MATH512 - Project 2

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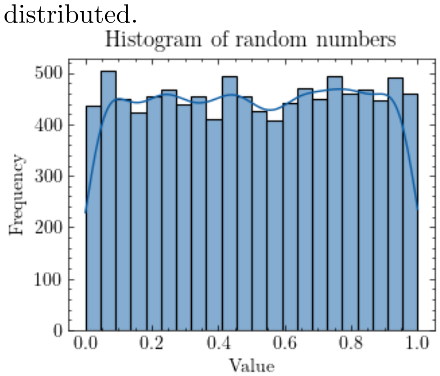
Question 1

(a) We set up a hypothesis test to determine if the data is uniformly distributed. We use the Kolmogorov-Smirnov test to test the null hypothesis that the data is uniformly distributed. The test statistic is given by

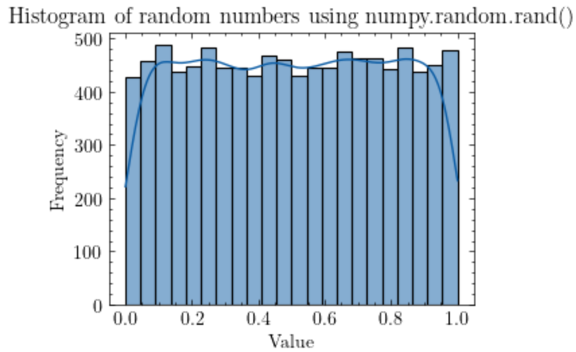
$$D = \max_{1 \leq i \leq n} \left\{ \frac{i}{n} - F(X_{(i)}) \right\}$$

where $F(X_{(i)})$ is the empirical distribution function of the data. We then compare the test statistic to the critical value of the Kolmogorov-Smirnov distribution to determine if we can reject the null hypothesis. We use a significance level of $\alpha = 0.05$.

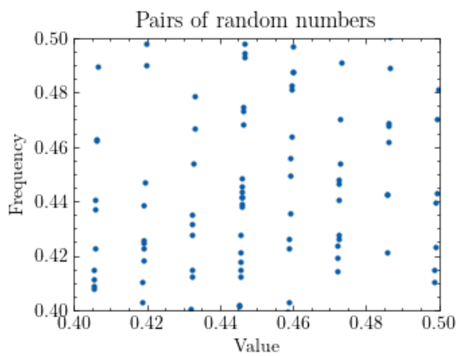
We find that the test statistic is $D = 0.010$ and a p-value of 0.20. Since the p-value is greater than the significance level, we fail to reject the null hypothesis. Thus, we conclude that the data is uniformly distributed. Note the histogram provided below. It appears that the data is uniformly distributed.



(b) For the first set of parameters, we notice that the data starts at $x_0 = 3$ and then generates [3.7.9.10.5.8.4.2.1.6.] for $n = 10$. The period is 11. For the second set of parameters, we notice that the data starts at $x_0 = 3$ and then stays at $x_n = 8$ for the rest of the data. The sequence is [3.8.8.8.8.8.8.8.8.8.]. This is not a random sequence and is not uniformly distributed. The period is 5.

(c) 

The histogram of the data generated by the linear congruential generator with the first set of parameters appears to be uniformly distributed, and quite similar to the histogram of the data generated by the numpy random number generator.

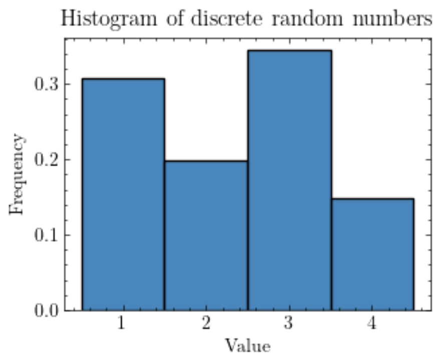
(d) 

I'm not seeing a pattern...

(e) We notice that a downfall of the linear congruential generator is that it can appear to be random with the right set of parameters, but as we saw with the second parameters in part *b*, the data can be very non-random. This is a problem because we want to be able to trust the data that we generate.

