A Permutation Test for Three-Dimensional Rotational Data

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1. Three-Dimensional Rotational Data

- Three-dimensional rotation data is commonly used in the study of human motion. One example is joint rotation (elbow, knee, etc.).
- This type of data can be displayed graphically on a three-dimensional sphere plot or represented mathematically as a 3x3 orthogonal rotational matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Figure 1. Generic 3x3 Orthogonal Rotational Matrix.

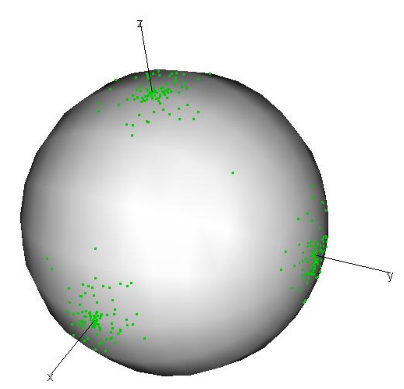

Figure 2. Generic Sphere Plot of Three-Dimensional Rotational Data. (k=20, n=100)

- Each column of a 3x3 orthogonal rotational matrix is orthogonal to all other columns.
- Therefore, $a_{1i}\,a_{1j} + a_{2i}\,a_{2j} + a_{3i}\,a_{3j} = 0$, for $i \neq j$
- Similarly, $(a_{i1})^2 + (a_{i2})^2 + (a_{i3})^2 = 1$, for $i=1,2,3$

- When simulating the data, the spread of the points is controlled by the concentration parameter, k.
- The size of the dataset is controlled by n.

2. Permutation Tests in General

- Non-parametric statistics is an area of statistics where distributional assumptions are not required.
- A permutation test is a test used in non-parametric statistics to conclude if two or more data sets are different in some way. For example, there are tests for differences in means (μ), variances (σ²), shapes, etc.

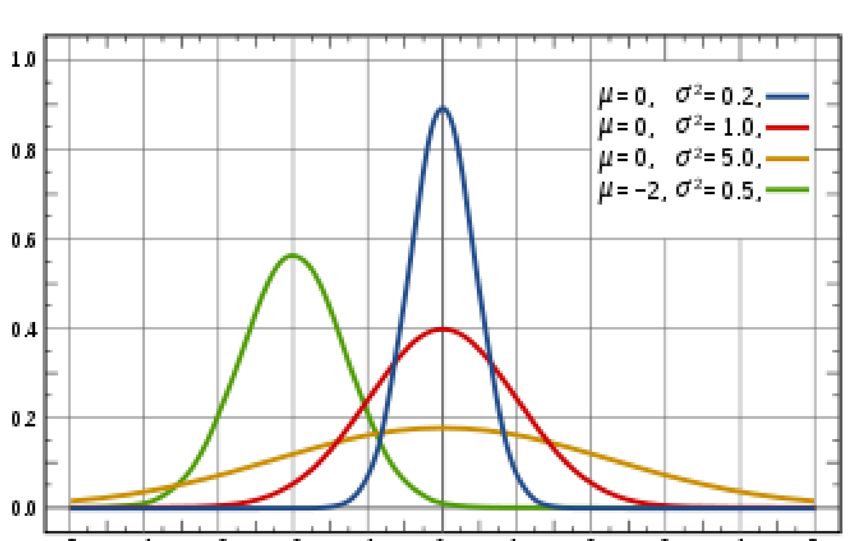

Figure 3. Generic distribution plot showing various datasets with different means, variances, and shapes.

Dataset 1	Dataset 2	Difference
12	7	5
15	8	7
9	12	-3
10	6	4
8	9	-1
11	11	0

Test Statistic: $\mu_D = 12$

Table 1. An example of a two dimensional observed test statistic μ_D

- The purpose of a permutation test, in general, is to discover if the observed test statistic for a set of specific data points is more extreme than what an expected test statistic would be for *any* permutation of those data points.
- A permutation test permutes the data a specified amount of times, calculates the test statistic for each permutation, and checks if it is more or less extreme than the observed test statistic.
- If the observed test statistic is proven to be rare (there were not many simulated test statistics which were more extreme than the observed) then the data sets would be considered different.

3. A Permutation Test for Three-Dimensional Rotations

- In three-dimensions, the mean is no longer just a number, it is an orientation (set of three orthogonal axes) describing a position in space. Therefore, a new method for determining a difference between the means of two datasets was needed.
- Angles can be used to quantify the difference between two three-dimensional rotations. A misorientation angle is defined as the angle needed to get from one orthogonal matrix to another via a spin about some axis. (See Figure 4.)
- The mean matrix can be found for each rotation dataset and the misorientation angle between the means will measure the "distance" between these two datasets. This misorientation angle is what is used as the test statistic in the 3-D Permutation Test.

