Math 311W
Worksheet for April 13-14

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Exercise 1) Let $A = \{1,2,3,4,5\}$. Consider the collection of all subsets of A, part of A, $\wp(A)$,(It is not necessary to list all of them explicitly, but it might help. There are 32 subsets.) and on it consider the relation

$$XRY \iff X \cap Y \neq \varnothing$$

Which of the four properties does this relation have or not have? Explain.

The powerset is :$\mathcal{P}(A) = \big\{\{1, 3, 5\}, \{1, 4\}, \{2, 3, 4\}, \{2, 3\}, \{4, 5\}, \{1\}, \{3, 4\}, \{1, 2, 3, 4, 5\}, \{2, 4\}, \{2, 3, 5\}, \{3, 5\}, \{3, 4, 5\}, \{1, 2, 3\}, \{1, 3\}, \{1, 3, 4\}, \{1, 2, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 5\}, \{1, 5\}, \{1, 4, 5\}, \{1, 2, 3, 4\}, \{5\}, \{4\}, \{2\}, \{1, 3, 4, 5\}, \{1, 2\}, \{1, 2, 4\}, \{1, 2, 3, 5\}, \{2, 4, 5\}, \{2, 5\}, \{3\}, \{\}\big\}$

The relation is not reflexive, because $\emptyset \not R \emptyset$

This relation is symmetric,

$$\begin{aligned} &\text{Assume } XRY \\ &\iff X \cap Y \neq \emptyset \\ &\iff Y \cap X \neq \emptyset \\ &\iff YRX \end{aligned}$$

This relation is not Transitive,

$$\begin{aligned} &\text{Let } X = \{1\}, Y = \{2\}, Z = \{2, 3\} \\ &YRZ \not\Rightarrow YRX \end{aligned}$$

This relation is not antisymmetric, consider $X = \{1, 2\} Y = \{2, 3\}$

$$\begin{aligned} XRY \wedge YRX \\ X \neq Y \end{aligned}$$

Exercise 2) Let U be a universal set. Define the following relation between the subsets of U:

$$ARB \iff B \text{ has at least as many elements as } A$$

$|A| \leq |B| \iff B$ has at least as many elements as A

Which of the four properties does this relation have or does not have? Explain.

- This relation is reflexive, for any A, $|A| = |A|$ Hence, ARA;
- It is not symmetric, Let $A = \{1\}, B = \{1, 2\}$ then $|A| \leq |B|, |B| \not\leq |A|$;

- transitive, If $|A| \leq |C| \leq |B| \iff |A| \leq |B|$
- It is not antisymmetric, If $A = \{1\}, B = \{2\}$ then $(ARB\ BRA) \wedge A \neq B$

Exercise 3) Consider the function $f : A \to B$, where $A = \{$all multiples of 2$\}$ and B = $\{$all multiples of 10$\}$ be defined as $f(n) = 5n$. Prove that f is one-to-one and onto.

One-to-One:
Let $x, y \in A$ with $f(x) = f(y)$

$$\begin{aligned} &f(x) = 5x,\ f(y) = 5y \\ \implies &x = y \end{aligned}$$

Onto:
Let $y \in B$

$$\begin{aligned} &y = 10n \\ \implies &f(2n) = y \\ &2n \in A \implies f \text{ is one-to-one} \end{aligned}$$

□

Exercise 4) There exists a unique prime number of the form $n^3 - 1$.

$$\begin{aligned} &n^3 - 1 \\ =&(n-1)(n^2+n+1) \\ \implies &(n-1) = 1 \vee (n^2+n+1) = 1 \\ \implies &(n-1) = 1 \\ \implies &n = 2 \\ \implies &2^3 - 1 = 7 \end{aligned}$$

□
