

# Week 2 Worksheet 137A Review; QM in 3D (and some spin)

Jacob Erlikhman

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## Exercise 0. Warm up.

- What is the momentum operator in the position representation in three dimensions?  
*Hint:* What is the canonical commutation relation?
- From (a), what is  $p^2 = \mathbf{p} \cdot \mathbf{p}$  in three dimensions in the position representation?
- Starting from the time-dependent Schrödinger equation in three dimensions for a potential  $V(\mathbf{x})$ , derive the time-independent version.  
*Hint:* Use separation of variables.

**Exercise 1. Rigged Hilbert Space.** A Hilbert space is mathematically defined as a *complete* vector space with an inner product. A vector space with an inner product is **complete** if it includes not only all finite sums of vectors in a basis, but also all limits of convergent sequences, i.e. given a sequence  $(v_n)$  of vectors in the Hilbert space,  $v$  is the **limit** of the sequence if  $\lim_{n \rightarrow \infty} \|v_n - v\| = 0$ , where  $\|v\| = \sqrt{v \cdot v}$ .

- Consider a Hilbert space  $\mathcal{H}$  that consists of all functions  $\psi(x)$  such that

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx < \infty.$$

Show that there are functions in  $\mathcal{H}$  for which  $\hat{x}\psi(x) = x\psi(x)$  is not in  $\mathcal{H}$ .

- Consider the function space  $\Omega$  in  $\mathcal{H}$  which consists of all  $\varphi(x)$  that satisfy the set of conditions

$$\int_{-\infty}^{\infty} |\varphi(x)|^2 (1 + |x|)^n dx < \infty,$$

for any  $n \in \{0, 1, 2, \dots\}$ . Show that for any  $\varphi(x)$  in  $\Omega$ ,  $\hat{x}\varphi(x)$  is also in  $\Omega$ .  $\Omega$  is called the **nuclear** space.

*Hint:* Binomial theorem.

- The **extended** space  $\Omega^\times$  consists of those functions  $\chi(x)$  which satisfy

$$(\chi, \varphi) = \int_{-\infty}^{\infty} \chi^*(x)\varphi(x) dx < \infty,$$

for any  $\varphi$  in  $\Omega$ , where  $(\cdot, \cdot)$  is the inner product on  $\mathcal{H}$ . Which of the following functions belong to  $\Omega$ , to  $\mathcal{H}$ , and/or to  $\Omega^\times$ ?

*Hints:* Note that in order to sit in  $\Omega$ , functions must vanish faster than any power of  $x$  as  $|x| \rightarrow \infty$ . Thus, as long as functions don't diverge at  $\infty$  more strongly than any power of  $|x|$ , they are in  $\Omega^\times$ .

**Remark.** The collection  $(\Omega, \mathcal{H}, \Omega^\times)$  is called “rigged Hilbert space,” and this is a rigorous way to include all the formalism (e.g. eigenvectors of position are delta functions, and hence can’t belong to an  $L^2$  space) into the Hilbert space formulation of quantum mechanics. Note that  $\Omega \subset \mathcal{H} \subset \Omega^\times$  (it’s easy to see this once you realize  $\mathcal{H} = \mathcal{H}^\times$ ). For more details, see Ballentine *Quantum Mechanics*, Chapter 1.

- i)  $\sin(x)$
- ii)  $\sin(x)/x$
- iii)  $x^2 \cos(x)$
- iv)  $e^{-ax}, a > 0.$
- v)  $\frac{\ln(1 + |x|)}{1 + |x|}$
- vi)  $e^{-x^2}$
- vii)  $x^4 e^{-|x|}$
- viii)  $\delta(x - a)$  for  $a$  real.

**Exercise 2. Harmonic Oscillator.** Solve the eigenvalue problem for the 3-D isotropic, harmonic oscillator, whose hamiltonian is  $H = p^2/2m + m\omega^2 x^2/2$ , where  $p^2 = \mathbf{p} \cdot \mathbf{p}$ ,  $x^2 = \mathbf{x} \cdot \mathbf{x}$  is the 3-D dot product.

*Hint:* There’s a way to do this without any calculations (if you remember the 1-D oscillator)!

**Exercise 3.** A particle of mass  $m$  is placed in a finite spherical well

$$V(r) = \begin{cases} -V_0, & r \leq a \\ 0, & r \geq a \end{cases}.$$

Find the equation that quantizes the energy (you don’t need to solve it), by solving the radial Schrödinger equation with  $\ell = 0$ . Explain how you could solve this equation and obtain the energies. Show that there is no bound state if  $V_0 a^2 < \pi^2 \hbar^2 / 8m$ .

*Hint:* Recall that the radial Schrödinger equation is identical to the time-independent, 1-dimensional Schrödinger equation with the wavefunction replaced by  $u(r) = rR(r)$  (where  $\psi(r, \theta, \varphi) = R(r)\Theta(\theta)\Phi(\varphi)$ ) and potential  $V_{\text{eff}}(r) = V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}$ .

**Exercise 4. Spin Representations.** a) Find the eigenvalues and eigenvectors of  $S_z$ .

- b) Do the same for  $S_y$ , and write them in terms of  $|\uparrow\rangle$  and  $|\downarrow\rangle$ , the eigenvectors of  $S_z$ .
- c) For a system of two spin 1/2 particles, starting with the “highest weight” state  $|\uparrow\uparrow\rangle$ , find all the states in the triplet.  
*Hint:* Apply the lowering operator.
- d) For a system of two spin 1/2 particles, are there any other states than the ones you found in (c)? If so, what are they? What is the action of  $S_-$ ,  $S_+$  on them?
- e) Describe how you would approach finding the Clebsch-Gordan coefficients for arbitrary spin systems.