Knizhnik-Zamolodchikov Equations: From Physics to Math and Back

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Abstract

The purpose of the present note is to discuss the physical origin of Wess-Zumino-Witten (WZW) models in conformal field theory, the derivation of the Knizhnik-Zamolodchikov (KZ) equations as differential equations for n-point correlation functions in these models, and then to describe the other side of the coin: The entirely mathematical derivation and utility of the KZ equations as equations for a certain function of vertex operators (a correlation function) in a vertex algebra with underlying simple Lie algebra $\mathfrak{g}$. We will start with the physical origin of the equations, then, after deriving them in this context, change to a vertex algebra formalism and derive them purely algebraically. Finally, we will show the equivalence of the two approaches. We note that much of the material of the "Physics" section follows [1], while much of the "Mathematics" section follows [2] (even if not explicitly stated).

1 Introduction

The natural physical setting for much of the discussion in the following sections is the following. Consider a (1+1)-dimensional quantum field theory. Passing to a euclidean spacetime (from the Minkowski original), we can define the complex variable $z = x + it$ and its complex conjugate. Suppose we have our theory defined on an infinite cylinder with space being the radial coordinate and time the axial, and suppose further that it is conformal (so there is a conformal symmetry in Minkowski spacetime or an $\mathrm{SL}(2, \mathbb{C})$ symmetry in $z, \bar{z}$). Mapping this onto the complex plane so that circles about the origin are at fixed time, we have obtained a natural time ordering by means of a "radial ordering." Indeed, suppose we are given two bosonic fields φ_1 and φ_2. Then time ordering of the fields is equivalent to radial ordering:

$$\mathcal{T}[\varphi_1(z)\varphi_2(w)] = \begin{cases} \varphi_1(z)\varphi_2(w), & |z| > |w| \\ \varphi_2(w)\varphi_1(z), & |z| < |w| \end{cases},$$

where $\mathcal{T}$ stands for the time-ordered product. This is one of the mathematical advantages of conformal field theories. The rest of the formalism needed (that was not covered in class) is elementary quantum field theory (including functional quantization).

2 The Physics

Consider a nonlinear sigma model given by the action

$$S_0 = \frac{1}{4g^2} \int \mathrm{d}^2 x \, \mathrm{tr}'(\partial^\mu \varphi^{-1} \partial_\mu \varphi),$$

where $g^2 > 0$ is a dimensionless coupling constant, tr' is a normalized trace such that $\mathrm{tr}'(t^a t^b) = 2\delta_{ab}$ for t^a (the image in a unitary representation of) the generators of the Lie algebra $\mathfrak{g}$ of the semisimple symmetry group G, and φ is a bosonic field taking values in a unitary G-representation V. Since φ is unitary, it's clear that $\varphi^{-1}\partial_\mu\varphi$ is antihermitian, which assures that $\mathrm{tr}'(\partial^\mu\varphi^{-1}\partial_\mu\varphi) = \mathrm{tr}'((\partial^\mu\varphi)^*\partial_\mu\varphi) \geq 0$, where * denotes the adjoint. We will show that this theory is not conformally invariant, and hence motivate a correction to the action to preserve conformal invariance, leading us to WZW models. We begin with a computational

Lemma 1. The Euler-Lagrange equation for S_0 is

$$\partial^\mu(\varphi^{-1}\partial_\mu\varphi) = 0. \tag{1}$$

Proof. Varying the action, we obtain

$$\delta S_0 = \frac{1}{2g^2}\int \mathrm{d}^2x\, \mathrm{tr}'\left(\varphi^{-1}\delta\varphi\partial^\mu(\varphi^{-1}\partial_\mu\varphi)\right),$$

where δf represents the variation of f.[1] To obtain this, we want to compute the functional derivative

$$\frac{\delta}{\delta\varphi}S_0 = \partial_\varphi\mathcal{L} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right),$$

which is the Euler-Lagrange equation ($\mathcal{L}$ is the Lagrangian density). This in turn leads us to $\frac{\delta}{\delta\varphi_{ij}}\mathrm{tr}'(f\varphi^{-1}g)$, where f and g are independent of φ. Computing this, we find

$$\frac{\delta}{\delta\varphi_{ij}}\mathrm{tr}'(f\varphi^{-1}g) = -\frac{1}{\sigma}\cdot(\varphi^{-1}gf\varphi^{-1})_{ij},$$

where $\sigma = \frac{\mathrm{tr}}{\mathrm{tr}'}$ is the normalization of the trace (so the whole derivative is independent of the specific representation). Thus, we have

$$\frac{\delta}{\delta\varphi_{ij}}\mathrm{tr}'(\partial^\mu\varphi^{-1}\partial_\mu\varphi) = -\frac{1}{\sigma}\partial^\mu(\varphi^{-1}\partial_\mu\varphi).$$

□

That this is not the theory we want can be seen by looking at the Noether currents, $J_\mu = \varphi^{-1}\partial_\mu\varphi$. Writing them in terms of the holomorphic and antiholomorphic variables, we can define

$$\begin{aligned} J_z &= \varphi^{-1}\partial_z\varphi \\ J_{\bar{z}} &= \varphi^{-1}\partial_{\bar{z}}\varphi. \end{aligned}$$

[1] We will take the physics approach to variational calculus here, since the ultimate goal—the calculation of the correlation functions and the KZ equations—will be derived in a later section by purely algebraic (and rigorous) methods. Of course, one could define the variation of φ as $\alpha : (-\varepsilon,\varepsilon)\times G \to \mathrm{GL}(V)$ such that $\alpha(0,x) = \varphi(x)$, factor it through the charts on G, identify $\mathrm{GL}(V)$ with $\mathrm{GL}(n,\mathbb{C})$, and consider the derivative

$$\left.\frac{\mathrm{d}S_0(\bar{\alpha}(u))}{\mathrm{d}u}\right|_{u=0},$$

where $\bar{\alpha}(u)$ denotes the function $t \mapsto \alpha(u,t)$ (see e.g. [6] Volume I Chapter 9 for details). We will not pursue this further here.

