# MATH 512 - Project 3

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# Question 1 Overview

- Let $W_t$ be a standard Wiener process, with drift parameter zero and variance parameter $\sigma^2 = 1$.
- We divide the interval $[0, 2]$ into $L$ subintervals $[t_i, t_{i+1}]$, where $t_i = i\delta t$ and $\delta t = 2/L$.
- Let $W_i = W(t_i)$ and $\delta W_i = W_{i+1} - W_i$.
- We verify numerically that:
  - $\sum_{i=0}^{L-1} |\delta W_i|$ is unbounded as $\delta t$ goes to zero.
  - $\sum_{i=0}^{L-1} \delta W_i^2$ converges to 2 in probability as $\delta t$ goes to zero.

# Question 1 Response

Refer to Figure 1

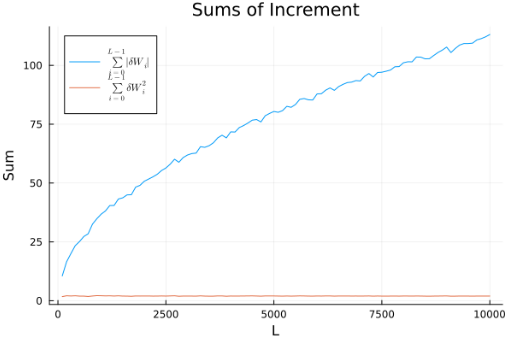



Figure: Stochastic Plots

Notice that as the $L$ parameter increases, the $|\delta W_i|$ term is unbounded while $\delta W_i^2$ converges to 2 in probability.

