MATH 512 - Project 3

Wasif Ahmed, Haoxiang Deng, Jacob Fein-Ashley,
Kanav Malhotra, Longzhe Yang

Question 1

- We wish to estimate the following expectation $\mathbb{E}[W_3^2 + \sin(W_3) + 2\exp W_3]$, where W_t is a standard Wiener process.
- We draw 20,000 pseudo-random samples in the range $[0, \sqrt{3}]$, with each entry as an element $\in W$ (W is a vector).
- Scale $W_3^2 + \sin(W_3) + 2\exp W_3$ and we take the sample mean, yielding $\boxed{11.73290903712649}$.

Question 1 Continued

We plot the iteration number vs. the expectation of the function, yielding the following graph.

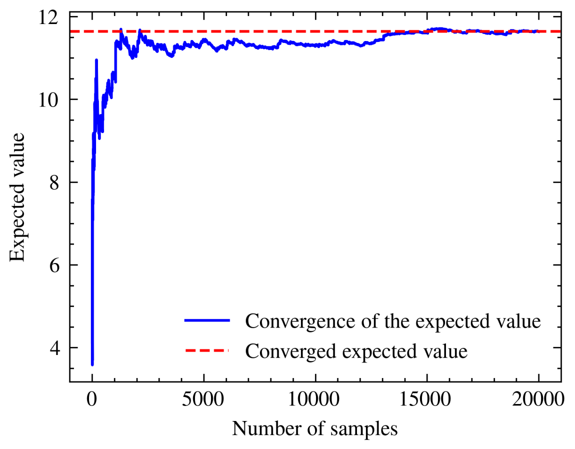

Figure: Convergence of the Given Expectation Over 20,000 Iterations

Question 2

With S_t as a Geometric Brownian Motion process, we have $S_t = S_0 e^{(\sigma W_t + (r - \frac{\sigma^2}{2})t)}$ where $r = 0.05$, $\sigma = 0.20$, $S_0 = 90$, and W_t is a standard Wiener process. We wish to estimate $\mathbb{E}(S_3)$.

Question 2 Continued

- We use a sufficiently large simulation size (20,000) to simulate $\mathbb{E}[S_3]$. The Wiener process is simulated using NumPy's builtin random number generator in the range $[0, \sqrt{t}]$, with $t = 3$.
- We calculate the expectation of S_t by simply taking $\mathbb{E}[S_t] = S_0 e^{rt}$.

Simulated $\mathbb{E}[S_3]$ is $\boxed{104.2625934286796}$
Expected $\mathbb{E}[S_3]$ is $\boxed{104.56508184554548}$

Question 3

Our goal is to evaluate the following expected value and probability:

- $\mathbb{E}(X_2^{0.6})$
- $\mathbb{P}(X^{0.6} > 2)$

The Ito's processes X evolve according to the following SDE:

$$dX_t = \left(\frac{1}{4} + \frac{1}{3}X_t\right) dt + \frac{3}{5} dW_t, \quad X_0 = 2$$

where W is a standard Wiener process.

Question 3 Continued

- We use a large sample size ($N = 1{,}000{,}000$) with a time step of $2/N = dt$ to simulate the process.
- We calculate the expected value of $X_2^{0.6}$ and the probability that $X_2^{0.6} > 2$.

We use the following algorithm to simulate the process:

for $i = 1, 2, \ldots, N$ **do**
$dW = \sqrt{dt}Z_i$
$X_{t_i} = X_{t_{i-1}} + \left(\frac{1}{4} + \frac{1}{3}X_{t_{i-1}}\right) dt + \frac{3}{5}dW$
Update $W_{t_i} = W_{t_{i-1}} + dW$
end for

and calculate the expected value and probability using the simulated values.

Question 3 Continued

The expecation is calculated using the sample mean and the probability is calculated using the sample mean of the indicator function. We do not have enough compute power to verify the results, but we conclude that $\mathbb{E}(X^2_{0.6})$ has a minimum value of ≈ 1.5 and is unbounded above as $N \to \infty$. The probability that $X^{0.6} > 2$ varies $\in [0, 1]$ according to the simulation's sample size, but approaches 1 as $N \to \infty$.

We verify this graphically in the coming slides.

