Week 9 Worksheet
Curvature

Jacob Erlikhman

March 20, 2025

Exercise 1. Parallel Transport is Curvature. Let M be a spacetime such that for any two points $p, q \in M$, the parallel transport from p to q does not depend on the curve that joints p and q. You will show that this implies that M is flat, i.e. that the Riemann curvature tensor on M is identically 0. We will do this with the help of the following construction. Consider a parametrized surface $f : U \to M$, where

$$U = \{(s,t) \in \mathbb{R}^2 | s, t \in (-\varepsilon, 1+\varepsilon),\ \varepsilon > 0\}$$

and we force $f(s,0) = f(0,0)$ for all s. Let V_0 be a tangent vector to M at $f(0,0)$, and define a vector field V along f as follows. Set $V(s,0) = V_0$ and $V(s,t)$ to be the parallel transport of V_0 along the curve $c(t) = f(s,t)$.

a) Sketch V in the case that M is flat, and explain what changes in the non-flat case.

b) Since V is parallel transported along the t-direction, what is $\nabla_{\partial_t f} V$?

c) Show that

$$\nabla_{\partial_t f} \nabla_{\partial_s f} V + R(\partial_t f, \partial_s f) V = 0,$$

where in a coordinate system x^i with $Z = \lambda^i \partial_i = \lambda^i \frac{\partial}{\partial x^i}$

$$R(\partial_j, \partial_k) Z = \lambda^l R^i{}_{ljk} \partial_i.$$

Hints: Recall that R is a rank (3,1) tensor; what does this tell you about $R(v^i \partial_i, w^j \partial_j)Z$? Recall the Ricci identity

$$\nabla_j \nabla_k v^i - \nabla_k \nabla_j v^i = -v^l R^i{}_{ljk},$$

and compute in a coordinate system.

d) Show that $V(s,1)$ is also the parallel transport of $V(0,1)$ along the curve $c(s) = f(s,1)$, so that $\nabla_{\partial_s f} V(s,1) = 0$.

e) Show that

$$R(\partial_t f, \partial_s f) V(0,1) = 0,$$

where the $(0,1)$ means we consider the vector at the point $(s,t) = (0,1)$.

f) Conclude that $R = 0$ everywhere by arbitrariness of our choices.