Math 311W
Worksheet for April 20-21

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Exercise 1) Let $f : \mathbb{N} \to \mathbb{Z} \setminus \{0\}$. be defined as $f(n) = \begin{cases} \frac{n+1}{2} & if\ n\ is\ odd \\ \frac{-n}{2} & if\ n\ is\ even \end{cases}$

Prove that f is a bijective function (one-to-one and onto).

(In words, f is the function that changes odd natural numbers into positive integers and even natural numbers into negative integers.)

Let $x, y \in \mathbb{N}$ such that $f(x) = f(y)$

Case 1, x,y odd:

$$\begin{aligned} f(x) &= \frac{x+1}{2} \\ f(y) &= \frac{y+1}{2} \\ \frac{x+1}{2} &= \frac{y+1}{2} \\ \implies x &= y \end{aligned}$$

Case 2, x,y even:

$$\begin{aligned} f(x) &= \frac{-x}{2} \\ f(y) &= \frac{-y}{2} \\ \frac{-x}{2} &= \frac{-y}{2} \\ \implies x &= y \end{aligned}$$

Therefore, the function is one-to-one

Onto: Let $x \in \mathbb{Z} \setminus \{0\}$

Case 1: Positive

consider $2x - 1 \in \mathbb{N}$

$$f(2x-1) = \frac{2x-1+1}{2} \qquad = x$$

Case 2: Negative

Consider $-2x \in \mathbb{N}$

$$f(-2x) = \frac{-(-2x)}{2} \qquad = x$$

Therefore, the function is onto. $\square$

Exercise 2) Let $f : \mathbb{Z} \to \mathbb{N}$ be the function defined as $f(n) = \begin{cases} 2^n & if\ n \geq 0 \\ 3^{-n} & if\ n < 0 \end{cases}$

Prove that f is a one-to-one function.
(You can use properties of exponential functions.)
(According to the theorem proved in class, this shows that $\mathbb{N}$ has at least as many elements as $\mathbb{Z}$)
Let $x, y \in \mathbb{Z}$ such that $f(x) = f(y)$
Case 1: $x, y \geq 0$

$$\begin{aligned} f(x) &= 2^x \\ f(y) &= 2^y \\ 2^x &= 2^y \\ x &= y \end{aligned}$$

Case 2: $x, y < 0$

$$\begin{aligned} f(x) &= 3^{-x} \\ f(y) &= 3^{-y} \\ 3^{-x} &= 3^{-y} \\ x &= y \end{aligned}$$

Therefore, the function is one-to-one. □

Exercise 3) Let n be an integer number with $n \geq 1$. the number $n^3 - n^2$ is always even.
Base Case:
$1 - 1 = 0$
It is true for the base case.
Assume there is some $k \geq 1$, $k^3 - k^2 = 2t$.
Is this true for $k + 1$?

$$\begin{aligned} &(k+1)^3 - (k+1)^2 \\ =&(k^3 + 3k^2 + 3k + 1) - (k^2 + 2k + 1) \\ =&(k^3 - k^2) + k^2 - k \\ =&2t + k(k-1) \end{aligned}$$

so it holds for all $k \geq 1$.
Therefore by P.M.I. it is true for all $n \geq 1$ □

Exercise 4) The graphs of the functions $f(x) = x^3$ and $g(x) = -x^2 - 2x$ have a unique point of intersection.

$$\begin{aligned} x^3 &= -x^2 - 2x \\ 0 &= -x^2 - 2x - x^3 \\ &= -x(x^2 + x + 2) \\ \implies x &= 0 \end{aligned}$$

Since the discriminant is negative,
there are no real solutions to the quadratic

This is unique because of the algebraic steps used to find the solution. □
