Math 311W
Worksheet for March 25

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Exercise 1) Let A, B, and C be three sets. Prove:

(a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

First we'll show that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$

$$\begin{aligned}
&\text{Let } (a, x) \in A \times (B \cap C) \\
&\text{Let } x \in B \cap C \\
&\implies x \in B \land x \in C \\
&\implies (a, x) \in A \times B \land (a, x) \in A \times C \\
&\implies (a, x) \in (A \times B) \cap (A \times C)
\end{aligned}$$

So, $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$

Now we'll show $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$

$$\begin{aligned}
&\text{Let } (a, x) \in (A \times B) \cap (A \times C) \\
&\implies (a, x) \in A \times B \land (a, x) \in A \times C \\
&\implies x \in B \land x \in C \\
&\implies x \in B \cap C \\
&\implies (a, x) \in A \times (B \cap C)
\end{aligned}$$

So, $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$

Therefore they are equivalent. $\square$

(b) $A \times (B \setminus C) \subseteq (A \times B) \setminus (A \times C)$

$$\begin{aligned}
&\text{Let } (a, x) \in A \times (B \setminus C) \\
&\implies x \in B \land x \notin C \\
&\implies (a, x) \in A \times B \land (a, x) \notin A \times C \\
&\implies (a, x) \in (A \times B) \setminus (A \times C)
\end{aligned}$$

Therefore $A \times (B \setminus C) \subseteq (A \times B) \setminus (A \times C)$ $\square$

Exercise 2) Let A = {3,5,7}, B = {1,2,3}, C= {3,7,9}, and U = {1,2,3,4,5,6,7,8,9,10}.

(a) Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$ by finding both sets.

$$\begin{aligned} A \times (B \cup C) =& \{3,5,7\} \times (\{1,2,3\} \cup \{3,7,9\}) \\ =& \{3,5,7\} \times \{1,2,3,7,9\} \\ =& \{(3,2),(5,1),(3,1),(3,7),(5,7),(5,3),(5,9),(3,3),(3,9),(7,1),(7,7),(7,3), \\ & (7,9),(7,2),(5,2)\} \\ (A \times B) \cup (A \times C) =& (\{3,5,7\} \times \{1,2,3\}) \cup (\{3,5,7\} \times \{3,7,9\}) \\ =& \{(3,2),(5,1),(3,1),(5,3),(3,3),(7,1),(7,3),(7,2),(5,2)\} \\ \cup& \{(5,7),(3,7),(5,3),(5,9),(3,3),(3,9),(7,7),(7,3),(7,9)\} \\ =& \{(3,2),(5,1),(3,1),(5,7),(3,7),(5,3),(5,9),(3,3),(3,9),(7,1),(7,7),(7,3), \\ & (7,9),(7,2),(5,2)\} \end{aligned}$$

(b) Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$ by finding both sets.

$$\begin{aligned} A \times (B \cap C) &= \{3,5,7\} \times (\{1,2,3\} \cap \{3,7,9\}) \\ &= \{3,5,7\} \times \{3\} \\ &= \{(7,3),(5,3),(3,3)\} \\ (A \times B) \cap (A \times C) &= (\{3,5,7\} \times \{1,2,3\}) \cap (\{3,5,7\} \times \{3,7,9\}) \\ &= \{(3,2),(5,1),(3,1),(5,3),(3,3),(7,1),(7,3),(7,2),(5,2)\} \\ &\cap \{(5,7),(3,7),(5,3),(5,9),(3,3),(3,9),(7,7),(7,3),(7,9)\} \\ &= \{(7,3),(5,3),(3,3)\} \end{aligned}$$

Exercise 3) Let A_1, A_2, A_3 be three subsets of a universal set U, such that $A_1 \cap A_2 = \emptyset, A_1 \cap A_3 = \emptyset, A_3 \cap A_2 = \emptyset$. These sets are disjoint when taken in pairs (pairwise disjoint). Find $A_1 \cap A_2 \cap A_3$, and explain how you reach your conclusion.

$A_1 \cap A_2 \cap A_3 = \emptyset$

Since none of the sets have any of the same elements from

$A_1 \cap A_2 = \emptyset, A_1 \cap A_3 = \emptyset, A_3 \cap A_2 = \emptyset$,

the intersection between all of the sets is an empty set.

Exercise 4) Let A and B be two finite sets of natural numbers.

(a) Explain why it is possible that $|A \cup B| \neq |A| + |B|$.

Claim: The equality holds if and only if $A \wedge B$ are disjoint.

Proof: Assume A and B are not disjoint, with $|A| = q, |B| = p$

$$\begin{aligned}
&\text{Let } x \in A \wedge x \in B \\
&\implies x \in (A \cup B) \\
&\text{Let } X = \{x\} \\
&\quad |X| = 1 \\
|(A \cup B)| =& (|A \setminus X| + |B \setminus X| + |X|) \textit{ (Because they are not disjoint)} \\
=& (q-1) + (p-1) + 1 \\
=& q + p - 1 \\
\neq& q + p = |A| + |B|
\end{aligned}$$

Therefore, when A and B are not disjoint, the equality does not hold. □

Now we'll show that when A and B are disjoint, the equality does hold:

Let B be a set with $|B| = p$.

Base Case:

$$\begin{aligned}
&\text{Let } |A| = 0 \\
&\quad |A| + |B| = 0 + p \\
&\quad |(A \cup B)| = |B| = p
\end{aligned}$$

Assume it works for some $|A| = n \in \mathbb{Z}$. Does it hold for n+1

$$\begin{aligned}
&\text{Let } x \in A \\
&\text{Let } X = \{x\} \\
&\quad |X| = 1 \\
&\quad |(A \cup B)| = |((A \setminus X) \cup X \cup B)| \\
&= |(A \setminus X) \cup B)| + 1 \\
&= |A \setminus X| + |B| + 1 \\
&= |A| + |B|
\end{aligned}$$

Therefore, it holds for the inductive step, so it holds for sets of all cardinalities.

(b) Under which conditions do you think that $|A \cup B| = |A| + |B|$? Make a conjecture, you don't have to prove it.

Proved in part A