Thus,

$$\partial_z J_{\bar{z}} + \partial_{\bar{z}} J_z = 0.$$

These need to be separately conserved, so that the two terms in the above equation must vanish separately. Hence, $\varepsilon^{\mu\nu} J_\nu$ must also be conserved ($\varepsilon^{\mu\nu}$ is the rank-two Levi-Civita tensor). However,

$$\begin{aligned}
&[J_\mu, J_\nu] + \partial_\mu J_\nu - \partial_\nu J_\mu = 0 \Longrightarrow \\
&\Longrightarrow \partial_\mu(\varepsilon^{\mu\nu} J_\nu) = -\varepsilon^{\mu\nu} J_\mu J_\nu \neq 0.
\end{aligned}$$

What we actually want are $J_z = \partial_z \varphi \varphi^{-1}$ and $J_{\bar{z}} = \varphi^{-1} \partial_{\bar{z}} \varphi$, which is exactly what we will obtain in the following sections.

2.1 Wess-Zumino-Witten Correction

In order to obtain conserved holomorphic/antiholomorphic currents, we need to add a correction to the action which enforces this conservation. It turns out that this correction is given by

$$\Gamma = \frac{-i}{24\pi} \int_M \mathrm{d}^3 y \varepsilon_{\alpha\beta\gamma} \operatorname{tr}' \left(\tilde{\varphi}^{-1} \partial^\alpha \tilde{\varphi} \tilde{\varphi}^{-1} \partial^\beta \tilde{\varphi} \tilde{\varphi}^{-1} \partial^\gamma \tilde{\varphi} \right),$$

where M is a 3-manifold such that ∂M is the compactification of the 2-dimensional spacetime over which the original CFT is defined, and $\tilde{\varphi}$ is the natural extension of $\tilde{\varphi}$ to M. now there is some ambiguity in Γ, implicit in the two choices for M. The difference between these two choices we can write as $\Delta\Gamma$, which is integrated over the whole compact three-dimensional space containing both M. Since it's compact, it is topologically just S^3, so

$$\Delta\Gamma = \frac{-i}{24\pi} \int_{S^3} \mathrm{d}^3 y \varepsilon_{\alpha\beta\gamma} \operatorname{tr}' \left(\tilde{\varphi}^{-1} \partial^\alpha \tilde{\varphi} \tilde{\varphi}^{-1} \partial^\beta \tilde{\varphi} \tilde{\varphi}^{-1} \partial^\gamma \tilde{\varphi} \right).$$

Now, $|\exp(\Delta\Gamma)| = 1$, which follows from the following

Proposition 1. Let G be complex semisimple. Then,

$$\Delta\Gamma \in 2\pi i \mathbb{Z}.$$

Let $G = \mathrm{SO}(3)$, then $\Delta\Gamma = \pi i$.

Proof. Note that given any semisimple complex group G, its Lie algebra $\mathfrak{g}$ is essentially made up of copies of $\mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{su}(2, \mathbb{C})$. Thus, the universal cover $\widetilde{G}$ of G is likewise made of copies of $\mathrm{SU}(2)$. It can be deduced from a theorem of Bott (which applies some Morse theory to the study of Lie groups) that it suffices to consider maps from S^3 to $\mathrm{SU}(2)$ subgroups of G [5]. We have thus reduced to the case $G = \mathrm{SU}(2)$. Consider the 2-dimensional representation of $\mathfrak{su}(2)$ given by

$$\tilde{\varphi}(y) = y^0 - i \sum_{k=1}^{3} y^k \sigma_k,$$

where σ_k are the Pauli matrices and $y \in S^3$. Now, the fundamental representation V_ϖ has index

$$\begin{aligned}\sigma_\varpi &= \frac{\dim V_\varpi \langle \varpi, 3\varpi \rangle}{2 \dim \mathfrak{su}(2)} \\ &= \frac{1}{2},\end{aligned}$$

where ϖ is the fundamental weight. Thus,

$$\operatorname{tr}' = 2 \operatorname{tr}.$$

Finally, note that this expression for $\tilde{\varphi}(y)$ defines a function $\tilde{\varphi}$ on all of S^3. Moreover, by evaluating the integrand of $\Delta\Gamma$ at some point, say $y_0 = (1, 0, 0, 0)$, we find that it equals

$$\sum_{\alpha,\beta,\gamma} \frac{1}{24\pi} \varepsilon_{\alpha\beta\gamma} \operatorname{tr}'(\sigma_\alpha \sigma_\beta \sigma_\gamma) = \frac{i}{\pi}.$$

But since our choice of "origin," i.e. the point $y \in S^3$ where $\tilde{\varphi}(y) = \mathbb{1}$, is arbitrary, we find that the integrand is *identically* equal to i/π. Hence,

$$\Delta\Gamma = 2\pi i.$$

Considering higher dimensional representations, we obtain n-coverings of $\mathrm{SU}(2)$ by $\mathrm{SU}(2)$ (or S^3 by S^3), which gives

$$\Delta\Gamma = 2\pi i n.$$

This proves the Proposition for the case of complex semisimple G. If G is real, for example $G = \mathrm{SO}(3)$, the index is instead $\sigma = 1$, so $\Delta\Gamma = \pi i$. □

Proposition 1 shows that the ambiguity inherent in the definition of Γ does not have any physical effect on the theory, since the partition function for the theory is well-defined up to a multiplicative ± 1 factor. Thus, we can define an action

$$S = S_0 + k\Gamma,$$

where $k \in \mathbb{Z}$. The variation of Γ is a 2-dimensional functional, which follows from Stokes' Theorem. Explicitly, we have

Lemma 2.

$$\delta\Gamma = \frac{i}{8\pi} \int \mathrm{d}^2 x \varepsilon_{\alpha\beta} \operatorname{tr}' \left(\varphi^{-1} \delta\varphi \partial^\alpha (\varphi^{-1} \partial^\beta \varphi) \right).$$

