MATH512 - Project 4

Wasif Ahmed, Haoxiang Deng, Jacob Fein-Ashley, Kanav Malhotra, Longzhe Yang

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Question 1

Refer to figure 1 and 2.

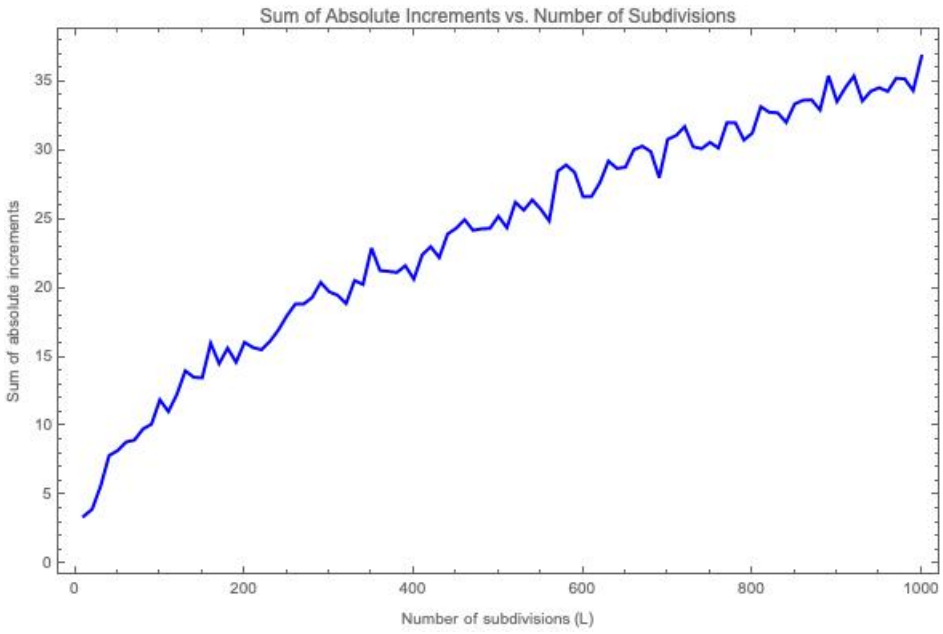


Figure 1: $|\delta W_i|$

