MATH 512 - Project 2

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Question 1 (a)

We use the *Kolmogorov-Smirnov* test to test for the uniformity of the random numbers generated by the LCG. The test statistic is given by

$$D_n = \max_{1 \leqslant i \leqslant n} \left(\frac{i}{n} - U_{(i)} \right) \vee \max_{1 \leqslant i \leqslant n} \left(U_{(i)} - \frac{i-1}{n} \right)$$

where $U_{(i)}$ is the i-th order statistic of the U_i's.

- $H_\circ$: the random numbers are uniformly distributed.

We find that $D_n = 0.0069$ and a p-value of 0.708. This means that we fail to reject $H_\circ$ at the 5% significance level and conclude that the random numbers are uniformly distributed.

Question 1 (b)

Parameters: $a = 6, m = 11, x_0 = 3, c = 0, n = 10$

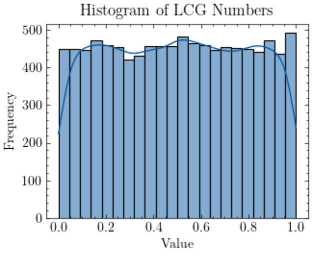
- The sequence is $\{3, 7, 9, 10, 5, 8, 4, 2, 1, 6\}$
- The period is 10

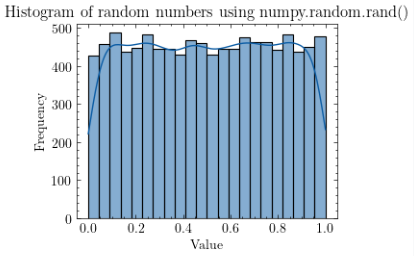
Parameters: $a = 6, m = 10, x_0 = 3, c = 0, n = 10$

- The sequence is $\{3, 8, 8, 8, 8, 8, 8, 8, 8, 8\}$
- The period is 1

We notice that even a *small* change in the parameters results in a seemingly non-random sample.

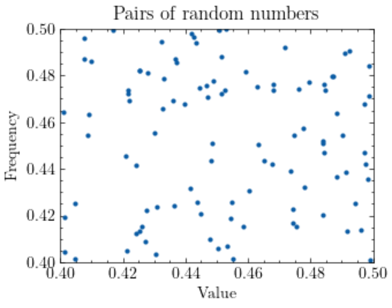
Question 1 (d)





- The two histograms look relatively similar, meaning both look relatively uniform.

Question 1 (e)



- The pairs of numbers seem to have a random pattern.

Question 1 (f)

Disadvantages of LCG:

- It can appear random with the right set of parameters, but as we saw, it can get “stuck” in a loop.
- The randomness depends on the choice of parameters.

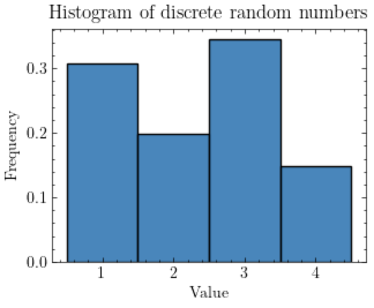
Question 2

$$P(X=k) = \begin{cases} 0.3 & \text{for } k=1 \\ 0.2 & \text{for } k=2 \\ 0.35 & \text{for } k=3 \\ 0.15 & \text{for } k=4 \end{cases}$$

We draw 10000 random Uniform random numbers using the following rule:

$$\begin{aligned} U \leqslant 0.3 &\implies X=1 \\ 0.3 < U \leqslant 0.5 &\implies X=2 \\ 0.5 < U \leqslant 0.85 &\implies X=3 \\ 0.85 < U \leqslant 1 &\implies X=4 \end{aligned}$$

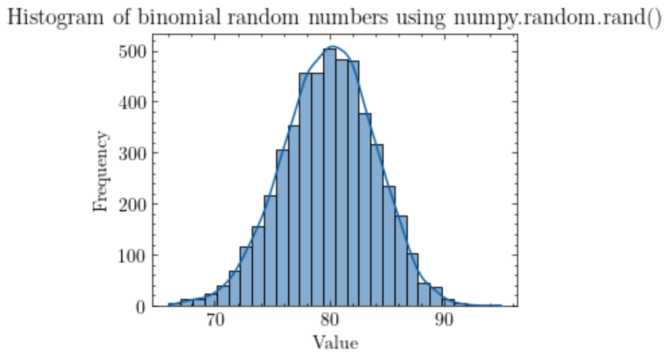
Question 2



Histogram of discrete random numbers
Frequency
0.0
0.1
0.2
0.3
1
2
3
4
Value

Question 3(a) NumPy r.v. Generator

- Observed Probability that $X < 50$: 0.00000000000.



