Project 4 Mathematics 512

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Due date: Friday April 5th

1.

Let W_t be a standard Wiener process, that is the drift parameter is zero and the Variance parameter $\sigma^2 = 1$. Suppose that we divide the interval $[0,2]$ $into\ L\ subintervals\ [t_i, t_{i+1}], with\ t_i = i\delta t\ and\ \delta t = 2/L$. Let $W_i = W(t_i)\ \ and\ \ \delta W_i = W_{i+1} - W_i$. Verify numerically that

a) $\sum_{i=0}^{L-1} |\delta W_i|$ is unbounded as δt goes to zero

b) $\sum_{i=0}^{L-1} \delta W_i{}^2$ converges to 2 in probability as δt goes to zero

2.

Evaluate numerically the stochastic integrals

a) Itô $\int_0^2 W(t)dW(t)$

b) Stratonovich $\int_0^2 W(t) \circ dW(t)$

c) $E\left[\int_0^2 W(t)dW(t)\right]$

d) $E\left\{\left[\int_0^2 W(t)dW(t)\right]^2\right\}$

e) $E\left[\int_0^2 W^2(t)dt\right]$

f) For $t \in [0,2]$ evaluate $\int_0^t W(t)dW(t)$, $\int_0^t W(t) \circ dW(t)$ and $\frac{1}{2}\int_0^t dt$ what do you observe.

3.

Consider the following SDE:

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t)\,,\quad X(0) = 3, \qquad \mu = 2\,, \qquad \sigma = 0.10$$

Where $t \in [0,1]$.

a) Show that the Euler Maruyama method has weak order of convergence equal to one. That is $|E[X_1] - E[X(1)]| = C\Delta t$. Here $X(1)$ is the exact solution at time 1 and X_1 is the computed solution at time 1.

b) Show that the Euler Maruyama method has strong order of convergence equal to one half. That is $E|X_1 - X(1)| = C\Delta t^{0.5}$. Here $X(1)$ is the exact solution at time 1 and X_1 is the computed solution at time 1.

4.

Consider the following SDE:

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t)\,,\;\; X(0) = 3, \qquad \mu = 2\,, \qquad \sigma = 0.10$$

a) Simulate (over the interval $[0,20]$) this stochastic process using an implicit method of the form

$$X_{n+1} = X_n + (1-\theta)\Delta t f(X_n) + \theta \Delta t f(X_{n+1}) + \sqrt{\Delta t}\alpha_n g(X_n)$$

b) Compare with the analytical solution.

c) For what values of $\mu \; and \; \sigma$ is the SDE mean-square stable.

d) For what values of θ is the implicit method mean-square stable.

e) For what values of $\mu \; and \; \sigma$ is the SDE asymptotically stable.

f) For what values of θ is the Implicit method asymptotically stable.

Verify your results using numerical simulations.

5.

Consider the following SDE:

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t)\,,\;\; X(0) = 2, \qquad \mu = 0.1, \qquad \sigma = 0.15$$

Let $a = 0.5 \;\; and \; b = 3$.
Compute the mean exit time function $v(x)$ for $x \in [0.5, 3]$