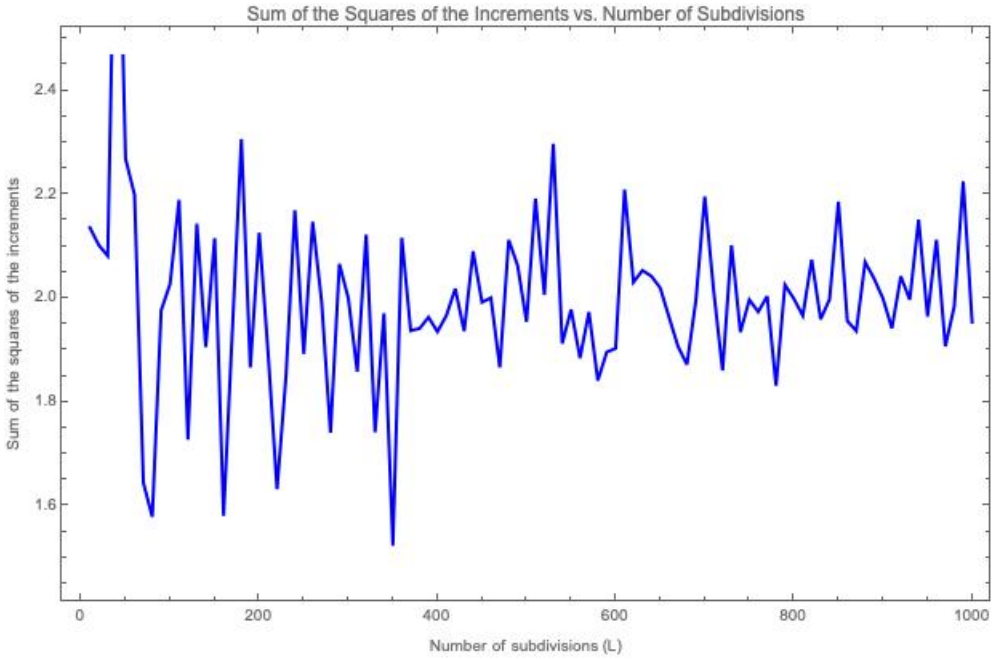


Figure 2: δW_i^2

Notice that as the L parameter increases, the $|\delta W_i|$ term is unbounded while δW_i^2 converges to 2 in probability.

Question 2

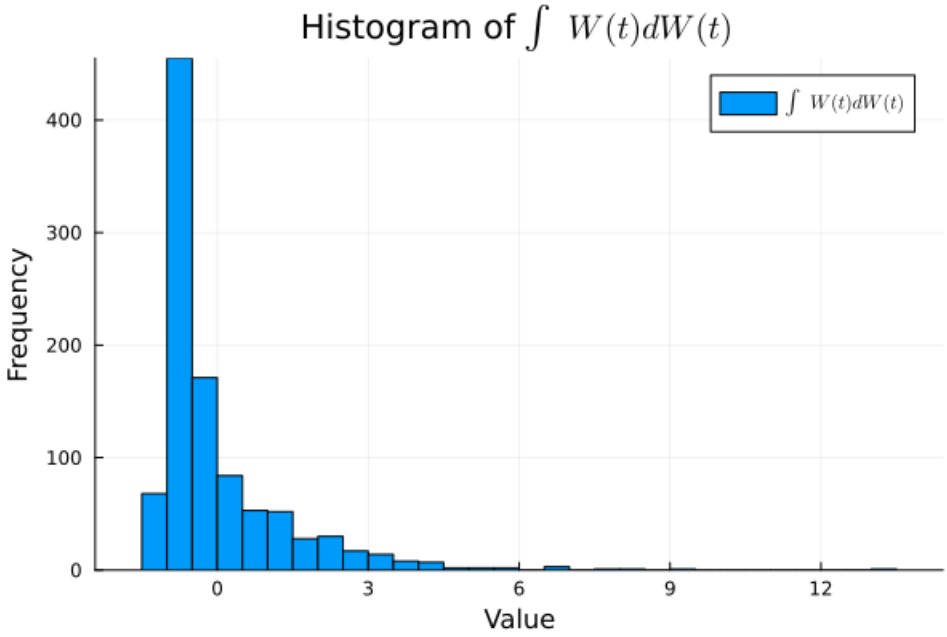


Figure 3: 2a

1.

