Midterm Review Problems Solutions

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Exercise 1. A solid conducting sphere of radius a is in a constant, uniform external electric field $\mathbf{E_0}$. It is cut in half into two identical halves with an infinitely thin cut, which is perpendicular to $\mathbf{E_0}$. What force $\mathbf{F}$ acts on each half? How will this force change if we turn off the external field $\mathbf{E_0}$?

Let the external field be along the z-axis, i.e. $\mathbf{E_0} = E_0\hat{z}$, and suppose the conductor has potential $V = 0$ (note that we can't put the zero of the potential at infinity, since the electric field is not 0 there). The potential was worked out in Example 3.8 in Griffiths (though you should be able to do this on your own!)—it is given by

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta.$$

The charge density is

$$\begin{aligned}\sigma =& -\varepsilon_0\frac{\partial V}{\partial n}\\ =&\varepsilon_0\left(E_0 + \frac{2R^3}{R^3}\right)\cos\theta\\ =&3\varepsilon_0 E_0\cos\theta.\end{aligned}$$

since $n = r$. Now, let's calculate the pressure on one of the halves. It is given by

$$\frac{1}{2\varepsilon_0}\int\sigma^2\cos\theta\,\mathrm{d}a,$$

where the integral is taken over the surface of one of the halves and we have a $\cos\theta$ term to project onto the z-axis (since the other directions will cancel each other out). We can calculate this explicitly as

$$\frac{9\varepsilon_0 {E_0}^2}{2}\int_0^{\pi/2}\mathrm{d}\theta\cos^3\theta\sin\theta\cdot 2\pi a^2 = -9\pi\varepsilon_0 {E_0}^2 a^2\frac{\cos^4\theta}{4}\bigg|_0^{\pi/2} = \frac{9\pi\varepsilon_0 {E_0}^2 a^2}{4}.$$

When we turn off the external field, all the residual charge on the conducting halves will run to the flat part where we cut them. Hence, the force will be the same as that due to two conducting plates in the shape of disks. In other words, the problem is now to find the force of attraction between two oppositely charged disks of radius a. Ignoring the fringe electric fields, we have an electric field between the disks given by σ/ε_0 with σ uniform. Initially, we had a charge of

$$\begin{aligned}q =& \int_0^{\pi/2}3\varepsilon_0 E_0\cos\theta\sin\theta\,\mathrm{d}\theta\cdot 2\pi a^2\\ =&3\pi\varepsilon_0 E_0 a^2.\end{aligned}$$

So now our $\sigma = q/\pi a^2$. Hence, the force is

$$\frac{1}{2\varepsilon_0}\int\left(\frac{q}{\pi a^2}\right)^2 \mathrm{d}a = \frac{1}{2\varepsilon_0}\frac{q^2}{\pi a^2} = \frac{9\pi\varepsilon_0 {E_0}^2 a^2}{2},$$

so the force after we turn off the external field is exactly *twice* that of the prior one.

Exercise 2. Recall the image solution to a point charge outside a grounded conducting sphere: a charge $q' = -qa/b$ at a distance $b' = a^2/b$ from the center of the sphere, where the charge q is at a distance b from the center of the sphere of radius a.

a) ***Griffiths 3.9.*** Find the image solution to the above configuration where the sphere is a *neutral* conducting sphere. Also find the force on the charge and the energy of the configuration.

b) Find the image solution to a point dipole with dipole moment $\mathbf{p}$ placed at a distance b from the center of a neutral conducting sphere of radius a in the two orientations: 1) The dipole points in the direction towards the center of the sphere; 2) the dipole is perpendicular to the previous direction.

a) Notice that if we place a point charge $q'' = 4\pi\varepsilon_0 a V_0$ at the center of the sphere, the sphere will have potential exactly V_0. Since we want the sphere to be neutral, we better take $q' = -q''$. Hence, we are in the following situation: There is a charge q, a charge q' at a distance $b - a^2/b$ from q, and a charge $q'' = -q'$ at a distance a^2/b from q' and b from q. All the charges sit on one line.

