MATH 512 - Project 5

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Question 1

- Estimate the historical volatility σ using the closing prices of the past 6 months.
- We used the formula:

$$\sigma = \sqrt{\frac{T}{N-1}\sum_{i=1}^{N}(r_i - \bar{r})^2}$$

- r_i is calculated as $\log(\frac{P_i}{P_{i-1}})$ where P_i is the price at time i.
- Where r_i is the return at time i, $\bar{r}$ is the mean return, $T = \frac{1}{2} \times 252$ and N is the number of returns.

We find that the historical volatility of TSLA is $\boxed{0.2676}$.

Question 2

- Use the binomial tree approach to estimate the price of a Jun 2024 European call option.
- Given $r = 0.05$, S_0 =current price, $K = S_0 + \$50$.
- An option expiring in June expires on the 3rd Friday of June, thus, we calculated the number of days to expiration.

Question 2 Continued

- We used the following formula to calculate the price of the European call option:

$$C_0 = e^{-rT}\left(p \times C_u + (1-p) \times C_d\right)$$

- Where $C_u = \max(S_u - K, 0)$, $C_d = \max(S_d - K, 0)$, $p = \frac{e^{rT}-d}{u-d}$, $u = e^{\sigma\sqrt{h}}$, $d = \frac{1}{u}$, $h = \frac{T}{N}$.
- $S_u = S_0 \times u$, $S_d = S_0 \times d$.

We find that the price of the European call option is $\boxed{0.1207}$.

Question 3

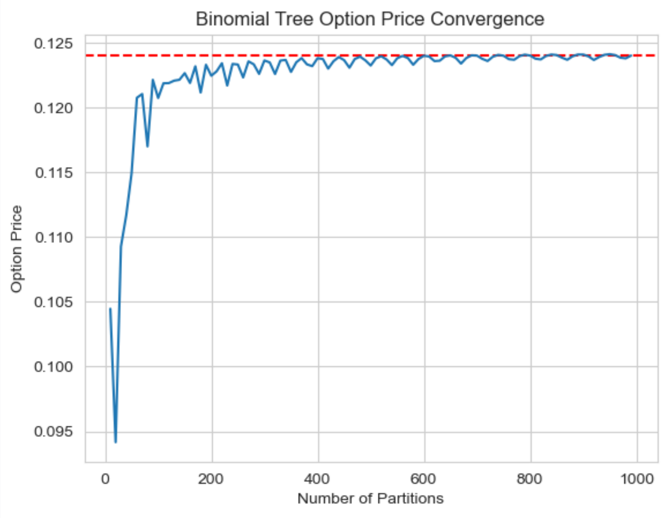


Figure: Convergence of the binomial tree method with the number of partitions.

Question 4

- Use the binomial tree approach to estimate the price of a Jun 2024 American put option.
- Given $r = 0.05$, S_0 =current price, $K = S_0 + \$50$.
- We used the same formula as the European call option, but with the following changes:
  - $C_u = \max(K - S_u, 0)$, $C_d = \max(K - S_d, 0)$.
  - We used the early exercise condition to calculate the price of the American put option.

> We find that the price of the American put option is $\boxed{47.78}$.

Question 5

- Given $r = 0.05$, S_0 =current price, $K = S_0 + \$50$.
- We used the following formula to calculate the price of the European call option:
- $C = S_0 \times N(d_1) - K \times e^{-rT} \times N(d_2)$.
- Where $d_1 = \frac{\log(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$.
- We used the put-call parity relation to calculate the price of the corresponding put option.

We find that the price of the European call option is $\boxed{0.1207}$. and the price of the corresponding put option is $\boxed{47.78}$. This is essentially the same result as the Binomial Tree method.

Question 5 Δ

$$\frac{\partial C}{\partial S} = \frac{BS_{Call}(S + \Delta S, K, T, r, \sigma) - BS_{Call}(S, K, T, r, \sigma)}{\Delta S}$$

This equations essentially calculates the change in the price of the call option with respect to the change in the price of the underlying asset.

We find that the Δ of the European call option is $\boxed{0.0185}$.