Proof. This is a simple computation. Consider the expression

$$\varepsilon_{\alpha\beta\gamma} \partial^\gamma \operatorname{tr}' \left(\tilde{\varphi}^{-1} \delta\tilde{\varphi} \partial^\alpha (\tilde{\varphi}^{-1} \partial^\beta \tilde{\varphi}) \right) = \varepsilon_{\alpha\beta\gamma} \operatorname{tr}' \left(\tilde{\varphi}^{-1} \partial^\gamma (\delta\tilde{\varphi}) \partial^\alpha \tilde{\varphi}^{-1} \partial^\beta \tilde{\varphi} + \partial^\gamma \tilde{\varphi}^{-1} \delta\tilde{\varphi} \partial^\alpha \tilde{\varphi}^{-1} \partial^\beta \tilde{\varphi} \right). \tag{2}$$

When evaluating this derivative, remember that terms which are symmetric under any two of the indices α, β, γ are 0 when summed over; these are the only surviving nonzero terms. Now, make the substitution $\tilde{\varphi} \to \tilde{\varphi} + \delta\tilde{\varphi}$ in Γ to obtain

$$\delta\Gamma = \frac{i}{24\pi} \int_M \mathrm{d}^3 y \cdot 3\varepsilon_{\alpha\beta\gamma} \operatorname{tr}' \left(\tilde{\varphi}^{-1} \partial^\gamma (\delta\tilde{\varphi}) \partial^\alpha \tilde{\varphi}^{-1} \partial^\beta \tilde{\varphi} - \delta\tilde{\varphi} \partial^\alpha \tilde{\varphi} \tilde{\varphi}^{-1} \partial^\beta \tilde{\varphi} \tilde{\varphi}^{-1} \partial^\gamma \tilde{\varphi} \right),$$

where we again use the cyclic property of the trace and the antisymmetry of the Levi-Civita tensor. Renaming indices and again using antisimmetry of the Levi-Civita tensor, we find that the integrand above is equal to the total derivative in Equation 2. Using Stokes' Theorem reduces $\delta\Gamma$ to the form given in the statement of the Lemma. $\square$

Using Lemma 2, we can read off the Euler-Lagrange equation for S:

$$\partial^\mu (\varphi^{-1} \partial_\mu \varphi) + \frac{g^2 i k}{4\pi} \varepsilon_{\mu\nu} \partial^\mu (\varphi^{-1} \partial^\nu \varphi) = 0.$$

Changing variables to the complex variables $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix^1$, we have

$$\left(1 + \frac{g^2 k}{4\pi}\right) \partial_z (\varphi^{-1} \partial_{\bar{z}} \varphi) + \left(1 - \frac{g^2 k}{4\pi}\right) \partial_{\bar{z}} (\varphi^{-1} \partial_z \varphi) = 0.$$

Thus, if we can find k such that $4\pi/g^2 = k$, we have

$$\partial_z (\varphi^{-1} \partial_{\bar{z}} \varphi) = 0.$$

Similarly, $k = -4\pi/g^2$ gives the conservation of the currents. This equation has immediate solution $\varphi(z, \bar{z}) = f(z) \bar{f}(\bar{z})$, for arbitrary functions f and $\bar{f}$.

Since both the holomorphic J_z and antiholomorphic $J_{\bar{z}}$ currents are conserved, the action is invariant under the transformation

$$\varphi(z, \bar{z}) \mapsto \Omega(z) \varphi(z, \bar{z}) \bar{\Omega}^{-1}(\bar{z}),$$

where $\Omega, \bar{\Omega} : U \to G$, $U \subset \mathbb{C}$ open. Considering values of Ω and $\bar{\Omega}$ near the identity, with their Lie algebra parts given by ω and $\bar{\omega}$, respectively, we have that

$$\delta_\omega \varphi = \omega\varphi \qquad\qquad \delta_{\bar{\omega}} \varphi = -\varphi\bar{\omega}.$$

Setting $g^2 = 4\pi/k$, we have that under $\varphi \mapsto \varphi + \delta_\omega \varphi + \delta_{\bar{\omega}} \varphi$, S has variation

$$\delta S = \frac{k}{2\pi} \int \mathrm{d}^2 x \operatorname{tr}' \left[\omega(z) \partial_{\bar{z}} (\partial_z \varphi \varphi^{-1}) - \bar{\omega}(\bar{z}) \partial_z (\varphi^{-1} \partial_{\bar{z}} \varphi) \right],$$

which vanishes if we integrate by parts. Thus, the global $G \times G$ symmetry is actually *local* for such a theory: It can be expressed in terms of the local functions $\Omega(z)$ and $\bar{\Omega}(\bar{z})$. Such theories, with $g^2 = 4\pi/k$ and action given by S, are called WZW models. We will now quantize it.

2.2 WZW Quantization

Rescale the currents

$$\begin{aligned} J(z) =& -k\partial_z\varphi\varphi^{-1} \\ \bar{J}(\bar{z}) =& k\varphi^{-1}\partial_{\bar{z}}\varphi, \end{aligned}$$

so that

$$\delta S = -\frac{1}{2\pi}\int \mathrm{d}^2x \left[\partial_{\bar{z}}\operatorname{tr}'(\omega(z)J(z)) + \partial_z \operatorname{tr}'(\bar{\omega}(\bar{z})\bar{J}(\bar{z}))\right].$$