   Calculating the force on q is now straightforward. Its magnitude is (dropping the $1/4\pi\varepsilon_0$)

$$q^2\frac{a}{b}\left(\frac{1}{(b-a^2/b)^2} - \frac{1}{b^2}\right) = q^2\frac{a}{b}\cdot\frac{2a^2 - a^4/b^2}{(b-a^2/b)^2 b^2} = \frac{q^2a^3}{b^3}\frac{2b^2-a^2}{(b^2-a^2)^2}.$$

   This force is an attractive force, since the charge is attracted to the neutral sphere.

   To calculate the energy, we can imagine first bringing in the first charge, q, then the second, q', and finally the third, q''. The first charge costs no work, the second costs

$$\frac{q^2a}{b}\int_\infty^{b-a^2/b}\frac{1}{x^2}\,\mathrm{d}x = -\frac{q^2a}{b^2-a^2}.$$

   The third costs

$$\left(\frac{qa}{b}\right)^2\int_\infty^{a^2/b}\frac{1}{x^2}\,\mathrm{d}x - \frac{q^2a}{b}\int_\infty^{b}\frac{1}{x^2}\,\mathrm{d}x = \frac{q^2}{b} - \frac{q^2a}{b^2}.$$

   Adding these together, we have

$$q^2\left(\frac{b-a}{b^2} - \frac{a}{b^2-a^2}\right),$$

   and you can add in the factor of $1/4\pi\varepsilon_0$.

b) Case 1. Consider the dipole as two charges $q, -q$ which are at a finite separation d from each other (we will later take $d \to 0$ and $q \to \infty$). Using part (a), we can find the image configurations for both charges: We have a charge $qa/(b-d/2)$ at a distance $a^2/(b-d/2)$ from the center of the sphere and a charge $-qa/(b-d/2)$ at the center of the sphere due to the negative charge $-q$, which is at a distance $b-d/2$ from the center of the sphere. Similarly, we have a charge $-qa/(b+d/2)$ at a distance $a^2/(b+d/2)$ from the center of the sphere and a charge $qa/(b+d/2)$ at the center of the sphere. Altogether, at the center of the sphere we have a charge $dqa/(b^2-d^2/4)$, and we have the two charges near the image location. Now, $qd = p$, and we will keep this fixed as we take $d \to 0$ and $q \to \infty$. Hence, the charge at the center will have magnitude pa/b^2. What about at the image location? The charges there are separated by a distance

$$\frac{a^2}{b-d/2} - \frac{a^2}{b+d/2} = \frac{a^2 d}{b^2-d^2/4}.$$

As we take the limit, they will induce a *dipole* at the image location with dipole moment magnitude

$$\frac{pa^3}{b^3}.$$

On the other hand, the charges at the image location have difference in charge

$$\frac{qa}{b-d/2} - \frac{qa}{b+d/2} = \frac{qad}{b^2-d^2/4},$$

and this will go to

$$pa/b^2$$

as we pass to the limit. Hence, at the image location, there is both a dipole with moment $\mathbf{p} = \mathbf{p}a^3/b^3$ *and* a charge of magnitdude pa/b^2. Also, we have the charge at the center of the sphere, altogether two charges and a dipole form the image to the dipole in case 1.

Case 2. We use the same method as in case 1, but this time the dipole is vertically situated. So the charges are at *equal* distances from the center of the sphere. In this case, the image point charges all cancel out (remember, it was the fact that they were at slightly different distances from the center of the sphere that gave us extra point charges). Hence, all we're left with is a single dipole at the image location with $\mathbf{p} = \mathbf{p}a^3/b^3$.

Exercise 3. The region between two parallel infinite conducting plates at $x = 0$ and $x = L$ is filled with charge of charge density $\rho = \rho_0 \sin(\pi x/L)$. Find the potential and electric field between the plates.