Question 5 Δ Continued

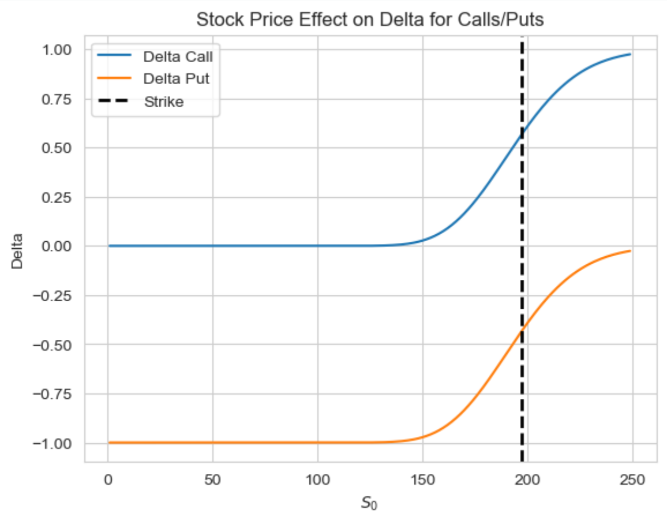


Figure: Convergence of the Δ changing the price of S_0

Question 5 Δ Continued

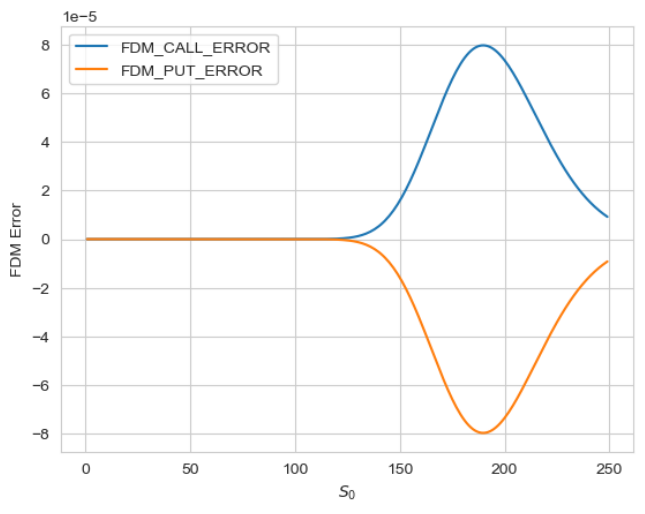


Figure: Error of the Δ changing the price of S_0

Question 5 Γ

$$\frac{\partial^2 C}{\partial S^2} = \frac{\partial^2 P}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{T}}$$

If Δ is essentiall the "speed" of the option, then Γ is the "acceleration" of the option.

We use a finite difference method (FDM) to calculate the Γ of the European call option.

$$\frac{\partial P}{\partial S} = \frac{BS_{Put}(S+\Delta S, K, T, r, \sigma) - 2BS_{Put}(S, K, T, r, \sigma) + BS_{Put}(S-\Delta S, K, T, r, \sigma)}{(\Delta S)^2}$$

We find that the Γ of the European call option is $\boxed{0.00184}$.

Question 5 Γ Continued

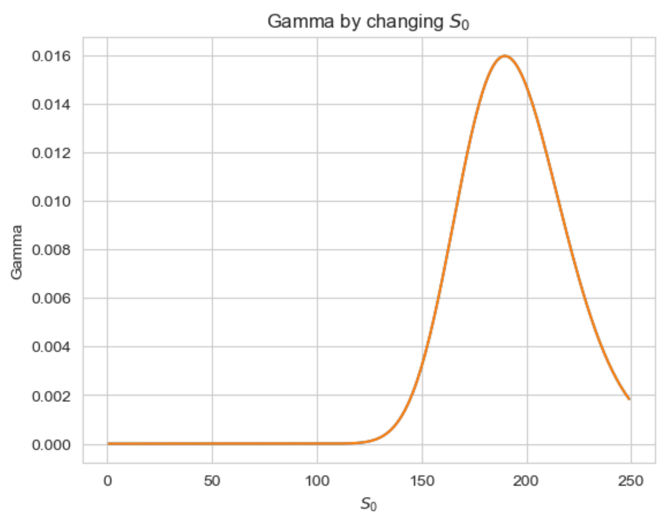


Figure: Γ changing the price of S_0

Question 5 Vega

$$\frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} = S\sqrt{T}N'(d1)$$

The Vega measures the sensitivity of the option price to changes in the volatility of the underlying asset.

Vega Finite Difference Method (FDM) is used to calculate the Vega of the European call option.

$$\frac{\partial C}{\partial \sigma} = \frac{BS_{Call}(S, K, T, r, \sigma) - BS_{Call}(S, K, T, r, \sigma - \Delta\sigma)}{\Delta\sigma}$$

We find that the Vega of the European call option is $\boxed{3.26089}$.

Question 5 Vega Continued

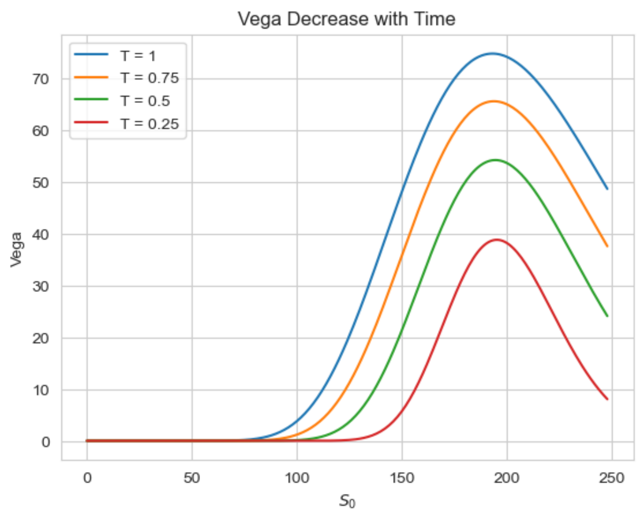


Figure: Vega changing the value of T

Question 5 Θ

$$\frac{\partial C}{\partial T} = \frac{\partial P}{\partial T} = -S_0 N(d_1)\sigma\frac{1}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

The Theta measures the sensitivity of the option price to changes in time.