- Theoretical Probability:

$$P(X < 50) = \sum_{k=0}^{49} \binom{100}{k} (0.8)^k (0.2)^{100-k} = 2.14 \times 10^{-11}$$

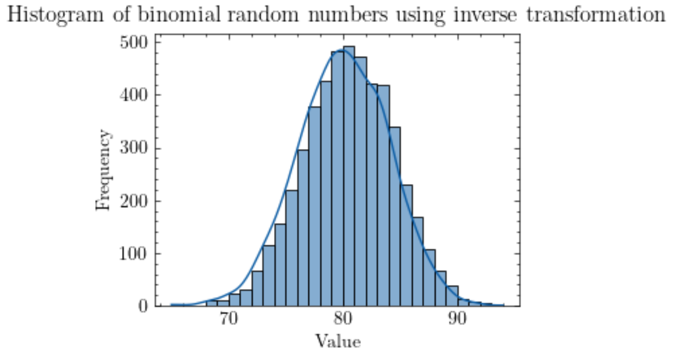
Question 3(b) Inverse Transformation Method

Generating a sum of random Bernoulli random variables is much faster. Nonetheless, we use the following algorithm for selecting a Binomial distribution:

Algorithm Binomial RV Generation

1. **for** $n = 1, \ldots, n_{sim}$ **do**
2. pmf = $[\binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$ for $k \in \{1, \cdots, k, k+1\}]$
3. cdf = $\sum_{i=1}^{k+1} pmf_i$
4. Draw $U \sim Uniform[0, 1]$
5. x[i] = arg max(cdf $> U$)
6. **end for**

Question 3(b) Inverse Transformation Method



Question 3(c)

- Time for NumPy r.v. Generator: 0.0726s
- Time for inverse transformation method: 0.8775s

Question 4

Formula:

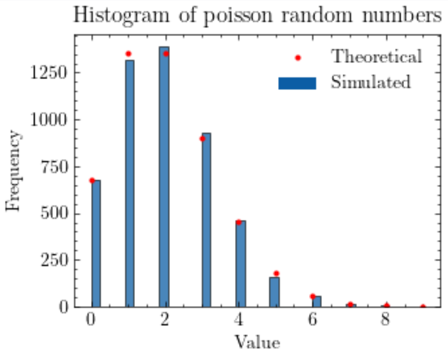
$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Poisson r.v. with $\lambda = 2$. We use the following algorithm:

Algorithm Poisson RV Generation

for $n = 1, \ldots, n_{sim}$ **do**
    $x[i] = 0$
    $j = 0$
    while $x[i] < \lambda$ **do**
        $x[i] -= \log(U)$
        $j += 1$
    end while
    $x[i] = j - 1$
end for

Question 4



Histogram of poisson random numbers
Theoretical
Simulated
Frequency
1250
1000
750
500
250
0
0
2
4
6
8
Value

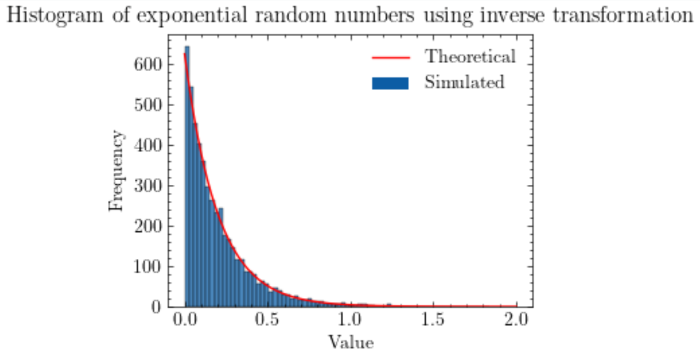
Question 5

Exponential r.v. with $\lambda = 5$.
Inv. Transformation Method:

$$F(x) = 1 - e^{-\lambda x}$$

$$F^{-1}(u) = -\frac{\log(1-u)}{\lambda}$$

Question 5


Histogram of exponential random numbers using inverse transformation
Theoretical
Simulated
Frequency
0
100
200
300
400
500
600
0.0
0.5
1.0
1.5
2.0
Value

Question 6

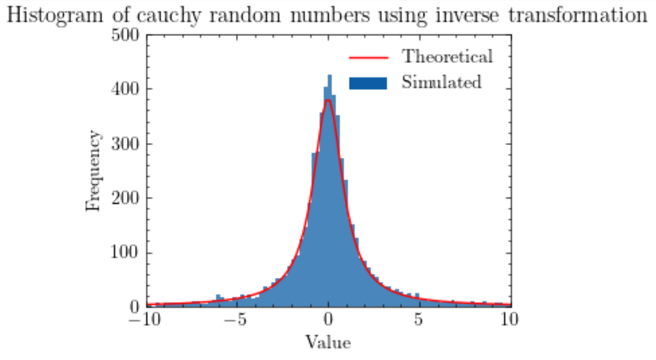
- The Cauchy r.v. has the following pdf:

$$f(x) = \frac{1}{\pi(1 + x^2)}$$

- We use the inverse transformation method to generate the Cauchy r.v.

