Math 311W
Worksheet for April 27-28

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Exercise 1) Construct the table for multiplication modulus 7 on $Z_7^*= \{[1],[2],[3],[4],[5],[6]\}$.

Assume that it is known that this operation is associative. Is (Z_7^*, x_7) a group? Provide explanations for your answers. (If you say yes, what is the identity? What is the inverse for each element?)

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

(Z_7^*, x_7) is a group:

Identity: 1

Inverse: 1 is 1, 2 is 4, 3 is 5, 4 is 2, 5 is 3, and 6 is 6.

Exercise 2) Construct the table for multiplication modulus 6 on $Z_6^*= \{[1],[2],[3],[4],[5]\}$

Assume that it is known that this operation is associative. Is (Z_6^*, x_6) a group? Provide explanations for your answers. (If you say yes, what is the identity? what is the inverse for each element?)

	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

(Z_6^*, x_6) is not a group:

There is no identity element.

Exercise 3) Prove the following statement.

Let $A = \{all\ integer\ multiples\ of\ 5\}$ and $B = \{all\ integer\ multiples\ of\ 4\}$.

There exists a bijective function between these two sets, and thus they have the same size. (construct a function, and show it is bijective)

Let $N = \{all\ integer\ multiples\ of\ n\}$:

$$f : N \to \mathbb{Z}$$
$$f(x) = \frac{x}{n}$$

We'll show that f is one-to-one:

$$\begin{aligned} f(a) = f(b) &\implies \frac{a}{n} = \frac{b}{n} \\ &\implies a = b \end{aligned}$$

We'll show that f is onto: Let $a \in \mathbb{Z}$

$$\begin{aligned} f(an) &= \frac{an}{n} \\ &= a \end{aligned}$$

Therefore for every n, there exists a bijection between the set of integer multiples of n and $\mathbb{Z}$.

Let $f : A \to \mathbb{Z}, g : \mathbb{Z} \to B$ be such bijections.

Then $g \circ f : A \to B$ is a bijection.(Note: $(g \circ f)(x) = \frac{4}{5}x$) $\square$

Exercise 4) Use a truth table to prove that the statement $A \to B$ and its contrapositive $(notB) \to (notA)$ are logically equivalent.

A	B	not A	not B	A→B	(not B)→(not A)
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

$\square$

Exercise 5) Prove the following statement: Let a and b be two rational numbers. if $a < b$, then there exists at least one rational number c with $a < c < b$.

Consider $\frac{a+b}{2}$

$$\begin{aligned} \frac{a+b}{2} &> \frac{a+a}{2} \\ &> a \\ \frac{a+b}{2} &< \frac{b+b}{2} \\ &< b \end{aligned}$$

$$\begin{aligned} \text{Let } a = \frac{p}{q},\ b &= \frac{s}{t} \text{ with } p, q, s, t \in \mathbb{Z} \\ \frac{a+b}{2} &= \frac{pt+sq}{2st} \\ &\implies \frac{a+b}{2} \in \mathbb{Q} \end{aligned}$$

$\square$

Exercise 6) Prove that for all natural numbers $n \geq 1$ the following inequality holds true:

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + ... + \sqrt{n} \leq n\sqrt{n}$$

Base Case, n=1:
$1 = 1\sqrt{1}$
So this holds true.
Assume there is some $k \geq 1$, such that $\sqrt{1} + \sqrt{2} + \sqrt{3} + ... + \sqrt{k} \leq k\sqrt{k}$
Is this true for $k+1$?

$$\begin{aligned}\sqrt{1} + \sqrt{2} + \sqrt{3} + ... + \sqrt{k} + \sqrt{k+1} &\leq k\sqrt{k} + \sqrt{k+1} \\ &< k\sqrt{k+1} + \sqrt{k+1} \\ &< (k+1)\sqrt{k+1}\end{aligned}$$

It holds for $k+1$
Therefore, by P.M.I. it holds for all $n \geq 1$. $\square$