We need to solve Poisson's equation $\nabla^2 V = \rho$ in the region between the plates. Since the situation is independent of y and z, we can just take this to be an ODE in one variable:

$$\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} = \rho_0 \sin(\pi x/L).$$

This has the solution (subject to the boundary conditions $V(0) = 0$ and $V(L) = 0$)

$$V(x) = -\frac{\rho_0 L^2}{\pi^2} \sin(\pi x/L).$$

Exercise 4. ***Griffiths 3.55.*** a) A long metal pipe of square cross-section (side a) is grounded on three sides, while the fourth (insulated from the rest) is maintained at constant potential V_0. Show that the net charge per unit length on the side opposite V_0 is

$$\lambda = -\frac{\varepsilon_0 V_0}{\pi} \ln 2.$$

b) A long metal pipe of circular cross-section of radius R is divided lengthwise into four equal sections, three of them grounded and the fourth maintained at constant potential V_0. Show that the net charge per unit length on the section opposite V_0 is the same as in (a).

a) Arrange the pipe so that the situation is independent of z and so that the boundary conditions give $V(0,y) = V_0$, $V(a,y) = 0$, $V(x,0) = 0$, $V(x,a) = V_0$. We can then solve Laplace's equation in x and y only, so that writing $V(x,y) = X(x)Y(y)$, we have

$$\begin{aligned} X &= A\sin(\alpha x) + B\cos(\alpha x) \\ Y &= Ce^{\alpha y} + De^{-\alpha y}. \end{aligned}$$

Imposing the boundary conditions at $x = 0$, $x = a$, and $y = 0$, we find that the ones for x give $B = 0$ and $\alpha = n\pi/a$, where $n \in \mathbb{N}$, while those for Y give that $C = -D$. It follows that we can write

$$\begin{aligned} Y &= C\sinh(\frac{n\pi}{a}y) \\ X &= A\sin\left(\frac{n\pi}{a}x\right) \end{aligned}$$

It follows that the full solution is given by

$$V(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right).$$

The only remaining boundary condition is that at $y = a$. It gives

$$V_0 = \sum A_n \sin\left(\frac{n\pi}{a}x\right)\sinh(n\pi).$$

Multiplying both sides by $\sin(m\pi x/a)$, where $m \in \mathbb{N}$, and integrating, we get that the sum on the right hand side will collapse by orthogonality of the sine functions. Thus, we'll have

$$-V_0\frac{a}{m\pi}\cos\left(\frac{m\pi}{a}x\right)\Big|_0^a = \frac{a}{2}A_m\sinh(m\pi).$$

Solving for A_m, we have

$$A_m = -\frac{2V_0}{m\pi\sinh(m\pi)}\left[(-1)^m - 1\right].$$

Now, the term in brackets is 0 if m is even and -2 if m is odd; hence,

$$A_m = \begin{cases} 0, & m \text{ even} \\ \frac{4V_0}{m\pi \sinh(m\pi)}, & m \text{ odd} \end{cases}.$$

Finally, we can write the full solution

$$V(x,y) = \sum_{k=0}^{\infty} \frac{4V_0}{(2k+1)\pi \sinh[(2k+1)\pi]} \sin\left(\frac{(2k+1)\pi}{a}x\right) \sinh\left(\frac{(2k+1)\pi}{a}y\right).$$

To find the surface charge on the side opposite V_0, we need to calculate

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial y}\bigg|_{y=0}.$$

This will give

$$\sigma = \sum_{k=0}^{\infty} \frac{4\varepsilon_0 V_0}{a \sinh[(2k+1)\pi]} \sin\left(\frac{(2k+1)\pi}{a}x\right).$$

