with the desired result.

Exercise 2. ***Griffiths 2.39.*** Two spherical cavities of radii a and b, respectively, are hollowed out from the interior of a neutral conducting sphere of radius R. At the center of each cavity is a point charge, q_a and q_b, respectively.

a) Find the surface charge densities σ_a, σ_b, and σ_R.

b) What is the field outside the conductor?

c) What is the field within each cavity?

d) What is the force on q_a and q_b?

e) Use the previous exercise to find the energy of this configuration.

a) The conductor will try to "cancel out" the charge it encloses. Hence, the charge around q_a will be $-q_a$, and likewise for q_b. It follows that

$$\begin{aligned}\sigma_a =& \frac{q_a}{4\pi a^2}\\ \sigma_b =& \frac{q_b}{4\pi b^2}.\end{aligned}$$

There is a residual charge of $q_a + q_b$ left in the conductor, and this charge must go to the surface. Hence,

$$\sigma_R = \frac{q_a + q_b}{4\pi R^2}.$$

b) We can use Gauss' law to determine the field. The charge enclosed is just $q_a + q_b$, so, using a Gaussian sphere of radius r, we have

$$\mathbf{E} = \frac{q_a + q_b}{4\pi\varepsilon_0 r^2}\hat{r}.$$

c) Within a cavity, a Gaussian sphere centered around the charge will give exactly analogously to part (b)

$$\mathbf{E} = \frac{q_a}{4\pi\varepsilon_0 {r_a}^2}\hat{r}_a,$$

where $\hat{r}_a$ denotes the radial direction from q_a and r_a denotes the radial distance from q_a. We have a similar result for q_b.

d) The force is 0, because inside the cavities we have electric field only due to the charges which are inside there. Since a charge cannot induce a force on itself, we have that it must be 0.

e) We integrate the electric field over all space to obtain infinity (from the integrals inside the cavities)! What happened? The problem is that we have calculated the amount of energy it takes to *assemble* the point charges, which is infinite, since they are infinitely small. It follows that we can't use the result of Exercise 1 when there are point charges present.