Week 14 Worksheet Solutions
Black Holes

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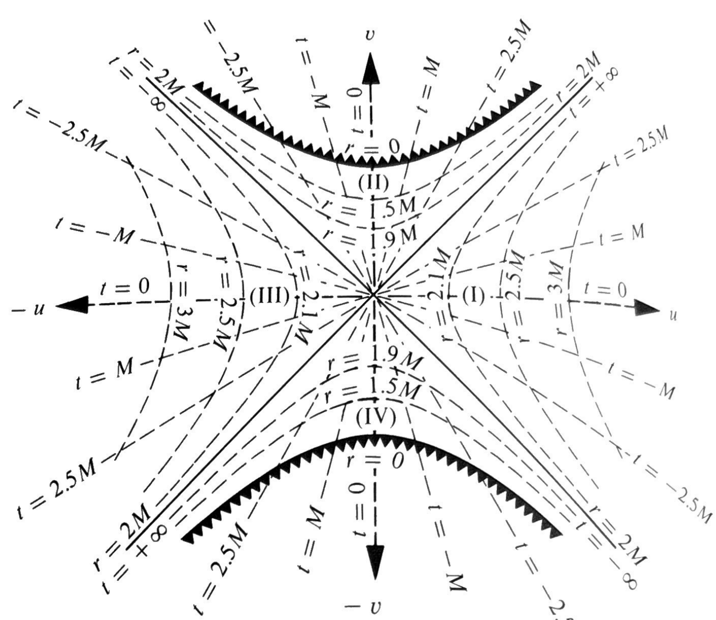
The Kruskal coordinates V, U are defined by

$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = U^2 - V^2$$
$$\frac{t}{2M} = \ln\left(\frac{V+U}{U-V}\right) = 2\tanh^{-1}(V/U).$$

The Schwarzschild metric in Kruskal coordinates is

$$ds^2 = \frac{32M^3 e^{-r/2M}}{r}(-dV^2 + dU^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

The Kruskal diagram of a black hole is



Kruskal-Szekeres

Exercise.

a) Identify the worldlines of photons traveling radially in a Kruskal diagram.

b) Show that the worldline of a photon traveling nonradially makes an angle of less than 45 degrees with the vertical axis of the Kruskal diagram.

c) Use part (b) to show that particles with finite mass always move at an angle less than 45 degrees with the vertical axis.

d) If someone falls past the radius $r = 2M$, he or she will always hit the singularity at $r = 0$.

e) Once someone has fallen past $r = 2M$, he or she can't send messages to friends located at $r > 2M$ but can still receive messages.

f) Show that once someone falling in reaches the gravitational radius $r = 2M$, then *no matter what he or she does subsequently—no matter in what direction, how long, and how hard he or she blasts his or her rocket engines*—he or she will be killed by the singularity at $r = 0$ in a proper time of

$$\tau < 1.54 \cdot 10^{-5} \frac{M}{M_\odot} \text{ seconds,}$$

where $M_\odot = 2 \cdot 10^{30}$ kg is the mass of the Sun (and $G = 6.7 \cdot 10^{-11}$ m^3/kg s^2).
Hint: Note that

$$\left(\frac{dr}{d\tau}\right)^2 = e^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right).$$

a) Radial null geodesics are those with $d\varphi = 0$ and $d\theta = 0$, so

$$0 = -dV^2 + dU^2.$$

Hence,

$$\frac{dV}{dU} = \pm 1$$

for such geodesics.

b) Write the line element for photons

$$0 = \frac{32M^3}{r} e^{-r/2M}[-(p^V)^2 + (p^U)^2] + r^2[(p^\theta)^2 + (p^\varphi)^2].$$

Since the second term in brackets is > 0 for nonradial travel, so $p^V > |p^U|$.

c) The null geodesics define the light cone, so if they have less than a 45 degree angle with the vertical axis, then necessarily so will timelike geodesics which correspond to massive particles.

d) Follows immediately from the diagram and part (c) together with causality. Namely, we have identified that

$$\left|\frac{dV}{dU}\right| \geq 1$$

for light rays; however, this does not tell us about the *direction* that light rays. If we identify that light rays travel “up” in the Kruskal diagram, then we will have identified the future/past light cones. We can do this by analyzing the formula

$$\frac{t}{2M} = \ln\left(\frac{V+U}{U-V}\right).$$

Consider two points (U_1, V_1) and (U_2, V_2). Then

$$t_2 - t_1 = 2M \ln\left(\frac{(U_2+V_2)(U_1-V_1)}{(U_2-V_2)(U_1+V_1)}\right).$$

Now, let $(U_1, V_1) = (0, 1)$ and $(U_2, V_2) = (1, 2)$. Then we find that

$$t_2 - t_1 > 0.$$

Similarly, if $(U_1, V_1) = (0, 1)$ and $(U_2, V_2) = (-1, 2)$, we again find that

$$t_2 - t_1 > 0.$$

It follows that on slope ± 1 lines light travels up (other such lines will be displaced by constants from these). Similarly, one checks that on slope > 1 lines light travels up with a Δt larger than in the slope 1 case, implying that light always travels up. This implies the same is true for massive particles, since we have found the future light cones to be pointing up.

e) Follows immediately from the diagram and part (b).

f) This problem is effectively on the homework, so I won’t post solutions to it.