Now, to calculate λ, we just need to integrate this over the x-direction. Thus,

$$\begin{aligned} \lambda &= \int_0^a \sigma \, \mathrm{d}x \\ &= \sum_{k=0}^{\infty} \frac{8\varepsilon_0 V_0}{(2k+1)\pi \sinh[(2k+1)\pi]}, \end{aligned}$$

since

$$\int_0^a \sin\left(\frac{(2k+1)\pi}{a}x\right) \mathrm{d}x = \frac{a}{(2k+1)\pi}\left[\cos((2k+1)\pi) - 1\right] = -\frac{2a}{(2k+1)\pi}.$$

Now, I wasn't able to figure out how to sum this analytically; however, Mathematica gave that

$$\sum_{k=0}^{\infty} \frac{8}{(2k+1)\sinh[(2k+1)\pi]} \approx 0.0866434$$

and

$$\ln(2)/8 \approx 0.0866434,$$

so they match exactly.

b) We need to solve essentially the same problem as in (a), but this time in cylindrical coordinates. Set up the problem so that the boundary conditions are

$$\begin{aligned} s = R,\ \varphi \in (-\pi/4, \pi/4) &\implies V = V_0 \\ s = R,\ \varphi \notin (-\pi/4, \pi/4) &\implies V = 0, \end{aligned}$$

where we take $\varphi \in [-\pi, \pi]$. The situation is independent of z, so we can solve Laplace's equation in cylindrical coordinates:

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 V}{\partial \varphi^2}=0.$$

Writing $V = S(s)\Phi(\varphi)$, we have

$$s^2\frac{S''}{S}+s\frac{S'}{S}+\frac{\Phi''}{\Phi}=0.$$

As in the case of cartesian coordinates, the sum of the first two terms is a constant, α^2, say, and the second term is likewise equal to $-\alpha^2$. Hence, we have two differential equations, and we can immediately write down the solution to the one for φ. It is

$$\Phi = A\sin(\alpha\varphi)+B\cos(\alpha\varphi).$$

Now, recall that $\Phi(\varphi) = \Phi(\varphi + 2\pi)$, so that α must be an integer, say $n \in \mathbb{Z}$. We can plug this in to the differential equation for s:

$$s^2S''+sS'=n^2S.$$

It's easy to see that this has solutions

$$s^{\pm n}$$

for $n \neq 0$. Now, if $n = 0$, then we end up with the equation

$$sS''=-S'.$$

We can solve this by writing $u = S'$, so that we instead need to solve

$$u'=-\frac{1}{s}u,$$

which is separable. Its solution is

$$\ln u = -\ln s + C,$$

so

$$S'=\frac{C}{s},$$

where we've redefined C. Solving this for S, we have

$$S = C\ln(s)+D.$$

Let's put it all together. For $V = \sum_n S_n\Phi_n$, our solutions are given by

$$V(s,\varphi) = C_0\ln(s)+D_0+\sum_{n=1}^{\infty}\left(C_ns^n+D_ns^{-n}\right)\left(A_n\sin(n\varphi)+B_n\cos(n\varphi)\right).$$

Now, we can finally use our boundary conditions. First of all, the solutions with s^{-n} are not physical, since they blow up at $s = 0$. Thus, $D_n = 0$. Similarly, $\ln(s)$ blows up at $s = 0$, so $C_0 = 0$ as well. Lastly, $\sin$ is antisymmetric in φ, while our boundary conditions are symmetric about the origin in $\varphi \in [-\pi, \pi]$; thus, $A_n = 0$. We are thus left with

$$V(s,\varphi) = A + \sum_{n=1}^{\infty} A_n s^n \cos(n\varphi),$$

where we've again redefined the constants. When $s = R$, we have

$$V(R,\varphi) = A + \sum_{n=1}^{\infty} A_n R^n \cos(n\varphi).$$

First, multiply both sides by $\cos(m\varphi)$, where $m \neq 0$, and integrate from $-\pi$ to π. We can split up the integral on the left hand side into two: one where $V(R,\varphi)$ is 0 and one where $V(R,\varphi) = V_0$. Hence, we have