# Question 2a

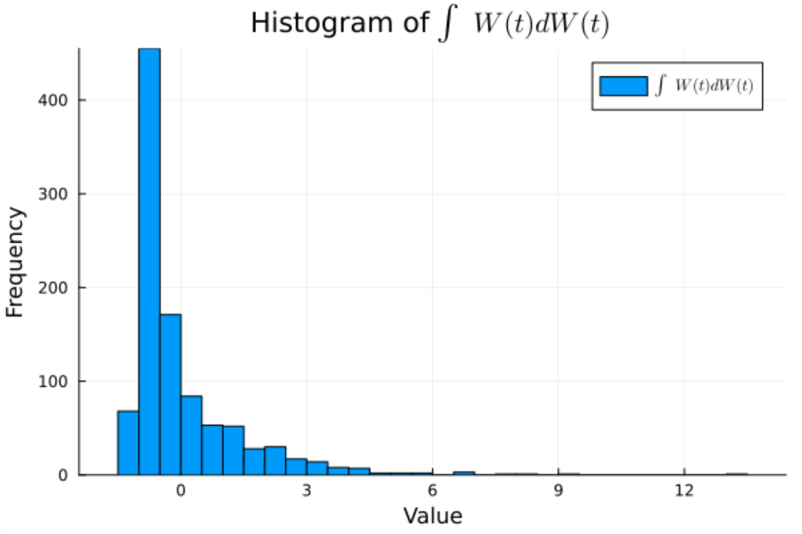



Figure: 2a

# Question 2b

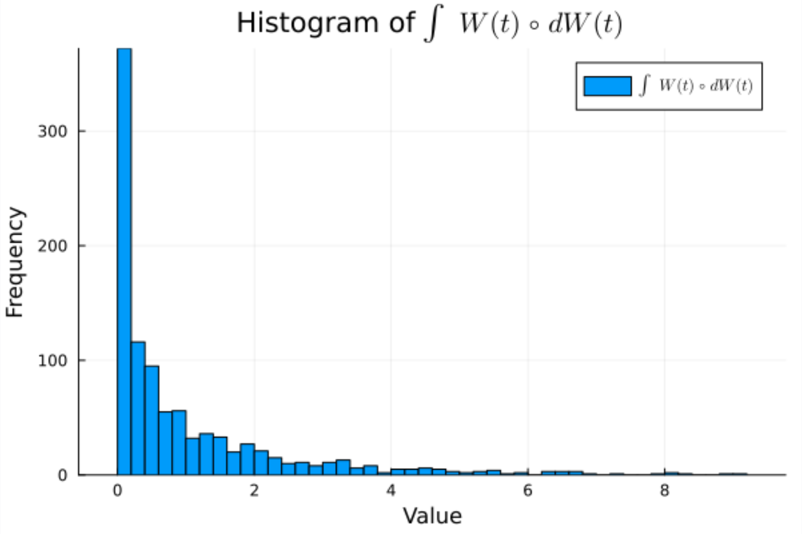



Figure: 2b

## Question 2c

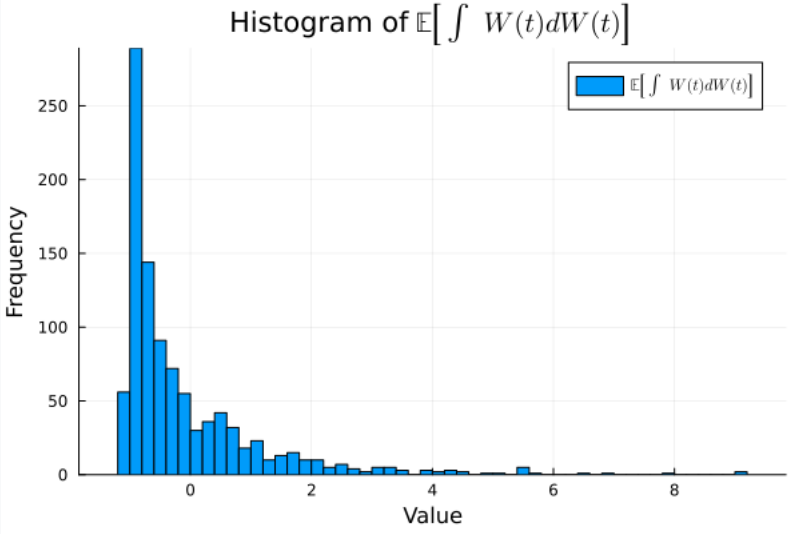



Figure: 2c

# Question 2d

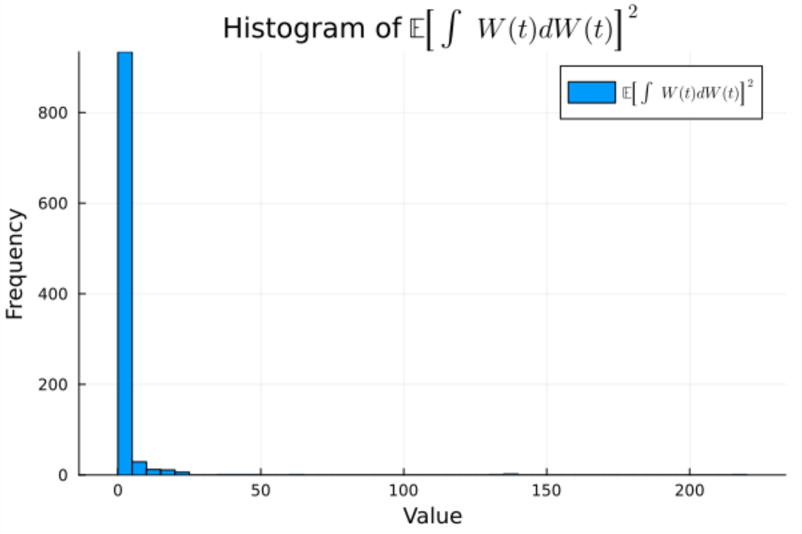



Figure: 2d

# Question 2e

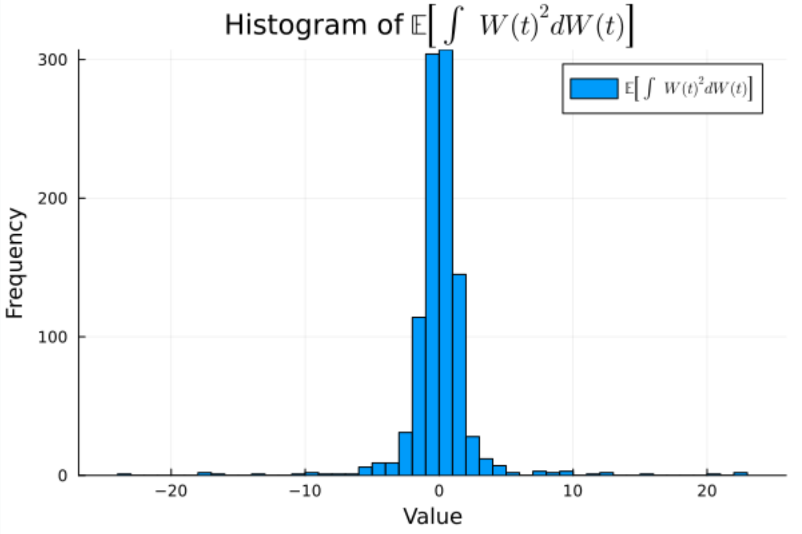



Figure: 2e

# Question 2f

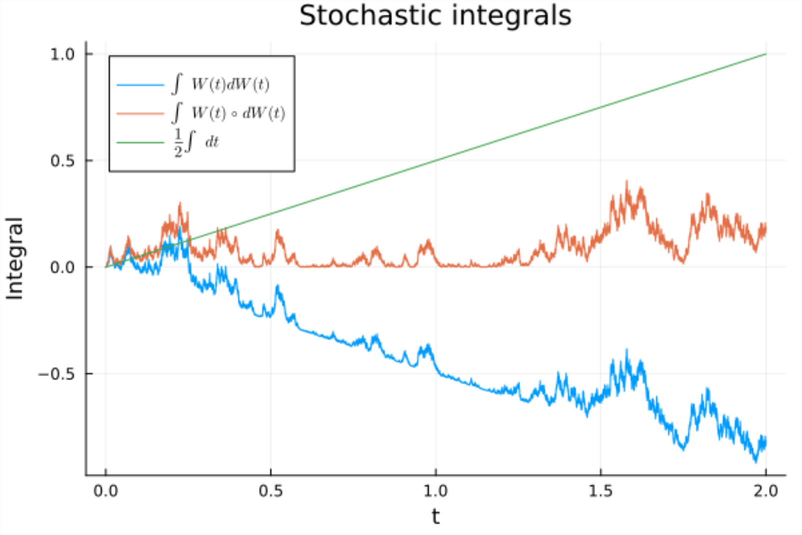



Figure: 2f

# Question 3: Weak and Strong Order of Convergence for EM Method

**Objective:** Analyze the weak and strong order of convergence of the Euler-Maruyama method for solving the given Stochastic Differential Equation (SDE):

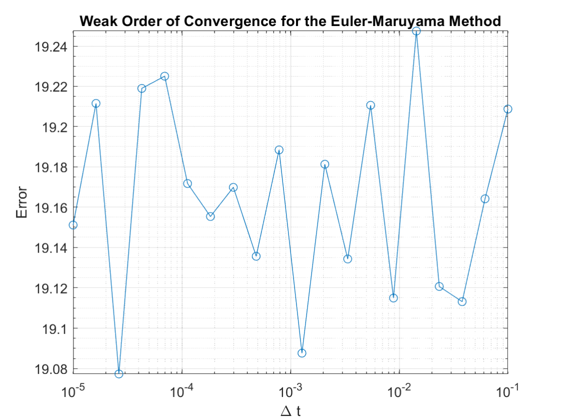
$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \quad X(0) = 3, \quad \mu = 2, \quad \sigma = 0.10$$

- **Weak order of convergence equal to 1**, indicating that the error in the expected value of the solution does not converge with the decrease of time step size, $\Delta t$.
- **Strong order of convergence equal to 0.5:**, meaning that the expected value of the absolute error between the numerical and exact solutions decreases with the square root of the time step size.

# Weak Order of Convergence Results

**Findings:** Through MATLAB simulations over various $\Delta t$, we observed:

- A plot of the error versus $\Delta t$ for the weak order of convergence of 1.



# Strong Order of Convergence Results

**Findings:** For the strong order of convergence:

- A strong order of convergence of 0.5.