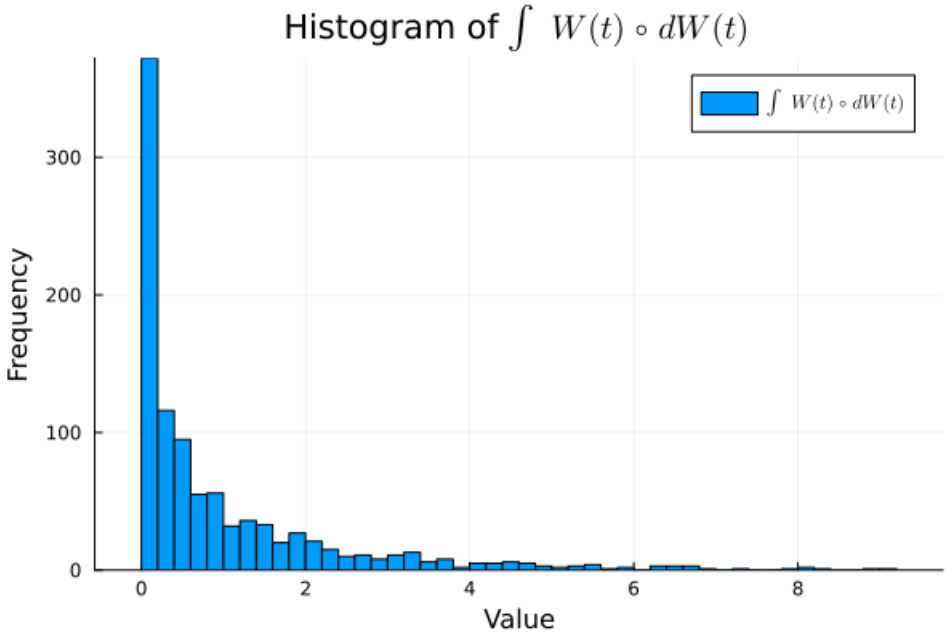

Histogram of $\int W(t) \circ dW(t)$
$\int W(t) \circ dW(t)$
Frequency
300
200
100
0
0
2
4
6
8
Value

Figure 4: 2b

2.

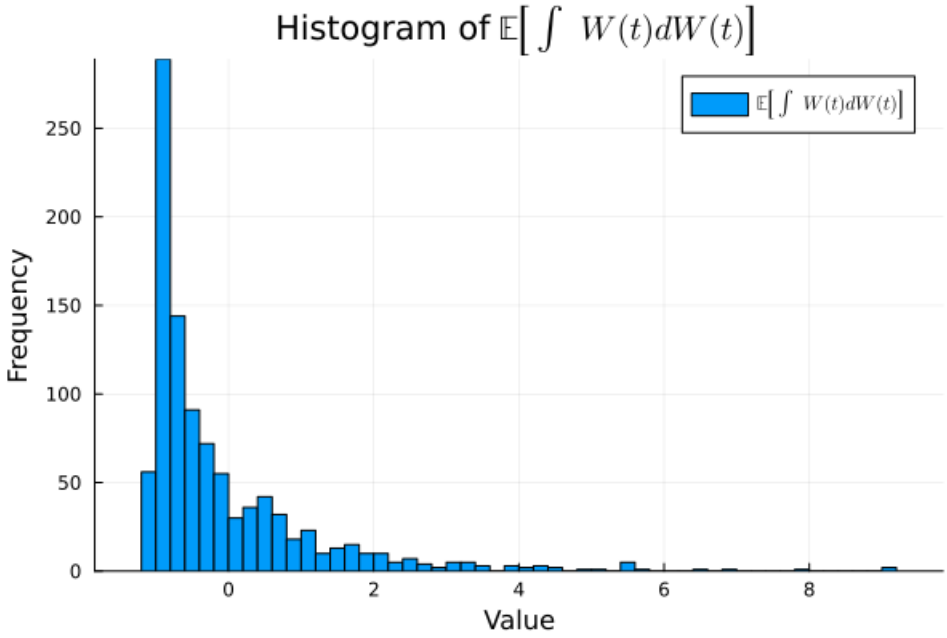

Histogram of $\mathbb{E}\left[\int W(t)dW(t)\right]$
$\mathbb{E}\left[\int W(t)dW(t)\right]$
Frequency
250
200
150
100
50
0
0
2
4
6
8
Value

Figure 5: 2c

3.