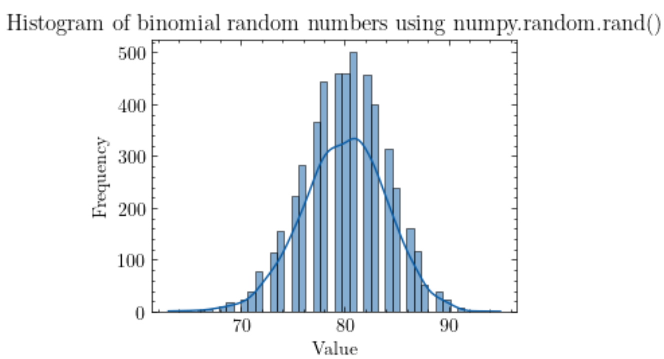
Question 2

(a)
```
x = np.random.rand(10000)
y = np.zeros(10000)
y[x < 0.3] = 1
y[(x >= 0.3) & (x < 0.5)] = 2
y[(x >= 0.5) & (x < 0.85)] = 3
y[(x >= 0.85)] = 4
```



(b)

Question 3

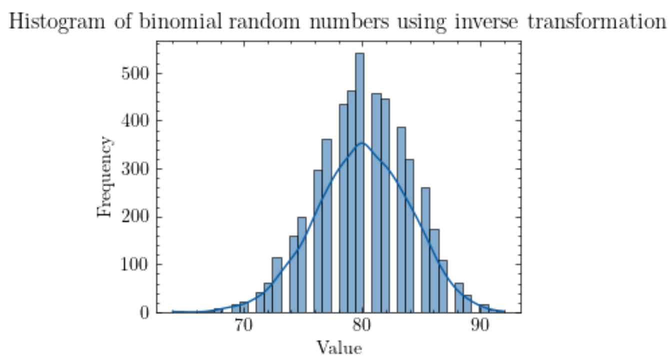


(a)

Time for generation: 0.0349 seconds. Probability that $X < 50$: 0.00000000. The theoretical answer for a Binomial Random Variable with $(n = 100, p = 0.8)$ can be found by calculating

$$
\begin{aligned}
P(X < 50) &= \sum_{k=0}^{49} \binom{100}{k} (0.8)^k (0.2)^{100-k} \\
&= 0.00000000
\end{aligned}
$$

which is approximately 0.00000000.

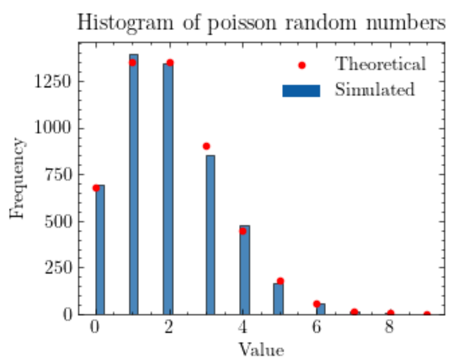


(b)

Time for generation: 0.0350 seconds.

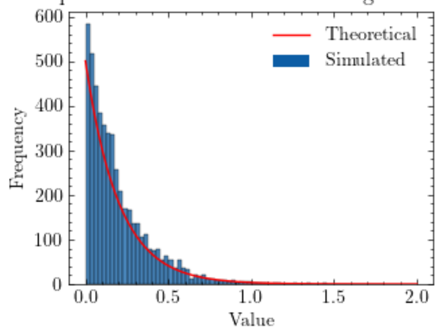
(c) The histograms and the times for generation for the inverse method and NumPy's random number generator are very similar. The inverse method is slightly faster than NumPy's random number generator.

Question 4



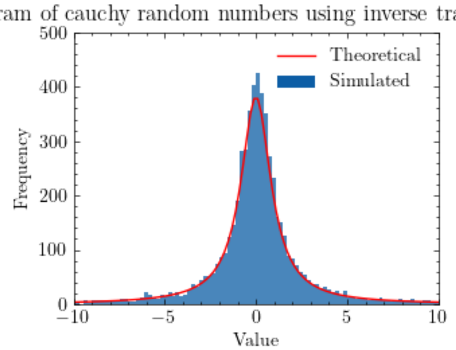
Question 5

Histogram of exponential random numbers using inverse transformation



Question 6

Histogram of cauchy random numbers using inverse transformation



Question 7

```
n = 100
n_sim = 10000

# Run simulation
hits = np.zeros(n_sim)
for i in range(n_sim):
    deck = np.arange(1, n+1)
    np.random.shuffle(deck)
    hits[i] = np.sum(deck == np.arange(1, n+1))

# Calculate expectation and variance
```

```
print('Expectation:', np.mean(hits))
print('Variance:', np.var(hits))
```

With a repetition of 10000 simulations, we find that the expectation is 1.0035 and the variance is 1.01.

We can calculate $\mathbb{E}[X]$ and $\mathbb{V}[X]$ exactly and compare them with our estimates. We have

$$\mathbb{E}[X] = \sum_{i=1}^{100} i \cdot \frac{1}{100} = 1$$
$$\mathbb{V}[X] = \sum_{i=1}^{100} (i-1)^2 \cdot \frac{1}{100} - 1 = 1$$

The estimates are very close to the exact values. This is expected because the number of simulations is large.

