MATH 512 - Project 4

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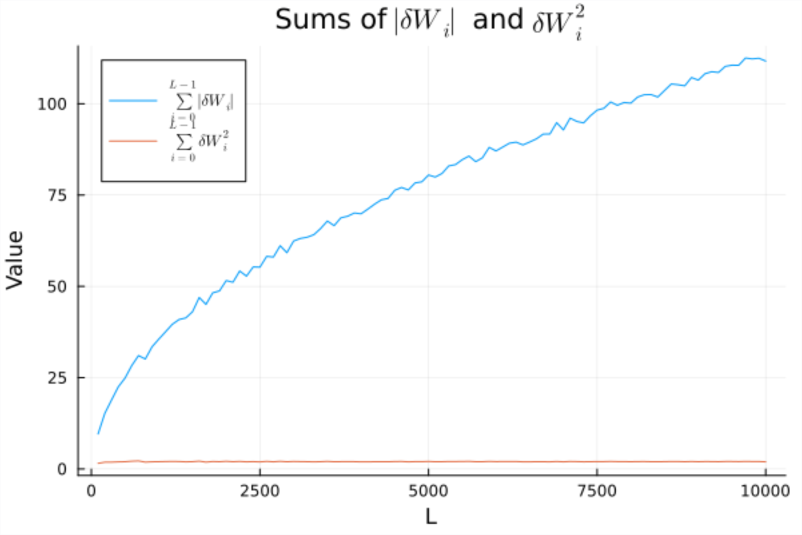
Question 1 Overview

- Let W_t be a standard Wiener process, with drift parameter zero and variance parameter $\sigma^2 = 1$.
- We divide the interval $[0, 2]$ into L subintervals $[t_i, t_{i+1}]$, where $t_i = i\delta t$ and $\delta t = 2/L$.
- Let $W_i = W(t_i)$ and $\delta W_i = W_{i+1} - W_i$.
- We verify numerically that:
  - $\sum_{i=0}^{L-1} |\delta W_i|$ is unbounded as δt goes to zero.
  - $\sum_{i=0}^{L-1} \delta W_i^2$ converges to 2 in probability as δt goes to zero.