Writing out the measure $\mathrm{d}^2x = -\frac{i}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$, integrating by parts, and picking a counterclockwise contour for the holomorphic part and a clockwise one for the antiholomorphic part, we get

$$\delta_{\omega,\bar{\omega}}S = \frac{i}{4\pi}\oint \mathrm{d}z\,\operatorname{tr}'(\omega(z)J(z)) - \frac{i}{4\pi}\oint \mathrm{d}\bar{z}\,\operatorname{tr}'(\bar{\omega}(\bar{z})\bar{J}(\bar{z})),$$

where both contours are counterclockwise (hence the $-$ sign). Writing this in components, we get

$$\delta_{\omega,\bar{\omega}}S = -\frac{1}{2\pi i}\oint \mathrm{d}z \sum_a \omega^a J^a + \frac{1}{2\pi i}\oint \mathrm{d}\bar{z} \sum_a \bar{\omega}^a \bar{J}^a,$$

where $J = \sum J^a t^a$ and $\omega = \sum \omega^a t^a$. Denote by $\langle\cdots\rangle$ an arbitrary n-point correlation function, so that

$$\delta_{\omega,\bar{\omega}}\langle\cdots\rangle = -\frac{1}{2\pi i}\oint \mathrm{d}z \sum_a \omega^a \langle J^a\cdots\rangle + \frac{1}{2\pi i}\oint \mathrm{d}\bar{z} \sum_a \bar{\omega}^a \langle \bar{J}^a\cdots\rangle.$$

Finally, one can do a direct computation to get the transformation law for the currents

$$\delta_\omega J^a = iC_{abc}\omega^b J^c - k\partial_z\omega^a,$$

where C_{abc} are the structure constants of $\mathfrak{g}$. Substituting this into the transformation law for the n-point function, we get

$$J^a(z)J^b(w) \sim \frac{k\delta_{ab}}{(z-w)^2} + iC_{abc}\frac{J^c(w)}{z-w}, \tag{3}$$

which is the operator product expansion (OPE) for the **current algebra**. Passing to the Laurent expansion

$$J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1}{J^a}_n,$$

we will now show that the commutation relations for the affine algebra $\hat{\mathfrak{g}}$ are equivalent to Equation 3. We calculate

$$[{J^a}_n, {J^b}_m] = -\frac{1}{4\pi}\oint \mathrm{d}w \oint \mathrm{d}z z^n w^m \left[\frac{k\delta^{ab}}{(z-w)^2} + iC_{abc}\frac{J^c(w)}{z-w}\right].$$

Evaluating the contour integrals, we get

$$[{J^a}_n, {J^b}_m] = kn\delta^{ab}\delta_{n+m,0} + iC_{abc}{J^c}_{n+m},$$

which is exactly the relation for $\hat{\mathfrak{g}}$ at level k. Of course, we also have the antiholomorphic affine algebra (generated by the operators in the Laurent expansion of $\bar{J}(\bar{z})$), which is another copy of $\hat{\mathfrak{g}}$. We have thus shown that, under quantization, the local G-symmetry induces (two copies of) a $\hat{\mathfrak{g}}$ current algebra.

2.3 The Segal-Sugawara Operator

Since the Segal-Sugawwara operator was discussed extensively in class, we will only translate the construction into the language of WZW models and conformal field theory. Recall that we defined the Segal-Sugawara operators

$$L_n = \frac{1}{2(k+h^\vee)} \sum_{a,m} : J^a{}_m J^a{}_{n-m} :, \tag{4}$$

acting on a highest-weight representation V of $\hat{\mathfrak{g}}$ of level $k \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number. We would like to extend the natural $\hat{\mathfrak{g}}$ action to one of $\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus d$, where $d = -L_0$. This is in fact a natural choice. Let $v \in V$ be a highest weight vector, so that it is annihilated by elements of the form $x_{(n)} = x \otimes t^n \in \hat{\mathfrak{g}}$ where $n > 0$. Thus,

$$L_0 v = \frac{1}{2(k+h^\vee)} \sum_a J^a J^a v = \frac{1}{2(k+h^\vee)} \Omega v = \frac{\langle \lambda, \lambda + 2\rho \rangle}{2(k+h^\vee)},$$

where Ω is the Casimir of $\mathfrak{g}$, which necessarily acts by the scalar $\langle \lambda, \lambda + 2\rho \rangle$, where λ is the highest weight corresponding to V and ρ is half the sum of the positive roots of $\mathfrak{g}$. We have thus shown that L_0 acts as a Casimir for $\tilde{\mathfrak{g}}$.

The connection to the physics is the following. The energy-momentum tensor has classical form $\frac{1}{2k} \sum_a J^a J^a$, which is just the scaled Casimir of $\mathfrak{g}$. A normal ordered version of this could be written

$$T(z) = \gamma \sum_a : J^a(z) J^a(z) :,$$

where γ is an unknown constant. It can be shown by calculating contractions (in the sense of quantum field theory) that $\gamma = 1/[2(k+h^\vee)]$ (the level k is renormalized to $k + h^\vee$; see [1], pp. 627-8). Thus, when we write out $J^a(z)$ in a mode expansion, we find that $T(z)$ can be expressed in terms of the L_n.

2.4 Primary Fields and the KZ Equation

We will call a field **primary** if it transforms covariantly under representations with highest weights λ and μ acting on the holomorphic and antiholomorphic variables, respectively. Explicitly, this means

$$J^a(z)\varphi(w,\bar{w}) \sim -\frac{t^a \varphi(w,\bar{w})}{z-w} \tag{a}$$

$$\bar{J}^a(\bar{z})\varphi(w,\bar{w}) \sim \frac{\varphi(w,\bar{w}) t^a}{\bar{z}-\bar{w}}, \tag{b}$$

where the t^a are the images of the generators t^a of $\mathfrak{g}$ in the λ-representation in (a), respectively the μ-representation in (b). Let, now, $\{\varphi_i | i \in \{1,\ldots,n\}\}$ be primary fields. Conformal invariance ($\mathrm{SL}(2,\mathbb{C})$ invariance in the complex variables z and $\bar{z}$) gives the Ward identities for the n-point function, which we will just state:

$$\sum_{i=1}^n z_i{}^m (z_i \partial_{z_i} + (m+1) h_i) \langle \varphi_1(z_1) \cdots \varphi_n(z_n) \rangle = 0 \qquad \text{for} \qquad m \in \{-1, 0, 1\}.$$

