MATH 512 - Project 3

Wasif Ahmed, Haoxiang Deng, Jacob Fein-Ashley,
Kanav Malhotra, Longzhe Yang

Question 1

- We wish to estimate the following expectation $\mathbb{E}[W_3^2 + \sin(W_3) + 2\exp W_3]$, where W_t is a standard Wiener process.
- We draw 20,000 pseudo-random samples in the range $[0, \sqrt{3}]$, with each entry as an element $\in W$ (W is a vector).
- Scale $W_3^2 + \sin(W_3) + 2\exp W_3$ and we take the sample mean, yielding $\boxed{11.73290903712649}$.

Question 1 Continued

We plot the iteration number vs. the expectation of the function, yielding the following graph.

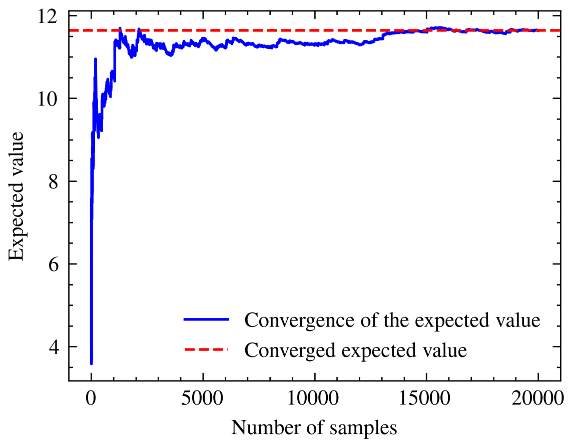

Figure: Convergence of the Given Expectation Over 20,000 Iterations

Question 2

With S_t as a Geometric Brownian Motion process, we have $S_t = S_0 e^{(\sigma W_t + (r - \frac{\sigma^2}{2})t)}$ where $r = 0.05$, $\sigma = 0.20$, $S_0 = 90$, and W_t is a standard Wiener process. We wish to estimate $\mathbb{E}(S_3)$.

Question 2 Continued

- We use a sufficiently large simulation size (20,000) to simulate $\mathbb{E}[S_3]$. The Wiener process is simulated using NumPy's builtin random number generator in the range $[0, \sqrt{t}]$, with $t = 3$.
- We calculate the expectation of S_t by simply taking $\mathbb{E}[S_t] = S_0 e^{rt}$.

Simulated $\mathbb{E}[S_3]$ is $\boxed{104.2625934286796}$
Expected $\mathbb{E}[S_3]$ is $\boxed{104.56508184554548}$

Question 3

Our goal is to evaluate the following expected value and probability:

- $\mathbb{E}(X_2^{0.6})$
- $\mathbb{P}(X^{0.6} > 2)$

The Ito's processes X evolve according to the following SDE:

$$dX_t = \left(\frac{1}{4} + \frac{1}{3}X_t\right) dt + \frac{3}{5} dW_t, \quad X_0 = 2$$

where W is a standard Wiener process.

Question 3 Continued

- We use a large sample size ($N = 1{,}000{,}000$) with a time step of $2/N = dt$ to simulate the process.
- We calculate the expected value of $X_2^{0.6}$ and the probability that $X_2^{0.6} > 2$.

We use the following algorithm to simulate the process:

for $i = 1, 2, \ldots, N$ **do**
$$dW = \sqrt{dt}Z_i$$
$$X_{t_i} = X_{t_{i-1}} + \left(\tfrac{1}{4} + \tfrac{1}{3}X_{t_{i-1}}\right) dt + \tfrac{3}{5}dW$$
Update $W_{t_i} = W_{t_{i-1}} + dW$
end for

and calculate the expected value and probability using the simulated values.

Question 3 Continued

The expectation and probability are calculated using the indicator function's sample mean. We do not have enough computing power to verify the results, but we conclude that $\mathbb{E}(X_2^{0.6})$ has a minimum value of ≈ 1.5 and is unbounded above as $N \to \infty$. The probability that $X^{0.6} > 2$ varies $\in [0, 1]$ according to the simulation's sample size, but approaches 1 as $N \to \infty$.

We verify this graphically in the coming slides.

Question 3 Continued

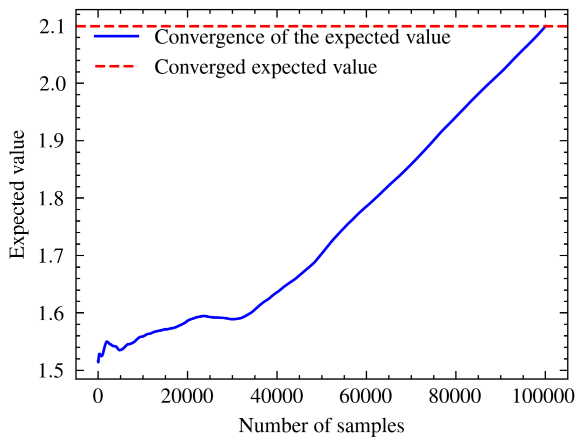

Figure: Convergence of the Expected Value and Probability Over 100,000 Iterations