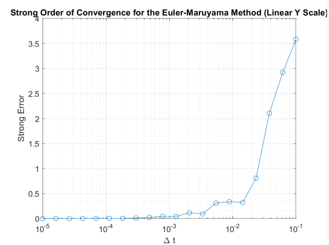



Figure: Weak Order of Convergence

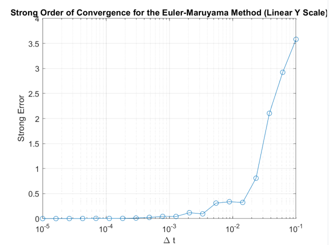



Figure: Strong Order of Convergence

**Note:** These results show the Euler-Maruyama method has a weak order of convergence equal to 0.5 and a strong order of convergence equal to 1.

# Question 4

We consider the following SDE:

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \quad X(0) = 3, \quad \mu = 2, \quad \sigma = 0.10$$

- Simulate this stochastic process over the interval [0, 20] using an implicit method of the form:

$$X_{n+1} = X_n + (1-\theta)\Delta t f(X_n) + \theta \Delta t f(X_{n+1}) + \sqrt{\Delta t}\alpha_n g(X_n)$$

- We use $\theta = 0.5$ and $\alpha_n = \sigma X_n$.
- We plot the results for $\Delta t = 0.01$ and $\Delta t = 0.001$.

# Question 4 Continued

We plot the explicit and implicit methods for $\Delta t = 0.01$ and $\Delta t = 0.001$:

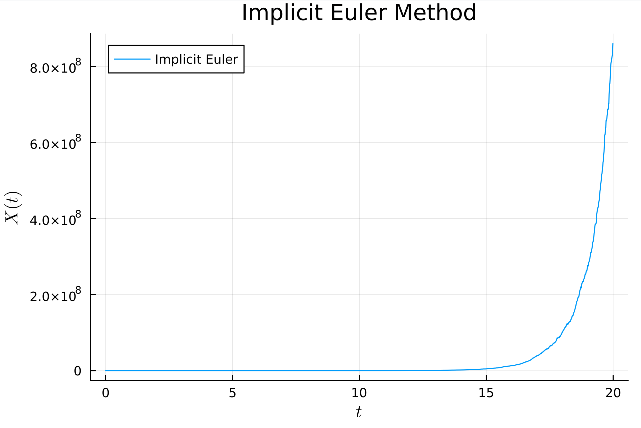



Figure: Implicit Euler Method

# Question 4 Continued

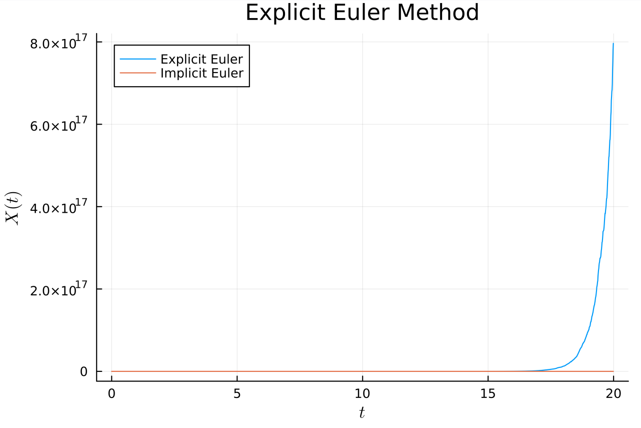



Figure: Explicit Euler Method

# Question 4 Continued

We determine the values of $\mu$ and $\sigma$ for which the SDE is mean-square stable:

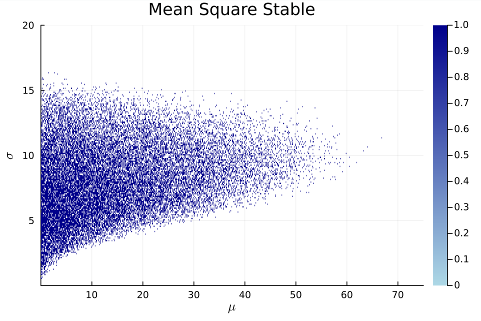



Figure: Mean-Square Stability

and we find that the SDE is not mean square stable for any values of $\theta$.