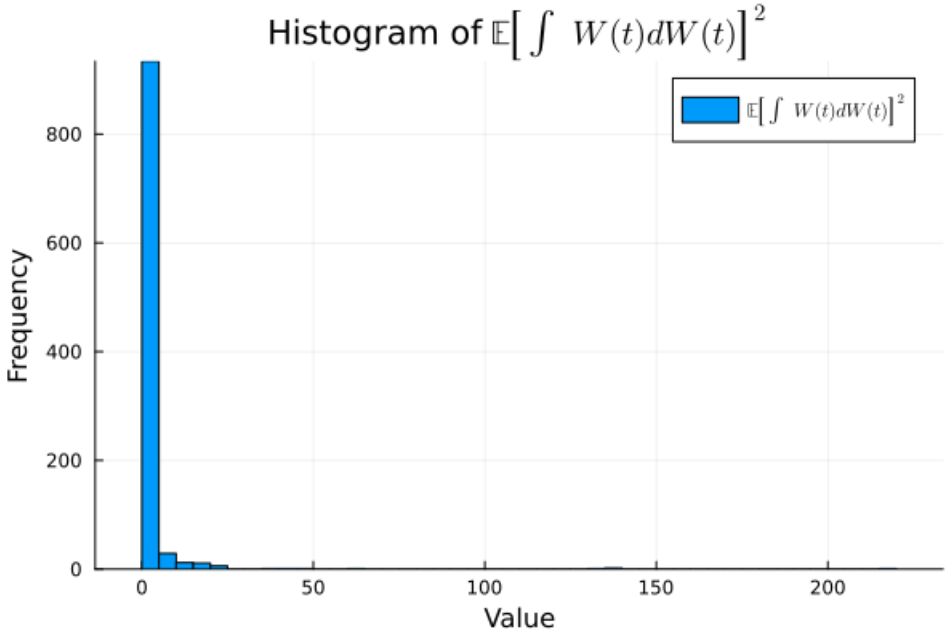

Histogram of $\mathbb{E}\left[\int W(t)dW(t)\right]^2$
$\mathbb{E}\left[\int W(t)dW(t)\right]^2$
Frequency
0
200
400
600
800
0
50
100
150
200
Value

Figure 6: 2d

4.

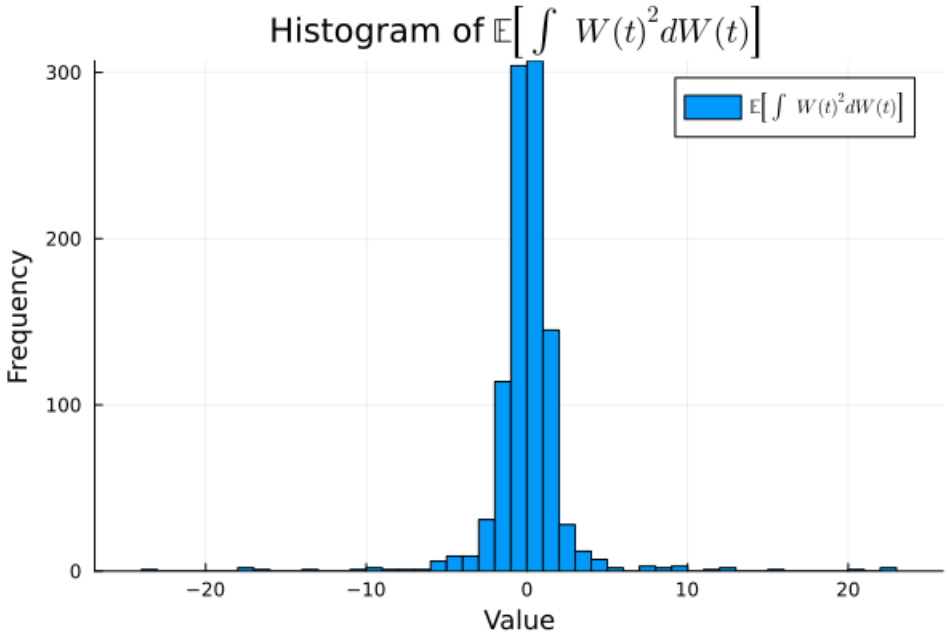

Histogram of $\mathbb{E}\left[\int W(t)^2 dW(t)\right]$
$\mathbb{E}\left[\int W(t)^2 dW(t)\right]$
Frequency
300
200
100
0
−20
−10
0
10
20
Value

Figure 7: 2e

5.