Question 1 Response



Question 1 Response

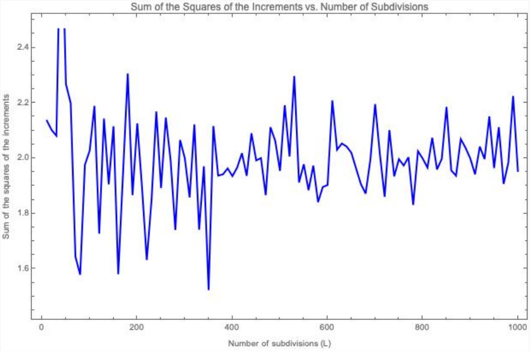


Figure: δW_i^2

Question 2a

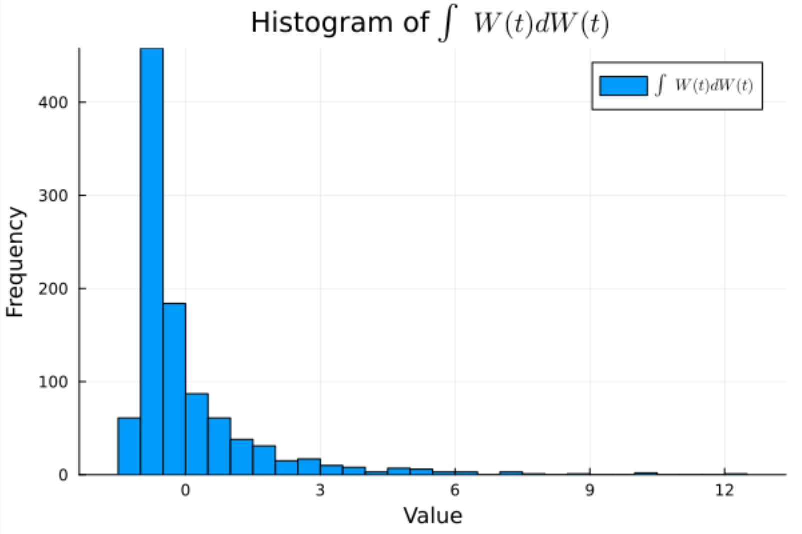


Figure: 2a

Question 2b

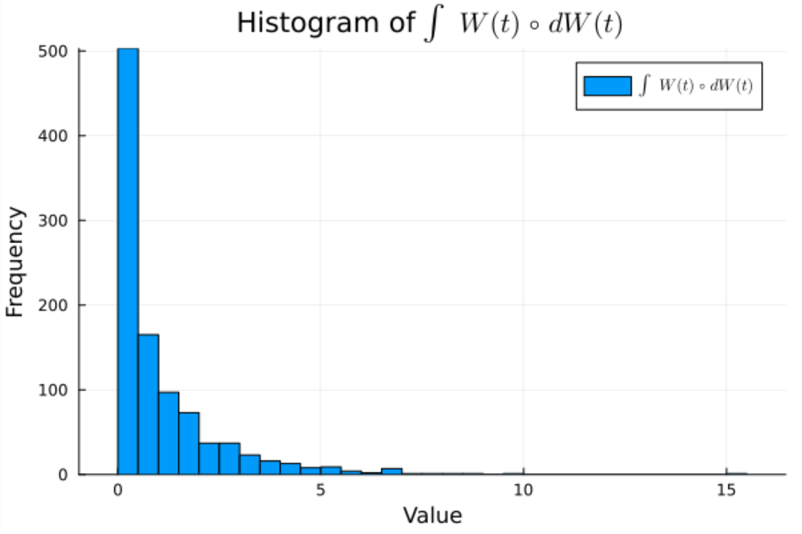


Figure: 2b

Question 2c

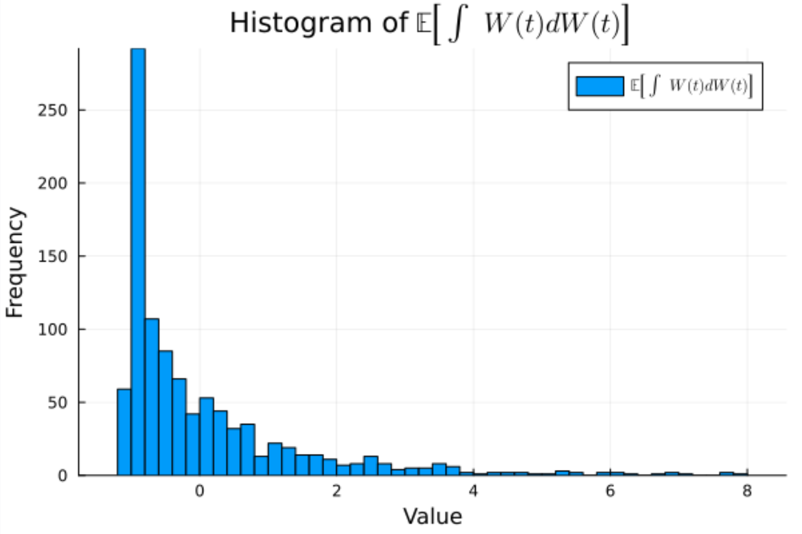


Figure: 2c

Question 2d

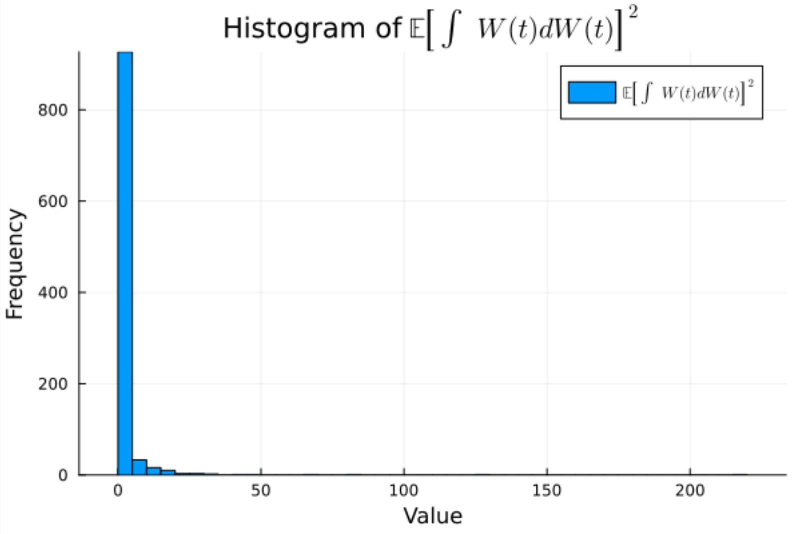


Figure: 2d

Question 2e

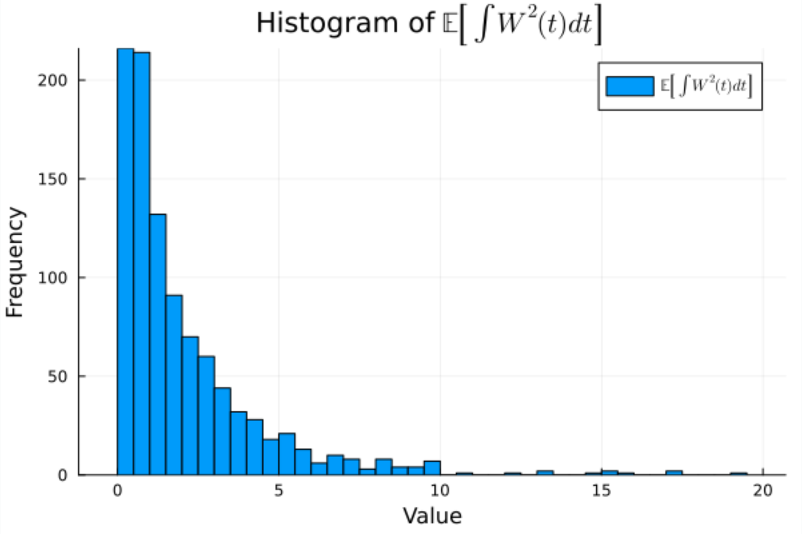


Figure: 2e

Question 2f

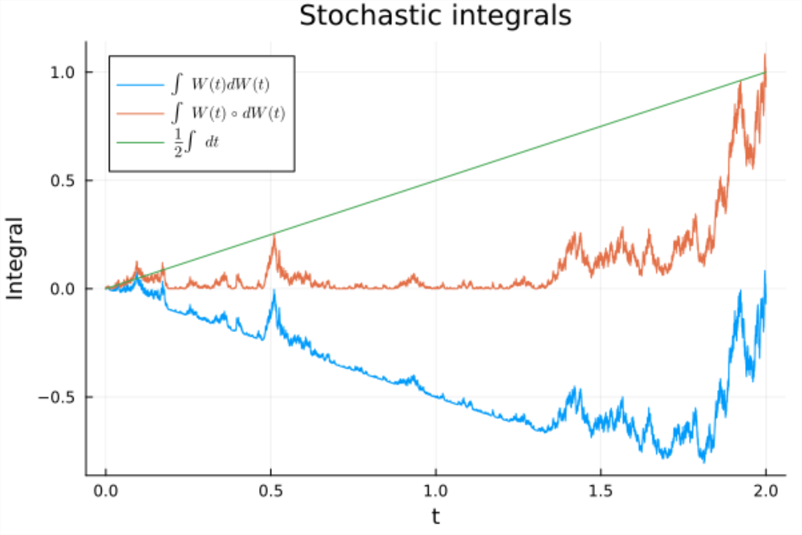


Figure: 2f

Question 3: Weak and Strong Order of Convergence for EM Method

SDE:

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \quad X(0) = 3, \quad \mu = 2, \quad \sigma = 0.10$$

- **Weak order of convergence equal to 1:** to show that