G-invariance requires

$$\delta_\omega \langle \varphi_1(z_1) \cdots \varphi_n(z_n) \rangle = 0.$$

Combining the above two relations, we get

$$\oint \mathrm{d}z \sum_a \omega^a \langle J^a(z) \varphi_1(z_1) \cdots \varphi_n(z_n) \rangle = -\sum_{i=1}^n \frac{1}{2\pi i} \oint \frac{\mathrm{d}z}{z - z_i} \sum_a \omega^a {t_i}^a \langle \varphi_1(z_1) \cdots \varphi_n(z_n) \rangle = 0,$$

where ${t_i}^a$ is the image in the representation labeled by i of the Lie algebra generators. Since ω is totally arbitrary, this reduces to

$$\sum_{i=1}^n {t_i}^a \langle \varphi_1(z_1) \cdots \varphi_n(z_n) \rangle = 0.$$

Theorem 1a: Knizhnik-Zamolodchikov Equation. Arbitrary n-point correlation functions obey the following differential equation,

$$\left[\partial_{z_i} + \frac{1}{k + h^\vee} \sum_{j \neq i} \frac{\sum_a {t_i}^a \otimes_{\mathbb{C}} {t_j}^a}{z_i - z_j} \right] \langle \varphi_1(z_1) \cdots \varphi_n(z_n) \rangle = 0.$$

Physics Proof. Consider the state vector (i.e. element of the state Hilbert space) associated to the primary field φ_i, denoted $|\varphi_i\rangle$, and act on it with L_{-1}, defined in Equation 4:

$$\begin{aligned} L_{-1} |\varphi_i\rangle =& \frac{1}{k + h^\vee} \sum_a {J^a}_{-1} {J^a}_0 |\varphi_i\rangle \\ =& -\frac{1}{k + h^\vee} \sum_a {J^a}_{-1} {t_i}^a |\varphi_i\rangle. \end{aligned}$$

Now, suppose we insert the 0 state vector,

$$|\chi\rangle = \left[L_{-1} + \frac{1}{k + h^\vee} \sum_a J^a_{-1} {t_i}^a \right] |\varphi_i\rangle,$$

into a correlation function. First, consider the insertion of ${J^a}_{-1}$. This gives

$$\begin{aligned} \langle \varphi_1(z_1) \cdots ({J^a}_{-1} \varphi_i)(z_i) \cdots \varphi_n(z_n) \rangle =& \frac{1}{2\pi i} \oint_{z_i} \frac{\mathrm{d}z}{z - z_i} \langle J^a(z) \varphi_1(z) \cdots \varphi_n(z) \rangle \\ =& \frac{1}{2\pi i} \oint_{z_j, j \neq i} \frac{\mathrm{d}z}{z - z_i} \sum_{j \neq i} \frac{{t_j}^a}{z - z_j} \langle \varphi_1(z_1) \cdots \varphi_n(z_n) \rangle \\ =& \sum_{j \neq i} \frac{{t_j}^a}{z_i - z_j} \langle \varphi_1(z_1) \cdots \varphi_n(z_n) \rangle, \end{aligned}$$

where the symbol $\oint_{z_i}$ denotes a contour containing z_i, we use the OPE (3) in the second equality (and reverse the contour to preserve the sign), and the last equality is just the evaluation of the contour integral by the residue theorem. We thus have that

$$\langle \varphi_1(z_1) \cdots \chi(z_i) \cdots \varphi_n(z_n) \rangle = \left[\partial_{z_i} + \frac{1}{k+h^\vee} \sum_{j \neq i} \frac{\sum_a t_i{}^a \otimes_{\mathbb{C}} t_j{}^a}{z_i - z_j} \right] \langle \varphi_1(z_1) \cdots \varphi_n(z_n) \rangle,$$

which vanishes. □

We will now derive a mathematically equivalent version of Theorem 1a. The proof that it is equivalent will occupy much of the final section.

3 The Mathematics

3.1 Intertwining Operators

Denote by L_{λ_i}, $i \in \{0, 1, \ldots\}$ highest-weight modules of a simple Lie algebra $\mathfrak{g}$ of highest weight λ_i, and let $k \in \mathbb{C}$. Suppose that for each λ_i, k is **generic**, i.e. it satisfies $k \notin \mathbb{Q}\langle \alpha^\vee, \lambda_i \rangle + \mathbb{Q}$, for all $\alpha \in \Delta^+$ (i.e. for all positive roots α). Further, let $\tilde{\mathfrak{g}}$ denote the extension of the affine algebra $\hat{\mathfrak{g}}$ by the exterior derivation. In other words, if $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ is the loop algebra and $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbb{1}$ its central extension, then $\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus \mathbb{C}d$, where $d = t\partial_t$. $\tilde{\mathfrak{g}}^+$ denotes the extension of $\mathfrak{g} \otimes \mathbb{C}[t]$. With this notation, we have

Proposition 1. If k is generic for $\lambda \in \mathfrak{h}^*$, then the induced representation $V_{\lambda,k} = \mathrm{Ind}_{\tilde{\mathfrak{g}}^+}^{\tilde{\mathfrak{g}}} L_\lambda$ is irreducible. Here, $\mathbb{1}$ acts by k and d acts by

$$\Delta(\lambda) = \frac{\langle \lambda, \lambda + 2\rho \rangle}{2(k + h^\vee)}, \tag{5}$$

where $\rho \in \mathfrak{h}^*$ is some (generally not unique) solution of $\langle \rho, \alpha_i{}^\vee \rangle = a_{ii}/2, \ \forall i$. The a_{ii} are the terms on the diagonal of the generalized Cartan matrix of $\hat{\mathfrak{g}}$.