# Question 5

An algorithm to simulate exit times for the SDE:

Choose a step size $\Delta t$
Choose a number of paths, $M$
**for** $s = 1$ to $M$ **do**
    Set $t_n = 0$ and $X_n = X_0$
    **while** $X_n > a$ and $X_n < b$ **do**
        Compute a $N(0, 1)$ sample $\xi_n$
        Replace $X_n$ by $X_n + \Delta t f(X_n) + \sqrt{\Delta t}\xi_n g(X_n)$
        Replace $t_n$ by $t_n + \Delta t$
    **end while**
    Set $T_s^{exit} = t_n - 1/2\Delta t$
**end for**
Set $a_M = \frac{1}{M}\sum_{s=1}^{M} T_s^{exit}$
Set $b_M^2 = \frac{1}{M-1}\sum_{s=1}^{M}(T_s^{exit} - a_M)^2$

# Question 5 Continued

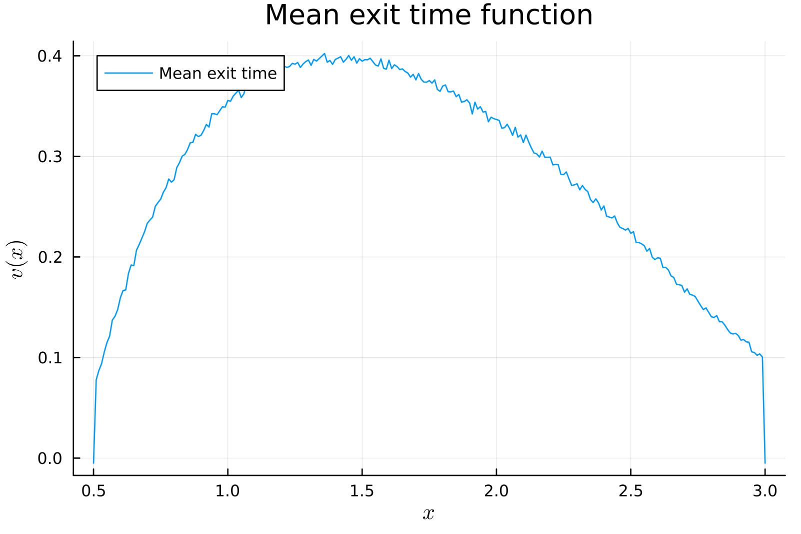



Figure: 5

# Question 5 Continued

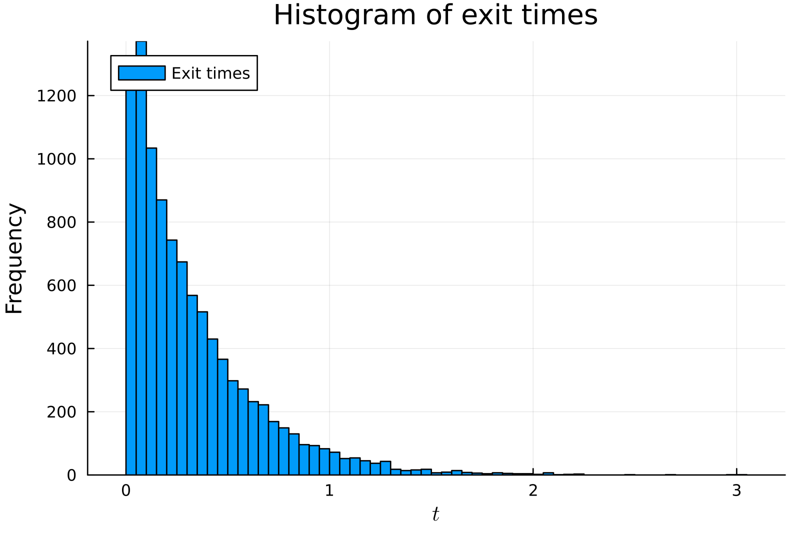



Figure: 5

Questions?