$$|E[X_1] - E[X(1)]| = C\Delta t$$

$$|X_0 e^{\mu * 1} - \frac{1}{numPaths} \sum_{j=1}^{numPaths} X_j(1)| = C\Delta t$$

- **Strong order of convergence equal to 0.5:** to show that

$$E|X_1 - X(1)| = C\Delta t^{0.5}$$

$$\frac{1}{numPaths} \sum_{j=1}^{numPaths} |X_0 e^{\mu * 1 + 0.5 * \sigma^2 * 1} - X_j(1)| = C\Delta t^{0.5}$$

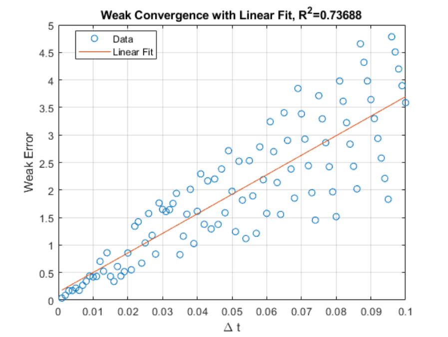
where X_1 is the exact solution and X(1) is the estimated solution.

Weak Order of Convergence Results

Weak Order of Convergence focuses on the **expected values** of the numerical solution compared to the exact solution

$$|E[X_1] - E[X(1)]| = C\Delta t$$

- A plot of the error versus Δt for the weak order of convergence of 1.

