Math 311W
Worksheet for April 6-7

Jacob Harkins
jah6863@psu.edu

May 20, 2024

Exercise 1) A budding mathematician was trying to remember the rules of operation between sets using the rules of operations between numbers as an example. Are the following "simplification" rules true or false? Investigate each one of them to decide whether it is true or false. Provide a proof or a counterexample.
Let A, B, C be three subsets of a universal set U.

(a) If $A \cap B = A \cap C, \implies B = C$

This is false, Consider:

$$\begin{aligned} A = \{2,3\}, B = \{3,4\}, C = \{3,8\} \\ A \cap B = A \cap C = \{3\} \\ B \neq C \end{aligned}$$

(b) If $A \cup B = A \cup C, \implies B = C$

This is false, Consider:

$$\begin{aligned} A = \{1,2\}, B\{1\}, C\{2\} \\ A \cup B = A \cup C = \{1,2\} \\ B \neq C \end{aligned}$$

(c) If $A \times B = A \times C, \implies B = C$

This is true, proof by double inclusion:

$$\begin{aligned} &\text{Let } x \in B, a \in A \\ &\implies (a,x) \in A \times B \\ &\implies (a,x) \in A \times C \\ &\implies x \in C \\ &\implies B \subseteq C \\ \\ &\text{Let } x \in C, a \in A \\ &\implies (a,x) \in A \times C \\ &\implies (a,x) \in A \times B \\ &\implies x \in B \\ &\implies C \subseteq B \\ \\ &\implies B = C \end{aligned}$$

$\square$

(d) If $B' = C'$, $\implies B = C$

This is true, proof by double inclusion:

$$\begin{aligned} &\text{Let } x \in B \\ &\implies x \notin B' \\ &\implies x \notin C' \\ &\implies x \in C \\ \\ &\text{Let } x \in C \\ &\implies x \notin C' \\ &\implies x \notin B' \\ &\implies x \in B \end{aligned}$$

$$\implies B = C$$

$\square$

Exercise 2) Let A={integer numbers}=¢. On it define the relation D as follows:
Let a and b be elements of ¢.

a D $b \iff a$ and b have the same number of digits(i.e. number of digits of a=number of digits of b).
Check which ones of the four properties (reflexive, symmetric, transitive, antisymmetric) D has. Provide a proof or a counterexample to support your conclusion.

Assume each element is expressed in base 1, then the number of digits in a number n, is exactly n. Let F(x)=x be a function which counts the number of digits in x.

(a) Reflexive

D is reflexive trivially:

$$F(a) = F(a)$$

(b) Symmetric

D is symmetric:

$$\begin{aligned} &\text{Assume } aDb \\ &\quad \implies F(a) = F(b) \\ &\quad \implies F(b) = F(a) \\ &\quad \implies bDa \end{aligned}$$

(c) Transitive

D is transitive:

$$\begin{aligned} &\text{Let } c \in A \\ &\text{Assume } aDb \wedge bDc \\ &\quad \implies F(a) = F(b) \wedge F(b) = F(c) \\ &\quad \implies F(a) = F(c) \\ &\quad \implies aDc \end{aligned}$$

(d) Antisymmetric

D is antisymmetric:

$$\begin{aligned}&\text{Assume } aDb \wedge bDa \\ &\implies F(a) = F(b) \wedge F(b) = F(a) \\ &\implies a = b\end{aligned}$$

Exercise 3) On the set A ={integer numbers} = ¢, define the relation S as follows:

Let a and b be elements of ¢.

$$a \text{ S } b \iff b = a^2$$

Check which ones of the three properties (reflexive, symmetric, transitive) S has. Provide a proof or a counterexample to support your conclusion.

(a) Reflexive

S is not Reflexive, Consider $a = 2$:

$$2 \neq 2^2 \implies a \not{S}\, a$$

(b) Symmetric

S is not Symmetric, Consider $a = 2, b = 4$:

$$\begin{aligned}4 = 2^2 &\implies aSb \\ 2 \neq 4^2 &\implies b \not{S}\, a\end{aligned}$$

(c) Transitive

S is not Transitive, Consider $a = 2, b = 4, c = 16$:

$$\begin{aligned}4 = 2^2 \wedge 16 = 4^2 &\implies aSb \wedge bSc \\ 16 \neq 2^2 &\implies a \not{S}\, c\end{aligned}$$

Exercise 4) Let A={all polynomials of degree 5 with real coefficients} On it define the relation Z as follows:

Let P and Q be elements of A .

$$PZQ \iff P \text{ has the same zeros as } Q$$

Check which ones of the four properties (reflexive, symmetric, transitive,antisymmetric) Z has. Provide a proof or a counterexample to support your conclusion.

(a) Reflexive

Z is reflexive trivially:

(b) Symmetric

Z is Symmetric, proof:

Assume PZQ

$$\begin{aligned}&\text{Let } p_1, p_2, p_3, p_4, p_5 \text{ be the roots of } P \\ &\implies p_1, p_2, p_3, p_4, p_5 \text{ are the roots of } Q \\ &\implies QZP\end{aligned}$$

(c) Transitive

Z is Transitive, proof:

Let $\Delta \in A$

Assume $PZQ \land QZ\Delta$

$$\begin{aligned} & p_1, p_2, p_3, p_4, p_5 \text{ are the roots of P} \\ & \implies p_1, p_2, p_3, p_4, p_5 \text{ are the roots of Q} \\ QZ\Delta & \implies p_1, p_2, p_3, p_4, p_5 \text{ are the roots of } \Delta \implies PZ\Delta \end{aligned}$$

(d) Antisymmetric

Z is not Antisymmetric, Consider Q = -P:

$$\begin{aligned} & PZQ \land QZP \\ & Q \neq P \end{aligned}$$