$$\int_{-\pi/4}^{\pi/4} V_0 \cos(m\varphi)\, \mathrm{d}\varphi = A_m R^m \pi,$$

so that

$$A_m = \frac{2V_0}{mR^m\pi} \sin\left(\frac{m\pi}{4}\right)$$

when $m \neq 0$. In the case $m = 0$, $\cos(m\varphi) = 1$, so just integrate both sides from 0 to 2π to kill the sum and obtain

$$V_0 \frac{\pi}{2} = 2\pi A.$$

Thus,

$$A = \frac{V_0}{4}.$$

We have our answer:

$$V(s,\varphi) = \frac{V_0}{4} + \sum_{n=1}^{\infty} \frac{2V_0 \sin(n\pi/4)}{nR^n\pi} s^n \cos(n\varphi).$$

To compute the surface charge on the side opposite V_0, we take

$$\begin{aligned}
\sigma &= -\varepsilon_0 \left.\frac{\partial V}{\partial s}\right|_{s=R,\ \varphi\in[3\pi/4,\pi]\cup[-\pi,-3\pi/4]} \\
&= -\sum_{n=1}^{\infty} \frac{2\varepsilon_0 V_0}{R\pi} \sin(n\pi/4)\cos(n\varphi).
\end{aligned}$$

To compute λ, we just integrate:

$$\begin{aligned}\lambda =& \int_{-\pi}^{-3\pi/4} \sigma \,\mathrm{d}\varphi + \int_{3\pi/4}^{\pi} \sigma \,\mathrm{d}\varphi \\ =& 2\int_{3\pi/4}^{\pi} \sigma \,\mathrm{d}\varphi \\ =& \sum_{n=1}^{\infty} \frac{4\varepsilon_0 V_0}{nR\pi} \sin(n\pi/4)\sin(3n\pi/4),\end{aligned}$$

since

$$\int_{3\pi/4}^{\pi} \sin(n\varphi)\,\mathrm{d}\varphi = -\sin(3n\pi/4).$$

Unlike the sum in (a), we can evaluate this sum analytically. Let's do so, but before then, we can check that we got the right answer with Mathematica. Indeed, we have that the sum is approximately equal to

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/4)\sin(3n\pi/4)}{n} \approx 0.173287 \approx \ln(2)/4.$$

OK, let's try and tackle this thing. We want to sum

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/4)\sin(3n\pi/4)}{n}.$$

Use the product identity to get that this is the same as

$$\sum \frac{\cos(n\pi/2) - \cos(n\pi)}{2n}.$$

Now,

$$\cos(n\pi/2) = \begin{cases} (-1)^{n/2}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases},$$

and

$$\cos(n\pi) = (-1)^n.$$

Hence,

$$\sum \frac{\cos(n\pi/2) - \cos(n\pi)}{2n} = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{(-1)^k}{2k} - \frac{(-1)^k}{k}\right),$$

where we've changed variables $n = 2k$ in the first term and set $n = k$ in the second. Simplifying, we get

$$\frac{1}{2}\sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{2k}-\frac{1}{k}\right) = -\frac{1}{4}\sum_{k=1}^{\infty}(-1)^k\frac{1}{k}.$$

Phew! We finally have a nice sum. Notice that by the alternating series test, the sum

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$$

is convergent! It must be equal to $-\ln(2)$ to give us the right answer. Recall (or look up) the power series for $\ln(1+x)$:

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n},$$

and notice that for $x = 1$ this series exactly matches negative the one we have obtained. Hence, it must sum to $-\ln(2)$. So we're done!

Remark. This problem is *much* harder than anything you'd be expected to do on the exam, so you will definitely be prepared if you solved it!