Question 8

```
dice = np.arange(1, 7)
n_sim = 10000

# Run simulation
rolls = np.zeros(n_sim)
for i in range(n_sim):
    n_rolls = 0
    outcomes = np.zeros(11)
    while np.sum(outcomes == 0) > 0:
        roll = np.random.choice(dice, 2)
        n_rolls += 1
        outcomes[np.sum(roll) - 2] = 1
    rolls[i] = n_rolls
```

We can model the number of rolls needed as a geometric random variable. We have

$$X \sim \text{Geometric}(p)$$

where p is the probability that all the possible outcomes have occurred at least once. We have

$$p = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6^6} = \frac{720}{46656} \approx 0.0154$$

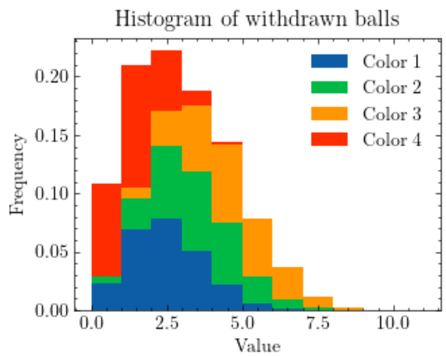
We can then calculate the expected value of X as

$$\mathbb{E}[X] = \frac{1}{p} = \frac{46656}{720} \approx 64.8$$

The simulation gives us an estimate of $\mathbb{E}[X] \approx 64.8$. Then, the variance of X is

$$\mathbb{V}[X] = \frac{1-p}{p^2} = \frac{1-0.0154}{0.0154^2} \approx 4151.63$$

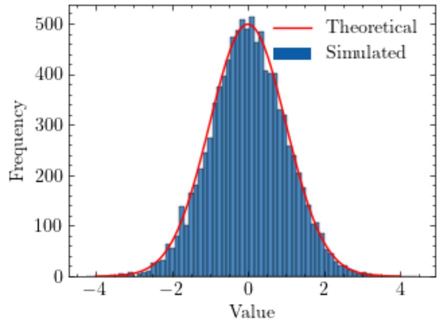
Question 9



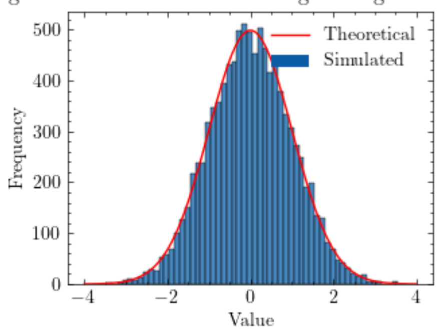

```
n_sim = 10000
n = 10
balls = np.array([20, 30, 40, 10])

# Run simulation
x = np.zeros((n_sim, 4))
for i in range(n_sim):
    urn = np.repeat(np.arange(4), balls)
    np.random.shuffle(urn)
    x[i] = np.bincount(urn[:n], minlength=4)
```

Question 10

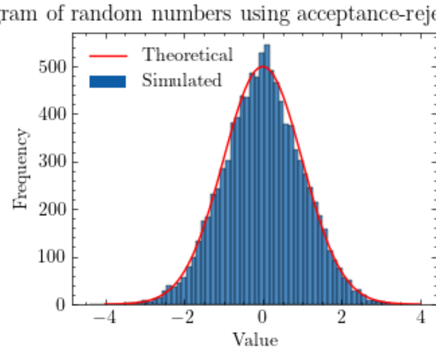
Histogram of random numbers using Box-Muller method

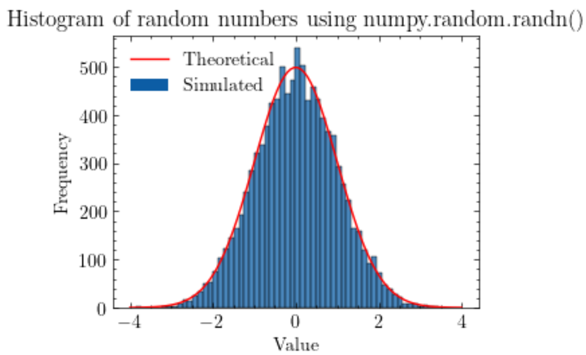


Histogram of random numbers using Marsaglia-Bray method



Histogram of random numbers using acceptance-rejection method





All of the methods seem to be very close to the theoereical normal distribution.