Math 311W
Worksheet for April 15-17

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Exercise 1) Prove or disprove. The number $3^n - 1$ is divisible by 2 for all natural numbers n

Base Case, $n = 0$:
$3^0 - 1 = 0 = 2(0)$
This holds true, Assume $3^n - 1 = 2k$ for some $n, k \geq 0$.
Is this true for $n + 1$? Lets find out!

$$\begin{aligned} &3^{n+1} - 1 \\ =&3(3^n) - 1 \\ =&3(3^n) - 3 + 2 \\ =&3(3^n - 1) + 2 \\ =&3(2k) + 2 \\ =&2(3k + 1) \end{aligned}$$

This is true for $n + 1$, Hence by P.M.I. this is true for all $n \geq 0$. □

Exercise 2) Prove or disprove. Let a be a positive integer number. If 3 does not divide a, then 3 divides $a^2 - 1$. Proof Since a is not divisible by 3, $a = 3k \pm 1$.

$$\begin{aligned} &a^2 - 1 \\ =&(3k \pm 1)^2 - 1 \\ =&(9k^2 \pm 6k + 1) - 1 \\ =&3(3k^2 \pm 2k) \end{aligned}$$

Therefore, $a^2 - 1$ is divisible by 3. □

Exercise 3 On the set of all integers $\mathbb{Z}$ consider the relation:

$$a \; R \; b \iff a - 3b \text{ is an even number}$$

Which of the four properties (reflexive, symmetric, transitive, and antisymmetric) does this relation have

- Reflexive: Let $a \in \mathbb{Z}$

$$\begin{aligned} &a - 3a \\ =&a(1 - 3) \\ =&2(-a) \\ \implies& a \; R \; a \end{aligned}$$

- Symmetric: Let $a, b, s \in \mathbb{Z}$ with $a\ R\ b$.

$$\begin{aligned} a - 3b &= 2s \\ a &= 2s + 3b \end{aligned}$$

$$\begin{aligned} b - 3a &= b - 3(2s + 3b) \\ &= 6s - 8b \\ &= 2(3s - 4b) \\ \implies &b\ R\ a \end{aligned}$$

- Transitive: Let $a, b, c \in \mathbb{Z}$ with $a\ R\ b \wedge b\ R\ c$.

$$\begin{aligned} b - 3c &= 2s \\ b &= 2s + 3c \end{aligned}$$

$$\begin{aligned} a - 3b &= 2n \\ a - 3(2s + 3c) &= 2n \\ a - 9c &= 6s + 2n \\ a - 3c &= 6s + 2n + 6c \\ a - 3c &= 2(3s + n + 3c) \\ \implies &a\ R\ c \end{aligned}$$

- R is not antisymmetric: Consider $a = 1, b = 3$.

$$\begin{aligned} 1 - 9 &= -8 \\ 3 - 3 &= 0 \end{aligned}$$

Therefore, $a\ R\ b \wedge b\ R\ a \wedge a \neq b$ $\square$

Exercise 4) Let $A = \{1, 2, 3, 4, 5, 6\}, B = \{4, 5, 6, 7\}$, and $C = \{0, 1, 2, 5\}$

(a) Use the roster notation to describe the following four sets(i.e. list all their elements)

  (a) $A_1 = A - (B \cup C)$
  $A_1 = \{3\}$
  (b) $A_2 = A \cap B \cap C$
  $A_2 = \{5\}$
  (c) $A_3 = A \cap (B - C)$
  $A_3 = \{4, 6\}$
  (d) $A_= A \cap (C - B)$
  $A_4 = \{1, 2\}$

(b) Prove that the collection $F = \{A_1, A_2, A_3, A_4\}$ is a partition of A.
F= {{3},{5},{4,6},{1,2}}

  (a) $\emptyset \notin F$
  (b) None of the parts in F share any elements, so they are pairwise disjoint.
  (c) $A_1 \cup A_2 \cup A_3 \cup A_4 = \{1, 2, 3, 4, 5, 6\} = A$