Question 3 Continued

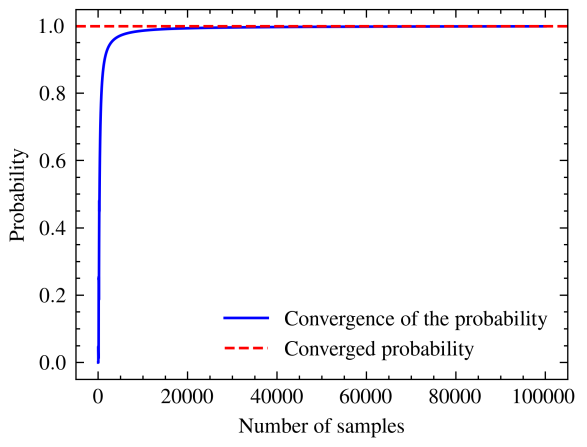

Figure: Convergence of the Probability Over 100,000 Iterations

Question 4

We consider the following SDE:

$$dX_t = aX_t dt + bX_t dW_t, \quad X_0 = 100, \quad a = 0.07, \quad b = 0.12$$

- We simulate this stochastic process using the discretization schemes of Euler-Maruyama.
- We compare the simulation with the analytical solution.

We use the following algorithm to simulate the process:

for $i = 1, 2, \ldots, N$ **do**
  $dW = \sqrt{dt} Z_i$
  $X_{t_i} = X_{t_{i-1}} + aX_{t_{i-1}} dt + bX_{t_{i-1}} dW$
  Update $W_{t_i} = W_{t_{i-1}} + dW$
end for

With the analytical solution given by $X_{\text{analytical}} = X_0 \exp((a - b^2/2)t + bW)$.

Question 4 Continued

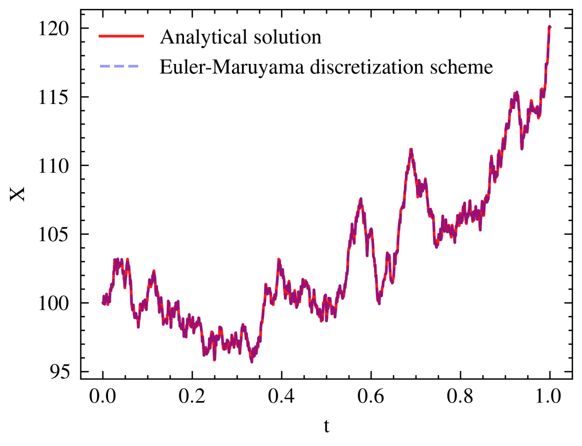

Figure: Comparison of the Analytical Solution and the Euler-Maruyama Method

Question 4 Continued

We get a very close match between the analytical solution and the Euler-Maruyama method. The Euler-Maruyama method is a good approximation for the analytical solution.We calculate the sum of the absolute difference between the two methods and find that the average difference

$$\text{Average Difference} = \frac{1}{N} \sum_{i=1}^{N} |X_{\text{analytical}} - X_{\text{Euler-Maruyama}}|$$

is $\approx 15.56230558918206$ and varies with the function.

Question 4 Continued

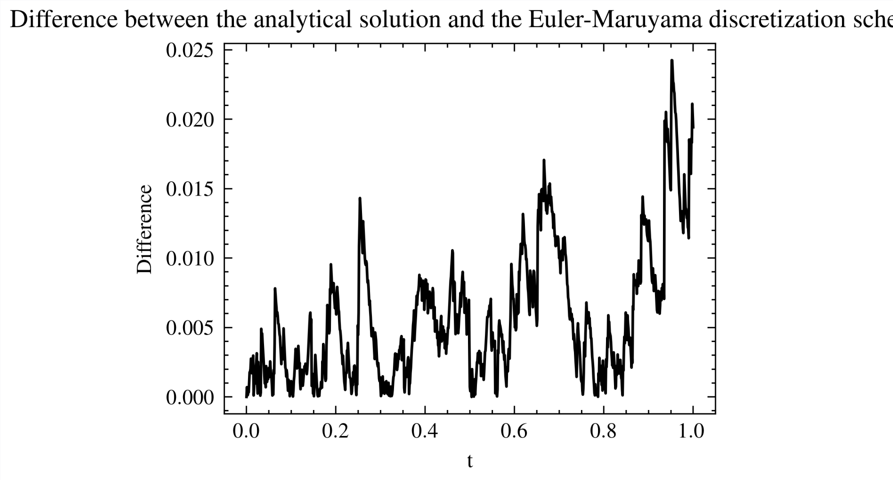

Figure: Difference Between the Analytical Solution and the Euler-Maruyama Method

Questions?