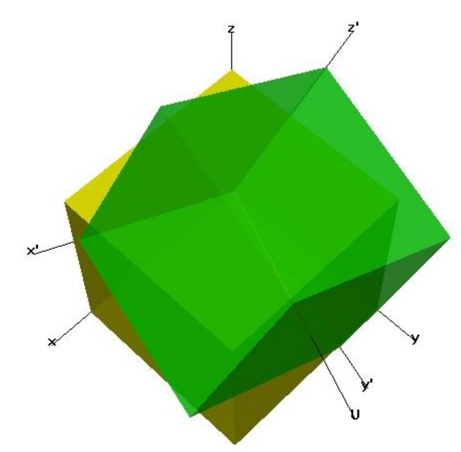

Figure 4. Two rotations that differ by a misorientation angle of 0.5 about the axis U

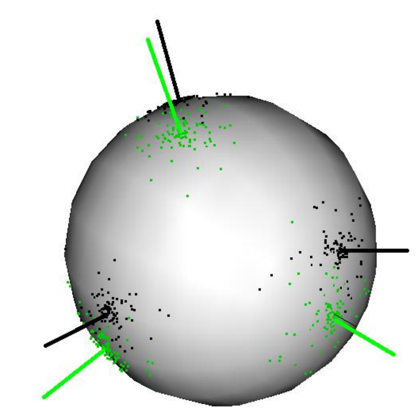

Figure 5. Sphere plot showing two sets of three-dimensional rotation data (k=20, n=100) with mean rotations represented by the axes

Outline of the Three-Dimensional Permutation Test

H_o (Null Hypothesis): No difference between the centers of the datasets
H_a (Alternative Hypothesis): There is a difference between the centers

1. **Calculate the observed misorientation angle from the actual data.**
2. **Permute the data 10000 times by randomly assigning the rotation data points to the two groups. Store the simulated misorientation angle each time.**
3. **Count the amount of times the simulated misorientation angle is greater than the observed misorientation angle.**
4. **Calculate p-value = (# times simulated angle > observed angle) /10000**
5. **If the p-value is less than 0.05, reject the null hypothesis and conclude that the centers of the two rotational datasets differ.**

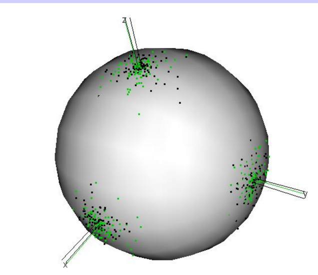

Figure 6. Example of two three-dimensional rotational datasets that do not differ. Fail to reject H_o. (k=20, n=100).

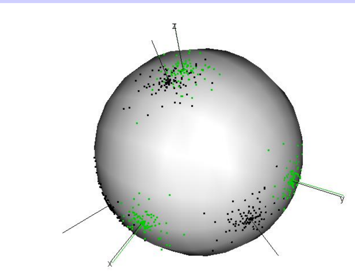

Figure 7. Example of two three-dimensional rotational datasets that differ. Reject H_o. (k=20, n=100).

4. Power

- The power of a statistical inference test is the probability that it will correctly reject the null hypothesis, H_o (i.e. reject the null when the null is really false).
- The power of the Three-Dimensional Permutation Test was analyzed using a simulation study which simulated rotation data while using different combinations of concentration parameter (k) and sample size (n).

Power (k = 5)

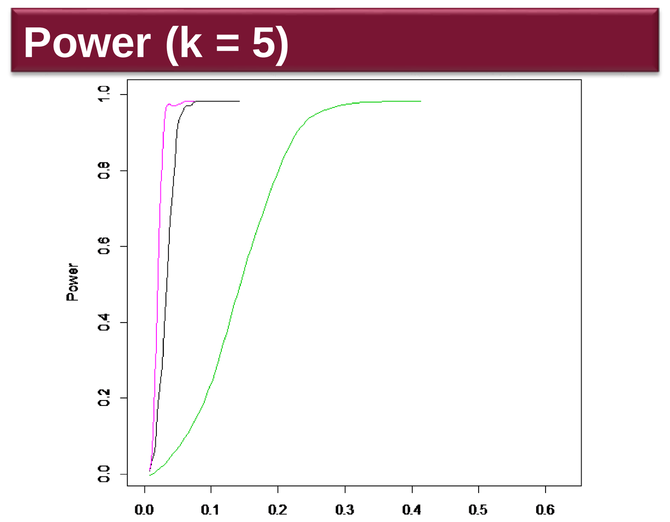

Figure 8. Power of the 3-D Permutation Test when k = 5.