$$F(x) = \frac{1}{\pi}\,\text{arctan}(x) + \frac{1}{2}$$
$$F^{-1}(u) = \tan(\pi(u - \frac{1}{2}))$$

Question 6



Histogram of cauchy random numbers using inverse transformation
Theoretical
Simulated
Frequency
Value
500
400
300
200
100
0
−10
−5
0
5
10

Question 7

Simulation:

- For x=1,2,...100, shuffle the position of x[i]
- After shuffling, if x[i]=i, count 1, o.w. 0.
- Sum up all the counting.

With a repetition of 10000 simulations, we find that the expectation is 1.00 and the variance is 1.01.

We can calculate $\mathbb{E}[X]$ and $\mathbb{V}[X]$ exactly based on Poisson Distribution and compare them with our estimates.

$$\mathbb{E}[X] = \sum_{i=1}^{100} i \cdot \frac{1^i \cdot e^{-1}}{i!} = 1$$

$$\mathbb{V}[X] = \sum_{i=1}^{100} i^2 \cdot \frac{1^i \cdot e^{-1}}{i!} - (\mathbb{E}[X])^2 = 1$$

The estimates are very close to the exact values. This is expected because the number of simulations is large.

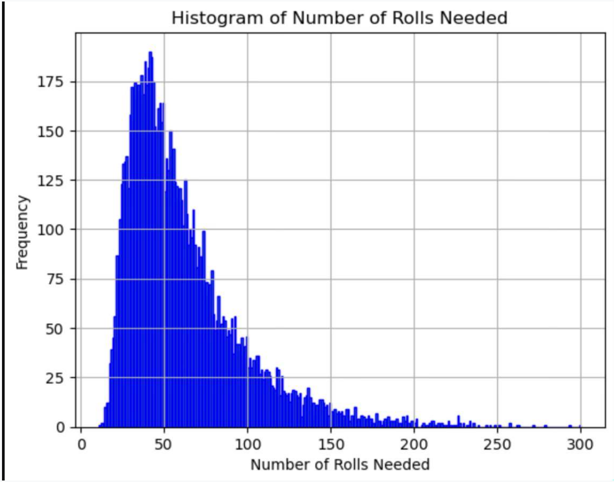
Question 8: Rolling Two Die

Algorithm:

- Generate two random integers in between [1, 6] and take the sum.
- Keep on generating the sum until all the integers in between [2, 12] are generated.
- Count the number of iterations it requires.
- Repeat the same process 10000 times.

Result: The expected number of rolls required is **60.8352**.
Note: Calculating expected values by hand would require some Markov Chain (simulation is easier!)

Question 8: Rolling Two Die



Question 9: Random Selection of 10 Balls

Algorithm

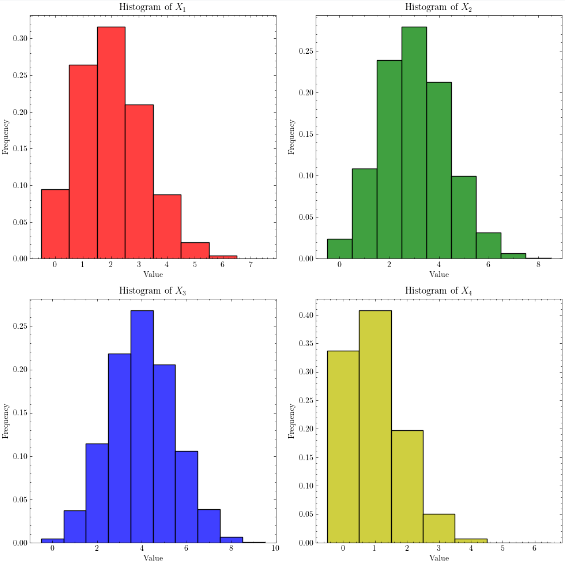
- Make an array of the following form:

$$urn = \{0, 0, \cdots 20 \text{ times}; 1, 1, \cdots 30 \text{ times};$$

$$2, 2, \cdots 40 \text{ times}; 3, 3, \cdots 10 \text{ times}\}$$

- Reshuffle urn.
- Pick the first 10 numbers and count the number of times we have $\{X_1, X_2, X_3, X_4\} = \{0, 1, 2, 3\}$.
- Repeat.

