Week 8 Worksheet Solutions
Boundary Value Problems and Mutlipoles

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Exercise 1. ***Griffiths 3.13.*** Two infinite grounded metal plates lie parallel to the (x,z)-plane. One is at $y=0$ and the other at $y=a$. The left end, at $x=0$, is closed off with an infinite strip insulated from the two plates and maintained at a potential

$$V_0(y)=\begin{cases} V_0, & y\in\left(0,\frac{a}{2}\right) \\ -V_0, & y\in\left(\frac{a}{2},a\right). \end{cases}$$

a) What are the boundary conditions for this problem?

b) Argue that the situation is independent of z, so that we can use the Laplace equation in the x and y coordinates only.

c) Qualitatively describe the behavior of the potential as a function of x for $x \gg 0$.

d) Write down Laplace's equation, and separate variables.

e) After obtaining something of the form

$$\frac{X''}{X}+\frac{Y''}{Y}=0,$$

argue that each term must be individually constant.
Hint: The second term is a function of y only, so, fixing $y=y_0$, it must remain constant as we vary x. What happens to the first term as we do this?

f) Write down the general form of the solutions for X and Y.

g) Enforce the boundary conditions on your solutions, and solve for the potential inside the slot.

a) Since the plates are grounded, we must have $V=0$ at $y=0$ and $y=a$ for $x>0$. We also have the boundary condition

$$V(0,y)=V_0(y).$$

Lastly, the potential must go to 0 as we get far away from the plate that is held at potential $\pm V_0$. Hence, $V\to 0$ as $x\to\infty$.

b) Any solution we obtain must be the same at any value of z, since the picture of the configuration that we obtain is independent of z. Said another way, translations along the z-axis do not change the boundary configuration; hence, they cannot affect the obtained solution to Laplace's equation V.

c) It should be exponentially decaying as we get farther from $x = 0$ (or at least decaying somehow).

d) Write $V(x, y) = X(x)Y(y)$, so that the Laplace equation is

$$\frac{X''}{X} + \frac{Y''}{Y} = 0.$$

Now, each of these terms must be individually equal to a constant. Indeed, the first term depends on x only, while the second on y only. Hence, as we vary x say, the second term would remain equal to some function of y, while the first term would change in general. But since the second term is constant, the first must be constant too. The same argument applies in the other direction, so both are equal to constants. Since they add up to 0, we can set one of them to be α^2 and the other $-\alpha^2$.

e) We can thus write without loss of generality

$$\begin{aligned}\frac{X''}{X} =& \alpha^2 \\ \frac{Y''}{Y} =& -\alpha^2,\end{aligned}$$

where we've set X to be equal to the positive constant since we want the potential to exponentially die off in the x-direction. Indeed, the potential should go to 0 as x goes to infinity. The solutions are then

$$\begin{aligned}X(x) =& Ae^{-\alpha x} + Be^{\alpha x} \\ Y(y) =& C\cos(\alpha y) + D\sin(\alpha y).\end{aligned}$$

f) Now, our boundary condition that the potential must die off at infinity says $B = 0$. The condition that $V = 0$ at $y = a$ and $y = 0$ tell us that $C = 0$ and that

$$\alpha = \frac{n\pi}{a},$$

where n is any integer. The general solution will be a linear combination of all allowed ones, so we need to sum over all n. We can thus write the general solution for V as

$$V(x, y) = \sum_{n=1}^{\infty} A_n e^{-n\pi x/a} \sin\left(\frac{n\pi}{a} y\right)$$

where we've taken the sum to start from 1 by noticing that $\sin(0) = 0$ and that we should ignore all negative n since they would cause the potential to blow up at $x \to \infty$ due to the exponential. It remains to determine what the A_n are, which we can do by considering the final boundary condition: the potential at $x=0$. Since the sine functions for different values of n are orthogonal, we can set

$$V(0,y) = V_0(y)$$

and then multiply both sides by $\sin(\frac{m\pi}{a}y)$ and integrate from $y=0$ to $y=a$ to determine A_m. On the right hand side, we get

$$\begin{aligned}
&V_0\left(\int_0^{a/2}\sin\left(\frac{m\pi}{a}y\right)\mathrm{d}y - \int_{a/2}^{a}\sin\left(\frac{m\pi}{a}y\right)\mathrm{d}y\right) = \\
&= \frac{V_0 a}{m\pi}\left[-\cos\left(\frac{m\pi}{2}\right) + 1 + (-1)^m - \cos\left(\frac{m\pi}{2}\right)\right] = \\
&= \frac{2V_0 a}{m\pi}\begin{cases} 1-(-1)^{m/2}, & m \text{ even} \\ 0, & m \text{ odd} \end{cases}.
\end{aligned}$$