Proof. This follows as a special case of Theorem 2 in [3]. □

Let $\varphi : V_{\lambda_1,k} \to V_{\lambda_0,k} \hat{\otimes} V(z)$ be a $\hat{\mathfrak{g}}$-homomorphism, where $z \in \mathbb{C}$, $V(z)$ denotes the representation of $\hat{\mathfrak{g}}$ obtained from some $\mathfrak{g}$-representation V by setting $x \otimes p(t) \cdot v = p(z)xv$ for any $v \in V$, and $\hat{\otimes}$ denotes the completed tensor product, so it can have infinite expressions of the form $\sum w_i \otimes v_i$ such that the w_i are homogeneous and $\deg(w_i) \to -\infty$. Such a φ is called **intertwining**: It satisfies

$$\varphi(x_{(n)}) = (x_{(n)} \otimes 1 + z^n \cdot 1 \otimes x)\varphi,$$

where $x_{(n)} \in \hat{\mathfrak{g}}$ denotes $x \otimes t^n$. We will show below that intertwining operators are primary fields in a WZW model with symmetry group the adjoint group of $\mathfrak{g}$. Along the way, we will see that these are also the vertex operators we had discussed in class.

3.2 Operator KZ Equation

We begin with a result from class:

Theorem 2. Let $\pi : L_{\lambda_1} \to L_{\lambda_0} \otimes V$ be a $\mathfrak{g}$-homomorphism. Then for $k \in \mathbb{C}$ generic, there exists a unique $\hat{\mathfrak{g}}$-intertwiner

$$\varphi^\pi(z) : V_{\lambda_1,k} \to V_{\lambda_0,k} \hat{\otimes} V(z),$$

such that for each $v \in V_{\lambda_1,k,(0)} = L_{\lambda_1}$, the degree 0 part of $\varphi^\pi(z)v$ is πv.

This theorem was (essentially) proved in class; we will not repeat the proof here, except to remark that this implies we can expand φ^π in a Laurent series,

$$\varphi^\pi(z) = \sum_{n \in \mathbb{Z}} \varphi^\pi{}_{(n)} z^{-n},$$

where each $\varphi^\pi{}_{(n)}$ is a homogeneous operator. Define, now, for $u \in V^*$, $\varphi^\pi_u(z)$ by the equation

$$\varphi^\pi_u(z)v = u(\varphi^\pi(z)v)$$

for $v \in V_{\lambda_1,k}$. We can then write the intertwining property as

$$[x_{(n)}, \varphi^\pi_u(z)] = z^n \varphi^\pi_{xv}(z). \tag{6}$$

Define formal sums associated to any $x \in \mathfrak{g}$

$$\begin{aligned} J_x(z) =& \sum_{n \in \mathbb{Z}} x_{(n)} z^{-n-1} \in \hat{\mathfrak{g}}[z, z^{-1}] \\ =& J_x{}^+(z) - J_x{}^-(z) \\ =& \sum_{n<0} x_{(n)} z^{-n-1} - \left[-\sum_{n \geq 0} x_{(n)} z^{-n-1} \right]. \end{aligned}$$

We will call these the **currents** associated to x. We remark that these have the same power series expansions as the Noether currents in WZW models, where we consider Noether currents parametrized by elements of $\mathfrak{g}$ rather than by a basis t^a of $\mathfrak{g}$ as in §2.2. Let $u \in V^*(z)$. Then

$$\begin{aligned} J_x{}^+(w)u =& \sum_{n \in \mathbb{N}} w^{n-1} x_{(-n)} u \\ =& \sum_{n \in \mathbb{N}} w^{n-1} z^{-n} x u \\ =& \frac{xu}{z-w}. \end{aligned}$$

The $J_x{}^-$-action has a similar power series expansion (with $n \leq 0$ and an overall minus sign); hence,

$$J_x{}^-(w)u = \frac{xu}{z-w},$$

as well. Thus, using the intertwining property in the form of Equation 6,

$$[J_x{}^{\pm}(w), \varphi^{\pi}{}_u(z)] = \frac{1}{z-w}\varphi^{\pi}_{xu}(z), \tag{7}$$

which is equivalent to the locality property for vertex operators and the defintion of primary fields above.

Let now V be a lowest-weight $\mathfrak{g}$-module with lowest weight $-\mu$. Define the operators

$$\hat{\varphi}^{\pi}(z) = z^{-\Delta(\mu)}\tilde{\varphi}^{\pi}(z) = \sum_{n\in\mathbb{Z}} \varphi^{\pi}{}_{(n)} z^{-n-\Delta},$$

where $\Delta = \Delta(\lambda_1) - \Delta(\lambda_0) + \Delta(\mu)$ and $\Delta(\lambda)$ is defined in Equation 5. Use the normal ordering we defined in class; namely, given $x \in \mathfrak{g}$,

$$: J_x(z)\hat{\varphi}^{\pi}{}_u(z) := J_x{}^{+}(z)\hat{\varphi}^{\pi}{}_u(z) - \hat{\varphi}^{\pi}{}_u(z) J_x{}^{-}(z).$$

We then have

Theorem 1b: Operator KZ Equation. Let $\mathscr{B} \subset \mathfrak{g}$ be an orthonormal basis. Then the operators $\hat{\varphi}^{\pi}{}_u(z)$ satisfy

$$(k + h^{\vee})\partial_z \hat{\varphi}^{\pi}{}_u(z) = \sum_{a\in\mathscr{B}} : J_a(z)\hat{\varphi}^{\pi}{}_{au}(z) : .$$

Remark. We note that the proof of Theorem 1a suggests that something like this must be true. In fact, this is derivable from Theorem 1a. However, we believe it more instructive to begin with this result, prove it algebraically, and then show that it leads to Theorem 1a.

