MATH 512 - Project 3

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Question 1

- We wish to estimate the following expectation $\mathbb{E}[W_3^2 + \sin(W_3) + 2\exp W_3]$, where W_t is a standard Wiener process.
- We draw 200,000 pseudo-random samples in the range $[0, \sqrt{3}]$, with each entry as an element $\in W$ (W is a vector).
- Scale $W_3^2 + \sin(W_3) + 2\exp W_3$ and we take the sample mean, yielding $\boxed{11.97068421176774}$.

Question 1 Continued

We plot the expected value of the function as it varies with the number of samples in Figure **??**.

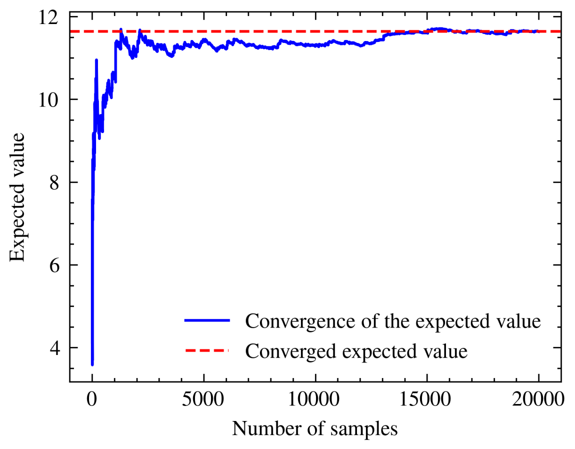

Figure: Convergence of the Given Expectation Over 200,000 Iterations

Question 1 Continued

Additionally, we plot a histogram for the function below.

imgs/histogram.png

Figure: Histogram of the Function

Question 1 Continued

Although there is a large density of values close to the value 0, we find a variance of the function of 1661.25! We find the contributing variable that explains the most variance in the following way:

Variable	Mean	Variance
W_t^2	2.97729902284579	17.592746124707148
$\sin(W_t)$	-0.0018507484598128696	0.4984857967100535
$2e^{W_t}$	8.85261926148378	1481.7866965481849

Thus, $2e^{W_t}$ contributes the most to the variance of the function with an extremely large variance.

Question 2

With S_t as a Geometric Brownian Motion process, we have $S_t = S_0 e^{(\sigma W_t + (r - \frac{\sigma^2}{2})t)}$ where $r = 0.05$, $\sigma = 0.20$, $S_0 = 90$, and W_t is a standard Wiener process. We wish to estimate $\mathbb{E}[S_3]$.

Question 2 Continued

- We use a sufficiently large simulation size (20,000) to simulate $\mathbb{E}[S_3]$. The Wiener process is simulated using NumPy's built-in random number generator in the range $[0, \sqrt{t}]$, with $t = 3$.
- We note that B_t is a GBM, i.e. B_t has moment generating function $\mathbb{E}[e^{uB_t}] = e^{\frac{u^2}{2}t}(1)$
- So using (1) with $u = \sigma$ we calculate our expected $\mathbb{E}[S_t] = S_0 e^{(r-\frac{\sigma^2}{2}t)} e^{(\frac{\sigma^2}{2}t)} = S_0 e^{rt}$

Simulated $\mathbb{E}[S_3]$ is $\boxed{104.2625934286796}$
Expected $\mathbb{E}[S_3]$ is $\boxed{104.56508184554548}$

Question 3

Our goal is to evaluate the following expected value and probability:

- $\mathbb{E}(X_2^{0.6})$
- $\mathbb{P}(X^{0.6} > 2)$

The Ito's processes X evolve according to the following SDE:

$$dX_t = \left(\frac{1}{4} + \frac{1}{3}X_t\right) dt + \frac{3}{5} dW_t, \quad X_0 = 2$$

where W is a standard Wiener process.

Question 3 Continued

- This is an Ornstein-Uhlenbeck problem with a drift.

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t$$

- The general solution can be written as:

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} dW_s$$

- The expectation value can be calculated as,

$$E[X_t] = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t})$$

Question 3 Continued

- In our case, the solution can be written as,

$$X_t = -\frac{3}{4} + \frac{5}{4}e^{\frac{t}{3}} + \frac{3}{5}\int_0^t e^{\frac{1}{3}(t-s)}\,dW_s.$$

- The stochastic part is,

$$N(0, \sqrt{\frac{3}{2}(-1 + e^{\frac{2t}{5}})}).$$

- The expectation value is,

$$E[X_t] = -\frac{3}{4} + \frac{5}{4}e^{\frac{t}{3}}$$

.

