

Week 9 Worksheet

Curvature

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Exercise 1. Parallel Transport is Curvature. Let M be a spacetime such that for any two points $p, q \in M$, the parallel transport from p to q does not depend on the curve that joins p and q . You will show that this implies that M is flat, i.e. that the Riemann curvature tensor on M is identically 0. We will do this with the help of the following construction. Consider a parametrized surface $f : U \rightarrow M$, where

$$U = \{(s, t) \in \mathbb{R}^2 | s, t \in (-\varepsilon, 1 + \varepsilon), \varepsilon > 0\}$$

and we force $f(s, 0) = f(0, 0)$ for all s . Let V_0 be a tangent vector to M at $f(0, 0)$, and define a vector field V along f as follows. Set $V(s, 0) = V_0$ and $V(s, t)$ to be the parallel transport of V_0 along the curve $c(t) = f(s, t)$.

- a) Sketch V in the case that M is flat, and explain what changes in the non-flat case.
- b) Since V is parallel transported along the t -direction, what is $\nabla_{\partial_t f} V$?
- c) Show that

$$\nabla_{\partial_t f} \nabla_{\partial_s f} V + R(\partial_t f, \partial_s f) V = 0,$$

where in a coordinate system x^i with $Z = \lambda^i \partial_i = \lambda^i \frac{\partial}{\partial x^i}$

$$R(\partial_j, \partial_k) Z = -\lambda^l R^i{}_{ljk} \partial_i.$$

Hints: Recall that R is a rank (3,1) tensor; what does this tell you about $R(v^i \partial_i, w^j \partial_j) Z$? Recall the Ricci identity

$$\nabla_j \nabla_k v^i - \nabla_k \nabla_j v^i = -v^l R^i{}_{ljk},$$

and compute in a coordinate system.

- d) Show that $V(s, 1)$ is also the parallel transport of $V(0, 1)$ along the curve $c(s) = f(s, 1)$, so that $\nabla_{\partial_s f} V(s, 1) = 0$.
- e) Show that

$$R(\partial_t f, \partial_s f) V(0, 1) = 0,$$

where the $(0, 1)$ means we consider the vector at the point $(s, t) = (0, 1)$.

- f) Conclude that $R = 0$ everywhere by arbitrariness of our choices.