Proof. We first need a technical

Lemma.

$$z^{-\Delta}\varphi^{\pi}(z) : V_{\lambda_1,k} \to V_{\lambda_0,k} \hat{\otimes} z^{-\Delta} V[z, z^{-1}]$$

is a $\tilde{\mathfrak{g}}$-homomorphism if and only if

$$\Delta = \Delta(\lambda_1) - \Delta(\lambda_0).$$

Proof of Lemma. By definition, $\tilde{\varphi}^{\pi}(z) = z^{-\Delta}\varphi^{\pi}(z)$ is a $\tilde{\mathfrak{g}}$-intertwiner if and only if

$$\tilde{\varphi}^{\pi}(z) d - (d \otimes 1)\tilde{\varphi}^{\pi}(z) = (1 \otimes z\partial_z)\,\tilde{\varphi}^{\pi}(z).$$

From the formal power series expansion $\tilde{\varphi}^{\pi}(z) = \sum_{n\in\mathbb{Z}} \varphi^{\pi}{}_{(n)} z^{-n-\Delta}$ with each $\varphi^{\pi}{}_{(n)}$ homogeneous of degree n, we need only check this at degree 0. But this exactly gives the expression for Δ in the statement of the Lemma. $\square$

Henceforth, $\tilde{\varphi}^{\pi}(z) = z^{-\Delta}\varphi^{\pi}(z)$ for Δ as in the Lemma. d-invariance of $\tilde{\varphi}^{\pi}(z)$ implies

$$z\partial_z \tilde{\varphi}^{\pi}{}_u(z) = -[d, \tilde{\varphi}^{\pi}{}_u(z)],$$

which can be written as

$$z\partial_z \hat{\varphi}^\pi{}_u(z) = -[d, \hat{\varphi}^\pi{}_u(z)] - \Delta(\mu)\hat{\varphi}^\pi{}_u(z).$$

Substitute $d = \mathcal{S}$, where

$$\mathcal{S} = \frac{1}{2(k+h^\vee)} \sum_{a\in\mathscr{B}} \sum_{n\in\mathbb{Z}} : a_{(n)} a_{(-n)} :$$

is the Segal-Sugawara operator defined in class. We thus have

$$z\partial_z \hat{\varphi}^\pi{}_u(z) = \frac{1}{2(k+h^\vee)} \sum_{a\in\mathscr{B}} \left[\sum_{n\le 0} [a_{(n)}a_{(-n)}, \hat{\varphi}^\pi{}_u(z)] + \sum_{n\in\mathbb{N}} [a_{(-n)}a_{(n)}, \hat{\varphi}^\pi{}_u(z)] \right] - \Delta(\mu)\hat{\varphi}^\pi{}_u(z).$$

Now, use the intertwining property in the form of Equation 7 of $\hat{\varphi}^\pi{}_u(z)$ to rewrite this as

$$\begin{aligned}
z\partial_z \hat{\varphi}^\pi{}_u(z) =& \frac{1}{2(k+h^\vee)} \sum_{a\in\mathscr{B}} \Big[\sum_{n\le 0} \big(z^{-n} a_{(n)} \hat{\varphi}^\pi{}_{au}(z) + z^n \hat{\varphi}^\pi{}_{au}(z) a_{(-n)} \big) + \\
&\qquad + \sum_{n\in\mathbb{N}} \big(z^n a_{(-n)} \hat{\varphi}^\pi{}_{au}(z) + z^{-n} \hat{\varphi}^\pi{}_{au}(z) a_{(n)} \big) \Big] - \\
&- \Delta(\mu)\hat{\varphi}^\pi{}_u(z) \\
=& \frac{1}{2(k+h^\vee)} \sum_{a\in\mathscr{B}} \big(2z J_a{}^+(z) \hat{\varphi}^\pi{}_{au}(z) - 2z \hat{\varphi}^\pi{}_{au}(z) J_a{}^-(z) + \\
&\qquad + a_{(0)} \hat{\varphi}^\pi{}_{au}(z) - \hat{\varphi}^\pi{}_{au}(z) a_{(0)} \big) \\
&- \Delta(\mu)\hat{\varphi}^\pi{}_u(z) \\
=& \frac{1}{k+h^\vee} \sum_{a\in\mathscr{B}} \left(z : J_a(z) \hat{\varphi}^\pi{}_{au}(z) : + \frac{1}{2} \hat{\varphi}^\pi{}_{a^2u}(z) \right) - \Delta(\mu)\hat{\varphi}^\pi{}_u(z).
\end{aligned}$$

Since $\sum_{a\in\mathscr{B}} \hat{\varphi}^\pi{}_{a^2u}(z) = \hat{\varphi}^\pi{}_{\Omega u}(z) = \langle \mu, \mu + 2\rho \rangle \hat{\varphi}^\pi{}_u(z)$, where Ω is the Casimir (and hence acts as $\langle \mu, \mu + 2\rho \rangle$), the last two terms cancel, which gives the desired equation. □

4 Equivalence of KZ Equations: Physics ↔ Mathematics

We will now show that this is equivalent to Theorem 1a. Let L_{λ_i} for $i \in \{0, \dots, n\}$ be irreducible highest-weight $\mathfrak{g}$-modules, and let V_{μ_i}, $i \in \{0, \dots, n\}$, be lowest-weight modules with lowest weights $-\mu_i$. Fix a level $k \in \mathbb{C}$ generic for each λ_i and $\mathfrak{g}$-homomorphisms $\pi_i : L_{\lambda_i} \to L_{\lambda_{i-1}} \otimes V_{\mu_i}$. Define the corresponding $\hat{\mathfrak{g}}$-intertwiners

$$\hat{\varphi}^{\pi_i}(z_i) : V_{\lambda_i,k} \to V_{\lambda_{i-1},k} \hat{\otimes} z^{-\Delta} V_{\mu_i}[z_i, z_i{}^{-1}].$$

Further, define the product

$$\Psi(z_1, \dots, z_n) = (\hat{\varphi}^{\pi_1}(z_1) \otimes \cdots \otimes 1) \cdots (\hat{\varphi}^{\pi_{n-1}}(z_{n-1}) \otimes 1) \cdot \varphi^{\pi_n}(z_n) : V_{\lambda_n,k} \to V_{\lambda_0,k} \hat{\otimes} V_{\mu_1} \hat{\otimes} \cdots \hat{\otimes} V_{\mu_n}.$$