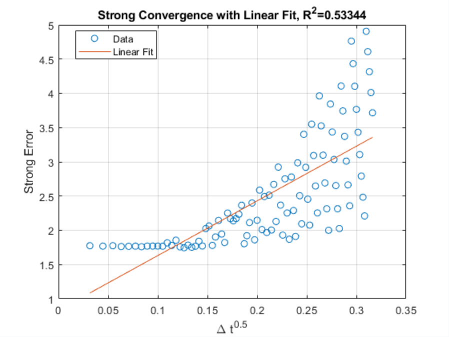


Strong Order of Convergence Results

Strong Order of Convergence to 0.5 is concerned with the **pathwise accuracy**, evaluating how closely the numerical solution follows individual realizations of the exact solution.

$$E|X_1 - X(1)| = C\Delta t^{0.5}$$

- A plot of the error versus $\Delta t^{0.5}$ for the strong order of convergence of 0.5.



Question 4

We consider the following SDE:

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \quad X(0) = 3, \quad \mu = 2, \quad \sigma = 0.10$$

- Simulate this stochastic process over the interval $[0, 20]$ using an implicit method of the form:

$$X_{n+1} = X_n + (1-\theta)\Delta t f(X_n) + \theta \Delta t f(X_{n+1}) + \sqrt{\Delta t}\alpha_n g(X_n)$$

- We use $\theta = 0.5$ and $\alpha_n = \sigma X_n$.
- We plot the results for $\Delta t = 0.01$ and $\Delta t = 0.001$.

Question 4 Continued

We plot the explicit and implicit methods for $\Delta t = 0.01$ and $\Delta t = 0.001$:

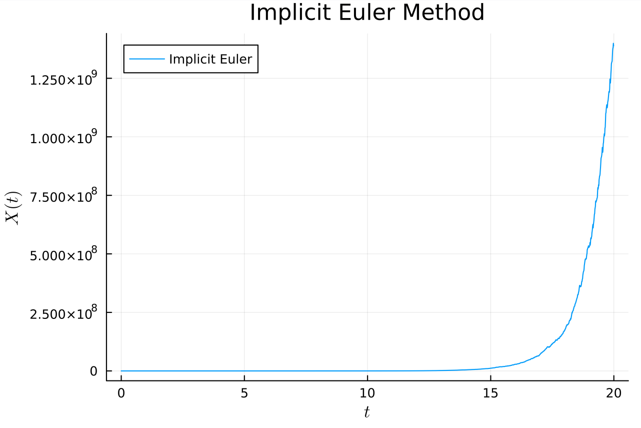


Figure: Implicit Euler Method

Question 4 Continued

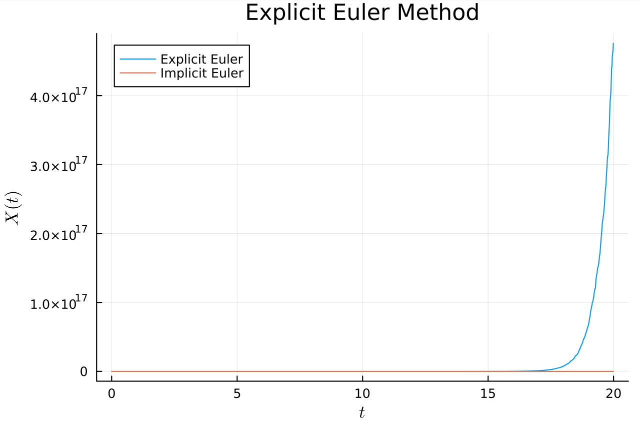


Figure: Explicit Euler Method

Question 4 Continued

We determine the values of μ and σ for which the SDE is mean-square stable:

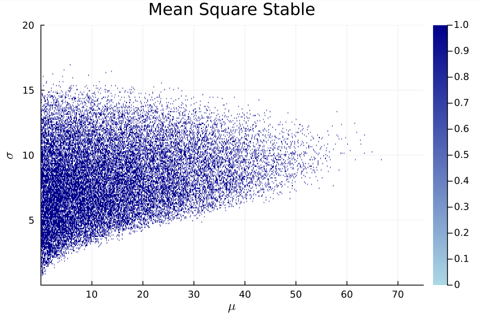


Figure: Mean-Square Stability

Question 4 Continued

We theoretically calculate the values of μ and σ for which the SDE is mean-square stable:

$$\begin{aligned} dX(t) &= \mu X(t)dt + \sigma X(t)dW(t) \\ &= \mu X(t)dt + \sigma X(t)\sqrt{dt}Z(t) \end{aligned}$$

where $Z(t) \sim N(0, 1)$.

