The Odds of Getting Slayer-Locked

Platpus3000

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Let p be the probability that you can complete any given task. Let $N(t)$ be the amount of points you would have on task t, allowing negative points. Let L be the event that you get locked at some point (that is, $N(t) < 0$ for some t). Let φ the amount of points awarded for completion, and let β be the cost to skip a task. This problem is interestingly much more difficult if φ and β are irrational (we will see why shortly); fortunately, this is not the case for any slayer master, at the moment[1]. We may further assume without loss of generality that φ and β are integers, as the probability of getting locked does not change if we multiply both φ and β by the product of their denominators, making both integers.

For any time t and $n \geq 0$,

$$\begin{aligned} P(L|N(t)=n) = {} & P(L|N(t)=n, N(t+1)=N(t)+\varphi)P(N(t+1)=N(t)+\varphi|N(t)=n) \\ & + P(L|N(t)=n, N(t+1)=N(t)-\beta)P(N(t+1)=N(t)-\beta|N(t)=n). \end{aligned}$$

Now, $N(t+1) - N(t)$ is independent of $N(t)$, and $P(N(t+1)=N(t)+\varphi) = p$, and $P(N(t+1)=N(t)-\beta) = 1-p$, so

$$P(L|N(t)=n) = pP(L|N(t)=n, N(t+1)=n+\varphi) + (1-p)P(L|N(t)=n, N(t+1)=N(t)-\beta).$$

This is wrong! Furthermore, whether you get locked depends only on your most recent number of points, so

$$P(L|N(t)=n) = pP(L|N(t+1)=n+\varphi) + (1-p)P(L|N(t+1)=n-\beta).$$

If $n \geq \beta$, then we can't say much more; if $n < \beta$, however, we know that $P(L|N(t+1)=n-\beta) = 1$, as you have already been locked. Let $a_n = P(L|N(t)=n)$. Then

$$a_n = pa_{n+\varphi} + (1-p)a_{n-\beta}$$

[1] It would be very strange, indeed, if point rewards or skip costs were *ever* irrational: imagine a slayer master that charged a reduced rate of $\sqrt{43}$ points to skip a task!

if $n \geq \beta$, and

$$a_n = pa_{n+\varphi} + 1 - p$$

if $n < \beta$. Since $P(L) = P(L|N(0)=0) = a_0$ (because $N(0)$ is always 0), we have

$$\begin{aligned}
a_0 = P(L) &= \sum_{n=-\infty}^{\infty} P(L|N(t)=n)P(N(t)=n) \\
&= \sum_{n=1}^{\infty} P(L|N(t)=-n)P(N(t)=-n) + \sum_{n=0}^{\infty} a_n P(N(t)=n) \\
&= \sum_{n=1}^{\infty} P(N(t)=-n) + a_0 P(N(t)=0) + \sum_{n=1}^{\infty} a_n P(N(t)=n) \\
&= P(N(t)<0) + a_0 P(N(t)=0) + \sum_{n=1}^{\infty} a_n P(N(t)=n).
\end{aligned}$$

for any t. Then, for any $t > 0$,

$$a_0 = \frac{1}{1 - P(N(t)=0)} \left[P(N(t)<0) + \sum_{n=1}^{\infty} a_n P(N(t)=n) \right].$$

The probability $P(N(t)<0)$ is actually not too hard to compute; $N(t) < 0$ if and only if you completed u tasks and could not complete v tasks and $N(t) = u\varphi - v\beta \leq -1$. Since $t = u + v$, this is equivalent to $u \leq \frac{\beta t - 1}{\beta + \varphi}$. The number u of doable tasks in t tasks is binomially distributed with parameters p and t, so we get

$$P(N(t)<0) = F_{p,t}\left(\frac{\beta t - 1}{\beta + \varphi}\right),$$

where $F_{p,t}$ is the CDF of a binomial random variable with parameters p and t.

We can also compute $P(N(t)=0)$ in a similar way. In fact, $N(t) = 0$ if and only if you completed u tasks and could not complete v, and $N(t) = u\varphi - \beta v = 0$. Again using the fact that $t = u + v$, this equivalent to $u = \frac{\beta t}{\beta + \varphi}$. Let $g = \gcd(\beta, \varphi)$, and write $\beta = bg$ and $\varphi = fg$. Since $b \nmid (b + f)$ because b and f must be relatively prime, it follows that $u = \frac{\beta t}{\beta + \varphi} = \frac{bt}{b+f}$ is an integer if and only if $(b + f) \mid t$. That is, $N(t) = 0$ can only occur when t is a multiple of $b + f$. Then we get

$$P(N(t) = 0) = \begin{cases} \binom{t}{\frac{\beta t}{\beta + \varphi}} p^{\frac{\beta t}{\beta + \varphi}} (1 - p)^{\frac{\varphi t}{\beta + \varphi}} & (b + f) \mid t, \\ 0 & (b + f) \nmid t. \end{cases}$$

Finally, we also compute $P(N(t)=n)$ for $n \geq 1$ using a similar technique. Similar to before, $N(t) = n$ if and only if you completed u tasks and could not complete

$v = t - u$, and $u\varphi - v\beta = n$, that is, $u = \frac{\beta t+n}{\beta+\varphi}$. Again, u is an integer, so we need $\frac{\beta t+n}{\beta+\varphi}$ also to be an integer. Let $n = dg + r$, where $0 \le r < g$. Then $N(t) = n$ if and only if $u = \frac{bt+d+\frac{r}{g}}{b+f}$, which can only be an integer if $r = 0$. Indeed, if $r \neq 0$, then $\frac{r}{g} \in (0, 1)$, so the numerator $(bt + d + \frac{r}{g})$ in the previous expression would be a non-integer, and a non-integer divided by an integer $(b + f)$ is still a non-integer by the closure of $\mathbb{Z}$ under multiplication.

