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Math 311W  
Worksheet for April 20-21

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**Exercise 1)** Let  $f : \mathbb{N} \rightarrow \mathbb{Z} \setminus \{0\}$ . be defined as  $f(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{-n}{2} & \text{if } n \text{ is even} \end{cases}$

Prove that  $f$  is a bijective function (one-to-one and onto).

(In words,  $f$  is the function that changes odd natural numbers into positive integers and even natural numbers into negative integers.)

Let  $x, y \in \mathbb{N}$  such that  $f(x) = f(y)$

Case 1, x,y odd:

$$\begin{aligned} f(x) &= \frac{x+1}{2} \\ f(y) &= \frac{y+1}{2} \\ \frac{x+1}{2} &= \frac{y+1}{2} \\ \implies x &= y \end{aligned}$$

Case 2, x,y even:

$$\begin{aligned} f(x) &= \frac{-x}{2} \\ f(y) &= \frac{-y}{2} \\ \frac{-x}{2} &= \frac{-y}{2} \\ \implies x &= y \end{aligned}$$

Therefore, the function is one-to-one

Onto: Let  $x \in \mathbb{Z} \setminus \{0\}$

Case 1: Positive

consider  $2x - 1 \in \mathbb{N}$

$$f(2x - 1) = \frac{2x - 1 + 1}{2} = x$$

Case 2: Negative

Consider  $-2x \in \mathbb{N}$

$$f(-2x) = \frac{-(-2x)}{2} = x$$

Therefore, the function is onto. □

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**Exercise 2)** Let  $f : \mathbb{Z} \rightarrow \mathbb{N}$  be the function defined as  $f(n) = \begin{cases} 2^n & \text{if } n \geq 0 \\ 3^{-n} & \text{if } n < 0 \end{cases}$

Prove that  $f$  is a one-to-one function.

(You can use properties of exponential functions.)

(According to the theorem proved in class, this shows that  $\mathbb{N}$  has at least as many elements as  $\mathbb{Z}$ )

Let  $x, y \in \mathbb{Z}$  such that  $f(x) = f(y)$

Case 1:  $x, y \geq 0$

$$f(x) = 2^x$$

$$f(y) = 2^y$$

$$2^x = 2^y$$

$$x = y$$

Case 2:  $x, y < 0$

$$f(x) = 3^{-x}$$

$$f(y) = 3^{-y}$$

$$3^{-x} = 3^{-y}$$

$$x = y$$

Therefore, the function is one-to-one. □

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**Exercise 3)** Let  $n$  be an integer number with  $n \geq 1$ . the number  $n^3 - n^2$  is always even.

Base Case:

$$1 - 1 = 0$$

It is true for the base case.

Assume there is some  $k \geq 1$ ,  $k^3 - k^2 = 2t$ .

Is this true for  $k + 1$ ?

$$\begin{aligned} & (k+1)^3 - (k+1)^2 \\ &= (k^3 + 3k^2 + 3k + 1) - (k^2 + 2k + 1) \\ &= (k^3 - k^2) + k^2 - k \\ &= 2t + k(k-1) \end{aligned}$$

so it holds for all  $k \geq 1$ .

Therefore by P.M.I. it is true for all  $n \geq 1$  □

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**Exercise 4)** The graphs of the functions  $f(x) = x^3$  and  $g(x) = -x^2 - 2x$  have a unique point of intersection.

$$\begin{aligned} x^3 &= -x^2 - 2x \\ 0 &= -x^2 - 2x - x^3 \\ &= -x(x^2 + x + 2) \\ \implies x &= 0 \end{aligned}$$

Since the discriminant is negative,

there are no real solutions to the quadratic

This is unique because of the algebraic steps used to find the solution. □

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