Exercise 5. ***Griffiths 3.28.*** A charge is distributed with uniform linear charge density λ over the circumference of a circle of radius R which lies in the (x, y)-plane with center at the origin.

a) Find the potential $V(z)$ on the z-axis.

b) Find the first three terms in the multipole expansion for $V(r, \theta)$.

a) The distance from the circle to a point on the z-axis is

$$\sqrt{R^2+z^2},$$

so the potential is

$$\begin{aligned} V &= \int_0^{2\pi}\frac{\lambda R\,\mathrm{d}\varphi}{\sqrt{R^2+z^2}} \\ &= \frac{2\pi\lambda R}{\sqrt{R^2+z^2}}. \end{aligned}$$

b) The first term is

$$\frac{1}{r}\int_0^{2\pi}\lambda R\,\mathrm{d}\varphi = \frac{2\pi\lambda R}{r}.$$

The second includes an angle α with $\cos\alpha = \hat{r}\cdot\hat{r}'$. We can write

$$\begin{aligned}\hat{r} =&(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\\ \hat{r}' =&(\cos\varphi', \sin\varphi', 0),\end{aligned}$$

so that

$$\cos\alpha = \sin\theta(\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi').$$

Now, we can do the integral with $r' = R$

$$\begin{aligned}V_{\text{dip}}(\mathbf{r}) =&\frac{1}{r^2}\int_0^{2\pi} R\sin\theta(\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi')\lambda R\,\mathrm{d}\varphi'\\ =&0.\end{aligned}$$

Similarly, we can do the quadrupole integral

$$\begin{aligned}V_{\text{quad}}(\mathbf{r}) =&\frac{1}{r^3}\int_0^{2\pi} R^2\left(\frac{3}{2}\sin^2\theta(\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi')^2 - \frac{1}{2}\right)\lambda R\,\mathrm{d}\varphi'\\ =&\frac{\lambda R^3}{2r^3}\int_0^{2\pi}\left(\frac{3}{2}\sin^2\theta(\cos^2\varphi\cos^2\varphi' + \sin^2\varphi\sin^2\varphi') - \frac{1}{2}\right)\mathrm{d}\varphi'\\ =&\frac{\lambda R^3}{4r^3}\left(3\pi\sin^2\theta - 2\pi\right).\end{aligned}$$

Exercise 6. Six equal by absolute value charges are placed at the vertices of a regular hexagon. The signs of any two neighboring charges are opposite. What kind of multipole does the following system form? By what power law does the potential decay at large distances r from the center of the hexagon?

The total charge of the configuration is 0, so there is no monopole term. Similarly, the dipole moment

$$\begin{aligned}\mathbf{p} =&\int \mathbf{r}'\rho(\mathbf{r}')\,\mathrm{d}V'\\ =&aq - aq + aq - aq + aq - aq = 0,\end{aligned}$$

where a is the distance from the origin to any of the charges and the charges have magnitude q.

What about the quadrupole term? You could compute this by explicitly working through the definitions, but there is an easier way. Since $P_n(\cos\alpha)$ is odd for n odd and even for n even, we have that $\sum_i q_i P_2(\cos\alpha_i) = 0$ for the hexagon, since it has odd symmetry (the regular hexagon itself has even symmetry, but the negative charges make it have odd symmetry in this example).

What about the octopole term? Since $P_3(\cos\alpha) = \frac{5\cos^3\alpha - 3\cos\alpha}{2}$, we see that the second term will vanish for the same reason that the dipole term vanishes. On the other hand, we need to only check that the first term does not vanish to see if we get an octopole. Let's place the hexagon in the (x,y)-plane, with two charges on the x-axis, and let's try to compute the octopole potential on the x-axis. Just looking at $\sum q_i\cos^3\alpha_i$, we have

$$q\left(1 - (-1) + \cos(\pi/3) - (-\cos(\pi/3)) + \cos(2\pi/3) - (-\cos(2\pi/3))\right) \neq 0.$$

Therefore, this system has an octopole moment and hence has potential that dies off as $1/r^4$ at large distances.