Question 3 Continued

imgs/stochastic_plot.jpeg

Figure: X_t as a function of time

Question 3 Continued

- data = RandomVariate[NormalDistribution[0, Sqrt[3/2 (-1 + Exp[4/5])]],10^4]; Mean[data]=0.0047327
- N[-3/4 + 5/4 Exp[2/3]]=1.68467

Question 4

We consider the following SDE:

$$dX_t = aX_t dt + bX_t dW_t, \quad X_0 = 100, \quad a = 0.07, \quad b = 0.12$$

- We simulate this stochastic process using the discretization schemes of Euler-Maruyama.
- We compare the simulation with the analytical solution.

We use the following algorithm to simulate the process:

for $i = 1, 2, \ldots, N$ **do**
  $dW = \sqrt{dt} Z_i$
  $X_{t_i} = X_{t_{i-1}} + aX_{t_{i-1}} dt + bX_{t_{i-1}} dW$
  Update $W_{t_i} = W_{t_{i-1}} + dW$
end for

With the analytical solution given by $X_{\text{analytical}} = X_0 \exp((a - b^2/2)t + bW)$.

Question 4 Continued

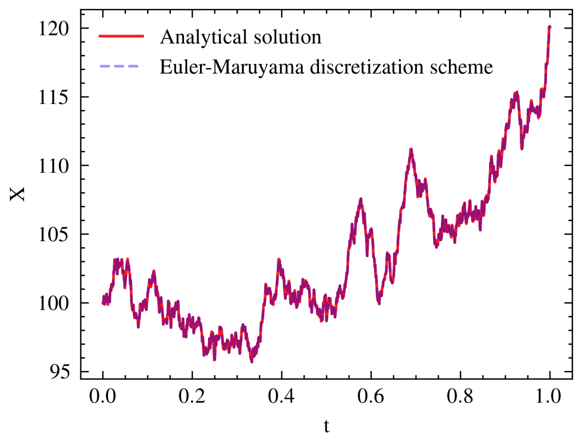

Figure: Comparison of the Analytical Solution and the Euler-Maruyama Method

Question 4 Continued

We get a very close match between the analytical solution and the Euler-Maruyama method. The Euler-Maruyama method is a good approximation for the analytical solution. We calculate the sum of the absolute difference between the two methods and find that the average percent difference

$$\text{Average Percent Difference} = \frac{1}{N}\sum_{i=1}^{N}\left|\frac{X_{\text{analytical}} - X_{\text{Euler-Maruyama}}}{X_{\text{analytical}}}\right|$$

is $\approx 0.01365803539\%$ and varies with the function.

Question 4 Continued

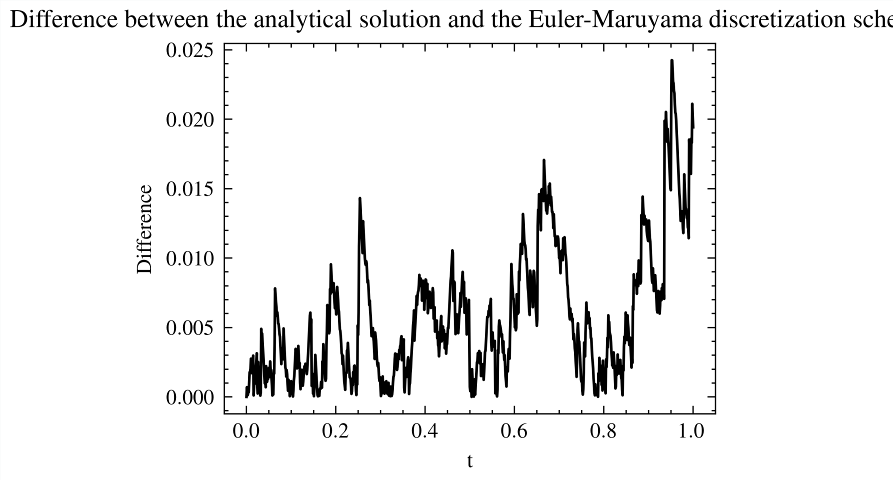

Figure: Difference Between the Analytical Solution and the Euler-Maruyama Method

Questions?