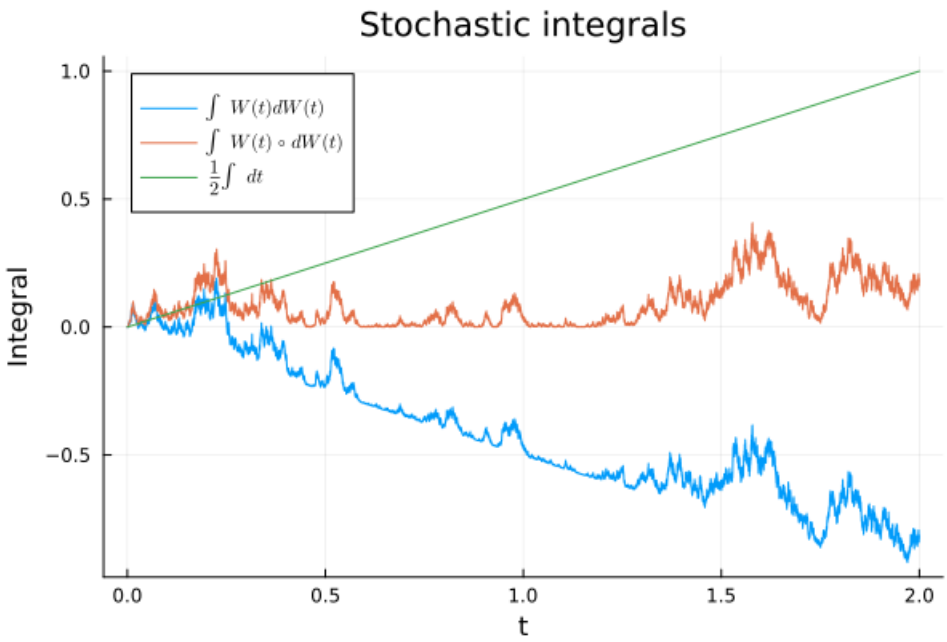


Figure 8: 2f

6.

Question 3

Objective: Analyze the weak and strong order of convergence of the EM method for solving the given SDE:

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \quad X(0) = 3, \quad \mu = 2, \quad \sigma = 0.10$$

- **Weak order of convergence equal to 1:** to show that

$$|E[X_1] - E[X(1)]| = C\Delta t$$

- **Strong order of convergence equal to 0.5:** to show that

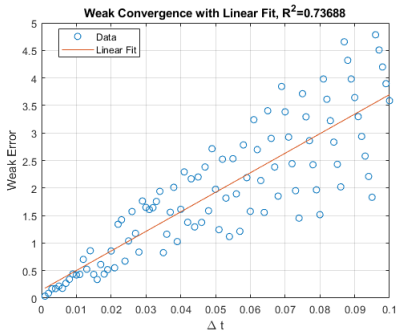
$$E|X_1 - X(1)| = C\Delta t^{0.5}$$

where X_1 stands for the exact solution and X(1) stands for the estimated solution.

Weak Order of Convergence focuses on the **expected values** of the numerical solution compared to the exact solution

$$|E[X_1] - E[X(1)]| = C\Delta t$$

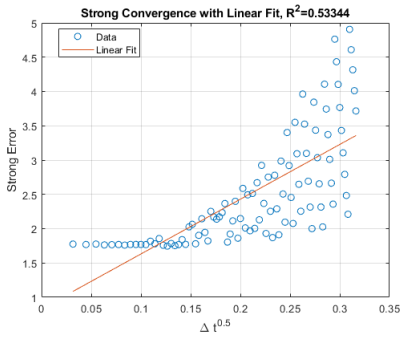
- A plot of the error versus Δt for the weak order of convergence of 1.



Strong Order of Convergence to 0.5 is concerned with the **pathwise accuracy**, evaluating how closely the numerical solution follows individual realizations of the exact solution.

$$E|X_1 - X(1)| = C\Delta t^{0.5}$$

- A plot of the error versus $\Delta t^{0.5}$ for the strong order of convergence of 0.5.



