Math 311W
Worksheet for March 30-31

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Exercise 1) The equality $\sum_{j=1}^{n} \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$ holds true for all $n \geq 1$. Base Case

$$\text{Let } n = 1$$
$$\frac{1}{1(2)} = \frac{1}{2}$$

This holds true.
Assume there is some $K \geq 1$ for which this holds true.
Is this true for $K+1$?

$$\begin{aligned}\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + ... + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\ &= \frac{(n+1)^2}{(n+1)(n+2)} \\ &= \frac{n+1}{n+2}\end{aligned}$$

So it holds true for $K+1$
Therefore, By the Principle of Math Induction it holds true for all $n \geq 1$. $\square$

Exercise 2) Consider the sequence of Fibonacci numbers defined as $f_1 = 1, f_2 = 1, f_{\text{n}} = f_{\text{n-2}} + f_{\text{n-1}}$ for all $n \geq 3$. Prove the following formula for the sum of the first n even placed Fibonacci numbers for all $n \geq 1$.

$$\underbrace{f_2 + f_4 + f_6 + ... + f_{\text{2n}}}_{\text{n terms}} = f_{\text{2n+1}} - 1$$

Base Case:

$$f_1 = 1$$
$$f_3 - 1 = 2 - 1 = 1$$

This holds true.
Inductive Hypotheis: Assume there exists a number $K \geq 1$ for which this is true.
Deductive step, is this true for $K+1$:

$$\begin{aligned}&f_1 + f_2 + ...f_{\text{k}} + f_{\text{k+1}} = f_{\text{k+2}} - 1 + f_{\text{k+1}} \\ &f_{\text{k}} = f_{\text{k-1}} + f_{\text{k-2}} \\ &f_{\text{k+3}} = f_{\text{k+2}} + f_{\text{k+2}} \\ \implies &f_1 + f_2 + ... + f_{\text{k}} + f_{\text{k+1}} = f_{\text{k+3}} - 1\end{aligned}$$

So it is true for $K+1$
By the principle of math induction, it is true for all $K \geq 1$. $\square$

Exercise 5) Let $A = \{1,2,3\}$ and $B = \{2,3\}$

(a) Construct $A \times B$

$$A \times B = \{(3,2),(2,3),(3,3),(1,3),(1,2),(2,2)\}$$

(b) Show that $A \times B \neq B \times A$

$$\begin{aligned} A \times B &= \{(3,2),(2,3),(3,3),(1,3),(1,2),(2,2)\} \\ B \times A &= \{(3,2),(3,1),(2,3),(3,3),(2,2),(2,1)\} \\ &\implies (1,3) \in A \times B \wedge (1,3) \notin B \times A \end{aligned}$$

Exercise 6) The number $8^n - 3^n$ is divisible by 5 for all $n \geq 2$.
Base Case:

$$\begin{aligned} &\text{Let } n = 2 \\ &8^2 - 3^2 = 55 \end{aligned}$$

It holds true for $n = 2$
Assume there exists a $K \geq 2$ for which $5t = 8^k - 3^k$ with $t \in \mathbb{N}$.
Is it true for $K+1$?

$$\begin{aligned} &\text{Let } s \in \mathbb{N} \\ &8^{K+1} - 3^{K+1} \\ =&8 \cdot 8^K - 3 \cdot 3^K \\ =&(5+3) \cdot 8^k - 3 \cdot 3^k \\ =&5 \cdot 8^k + 3 \cdot 8^k - 3 \cdot 3^k \\ =&5 \cdot 8^k + 3 \cdot (8^k - 3^k) \\ =&5 \cdot 8^k + 3 \cdot (5t) \\ =&5 \cdot (8^k + 3t) \end{aligned}$$

So it holds true for $K+1$.
By the princple of mathematical induction $8^n - 3^n$ is divisible by 5 for all $n \geq 2$. $\square$

Exercise 7) Find the parts (or power sets) of:

(a) $\emptyset$

$$P(\emptyset) = \{\}$$

(b) $A = \{1, c\}$

$$P(A) = \{\{\},\{1\},\{c\},\{1,c\}\}$$

Exercise 8) There exists a unique second degree polynomial $P(x) = a_2x^2 + a_1x + a_0$ such that $P(0) = 2, P(1) = -2, P(-1) = 0$

$$\begin{aligned}
P(0) &= a_2(0)^2 + a_1(0) + a_0 = 2 \\
P(0) &= a_0 = 2 \\
\implies P(x) &= a_2x^2 + a_1x + 2
\end{aligned}$$

$$\begin{aligned}
P(1) &= a_2 + a_1 + 2 = -2 \\
P(-1) &= a_2 - a_1 + 2 = 0 \\
P(1) + P(-1) &= 2a_2 + 4 = -2 \\
\implies a_2 &= -3 \\
\implies a_1 &= 1
\end{aligned}$$

So, there exists a second degree polynomial $P(x) = a_2x^2+a_1x+a_0$ such that $P(0) = 2, P(1) = -2, P(-1) = 0$
Since $a_2 = -3, a_1 = 1, a_2 = 0$ is the unique solution to the initial conditions, P(x) is unique. $\square$

Exercise 9) Suppose you are reading the proof of the statement "If n is a multiple of a and b, then n is a multiple of $a + b$."

(a) The beginning of the first sentence reads:
Since n is not a multiple of $a + b$ and n is a multiple of a...
What kind of proof is the writer using(contradiction, contrapositive,direct, incorrect)?

The writer is using proof by contrapositive

(b) The beginning of the first sentence reads:
Assume that n is a multiple of $a + b$ and n is not a multiple of a.
What kind of proof is the writer using (contradiction, contrapositive, direct, incorrect)?

The writer is using an incorrect proof

Exercise 10) State the Fundamental Theorem of Arithmetic.

The Fundamental Theorem of Arithmetic states that any positive integer $n \geq 1$ can be expressed as the product of one or more prime numbers.