Power (k = 20)

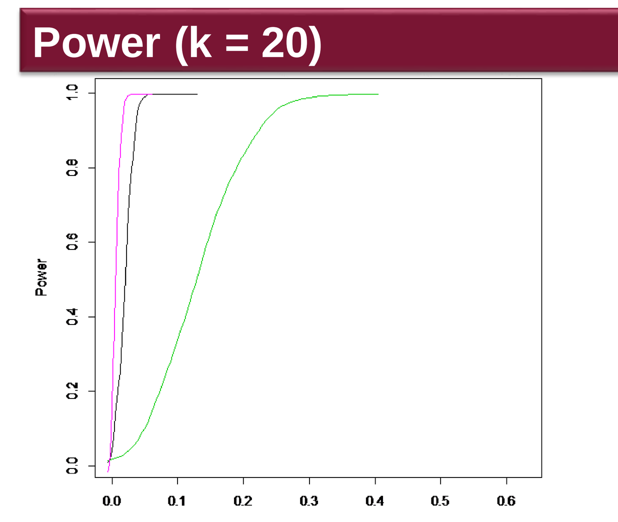

Figure 9. Power of the 3-D Permutation Test when k = 20.

Power (k = 50)

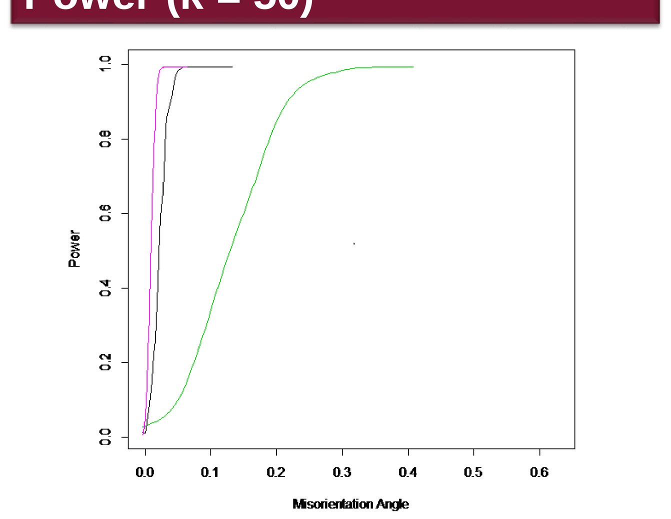

Figure 10. Power of Misorientation Angle Permutation Test when k = 5

Power (k = 100)

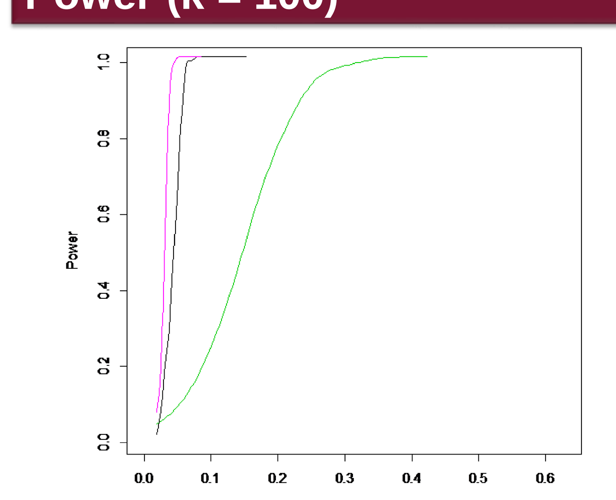

Figure 11. Power of Misorientation Angle Permutation Test when k = 100

Key: For Figures 8-11

n = 10 n = 100

n = 50

- The faster the power curve approached 1, the more powerful the test will be at rejecting the null hypothesis.
- Therefore, as n increases the test is more likely to reject H_o.

- While the sample size (n) has a noticeable effect on the power of the Three-Dimensional Permutation Test ,the concentration parameter (k) does not.

5. Future Applications/Acknowledgments

- In the future, Dr. Bingham and I hope to apply the Three-Dimensional Permutation Test to rotational data collected by the Physical Therapy Department at UW-La Crosse.
- This research was funded by NSF grant #1104409.