Thus, $P(N(t){=}n)$ is nonzero if and only if $n = dg$ is a multiple of g, and $u = \frac{bt+d}{b+f}$ is an integer. Then we require that $(b + f) \mid (bt + d)$, so $bt + d = k(b + f)$ for some integer k, or $d = (k - t)b + kf$. Since u is distributed binomially with parameters p and t, we also require that $0 \le u \le t$ to obtain a nonzero probability, which is equivalent to saying that $1 \le n \le \varphi t$, and $1 \le d \le ft$, which is equivalent to $\frac{bt+1}{b+f} \le k \le t$. Thus, $P(N(t) = n)$ is nonzero if and only if $n = dg = (k - t)\beta + k\varphi$ for $\left\lceil \frac{bt+1}{b+f} \right\rceil \le k \le t$. Hence,

$$P(N(t){=}n) = \begin{cases} \binom{t}{\frac{\beta t+n}{\beta+\varphi}} p^{\frac{\beta t+n}{\beta+\varphi}} (1-p)^{\frac{\varphi t-n}{\beta+\varphi}} & n = k(\beta+\varphi) - \beta t,\ \left\lceil \frac{bt+1}{b+f} \right\rceil \le k \le t, \\ 0 & \text{otherwise.} \end{cases}$$

Combining these facts with the original equation gives, for any $t > 0$,

$$a_0 = \frac{1}{1 - P(N(t){=}0)} \left[P(N(t){<}0) + \sum_{k=\left\lceil \frac{bt+1}{b+f} \right\rceil}^{t} a_{(k-t)\beta+k\varphi} P(N(t){=}(k-t)\beta + k\varphi) \right]. \tag{1}$$

In this equation, we know everything except for $\{a_n\}$. Some basic facts, however may be derived. It is obvious that $P(N(t){=}0) \to 0$ as $t \to \infty$. Since $a_n \ge 0$ for all n (by virtue of being a probability), it follows that

$$a_0 \ge \lim_{t\to\infty} P(N(t){<}0) = \lim_{t\to\infty} F_{p,t}\left(\frac{\beta t - 1}{\beta + \varphi}\right).$$

By the central limit theorem, we have

$$F_{p,t}(x) \to \Phi\left(\frac{x - tp}{\sqrt{tp(1-p)}}\right)$$

uniformly in x as $t \to \infty$, where Φ is the CDF of the standard normal distribution. Then

$$a_0 \ge \lim_{t\to\infty} \Phi\left(\frac{\frac{\beta t-1}{\beta+\varphi} - tp}{\sqrt{tp(1-p)}}\right) = \begin{cases} 1 & \frac{\beta}{\beta+\varphi} > p, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Since $a_0 = P(L)$ is a probability, we also have $a_0 \leq 1$; thus, you are guaranteed to get slayer-locked at some point if $\frac{\beta}{\beta+\varphi} > p$.

If $\frac{\beta}{\beta+\varphi} \leq p$, then (2) tells us nothing, as we already know that the probability $a_0 \geq 0$ in any case. We can also use the fact that $a_n \leq 1$ for all n (by virtue of being a probability) to obtain the upper bound

$$a_0 \leq \frac{1}{1 - P(N(t){=}0)} \left[P(N(t){<}0) + \sum_{k=\left\lceil \frac{bt+1}{b+f} \right\rceil}^{t} P(N(t){=}(k-t)\beta + k\varphi) \right].$$

We can again take the limit as $t \to \infty$, in which case we again have $P(N(t){=}0) \to 0$, and, by the assumption $\frac{\beta}{\beta+\varphi} \leq p$, we also have $P(N(t){<}0) \to 0$. Thus,

$$\begin{aligned}
a_0 &\leq \lim_{t\to\infty} \sum_{k=\left\lceil \frac{bt+1}{b+f} \right\rceil}^{t} P(N(t){=}(k-t)\beta + k\varphi) \\
&\leq \lim_{t\to\infty} \sum_{k=\left\lceil \frac{bt+1}{b+f} \right\rceil}^{t} \binom{t}{k} p^k (1-p)^{t-k} \\
&\leq \lim_{t\to\infty} \left[1 - F_{p,t}\left(t - \left\lceil \frac{bt+1}{b+f} \right\rceil - 1 \right) \right] \\
&\leq 1 - \lim_{t\to\infty} \Phi\left(\frac{t - \left\lceil \frac{bt+1}{b+f} \right\rceil - 1 - tp}{\sqrt{tp(1-p)}} \right)
\end{aligned}$$