On the left hand side, note that

$$\int_0^a \sin^2\left(\frac{m\pi}{a}y\right)\mathrm{d}y = \frac{a}{2},$$

so that we get

$$\frac{A_m a}{2} = \frac{2V_0 a}{m\pi}\begin{cases} 1-(-1)^{m/2}, & m \text{ even} \\ 0, & m \text{ odd} \end{cases}.$$

Set $m = 2k$, so that

$$A_{2k} = \frac{2V_0}{k\pi}[1-(-1)^k].$$

Finally, we can write our solution as

$$\begin{aligned}
V(x,y) &= \sum_{k=1}^{\infty}\frac{2V_0}{k\pi}[1-(-1)^k]e^{-2k\pi x/a}\sin\left(\frac{2k\pi}{a}y\right) \\
&= \sum_{k=0}^{\infty}\frac{4V_0}{(2k+1)\pi}e^{-2(2k+1)\pi x/a}\sin\left(\frac{2(2k+1)\pi}{a}y\right).
\end{aligned}$$

Exercise 2. A half-infinite dipole string with linear dipole moment density $\mathbf{p}_l = p_l\hat{x}$ is placed along the negative z-axis.

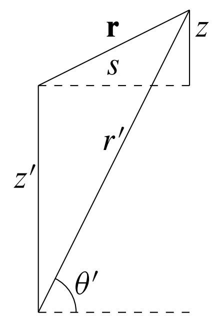
a) The potential due to a dipole *at the origin* is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{r}}{r^2}.$$

Generalize this to find the potential $V(x,y,z)$ due to the string. This is probably easier if you use cylindrical coordinates.

b) Investigate the behavior of the potential from (a) as you approach the z-axis in the regions $z < 0$ and $z > 0$.

a) In general, r will be the distance from a point on the negative z-axis to a point where we'd like to calculate the potential. Similarly, $\mathbf{p} \cdot \hat{r}$ will be $\cos(\theta') p_l$, where θ' is the angle between a vector $\mathbf{r}'$ from a point on the negative z-axis to the point where we'd like to calculate the potential and the x-axis (since $\mathbf{p}$ is along $\hat{x}$). To compute the total potential, we'd then integrate along the negative z-axis. Let's calculate each of these in cylindrical coordinates, but it helps to first draw a picture:



Notice that from the picture we can read off that $\cos\theta' = \frac{s}{r'}$ and $r'^2 = s^2 + (z+z')^2$. Hence, we have

$$\begin{aligned} V(\mathbf{r}) &= \int_0^{-\infty} \frac{p_l s}{r'^3} \, \mathrm{d}z' \\ &= \int_0^{-\infty} \frac{p_l s}{\left[s^2 + (z+z')^2\right]^{3/2}} \, \mathrm{d}z' \\ &= \frac{z'+z}{s\sqrt{s^2 + (z+z')^2}} \Bigg|_0^{-\infty} \\ &= -\frac{1}{s} - \frac{z}{s\sqrt{s^2+z^2}}. \end{aligned}$$

To convert back to cartesian coordinates, we need only write $s = \sqrt{x^2+y^2}$, so that

$$V(\mathbf{r}) = -\frac{1}{\sqrt{x^2+y^2}} - \frac{z}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}}.$$

b) Now, let's investigate what happens as we approach the z-axis. This is easier to do with the cylindrical coordinate version of our result above. Notice that the function we obtain has a pole of order 1 in s (for any value of z). Now, if $z \le 0$, then our function approaches ∞ as $s \to 0^-$ and it approaches $-\infty$ as $s \to 0^+$. Hence, there is an infinite discontinuity at the z-axis, when $s = 0$. On the other hand, if $z > 0$, then our function approaches 0 as $s \to 0$ from either the positive direction or the negative one. Hence, the function is *continous at $s = 0$ in this case*! This means that the infinite line of dipoles only causes a divergence in our potential when we get near it—if we're in the region $z > 0$ there is no divergence at all.

Remark. What we are witnessing in this exercise is a breakdown of the space in which we are doing physics. Usually, we are able to define a *continuous* potential function on all of $\mathbb{R}^3$; however, it seems that now the negative z-axis is "not allowed." This means that the topology is no longer $\mathbb{R}^3$, but $\mathbb{R}^3 \setminus \mathbb{R}^+$, or something like that. Later on, we'll talk about solenoids: These have a magnetic field inside but none outside. In the Aharanov-Bohm effect, an electron will "feel the effect" of the magnetic field inside a solenoid via the vector potential (i.e. a vector $\mathbf{A}$ such that $\nabla \times \mathbf{A} = \mathbf{B}$, where $\mathbf{B}$ is the magnetic field), which is *not* zero outside the solenoid. This is because the electron somehow knows that the topology of space changed from $\mathbb{R}^3$ to $\mathbb{R}^3$ with a hole in it (the hole is the solenoid). This has deep connections to magnetic monopoles and other interesting phenomena, and this exercise is a toy example of this.