Question 9: Random Selection of 10 Balls



Question 10(a): Box-Muller Method

Algorithm:

- Generate two independent $U[0, 1]$ set of random numbers: $u1$ and $u2$.
- Define:

$$R = \sqrt{-2\log(u1)}$$

$$x = \cos(2\pi u2)$$

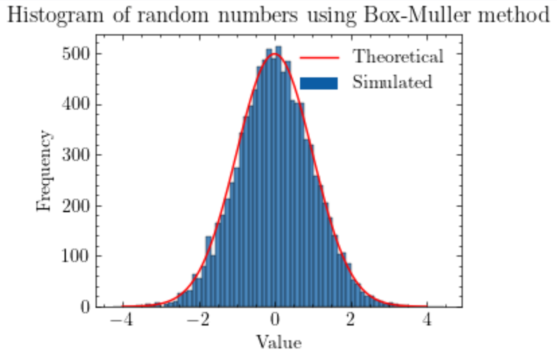
$$y = \sin(2\pi u2)$$

$$z1 = r * x$$

$$z2 = r * y$$

- Return $z1$ and $z2$.

Question 10 (a): Box-Muller Method



Question 10(b): Marsaglia-Bray Method

Algorithm:

- Generate two independent $U[0, 1]$ set of random numbers: $u1$ and $u2$.
- Define:

$$w1 = (2u1) - 1$$

$$w2 = (2u2) - 1$$

$$s = w1^2 + w2^2$$

- if $s < 1$,

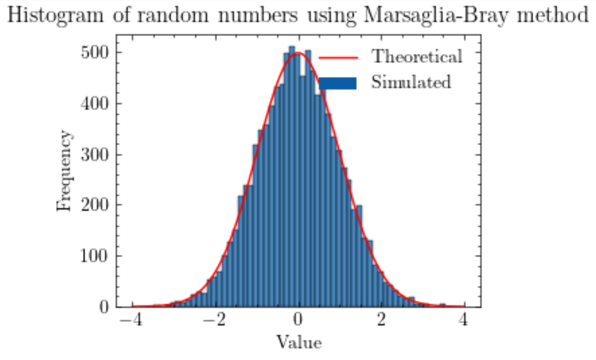
$$t = \sqrt{-2\frac{\log(s)}{s}}$$

$$z1 = w1 * t$$

$$z2 = w2 * t$$

- Return $z1$ and $z2$.

Question 10(b): Marsaglia-Bray Method

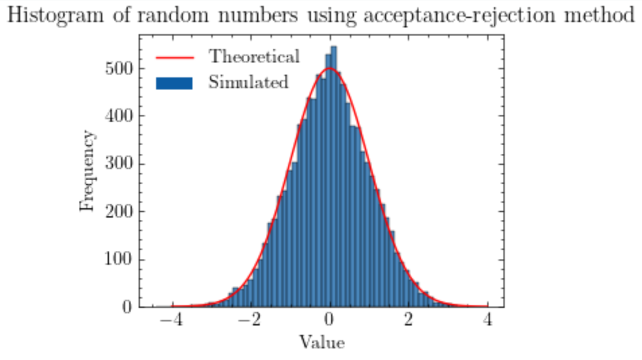


Question 10(c): Acceptance-Rejection Method

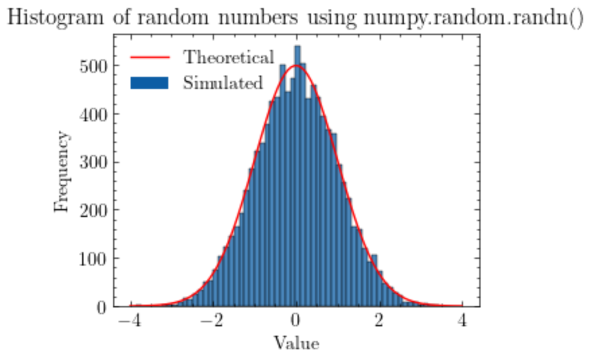
Algorithm:

- Define:
  Normal Density Function (NDF): $\exp(\frac{-x^2}{2}/2\pi)$
  Proposal Density Function (PDF): $\exp(-x^2/2)$
- Generate $u \sim U[0, 1]$ and x ~PDF.
- $\sim 1.32 = \sqrt{2e/\pi}$
- if $u < \frac{\mathrm{NDF}[x]}{1.32 * \mathrm{PDF}[x]}$, then accept x.

Question 10(c): Acceptance-Rejection Method



Question 10(d): *random.randn()*



All of the methods are very close to the theoretical normal distribution.

Question 10: Timing

Table: Execution Times of Various Random Number Generation Methods

Method	Time (ms)	Comments
Box-Muller Method	1.2269	Fast execution
Marsaglia-Bray Method	16.712	Moderate execution
Acceptance-Rejection Method	75.590	Slow execution
$random.randn()$	0.1469	Very fast execution

Questions?