Week 12 Worksheet
Magnets!

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Exercise 1. Suppose a magnetized object has a *uniform* magnetization $\mathbf{M}$.

a) What does $\int \mathbf{M}\,\mathrm{d}V$ calculate, where the integral is taken over the volume of the object?

b) Suppose a little current loop creates one of the infinitesimal dipole moments. Let it have area a and thickness t. What is m for this loop?

c) Since $m = Ia$, find an equation for the current I, and hence for the bound surface current K_b.
Hint: You can write K_b in terms of I and the dimension(s) of the loop; then, you can eliminate the dimensions using the previous results.

d) Deduce that

$$\mathbf{K}_b = \mathbf{M} \times \hat{n}.$$

This is done in Griffiths.

Exercise 2. Suppose a magnetized object has a *non-uniform* magnetization $\mathbf{M}$. We already know how to calculate the surface current density, so we would now like to calculate the volume current density.

a) Consider two adjacent infinitesimal chunks of magnetized material, one at (x, y, z) and the other at $(x, y + \mathrm{d}y, z)$. On the surface where the chunks touch, which way does the net current point? Show that its magnitude is

$$\frac{\partial M_z}{\partial y}\,\mathrm{d}y\,\mathrm{d}z.$$

b) What is the net bound current density due the touching surface for the chunks considered in (b)?

c) Now, consider two chunks situated at (x, y, z) and $(x, y, z + \mathrm{d}z)$, and consider M_y. Obtain that the net bound current density due to the chunks in (c) and the new chunks is

$$(J_b)_x = (\boldsymbol{\nabla} \times \mathbf{M})_x.$$

This is also done in Griffiths.

Exercise 3. We can write $\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f$. Since $\frac{1}{\mu_0}\nabla\times\mathbf{B} = \mathbf{J}$, write $\mathbf{J}_b = \nabla\times\mathbf{M}$ to obtain a definition for $\mathbf{H}$, where

$$\nabla\times\mathbf{H} = \mathbf{J}_f.$$

Likewise.

Exercise 3. ***Griffiths 6.13.*** Suppose the field inside a large piece of magnetic material is $\mathbf{B}_0$, so that $\mathbf{H}_0 = \frac{1}{\mu_0}\mathbf{B}_0 - \mathbf{M}$, where $\mathbf{M}$ is a "frozen-in" magnetization. Assume the cavities are small enough so that $\mathbf{M}$, $\mathbf{B}_0$, and $\mathbf{H}$ are essentially constant inside them.
Hint: What is the magnetization inside the cavity? Is there a way you could induce that magnetization by using a different configuration and the property of superposition?

a) Now, a small spherical cavity is hollowed out of the material. Find the field at the center of the cavity in terms of $\mathbf{B}_0$ and $\mathbf{M}$. Also find $\mathbf{H}$ at the center of the cavity in terms of $\mathbf{H}_0$ and $\mathbf{M}$.

b) Do the same for a long needle-shaped cavity running parallel to $\mathbf{M}$.

c) Do the same for a thin wafer-shaped cavity perpendicular to $\mathbf{M}$. (A wafer is a disk with a small thickness relative to its radius).

a) Following the hint, we can deduce that the magnetization inside the cavity is 0 (there isn't any material there to be magnetized). The second part of the hint says that we can achieve the same magnetization by instead considering the object without a cavity, and superimpose a spherical object over the cavity area with magnetization $-\mathbf{M}$. Doing this, we'd find that the magnetization in the area where the cavity is supposed to be is exactly 0, so it's the same problem. I gave the field inside a uniformly magnetized sphere in discussion, which is

$$\mathbf{B} = \frac{2}{3}\mu_0\mathbf{M}.$$

Hence, we can compute the new field at the center of the cavity:

$$\mathbf{B} = \mathbf{B}_0 - \frac{2}{3}\mu_0\mathbf{M}.$$

Since

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

and there is no magnetization inside the cavity, we can compute

$$\begin{aligned}\mathbf{H} &= \frac{1}{\mu_0}\mathbf{B} \\ &= \frac{1}{\mu_0}\left(\mathbf{B}_0 - \frac{2}{3}\mu_0\mathbf{M}\right) \\ &= \mathbf{H}_0 + \frac{1}{3}\mathbf{M}.\end{aligned}$$

b) We take the same approach as in (a), and superimpose a needle-shaped material with magnetization $-\mathbf{M}$ in the area where the cavity is supposed to be. Now, if we approximate the needle as a cylinder, notice that since it has uniform magnetization along its axis and a very small radius, the bound surface current will flow along the surface of the cylinder in the $\hat{\varphi}$ direction (if the axis is along $\hat{z}$, say). But this is just a solenoid! We can solve this problem as usual by using an amperian loop which goes along the axis of the cylinder (but is localized near the center, so that we can ignore fringe fields) to find that

$$B\ell = \mu_0 K \ell.$$

Using the fact that $K = M$ from Exercise 1, we have $\mathbf{B} = \mu_0 \mathbf{M}$. Thus, the field inside the cavity is

$$\mathbf{B} = \mathbf{B}_0 - \mu_0 \mathbf{M}.$$

It follows that

$$\mathbf{H} = \mathbf{H}_0.$$

So in this case $\mathbf{H}$ is the same as the original one.

c) This is the opposite limit of the needle in (b). We have a large radius but small thickness. Thus, we can think of this as a single current loop which flows along $\hat{\varphi}$ with radius equal to the radius of the disk. I claim that the field at the center due to this object is 0. Consider first a solenoid. Near the ends of the cylinder, there are fringe fields, and the field at the center of the fringes is small. As you take the length of the cylinder to 0, the dipole field becomes more and more spread out, and the field at the center of the fringes becomes 0. Moreover, as the length goes to 0, the location of the center of the fringes *becomes the center of the disk*. Hence, the field at the center of the disk is 0. It's now easy to solve the problem: The field in the cavity will be exactly

$$\mathbf{B} = \mathbf{B}_0,$$

so in this case it is unchanged. On the other hand, $\mathbf{M} = \mathbf{0}$ in the cavity, so

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{M}.$$