Question 4

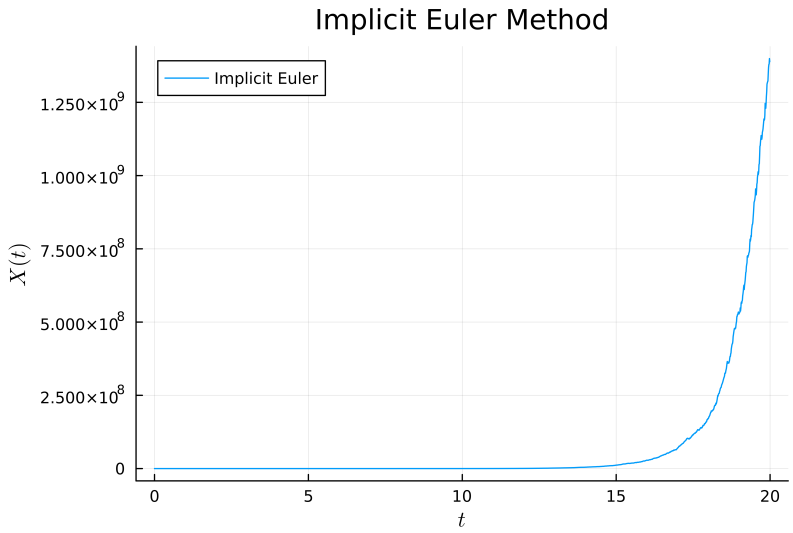


Figure 9: Implicit Euler

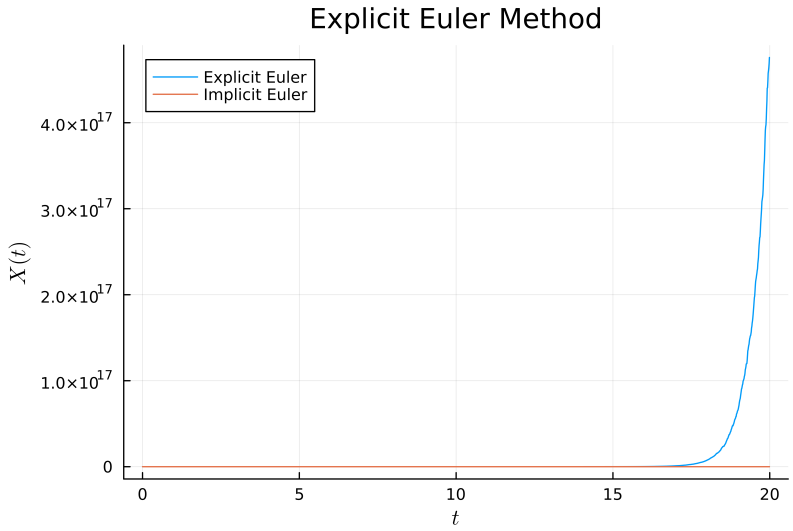


Figure 10: Explicit Euler

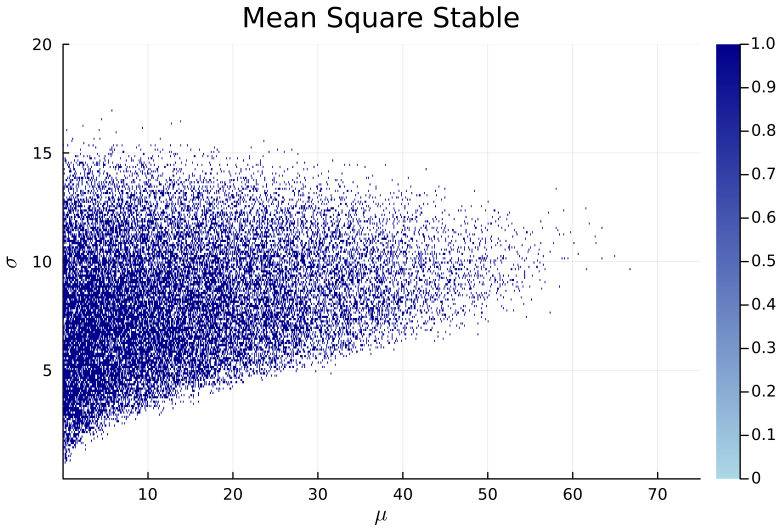


Figure 11: Mean Square Stable Values

Question 5

An algorithm to simulate exit times for the SDE:

Choose a step size Δt
Choose some paths, M
for $s = 1$ to M **do**
    Set $t_n = 0$ and $X_n = X_0$
    while $X_n > a$ and $X_n < b$ **do**
        Compute a $N(0,1)$ sample ξ_n
        Replace X_n by $X_n + \Delta t f(X_n) + \sqrt{\Delta t}\xi_n g(X_n)$
        Replace t_n by $t_n + \Delta t$
    end while
    Set $T_s^{exit} = t_n - 1/2\Delta t$
end for
Set $a_M = \frac{1}{M}\sum_{s=1}^{M} T_s^{exit}$
Set $b_M^2 = \frac{1}{M-1}\sum_{s=1}^{M}(T_s^{exit} - a_M)^2$

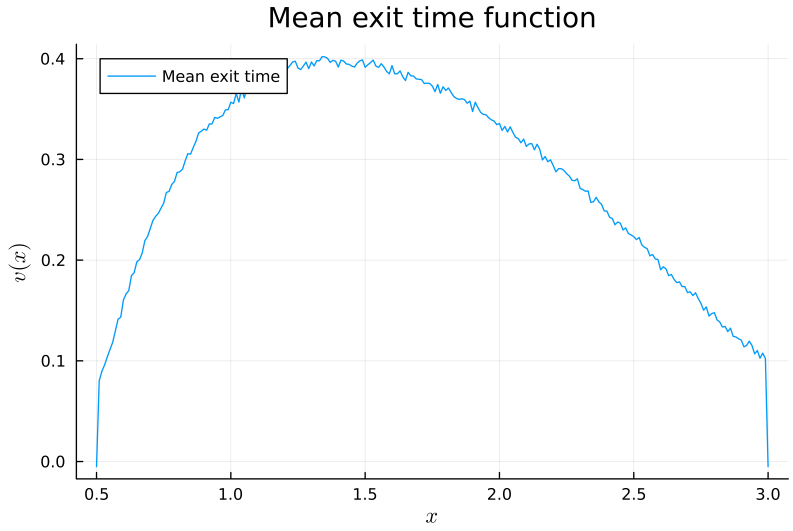


Figure 12: 5

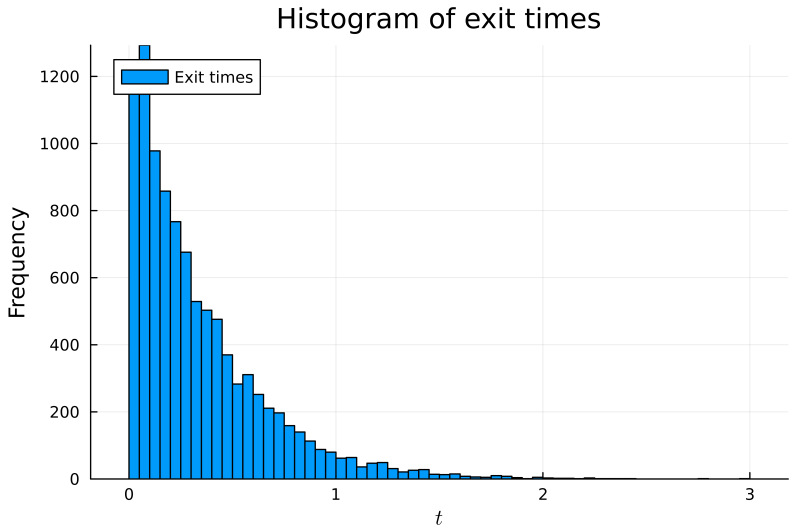


Figure 13: 5