Question 3 Continued

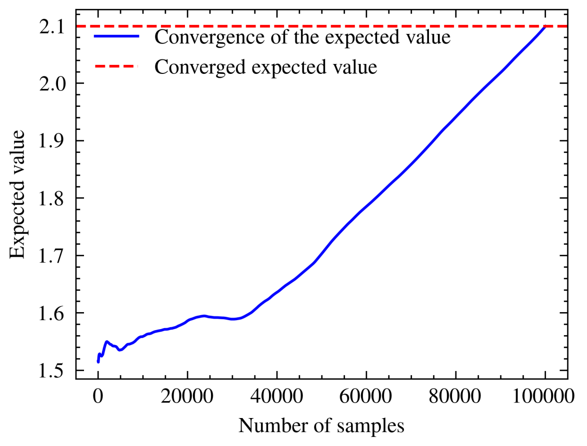

Figure: Convergence of the Expected Value and Probability Over 100,000 Iterations

Question 3 Continued

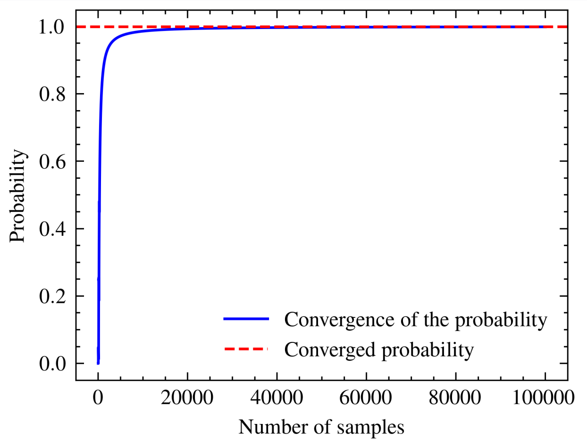

Figure: Convergence of the Probability Over 100,000 Iterations

Question 4

We consider the following SDE:

$$dX_t = aX_t dt + bX_t dW_t, \quad X_0 = 100, \quad a = 0.07, \quad b = 0.12$$

- We simulate this stochastic process using the discretization schemes of Euler-Maruyama.
- We compare the simulation with the analytical solution.

We use the following algorithm to simulate the process:

for $i = 1, 2, \ldots, N$ **do**
  $dW = \sqrt{dt} Z_i$
  $X_{t_i} = X_{t_{i-1}} + aX_{t_{i-1}} dt + bX_{t_{i-1}} dW$
  Update $W_{t_i} = W_{t_{i-1}} + dW$
end for

With the analytical solution given by
$X_{\text{analytical}} = X_0 \exp((a - b^2/2)t + bW)$.

Question 4 Continued

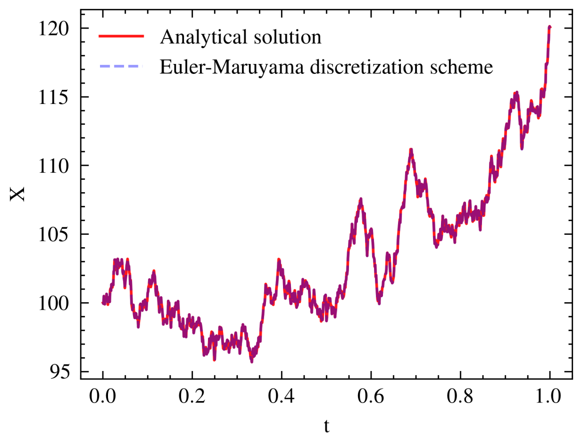

Figure: Comparison of the Analytical Solution and the Euler-Maruyama Method

Question 4 Continued

We get a very close match between the analytical solution and the Euler-Maruyama method. The Euler-Maruyama method is a good approximation for the analytical solution.We calculate the sum of the absolute difference between the two methods and find that the average difference is $\approx 16.963964394250993$ and varies with the function.

Question 4 Continued

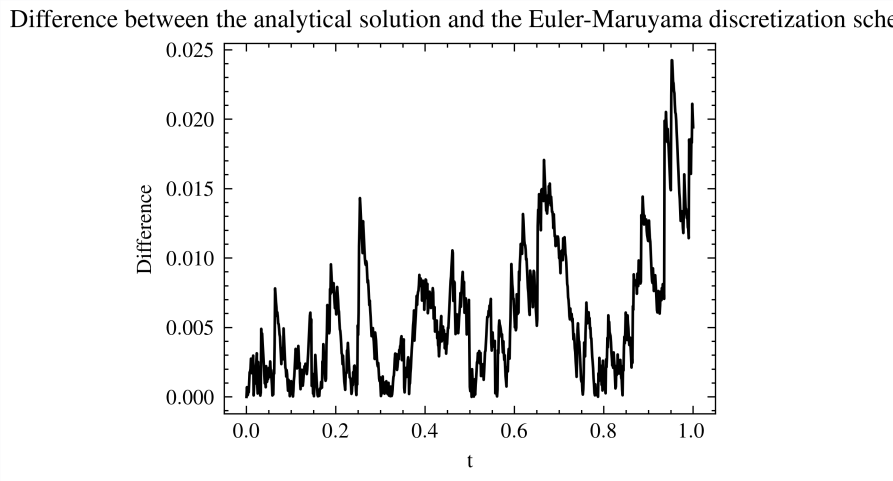

Figure: Difference Between the Analytical Solution and the Euler-Maruyama Method

Questions?