We find that the values of μ and σ for which the SDE is mean-square stable are calculated as:

$$\begin{aligned} \mathbb{E}[(dX(t))^2] &= \mathbb{E}[(\mu X(t)dt + \sigma X(t)\sqrt{dt}Z(t))^2] \\ &= \mu^2 X(t)^2 dt^2 + \sigma^2 X(t)^2 dt \end{aligned}$$

which implies that $\mu^2 dt^2 + \sigma^2 dt \leqslant 0$.

Question 4 Continued

We also find that the asymptotic stability of the SDE is determined by the drift term μ and the diffusion term σ.

We find that the SDE is mean-square stable in the range calculated:

$$\lim_{n\to\infty} |Y_n| = 0, \text{ with probability one, for any } X_0$$

and the region is similar to the region of mean-square stability.

Question 5

An algorithm to simulate exit times for the SDE:

Choose a step size Δt
Choose some paths, M
for $s = 1$ to M **do**
  Set $t_n = 0$ and $X_n = X_0$
  while $X_n > a$ and $X_n < b$ **do**
    Compute a $N(0, 1)$ sample ξ_n
    Replace X_n by $X_n + \mu X_n \Delta t + \sigma X_n \xi_n$
    Replace t_n by $t_n + \Delta t$
  end while
  Set $T_s^{exit} = t_n - 1/2\Delta t$
end for
Set $a_M = \frac{1}{M} \sum_{s=1}^{M} T_s^{exit}$
Set $b_M^2 = \frac{1}{M-1} \sum_{s=1}^{M} (T_s^{exit} - a_M)^2$

Question 5 Continued

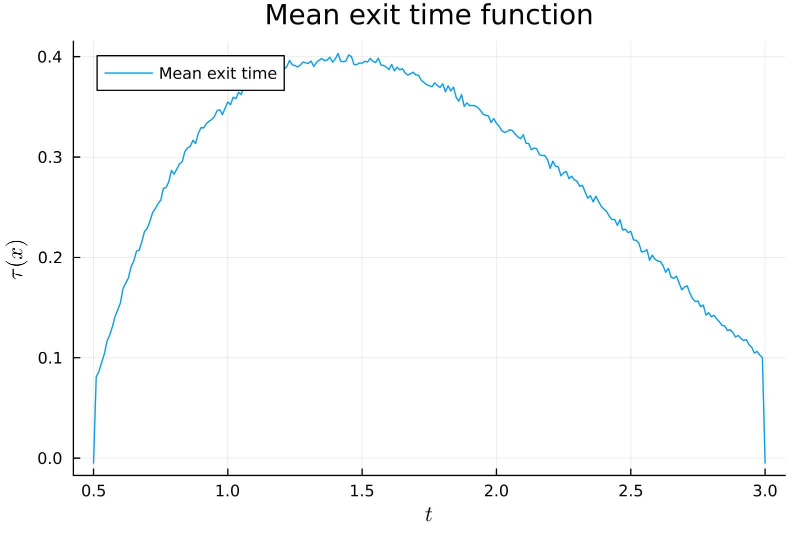


Figure: 5

Question 5 Continued

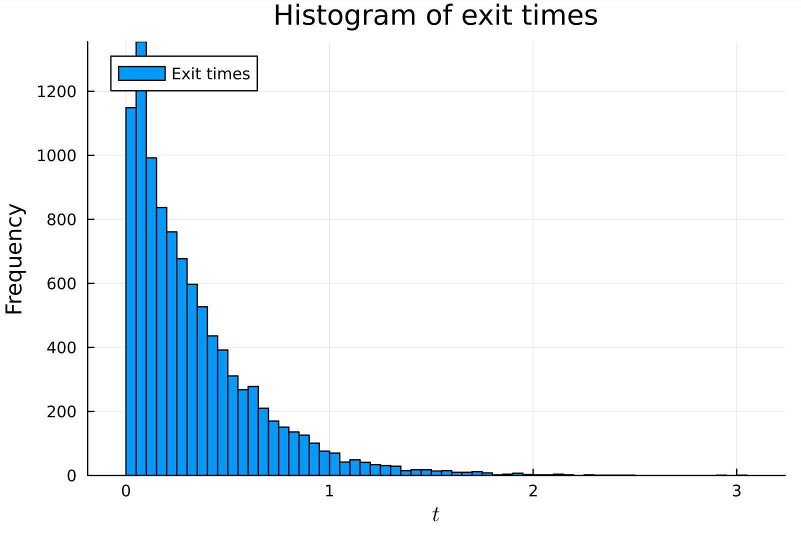


Figure: 5

Questions?