We note that this is already almost a correlation function; all that's left is to take for $u_0 \in (V_{\lambda_0,k,(0)})^* = {L_{\lambda_0}}^*$, $u_{n+1} \in V_{\lambda_n,k,(0)} = L_{\lambda_n}$,

$$\langle u_0, \Psi(z_1, \dots, z_n)u_{n+1}\rangle \in V_{\mu_1} \otimes \cdots \otimes V_{\mu_n}. \tag{8}$$

It should be clear from the form of this function that it is just the n-point correlation function, if we consider the intertwiners as primary fields in a WZW model. Explicitly, one would write out the mode expansions for the fields in the product, act on u_{n+1}, and take the inner product, which amounts to integrating over a spacetime contour that connects the modules $V_{\lambda_n,k}$ and $V_{\lambda_0,k}$. Let $V = V_{\mu_1} \otimes \cdots \otimes V_{\mu_n} \otimes {L_{\lambda_n}}^*$. For fixed u_0, we can define a V-valued function

$$\psi(z_1, \dots, z_n) = \langle u_0, \Psi(z_1, \dots, z_n)\cdot\rangle.$$

We can now state

Theorem 1c: The KZ Equation, 2**nd** **Version.** $\psi(z_1, \dots, z_n)$ satisfies

$$(k + h^\vee)\partial_{z_i}\psi = \left(\sum_{j=1, j\neq i}^{n} \frac{\Omega_{ij}}{z_i - z_j} + \frac{\Omega_{i,n+1}}{z_i}\right)\psi,\ \forall i \in \{1, \dots, n\}. \tag{9}$$

Remark. This is equivalent to the form of the KZ equation in Theorem 1a. There, the equation reads

$$(k + h^\vee)\partial_{z_i}\psi = \sum_{j=1, j\neq i}^{n+1} \frac{\Omega_{ij}}{z_i - z_j}\psi,\ \forall i \in \{1, \dots, n+1\}, \tag{10}$$

where Ω_{ij} is the same operator as $\sum_a {t_i}^a \otimes {t_j}^a$ in Theorem 1a. To see this, note that the action of $\sum_a {t_i}^a \otimes {t_j}^a$ is effectively the insertion of the current modes into the correlation function. From the proof below, we see that Ω_{ij} defines the same insertion. To see the equivalence of the two forms above,note that if $\psi(z_1, \dots, z_{n+1})$ is a solution of Equation 10, then clearly $\psi(z_1, \dots, z_n, 0)$ satisfies Equation 9. Conversely, if $\psi(z_1, \dots, z_n)$ is a solution of Equation 9, then $\tilde{\psi}(z_1, \dots, z_n, z_{n+1}) = \psi(z_1 - z_{n+1}, \dots, z_n - z_{n+1})$ satisfies the Equation 10

Proof. Choose $u_1 \in {V_{\mu_1}}^*, \dots, u_n \in {V_{\mu_n}}^*$, and define

$$\psi_{u_1,\dots,u_{n+1}}(z_1, \dots, z_n) = \langle u_0, {\hat{\varphi}^{\pi_1}}_{u_1}(z_1) \cdots {\hat{\varphi}^{\pi_n}}_{u_n}(z_n)u_{n+1}\rangle \in \mathbb{C}.$$

By Theorem 1b, we get

$$\begin{aligned}
&(k{+}h^\vee)\partial_{z_i}\psi_{u_1,\dots,u_{n+1}}(z_1, \dots, z_n) = \\
&\quad=\langle u_0, \hat{\varphi}_{u_1}(z_1 \cdots (k + h^\vee)\partial_{z_i}\hat{\varphi}_{u_i}(z_i) \cdots \hat{\varphi}_{u_n}(z_n)u_{n+1}\rangle \\
&\quad= \sum_{a\in\mathscr{B}} \langle u_0, \hat{\varphi}_{u_1}(z_1) \cdots \left({J_a}^+(z_i)\hat{\varphi}_{au_i}(z_i) - \hat{\varphi}_{au_i}{J_a}^-(z_i)\right) \cdots \hat{\varphi}_{u_n}(z_n)u_{n+1}\rangle,
\end{aligned}$$

where we use the intertwining relation to get the third line. Now, use the commutation relation (7) and pull all the J^- to the right and all the J^+ to the left to get

$$\begin{aligned}
&(k{+}h^\vee)\partial_{z_i}\psi_{u_1,\dots,u_{n+1}}(z_1, \dots, z_n) = \\
&\quad= \sum_{i\neq j}\sum_{a\in\mathscr{B}} \frac{1}{z_i - z_j}\langle u_0, \hat{\varphi}_{u_1}(z_1) \cdots \hat{\varphi}_{au_j}(z_j) \cdots \hat{\varphi}_{au_i}(z_i) \cdots \hat{\varphi}_{u_n}(z_n)u_{n+1}\rangle + \\
&\qquad+ \sum_{a\in\mathscr{B}} \frac{1}{z_i}\langle u_0, \hat{\varphi}_{u_1}(z_1) \cdots \hat{\varphi}_{au_i}(z_i) \cdots \hat{\varphi}_{u_n}(z_n)au_{n+1}\rangle,
\end{aligned}$$

where we use that $J_a{}^-(z)u_{n+1} = -au_{n+1}/z$, and $\langle u_0, J_a{}^+(z)v\rangle = -\langle J_a{}^+(z)u_0, w\rangle = 0$ for any $v \in V_{\lambda_0,k}$ (since both u_0 and u_{n+1} are zero degree vectors). $\square$

We have thus shown that quantum WZW models carry the natural structure of a vertex algebra, with primary fields corresponding to vertex operators. Furthermore, we can define all objects arising naturally in the field theoretic description bypassing and generalizing the physical picture.

If I had more time and/or space, calculating some correlation functions for the simplest case, $\mathfrak{g} = \mathfrak{sl}(2)$, would be interesting. Showing that the Gauss hypergeometric function arises naturally out of the KZ equations would also be an interesting next stopping point. Finally, I really wanted to get to quantum groups and the deformed KZ equations but did not have time.

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