Theta Finite Difference Method (FDM) is used to calculate the Theta of the European call option.

$$\frac{\partial C}{\partial T} = \frac{BS_{Call}(S, K, T, r, \sigma) - BS_{Call}(S, K, T - \Delta T, r, \sigma)}{\Delta T}$$

We find that the Theta of the European call option is $\boxed{-1.963}$.

Question 5 Θ Continued

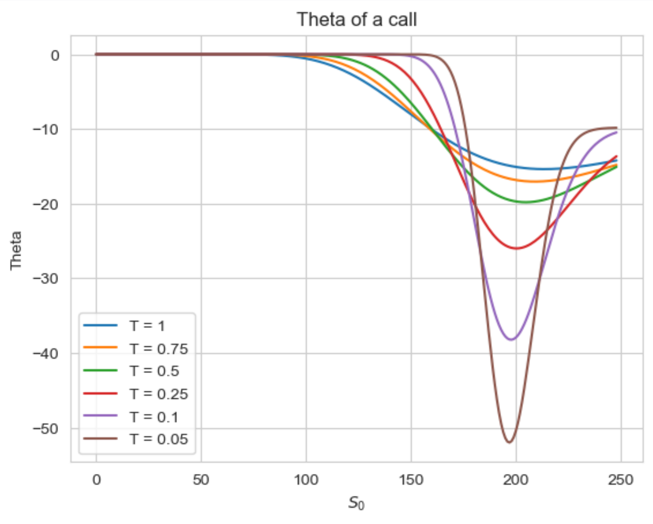


Figure: Theta changing the value of T

Question 5 ρ

$$\frac{\partial C}{\partial r} = \frac{\partial P}{\partial r} = -KTe^{-rT}N(d_2)$$

The Rho measures the sensitivity of the option price to changes in the risk-free interest rate.

Rho Finite Difference Method (FDM) is used to calculate the Rho of the European call option.

$$\frac{\partial C}{\partial r} = \frac{BS_{Call}(S, K, T, r, \sigma) - BS_{Call}(S, K, T, r - \Delta r, \sigma)}{\Delta r}$$

We find that the Rho of the European call option is $\boxed{0.620}$.

Question 6

- Use a simple Monte Carlo method to estimate the price of a Jun 2024 European call option.
- Given $r = 0.05$, $S_0 =$current price, $K = S_0 + \$50$.
- We used the following formula to calculate the price of the European call option:

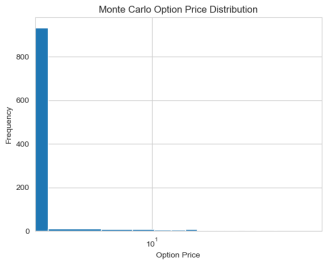
$$C = e^{-rT}\left(\frac{1}{N}\sum_{i=1}^{N} \max(S_i - K, 0)\right)$$

- Where $S_i = S_0 \times e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z_i}$, $Z_i \sim N(0,1)$.

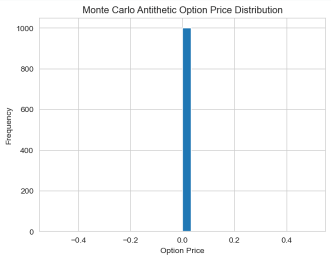
Question 6 Continued

We find that the price of the European call option is not as accurate as the other methods. Most values are around \$0, which is different. We can take the mean of the values to get a better estimate, and we find that the price of the European call option is $\boxed{1.3675}$, whereas the other methods gave us a price of \$0.1207.

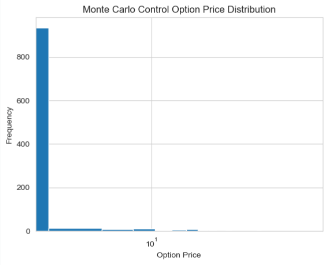
Question 6



(a) Simple Monte Carlo



(b) Antithetic Monte Carlo



(c) Control Variate Monte Carlo

Question 7

- Estimate the implied volatility for the problem in part e.
- We used the following formula to calculate the implied volatility:

$$\sigma = \text{optimize.newton}(f, 0.5)$$

- Where
$f(\sigma) = \text{black\_scholes}(S_0, K, T, r, \sigma, \text{option\_type}) - \text{price}.$

We find that the implied volatility of the European call option is $\boxed{0.163}$, which is around half of the historical volatility.

Questions?